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On the Defect Group of a 6D SCFT

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Abstract

We use the F-theory realization of 6D superconformal field theories (SCFTs) to study the corresponding spectrum of stringlike, i.e. surface defects. On the tensor branch, all of the stringlike excitations pick up a finite tension, and there is a corresponding lattice of string charges, as well as a dual lattice of charges for the surface defects. The defect group is data intrinsic to the SCFT and measures the surface defect charges which are not screened by dynamical strings. When non-trivial, it indicates that the associated theory has a partition vector rather than a partition function. We compute the defect group for all known 6D SCFTs, and find that it is just the abelianization of the discrete subgroup of $U(2)$ which appears in the classification of 6D SCFTs realized in F-theory. We also explain how the defect group specifies defining data in the compactification of a $(1,0)$ SCFT.

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1 Introduction

An outstanding open problem of theoretical physics is to fully characterize the defining data of a quantum field theory. One important insight from recent work on non-perturbative aspects is that specifying the spectrum of local operators is typically not enough to complete this characterization. Rather, additional information involving the spectrum of extended objects must also be included. For example, in the context of four-dimensional theories, the spectrum of line operators and surface operators must be specified [1–6]. A notable example of this type are the 4D $\mathcal{N} = 2$ theories of class $\mathcal{S}$ [7–10] obtained by compactifications of the 6D $(2, 0)$ theories on Riemann surfaces [11–15].

In this note we begin the study of extended defects for six-dimensional superconformal field theories (SCFTs) with minimal supersymmetry.[1] Recently, steady progress on the classification of such theories has been made [17–23], with a recently announced classification of theories (which is quite possibly complete) which can arise from compactifications of F-theory [22]. A hallmark of these systems is the existence of tensionless strings in the low energy effective field theory. As these are extended objects, it is natural to also expect a physically rich spectrum of defects. For earlier work on realizing 6D SCFTs in string theory, see e.g. [24–37].

In the F-theory realization of these theories, we have an elliptically fibered Calabi-Yau threefold $X \to B$ over a non-compact base. The tensionless strings arise from D3-branes wrapping contracting spheres of B, and the associated lattice of string charges is $\Lambda_{\text{string}} = H_2^{\text{cpct}}(B, \mathbb{Z})$. Geometrically, the dual lattice $\Lambda^*_{\text{string}}$ is associated with a basis of non-compact two-cycles canonically paired with elements of Λ_{string}. Wrapping a D3-brane over such a non-compact cycle leads to a surface defect in the effective 6D theory. Physically then, we can view Λ_{string} as the lattice of charges for stringlike excitations and $\Lambda^*_{\text{string}}$ as the lattice of charges for stringlike defects, i.e. "surface operators". This singles out the unimodular SCFTs, i.e. those theories with $\Lambda_{\text{string}} = \Lambda^*_{\text{string}}$. Unimodularity of the charge lattice is a necessary condition for the given 6D model to have a partition function on curved manifolds (see e.g. [16, 21, 6]).

But in general, there can be a mismatch between the lattices Λ_{string} and $\Lambda^*_{\text{string}}$, and this mismatch is measured by the abelian "defect group" of the theory:

$$\mathcal{C} \equiv \Lambda^*_{\text{string}}/\Lambda_{\text{string}}. \tag{1.1}$$

This group also determines the obstruction to defining a partition function for the 6D theory [16]—when $\mathcal{C}$ is non-trivial, the 6D theory is best thought of as a relative quantum field theory [11, 14, 38–46].

The physical interpretation of $\mathcal{C}$ is the natural generalization of the discrete electric / magnetic 't Hooft fluxes present in non-abelian gauge theory [47, 48], but now in the context

[1] The role of the global structure of extended defects in six-dimensional supergravity theories was discussed in [16].

of a 6D theory. Recall that for gauge theory, we can move to a phase where the effects of the non-abelian force carriers (i.e. the gluons) are screened at long distances. Then, the spectrum of defects, i.e. the line operators organize according to "N-ality", i.e. the discrete charge carried by the center of the gauge group. Though it is still an open problem to determine a formulation of the tensionless non-abelian force carriers for a 6D SCFT, we do know that on the tensor branch their effects are screened. What the defect group measures is the charges of surface operators which cannot be screened by the dynamical strings of our theory.

This discrete data shows up most clearly when we consider our 6D theory on a spacetime other than $\mathbb{R}^{5,1}$. In flat space, the locations of defects correspond to deleting certain 2D subspaces from the spacetime. More generally, we can work on a six-manifold M_6. The choice of a background flux sector is captured by the cohomology $H^3(M_6, \mathcal{C})$. Then, for any given model each choice of a maximal set of mutually local observables $L \subset H^3(M_6, \mathcal{C})$ corresponds to a set of generalized conformal blocks Z_a indexed by $a \in H^3(M_6, \mathcal{C})/L$. For the analogous statement in the case of the 6D (2,0) theories, see e.g. references [11,14,40,45,49].

In this note we compute the defect group for all known 6D SCFTs. An important result from this computation is that it does not involve a specific choice of resolution of the associated F-theory geometry. Rather, we find that for F-theory on a base obtained by blowing up $\mathbb{C}^2/\Gamma$ for Γ an appropriate discrete subgroup of $U(2)$, the defect group is always the abelianization of Γ:

$$\mathcal{C} = \mathrm{Ab}[\Gamma]. \tag{1.2}$$

All 6D SCFTs which can be generated in F-theory arise from blow ups of $\mathbb{C}^2/\Gamma$, and possibly making the elliptic fiber more singular. Further, the explicit list of possible $\Gamma \subset U(2)$ which can appear has been worked out in the generalized ADE classification of reference [17]. The fact that the defect group is independent of the geometric resolution parameters indicates that this data is intrinsic to the SCFT itself (i.e. it does not depend on the specific values of the string tensions). Summarizing our findings, the class of defect groups we uncover are as follows:

	A-type	D-type$_{(\text{even})}$	D-type$_{(\text{odd})}$	E_6	E_7	E_8
$\mathcal{C}$	$\mathbb{Z}_p$	$\mathbb{Z}_2 \times \mathbb{Z}_{2p}$	$\mathbb{Z}_{4p}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	1

, (1.3)

where there turn out to be two D-type defect groups. Let us note that in the special case of the $(2,0)$ theories, $\mathcal{C}$ is nothing but the center of the corresponding simply connected Lie group.[2]

The rest of this note is organized as follows. In section 2 we discuss in more detail the defect group of a 6D SCFT. In section 3, we review some elements of the classification of 6D SCFTs, and in particular the correspondence with generalized ADE subgroups of $U(2)$. In this same section we show that for a given F-theory model, performing blowups /

[2]Indeed, the center of the Lie group is given by the quotient of the weight lattice by the root lattice, i.e. $\Lambda_{\text{weight}}/\Lambda_{\text{root}}$, and its first homotopy group is given by the quotient of the co-weight lattice by the co-root lattice, i.e. $\Lambda^\vee_{\text{weight}}/\Lambda^\vee_{\text{root}}$. See e.g. references [50,51].

blowdowns of a base does not change the defect group. We then proceed in section 4 to an explicit determination of all possible defect groups, and show that it always corresponds to the abelianization of an associated discrete subgroup of $U(2)$. In section 5 we discuss the remnants of this discrete three-form flux data upon compactification to a lower-dimensional theory. We present our conclusions in section 6.

2 The Defect Group

In this section we introduce the notion of a defect group, and explain in more detail why we expect it to capture important discrete data of a 6D SCFT. As we shall be relying on their geometric characterization, let us first recall some additional details on the F-theory construction of a 6D SCFT.

Recall that in F-theory, we work with a non-compact elliptically fibered Calabi-Yau threefold $X \to B$, in which the base B is a non-compact Kähler surface. Recent work in [22] has led to a classification of all possible base geometries B, as well as all possible elliptic fibrations over a given base.

The charge lattice for dynamical strings is captured by $\Lambda_{\text{string}} = H_2^{\text{cpct}}(B, \mathbb{Z})$, and comes equipped with an intersection pairing, which in turn defines a natural dot product. Indeed, given a basis of two-cycles $e_i = [\Sigma_i]$ for $i = 1, ..., r$, we can then speak of the adjacency matrix for the lattice:

$$A_{ij} = e_i \cdot e_j = -\Sigma_i \cap \Sigma_j, \tag{2.1}$$

in the obvious notation. We can also introduce the dual lattice $\Lambda^*_{\text{string}}$ with generators w^i such that $w^i \cdot e_j = \delta^i{}_j$. Geometrically, these generators are associated with the non-compact two-cycles dual to the compact ones. Physically, this symmetric pairing is just the Dirac pairing in six dimensions [52]. The fact that $\Lambda^*_{\text{string}}$ is the lattice of defect charges follows by requiring these are labeled by a maximal set consistent with Dirac quantization. Surface defects are thus labeled by their charge, the surface they wrap and the supersymmetry they preserve.

Let us now turn to the F-theory realization of these defects. Consider a D3-brane wrapped over such a non-compact two-cycle. In the six-dimensional effective theory, we get a non-dynamical effective string with formally infinite tension (as it wraps a non-compact cycle). We refer to this as a surface operator. This should be viewed as a heavy probe of the 6D theory, and is the higher-dimensional generalization of a line operator. Now, in the context of 4D gauge theory, it is well-known that the lattice of defect charges can be screened, leaving us with just the center of the gauge group [47, 48]. Though we do not have the 6D analogue of non-abelian gauge theory for two-form potentials, we can still determine the effects of screening using the geometric characterization of these theories.[3]

[3]We thank C. Vafa for helpful comments.

Along these lines, let us first briefly discuss the effects of screening for 4D gauge theory from the perspective of geometric engineering. For concreteness, consider type IIB string theory on $T^2 \times \mathbb{C}^2/\Gamma_{ADE}$ for Γ_{ADE} a discrete ADE subgroup of $SU(2)$. In the four uncompactified directions, we get 4D $\mathcal{N} = 4$ Super Yang-Mills theory with gauge group G_{ADE} the simply connected Lie group of ADE type. Now, one way to understand the non-abelian gauge bosons is to take the small resolution of this singularity. Wrapping D3-branes over the compact two-cycles associated with the roots of the corresponding algebra and the A-cycle of the T^2, we see explicitly all the massive states which are going to assemble into the gluons of the theory. Introducing the root lattice $\Lambda_{\text{root}} = H_2^{cpct}(B, \mathbb{Z})$, the dual lattice Λ^*_{root} tells us about the possible line operators in this theory. Precisely because the force carriers are massive, there is a screening of the charge of these line operators. The charges which cannot be screened are simply $\Lambda^*_{\text{root}}/\Lambda_{\text{root}}$. In group theoretic terms, it is also known that this quotient is nothing but $\mathcal{Z}(G_{ADE})$, the center of the corresponding simply connected Lie group of ADE type. Observe that in the special case $G = SU(N)$, the center is the N-ality group $\mathbb{Z}_N$.

Consider next the case of our 6D SCFTs. Here, we do not have a formulation of interacting non-abelian two-form potentials. Nevertheless, from the geometry, we can still see that on the tensor branch, the associated string-like excitations have picked up a non-zero tension. Indeed, the reason it is possible to say anything concrete at all about the tensor branch is that the associated "off-diagonal" force carriers are necessarily screened (as they have picked up a tension). From this perspective, we can follow the same analysis as we did for the 4D gauge theory: We take the lattice of compact two-cycles Λ_{string}, introduce its dual $\Lambda^*_{\text{string}}$ for the surface defects, and the quotient group:

$$\mathcal{C} = \Lambda^*_{\text{string}}/\Lambda_{\text{string}} \tag{2.2}$$

tells us what charges for the surface operators cannot be screened.

Inserting a surface defect in six dimensions can be viewed as deleting its 2D worldvolume from our 6D spacetime. This is why the global topology of the spacetime figures into our discussion of surface defects. On a six-manifold M_6, the discrete three-form fluxes are given by $H^3(M_6, \mathcal{C}) \simeq H^3(M_6, \mathbb{Z}) \otimes \mathcal{C}$.[4]

When $\mathcal{C}$ is non-trivial, the 6D theory is a relative quantum field theory, rather than a conventional quantum field theory. Much of what we say in the rest of this section has been stated in various forms in the literature for the $(2, 0)$ SCFTs and, for that matter, relative quantum field theories in general [11, 14, 38–46]. The main novelty here is to show how these same structures naturally persist in the context of the $(1, 0)$ theories.

The theory on the background Euclidean spacetime M_6 is not expected to have a well-defined partition function. Rather, it has a collection of "partition functions" that can be

[4] Throughout, we assume that $H^3(M_6, \mathbb{Z})$ does not have a torsion group. Issues related to the torsion of $H^3(M_6, \mathbb{Z})$ have been discussed at length, for example, in [40].

organized into a "partition vector." It is natural to speculate that elements of this vector space can be understood as states in the Hilbert space of some 7D topological field theory, as has been worked out for some of the $(2,0)$ theories. The Hilbert space is then given by an irreducible representation of a particular Heisenberg group, which we now define.

The Heisenberg group is defined by noticing that there is a natural pairing $E : H^3(M_6, \mathcal{C}) \times H^3(M_6, \mathcal{C}) \to U(1)$ which takes $h, h' \in H^3(M_6, \mathcal{C})$ to $\exp(2\pi i \langle h, h' \rangle)$, in the obvious notation [14].[5] This enables us to define an extension, the Heisenberg group $\underline{H}^3(M_6, \mathcal{C})$ via the exact sequence:

$$1 \to U(1) \to \underline{H}^3(M_6, \mathcal{C}) \to H^3(M_6, \mathcal{C}) \to 0. \tag{2.4}$$

In more practical terms, to any given element $h \in H^3(M_6, \mathcal{C})$, we can assign a corresponding quantum flux $\Phi(h) \in \underline{H}^3(M_6, \mathcal{C})$ [43]. These elements are subject to the comutation relation:

$$\Phi(h)\Phi(h') = \exp(2\pi i \langle h, h' \rangle)\Phi(h')\Phi(h). \tag{2.5}$$

By a famous theorem by Stone, von Neumann and Mackey, the Heisenberg group $\underline{H}^3(M_6, \mathcal{C})$ has a unique irreducible representation (see, e.g., [53, 54]). The vector space of this representation is the Hilbert space of the seven-dimensional topological quantum field theory, or equivalently, the "partition vector space." This vector space can be built out of a maximal isotropic subgroup L of $H^3(M_6, \mathcal{C})$ [14, 40, 53]. First, there is a unique ray in the vector space, which we represent by the vector Z_0^L that is invariant under the action of L. Denoting the coset

$$L^\perp \equiv H^3(M_6, \mathcal{C})/L\,, \tag{2.6}$$

the rest of the basis vectors are obtained by acting on Z_0^L by elements $v \in L^\perp$:

$$Z_v^L = \Phi(v) Z_0^L\,. \tag{2.7}$$

Note that Z_v^L are eigenvectors for the elements $\Phi(w)$ with $w \in L$:

$$\Phi(w) Z_v^L = \exp(2\pi i \langle w, v \rangle) Z_v^L\,. \tag{2.8}$$

In particular, while the overall normalization of Z_0^L is not fixed, the normalization of Z_v^L is fixed with respect to Z_0^L. The theorem of Stone, von Neumann and Mackey states that given two maximal isotropic subgroups L and L', the two representations constructed this way are isomorphic. In particular, there is an invertible linear transformation R such that

[5] This pairing is derived from a canonical pairing $\mathcal{C} \times \mathcal{C} \to U(1)$ for elements of group $\mathcal{C}$. For example, when $\mathcal{C} = \mathbb{Z}_N$, the pairing is given by

$$\exp(2\pi i \langle n, m \rangle) = \exp\left(\frac{2\pi i n m}{N}\right)\,. \tag{2.3}$$

Notice that it is convenient to present $\mathcal{C}$ as an additive abelian group for such purposes—we shall use this same convention in section 5.

for all $v' \in L'^{\perp}$,

$$Z_{v'}^{L'} = \sum_{v \in L^{\perp}} R_{v'}^{\ v} Z_v^L \,. \tag{2.9}$$

In physical terms, we must demand that our partition vector can be decomposed into eigenfunctions of discrete fluxes which can all be simultaneously measured, the condition of mutal locality of the fluxes being encoded in the pairing $E : H^3(M_6, \mathcal{C}) \times H^3(M_6, \mathcal{C}) \to U(1)$. That means we must pick a sublattice $L \subset H^3(M_6, \mathcal{C})$ such that for all pairs of elements $h, h' \in L$, $E(h, h') = 1$. This is precisely the definition of an isotropic subgroup of $H^3(M_6, \mathcal{C})$. The physical interpretation of L is that it defines a maximal collection of mutually local discrete fluxes.[6] We stress that any maximally mutually local sublattice L would work, and each such choice gives rise to a different realization of the generalized conformal blocks of the theory.

3 Defects and Generalized ADE

Having introduced the defect group of a 6D SCFT, we now determine some of its general properties. The main result from this section will be that the generalized ADE classification of 6D SCFTs (with an F-theory realization) in terms of specific discrete subgroups $\Gamma \subset U(2)$ is enough to determine the list of possible defect groups. In geometric terms, this is the statement that in F-theory, all of the bases are obtained from blowups of $\mathbb{C}^2/\Gamma$ and that the defect group is invariant under such blowups. A corollary of this result is that the data of the defect group is insensitive to the geometric resolution parameters and thus to the specific tensions for our effective strings, as one should expect for data intrinsic to the SCFT itself.

To frame the discussion to follow, we review some elements of how to build and classify 6D SCFTs in F-theory, and in particular the correspondence with generalized ADE discrete subgroups of $U(2)$. One of the results from reference [22] is a full classification of possible bases. These are composed of configurations of $\mathbb{P}^1$'s with self-intersection $-n$ with $1 \leq n \leq 12$. Importantly, each -1 curve intersects at most two other curves. On the diagonal of the adjacency matrix A_{ij}, the entries are all positive, and the only off-diagonal entries are 0 or -1 and the structure of the graph does not contain any closed loops. Further, it turns out that the resulting structures of possible configurations of curves are remarkably constrained, and have the form of a long "spine" with a small amount of decoration on the ends [22].

The building blocks for these bases involve the so-called non-Higgsable clusters (NHCs) of reference [55], which include the important observation that there is a minimal canonical singular fiber type associated with each such cluster. The classification result of [22] also involves determining possible enhancements in the elliptic fibration over a given base. Observe that if the base is fixed, this additional data does not affect the lattice of string charges. It will therefore not figure in to our considerations.

[6] It is helpful to recall the distinction between gauge theory with group $SU(N)$ and $SU(N)/\mathbb{Z}_N$. In the latter case, we impose a restriction on the admissible Wilson lines we consider, i.e. we restrict the possible representations, and in particular exclude the fundamental representation.

Given a base, a natural geometric operation is to consider the blowdown of all -1 curves in the geometry. This shifts the self-intersection numbers of neighboring curves according to the rule:

$$(n+1, 1, m+1) \to (n, m). \tag{3.1}$$

Continuing in this way, we can iteratively blow down all -1 curves until we reach an "endpoint" geometry in which all curves have self-intersection -2 or less. Quite remarkably, it turns out that these geometries are minimal resolutions of the orbifold singularities $\mathbb{C}^2/\Gamma$ where Γ is a discrete subgroup of $U(2)$.[7] One of the results of reference [17] is that there is a generalized ADE classification of $(1,0)$ theories. Their endpoint configurations are:

$$\text{A-type Endpoint:} \quad n_1, ..., n_k \tag{3.2}$$

$$\text{D-type Endpoint:} \quad 2, \overset{2}{m_1}, ..., m_l \tag{3.3}$$

$$\text{E-type Endpoint:} \quad \left\{ \begin{array}{l} 2,2,\overset{2}{2},2,2 \\ 2,2,\overset{2}{2},2,2,2 \\ 2,2,\overset{2}{2},2,2,2,2 \end{array} \right\}, \tag{3.4}$$

where the n_i and m_j correspond to curves of self-intersection $-n_i$ and $-m_j$, in the obvious notation. Not all values of the n's and m's occur, but the full list which do appear can be found in reference [17]. Note that in the context of F-theory, the E-type subgroups are always a discrete subgroup of $SU(2)$. Finally, we can of course also have bases which blow down to $\mathbb{C}^2$. For example a configuration of curves such as $1, 2, ..., 2$ is of this type.

Now, given an endpoint there is always a minimal set of blowups required to reach a consistent F-theory model. Additionally, other "unforced blowups" can sometimes be included. In more detail, the operation of blowing up the intersection point involves the following procedure. Given curves Σ_L and Σ_R of self-intersection $-n$ and $-m$ which intersect at one point, blowing up this point introduces a new exceptional divisor Σ_{new} and shifts the homology classes as follows:

$$[\Sigma_L] \to [\Sigma_L] + [\Sigma_{\text{new}}] \quad \text{and} \quad [\Sigma_R] \to [\Sigma_R] + [\Sigma_{\text{new}}]. \tag{3.5}$$

Let us now show that blowing up the base does not alter the group $\Lambda^*_{\text{string}}/\Lambda_{\text{string}}$. To establish this, we proceed inductively. Suppose that we have a lattice Λ_{string}, and that we then perform another blowup on a point of one of the curves of our base. This introduces an additional exceptional divisor Σ_{new}, so the new lattice of string charges $\Lambda^{(1)}_{\text{string}}$ has increased in rank by one. Additionally, a divisor class which touches the new curve will shift as $[\Sigma] \to [\Sigma] + [\Sigma_{\text{new}}]$.

In fact, by an integral change of basis, we see that the new lattice of string charges is

[7] Note that to define an F-theory background, further blowups in the base are required.

really just $\Lambda^{(1)}_{\text{string}} \simeq \Lambda_{\text{string}} \oplus \mathbb{Z}$, where the additional factor is generated by the new class $[\Sigma_{\text{new}}]$. Since $[\Sigma_{\text{new}}]$ has self-intersection -1, this additional factor is a one-dimensional self-dual lattice. By a similar token, if we consider b blowups of a base, we reach a lattice $\Lambda^{(b)}_{\text{string}}$ which is:

$$\Lambda^{(b)}_{\text{string}} \simeq \Lambda_{\text{string}} \oplus \underbrace{\mathbb{Z} \oplus ... \oplus \mathbb{Z}}_{b}. \tag{3.6}$$

Now, because the lattice $\mathbb{Z}^{\oplus b}$ is already self-dual, the dual lattice is:

$$\Lambda^{(b)*}_{\text{string}} \simeq \Lambda^{*}_{\text{string}} \oplus \underbrace{\mathbb{Z} \oplus ... \oplus \mathbb{Z}}_{b}. \tag{3.7}$$

So, we have the important fact that the defect group is unchanged by the number of blowups:

$$\Lambda^{(b)*}_{\text{string}}/\Lambda^{(b)}_{\text{string}} \simeq \Lambda^{*}_{\text{string}}/\Lambda_{\text{string}} \quad \text{for all } b. \tag{3.8}$$

Recall, however, that all of the F-theory bases for 6D SCFTs arise from blowups of generalized ADE orbifolds of $\Gamma \subset U(2)$. What this means is that for the purposes of calculating $\Lambda^{*}_{\text{string}}/\Lambda_{\text{string}}$, we can confine our analysis to the minimal resolution of the orbifold singularity $\mathbb{C}^2/\Gamma$, as described by lines (3.2)-(3.4). To sum up, we see that for the purposes of determining the defect group $\Lambda^{*}_{\text{string}}/\Lambda_{\text{string}}$, it is enough to focus on the geometry of $\mathbb{C}^2/\Gamma$.

4 Defect Groups of $(1,0)$ Theories

In the previous section we established that the generalized ADE-type of a $(1,0)$ theory determines the defect group $\mathcal{C} = \Lambda^{*}_{\text{string}}/\Lambda_{\text{string}}$. In this section we compute the explicit form of $\mathcal{C}$.

To begin, we note that the order of the defect group is readily computed from the adjacency matrix. The adjacency matrix A defines an embedding of the lattice Λ_{string} in $\Lambda^{*}_{\text{string}}$:

$$A : \Lambda_{\text{string}} \to \Lambda^{*}_{\text{string}}. \tag{4.1}$$

As a consequence, the order of the defect group satisfies $|\mathcal{C}| = \det A$.

We can also determine the abstract form of this group. For Λ_{string} a rank k lattice, the group $\Lambda^{*}_{\text{string}}/\Lambda_{\text{string}}$ is given by the abelian group on k commuting generators $a_1, ..., a_k$ subject to the relations:

$$\prod_{j=1}^{k} a_j^{A_{ij}} = 1 \tag{4.2}$$

for $i = 1, ..., k$. Similar formulae appear in the computation of D-brane spectra in Landau-Ginzburg vacua. This is of course not an accident, since the computation of the spectrum of defects is quite parallel to that situation. Indeed, we will also establish that the defect group is the abelianization Ab[Γ] for the associated discrete group $\Gamma \subset U(2)$. This is again suggestive of a larger structure in the context of F-theory backgrounds, on which we comment further in section 6.

Our plan in this section will be to compute the precise form of the defect groups for all 6D SCFTs. To this end, we first compute this data in the case of an endpoint which is an ADE subgroup of $SU(2)$. Here, we observe that $\mathcal{C}$ is nothing but the center of the corresponding simply connected ADE Lie group. We then establish similar formulae for the generalized A-type and D-type theories.

4.1 The ADE Bases

Our plan in this section will be to consider F-theory models with base an ADE singularity. The minimal resolution of the base gives a bouquet of -2 curves which organize according to the respective Dynkin diagram. When the elliptic fibration is trivial, i.e. when the F-theory background is of the form $\mathbb{C}^2/\Gamma \times T^2$ for Γ an ADE subgroup of $SU(2)$, we reach a 6D $(2,0)$ theory. When the elliptic fibration is non-trivial over the compact curves, we instead reach a $(1,0)$ theory.

Now, in the special context of the $(2,0)$ theories, it is well known that the center of the associated simply connected Lie group also determines the obstruction to specifying a partition function, and thus the defect group. Not coincidentally, the centers of the associated simply connected Lie group of the same ADE type exactly match the defect group, as well as the abelianization of the associated discrete subgroup of $SU(2)$:

	A_k	D_{2k}	D_{2k+1}	E_6	E_7	E_8
$\mathcal{Z}(G_{ADE}) = \mathcal{C}(\Gamma_{ADE}) = \text{Ab}[\Gamma_{ADE}]$	$\mathbb{Z}_{k+1}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	1

. (4.3)

Our plan in this subsection will be to perform a direct computation to verify the equivalence of the defect group and the abelianization of the quotient group, which holds for a general F-theory base on an ADE singularity. The material of this subsection collects well-known results, which are encapsulated by the classical McKay correspondence [56]. Our main purpose here is to establish notation and set the stage for the computation for all $(1,0)$ theories.

A-type Theories

Consider first the A-type bases. These are realized in F-theory on a base consisting of k curves of self-intersection -2 arranged as in the corresponding Dynkin diagram. Returning to our general formula in equation (4.2), we have:

$$(a_1)^2 = a_2 \tag{4.4}$$
$$(a_i)^2 = a_{i-1}a_{i+1} \quad \text{for} \quad 1 < i < k \tag{4.5}$$
$$(a_k)^2 = a_{k-1}. \tag{4.6}$$

Iteratively solving these constraints, we learn:

$$a_i = (a_1)^i \quad \text{and} \quad (a_1)^{k+1} = 1. \tag{4.7}$$

So in other words, the group is generated by a_1, an element of order $k+1$. We therefore learn that $\mathcal{C}(A_k) = \mathbb{Z}_{k+1}$. Note that the abelianization $\mathrm{Ab}[\Gamma_{A_k}] = \mathbb{Z}_{k+1}$, since the group is abelian.

D-type Theories

Consider next the D-type bases. These are realized in F-theory on a base consisting of k curves of self-intersection -2 arranged as in the corresponding Dynkin diagram. In this case, it is convenient to observe that $\det A = 4$ for all of the D-type Cartan matrices, so the defect group is necessarily of order four. Performing a similar computation to that given for the A-type theories, we learn that $\mathcal{C}$ is either the cyclic group of order four, or the Klein four-group. The particular case which is realized depends on whether we have an even number of -2 curves or an odd number. For k even, we get $\mathbb{Z}_2 \times \mathbb{Z}_2$, while for k odd, we get $\mathbb{Z}_4$.

We can also determine that the abelianization $\mathrm{Ab}[\Gamma_{D_k}]$ matches to these choices. Recall that the D-type subgroups of $SU(2)$ are the binary dihedral groups. The binary dihedral group and its abelianization are given by the following abstract group with two generators:

$$\Gamma_{D_k} = \left\langle x, a | a^{2k-4} = 1, \quad x^2 = a^{k-2}, \quad xax^{-1} = a^{-1} \right\rangle \tag{4.8}$$
$$\mathrm{Ab}[\Gamma_{D_k}] = \left\langle x, a | a^{2k-4} = 1, \quad x^2 = a^{k-2}, \quad a^2 = 1 \right\rangle_{\mathrm{comm}}. \tag{4.9}$$

Here, the subscript "comm" amounts to imposing the relations $gh = hg$ for all elements g and h of the group. Depending on whether k is even or odd, we get two different abelianizations:

$$\mathrm{Ab}[\Gamma_{D_k}] = \left\{ \begin{array}{ll} \mathbb{Z}_2 \times \mathbb{Z}_2 & k \text{ even} \\ \mathbb{Z}_4 & k \text{ odd} \end{array} \right\}. \tag{4.10}$$

E-type Theories

Finally, consider the E-type theories. These are realized by F-theory on a base consisting of -2 curves arranged according to the corresponding E_6, E_7 and E_8 Dynkin diagrams. To compute the defect group in these cases, we observe that the determinant of the adjacency matrix is respectively 3, 2, and 1. We thus learn that the defect groups are:

	E_6	E_7	E_8
$\mathcal{C}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	1

. (4.11)

Turning next to the abelianization, we recall that the binary tetrahedral group (i.e. E_6), the binary icosahedral group (i.e. E_7) and the binary octahedral group (i.e. E_8) and their respective abelianizations are:

$$\Gamma_{E_6} = \left\langle a, b, c | a^3 = b^3 = c^2 = abc \right\rangle \quad \text{and} \quad \mathrm{Ab}[\Gamma_{E_6}] = \left\langle \zeta | \zeta^3 = 1 \right\rangle \tag{4.12}$$
$$\Gamma_{E_7} = \left\langle a, b, c | a^4 = b^3 = c^2 = abc \right\rangle \quad \text{and} \quad \mathrm{Ab}[\Gamma_{E_7}] = \left\langle \zeta | \zeta^2 = 1 \right\rangle \tag{4.13}$$
$$\Gamma_{E_8} = \left\langle a, b, c | a^5 = b^3 = c^2 = abc \right\rangle \quad \text{and} \quad \mathrm{Ab}[\Gamma_{E_8}] = \left\langle \zeta | \zeta = 1 \right\rangle , \tag{4.14}$$

and as expected, there is an exact match between the two characterizations.

4.2 Generalized A-type Bases

Let us now turn to the computation of the defect group for the generalized A-type bases. Recall that these are given by blowups of a configuration of curves of self-intersection $-n_1, ..., -n_k$ which intersect pairwise, forming a single chain of curves. Contracting these curves leads to an orbifold singularity $\mathbb{C}^2/\Gamma$ where the group action is [57–60]:

$$(u, v) \to (\omega u, \omega^q v) \quad \text{with} \quad \omega^p = 1, \tag{4.15}$$

and the integers p and q are determined by the Hirzebruch-Jung continued fraction:

$$\frac{p}{q} = n_1 - \frac{1}{n_2 - \frac{1}{n_3 - \ldots \frac{1}{n_k}}}. \tag{4.16}$$

Clearing denominators in the continued fraction, we can extract the corresponding values of p and q. These are given by the determinants of the adjacency matrix A, as well as $A^{(1)}$, the matrix obtained by deleting the first row and column. (i.e. those containing the entry n_1):

$$p = \det A \quad \text{and} \quad q = \det A^{(1)}. \tag{4.17}$$

From this, we observe that the order of the defect group is $|\mathcal{C}| = \det A = p$.

Iteratively solving the group relations in equation (4.2), we also see that either a_1 or a_k can serve as a generator for the entire group. That is, we have a cyclic group of order $\det A = p$. This establishes the claim that $\mathcal{C} \simeq \mathbb{Z}_p$. Moreover, since our Γ is already abelian, the abelianization is clearly $\mathbb{Z}_p$. Finally, note that in the special case where the base is just a collection of -2 curves, we recover the special case of the defect group for the A-type $(2, 0)$ theories.

4.3 Generalized D-type Bases

Finally, we come to the D-type bases. Recall that these are given by a configuration of curves arranged according to the generalized Dynkin diagram:

$$\text{D-type Endpoint:} \quad 2, \overset{2}{n}, m_1, ..., m_l, \tag{4.18}$$

so that the adjacency matrix takes the form:

$$A = \begin{pmatrix} 2 & & -1 & & \cdots & & \\ & 2 & -1 & & \cdots & & \\ -1 & -1 & n & -1 & \cdots & & \\ & & -1 & m_1 & \cdots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & & \cdots & m_{l-1} & -1 \\ & & & & \cdots & -1 & m_l \end{pmatrix}. \tag{4.19}$$

Contracting all of these curves leads us to the orbifold singularity $\mathbb{C}^2/\Gamma$ where $\Gamma = D_{p+q,q}$ is a D-type discrete subgroup of $U(2)$, where p and q are relatively prime positive integers given by the continued fraction (see e.g. [59–61]):

$$\frac{p}{q} = (n-1) - \frac{1}{m_1 - \frac{1}{m_2 - \dots \frac{1}{m_l}}}. \tag{4.20}$$

The integers p and q are given by the determinants of the reduced adjacency matrices B, and $B^{(1)}$, the matrix obtained by deleting the first row and column:

$$p = \det B \quad \text{and} \quad q = \det B^{(1)}, \tag{4.21}$$

where:

$$B = \begin{pmatrix} n-1 & -1 & \cdots & & \\ -1 & m_1 & -1 & & \\ & -1 & \ddots & -1 & \\ & & -1 & m_{l-1} & -1 \\ & & & -1 & m_l \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} m_1 & -1 & & \\ -1 & \ddots & -1 & \\ & -1 & m_{l-1} & -1 \\ & & -1 & m_l \end{pmatrix}. \tag{4.22}$$

The specific orbifold group depends on whether p is even or odd:

$$D_{p+q,q} = \left\{ \begin{array}{ll} \langle \psi_{2q}, \varphi_{2p}, \tau \rangle & p \text{ odd} \\ \langle \psi_{2q}, \lambda_{4p} \rangle & p \text{ even} \end{array} \right\}, \tag{4.23}$$

where we have introduced the generators:

$$\psi_k = \begin{bmatrix} e^{2\pi i/k} & \\ & e^{-2\pi i/k} \end{bmatrix}, \quad \varphi_k = \begin{bmatrix} e^{2\pi i/k} & \\ & e^{2\pi i/k} \end{bmatrix} \tag{4.24}$$

$$\tau = \begin{bmatrix} & i \\ i & \end{bmatrix}, \quad \lambda_k = \varphi_k \tau = \begin{bmatrix} & ie^{2\pi i/k} \\ ie^{2\pi i/k} & \end{bmatrix}. \tag{4.25}$$

As an abstract group, $D_{p+q,q}$ for p odd and even is:

$$D_{p+q,q} = \left\{ \begin{array}{ll} \langle \psi, \varphi, \tau | \varphi^{2p} = 1,\, \tau^2 = \psi^q = \varphi^p,\, \varphi\psi = \psi\varphi,\, \varphi\tau = \tau\varphi,\, \psi\tau = \tau\psi^{-1} \rangle & p \text{ odd} \\ \langle \psi, \lambda | \lambda^{4p} = 1,\, \psi^q = \lambda^{2p},\, \psi\lambda = \lambda\psi^{-1} \rangle \quad p \text{ even} & \end{array} \right\}. \tag{4.26}$$

Let us now turn to the abelianization of $D_{p+q,q}$:

$$\mathrm{Ab}[D_{p+q,q}] = \left\{ \begin{array}{ll} \langle \psi, \varphi, \tau | \varphi^{2p} = 1,\, \tau^2 = \psi^q = \varphi^p,\, \psi^2 = 1 \rangle_{\text{comm}} & p \text{ odd} \\ \langle \psi, \lambda | \lambda^{4p} = 1,\, \psi^q = \lambda^{2p},\, \psi^2 = 1 \rangle_{\text{comm}} & p \text{ even} \end{array} \right\}. \tag{4.27}$$

Since p and q are relatively prime, the pair (p, q) has three possibilities: (even,odd), (odd, odd) and (odd, even). Let us treat each of these cases in turn.

For $(p, q) =$ (odd,even), we see that since $\psi^2 = 1$, we also have $\psi^q = 1$, so we also have $\tau^2 = \psi^q = \varphi^p = 1$. In other words, the group is isomorphic to $\mathbb{Z}_2^{(\psi)} \times \mathbb{Z}_p^{(\varphi)} \times \mathbb{Z}_2^{(\tau)} \simeq \mathbb{Z}_{2p} \times \mathbb{Z}_2$, where on the lefthand side we have indicated the explicit generators by a superscript, and in the isomorphism we used the fact that p is odd.

For $(p, q) =$ (odd,odd), it is helpful to write $q = 2s + 1$. Then, since $\psi^2 = 1$, we learn that $\tau^2 = \psi = \varphi^p$. So in other words, the independent elements are $\tau^a\varphi^b$ for $a = 0, 1$ and $b = 0, ..., 2p - 1$, and our group is of order $4p$. Let us proceed by showing it is a cyclic group. To see this, let us show that it has an element of order $4p$. We claim it is $\tau\varphi$, indeed $(\tau\varphi)^{2p} = \tau^{2p} = \varphi^{p^2} = \varphi^p \neq 1$, as p is odd. Let k be the smallest natural number with $(\tau\varphi)^k = 1$: $k|4p$ but it cannot divide $2p$. As $\gcd(2p, 4p) = 2p$, this forces $k = 4p$. So in this case we find $\mathbb{Z}_{4p}$.

Finally, for $(p, q) =$ (even,odd), we can again write $q = 2s + 1$. Then, we have $\psi^q = \psi = \lambda^{2p}$, so the group is generated by λ, an element of order $4p$.

Summarizing, we learn that the type of group is actually dictated by whether q is even or odd:

$$\mathrm{Ab}[D_{p+q,q}] = \left\{ \begin{array}{ll} \mathbb{Z}_{4p} & q \text{ odd} \\ \mathbb{Z}_{2p} \times \mathbb{Z}_2 & q \text{ even} \end{array} \right\}. \tag{4.28}$$

We now show that this same structure is reproduced by a direct computation of the defect group. To this end, we first observe that the determinant of the adjacency matrix satisfies:

$$\det A = 4p. \tag{4.29}$$

As a consequence, we always have $\left|\Lambda^*_{\text{string}}/\Lambda_{\text{string}}\right| = 4p$. To determine the defect group $\mathcal{C} = \Lambda^*_{\text{string}}/\Lambda_{\text{string}}$, we work in a basis where the generators are $v = (b_1, b_2, c, a_1, ..., a_l)$. Recursively solving equation (4.2), we learn that the other generators are all obtained from appropriate powers of $a \equiv a_l$ and $b \equiv b_1$. Moreover, since $b^2 = (b_2)^2 = c$, we also find:

$$a^{2p} = 1 \quad \text{and} \quad b^2 = a^q. \tag{4.30}$$

Hence, the defect group is given by:

$$\mathcal{C} = \left\langle a, b | a^{2p} = 1,\, b^2 = a^q \right\rangle_{\text{comm}}. \tag{4.31}$$

When $q = 2s + 1$, $(ba^{-s})^2 = a$, so the group is an order $4p$ cyclic group generated by the element ba^{-s}. On the other hand, when $q = 2s$, the group may be written as

$$\mathcal{C} = \left\langle a, \beta | a^{2p} = 1,\, \beta^2 = 1 \right\rangle_{\text{comm}} \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2 \tag{4.32}$$

with $\beta \equiv ba^{-s}$. Summarizing, we learn:

$$\mathcal{C} = \text{Ab}[D_{p+q,q}] = \left\{ \begin{array}{ll} \mathbb{Z}_{4p} & q \text{ odd} \\ \mathbb{Z}_{2p} \times \mathbb{Z}_2 & q \text{ even} \end{array} \right\}. \tag{4.33}$$

List of Defect Groups for D-type Theories

Let us now apply these general considerations to determine the range of possible defect groups for the generalized D-type orbifold groups. First of all, we recall that the generalized D-type endpoints are all of the form [17]:

$$2\overset{2}{3}2, \quad 2\overset{2}{2}\underbrace{2...24}_{l\geq 1}, \quad 2\overset{2}{2}\underbrace{2...23}_{l\geq 1}, \quad 2\overset{2}{2}\underbrace{2...32}_{l\geq 2}. \tag{4.34}$$

For these cases, the associated values of p and q are:

$$2\overset{2}{3}2: \quad \frac{p}{q} = \frac{3}{2} \tag{4.35}$$

$$2\overset{2}{2}\underbrace{2...24}_{l\geq 1}: \quad \frac{p}{q} = \frac{3}{3l+1} \tag{4.36}$$

$$2\overset{2}{2}\underbrace{2...23}_{l\geq 1}: \quad \frac{p}{q} = \frac{2}{2l+1} \tag{4.37}$$

$$2\overset{2}{2}\underbrace{2...32}_{l\geq 2}: \quad \frac{p}{q} = \frac{3}{3l-1}. \tag{4.38}$$

So in other words, the only defect groups for the D-type theories are:

$$C = \left\{ \begin{array}{ll} \mathbb{Z}_4, \mathbb{Z}_8, \text{ or } \mathbb{Z}_{12} & l \text{ even} \\ \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4, \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_6 & l \text{ odd} \end{array} \right\}. \tag{4.39}$$

where for completeness we have included the case of the D-type orbifold subgroups of $SU(2)$.

5 Compactification

So far, our discussion has focussed on the role of the defect group in six dimensions, and in particular how it serves to characterize discrete three-form fluxes via the abelian group $H^3(M_6, \mathcal{C})$, for an SCFT on a six-manifold M_6. It is natural to ask how this data shows up when we compactify a $(1,0)$ theory to lower dimensions. Much of what we say in this section has been stated in various forms in the literature for the $(2,0)$ SCFTs. Our discussion will most closely follow that given in references [11, 14]. As far as we are aware, however, the extension to $(1,0)$ theories has not been previously worked out.

Upon compactification on a $(6-d)$-dimensional Riemannian manifold Σ_d, we reach a d-dimensional theory. The tensor multiplet will descend to a collection of vector multiplets. For example, by compactifying on a circle, our two-form potential converts to a standard gauge field. In the case of compactification on a Riemann surface, each of the one-cycles similarly leads to an additional vector multiplet. Now, in the context of 6D $(1,0)$ SCFTs, there will generically be additional 6D vector multiplets, which upon reduction will also contribute to the field content of the lower-dimensional theory. Importantly, this data is independent of $\mathcal{C}$, and in particular does not appear to lead to additional discrete flux data. We shall therefore neglect it in what follows.

So let us now study how the defect group shows up in compactifications to lower dimensions $d < 6$. Let us assume that some d-dimensional theory is obtained by compactifying the six-dimensional theory on a manifold Σ of dimension $(6-d)$. In order for the d-dimensional theory to have a well-defined partition function, we must be able to assign an element $Z_\Sigma(M_d)$ in the partition vector space of the six-dimensional theory for any M_d. Naively, when $\mathcal{C}$ is non-trivial, this is not possible, since the vector space would typically be multi-dimensional. However, we expect that upon specifying some additional data, a "flux ensemble" F, it is possible to pick out (up to an overall constant of proportionality) a unique vector $Z_{\Sigma,F}(M_d)$ in the partition vector space. Essentially, a flux ensemble F provides a canonical mapping L_F which assigns to each M_d a maximal isotropic subgroup $L_F(M_d)$ of $H^3(M_d \times \Sigma, \mathcal{C})$. We can then define the partition function $Z_{\Sigma,F}(M_d)$ to be the partition vector invariant under the maximal isotropic subgroup $L_F(M_d)$, i.e.,

$$Z_{\Sigma,F}(M_d) \equiv Z_0^{L_F(M_d)}(M_d)\,. \tag{5.1}$$

This picks out a unique partition function up to a constant, since we know that for any maximal isotropic subgroup of $H^3(M_d \times \Sigma, \mathcal{C})$, there is a unique ray in the partition vector space that is invariant under it. Since the definition is rather abstract, let us demonstrate the above by finding the set of allowed flux ensembles $\mathcal{F} = \{F\}$ for specific compactifications.

First of all, let us compactify on a circle to reach a 5D effective theory. That is, we take the special case $M_6 = M_5 \times S^1$. The compactification of a $(1,0)$ theory yields an $\mathcal{N} = 1$ theory in five dimensions. We study this compactification in detail to illustrate how the partition function is fixed using flux ensembles. Now the discrete three-form fluxes decompose as:

$$H^3(M_5 \times S^1, \mathcal{C}) = H^3(M_5, \mathcal{C}) \oplus H^2(M_5, \mathcal{C}). \tag{5.2}$$

We claim that

$$\mathcal{F} \equiv \{F = (G_1, G_2) : G_1, G_2 \subset \mathcal{C},\ G_1 = G_2^{\perp},\ G_2 = G_1^{\perp}\}\,. \tag{5.3}$$

where

$$G^{\perp} = \{g \in \mathcal{C} : \exp(2\pi i \langle g, g' \rangle) = 1 \text{ for all } g' \in G\}\,. \tag{5.4}$$

For example, when $\mathcal{C} = \mathbb{Z}_N$ generated by the order-N element 1,

$$G_1 = [p] \cong \mathbb{Z}_q, \quad G_2 = [q] \cong \mathbb{Z}_p \tag{5.5}$$

satisfy these relations when $pq = N$. Here we have used the standard notation where $[a]$ is the additive group generated by the element a. Now given a flux ensemble $F = (G_1, G_2) \in \mathcal{F}$, the map L_F is given by:

$$L_F(M_5) = H^3(M_5, G_1) \oplus H^2(M_5, G_2)\,. \tag{5.6}$$

$L_F(M_5)$ is maximal and isotropic, and hence $L_F(M_5)$ defines a unique partition vector $Z_0^{L_F}$. We will see that the set of discrete data $\mathcal{F}$, in the case of $(2,0)$ theories, boils down to the choice of the gauge group of the 5D gauge theory [11].

To be more concrete, let us consider the case when $\mathcal{C} = \mathbb{Z}_N$. In this case, a divisor p of N specifies the flux ensemble needed to define the 5D theory:

$$F_p = ([p], [N/p])\,. \tag{5.7}$$

Note that $H^3(M_5 \times S^1, \mathcal{C}) = F_1 \oplus F_N$ as

$$L_{F_1}(M_5) = H^3(M_5, \mathbb{Z}_N), \quad L_{F_N}(M_5) = H^2(M_5, \mathbb{Z}_N)\,. \tag{5.8}$$

Each L_{F_p} with $p|N$ singles out a ray

$$Z_{S^1, F_p} = Z_0^{L_{F_p}} \tag{5.9}$$

in the partition vector space of the theory. We can express the partition function Z_{S^1,F_p} using the basis of vectors

$$Z_v \equiv Z_v^{L_{F_1}}, \quad v \in H^3(M_5 \times S^1, \mathcal{C})/L_{F_1} = H^2(M_5, \mathbb{Z}_N)\,. \tag{5.10}$$

It is given by

$$Z_{S^1,F_p} = \sum_{v \in H^2(M_5,[N/p])} Z_v\,. \tag{5.11}$$

It is simple to check that this element of the partition vector space is invariant under the action of the Heisenberg group operators $\Phi(v)$ for $v \in H^3(M_5,[p]) \oplus H^2(M_5,[N/p])$.

When the six-dimensional theory is an A_{N-1} (2,0) theory, $v \in H^2(M_5,[N/p])$ are the Stiefel-Whitney classes of a $SU(N)/\mathbb{Z}_p$ gauge bundle, where $\mathbb{Z}_p = [N/p]$ is a subgroup of the center $\mathbb{Z}_N$ of $SU(N)$. The basis vectors Z_v then have the interpretation as the partition function of a five-dimensional $SU(N)/\mathbb{Z}_p$ gauge theory restricted to gauge bundles of Stiefel-Whitney class v [11]. It follows that the partition function Z_{S^1,F_p} is the partition function of 5D $\mathcal{N} = 2$ $SU(N)/\mathbb{Z}_p$ super-Yang-Mills theory. We therefore see that the discrete data in this case merely specifies the global structure of the gauge group of the five-dimensional theory, as claimed. For compactifications of $(1,0)$ theories, such an interpretation does not (yet) exist, although the partition function is defined by the same formula (5.11) when $\mathcal{C} = \mathbb{Z}_N$.

Next, consider compactification on Σ a genus g Riemann surface, i.e. the case $M_6 = M_4 \times \Sigma$. When a $(1,0)$ theory is compactified on a torus, it has $\mathcal{N} = 2$ supersymmetry. For other compactification manifolds, the supersymmetry of the four-dimensional effective theory is $\mathcal{N} = 1$, i.e. four real supercharges. The discrete data that specify the lower dimensional theory is given by a choice of a maximal isotropic subgroup F of $H^1(\Sigma,\mathcal{C}) \simeq \mathcal{C}^{2g}$ when Σ is compact [11, 14]. Given this, a maximal isotropic subgroup $L_F(M_4)$ of $H^3(M_4 \times \Sigma, \mathcal{C})$ is singled out, and hence so is a partition function, as explained in detail in [14].

Similar analyses should apply to compactifications of a $(1,0)$ theory on a three-manifold and four-manifold. It would be quite interesting to work out further physical consequences of the defect group in such cases.

6 Conclusions

In this note we have introduced and computed the defect group of all known (and quite possibly all) 6D SCFTs. Quite remarkably, this data is fully captured by the abelianization of the discrete orbifold subgroups of $U(2)$ which appear in the classification of 6D SCFTs. We have determined the general pattern of possible defect groups, and have also taken some preliminary steps in the study of compactifying 6D SCFTs. In the remainder of this section we discuss some avenues for further investigation.

The appearance of the abelianization of an orbifold group Γ gives a physical explanation for the appearance of these discrete groups in the classification of 6D SCFTs. This is quite suggestive of a further role for the theory of $\Gamma \otimes SL(2, \mathbb{Z})$ equivariant K-theory in the study of brane charges in an F-theory compactification. Developing such a correspondence would dovetail with the mathematical structures observed in earlier work on the physically different case of tachyon condensation on non-supersymmetric orbifolds (see e.g. [62]). It would likely also point the way to a more algebraic characterization of F-theory vacua.

We have also seen that a suitable generalization of the topological data for $(2, 0)$ theories can be carried over to the case of $(1, 0)$ theories. In this spirit, it is widely suspected that there is a 7D topological field theory which governs the structure of conformal blocks for a given $(2, 0)$ theory. At a formal level, a similar structure must exist for the $(1, 0)$ theories. Determining its explicit form would be most instructive.

Finally, it is natural to ask how the topological data of the defect group carries over to those $(1, 0)$ SCFTs with a holographic dual (for recent work see e.g. [18,19]). In the case of the A-type $(2, 0)$ theories, it is well-known how this data descends to lower-dimensional systems. The fact that there are also discrete choices in the holographic duals for compactifications of $(1, 0)$ systems [63] is quite suggestive, and would be interesting to study further.

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