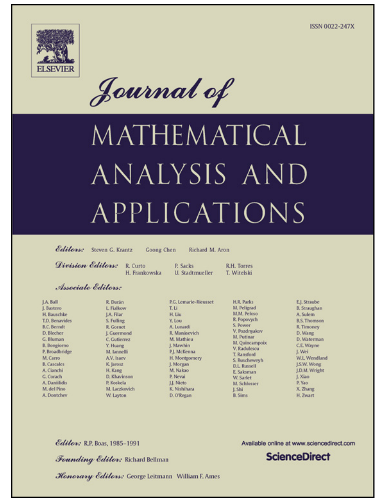
Accepted Manuscript



Classes of operators preserved by extensions or liftings

Jesús M.F. Castillo, Manuel González, Antonio Martínez-Abejón

PII: S0022-247X(18)30128-8
DOI: https://doi.org/10.1016/j.jmaa.2018.02.011
Reference: YJMAA 22023

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 28 March 2017

Classes of operators preserved by extensions or liftings

Jesús M. F. Castillo, Manuel González, and Antonio Martínez-Abejón

Abstract. A standard way to obtain extensions (resp. liftings) of operators is by making the so-called operations of push-out (resp. pull-back). In this paper we study the preservation of some classes of operators associated with an operator ideal $\mathcal{A}$ under push-out extensions or pull-back liftings. We show several examples of classical operator ideals whose associated classes are preserved, we prove that the preservation of those classes under push-out extension or pull-back lifting implies that the space ideal of $\mathcal{A}$ satisfies the 3-space property, and we derive some results for $\mathcal{A}$ that are useful in the study of commutative diagrams of operators.

1. Introduction

We study the preservation of some classes of operators under the canonical methods of push-out and pull-back (PO and PB in short). We refer to Section 2 for unexplained terminology. A germinal result was proved in [**13, 14**] showing that the tauberian operators are preserved under PO-extensions and the cotauberian operators are preserved under PB-liftings.

In Section 3 we prove several stability results under PO-extensions or PB-liftings for the semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ associated with an operator ideal $\mathcal{A}$ (see [**1**]). Some of these results are valid for $\mathcal{A}$ injective or surjective. Moreover, if $\mathcal{A}_+$ is always preserved by PO-extensions (i.e., it satisfies the 3S-PO property) or $\mathcal{A}_-$ is always preserved by PB-liftings (i.e., it satisfies the 3S-PB property), then the space ideal of $\mathcal{A}$ has the 3-space property. Some concrete examples are provided by the semigroups $\mathcal{A}_+$ characterized in terms of sequences in Proposition 8: $\mathcal{A}_+$ satisfies the 3S-PO property and its dual class $(\mathcal{A}_+)^d = (\mathcal{A}^d)_-$ satisfies the 3S-PB property.

In general, operator ideals are stable under PO-extensions or PB-liftings only if we impose some additional conditions. This is what we do in Section 4 where we introduce the $3S_-$ property and the $3S_+$ property in terms of exact sequences.

Many operator ideals satisfy those restricted stability properties. For example, the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{U}$, $\mathcal{C}$ and $\mathcal{WC}$, introduced in Subsection 3.1, satisfy the $3S_+$

2010 Mathematics Subject Classification. Primary: 46B03, 46B08, 46B10.
Keywords: extending operators; lifting operators; operator ideal.
The research of the first author has been supported in part by Project IB16056 and Ayuda a Grupos GR15152 de la Junta de Extremadura; that of the three authors has been supported in part by MINECO (Spain), Project MTM2016-76958.

property, their dual operator ideals satisfy the $3S_-$ property, and the p-converging operators ($1 \le p \le \infty$) satisfy both properties.

To know that an operator ideal $\mathcal{A}$ satisfies one of the properties $3S_-$ or $3S_+$ is useful when studying complex situations in commutative diagrams. For example, let us consider the push-out diagram in Proposition 14. It is shown in [**11**, Proposition 3.3] that the operators q_g, q_j are strictly singular if and only if so are q_i, q_f; and [**7**, Lemma 8] can be restated by saying that the operator ideal of strictly singular operators satisfy the $3S_-$ property. We show that such results can be framed in a general study about the behavior of operator ideals. We also show that the space ideal of $\mathcal{A}$ has the 3-space property when $\mathcal{A}$ has one of the properties $3S_-$ or $3S_+$.

Our notations are standard like those of [**2**] and [**10**]. An *operator* is a bounded linear map acting between Banach spaces, and an *embedding* is an operator with continuous inverse, not necessarily surjective. We denote by $\operatorname{Ker} T$ and $\operatorname{Ran} T$ the *kernel* and the *range* of an operator T. The set of all operators acting from X into Y is denoted $\mathcal{L}(X,Y)$, given a class $\mathcal{A}$ of operators, its component in $\mathcal{L}(X,Y)$ is $\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y)$, and we write $\mathcal{A}(X)$ instead of $\mathcal{A}(X,X)$. A sequence of operators

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0$$

is an *exact sequence* if the kernel of each operator coincides with the range of the previous one, that is, j is an embedding, q is surjective, and $\operatorname{Ker} q = \operatorname{Ran} j$.

2. Preliminaries and basic results

We review the push-out and pull-back constructions for a pair of operators when one of them is either an embedding or a surjection.

Given $T \in \mathcal{L}(X,Y)$ and an embedding $J \in \mathcal{L}(X,Z)$ then $\Delta := \{(Tx, -Jx) : x \in X\}$ is a closed subspace of $Y \oplus_1 Z$, and the *push-out space* $\operatorname{PO}(T,J)$ of the pair (T,J) is defined as the quotient space $\operatorname{PO}(T,J) = (Y \oplus_1 Z)/\Delta$. The operators $\overline{J} : Y \to \operatorname{PO}(T,J)$ and $\overline{T} : Y \to \operatorname{PO}(T,J)$ defined as $\overline{J}(y) = (y,0) + \Delta$ and $\overline{T}(z) = (0,z) + \Delta$ yield the following commutative diagram which is called the *PO-diagram of* (T,J)*:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\ J\ } & Z & \xrightarrow{\ q\ } & Z/J(X) & \longrightarrow & 0 \\
 & & {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \overline{T}} & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{\ \overline{J}\ } & \operatorname{PO}(T,J) & \xrightarrow{\ p\ } & Z/J(X) & \longrightarrow & 0
\end{array} \tag{1}$$

where q is the quotient map, and $p\big((y,z)+\Delta\big) = z + J(X)$.

Since $\overline{J}$ is an embedding (Proposition 1), both rows in (1) are exact sequences, and $\overline{T}$ can be regarded as an extension of T to the superspace Z, which we call the *PO-extension of* T *through* J.

Similarly, given $S \in \mathcal{L}(Y,X)$ and a surjective operator $Q \in \mathcal{L}(Z,X)$, the *pull-back space* of (S,Q) is defined as the space $\operatorname{PB}(S,Q) = \{(y,z) \in Y \oplus_\infty Z \colon Sy = Qz\}$. The operators $\underline{S} : \operatorname{PB}(S,Q) \to Z$ and $\underline{Q}\colon \operatorname{PB}(S,Q) \to Y$, given by $\underline{S}(y,z) = z$ and $\underline{Q}(y,z) = y$ provide a commutative diagram which is called the *PB-diagram of* (S,Q)*:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} Q & \xrightarrow{\ i\ } & PB(S,Q) & \xrightarrow{\ Q\ } & Y & \longrightarrow & 0 \\
 & & \| & & \underline{S}\downarrow & & \downarrow S & & \\
0 & \longrightarrow & \operatorname{Ker} Q & \xrightarrow{\ j\ } & Z & \xrightarrow{\ Q\ } & X & \longrightarrow & 0
\end{array} \tag{2}
$$

where j is the natural embedding, and $i(z) = (0, z)$.

Since $\underline{Q}$ is surjective (Proposition 3), both rows in (2) are exact sequences, and $\underline{S}$ can be regarded as a lifting of S to Z, which we call the *PB-lifting of S by Q.*

Sometimes we will simply write PO and PB instead of PO(T, J) and PB(S, Q).

The proof of the following result is not difficult.

Lemma 1. *An operator $T \in \mathcal{L}(X, Y)$ has closed range if and only if each bounded sequence $(x_n)_{n=1}^{\infty}$ in X such that $\lim Tx_n = 0$ satisfies $\lim_{n\to\infty} \operatorname{dist}(x_n, \operatorname{Ker} T) = 0$.*

Next we show some properties of the PO-extensions and PB-liftings.

Proposition 1. *In the PO-diagram (1), where J is an embedding, one has:*

(i) *$\overline{J}$ is an embedding;*
(i) $\dim \operatorname{Ker} T = \dim \operatorname{Ker} \overline{T}$*;*
(ii) *T has closed range if and only if so has $\overline{T}$.*

Proof. Statement *(i)* is proved in [**10**, Lemma 1.3.b], and *(ii)* is a consequence of $\operatorname{Ker} \overline{T} = J\big(\operatorname{Ker} T\big)$.

To prove *(iii)*, we apply Lemma 1. Assume $\overline{T}$ has closed range and let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in X such that $\lim Tx_n = 0$. Thus $\lim \overline{T}Jx_n = 0$, and therefore, $\lim \operatorname{dist}(Jx_n, \operatorname{Ker} \overline{T}) = 0$. But J is an embedding and $\operatorname{Ker} \overline{T} = J(\operatorname{Ker} T)$, hence also $\lim \operatorname{dist}(x_n, \operatorname{Ker} T) = 0$, which proves that T has closed range.

Conversely, assume T has closed range and let $(z_n)_{n=1}^{\infty}$ be a bounded sequence in Z such that $\lim \overline{T}(z_n) = 0$. Then there exists a sequence $(Tx_n, -Jx_n)_{n=1}^{\infty}$ in Δ such that $\lim(Tx_n, z_n - Jx_n) = 0$. Since $Tx_n \to 0$ and T has closed range, $\lim \operatorname{dist}(x_n, \operatorname{Ker} T) = 0$. Moreover $\operatorname{Ker} \overline{T} = J\big(\operatorname{Ker} T\big)$ and $\lim(z_n - Jx_n) = 0$ means that $\lim \operatorname{dist}(z_n, \operatorname{Ker} \overline{T}) = 0$. Hence the range of $\overline{T}$ is closed. □

The push-out and pull-back constructions satisfy the following relations.

Proposition 2 (Duality relations).

(a) *Given $T \in \mathcal{L}(X, Y)$ and an embedding $J \in \mathcal{L}(X, Z)$, the dual of the PO-diagram of (T, J) can be identified with the PB-diagram of (T^*, J^*).*
(b) *Given $S \in \mathcal{L}(Y, X)$ and a surjective operator $Q \in \mathcal{L}(Z, X)$, the dual of the PB-diagram of (S, Q) can be identified with the PO-diagram of (S^*, Q^*).*

Proof. (a) The operator J^* is surjective with kernel $J(X)^{\perp}$. Since $\mathrm{PO}(T, J) = (Y \oplus_1 Z)/\Delta$ and $\Delta^{\perp} = \mathrm{PB}(T^*, J^*)$, we get $\mathrm{PO}(T, J)^* \equiv \mathrm{PB}(T^*, J^*)$. Let $(y^*, z^*) \in \mathrm{PB}(T^*, J^*) \equiv \Delta^{\perp}$. For each $z \in Z$,

$$
\begin{aligned}
\langle z, (\overline{T})^*(y^*, z^*)\rangle &= \langle \overline{T}z, (y^*, z^*)\rangle = \langle (0, z) + \Delta, (y^*, z^*)\rangle \\
&= \langle z, z^*\rangle = \langle z, \overline{(T^*)}(y^*, z^*)\rangle.
\end{aligned}
$$

Thus $(\overline{T})^* = \overline{(T^*)}$, and similarly we can prove $(\overline{J})^* = \overline{(J^*)}$ and the remaining identifications.

(b) The operator Q^* is an embedding and $\mathrm{PB}(S,Q)^\perp = \{(S^*x^*, -Q^*x^*) : x^* \in X^*\}$. Indeed, it is easy to check that $\mathrm{PB}(S,Q)$ coincides with the annihilator ${}^\perp\{(S^*x^*, -Q^*x^*) : x^* \in X^*\}$. So the equality follows from the fact that $\{(S^*x^*, -Q^*x^*) : x^* \in X^*\}$ is a weak*-closed subspace of $(Y \oplus_\infty Z)^*$. Thus $\mathrm{PB}(S,Q)^* \equiv \mathrm{PO}(S^*, Q^*)$.

To prove $(\underline{S})^* = \overline{(S^*)}$, $(\underline{Q})^* = \overline{(Q^*)}$ and the remaining identifications, we can proceed as we did for part (a). □

Propositions 1 and 2 imply the next result.

Proposition 3. *In the PB-diagram (2), where Q is surjective, one has:*

(i) *$\underline{Q}$ is surjective;*
(ii) $\dim X/\overline{\mathrm{Ran}\, S} = \dim Z/\overline{\mathrm{Ran}\, \underline{S}}$*;*
(iii) *S has closed range if and only if so has $\underline{S}$.*

3. Semigroups associated with an operator ideal

In [**12**, Definition 6.1.1] (see also [**1**]), certain classes of operators, called *semigroups,* were introduced, and it was proved that each operator ideal $\mathcal{A}$ has two associated semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ whose components are defined as follows:

$$\mathcal{A}_+(X,Y) := \{T \in \mathcal{L}(X,Y) \colon S \in \mathcal{L}(Z,X), TS \in \mathcal{A} \Rightarrow S \in \mathcal{A}\},$$
$$\mathcal{A}_-(X,Y) := \{T \in \mathcal{L}(X,Y) \colon S \in \mathcal{L}(Y,Z), ST \in \mathcal{A} \Rightarrow S \in \mathcal{A}\}.$$

Among the basic properties of $\mathcal{A}_+$ and $\mathcal{A}_-$ one has:

Proposition 4. [**12**, Propositions 6.1.8 and 6.1.14] *Let $\mathcal{A}$ be an operator ideal, let $T \in \mathcal{L}(X,Y)$ and let $S \in \mathcal{L}(Y,Z)$.*

(1) $S, T \in \mathcal{A}_+ \Rightarrow ST \in \mathcal{A}_+ \Rightarrow T \in \mathcal{A}_+$,
(2) $S, T \in \mathcal{A}_- \Rightarrow ST \in \mathcal{A}_- \Rightarrow S \in \mathcal{A}_-$.

Note that two different operator ideals $\mathcal{A}$ and $\mathcal{B}$ can have the same associated semigroups, say $\mathcal{A}_+ = \mathcal{B}_+$ (when $\mathcal{A}$ stands for the compact or the strictly singular operators, $\mathcal{A}_+$ is the class of upper semi-Fredholm operators) or $\mathcal{A}_- = \mathcal{B}_-$ (when $\mathcal{A}$ stands for the compact or the strictly cosingular operators, $\mathcal{A}_-$ is the class of lower semi-Fredholm operators).

Recall that an operator ideal $\mathcal{A}$ is called *injective* if given $S \in \mathcal{L}(X,Y)$ and an embedding $J \in \mathcal{L}(Y,Z)$, $JS \in \mathcal{A}$ implies $S \in \mathcal{A}$ (i.e., all embeddings are in $\mathcal{A}_+$). The operator ideal $\mathcal{A}$ is called *surjective* if given $S \in \mathcal{L}(X,Y)$ and a surjective operator $Q \in \mathcal{L}(Z,X)$, $SQ \in \mathcal{A}$ implies $S \in \mathcal{A}$ (i.e., all surjective operators are in $\mathcal{A}_-$).

Proposition 5. *Let $\mathcal{A}$ be an operator ideal and consider $T \in \mathcal{L}(X,Y)$ and an embedding $J \in \mathcal{L}(X,Z)$.*

(1) *If $\mathcal{A}$ is surjective and $T \in \mathcal{A}_-$ then $\overline{T} \in \mathcal{A}_-$.*
(2) *If $\mathcal{A}$ is injective and $\overline{T}J \in \mathcal{A}_-$ then T, $\overline{J} \in \mathcal{A}_-$.*
(3) *If $\mathcal{A}$ is injective and $\overline{T} \in \mathcal{A}_+$ then $T \in \mathcal{A}_+$.*

Proof. (1) Let $S \in \mathcal{L}(\mathrm{PO}, W)$ and suppose that $S\overline{T} \in \mathcal{A}$. Since $T \in \mathcal{A}_-$ and $S\overline{T}J = S\overline{J}T \in \mathcal{A}$, we obtain $S\overline{J} \in \mathcal{A}$. Let $Q : Y \oplus_1 Z \to \mathrm{PO}$ be the natural quotient map. Then $S\overline{T}, S\overline{J} \in \mathcal{A}$ and $SQ(y, z) = S\overline{J}y + S\overline{T}z$, hence $SQ \in \mathcal{A}$, thus $S \in \mathcal{A}$ because $\mathcal{A}$ is surjective.

(2) Note that $\overline{J}\,T = \overline{T}J \in \mathcal{A}_-$ implies $\overline{J} \in \mathcal{A}_-$. For the other part, let $\tau \in \mathcal{L}(Y, W)$, suppose that $\tau T \in \mathcal{A}$, and look at the diagram

$$\begin{array}{ccccc} X & \xrightarrow{T} & Y & \xrightarrow{\tau} & W \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle \overline{J}} & & \downarrow{\scriptstyle \overline{J}\,'} \\ Z & \xrightarrow[\overline{T}]{} & \mathrm{PO} & \xrightarrow[\tau']{} & \mathrm{PO}' \end{array}$$

where $\mathrm{PO}' = PO(\tau, \overline{J})$ and τ' and $\overline{J}'$ are the corresponding extensions.

From $\overline{T}J \in \mathcal{A}_-$ and $\overline{J}\,'\tau T = \tau'\overline{T}J \in \mathcal{A}$ we get $\tau' \in \mathcal{A}$, and thus $\overline{J}\,'\tau = \tau'\overline{J} \in \mathcal{A}$. Since $\overline{J}\,'$ is an embedding and $\mathcal{A}$ is injective, $\tau \in \mathcal{A}$. Thus, $T \in \mathcal{A}_-$.

(3) $\mathcal{A}$ injective implies $J \in \mathcal{A}_+$. Thus $\overline{T}J = \overline{J}T \in \mathcal{A}_+$ which implies $T \in \mathcal{A}_+$. □

An argument dual to the one in the proof of Proposition 5 provides the following result.

Proposition 6. *Let $\mathcal{A}$ be an operator ideal and consider $S \in \mathcal{L}(Y, X)$ and a surjective operator $Q \in \mathcal{L}(Z, X)$.*

(1) *If $\mathcal{A}$ is injective and $S \in \mathcal{A}_+$ then $\underline{S} \in \mathcal{A}_+$.*
(2) *If $\mathcal{A}$ is surjective and $Q\underline{S} \in \mathcal{A}_+$ then S, $\underline{Q} \in \mathcal{A}_+$.*
(3) *If $\mathcal{A}$ is surjective and $\underline{S} \in \mathcal{A}_-$ then $S \in \mathcal{A}_-$.*

Recall that a class of Banach spaces $\mathbb{A}$ satisfies the 3-*space property* if X is in $\mathbb{A}$ when X contains a closed subspace Y such that $Y, X/Y \in \mathbb{A}$. We refer to [**10**] for examples of classes of Banach spaces satisfying or failing the 3-space property. Also the *space ideal* $Sp(\mathcal{A})$ associated with an operator ideal $\mathcal{A}$ is the class of all Banach spaces X such that the identity I_X belongs to $\mathcal{A}$ [**21**].

Note that $X \in Sp(\mathcal{A})$ if and only if $\mathcal{A}(X, Y) = \mathcal{A}_+(X, Y) = \mathcal{L}(X, Y)$ for every Y, and similarly $Y \in Sp(\mathcal{A})$ if and only if $\mathcal{A}(X, Y) = \mathcal{A}_-(X, Y) = \mathcal{L}(X, Y)$ for every X.

Proposition 7. *Let $\mathcal{A}$ be an operator ideal. Suppose that*

(a) *every PO-extension of an operator in $\mathcal{A}_+$ belongs to $\mathcal{A}_+$, or*
(b) *every PB-lifting of an operator in $\mathcal{A}_-$ belongs to $\mathcal{A}_-$.*

Then $Sp(\mathcal{A})$ satisfies the 3-space property.

Proof. Let X be a Banach space with a closed subspace Y such that $Y, X/Y \in Sp(\mathcal{A})$. Let us denote by J and Q the embedding of Y into X and the quotient map from X onto X/Y respectively.

(a) When every PO-extension of an operator in $\mathcal{A}_+$ belongs to $\mathcal{A}_+$, we consider the PO-diagram of $(0_Y, J)$, where 0_Y is the zero operator on Y. Thus $\mathrm{PO} = Y \oplus_1 X/Y$ and $\overline{0_Y}x = (0, Qx)$.

We have $\overline{0_Y} \in \mathcal{A}(X, \mathrm{PO}) \cap \mathcal{A}_+(X, \mathrm{PO})$ because $\mathrm{PO} \in Sp(\mathcal{A})$ and $0_Y \in \mathcal{A}_+$. Hence $X \in Sp(\mathcal{A})$.

(b) When every PB-lifting of an operator in $\mathcal{A}_-$ belongs to $\mathcal{A}_-$, we consider the PB-diagram of $(0_{X/Y}, Q)$. Thus $\mathrm{PB} = X/Y \oplus_\infty Y \in Sp(\mathcal{A})$ and $\underline{0_{X/Y}}(Qx, y) = y$.

We have $\underline{0_{X/Y}} \in \mathcal{A}(\mathrm{PB}, X) \cap \mathcal{A}_-(\mathrm{PB}, X)$ because $\mathrm{PB} \in \overline{Sp(\mathcal{A})}$ and $0_{X/Y} \in \mathcal{A}_-$. Hence $X \in Sp(\mathcal{A})$. □

Proposition 7 suggests to introduce the following properties:

Definition 1. *Let $\mathcal{A}$ be an operator ideal.*

(a) *We say that $\mathcal{A}_+$ satisfies the* 3S-PO property *if every PO-extension of an operator in $\mathcal{A}_+$ belongs to $\mathcal{A}_+$.*
(b) *We say that $\mathcal{A}_-$ satisfies the* 3S-PB property *if every PB-lifting of an operator in $\mathcal{A}_-$ belongs to $\mathcal{A}_-$.*

As we mentioned before, for different operator ideals $\mathcal{A}$ and $\mathcal{B}$ it may happen that $\mathcal{A}_+ = \mathcal{B}_+$ or $\mathcal{A}_- = \mathcal{B}_-$. For this reason the notions introduced in Definition 1 cannot be properly described as properties of the operator ideal.

3.1. Semigroups admitting a sequential characterization. Here we show several concrete examples of semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ for which the previous results in this section are applicable.

We denote by $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{U}$, $\mathcal{C}$ and $\mathcal{WC}$ the operator ideals of compact, weakly compact, Rosenthal, unconditionally convergent, completely continuous, and weakly completely continuous operators (see [**21**] for their definitions). The corresponding semigroups $\mathcal{A}_+$ admit a sequential characterization:

Proposition 8. [**15**], [**1**, Section 3.4] *Let $T \in \mathcal{L}(X, Y)$.*

(a) *$T \in \mathcal{K}_+$ if and only if each bounded sequence (x_n) with (Tx_n) convergent has a convergent subsequence.*
(b) *$T \in \mathcal{R}_+$ if and only if each bounded sequence (x_n) with (Tx_n) weakly Cauchy has a weakly Cauchy subsequence.*
(c) *$T \in \mathcal{W}_+$ if and only if each bounded sequence (x_n) with (Tx_n) weakly convergent has a weakly convergent subsequence.*
(d) *$T \in \mathcal{U}_+$ if and only if each w.u.C. series $\sum_n x_n$ with $\sum_n Tx_n$ unconditionally convergent is unconditionally convergent.*
(e) *$T \in \mathcal{C}_+$ if and only if each weakly Cauchy sequence (x_n) with (Tx_n) convergent is convergent.*
(f) *$T \in \mathcal{WC}_+$ if and only if each weakly Cauchy sequence (x_n) with (Tx_n) weakly convergent is weakly convergent.*

The semigroups characterized in Proposition 8 were studied in [**1, 4, 18, 20, 6, 16**]. Also note that $\mathcal{W}_+$ is the class of tauberian operators and $\mathcal{W}_-$ is the class of cotauberian operators introduced in [**18**] and [**22**], respectively.

Given a class $\mathcal{A}$ of operators, its dual class $\mathcal{A}^d$ is defined by its components

$$\mathcal{A}^d(X, Y) := \{T \in \mathcal{L}(X, Y) : T^* \in \mathcal{A}\}.$$

Observe that $\mathcal{K} = \mathcal{K}^d$, $\mathcal{W} = \mathcal{W}^d$, and that the dual class of an operator ideal is also an operator ideal [**21**].

REMARK 1. Let $\mathcal{A}$ be one of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{U}$, $\mathcal{C}$ and $\mathcal{WC}$. Then $\mathcal{A}$ is injective and $\mathcal{A}^d$ surjective [**21**]. Moreover $(\mathcal{A}^d)_- = (\mathcal{A}_+)^d$ [**12**, Section 3.5].

THEOREM 1. *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{U}$, $\mathcal{C}$ and $\mathcal{WC}$. Then $\mathcal{A}_+$ satisfies the* 3S-PO *property and $(\mathcal{A}^d)_-$ satisfies the* 3S-PB *property.*

PROOF. We begin with the result for $\mathcal{A}_+$, in which we have to consider separately the following different cases. Recall that Δ is the closed subspace $\{(Tx, -Jx) : x \in X\}$ of $Y \oplus_1 Z$.

Suppose $\overline{T} \notin \mathcal{C}_+$. Then Z contains a weakly Cauchy sequence (z_n) with no convergent subsequence such that $(\overline{T}z_n)$ is convergent. Thus there exists $(y, z) \in Y \oplus_1 Z$ such that $\operatorname{dist}\big((y, z - z_n), \Delta\big) \to 0$. Hence, for every n, we can select $(Tx_n, -Jx_n) \in \Delta$ such that

$$\|(y, z - z_n) - (Tx_n, -Jx_n)\|_1 = \|y - Tx_n\| + \|z - z_n + Jx_n\| \to 0.$$

Since $\|z - z_n + Jx_n\| \to 0$, (x_n) is a weakly Cauchy sequence having no convergent subsequences, and since $\|y - Tx_n\| \to 0$, (Tx_n) is convergent. Hence $T \notin \mathcal{C}_+$.

Suppose $\overline{T} \notin \mathcal{WC}_+$. Then there exists a weakly Cauchy sequence (z_n) in Z having no weakly convergent subsequence and such that $(\overline{T}z_n)$ is weakly convergent. Since for convex sets the norm closure and the weak closure coincide, we can find a sequence (w_k) of successive convex combinations of (z_n) such that (w_k) is weakly Cauchy and has no weakly convergent subsequence, and $(\overline{T}w_k)$ is convergent. Applying to (w_k) the argument we gave in the previous case we conclude that $T \notin \mathcal{WC}_+$.

Suppose $\overline{T} \notin \mathcal{R}_+$. Following the proof of Theorem 1 in [**15**], there exists a sequence (z_n) in Z equivalent to the unit vector basis of ℓ_1 such that $(\overline{T}z_n)$ converges to 0. Thus $\operatorname{dist}\big((0, -z_n), \Delta\big) \to 0$, and repeating the argument of the case of $\mathcal{C}_+$ (with $(y, z) = (0, 0)$ here) we obtain a sequence (x_n) in X equivalent to the unit vector basis of ℓ_1 such that (Tx_n) converges to 0. Hence $T \notin \mathcal{R}_+$.

Suppose that $\overline{T} \notin \mathcal{U}_+$. Then there exists a sequence (z_n) in Z equivalent to the unit vector basis of c_0 such that $(\overline{T}z_n)$ converges to 0, and the argument of the case of $\mathcal{R}_+$ allows us to conclude that $T \notin \mathcal{U}_+$.

For $\mathcal{W}_+$ we can give a similar argument (a weakly convergent sequence admits a sequence of successive convex blocks which is convergent). Moreover, since $\mathcal{K}_+$ is the class of operators with closed range and finite dimensional kernel, we can apply Proposition 1.

The result for $(\mathcal{A}^d)_-$ is a direct consequence of the result for $\mathcal{A}_+$, since $(\mathcal{A}^d)_- = (\mathcal{A}_+)^d$ and, by Proposition 2, we can identify the operators $(\underline{T})^*$ and $\overline{(T^*)}$. Moreover, since $\mathcal{K}_-$ is the class of operators with closed range and finite codimensional range, this case is a direct consequence of Proposition 3. □

4. Preservation of operator ideals by extension or lifting

We now consider operator ideals $\mathcal{A}$ such that, given an exact sequence

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0$$

satisfying some conditions and an operator $T : X \to W$, to check that $T \in \mathcal{A}$ it is enough to show that the restriction Tj is in $\mathcal{A}$. Clearly this property is equivalent to $j \in \mathcal{A}_-$.

We are also interested in operator ideals $\mathcal{A}$ for which, under some conditions on the exact sequence, $q \in \mathcal{A}_+$. Equivalently, an operator $S : W \to X$ is in $\mathcal{A}$ when $qS \in \mathcal{A}$.

The reason for the name of the following concept will be clear in Proposition 18.

Definition 2. *We say that an operator ideal $\mathcal{A}$ satisfies the* $3S_-$ property *if given an exact sequence*

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0$$

with $q \in \mathcal{A}$, we have $j \in \mathcal{A}_-$.

The operator ideal $\mathcal{K}$ satisfies the $3S_-$ property, but this is a trivial example because q compact implies Z finite dimensional.

Proposition 9. *The operator ideals $\mathcal{W}$, $\mathcal{R}^d$, $\mathcal{U}^d$, $\mathcal{C}^d$ and $\mathcal{WC}^d$ satisfy the $3S_-$ property.*

Proof. We consider an exact sequence

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0$$

and assume first that $q \in \mathcal{W}$. Since $\mathcal{W}$ is surjective, $Z \simeq X/j(Y)$ is reflexive. Hence $j \in \mathcal{W}_-$ by [**12**, Proposition 3.1.5], and $\mathcal{W}$ satisfies the $3S_-$ property.

Let $\mathcal{A}^d$ denote $\mathcal{R}^d$, $\mathcal{U}^d$, $\mathcal{C}^d$ or $\mathcal{WC}^d$. Since $\mathcal{A}^d$ is also surjective, using the characterization of $j \in (\mathcal{A}^d)_-$ given in [**12**, Proposition 3.5.12], we can apply the argument we used for $\mathcal{W}$. □

Proposition 10. *An operator ideal $\mathcal{A}$ satisfies the $3S_-$ property if and only if given a commutative diagram*

$$\begin{CD}
0 @>>> Y @>{j}>> X @>{q}>> Z @>>> 0\\
@. @V{T}VV @VV{\widehat{T}}V @| @.\\
0 @>>> Y' @>{i}>> X' @>{p}>> Z @>>> 0
\end{CD}$$

whose rows are exact sequences, we have $\widehat{T} \in \mathcal{A}$ when q and T belong to $\mathcal{A}$.

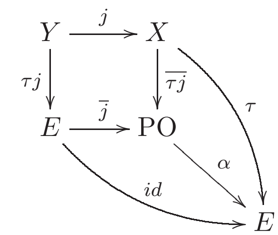
Proof. Suppose that $\mathcal{A}$ satisfies the $3S_-$ property and $q, T \in \mathcal{A}$. Then $j \in \mathcal{A}_-$ and $iT = \widehat{T}j \in \mathcal{A}$, hence $\widehat{T} \in \mathcal{A}$.

For the converse implication, suppose that $q \in \mathcal{A}$, and let $\tau \in \mathcal{L}(X, E)$ such that $\tau j \in \mathcal{A}$. A look at the PO-diagram

$$\begin{CD}
0 @>>> Y @>{j}>> X @>{q}>> Z @>>> 0\\
@. @V{\tau j}VV @VV{\overline{\tau j}}V @| @.\\
0 @>>> E @>>{\overline{j}}> \mathrm{PO} @>>> Z @>>> 0
\end{CD}$$

shows that the hypothesis yields that $\overline{\tau j} \in \mathcal{A}$. The universal property of the push-out [**5**, Section A.4.1] implies the existence of an operator α making a commutative diagram

(3)



Therefore $\tau = \alpha\overline{\tau j} \in \mathcal{A}$, and we conclude $j \in \mathcal{A}_-$. □

Remark 2. The proof of Proposition 10 makes clear that in its statement it is enough to consider the PO-diagrams described in Formula (1).

An operator $T \in \mathcal{L}(X,Y)$ is *strictly singular* (denoted $T \in \mathcal{SS}$) if no restriction of T to an infinite dimensional subspace of X is an embedding; and T is *strictly cosingular* (denoted $T \in \mathcal{SC}$) if there is no infinite codimensional closed subspace N of Y such that $Q_N T$ is surjective, where $Q_N : Y \to Y/N$ is the quotient map. Both $\mathcal{SS}$ and $\mathcal{SC}$ are operator ideals, $\mathcal{SS}$ is injective, not surjective, and $\mathcal{SC}$ is surjective, not injective [**21**].

Note also that the existence of an exact sequence

$$0 \longrightarrow \ell_2 \xrightarrow{\ j\ } Z_2 \xrightarrow{\ q\ } \ell_2 \longrightarrow 0$$

with j strictly cosingular and q strictly singular was proved in [**17**, Theorem 6.4], and other examples of such exact sequences can be found in [**7**].

The following result is a direct consequence of [**7**, Lemma 8] and Proposition 10.

Proposition 11. *The operator ideal $\mathcal{SS}$ satisfies the $3S_-$ property.*

A Banach space $\mathcal{X}$ is called *minimal* if every infinite dimensional subspace of $\mathcal{X}$ contains a subspace isomorphic to $\mathcal{X}$. Example of minimal spaces are c_0, ℓ_p $(1 \le p < \infty)$ and Tsirelson's space T^*. We refer to [**8**] for other examples of minimal spaces.

Definition 3. *Let $\mathcal{X}$ be a minimal space. We say that $T \in \mathcal{L}(X,Y)$ is* $\mathcal{X}$-singular *if there exists no subspace M of X isomorphic to $\mathcal{X}$ such that the restriction of T to M is an isomorphism.*

The arguments in the proof of [**23**, Proposition on p. 289] show that the $\mathcal{X}$-singular operators form an operator ideal, which we denote $\mathcal{X}$-$\mathcal{S}$. Clearly $\mathcal{X}$-$\mathcal{S}$ is an injective operator ideal. Moreover ℓ_1-$\mathcal{S} = \mathcal{R}$, the Rosenthal operators, and c_0-$\mathcal{S} = \mathcal{U}$, the unconditionally converging operators [**12**, Proposition 3.5.4].

Proposition 12. *Let $\mathcal{X}$ be a minimal Banach space. Then the operator ideal $\mathcal{X}$-$\mathcal{S}$ satisfies the $3S_-$ property.*

Proof. Suppose in the statement of Proposition 10 that $T \in \mathcal{X}$-$\mathcal{S}$ but $\widehat{T} \notin \mathcal{X}$-$\mathcal{S}$. Then there exists a subspace M of X isomorphic to $\mathcal{X}$ such that the restriction of $\widehat{T}$ to M is an isomorphism.

From $T \in \mathcal{X}$-$\mathcal{S}$ we derive that $M \cap j(Y)$ is finite dimensional and $M + j(Y)$ is closed. Otherwise, a standard perturbation argument would allow us to show that there exist a subspace N of M isomorphic to $\mathcal{X}$ and a nuclear operator $K : X \to X$ with $\|K\|$ arbitrarily small norm such that $(I - K)(N) \subset j(Y)$ and Tj an isomorphism on $(I - K)(N)$, which is not possible. And from $M \cap j(Y)$ finite dimensional and $M + j(Y)$ closed we derive $q \notin \mathcal{X}\text{-}\mathcal{S}$, concluding the proof. □

We say that a sequence (x_n) in a Banach space X is *weakly p-summable* $(1 \leq p < \infty)$ when, for each $f \in X^*$, $\sum_{n=1}^{\infty} |f(x_n)|^p < \infty$. We say that (x_n) is *weakly ∞-summable* if it is weakly null.

Definition 4. *Let $1 \leq p \leq \infty$. We say that an operator $T \in \mathcal{L}(X, Y)$ is* p-converging *if $\|Tx_n\| \to 0$ for each weakly p-convergent sequence (x_n) in X.*

We denote by $\mathcal{C}_p$ the class of p-converging operators. Note that $\mathcal{C}_p$ $(1 \leq p \leq \infty)$ is an injective, non-surjective, operator ideal [**9**, Lemma 2], $\mathcal{C}_\infty = \mathcal{C}$, the completely continuous operators, and $\mathcal{C}_1 = \mathcal{U}$, the unconditionally converging operators.

Proposition 13. *For $1 \leq p \leq \infty$, the operator ideal $\mathcal{C}_p$ satisfies the $3S_-$ property.*

Proof. Suppose in the statement of Proposition 10 that $q, T \in \mathcal{C}_p$ and let (x_n) be a weakly p-convergent sequence in X. It is enough to show that a subsequence of $(\widehat{T}x_n)$ converges in norm to 0.

Since $q \in \mathcal{C}_p$ we get $\|qx_n\| \to 0$, and passing to a subsequence we can pick $y_n \in Y$ so that $\|x_n - y_n\| \leq 2^{-n}$. Thus (y_n) is also weakly p-summable in Y, and $T \in \mathcal{C}_p$ implies $\|Ty_n\| \to 0$. Hence $\|\widehat{T}x_n\| \to 0$. □

The choice $\mathcal{A} = \mathcal{SS}$ in the following result yields [**11**, Proposition 3.3].

Proposition 14. *Let $\mathcal{A}$ be an operator ideal satisfying the $3S_-$ property. Consider the following commutative diagram*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Y & \xrightarrow{f} & X & \xrightarrow{q_f} & Z & \longrightarrow & 0 \\
 & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} & & \| & & \\
0 & \longrightarrow & X' & \xrightarrow{g} & P & \xrightarrow{q_g} & Z & \longrightarrow & 0 \\
 & & {\scriptstyle q_j}\downarrow & & \downarrow{\scriptstyle q_i} & & & & \\
 & & Z' & = & Z' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

whose rows and columns are exact sequences. Then $q_g, q_j \in \mathcal{A}$ if and only if $q_i, q_f \in \mathcal{A}$.

Proof. Suppose $q_g, q_j \in \mathcal{A}$. Then $q_f = iq_g \in \mathcal{A}$. Moreover, by Proposition 10, we have $g, j \in \mathcal{A}_-$. Then $gj = if \in \mathcal{A}_-$, hence $i \in \mathcal{A}_-$ which implies $q_i \in \mathcal{A}$. The proof of the converse implication is identical. □

Next we consider a dual version of Definition 2.

DEFINITION 5. *We say that an operator ideal $\mathcal{A}$ satisfies the* $3S_+$ property *if given an exact sequence*

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0$$

with $j \in \mathcal{A}$, we have $q \in \mathcal{A}_+$.

Again $\mathcal{K}$ is a trivial example of operator ideal satisfying the $3S_+$ property. The following results gives some other examples.

PROPOSITION 15. *The operator ideals $\mathcal{W}$, $\mathcal{R}$, $\mathcal{U}$, $\mathcal{C}$ and $\mathcal{WC}$ satisfy the $3S_+$ property.*

PROOF. Let $\mathcal{A}$ denote $\mathcal{W}$, $\mathcal{R}$, $\mathcal{U}$, $\mathcal{C}$ or $\mathcal{WC}$. We consider an exact sequence

$$0 \longrightarrow Y \xrightarrow{\ j\ } X \xrightarrow{\ q\ } Z \longrightarrow 0$$

and assume that $j \in \mathcal{A}$. Since $\mathcal{A}$ is injective, we get $Y \equiv \ker q \in Sp(\mathcal{A})$, and applying the characterizations of $\mathcal{A}_+$ in [**12**, Theorem 2.1.5 and Proposition 3.5.12], we get $q \in \mathcal{A}_+$. $\square$

A dual version of the proof of Proposition 10 proves the following result and shows that in its statement it is enough to consider the PB-diagrams described in Formula (2).

PROPOSITION 16. *An operator ideal $\mathcal{A}$ satisfies the $3S_+$ property if and only if, given a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\ i\ } & X' & \xrightarrow{\ p\ } & Z' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \widetilde{T}} & & \downarrow{\scriptstyle T} & & \\
0 & \longrightarrow & Y & \xrightarrow{\ j\ } & X & \xrightarrow{\ q\ } & Z & \longrightarrow & 0
\end{array}$$

whose rows are exact sequences, we have $\widetilde{T} \in \mathcal{A}$ when j and T belong to $\mathcal{A}$.

PROPOSITION 17. *For $1 \leq p \leq \infty$, the operator ideal $\mathcal{C}_p$ satisfies the $3S_+$ property.*

PROOF. We consider the commutative diagram in the statement of Proposition 16, and assume that $j, T \in \mathcal{C}_p$. Let (x_n) be a weakly p-convergent sequence in X'. It is enough to show that a subsequence of $(\widetilde{T}x_n)$ converges in norm to 0.

Since $q\widetilde{T} = Tp \in \mathcal{C}_p$, the sequence $(q\widetilde{T}x_n)$ converges in norm to 0, and passing to a subsequence we can pick $y_n \in Y$ so that $\|\widetilde{T}x_n - jy_n\| \leq 2^{-n}$. Thus (jy_n), and also (y_n), are weakly p-summable, and $j \in \mathcal{C}_p$ implies $\|jy_n\| \to 0$. Hence $\|\widetilde{T}x_n\| \to 0$. $\square$

It follows from Proposition 16 and [**7**, Lemma 10] that $\mathcal{SC}$ is a non-trivial example of operator ideal which satisfies the $3S_+$ property.

PROPOSITION 18. *Let $\mathcal{A}$ be an operator ideal satisfying one of the properties $3S_-$ or $3S_+$. Then $Sp(\mathcal{A})$ satisfies the 3-space property.*

PROOF. Suppose that $Sp(\mathcal{A})$ fails the 3-space property, take a Banach space $X \notin Sp(\mathcal{A})$ with a subspace M such that $M, X/M \in Sp(\mathcal{A})$, and consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\ j\ } X \xrightarrow{\ q\ } X/M \longrightarrow 0.$$

Then $j, q \in \mathcal{A}$ because $M, X/M \in Sp(\mathcal{A})$, but $I_X \notin \mathcal{A}$. Thus $j \notin \mathcal{A}_-$ and $q \notin \mathcal{A}_+$. Hence $\mathcal{A}$ fails both the $3S_-$ and the $3S_+$ properties. □

We can also state the following dual version of Proposition 14.

Proposition 19. *Let $\mathcal{A}$ be an operator ideal satisfying the $3S_+$ property. Consider the following commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longleftarrow & Z & \overset{q_f}{\longleftarrow} & X & \overset{f}{\longleftarrow} & Y & \longleftarrow & 0 \\
 & & {\scriptstyle q_j}\uparrow & & {\scriptstyle q_i}\uparrow & & \| & & \\
0 & \longleftarrow & X' & \overset{q_g}{\longleftarrow} & P & \overset{g}{\longleftarrow} & Y & \longleftarrow & 0 \\
 & & {\scriptstyle j}\uparrow & & {\scriptstyle i}\uparrow & & & & \\
 & & Y' & = & Y' & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

whose rows and columns are exact sequences. Then $g, j \in \mathcal{A}$ if and only if $i, f \in \mathcal{A}$.

We leave the details of the proof to the interested reader.

References

[1] P. Aiena, M. González, A. Martínez-Abejón. *Operator semigroups in Banach space theory.* Boll. Un. Mat. Ital. **8** 4-B (2001) 157–205.

[2] F. Albiac, N.J. Kalton. *Topics in Banach space theory.* Graduate Texts in Math. **233**. Springer, 2006.

[3] T. Alvarez, M. González. *Some examples of tauberian operators.* Proc. Amer. Math. Soc. **111** (1991) 1023–1027.

[4] K. Astala, H.-O. Tylli. *Seminorms related to weak compactness and to tauberian operators.* Math. Proc. Cambridge Phil. Soc. **107** (1990) 365–375.

[5] A. Avilés, F. Cabello Sánchez, J.M.F. Castillo, M. González, Y. Moreno. *Separably injective Banach spaces.* Lecture Notes in Math. **2132**. Springer, 2016.

[6] F. Bombal, B. Hernando. *A double-dual characterization of Rosenthal and semi-tauberian operators.* Proc. R. Ir. Acad. **95A** (1995) 69–75.

[7] F. Cabello Sánchez, J.M.F. Castillo, N.J. Kalton. *Complex interpolation and twisted twisted Hilbert spaces.* Pacific. J. Math. **276** (2015) 287–307.

[8] P.G. Casazza, T.J. Shura. *Tsirel'son's space.* Lecture Notes in Math. **1363**. Springer, 1989.

[9] J.M.F. Castillo. *On Banach spaces X such that $L(L_p, X) = K(L_p, X)$.* Extracta Math. **10** (1995) 27–36.

[10] J.M.F. Castillo, M. González. *Three-space problems in Banach space theory.* Lecture Notes in Math. **1667**. Springer, 1997.

[11] J.M.F. Castillo, M. Simões, J. Suárez. *On a question of Pełczyński about the duality problem for weakly compact strictly singular operators.* Bull. Polish Acad. Sci. Math. **60** (2012) 27–36.

[12] M. González, A. Martínez-Abejón. *Tauberian operators.* Operator Theory: Advances and Applications **194**. Birkhäuser, 2010.

[13] M. González, A. Martínez-Abejón. *Examples of tauberian operators acting on* $C[0,1]$. J. Math. Anal. Appl. **410** (2014) 323-329.

[14] M. González, A. Martínez-Abejón. *Cotauberian operators on* $L_1(0,1)$ *obtained by lifting.* Integral Equations Operator Theory **80** (2014) 107–116.

[15] M. González, V. M. Onieva. *Characterizations of tauberian operators and other semigroups of operators.* Proc. Amer. Math. Soc. **108** (1990) 399–405.

[16] B. Hernando. *Some properties of the second conjugate of a tauberian operator.* J. Math. Anal. Appl. **228** (1998) 60–65.

[17] N.J. Kalton, N.T. Peck. *Twisted sums of sequence spaces and the three space problem.* Trans. Amer. Math. Soc. **255** (1979) 1–30.

[18] N.J. Kalton, A. Wilansky. *Tauberian operators on Banach spaces.* Proc. Amer. Math. Soc. **57** (1976) 251–255.

[19] A. Lebow, M. Schechter. *Semigroups of operators and measures of non-compactness.* J. Funct. Anal. **7** (1971) 1–26.

[20] D. H. Martin, J. Swart. *A characterization of semi-Fredholm operators defined on almost reflexive spaces.* Proc. R. Ir. Acad. **86A** (1986) 91–93.

[21] A. Pietsch. *Operator ideals.* North-Holland, 1980.

[22] D. G. Tacon. *Generalized semi-Fredholm transformations.* J. Austral. Math. Soc. **34** (1983) 60–70.

[23] L. Weis. *On perturbations of Fredholm operators on* $L_p(\mu)$*-spaces.* Proc. Amer. Math. Soc. **67** (1977) 287–292.

Departamento de Matemáticas, Universidad de Extremadura, E-06011 Badajoz, Spain.
E-mail address: castillo@unex.es

Departamento de Matemáticas, Universidad de Cantabria, E-39071 Santander, Spain.
E-mail address: manuel.gonzalez@unican.es

Departamento de Matemáticas, Universidad de Oviedo, E-33007 Oviedo, Spain.
E-mail address: ama@uniovi.es