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Twin Solutions of Even Order Boundary Value Problems for Ordinary Differential Equations and Finite Difference Equations

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Twin Solutions of Even Order Boundary Value Problems for Ordinary Differential Equations and Finite Difference Equations

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ABSTRACT

The Avery-Henderson fixed-point theorem is first applied to obtain the existence of at least two positive solutions for the boundary value problem

$$(-1)^n y^{(2n)} = f(y),\ n = 1, 2, 3 \cdots \quad \text{and } t \in [0, 1],$$

with boundary conditions

$$\begin{cases} y^{(2k)}(0) = 0 \\ y^{(2k+1)}(1) = 0 \text{ for } k = 0, 1, 2 \cdots, n-1. \end{cases}$$

This theorem is subsequently used to obtain the existence of at least two positive solutions for the dynamic boundary value problem

$$(-1)^n \Delta^{(2n)} u(k) g(u(k)),\ n = 1, 2, 3 \cdots \quad \text{and } k \in \{0, \cdots N\},$$

with boundary conditions

$$\begin{cases} \Delta^{(2k)} u(0) = 0 \\ \Delta^{(2k+1)} u(N+1) = 0 \text{ for } k = 0, 1, 2 \cdots, n-1. \end{cases}$$

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Contents

1. Introduction

Green's functions are well-known mathematical tools which are important in the theory of boundary-value problems. They are also mathematical characterizations of important physical concepts. The Green's functions can be introduced in both mathematical and physical ways, but it might be more lively and easier to understand if they are developed via their physical counterparts. In this part, the Green's function will be derived from the following model, which can also be found in C. Ray Wylie, Mcgraw's book *Differential Equations*, see [19].

In the model, suppose that we have a perfectly flexible elastic string whose length is l after stretched under tension T. We also have two assumptions: firstly, we assume that the string tolerances a distributed load per unit length $w(x)$ where the weight of the string itself is included; furthermore, we also suppose that the deflections after loaded are all perpendicular to the original position of the string and all the forces are in the same plane. So for two different values of x in $[0, l]$, the forces on the portion of the string between different two points are all the same with or without the string deflecting. For any tiny part of the string, we have the forces shown in Figure (2). Since the deflected string is in equilibrium state, both of the net horizontal and the vertical force on the tiny part should be zero. So we have

$$\begin{aligned} F_1 \cos \alpha_1 &= F_2 \cos \alpha_2 \\ F_2 \sin \alpha_2 &= F_1 \sin \alpha_1 - w(x)\Delta x \end{aligned}$$

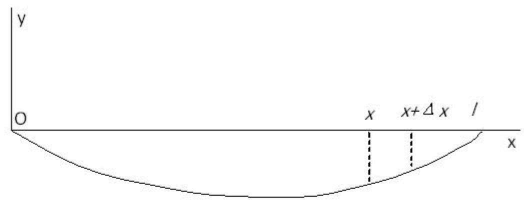

Figure 1

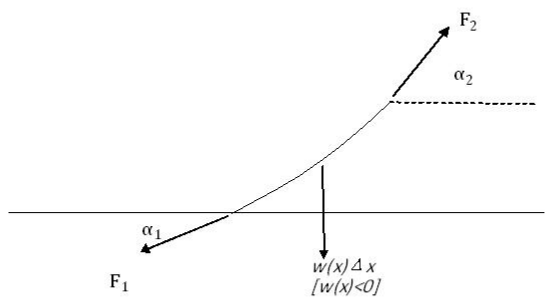

Figure 2

from the first equation, we know that the horizontal component force in the string is a constant, and because the deflections are so small, we can further assume that the constant horizontal component is the same as the tension T in the string before loaded. Then, we can plug
$F_1 = \frac{T}{\cos\alpha_1}$ and $F_2 = \frac{T}{\cos\alpha_2}$ into the second equation. After simplifying, we have

$$\tan\alpha_2 = \tan\alpha_1 - \frac{w(x)\Delta x}{T}. \tag{1}$$

From Fig.2, we know that $\tan\alpha_1$ is the slope of the deflection curve at the point x and $\tan\alpha_2$ is the slope of the deflection curve at the point $x + \Delta x$.

Then, we can rewrite equation (1) as

$$\frac{y'(x+\Delta x)-y'(x)}{\Delta x}=-\frac{w(x)}{T}.$$

As $\Delta x \to 0$, we have the differential equation

$$Ty''=-w(x), \tag{2}$$

satisfied by the curve of the string.

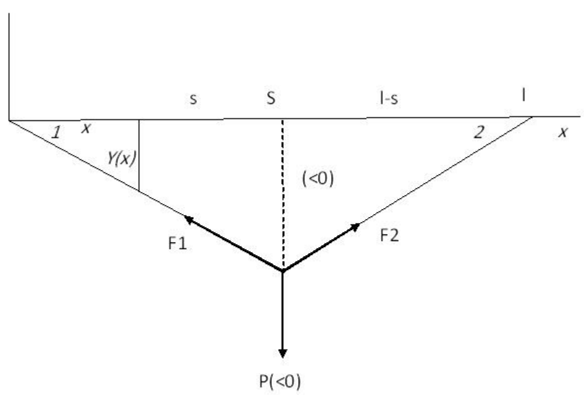

Figure 3 A stretched string deflected by a concentrated load

Now we turn to the deflection of the string, bearing a concentrated but not a distributed load. Equation (2) implies that y'' is zero at all points of the string without distributed load. Hence, from $y''=0$ we know that y is a linear function, so we can consider that the deflection curve of the string with load P consists of two linear sections, one is horizontal and the other one is vertical, see Fig.3. As discussed before, we have

$$\begin{cases} F_1\cos\alpha_1 = F_2\cos\alpha_2 = T \\ F_1\sin\alpha_1 + F_2\sin\alpha_2 = -P \end{cases}.$$

From above, the equation

$$\tan\alpha_1 + \tan\alpha_2 = \frac{-P}{T}$$

implies $-\frac{-\delta}{s} + \frac{-\delta}{l-s} = \frac{-P}{T}$ and so, $\delta = \frac{P(l-s)s}{Tl}$, where δ denotes the vertical distance from the load to the wall that is distance s from the origin.

Suppose the deflection δ is known, it is not difficult to find the deflection of the string at any point x with similar triangles. The results are

$$y = \begin{cases} \frac{P(l-s)x}{Tl} \text{ if } 0 \le x \le s \\ \frac{P(l-x)s}{Tl} \text{ if } s \le x \le l \end{cases}. \tag{3}$$

When P is a unit load, which means $P = 1$, $G(x, s)$ could be used as a new name for the corresponding function $y(x, s)$ defined by (3). Obviously, we can find that $G(x, s)$ is the same if you swap the two variables x and s; that is,

$$G(x, s) = G(s, x).$$

$G(x, s)$ is generally mentioned as an **influence function** because it describes the influence at any point x of the string on which there is a unit load concentrated at the point s.

Actually, we can find $G(x, s)$ even though we do not solve equation (2). To do this, we first divide the interval $[0, l]$ into n small subintervals by the points $s_0 = 0, s_1 = \frac{l}{n}, s_2 = \frac{2l}{n}, \ldots, s_n = l$ with $\Delta s_i = s_i - s_{i-1}$; and let ϵ_i be an arbitrary point in Δs_i. Here we have another assumption that we let the portion of the distributed load acting on the subinterval Δs_i, which is $w(\epsilon_i)\Delta s_i$, be concentrated at the point $s = \epsilon_i$. Then the deflection at each point x by this load is

$$[w(\epsilon_i)\Delta s_i]G(x,\epsilon_i).$$

We can obtain the sum

$$\sum_{i=1}^{n} w(\epsilon_i)G(x,s)\Delta s_i,$$

if we add up all the deflections at the point x, which together approximate the actual distributed load. As $\Delta s_i \to 0$, the deflection at an arbitrary point x is

$$y(x) = \int_0^t w(\epsilon_i)G(x,s)ds. \tag{4}$$

Thus, once the function $G(x,s)$ is known, we can find the deflection for any piecewise distributed load by the integral (4).

As we discussed above, the influence function

$$G(x,s) = \begin{cases} \frac{(l-s)x}{Tl} & 0 \le x \le s \\ \frac{(l-x)s}{Tl} & s \le x \le l \end{cases},$$

corresponding to the differential equation $Ty'' = -w(x)$ is symmetric for x and s. There are some other properties of $G(x,s)$. Firstly, it is easy to see that $G(x,s)$ satisfies the boundary conditions of the problem at the point $x = 0$ and $x = l$; secondly, it is not difficulty to verify that $G(x,s)$ is continuous for x on the interval $[0,l]$. From the expression of $G(x,s)$, it is obvious except at the point $x = s$, where we need to figure out both the left and right limits of $G(x,s)$. The limits

$$\lim_{x\to s^-}\frac{(l-s)x}{Tl} \quad = \quad \frac{(l-s)s}{Tl}$$

and

$$\lim_{x\to s^+}\frac{(l-x)s}{Tl} \quad = \quad \frac{(l-s)s}{Tl}$$

are both equal to $G(s,s)$. On the other hand, $G'(x,s)$ has a jump of $-\frac{1}{T}$ at $x=s$. This is a point of discontinuity. To verify this, first we notice that $G(x,s)$ is differentiable for the whole interval $[0,l]$ except at $x=s$. Therefore, we check that

$$\lim_{x\to s^-} G_x(x,s) = \lim_{x\to s^-}\frac{l-s}{Tl} \quad = \quad \frac{l-s}{Tl}$$

and

$$\lim_{x\to s^+} G_x(x,s) = \lim_{x\to s^+} -\frac{s}{Tl} \quad = \quad -\frac{s}{Tl}$$

So, their difference is

$$-\frac{s}{Tl}-\frac{l-s}{Tl}=-\frac{1}{T}.$$

Finally, we note that $G(x,s)$ satisfies the homogeneous differential equation $Ty''=0$ at all point of the interval $[0,l]$ and $G_{xx}(x,s)$ does not exist because $G_x(x,s)$ is discontinuous at that point.

Actually, we can call any function that satisfies all the properties we discussed above the Green's function for the associated differential equation with boundary conditions. Now we will give the definition of the Green's function.

Definition 1.1. For a differential equation

$$a_0(x)y''+a_1(x)y'+a_2(x)y=0$$

with the homogeneous boundary conditions $\alpha_1 y(a) = \alpha_2 y'(a)$, $\beta_1 y(b) = \beta_2 y'(b)$, where at least one of α_1, α_2 is not zero and at least one of β_1, β_2 is not zero. A function $G(x, s)$ with the property that

1. $G(x, s)$ satisfies the differential equation for $a \leq x \leq s$ and for $s \leq x \leq b$,
2. $G(x, s)$ satisfies the boundary condition of the associated problem, that is $\alpha_1 G(a, s) = \alpha_2 G_x(a, s)$, $\beta_1 G(b, s) = \beta_2 G_x(b, s)$ for $a \leq s \leq b$,
3. $G(x, s)$ is continuous function of x for $a \leq x \leq b$,
4. $G_x(s, x)$ is continuous for $a \leq x < s$ and for $s < x \leq b$ but has a step discontinuity of magnitude $-\frac{1}{a_0(s)}$ at $x = s$,

is called the **Green's function** of the problem defined by the given differential equation with its boundary conditions.

2. The Green's Functions for Even Ordered Problems

In this section, we will discussed the Green's function for some even order problems. For the second order problem,

$$(-1)y'' = f(y), \quad t \in [0,1] \text{ where } f(y) > 0$$

with boundary condition $y(0) = 0$, $y'(1) = 0$, the associated Green's function of

$$(-1)y'' = 0$$

satisfying the above boundary conditions is

$$G_2(t,s) = \begin{cases} t \text{ if } 0 \leq t \leq s \leq 1 \\ s \text{ if } 0 \leq s \leq t \leq 1 \end{cases}.$$

Now, for the fourth order differential equation

$$(-1)^2 y'''' = f(y), \quad t \in [0,1] \text{ where } f(y) > 0 \tag{5}$$

with the boundary conditions

$$\begin{cases} y(0) = y''(0) = 0 \\ y'(1) = y'''(1) = 0 \end{cases}, \tag{6}$$

the associated Green's function of

$$(-1)^2 y'''' = 0 \tag{7}$$

satisfying (6) is

$$G_4(t,s) = \begin{cases} -\frac{t^3}{6} - \frac{s^2 t}{2} + ts \text{ if } 0 \leq t \leq s \leq 1 \\ -\frac{s^3}{6} - \frac{st^2}{2} + ts \text{ if } 0 \leq s \leq t \leq 1 \end{cases}.$$

Now, we derive the Green's function for the even order problems.

Theorem 2.1. *For the 2nth order differential equation*

$$(-1)^n y^{(2n)}(t) + f(y) = 0, n = 1, 2, 3 \cdots \ t \in [0, 1], \tag{8}$$

with boundary conditions

$$\begin{cases} y^{(2k)}(0) = 0 \\ y^{(2k+1)}(1) = 0 \text{ for } k = 0, 1, 2, \cdots, n-1 \end{cases}, \tag{9}$$

the Green's function of

$$(-1)^n y^{(2n)} = 0$$

satisfying the above conditions is

$$G_{2n}(t, s) = \int_0^1 G_2(t, w) G_{2n-2}(w, s) dw.$$

Proof. Firstly, we will prove that the associated Green's functions of

$$(-1)^n y^{(2n)} = f(y), \ \ n = 1, 2, 3 \cdots$$

satisfies

$$G_{2n}(t, s) = \int_0^1 G_2(t, w) G_{2n-2}(w, s) dw$$

Suppose $G_2(t, s)$ is the Green's function of $-y''(t) = 0$ satisfying $y(0) = y'(1) = 0$. Then

$$-y''(t) = g \quad \text{implies} \quad y(t) = \int_0^1 G_2(t, s) g(s) \, ds.$$

Therefore, if

$$\begin{aligned} y''''(t) &= g \text{ that is } (y'')'' = g \\ \text{then } \; y''(t) &= -\int_0^1 G_2(t,s)g(s)\,ds = -H(t) \\ \text{so that } \; y(t) &= \int_0^1 G_2(t,w)H(w)\,dw \\ &= \int_0^1 G_2(t,w)\left\{\int_0^1 G_2(w,s)g(s)\,ds\right\} dw \\ &= \int_0^1\int_0^1 G_2(t,w)G_2(w,s)g(s)\,ds\,dw \\ &= \int_0^1 \left\{\int_0^1 G_2(t,w)G_2(w,s)\,dw\right\} g(s)\,ds \\ &= \int_0^1 G_4(t,s)g(s)\,ds \end{aligned}$$

where

$$G_4(t,s) = \int_0^1 G_2(t,w)G_2(w,s)\,dw.$$

By definition of $G_2(t,s)$,

$$G_4(t,s) = \int_0^1 G_2(t,w)G_2(w,s)\,dw$$

$$\text{implies } \; G_4''(t,s) = -G_2(t,s),$$

which implies that y'' satisfies the boundary conditions of the $2nd$ order problem, i.e.; $y(0) = 0$, $y'(1) = 0$.

Likewise, $G_4(t,s)$ satisfies the boundary conditions (6) so that $y(t)$ satisfies the boundary conditions

$$\begin{aligned} y(0) &= 0, \\ y'(1) &= 0, \\ y''(0) &= 0, \\ y'''(1) &= 0. \end{aligned}$$

So, $G_4(t,s)$ is the Green's function for the equation

$$(-1)^2 y''''(t) = 0$$

with boundary conditions

$$\begin{aligned} y(0) &= 0, \\ y'(1) &= 0, \\ y''(0) &= 0, \\ y'''(1) &= 0. \end{aligned}$$

Also,

$$(-1)^3 y^{(6)}(t) = g(t)$$

implies that

$$\begin{aligned} -(y'')^{(4)}(t) &= g(t). \\ \text{So we know that} \quad -y''(t) &= \int_0^1 G_4(t,s)g(s)\,ds = H(t) \\ \text{so that} \quad y(t) &= \int_0^1 G_2(t,w)H(w)\,dw \\ &= \int_0^1 G_2(t,w)\int_0^1 G_4(w,s)g(s)\,ds\,dw \\ &= \int_0^1 \left\{\int_0^1 G_2(t,w)G_4(w,s)g(s)\,dw\right\} ds \end{aligned}$$

$$= \int_0^1 G_6(t,s)g(s)\,ds$$

where

$$G_6(t,s) = \int_0^1 G_2(t,w)G_4(w,s)g(s)\,dw. \tag{10}$$

By definition of $G_2(w,s)$,(9) implies

$$G_6''(t,s) = -G_4(t,s)$$

which means that y'' satisfies the boundary conditions above, that is,

$$\begin{aligned} y(0) &= 0, \\ y'(1) &= 0, \\ y''(0) &= 0, \\ y'''(1) &= 0. \end{aligned}$$

Also, $G_6''(t,s)$ satisfies the boundary conditions for the 6th order differential equation so that $y(t)$ satisfies the boundary conditions

$$\begin{aligned} y(0) &= 0, \\ y'(1) &= 0, \\ y''(0) &= 0, \\ y'''(1) &= 0, \\ y''''(0) &= 0, \\ y^{(5)}(1) &= 0. \end{aligned}$$

Continuing in this way, we find out that

$$G_{2n}(t,s) = \int_0^1 G_2(t,w)G_{2n-2}(w,s)\,dw \quad n \in \mathbb{N} \tag{11}$$

is the Green's function for

$$(-1)^n y^{2n}(t) = 0, \quad n \in \mathbb{N}$$

with boundary conditions

$$\begin{cases} y^{(2k)} & = 0 \\ y^{(2k+1)} & = 0, \quad k = 0, 1, 2, \cdots, n-1 \end{cases}.$$

□

The following theorem gives the bound of G_{2n} which will be used in Chapter 3.

Theorem 2.2. *For* $\frac{1}{2} \leq t \leq 1$ *and* $0 \leq s \leq 1$,

$$\frac{3^{n-1}}{2^{4n-3}} s \leq G_{2n}(t,s) \leq 1.$$

Proof. We will prove this with induction and we will prove that $G_{2n}(t,s) \geq 0$ first. For $n = 1$, we know that

$$G_2(t,s) = \begin{cases} t \text{ if } 0 \leq t \leq s \leq 1 \\ s \text{ if } 0 \leq s \leq t \leq 1 \end{cases},$$

so $G_2(t,s) \geq 0$ and from Theorem 1.2,

$$G_4(t,s) = \int_0^1 G_2(t,w) G_2(w,s)\, dw.$$

Obviously, $G_4 \geq 0$, Similarly,

$$G_6(t,s) = \int_0^1 G_2(t,w) G_4(w,s)\, dw \geq 0,$$

continuing in this way, we know that for all n,

$$G_{2n}(t,s) \geq 0.$$

For $n = 1$, it is obvious that

$$\frac{1}{2} s \leq G_2(t,s) \leq G_2(s,s) = s \leq 1.$$

Suppose that for $n = k$,

$$\frac{3^{k-1}}{2^{4k-3}} s \leq G_{2k}(t,s) \leq 1$$

is also true. Then for $n = k+1$, from the previous part,

$$\begin{aligned}
G_{2k+2}(t,s) &= \int_0^1 G_2(t,w)G_{2k}(w,s)\,dw \\
&\geq \int_{\frac{1}{2}}^1 \frac{1}{2}w\frac{3^{k-1}}{2^{4k-3}}s\,dw \\
&= \frac{1}{2}\frac{3^{k-1}}{2^{4k-3}}s\int_{\frac{1}{2}}^1 w\,dw \\
&= \frac{3^k}{2^{4k+1}}s,
\end{aligned}$$

For the second inequality, we need to prove that $G_{2n}(s,s) \leq 1$, for $n \in \mathbb{N}$. When $n = 1$, $G_2(s,s) = s \leq 1$. Suppose for $n = k$, $G_{2k}(s,s) \leq 1$ is also true. Then, for $n = k+1$, since $G_2(s,s) \leq 1$ and $G_{2k}(s,s) = 1$,

$$G_{2k+2}(s,s) = \int_0^1 G_2(s,w)G_{2k}(w,s)\,dw \leq \int_0^1 1 \cdot 1\,dw = \int_0^1 1\,dw \leq 1,$$

which is also true. So, finally we have

$$\frac{3^{n-1}}{2^{4n-3}}G_{2n}(s,s) \leq G_{2n}(t,s) \leq 1,$$

for $\frac{1}{2} \leq t \leq 1$ and $0 \leq s \leq 1$. □

3. Background Definitions and a Twin Fixed-point Theorem

In this section, we will provide some background materials from the theory of cones in Banach spaces, and we will use a twin-fixed point theorem for a cone preserving operator.

Definition 3.1. Let $(B, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $\rho \subset B$ is said to be a *cone* provided the following are satisfied
(a) if $y \in \rho$ and $\lambda \geq 0$, then $\lambda y \in \rho$
(b) if $y \in \rho$ and $-y \in \rho$, then $y = 0$.

For every cone in B, we define

$$x \leq y \text{ if and only if } y - x \in \rho.$$

Definition 3.2. Given a cone ρ in real Banach space B, a functional $\psi : \rho \to \mathbb{R}$ is said to be non-decreasing on ρ, if $\psi(x) \leq \psi(y)$, for all $x, y \in \rho$ with $x \leq y$.

Definition 3.3. Given a nonnegative continuous functional γ on a cone ρ of a real Banach space B, we define for each $d > 0$, the set

$$\rho(\gamma, d) = \{x \in \rho | \gamma(x) < d\}.$$

Next, we are going to introduce a fixed-point theorem due to Avery and Henderson to discuss the multiple positive solutions of the $2nth$ order differential equation. The method requires that we determine whether the differential problems with boundary conditions after transformed by the operator A satisfy Avery and Henderson's Fixed Point Theorem. If it does, then we can say that the boundary value problems have at least two positive solutions because the fixed points are the solutions.

Theorem 3.4. *(Avery-Henderson Fixed Point Theorem [1]) Let ρ be a cone in a real Banach space B. Let α and γ be nondecreasing, nonnegative, continuous functionals on ρ, and let θ be a nonnegative continuous functional on ρ with $\theta(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \text{ and } \|x\| \leq M\gamma(x),$$

for all $x \in \overline{\rho(\gamma, c)}$. Suppose there exist a completely continuous operator $A : \overline{\rho(\gamma, c)} \to \rho$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x), \text{ for } 0 \leq \lambda \leq 1 \text{ and } x \in \partial\rho(\theta, b),$$

and

(i) $\gamma(Ax) > c$, *for all* $x \in \partial\rho(\gamma, c)$;

(ii) $\theta(Ax) < b$, *for all* $x \in \partial\rho(\theta, b)$;

(iii) $\rho(\alpha, a) \neq \emptyset$, *and* $\alpha(Ax) > a$, *for all* $x \in \partial\rho(\alpha, a)$.

Then A has at least two fixed points x_1 and x_2 belonging to $\overline{\rho(\gamma, c)}$ such that

$$\begin{cases} a < \alpha(x_1), \text{ with } \theta(x_1) < b \\ b < \theta(x_2), \text{ with } \gamma(x_2) < c \end{cases}.$$

4. Twin Positive Solutions of Even Order Differential Equations with Boundary Conditions

In this section, we will apply Theorem 2.4 to establish the existence of twin positive solutions of (8). Suppose $G_{(2n)}(t,s)$ is the Green's function for $(-1)^n y^{(2n)} = 0$. satisfying the boundary condition (9). As we discussed in Theorem (1.3), for

$$\frac{1}{2} \leq t \leq 1, 0 \leq s \leq 1,$$

the Green's function for (8)(9) satisfies

$$\frac{3^{n-1}}{2^{4n-3}} s \leq G_{2n}(t,s) \leq 1.$$

Next, let the Banach space $B = C[0,1]$ be endowed with the norm $\|y\| = \max\limits_{0 \leq t \leq 1} \{y(t)\}$, and choose the cone $\rho \subset B$ defined by

$$\rho = \{y \in B \mid y'' < 0, \ y' \geq 0 \text{ and } y \geq 0 \text{ on } \ [0,1]\}.$$

We fix

$$\frac{1}{2} < r < 1,$$

and define the nonnegative, nondecreasing, continuous functionals, $\gamma, \theta,$ and α, by

$$\begin{aligned}
\gamma(y) &= \min_{\frac{1}{2} \leq t \leq r} y(t) = y(\frac{1}{2}), \\
\theta(y) &= \max_{0 \leq t \leq \frac{1}{2}} y(t) = y(\frac{1}{2}), \\
\alpha(y) &= \max_{0 \leq t \leq r} y(t) = y(r).
\end{aligned}$$

We know that, for each $y \in \rho$,

$$\gamma(y) = \theta(y) \leq \alpha(y). \tag{12}$$

In addition, for each $y \in \rho$, we have $\gamma(y) = y(\frac{1}{2}) \geq \frac{1}{2}y(1) = \frac{1}{2}\|y\|$. Thus,

$$\|y\| \leq 2\gamma(y), \text{ for all } y \in \rho. \tag{13}$$

Finally, we also observe that,

$$\theta(\lambda y) = \lambda\theta(y), 0 \leq \lambda \leq 1 \text{ and } y \in \partial\rho(\theta, b). \tag{14}$$

Now, we state growth conditions on f so that (8), (9) has at least two positive solutions.

Theorem 4.1. *Let $0 < a < \frac{3^{n-1}r^2}{2^{4n-2}}b < \frac{3^{n-1}r^2}{2^{4n-1}}c$, and suppose that f satisfies the following conditions:*
(A) $f(w) > \frac{2^4 n}{3^n}c$ if $c \leq w \leq 2c$,
(B) $f(w) < b$ if $0 \leq w \leq 2b$,
(C) $f(w) > \frac{2^{4n-2}}{3^{n-1}r^2}a$ if $0 \leq w \leq a$.
Then, the boundary value problem (8) has at least two positive solutions, x_1 and x_2, such that

$$a < \max_{0 \leq t \leq r} x_1(t), \text{ with } \max_{0 \leq t \leq \frac{1}{2}} x_1(t) < b,$$

$$b < \max_{0 \leq t \leq \frac{1}{2}} x_2(t), \text{ with } \min_{\frac{1}{2} \leq t \leq r} x_2(t) < c.$$

Proof. We begin the proof by defining the completely continuous integral operator $A : B \to B$ by

$$Ax(t) = \int_0^1 G_{2n}(t, s) f(x(s))\, ds, x \in B, 0 \leq t \leq 1.$$

It is well known that the solution of (8),(9) is the fixed points of A and conversely. We will show that the conditions of Theorem 2.4 are satisfied.

First, let $x \in \overline{\rho(\gamma, c)}$. By the nonnegativity of f and G_{2n}, for $0 < t < 1$,

$$Ax(t) = \int_0^1 G_{2n}(t,s) f(x(s))\, ds \geq 0.$$

In addition, $(Ax)'' = -f(x(t)) \leq 0$. This implies $(Ax)(t)$ is concave down on $[0,1]$, and also $(Ax)'$ is nonincreasing. Since $G_{2n}(t,s)$ satisfies boundary conditions of (9), we know that $(Ax)'(1) = 0$, and so $(Ax)'(t) \geq 0$ on $[0,1]$. Consequently, $Ax \in \rho$. We conclude $A : \overline{\rho(\gamma, c)} \to \rho$.

Now, we turn to property (i) in Theorem 2.4. Choose $x \in \partial\rho(\gamma, c)$, then we have $\gamma(x) = \min\limits_{\frac{1}{2} \leq t \leq r} x(t) = x(\frac{1}{2}) = c$. Since $x \in \rho$, $x(t) \geq c$, $\frac{1}{2} \leq t \leq 1$. Because $\|x\| \leq 2\gamma(x) = 2c$, we have

$$c \leq x(t) \leq 2c, \quad \frac{1}{2} \leq t \leq 1.$$

As a consequence of condition (A),

$$f(x(s)) > \frac{2^4 n}{3^n} c, \;\; \frac{1}{2} \leq s \leq 1.$$

Also, $Ax \in \rho$, so

$$\begin{aligned}
\gamma(Ax) &= Ax(\frac{1}{2}) \\
&= \int_0^1 G_{2n}(\frac{1}{2}, s) f(x(s)) ds \\
&\geq \int_{\frac{1}{2}}^1 G_{2n}(\frac{1}{2}, s) f(x(s)) ds.
\end{aligned}$$

From Theorem 1.3

$$\begin{aligned}
\gamma(Ax) &\geq \int_{\frac{1}{2}}^{1} \frac{3^{n-1}}{2^{4n-3}} s f(x(s)) ds \\
&> \frac{2^{4n}}{3^{n}} c \frac{3^{n-1}}{2^{4n-3}} \int_{\frac{1}{2}}^{1} s ds \\
&= c.
\end{aligned}$$

So (i) of Theorem 2.4 is satisfied.

Next, we will address (ii) of Theorem 2.4. We choose $x \in \partial\rho(\theta, b)$. Then $\theta(x) = \max_{0 \leq t \leq \frac{1}{2}} x(t) = x(\frac{1}{2}) = b$. This implies $0 \leq x(t) \leq b, 0 \leq t \leq \frac{1}{2}$, and since $x \in \rho$, we also have $b \leq x(t) \leq \|x\| = x(1),\ \frac{1}{2} \leq t \leq 1$. Moreover, $\|x\| \leq 2\gamma(x) \leq 2\theta(x) = 2b$, so,

$$0 \leq x(t) \leq 2b,\ 0 \leq t \leq 1.$$

From condition (B), we have $f(x(s)) < b,\ 0 \leq s \leq 1$ and so

$$\begin{aligned}
\theta(Ax) &= Ax(\frac{1}{2}) \\
&= \int_0^1 G(\frac{1}{2}, s) f(x(s)) ds \\
&< b \int_0^1 G(\frac{1}{2}, s) ds \\
&\leq b \cdot \int_0^1 1 ds = b \cdot 1 = b.
\end{aligned}$$

Hence, (ii) of Theorem 2.4 holds. For the final part, we turn to (iii) of Theorem 2.4. We define $y(t) = \frac{a}{2}$, for all $0 \leq t \leq 1$, then

$\alpha(y) = \frac{a}{2} < a$, and $\rho(\alpha, a) \neq \emptyset$ We now choose $x \in \partial\rho(\alpha, a)$, which means

$\alpha(x) = \max\limits_{0 \le t \le r} x(l) = x(r) = a$. This implies

$$0 \le x(t) \le a,\ 0 \le t \le r$$

From assumption (C),

$$f(x(s)) > \frac{2^{4n-2}}{3^{n-1}r^2}a, \quad 0 \le s \le r.$$

Then, the same as before, $Ax \in \rho$, and so

$$\begin{aligned}
\alpha(Ax) &= (Ax)(r) \\
&= \int_0^1 G_{2n}(r,s)f(x(s))ds \\
&\ge \int_0^r G_{2n}(r,s)f(x(s))ds \\
&> \frac{2^{4n-2}}{3^{n-1}r^2}a \int_0^r \frac{3^{n-1}}{2^{4n-3}} sds \\
&= \frac{2^{4n-2}}{3^{n-1}r^2}a \frac{3^{n-1}}{2^{4n-3}} \int_0^r sds \\
&= \frac{2^{4n-2}}{3^{n-1}r^2}a \frac{3^{n-1}}{2^{4n-3}} (\frac{r^2}{2}) \\
&= a.
\end{aligned}$$

Thus, (iii) of Theorem 2.4 is satisfied. Hence, there are at least two fixed points of A which are positive solution belonging to $\overline{\rho(\gamma, c)}$, of the *2nth* order boundary value problem such that

$$a < \alpha(x_1),\ \text{with}\ \theta(x_1) < b$$

and

$$b < \theta(x_2),\ \text{with}\ \gamma(x_2) < c.$$

$\square$

5. Background of Definitions and Theorems for Time Scale

In this section, I would introduce some definitions and theorems on time scale introduced from Bohner, Martin and Peterson, Allen's book *Dynamic Equations on Time Scales* Birkhauser, Boston, [2].

A time scale is an arbitrary non-empty closed subset of the real numbers. It is usually denoted by $\mathbb{T}$. Thus $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0$ are some examples of time scales. But $\mathbb{Q}, \mathbb{R} - \mathbb{Q}\,\{irrationals\}\,, \mathbb{C}$ and $(0,1)$,i.e., the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1, are not time scales. We move through the time scale using forward and backward jump operators. The gaps in the time scale are measured by a function μ, defined in terms of forward jump operator, σ.

Definition 5.1. *Forward jump operator* Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$. we define :

Forward jump operator : An operator $\sigma : \mathbb{T} \to \mathbb{T}$, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

Backward jump operator : An operator $\rho : \mathbb{T} \to \mathbb{T}$, by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

Note 1: If $\sigma(t) > t$, we say that t is $right-scattered$, while if $\rho(t) < t$ we say that t is $left-scattered$. The points which are both right-scattered and left-scattered are called isolated.

Note 2 : If $t <\ \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense.

Note 3 : If $t >\ \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense.

The forward jump operator defined on t, $\sigma(t)$, is not always equal to t. The difference between $\sigma(t)$ and t is called graininess.

Definition 5.2. The Graininess of a time scale, $\mathbb{T}$, $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$.

Note 4: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$. That is,

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - (\rho(\sup\ \mathbb{T}), sup\mathbb{T}] & \text{if } \sup T < \infty \\ \mathbb{T} & \text{if } \sup\ \mathbb{T} = \infty. \end{cases}$$

Note 5: Let $f : \mathbb{T} \to \mathbb{R}$ be a function, then we define the function, $f^\sigma : \mathbb{T} \to \mathbb{R}$, by $f^\sigma(t) = f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}, \qquad i.e., \ f^\sigma = f \circ \sigma.$

Using σ we define the delta derivative of a function f in a natural way.

Definition 5.3. Differentiation: Assume $f : \mathbb{T}^k \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\triangle(t)$ to be the number(provided it exists) with the property that given any $\epsilon > 0$ there exists a neighborhood $U = (t-\delta, t+\delta) \cap \mathbb{T}$ of t for some $\delta > 0$ such that

$|[f(\sigma(t)) - f(s)] - f^\triangle(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$ for all $s \in U$

where $f^\triangle(t)$ is called delta derivative of f at t.

Using the limit definition, assume $f : \mathbb{T} \to \mathbb{R}$ is continuous and let $t \in \mathbb{T}^k$. Then we define

$$f^\triangle(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s},$$

provided the limits exist.

We will introduce the delta derivative $f^\triangle$ for a function f defined on $\mathbb{T}$. It is expressed as

(i) $f^{\triangle} = f'$ (is the usual derivative) if $\mathbb{T} = \mathbb{R}$ and

(ii) $f^{\triangle} = \triangle f$ (is the forward difference operator) if $\mathbb{T} = \mathbb{Z}$.

Theorem 5.4. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following.*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If t is right scattered and f is continuous at t, then f is differentiable at t with

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right- dense, then f is differentiable at t iff the limit

$$\lim_{s \to\ t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\triangle}(t) = \lim_{s \to\ t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t,then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\triangle}(t).$$

Now we introduce the most powerful fundamentals of derivatives: the sum rule, product rule, quotient rule and the transformation of the sigma function in terms of original function and its derivative.

Theorem 5.5. *Assume f, g: $\mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$\begin{aligned}(fg)^{\triangle}(t) &= f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t) \\ &= f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t)).\end{aligned}$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$\left\{\frac{1}{f}\right\}^{\triangle}(t) = \frac{-f^{\triangle}(t)}{f(t)f(\sigma(t))}$$

(v) If $g(t)g(\sigma(t)) \neq 0$ then $\frac{f}{g}$ is differentiable at t and

$$\left\{\frac{f}{g}\right\}(t) = \frac{f^{\triangle}(t)g(t) - f(t)g^{\triangle}(t)}{g(t)g(\sigma(t))}$$

In addition to the differentiability we need couple of more conditions for integrability of the function.

Definition 5.6. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated*, provided its right-sided limits exists(finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exists(finite) at all left-dense points in $\mathbb{T}$. The set of such function is denoted by R.

Definition 5.7. A function $f : \mathbb{T} \to \mathbb{R}$ is called $rd-$*continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. It is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 5.8. A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable in the region of differentiation D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k - D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each each $t \in D$.

Definition 5.9. Assume $f : \mathbb{T} \to \mathbb{T}$ is regulated function. Any function F is called a $pre-antiderivative$ of f if $F^{\triangle}(t) = f(t)$.

Theorem 5.10. *Existence of Pre-Antiderivative Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that $F^{\triangle}(t) = f(t)$ holds for all $t \in D$.*

The indefinite integral of a regulated function f is given by

$$\int f(t)\triangle t = F(t) + C$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_r^s f(t)\triangle t = F(s) - F(r)$$

for all $r, s \in \mathbb{T}$.

Definition 5.11. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided
$F^{\triangle}(t) = f(t)$ holds for all $\quad t \in \mathbb{T}^k$.

TABLE: Time scale derivative and Antiderivative for $\quad \mathbb{T} = \mathbb{R} \quad or \quad \mathbb{T} = \mathbb{Z}$

Time $\mathbb{T}$	symbol	$\mathbb{R}$	$\mathbb{Z}$
Backward jump operator	$\rho(t)$	t	$t-1$
Forward jump operator	$\sigma(t)$	t	$t+1$
Graininess	$\mu(t)$	0	1
Derivative	$f^{\triangle}(t)$	$f'(t)$	$\triangle f(t)$
Integral	$\int_a^b f(t)\triangle t$	$\int_a^b f(t)dt$	$\sum\limits_{t=a}^{b-1} f(t)(\text{if} \quad a<b)$
Rd-continuous	f	continuous f	any f

Theorem 5.12. *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)\triangle\tau = \mu(t)f(t).$$

Some fundamental laws of integration are summarized in the following theorem including two laws of integration by parts.

Theorem 5.13. *If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*
(i) $\int_a^b (f(t)+g(t))\triangle t = \int_a^b f(t)\triangle t + \int_a^b g(t)\triangle t$;
(ii) $\int_a^b (\alpha f)(t)\triangle t = \alpha \int_a^b f(t)\triangle t$;
(iii) $\int_a^b f(t)\triangle t = -\int_b^a f(t)\triangle t$;
(iv) $\int_a^b f(t)\triangle t = \int_a^c f(t)\triangle t + \int_c^b f(t)\triangle t$
(v) $\int_a^b f(\sigma(t))g(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(t)\triangle t$;
(vi) $\int_a^b f(t)g^{\triangle}(t)\triangle t = (fg)(b) - (fg)(a) - \int_a^b f^{\triangle}(t)g(\sigma(t))\triangle t$;
(vii) $\int_a^a f(t)\triangle t = 0$
(viii) If $|f(t)| \le g(t)$ *on* $[a, b)$, *then* $|\int_a^b f(t)\triangle t| \le \int_a^b g(t)\triangle t$;
(ix) If $f(t) \ge 0$ *for all* $a \le t \le b$, *then* $\int_a^b f(t)\triangle t \ge 0$.

The interesting part of time scale calculus is that the integration can also be performed if the domain of the function is a subset of the integers. Thus integration of any function depends upon the domain of the function.

Theorem 5.14. *Let $a, b, c \in \mathbb{T}$ and $f \in C_{rd}$.*

(i) If $\mathbb{T} = \mathbb{R}$,

$$\int_a^b f(t)\triangle t = \int_a^b f(t)dt$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) If [a,b] consists of only isolated points, then

$$\int_a^b f(t)\triangle t = \begin{cases} \sum\limits_{t\in[a,b)} \mu(t)f(t) & \text{if } a < b \\ 0 & \text{if } a{=}b \\ -\sum\limits_{t\in[b,a)} \mu(t)f(t) & \text{if } a > b. \end{cases}$$

(iii) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\triangle t = \begin{cases} \sum\limits_{t=a}^{b-1} f(t) & \text{if } a < b \\ 0 & \text{if } a{=}b \\ -\sum\limits_{t=b}^{a-1} f(t) & \text{if } a > b. \end{cases}$$

Now we have gone through some basic definitions and theorems on time scale which will be refered to in this thesis. In the next section, we will use the integral on time scale when $\mathbb{T} = \mathbb{Z}$.

6. Twin Positive Solutions of Even Order Difference Equation with Boundary Conditions

In this section, by applying Theorem 2.4, we will prove that the following dynamic equations

$$\Delta^{(2n)}u(k) + g(u(k)) = 0, n = 1, 2, 3 \cdots \;\; k \in \{0, \cdots N\} \tag{15}$$

with boundary conditions

$$\begin{cases} \Delta^{(2k)}u(0) = 0 \\ \Delta^{(2k+1)}u(N+1) = 0 \;\; \text{for} \;\; k = 0, 1, 2 \cdots, n-1 \end{cases} \tag{16}$$

has twin solutions. For $n = 1$, the associated Green's function $H_2(k, l)$ is

$$H_2(k,l) = \begin{cases} k, \;\; k \in \{0, 1, 2 \cdots, l\} \\ l+1, \;\; k \in \{l+1 \cdots, N+2\}. \end{cases}$$

For the *2nth* order difference equation, the Green's function is

$$H_{2n+2} = \int_0^N H_2(k, w) H_{2n}(w, s) \Delta w.$$

To find out the range of $H_{2n}(k, l)$, we set

$$h = [\frac{N+2n}{2}], \text{ which denote the integer part of } \frac{N+2n}{2},$$

then we can get the following theorem.

Theorem 6.1. *For $l = \{0, 1, 2, \cdots, N\}$, the Green's function of (15), (16) satisfies*

$$\begin{aligned} H_{2n}(k,l) &\leq (l+1)(\frac{N^2+3N+2}{2})^{n-1} \; \textit{for } k = \{0, 1, 2 \cdots, N+2n\}, \\ H_{2n}(k,l) &\geq \frac{l+1}{2} \frac{[(h+N+2)(N-h+1)]^{n-1}}{4^{n-1}} \; \textit{for } k = \{h \cdots, N+2n\}. \end{aligned}$$

Proof. We will prove the first inequality in the statement of the theorem by induction. First, for $n = 1$,

$$H_2(k,l) = \begin{cases} k, \ k \in \{0,1,2\cdots,l\} \\ l+1, \ k \in \{l+1\cdots,N+2\}. \end{cases}$$

$$H_2(k,l) = \begin{cases} k \leq l < l+1 \text{ when } k \in \{0,1,2,\cdots,l\} \\ l+1 \leq l+1 \text{ when } k \in \{l+1,l+2,\cdots,N+2\} \end{cases} .$$

So it is verified.

Suppose that for $n = m$ is also true. Then for $n = m+1$,

$$\begin{aligned} H_{2m+2}(k,l) &= \int_0^N H_2(k,w)H_{2m}(w,l)\Delta w \\ &\leq \int_0^N (w+1)(l+1)(\frac{N^2+3N+2}{2})^{m-1}\Delta w \\ &= (l+1)(\frac{N^2+3N+2}{2})^{m-1}\int_0^N (w+1)\Delta w \\ &= (l+1)(\frac{N^2+3N+2}{2})^{m-1}\sum_{w=0}^{N} w+1 \\ &= (l+1)(\frac{N^2+3N+2}{2})^{m-1}(\frac{N^2+3N+2}{2}) \\ &= (l+1)(\frac{N^2+3N+2}{2})^{m}. \end{aligned}$$

So, the inequality is also true for $n = m+1$. That is

$H_{2n}(k,l) \leq (l+1)(\frac{N^2+3N+2}{2})^{n-1}$ for $k = \{0,1,2\cdots,N+2n\}$, $l = \{0,1,2,\cdots,N\}$ holds. For the second inequality, for $n = 1$, obviously it is true. Suppose for

$n = m$ it is also true, Then for $n = m + 1$,

$$\begin{aligned}
H_{2m+2}(k,l) &= \int_0^N H_2(k,w)H_{2m}(w,l)\Delta w \\
&\geq \int_h^N H_2(k,w)H_{2m}(w,l)\Delta w \\
&\geq \int_h^N \frac{w+1}{2}\frac{l+1}{2}\frac{[(h+N+2)(N-h+1)]^{m-1}}{4^{m-1}}\Delta w \\
&= \frac{l+1}{2}\frac{[(h+N+2)(N-h+1)]^{m-1}}{4^{m-1}}\int_h^N \frac{w+1}{2}\Delta w \\
&= \frac{l+1}{2}\frac{[(h+N+2)(N-h+1)]^{m-1}}{4^{m-1}}\frac{(N+h+2)(N-h+1)}{4} \\
&= \frac{l+1}{2}\frac{[(h+N+2)(N-h+1)]^{m}}{4^{m}}.
\end{aligned}$$

So, the inequality is also true for $n = m + 1$ Therefore, we have

$$H_{2n}(k,l) \geq \frac{l+1}{2}[\frac{(h+N+2)(N-h+1)}{4}]^{n-1} \text{ for } k = \{h \cdots, N+2n\}, l = \{0,1,2,\cdots,N\}.$$

The theorem is proved. $\square$

Next, let the Banach Space $B = \{v : \{0,\cdots,N+2n\} \to \mathbb{R}\}$ be endowed with the norm $\|v\| = \max_{k\in\{0,\cdots,N+2n\}} |v(k)|$, and choose the cone $\rho \subset B$ defined by

$$\rho = \{v \in B \mid v(k) \geq 0, v^{\Delta}(k) \geq 0 \text{ on } \{0,1,\cdots,N+2n\}, \text{ and } \Delta^2 v(k) \leq 0, k \in \{0,\cdots,N\}\}$$

For $v \in \rho$,

$$v(k) \geq \frac{1}{2}\|v\| = \frac{1}{2}v(N+2n), \text{ for } k \in \{h,\cdots,N+2n\}.$$

Also, for the remainder of this section, fix an integer r with

$$h < r < N + 2n - 1,$$

and define the nonnegative, nondecreasing, continuous functionals, γ, θ, and α on ρ, by

$$\begin{aligned}\gamma(v) &= \min_{k\in\{h,\cdots,r\}} v(k) = v(h),\\ \theta(v) &= \max_{k\in\{0,\cdots,h\}} v(k) = v(h) \text{ and}\\ \alpha(v) &= \max_{k\in\{0,\cdots,r\}} v(k) = v(r).\end{aligned}$$

We observe that, for each $v \in \rho$,

$$\gamma(v) = \theta(v) \leq \alpha(v).$$

In addition, for each $v \in \rho$, $\gamma(v) \geq \frac{1}{2}v(N+2n) = \frac{1}{2}\|v\|$, so that

$$\|v\| \leq 2\gamma(v), \text{ for all } v \in \rho.$$

Finally, we again have

$$\theta(\lambda v) = \lambda\theta(v), 0 \leq \lambda \leq 1 \text{ and } v \in \partial\rho(\theta, b).$$

As in the Section 4, we now will put growth condition on g such that (15) and (16) has at least two positive solutions belonging to the cone ρ.

Theorem 6.2. *Let* $0 < a < \frac{[(h+N+2)(N-h+1)]^{n+1}(r+1)(r+2)}{[2(N^2+3N+2)]^n} b < \frac{[(h+N+2)(N-h+1)]^{n+1}(r+1)(r+2)}{2[2(N^2+3N+2)]} c$, *and suppose that g satisfies the following conditions:*

$$\begin{aligned}(A)\ g(w) &> [\frac{4}{(h+N+2)(N-h+1)}]^n c, \ \text{if } c \leq w \leq 2c,\\ (B)\ g(w) &< (\frac{2}{N^2+3N+2})^n b, \ \text{if } 0 \leq w \leq 2b,\\ (C)\ g(w) &> [\frac{4}{(h+N+2)(N-h+1)}]^{n-1} \frac{4}{(r+1)(r+2)} a, \ \text{if } 0 \leq w \leq a.\end{aligned}$$

Then, the dynamic equation (15) and (16) has at least two positive solutions, u_1 and u_2, such that

$$a < \max_{k\in\{0,1,\cdots,r\}} u_1(k), \ \textit{with} \ \max_{k\in\{0,\cdots,h\}} u_1(k) < b,$$

$$b < \max_{k\in\{0,1,\cdots,h\}} u_2(k), \ \textit{with} \ \min_{k\in\{h,\cdots,r\}} u_2(k) < c.$$

Proof. Define the operator $A : B \to B$ by

$$Au(k) = \int_0^N H_{2n}(k,l)g(u(l))\Delta k$$

From Theorem 5.3, it follows that

$$Au(k) = \sum_{l=0}^{N} H_{2n}(k,l)g(u(l)), \quad u \in B, \quad k \in \{0,\cdots,N+2n\}$$

So, A is completely continuous, and it is well known that $u \in B$ is a solution of (11) if and only if u is a fixed point of A. Now we will show that the conditions of Theorem 2.4 hold with respect to A. If we choose $u \in \overline{\rho(\gamma,c)}$, then $Au(k) = \sum_{t=0}^{N} H(k,l)g(u(l)) \geq 0$ on $\{0,\cdots,N+2n\}$, in addition, $\Delta^2(Au) = -g(u(k)) \leq 0$, and so $\Delta(Au)$ is nonincreasing on $\{0,\cdots,N+2n-1\}$. Thus, $Au(k)$ is nondecreasing on $\{0,\cdots,N+2n\}$. In addition, $(Au)(0) = 0$, and so $(Au)(0) \geq 0$. Hence, we know that $Au \in \rho$, so $A : \overline{\rho(\gamma,c)} \to \rho$.

For (i) of Theorem 2.4, we choose $u \in \partial\rho(\gamma,c)$. Then, $\gamma(u) = \min_{k\in h,\cdots,r} u(k) = u(h) = c$. This implies $u(k) \geq c, k \in \{h,\cdots,N+2n\}$. And because $\|u\| \leq 2\gamma(u) = 2c$, we have

$$c \leq u(k) \leq 2c, \ k \in \{h,\cdots,N+2n\}.$$

For condition (A),

$$g(u(l)) > [\frac{4}{(h+N+2)(N-h+1)}]^n c,\ l \in \{h, \cdots, N+2\}$$

Since $Au \in \rho$, we have

$$\begin{aligned}
\gamma(Au) &= (Au)(h) \\
&= \sum_{l=0}^{N} H_{2n}(h,l) g(u(l)) \\
&\geq \sum_{l=h}^{N} H_{2n}(h,l) g(u(l)) \\
&> [\frac{4}{(h+N+2)(N-h+1)}]^n c [\frac{(h+N+2)(N-h+1)}{4}]^{n-1} \cdot \\
&\quad \sum_{l=h}^{N} \frac{l+1}{2} \\
&= [\frac{4}{(h+N+2)(N-h+1)}]^n c [\frac{(h+N+2)(N-h+1)}{4}]^{n-1} \\
&\quad \frac{(h+N+2)(N-h+1)}{4} \\
&= [\frac{4}{(h+N+2)(N-h+1)}]^n c [\frac{(h+N+2)(N-h+1)}{4}]^{n} \\
&= c.
\end{aligned}$$

Thus, part (i) of Theorem 2.4 is satisfied. To verify (ii) of Theorem 2.4 is satisfied, we choose $u \in \partial\rho(\theta, b)$. Then, $\theta(u) = \max_{k \in \{0,\cdots,h\}} u(k) = u(h) = b$. So $0 \leq u(k) \leq b,\ k \in \{0, \cdots, h\}$, and since $u \in \rho$, it follows that $b \leq u(k) \leq \|u\| = u(N+2n),\ k \in \{h+1, \cdots, N+2n\}$. Recall that $\|u\| \leq 2\gamma(u) \leq 2\theta(u) = 2b$. So,

$$0 \leq u(k) \leq 2b,\ k \in \{0, \cdots, N+2n\}.$$

From condition (B),

$$g(u(l)) < (\frac{2}{N^2+3N+2})^n b,\ l \in 0,\cdots,N.$$

Then,

$$\begin{aligned}
\theta(Au) &= (Au)(h) \\
&= \sum_{l=0}^{N} H_{2n}(h,l)g(u(l)) \\
&< (\frac{2}{N^2+3N+2})^n b \sum_{l=0}^{N} H_{2n}(h,l) \\
&< (\frac{2}{N^2+3N+2})^n b \sum_{l=0}^{N} (l+1)(\frac{N^2+3N+2}{2})^{n-1} \\
&= (\frac{2}{N^2+3N+2})^n b(\frac{N^2+3N+2}{2})^{n-1} \sum_{l=0}^{N} (l+1) \\
&= (\frac{2}{N^2+3N+2})^n b(\frac{N^2+3N+2}{2})^{n-1} \frac{N^2+3N+2}{2} \\
&= (\frac{2}{N^2+3N+2})^n b(\frac{N^2+3N+2}{2})^{n} \\
&= b.
\end{aligned}$$

In particular, (ii) of Theorem 2.4 is satisfied.

Now we turn to (iii) of Theorem 2.4. We observe that $v(k) = a$, $\frac{a}{2} \in \rho(\alpha, a)$, so $\rho(\alpha, a) \neq \emptyset$. Let $u \in \partial\rho(\alpha, a)$. Then $\alpha(u) = \max_{k \in \{0,\cdots,r\}} u(k) = u(r) = a$. So

$$0 \le u(k) \le a,\ k \in \{0,\cdots,r\}.$$

Using hypothesis (C), we have

$$g(u(l)) > [\frac{4}{(h+N+2)(N-h+1)}]^{n-1} \frac{4}{(r+1)(r+2)} a,\ l \in \{0,\cdots,r\}$$

from which we obtain

$$
\begin{aligned}
\alpha(Au) &= (Au)(r) \\
&= \sum_{l=0}^{N} H_{2n}(r,l)g(u(l)) \\
&\geq \sum_{l=0}^{r} H_{2n}(r,l)g(u(l)) \\
&> [\frac{4}{(h+N+2)(N-h+1)}]^{n-1}\frac{4}{(r+1)(r+2)}a\sum_{l=0}^{r} H_{2n}(r,l) \\
&= [\frac{4}{(h+N+2)(N-h+1)}]^{n-1}\frac{4}{(r+1)(r+2)}a[\frac{(h+N+2)(N-h+1)}{4}]^{n-1}\sum_{l=0}^{r}\frac{l+1}{2} \\
&= [\frac{4}{(h+N+2)(N-h+1)}]^{n-1}\frac{4}{(r+1)(r+2)}a[\frac{(h+N+2)(N-h+1)}{4}]^{n-1}\frac{(r+1)(r+2)}{4} \\
&= a.
\end{aligned}
$$

So, part(iii) of Theorem 2.4 holds. Thus the dynamic equation (14)(15) has at least two positive solutions, u_1 and u_2, such that

$$a < \max_{k\in\{0,1,\cdots,r\}} u_1(k), \text{ with } \max_{k\in\{0,\cdots,h\}} u_1(k) < b$$

$$b < \max_{k\in\{0,1,\cdots,h\}} u_2(k), \text{ with } \min_{k\in\{h,\cdots,r\}} u_2(k) < c.$$

The proof is completed. $\square$

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