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Integer Programming, Marginal Revenue Productivity and Pricing of Resources

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Let B be an n-dimensional, square, integer matrix and N an integer matrix of order nxm. The matrices B and N are assumed to contain n unit vectors among their columns. Let c_B and c_N be n and m-dimensional integer row vectors, respectively, t and b be n-dimensional integer column vectors, and x be an m-dimensional integer column vector. Then consider the following linear programming problem:

(1) $\text{Max } (z = c_B \cdot t + c_N \cdot x)$

subject to

(2) $B \cdot t + N \cdot x = b$

$$t, x \geqq 0^{**}$$

The dual problem is:

(3) $\text{Min } (v = b \cdot p)$

subject to

(4) $$\begin{aligned} B' \cdot p &\leqq c_B \\ N' \cdot p &\leqq c_N \\ p &\geqq 0 \end{aligned}$$

where B' and N' are the transposes of B and N.

Assume that the optimal solution to the primal problem ((1) and (2)) is given by

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*This paper is an elaboration of part of my Ph.D. dissertation done for Princeton University in 1963. I would like to thank Professors Harold Kuhn and William Baumol for their help.

**Although all the vector and matrix constants are assumed to be integer, this entails only a slight loss of generality since any linear program with non-integer but rational constants is equivalent to a linear program obtained by multiplying all the constants by a common denominator.

$$(5)\quad t^* = B^{-1}\cdot b - B^{-1}\cdot x^*$$

$$x^* = 0$$

$$z^* = c_B\cdot B^{-1}\cdot b - (-c_N + c_B\cdot B^{-1}\cdot N)\cdot x^*$$

The optimal solution to the dual is

$$(6)\quad p^* = c_B\cdot B^{-1}$$

The matrix B is called the basis of the optimal solution. Variables t are basic variables and x are non-basic variables in the optimal solution.

Many properties of such a linear programming problem and its dual are well known. For example, the solution to the dual problem provides a set of prices (p^*) on the resources (b) which if applied to the amounts of resources used up by each activity result in zero profitability for activities (t) which enter the optimal solution at a positive level (i.e., $c_B - p^*\cdot B = c_B - c_B\cdot B^{-1}\cdot B = 0$) and non-positive profitability for activities (x) which do not enter the optimal solution (i.e., $c_N - p^*\cdot N = c_N - c_B\cdot B^{-1}\cdot N \leqq 0$). Furthermore, if the resources are changed (i.e., $b' = b + \Delta b$), the change in the objective function ($\Delta z(b) = z(b + \Delta b) - z(b)$) is less than or equal to the increased cost of resources when valued at the dual prices (i.e., $\Delta z(b) \leq p^*\cdot\Delta b$). (If the basis of the optimal solution remains the same under the transformation of b, then $\Delta z(b) = p^*\cdot\Delta b$.). This property of the dual prices ensures that it is never profitable to hire more or less resources if they are available at the dual prices (p^*). Finally, if only one resource is changed (say b_i is increased) then the marginal revenue productivity of that resource (the increase in the objective function z per unit increase in b_i for very small increases in b_i) is equal to the dual price of that resource (p_i^*) except in the case of degeneracy (i.e., if the basis of the optimal solution changes for any increase in b_i no matter how small). In the degenerate case, the marginal revenue productivity is less than or equal to the dual price (p_i^*) of the resource.

Some of the above properties do not apply to the case in which the variables t and x are required to be integer, i.e., to integer programming problems. There does exist, however, a set of prices for the resources and for the activities in an integer program which makes every activity profitless and results in zero profit for every activity which enters the optimal solution at a positive level. This in itself is a trivial conclusion. More importantly, Gomory and Baumol [4] show how these prices may be computed by allocating to the resources and activities in a natural way the dual prices assigned to the Gomory cutting planes (generated in the process of computing an optimal solution to the integer programming problem). In cases of non-degeneracy, where non-degeneracy is defined in a special way (to be discussed later), the Gomory-Baumol prices on the resources are identical to the regular linear programming dual prices.

The Gomory-Baumol prices do not, however, give the marginal revenue productivity of resources. The marginal revenue productivity of resource i in an integer program can be defined as the increase in the objective function for a unit increase of resource i rather than the per unit increase in the objective function for small increases in resource i since only unit changes can give any positive increment in the objective function. With integer programs, the marginal revenue productivity of a resource as a function of the amount of the resource available is not continuous, monotonic, nor constant in the range of non-degeneracy as is the case with linear programs. Furthermore, with integer programs, unlike with linear programs, the marginal revenue productivity of resource i is dependent on the amount available of resource j in non-degenerate cases. Finally, for an integer program in the range of non-degeneracy, the total increase in the objective function due to increases in resources i and j is greater than or equal to (rather than equal to as with linear programs) the sum of the marginal revenue productivities of increases in resources i and j individually.

The purpose of this paper is to determine functional form, applicable for certain cases of non-degeneracy, relating increases in the objective function to changes in the resource endowment in the integer programming case and to determine a simple pricing system which not only makes it profitless to change any level of activity but makes it profitless to hire any different combination of resources. A by-product of our analysis is a method of parametric programming which computes all optimal programs in a very quick and straightforward manner for a certain range of values of the resource endowment vector b.

I.

Let us rewrite the optimal solution (5) to the linear programming problem as follows:

$$(7)\quad t^* = B^{-1}\cdot b - B^{-1}\sum_{i=1}^{m} n_i\cdot x_i^*$$

$$z^* = \pi_0 - \sum_{i=1}^{m} \pi_i\cdot x_i$$

where n_i $(i = 1,\ldots,m)$ are the columns of N and π_i are non-negative scalars. If any of the constants in (7) are non-integer, a Gomory constraint (cutting plane) s_1 can be derived as follows:

$$(8)\quad s_1 = \frac{-\{\lambda(1)\ (B^{-1})^*\ b\}_D}{D} + \sum_{i=1}^{m} \frac{\{\lambda(1)(B^{-1})^*\ n_i\}_D}{D}\, x_i$$

$$= \frac{\eta_{01}}{D} + \sum_{i=1}^{m} \frac{\eta_{i1}}{D}\, x_i$$

where

(a) $\lambda(1) = (\lambda_1(1), \lambda_2(1),\ldots, \lambda_n(1))$ is a row vector with arbitrary non-negative integer elements (see Gomory [2]),

(b) $D = |\det B|$,

(c) B^{-1} is the inverse of B and $(B^{-1})^* = B^{-1}\cdot D$, and

(d) $\{a\}_D$ stands for the operation which transforms the elements of the matrix a into the corresponding numbers modulo D. For example,

$$\{-5\}_4 = 3$$

$$\left\{\begin{vmatrix} -2 & 6 \\ 10 & 3 \end{vmatrix}\right\}_3 = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$\{(-\tfrac{3}{10}, \tfrac{12}{5})\}_1 = (\tfrac{7}{10}, \tfrac{2}{5})$$

A Gaussian elimination can be performed on s_1 in such a way that we pivot on that x_i for which $\frac{\Pi_i}{\eta_{i1}}$ is a minimum.* Let us assume without loss of generality that $\min_{i=1,\ldots,m} \left(\frac{\eta_i}{\eta_{i1}}\right) = \frac{\eta_1}{\eta_{11}}$. After

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* That is, we perform a pivoting operation using the dual simplex method. See Dantzig, Ford, and Fulkerson [1].

performing the Gaussian elimination on the equations in (7) and (8) by pivoting on s_1 and x_1, we obtain

$$(9)\quad t = B^{-1}\cdot b - \frac{B^{-1}\cdot n_1\ \eta_{01}}{\eta_{11}} - \frac{B^{-1}\cdot n_1\cdot D}{\eta_{11}}\ s_1 - \sum_{i=2}^{m}\left(B^{-1}\cdot n_i - B^{-1}\cdot n_1\cdot\frac{\eta_{i1}}{\eta_{11}}\right)x_i$$

$$z = \pi_0 - \frac{\pi_1\cdot\eta_{01}}{\eta_{11}} - \frac{\pi_1\cdot D}{\eta_{11}}\ s_1 - \sum_{i=2}^{m}\left(\pi_i - \pi_1\cdot\frac{\eta_{i1}}{\eta_{11}}\right)x_i$$

$$x_1 = \frac{\eta_{01}}{\eta_{11}} + \frac{D}{\eta_{11}}\ s_1 - \sum_{i=2}^{m}\frac{\eta_{i1}}{\eta_{11}}\ x_i$$

The effect of the Gaussian elimination is to transform the constants in (7) which are expressible in terms of integers divided by D to the constants in (9) which are integers divided by η_{11}. The common denominator η_{11} is less than D. If η_{11} is not unity, a second Gomory constraint s_2 may be derived and a second Gaussian elimination performed. The process is continued as long as the common denominator of all coefficients is not unity. Since the common denominator is monotonically decreasing, however, the process need only be continued a finite number K (< D) of steps. Without loss of generality, we may assume that successive pivots on the Gomory constraints occur on variables $x_1, x_2, \ldots, x_K$*. Each pivot occurs in such a manner that the coefficients of the s_i and the x_i in the z equation remain non-negative. The final result is the following:

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*This, of course, rules out the possibility of pivoting on one of the previously introduced variables s_i. One may show, however, that a pivot on such a variable can be avoided by the proper choice of the arbitrary vector $\lambda(i)$. This will be illustrated in an example. Thus there is no real loss of generality.

(10) $$x_k = x_k\ (K) - \sum_{i=1}^{K} \alpha_{ik}\ (K)\ s_i - \sum_{i=K+1}^{m} \alpha_{ik}\ (K)\ x_i \quad \text{for } K=K,K-1,\ldots,1$$

$$t = t(K) - \sum_{i=1}^{K} \beta_i\ (K)\ s_i - \sum_{i=K+1}^{m} \beta_i\ (K)\ x_i$$

$$z = z\ (K) - \sum_{i=1}^{K} \gamma_i\ (K)\ s_i - \sum_{i=K+1}^{m} \gamma_i\ (K)\ x_i$$

where the α_{ik} (K), $\beta_i(K)=(\beta_{i1}(K),\ldots,\beta_{in}(K))$, and $\gamma_i(K)$ for $i = 1,\ldots,m$ are the coefficients of the non-basic variables after the pivot on the K^{th} Gomory constraint s_K. The x_k (K), t (K) and z (K) are determined recursively as follows:

(11) $$x_K\ (K) = \frac{\eta_{0K}}{\eta_{KK}}$$

$$x_k\ (K) = \frac{\eta_{0k}}{\eta_{kk}} - \sum_{i=k+1}^{K} \frac{\eta_{ik}}{\eta_{kk}} \cdot x_i\ (K) \quad \text{for } K = K-1,\ K-2,\ldots,1$$

$$t\ (K) = B^{-1} \cdot b - \sum_{k=1}^{K} B^{-1} n_k \cdot x_k\ (K)$$

$$z\ (K) = \pi_0 - \sum_{k=1}^{K} \pi_k \cdot x_k\ (K)$$

where the η_{ik} for $i = 1,\ldots,m$ are derived from the kth Gomory constraint.

(12) $$s_k = - \frac{\eta_{0k}}{\eta_{k-1,k-1}} + \sum_{i=1}^{k-1} \frac{\eta_{ik}}{\eta_{k-1,k-1}} s_i + \sum_{i=k}^{m} \frac{\eta_{ik}}{\eta_{k-1,k-1}} x_i$$

The η_{ik} are determined recursively from (8) and

(13) $\eta_{0k} = \{\lambda(k)\cdot B^{-1}b\cdot\eta_{k-1,k-1} - \eta_{k-1,k-1}\sum_{j=1}^{k-1}[\lambda(k)\ B^{-1}\cdot\eta_j-\delta_j(k)]$

$[\frac{\eta_{0j}}{\eta_{jj}} - \sum_{i=j+1}^{k-1}\frac{\eta_{ij}}{\eta_{jj}}x_i\ (k-1)]\}_{\eta_{k-1,\ k-1}}$ for $k=2,\ldots,K$

(14) $x_j\ (k) = \frac{\eta_{0j}}{\eta_{jj}} - \sum_{i=j+1}^{k}\frac{\eta_{ij}}{\eta_{jj}}\ x_j\ (k)$ for $j=1,\ldots,k$ $k=2,\ldots,K$

(15) $\eta_{ik} = \{\sum_{j=1}^{n}\lambda_j\ (k)\beta_{ij}\ (k-1) + \sum_{j=1}^{k-1}\delta_j\ (k)\ \alpha_{ij}\ (k-1)\}_{\eta_{k-1,\ k-1}}$

for $i=1,\ldots,m,\ k=2,\ldots,K$

The $\lambda(k) = (\lambda_1\ (k),\ldots,\ \lambda_n\ (k))$ and $\delta_j\ (k)$ for $j = 1,\ldots,\ k-1$ are arbitrary non-negative integer vectors.

Now (11) gives an optimal solution to the integer programming problem if $t(K) \geqq 0$ and $x_k(K) \geqq 0$ for $k=1,\ldots,K$. The $t(K)$ and $x_k(K)$ are integer since η_{KK}, the common denominator, is integer. Because of the dual simplex algorithm pivoting rule, $\delta_i(K) \geqq 0$ for $i=1,\ldots,m$. While the basic variables of the linear programming problem are those of the t vector, the integer programming problem has as its basic variables the t variables and the x_k for $k=1,\ldots,K$ variables. The new basic variables arise as the result of the introduction of the Gomory cutting planes s_i for $i=1,\ldots,K$. The equations in (11) give the optimal solution to the integer programming problem for any resource endowment b for which the integer basis remains the same, i.e., non-negative.*

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*As Gomory [ ] has shown the basic t variables remain the same as long as . The x_k variables may or may not remain the same for changes in b which keep the same basic t variables.

II.

The analysis to this point suggests a computational technique for parametric integer programming in cases where the integer basis consists of the variables t and x_k for k=1,...,K. First, solve the integer programming problem for any resource endowment b. From (12)-(15), we note that the η_{0k} for k = 1,..,n are dependent on b while the η_{ik} (and $i \geq k$) for $i \neq 0$ are independent of b. Thus, if the η_{ik} for $i \neq 0$ are (and $i \geq k$) are recorded as the solution for any b is obtained, only the η_{0k} need be determined for each b, using equations (13) and (14). Then (11) may be used to compute recursively, the values of the variables x_k, for k= K, K-1,...,1, t, and z in that order.

If for any particular value of b, any of the t, z, and x_k variables as computed by (11) are negative, then, of course, the proposed method does not work. In such a case, we suggest the following procedure:

(a) If some of the x_k are negative, pivot on the negative x_k until they are all non-negative. Then add additional Gomory constraints until all variables are integer.

(b) If none of the x_k are negative, but some of the t variables are negative, pivot on the t variables until all are non-negative. Add additional Gomory constraints until all variables are integer.

(c) If after performing step (a) or step (b), all variables are non-negative, a solution has been reached. Otherwise, repeat step (a) or step (b), whichever is appropriate.

Since this is a variation of the technique proposed by Gomory [2], it is easy to show that it converges to the optimal solution.

III.

The variables $x_1,\ldots,x_K$ can be eliminated from equation (11) by solving the first K equations. The solution can be written in terms of the cofactors of the following triangular matrix.

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\eta_{K,K-1}}{\eta_{K-1,K-1}} & 1 & & \\ \frac{\eta_{K,K-2}}{\eta_{K-2,K-2}} & \frac{\eta_{K-1,K-2}}{\eta_{K-2,K-2}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \frac{\eta_{K,1}}{\eta_{11}} & \frac{\eta_{K-1,1}}{\eta_{11}} & \cdots & 1 \end{bmatrix}$$

This matrix is independent of b. Now

$$(16) \quad x_k = \sum_{i=1}^{K} E_{K+1-i,\ K+1-k} \frac{\eta_{0i}}{\eta_{ii}}$$

where E_{ij} is the cofactor of the i^{th} row and the j^{th} column of E. If the solution in (16) is substituted into (11), one obtains for t and z

$$(17) \quad t\ (K) = B^{-1}\cdot b - \sum_{k=1}^{K} B^{-1}\cdot n_k \ (\sum_{i=1}^{K} E_{K+1-i,K+1-k} \frac{\eta_{0i}}{\eta_{ii}})$$

$$z\ (K) = \pi_0 - \sum_{k=1}^{K} \Pi_k \ (\sum_{i=1}^{K} E_{K+1-i,K+1-k} \frac{\eta_{0i}}{\eta_{ii}})$$

Since the η_{0i} are dependent on b as indicated by (13) and (14), we may write for (16) and (17):

(18) $x_k(b) = \phi_k(b)$ for $k=1,\ldots,K$

$$t(b) = B^{-1}b - \phi_t(b)$$

$$z(b) = c_B \cdot B^{-1}\, b - \phi_z(b) = p^* \cdot b - \phi_z(b)$$

The ϕ functions may be interpreted as the difference between the linear programming and the integer programming solutions.

The functions $\phi_k(b)$, $\phi_t(b)$, and $\phi_z(b)$ have several interesting properties, all of which are easily proved.

Property 1. If $B^{-1}b$ is an integer vector, then $\phi_k(b) = \phi_t(b) = \phi_z(b) = 0$.

Property 2. If δ is an integer column vector, then

$$\phi_k(b+\delta D) = \phi_k(b)$$

$$\phi_t(b+\delta D) = \phi_t(b)$$

$$\phi_z(b+\delta D) = \phi_z(b)$$

Property 3. If δ_i is an integer scalar and β_i is a column of B, then

$$\phi_k(b+\delta_i \beta_i) = \phi_k(b)$$

$$\phi_t(b+\delta_i \beta_i) = \phi_t(b)$$

$$\phi_z(b+\delta_i \beta_i) = \phi_z(b)$$

Property 4. If $b' \equiv b \pmod{B}$, then

$$\phi_k(b') = \phi_k(b)$$

$$\phi_t(b') = \phi_t(b)$$

$$\phi_z(b') = \phi_z(b)$$

Property 5. The functions $\phi_k(b)$, $\phi_t(b)$ and $\phi_z(b)$ assume at most D different values each.

Property 1 is merely a statement that if the optimal linear programming solution is integer then the linear programming solution is the integer programming solution. Property 5 is a statement of the periodicity of the ϕ functions while properties 2 through 4 indicate the various ways in which this periodicity may be expressed. First (Property 2), if one adds D units to any one of the resources, the difference between the integer programming solution and the linear programming solution remains the same. Secondly, (Property 3), if one adds amounts of all resources sufficient to increase one of the basic t activities exactly by one unit (or any integer number of units), the difference between the integer programming and linear programming solution remains the same. Finally, Property 4 says that the difference between the linear programming and integer programming solution remains the same if $b' \equiv b \pmod{B}$. This means that there exist vectors $[b']$ and $[b]$ such that (a) $f_{b'} = b' - [b'] = b - [b] = f_b$ (b) $[b']$ and $[b]$ are integer combinations of vectors in the basis B, and (c) $f_b = f_{b'} = B\cdot\lambda$ for some vector λ such that $0 \leqq \lambda < 1$. It is well known that $b(\text{mod}B) = b - [b]$ may assume at most D different values which give rise to Property 5.

We may rewrite Property 4 in the following way

$$\phi_k(b) = \phi_k(f_b)$$

$$\phi_t(b) = \phi_t(f_b)$$

$$\phi_z(b) = \phi_z(f_b)$$

where, as above, $f_b = b - [b]$. Let us assume that $(c_B, c_N) \geqq 0$. Then $z(0) = 0$ and

(19) $z(f_b) = p^* \cdot f_b - \phi_z(f_b) \geq 0$ or

(20) $\phi_z(b) = \phi_z(f_b) \leq p^* \cdot f_b$

Since f_b can only assume a finite number of values, the difference between the linear programming and the integer programming optimum value of the objective function is limited to some finite number. Furthermore, one can easily show that $f_b \leq \sum_{i=1}^{n} \beta_i$ so that

(21) $\phi_z(b) \leq p^* \sum_{i=1}^{n} \beta_i$

where β_i is a column of the basis B. We may interpret $\phi_z(b)$ as the loss which would accrue to the integer programming optimal solution if resources were given their linear programming dual prices p^*. The inequality (21) says that this loss never exceeds the cost of operating each and every activity in the linear programming basis at a unit level of activity.

IV

Given the above five properties of $\phi_z(b)$, we may also state several interesting properties of the marginal revenue productivity (MRP) of resources. The MRP_i of a resource i is

$$(22) \quad MRP_i = z\begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_i+1 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} - z\begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_i \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} = c_B \cdot B^{-1}\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} - (\phi_z\begin{pmatrix} b_i \\ \cdot \\ \cdot \\ \cdot \\ b_i+1 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} - \phi_z\begin{pmatrix} b_i \\ \cdot \\ \cdot \\ \cdot \\ b_i \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix})$$

if the basis t and $x_1,\ldots,x_k$ remain the same when b_i is increased one unit. Note that the first term on the right hand side of (22) is nothing more than the linear programming dual price from which is subtracted a periodic function of b, i.e.,

$$(23)\quad MRP_i = p_i^* - \psi_i(b)$$

where $\psi_i(b)$ may be either positive or negative and is a periodic function satisfying all of Properties 1 through 5. Since the increase in the objective function resulting from a unit increase in resource i is obviously non-negative, the following property must also hold

$$(24)\quad \psi_i(b) \leqq p_i^*$$

Let us define the marginal revenue productivity of a group of resources b which is incremented by $\Delta b = (\Delta b_1, \ldots, \Delta b_n)$ as follows:

$$\begin{aligned}(25)\quad MRP_{\Delta b}(b) &= z(b+\Delta b) - z(b)\\ &= c_B \cdot B^{-1} \cdot \Delta b - (\phi_z(b+\Delta b) - \phi_z(b))\\ &= p^* \cdot \Delta b - (\phi_z(b+\Delta b) - \phi_z(b))\\ &= p^* \cdot \Delta b - \psi_{\Delta b}(b)\end{aligned}$$

if the integer basis t and $x_1,\ldots,x_K$ remain the same. Furthermore, one can show:

$$(26)\quad \text{if } \Delta b > 0, \text{ then } p^* \cdot \Delta b \geqq \psi_{\Delta b}(b)$$

and

$$(27)\quad \text{if } \Delta b < 0, \text{ then } p^* \cdot \Delta b \leqq \psi_{\Delta b}(b)$$

The periodic function $\psi_{\Delta b}(b)$ also satisfies Properties 1 through 5.

Finally, the following relationship between the MRP of single resources and the MRP of an increment in a group of resources holds if $\Delta b \geqq 0$:

$$\text{(28)} \qquad MRP_{\Delta b}\ (b) \geqq \sum_{i=1}^{n} MRP_i \cdot \Delta b_i \text{ and}$$

$$\psi_{\Delta b}(b) \geqq \sum_{i=1}^{n} \psi_i(b)$$

V.

It is not always possible to find a set of prices for the resources (b) such that it both never pays to change the optimal activity levels nor to purchase any different combination of resources (See Gomory and Baumol [4]). There is an alternative method of pricing, however, which will achieve this result. Let us suppose that there is a dual pricing system applied to each resource. The first b_i^* units of resource i cost p_i' per unit and any purchases in excess of b_i^* cost p_i'' per unit. Then the profit accruing to any feasible integer program (t,x,-b) is

$$\text{(29)} \qquad \pi(t,x,-b) = c_B \cdot t + c_N \cdot x - p' \cdot b^* - p'' \cdot d''(b-b^*) - p' \cdot d'(b-b^*)$$

where d' and d" are diagonal matrices with elements d_i' and d_i'', respectively, along the diagonals such that

$$\text{(30)} \qquad d_i' = \begin{cases} 1 \text{ if } & b_i - b_i^* \leqq 0 \\ 0 \text{ if } & b_i - b_i^* > 0 \end{cases}$$

$$d_i'' = \begin{cases} 0 \text{ if } & b_i - b_i^* \leqq 0 \\ 1 \text{ if } & b_i - b_i^* > 0 \end{cases}$$

(31) $\text{Max } \pi(t,x,-b) = \underset{b}{\text{Max}} \ (z(b) - p'\cdot b^* - p''\cdot d''(b-b^*) - p'\cdot d'(b-b^*))$

$$= \underset{b}{\text{Max}} \ \pi(b)$$

Next, let us define efficiency as follows:

DEFINITION: A feasible point $(t,x,-b)$ is efficient if and only if there exists no other feasible point $(t', x',-b')$ such that $(t', x', -b') \geq (t,x,-b)$ with $t'_i > t_i$, $x'_i > x_i$, or $-b'_i > -b'_i$ for at least one i.

Let us call semi-optimal points, all points $(t,x,-b)$ in which, given b, t and x are an optimal integer program. The following lemma is easily proved:

LEMMA: A semi-optimal point $(t,x,-b)$ is efficient if and only if there exists no other semi-optimal point $(t',x',-b')$ such that $(z(b'), -b') \geq (z(b), -b)$ and either $z(b') > z(b)$ or $-b'_i > -b_i$ for some i.

Now we will show that for each efficient semi-optimal point there exists a set of dual prices $p' > 0$ and $p'' > 0$ such that the efficient semi-optimal point is also optimal, i.e.,

THEOREM: If $(t^*,x^*,-b^*)$ is semi-optimal and efficient and if the linear programming dual prices p^* exist and are all positive, then there exist price vectors $p' > 0$ and $p'' > 0$ such that

$$\underset{t,x,b}{\text{Max}} \ \pi(t,x,-b) = \underset{b}{\text{Max}} \ \pi(b) = \pi(b^*)$$

PROOF: Let us define the price vectors p' and p'' in terms of the basis B to the linear programming problem resource vector b. Now $p^* = c_B \cdot B^{-1}$ is the vector of dual prices corresponding to the basis B. Let

(32) $$p' = p^* - v_s$$

$$p'' = p^* + v_t$$

We then need to demonstrate the existence of a $v_t \geqq 0$ and a $v_s \geqq 0$ such that $v_s < v_p$. It is sufficient to show that for all $b = b^* + \Delta b^*$ for which an optimal solution to the integer programming problem exists $\Delta\pi(b^*) = \pi(b) - \pi(b^*) \leqq 0$ or

$$z(b^*+\Delta b^*) - (p^*-v_s)\cdot b^* - (p^*+v_t)\, d''\Delta b^* - (p^*-v_s)d'\,\Delta b^* - z(b^*) + (p^*-v_s)\, b^* \leqq 0.$$

Collecting terms, we get:

(33) $\Delta\pi(b^*) = z(b^*+\Delta b^*) - p^*\,\Delta b^* - z(b^*) - v_t d''\Delta b^* + v_s d'\Delta b^* \leqq 0$

Now

(34) $z(b^*) = p^* \cdot b^* - \phi_z(b^*) - \delta(b^*)$ and $z(b^*+\Delta b^*) = p^*(b^*+\Delta b^*) - \phi_z(b^*+\Delta b^*) - \delta(b)$

where $\delta(b^*)$ and $\delta(b)$ are non-negative correction factors indicating the difference between the value of z if (11) provides an optimal solution to the integer programming problem for b^* and $b^*+\Delta b^*$, respectively, and the value of z if (11) does not give an optimal solution and further dual pivoting steps must be performed to obtain the optimal solution. Substituting (34) into (33), we obtain

(35) $\Delta\pi(b^*) = -\phi_z(b^*+\Delta b^*) + \phi_z(b^*) - \delta(b) + \delta(b^*) - v_t d''\,\Delta b^* + v_s d'\Delta b^* \leqq 0.$

Now all terms on the left hand side of the inequality in (35) except for $\phi_z(b^*)$ and $\delta(b^*)$ are non-positive. Thus, $\phi_z(b^*) + \delta(b^*)$ is an upper bound on $\Delta\pi(b^*)$. If Δb^* contains any positive elements, however, since Δb^* is integer, $\Delta\pi(b^*)$ can be made non-positive by setting $v_t = (\phi_z(b^*) + \delta(b^*), \ldots, \phi_z(b^*) + \delta(b^*))$. If $\Delta b^* = 0$, then obviously (35) is satisfied. Thus, we need only consider those $\Delta b^* \leqq 0$ for which $\Delta b^* \neq 0$. With this information we may rewrite the inequality (33) as follows:

(36) $-z(b^* + \Delta b^*) + z(b^*) \geqq -(p^*-v_s)\,\Delta b^*$

We need not consider any Δb^* such that $b^* + \Delta b^* \equiv b^*$ (mod B) for then $-z(b^* + \Delta b^*) + z(b^*) = - p^* \cdot \Delta b^*$ and the inequality in (36) must hold for $\Delta b^* \leqq 0$. Furthermore, if the inequality (36) holds for $\Delta b^* = a$, then it holds for $\Delta b^* = a' \leqq a$ where $a' \equiv a$ (mod B). To show this to be true, let us write $z(b^*+a) = z(b^*+a') + p^*(a'-a)$ which follows from the fact that $b^* + a \equiv b^* + a'$ (mod B) and $\phi_z(b^* + a) = \phi_z(b^* + a')$ (See Property 4 in section IV). Since (36) holds for $\Delta b^* = a$, we have

$$-z(b^* + a) + z(b^*) \geqq -(p^*-v_s)\, a \text{ or } -z(b^*+a') + p^*(a'-a) + z(b^*) \geqq -(p^*-v_s)a \text{ or}$$

$$-z(b^*+a') + z(b^*) \geqq -p^*a' + v_s a$$

Since $a \geqq a'$ and $v_s \geqq 0$, it follows that

$$-z(b^*+a) + z(b^*) \geqq -(p^* - v_s)\, a'$$

Thus, (36) holds for $a' \leqq a$ and since a (mod B) can only take on a finite member D of values, there is a lower bound on the $\Delta b^* \leqq 0$ which must be considered. In fact one can easily show that we need only consider $\Delta b^* \geqq -f_{b^*} \geqq -(D-1, D-1, \ldots, D-1)$ where f_{b^*} is defined as above. The lower bound on the Δb^* implies an upper bound on the right hand side of (36) where $p^*-v_s > 0$. The left-hand side of (36) has a lower bound of unity for otherwise b^* would not be efficient according to the above lemma. Thus, v_s can assume some finite non-negative value such that $(p^*-v_s) > 0$ and (36) holds for all possible Δb^*. In particular if

$$v_{is} = p^*_i - \frac{1}{n f_{b^*_i}} \geqq 0$$

for $i = 1, \ldots, n$

where $f_{b^*_i}$ and v_{is} are the i^{th} components of f_{b^*} and v_s, then the inequality (36) is always satisfied. q.e.d.

The quantities v_s and v_t may be regarded as subsidies (v_s) and taxes (v_t) on the resources. If up to and including b_i^* units of resource i are purchased a subsidy of v_{is} per unit is paid. More than b_i^* units are taxed at a rate v_{it} per unit.

VI

To illustrate some of the above ideas, let us consider the following example:

(37) $\text{Max } 4t_1 + 5t_2 + t_3$

Subject to

(38)
$$\begin{aligned} 3t_1 + 2t_2 \quad\quad + x_1 \quad\quad &= 28 \\ t_1 + 4t_2 \quad\quad\quad + x_2 \quad &= 27 \\ 3t_1 + 3t_2 + t_3 \quad\quad + x_3 &= 36 \end{aligned}$$

$t_1, t_2, t_3, x_1, x_2, x_3 \geqq 0$ and integer. The revised simplex method yields the following matrix with the optimal program to the linear programming problem in the first column and the last row containing the first Gomory constraint s_1 which is introduced.

TABLE 1

	1	$-x_1$	$-x_2$	$-x_3$
z	52 4/10	2/10	4/10	1
t_1	5 8/10	4/10	-2/10	0
t_2	5 3/10	-1/10	3/10	0
t_3	2 7/10	-9/10	-3/10	1
s_1	-4/10	-2/10*	-4/10	0

The determinant D of the basis is 10. Thus, the constraint s_1 given in the matrix above is only one of 10 possible Gomory constraints. (See Gomory [2]). The coefficients of the 10 possible Gomory constraints s_1 are generated by letting $\lambda(1)$ an arbitrary non-negative integer vector assume various values in the following vector of coefficients:

$$\frac{\{(\lambda_1(1),\lambda_2(1),\lambda_3(1),\lambda_4(1))\cdot 10\ A\}_{10}}{10}$$

where A represents the above simplex matrix in Table 1. In particular, the 10 possible constraints may be generated by letting $\lambda_1(1)=\lambda_2(1)=\lambda_4(1)=0$ and $\lambda_3(1)=1,2,\ldots,10$. The result is

		$\lambda_3(1)$
(39)	(7/10, 1/10, 7/10, 0)	1
	(4/10, 2/10, 4/10, 0)	2
	(1/10, 3/10, 1/10, 0)	3
	(8/10, 4/10, 8/10, 0)	4
	(5/10, 5/10, 5/10, 0)	5
	(2/10, 6/10, 2/10, 0)	6
	(9/10, 7/10, 9/10, 0)	7
	(6/10, 8/10, 6/10, 0)	8
	(3/10, 9/10, 3/10, 0)	9
	(0, 0, 0, 0)	10

If we perform a pivot on the element of the simplex matrix in Table 1 marked with an asterisk we obtain the following matrix:

TABLE 2

	1	$-s_1$	$-x_2$	$-x_3$
z	52	1	0	1
t_1	5	2	-1	0
t_2	5 1/2	-1/2	1/2	0
t_3	4 1/2	-9/2	3/2	1
x_1	2	-5	2	0
s_2	-1/2	-1/2	-1/2*	0

Here the common denominator of all elements in the matrix is $\eta_{11}=2$ and there are two possible Gomory constraints. The particular constraint chosen can be generated by setting

$$\lambda(2) = (\lambda_1(2), \lambda_2(2), \lambda_3(2), \lambda_4(2)) = (0,1,0,0) \text{ and}$$

$$\delta_1(2) = 0.$$

Pivoting again, one obtains

TABLE 3

	1	$-s_1$	$-s_2$	$-x_3$
z	52	1	0	1
t_1	6	3	-2	0
t_2	5	-1	1	0
t_3	3	-3	3	1
x_1	0	-7	4	0
x_2	1	1	-2	0

The first column gives the optimal solution to the integer programming problem. From equation (11) we can derive an expression for the optimal values of the variables t_1, t_2, t_3, x_1 and x_2.

(40) $$x_2\ (2) = \frac{\eta_{02}}{1}$$

$$x_1\ (2) = \frac{\eta_{01}}{2} - \frac{4}{2}\, x_2(2)$$

$$t_1\ (2) = \frac{(4b_1 - 2b_2 + 0b_3)}{10} - \frac{4}{10}\, x_1(2) + 2/10\ x_2(2)$$

$$t_2\ (2) = \frac{(-1b_1+3b_2+0b_3)}{10} + 1/10\ x_1(2) - 3/10\ x_2(2)$$

$$t_3\ (2) = \frac{(-9b_1-3b_2+10b_3}{10} + 9/10\ x_1\ (2) + 3/10\ x_2\ (2)$$

$$z\ (2) = \frac{(2b_1+4b_2+10b_3)}{10} - 2/10\ x_1\ (2) - 4/10\ x_2\ (2)$$

where

(41) $\eta_{01} = \{2(-9b_1 - 3b_2 + 10b_3)\}_{10} = \{2b_1 + 4b_2\}_{10}$

$\eta_{02} = \{\frac{1(-1b_1 + 3b_2 + 0b_3)2}{10} + 2 \cdot 1 \cdot \frac{1}{10} \cdot x_1(1)\}_2 =$

$\{\frac{18b_1 + 6b_2}{10} + \frac{1}{10}\eta_{01}\}_2$

Solving (40) for x_1 (2) and x_2 (2), we obtain

(42) $x_2(2) = \eta_{02}$

$x_1(2) = \frac{\eta_{01}}{2} - \frac{4}{2}\eta_{02}$

$t_1(2) = \frac{(4b_1 - 2b_2)}{10} - \frac{2}{10}\eta_{01} + \eta_{02}$

$t_2(2) = \frac{(-1b_1 + 3b_2)}{10} + \frac{1}{20}\eta_{01} - \frac{1}{2}\eta_{02}$

$t_3(2) = \frac{(-9b_1 - 3b_2 + 10b_3)}{10} + \frac{9}{20}\eta_{01} - \frac{3}{2}\eta_{02}$

$z(2) = \frac{(2b_1 + 4b_2 + 10b_3)}{10} - \frac{1}{10}\eta_{01}$

where

(43) $\phi_1(b) = \eta_{02}$,

$\phi_2(b) = \eta_{01} - 2y_{02}$,

$$\phi_t(b) = \begin{pmatrix} \frac{1}{5} \eta_{01} - \eta_{02} \\ -\frac{1}{20} \eta_{01} + 1/2 \ \eta_{02} \\ -\frac{9}{20} \eta_{01} + 3/2 \ \eta_{02} \end{pmatrix} \text{ and}$$

$$\phi_z(b) = \frac{1}{10} \eta_{01}$$

From (42) and (41), we may determine the marginal revenue productivity of resources 1 and 2 as follows:

$$(44) \quad MRP_1(b) = \frac{2}{10} - \frac{1}{10} [\{2b_1 + 4b_2 + 2\}_{10} - \{2b_1 + 4b_2\}_{10}]$$

$$MRP_2(b) = \frac{4}{10} - \frac{1}{10} [\{2b_1 + 4b_2 + 4\}_{10} - \{2b_1 + 4b_2\}_{10}]$$

Note that the first terms in both of these expressions are the respective linear programming dual prices.

Let us calculate the optimal values of the variables x_1, x_2, t_1, t_2, t_3 and z for b_1 ranging from 22 to 31 with b_2 and b_3 hold constant at 27 and 36, respectively. In this particular case we may rewrite equations (41).

$$(45) \quad \eta_{01} = \{-2b_1 + 4\cdot 27\}_{10} = \{2b_1 + 8\}_{10}$$

$$= \{2b_1 + 8\}_{10}$$

$$\eta_{02} = \frac{\{-18b_1 + b\cdot 27 + \eta_{01}\}}{10}_2 = \frac{\{18b_1 + 2 + \eta_{01}\}}{10}_2$$

To further make computation less difficult we can calculate η_{01} and η_{02} on the basis of b_1 varying from 2 to 1b the result will be the same.

Performing the computations in (45) and substituting back into (42), the following results are obtained:

TABLE 4

b_1	η_{01}	η_{02}	x_1	x_2	t_1	t_2	t_3	z	MRP_1
22	2	0	1	0	3	6	9	51	0
23	4	0	2	0	3	6	9	51	0
24	6	0	3	0	3	6	9	51	0
25	8	0	4	0	3	6	9	51	1
26	0	1	-2	1	6	5	3	52	0
27	2	1	-1	1	6	5	3	52	0
28	4	1	0	1	6	5	3	52	0
29	6	1	1	1	6	5	3	52	0
30	8	1	2	1	6	5	3	52	1
31	0	0	0	0	7	5	0	53	0

Note that for $b_1 = 26$ and $b_1 = 27$, the proposed method of calculating the optimal solution does not work because x_1 is negative. Since there is a great deal of disgression in choosing the Gomory constraint at each stage in the process, a natural question is: would a different choice of the cutting planes s_1 and s_2 have resulted in a method which is valid for all changes in b_1 over the range from 22 to 31? The answer is yes. In fact if

$$(46) \quad s_1 = -\frac{\eta_{01}}{10} + \frac{1}{10}x_1 + \frac{7^*}{10}x_2 + 0x_3 \text{ and}$$

$$s_2 = -\frac{\eta_{02}}{10} + \frac{1^*}{7}x_1 + \frac{4}{7}x_2 + 0x_3$$

where the asterisk indicates the pivot variable, the following results are obtained:

TABLE 5

b_1	η_{01}	η_{02}	x_1	x_2	t_1	t_2	t_3	z	MRP_1
22	1	1	1	0	3	6	9	51	0
23	2	2	2	0	3	6	9	51	0
24	3	3	3	0	3	6	9	51	0
25	4	4	4	0	3	6	9	51	0
26	5	5	5	0	3	6	9	51	0
27	6	6	6	0	3	6	9	51	1
28	7	0	0	1	6	5	3	52	0
29	8	1	1	1	6	5	3	52	0
30	9	2	2	1	6	5	3	52	1
31	0	0	0	0	7	5	0	53	0

Another choice of Gomory constraints leading to non-negative values for x_1 and x_2 is

(47) $$s_1 = \frac{\eta_{01}}{10} + \frac{3}{10} x_1 + \frac{1}{10} x_2 + 0\, x_3 \text{ and}$$

$$s_2 = \frac{-\eta_{02}}{3} + \frac{2}{3} x_1 + \frac{1}{3} x_2 + 0 x_3.$$

The results are:

TABLE 6

b_1	η_{01}	η_{02}	x_1	x_2	t_1	t_2	t_3	z	MRP_1
22	3	0	1	0	3	6	9	51	0
23	6	0	2	0	3	6	9	51	0
24	9	0	3	0	3	6	9	51	0
25	2	2	0	2	5	5	6	51	0
26	5	2	1	2	5	5	6	51	0
27	8	2	2	2	5	5	6	51	1
28	1	1	0	1	6	5	3	52	0
29	4	1	1	1	6	5	3	52	0
30	7	1	2	1	6	5	3	52	1
31	0	0	0	0	7	5	0	53	0

Thus, for this particular problem, the optimal solution to the integer programming problem is not always unique.

A different choice of Gomory constraints leads to another problem. For example, if

$$(48) \qquad s_1 = - \frac{\eta_{01}}{10} + \frac{4^*}{10} x_1 + \frac{8}{10} x_2 + 0x_3$$

then the pivoting rule leads to a re-introduction of s_1 later on as a basic variable. The analysis in this paper breaks down in such a case. This can be avoided, however, if one uses the following criterion:

Criterion for Choosing s_k. Choose the non-zero s_k which gives the smallest value of η_{ik} for a variable x_i with the smallest entry in the first row of the simplex matrix. If more than one s_k satisfies this criterion, choose among them the s_k with the smallest η_{ik} for a variable x_i with the next smallest entry in the first row of the simplex matrix, and so on.

When pivoting, always pivot on a variable with the smallest entry in the first row of the simplex matrix if there is a choice of pivots.

Since our results are so sensitive to the choice of the Gomory constraint introduced at each step, it would be desirable to formulate a decision rule which would insure non-negative integer programming basic variables $x_1,\ldots,x_k$ no matter what the value of b. Good results have been obtained using the above criterion but one can construct examples for which this criterion does not work.* If this criterion or some other always results in non-negative $x_1, \ldots, x_K$, then (11) gives the optimal integer program so long as the t variables are non-negative.

The above criterion may be illustrated with reference to the first simplex matrix (Table 1) for our example, (37) and (38). All possible Gomory constraints are represented in (39). Since x_1 has the smallest number in the first row of the simplex matrix (Table 1), according to the above criterion the constraint represented by the first vector in (39) is the one to use.

Next let us specify the price vectors p' and p'' which make profit $\Pi(t, x, -b)$ a maximum at $(t_1^*, t_2^*, t_3^*, x_1^*, x_2^*, x_3^*, -b_1^*, -b_2^*, -b_3^*)$ = (6,5,3,0,1,0,-28,-27,-36), where (t^*, x^*) is the optimal program in (37) and (38). Now from (31), we have

(49) $\Pi(b) = z(b) - (p^* - v_s)b^* - (p^* + v_t)d''(b-b^*) - (p^* - v_s)d'(b-b^*)$

where

(50) $p' = p^* - v_s = (\frac{2}{10}, \frac{4}{10}, 1) - (v_{1s}, v_{2s}, v_{3s})$

$p'' = p^* + v_t = (\frac{2}{10}, \frac{4}{10}, 1) + (v_{1t}, v_{2t}, v_{3t})$

From (43), we know that $\phi_z(b) = \frac{1}{10}\eta_{01}$ and in particular when $b^* = (28,27,36)$ we have $\phi_z(b^*) = \frac{4}{10}$ (See equations (41)). From the proof to the theorem

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*I can prove that this criterion always works for $m \leqq 2$ or for K = 1 with m arbitrary if an integer programming solution exists. Briefly, if K = 1 the proof is trivial. If K = 2, we have $x_1 = \frac{\eta_{01}}{\eta_{11}} - \frac{\eta_{21}}{\eta_{11}}\eta_{02}$ and $x_2 = \eta_{02} \geqq 0$. If m = 2, one can show that either $\eta_{21} = 0$ in which case the proof is trivial or $\eta_{21} = 1$. Since $\eta_{02} < \eta_{11}$, it follows that $\frac{\eta_{21}}{\eta_{11}}\eta_{02} < 1$. Since x_1 must be integer, it must be non-negative.

in section V, we know that we can set

(49) $v_{1t} = v_{2t} = v_{3t} = \frac{4}{10}$

and

(50) $v_{is} = p_i^* - \frac{1}{3 \cdot f_{b_i^*}}$ for i = 1, 2, 3 if (50) gives non-negative v_{is} for i = 1, 2, 3.

The way in which f_{b^*} may be calculated is to take the fractional parts of the optimal linear programming values given in the first column of Table 1 and substitute these fractional parts $(\frac{8}{10}, \frac{3}{10}, \frac{7}{10})$ into (37) and (38). The result is

(51) $f_{b_1^*} = 3 \cdot \frac{8}{10} + 2 \cdot \frac{3}{10} = 3$

$f_{b_2^*} = \frac{8}{10} + 4 \cdot \frac{3}{10} = 2$

$f_{b_3^*} = 3 \cdot \frac{8}{10} + 3 \cdot \frac{3}{10} + \frac{7}{10} = 4$

Substituting (51) in (50), we obtain

(52) $v_{1s} = \frac{2}{10} - \frac{1}{9}$

$v_{2s} = \frac{4}{10} - \frac{1}{6}$

$v_{3s} = 1 - \frac{1}{12}$

From (48) we may write

(53) $p' = (\frac{1}{9}, \frac{1}{6}, \frac{1}{12})$

$p'' = (\frac{6}{10}, \frac{8}{10}, 1\frac{4}{10})$

This system of dual prices ensures that $\max_b \Pi(b) = \Pi(b^*)$.

VII

The properties of integer programs are not nearly as easy to determine as those of linear programs. As Gomory has shown [5], however, integer programs are intimately related to linear programming solutions for a large class of cases via a set of periodic functions of b, the resource vector. This paper shows how in certain cases these periodic functions may be derived explicitly which results in a method of parametric programming and enables one to express the marginal revenue productivity of any resource as a function of the linear programming dual price plus a periodic function. The method proposed in this paper does not always work, however, when K = 1 or when m, the number of non-basic linear programming variables is greater than 2, although the criterion which is proposed in this paper seems to give good results for m72. More experiments need to be performed to determine how often the criterion fails. Further research also needs to be done to determine whether there exists a criterion for choosing sk which ensures non-negative X_k for any b, or failing that, for any particular b.

It is impossible to specify a single set of prices for the resources such that the integer programming optimal solution gives maximum profits. Our proposal is a two part pricing system to make an efficient point a profit maximizing point. This proposal bears a strong resemblance to the pricing proposals in much of the literature on pricing in public utilities where individisibilities are present and such things as taxes and subsidies, two part tariffs, discriminatory pricing, etc., are the order of the day.

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