TOPOLOGY OF EXCEPTIONAL ORBIT HYPERSURFACES OF PREHOMOGENEOUS SPACES

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Abstract. We consider the topology for a class of hypersurfaces with highly nonisolated singularites which arise as "exceptional orbit varieties" of a special class of prehomogeneous vector spaces, which are representations of linear algebraic groups with open orbits. These hypersurface singularities include both determinantal hypersurfaces and linear free (and free*) divisors. Although these hypersurfaces have highly nonisolated singularities, we determine the topology of their Milnor fibers, complements and links. We do so by using the action of linear algebraic groups beginning with the complement, instead of using Morse-type arguments on the Milnor fibers. This includes replacing the local Milnor fiber by a global Milnor fiber which has a "complex geometry" resulting from a transitive action of an appropriate algebraic group, yielding a compact "model submanifold" for the homotopy type of the Milnor fiber. The topology includes the (co)homology (in characteristic 0, and 2-torsion in one family) and homotopy groups, and we deduce the triviality of the monodromy transformations on rational (or complex) cohomology.

Unlike isolated singularities, the cohomology of the Milnor fibers and complements are isomorphic as algebras to exterior algebras or for one family, modules over exterior algebras; and cohomology of the link is, as a vector space, a truncated and shifted exterior algebra, for which the cohomology product structure is essentially trivial. We also deduce from Bott's periodicity theorem, the homotopy groups of the Milnor fibers for determinantal hypersurfaces in the "stable range" as the stable homotopy groups of the associated infinite dimensional symmetric spaces. Lastly, we combine the preceding with a Theorem of Oka to obtain a class of "formal linear combinations" of exceptional orbit hypersurfaces which have Milnor fibers which are homotopy equivalent to joins of the compact model submanifolds. It follows that Milnor fibers for all of these hypersurfaces are essentially never homotopy equivalent to bouquets of spheres (even allowing differing dimensions).

Introduction

In this paper we investigate the topology of a class of highly nonisolated hypersurface singularities $\mathcal{E}$, each of which arises as the hypersurface of exceptional orbits, the *exceptional orbit variety*, for a rational representation of connected complex linear algebraic group $\rho : G \to GL(V)$ with an open orbit $\mathcal{U}$. Such a space with group action has been studied by Sato and Kimura [So], [SK] and is called by

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them a *prehomogeneous vector space*, which we will shorten in this paper to just *prehomogeneous space*. Both *determinantal hypersurfaces* and *linear free divisors* belong to this class. We consider how the topology of such singularities can be determined.

For nonisolated singularities with small dimensional singular set, a body of work by Siersma [Si], [Si2], [Si3], Tibar [Ti], Nemethi [Ne], Zaharia [Z], etc. has used Morse-theoretic methods to extend the results for isolated singularities, and show that the Milnor fiber is still homotopy equivalent to a bouquet of spheres, except now the spheres may have different dimensions. One might ask to what extent such results apply to these hypersurfaces of exceptional orbits, which now have highly nonisolated singularities. Complex Morse theory has been applied to determine the vanishing topology of "nonlinear sections" of these hypersurfaces in terms of "singular Milnor fibers" in results by Mond, Goryunov, Bruce, Pike, and this author, see e.g. [DM], [Br], [D1, D2], [GM], and [DP3]. However, these results provide little information about the topology of the hypersurface singularities themselves. We shall see that the structure of these singularities are quite different from those studied in the above work. In fact, to determine their topology we take a very different approach which makes substantial use of their representations via prehomogeneous spaces.

For the exceptional orbit varieties, we will be concerned with the topology of the Milnor fiber, the complement and the link; and we determine their homotopy types along with the (co)homology structure, homotopy groups, and the monodromy action. The changes in approach which make this possible are:

i) reversing the usual approach from first determining the Milnor fiber and monodromy to then compute the topology of the link and complement by beginning instead with the complement and deducing the topology of the link and Milnor fiber.
ii) replacing the local Milnor fiber by a global Milnor fiber, which is a smooth affine hypersurface that has a "model complex geometry" resulting from the transitive action of an associated linear algebraic group, yielding as a deformation retract a compact submanifold;
iii) using the relation between the two algebraic group actions and the topology of maximal compact subgroups to deduce the cohomological triviality of an associated fibration of the groups; and
iv) using the preceding to determine the topology and cohomology of the Milnor fiber.

We will explicity compute the cohomology of the Milnor fiber, the complement and the link for two classes: determinantal hypersurfaces, which are varieties of singular matrices in the spaces of $m \times m$ matrices which may be symmetric, general, or skew-symmetric (with m even); and exceptional orbit varieties in the general equidimensional case. This second class will in particular apply to both linear free (and free*) divisors, introduced for reductive groups by Buchweitz-Mond [BM] and determined for "block representations" of solvable groups in Damon-Pike [DP], [DP2].

We express the cohomology algebras of the Milnor fibers as either exterior algebras, or for the one case of symmetric $m \times m$ matrices with m even, modules over an exterior algebra on two generators, see Theorems 3.1 and 5.1. For determinantal hypersurfaces, we specifically show, Theorem 3.1, that the Milnor fibers are homotopy equivalent to compact classical symmetric spaces of Cartan; and besides obtaining their cohomology we give their homotopy groups in a stable range using Bott periodicity. As well, certain of the Milnor fibers exhibit 2-torsion in their cohomology. We further show that for either class, excluding the symmetric $m \times m$ matrices with m even, the monodromy acts trivially on the (rational or) complex cohomology of the Milnor fiber. For the links in either case, we compute the cohomology vector space as a truncated and shifted exterior algebra and determine that its cohomology product structure is "almost trivial", Theorems 3.2 and 4.5.

For the complex cohomology of the complement for reductive groups these results extend those obtained in Granger-Mond et al. [GMNS], although in that paper they also prove the cohomology is computed by the logarithmic complex, which we do not. The results here also extend the results obtained for the complement and Milnor fiber in the case of solvable algebraic groups in Damon-Pike [DP].

Lastly, we combine in §6 the results on exceptional orbit hypersurfaces we have described together with a Theorem of Mutsuo Oka [Ok] and the results of Siersma et al to compute the topology of a class of hypersurface singularities which are formed as "formal sums" of hypersurface singularities each of which is either an exceptional orbit hypersurface which we consider or is a weighted homogeneous nonisolated hypersurface singularity considered by the Siersma group. These yield hypersurface singularities whose Milnor fibers are homotopy equivalent to joins of compact manifolds, or more generally to a bouquet of suspensions of such joins of compact manifolds.

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1. Prehomogeneous Spaces and the Wang Sequence

A Special Class of Prehomogeneous Spaces. We consider a special class of representations of a connected complex linear algebraic groups $\rho : G \to \mathrm{GL}(V)$ which have an open orbit $\mathcal{U}$. We specifically consider the cases where the *exceptional orbit variety*, which is the union of the orbits of positive codimension, is a hypersurface $\mathcal{E}$. If H is the isotropy subgroup for a point $v_0 \in \mathcal{U}$, then H is a closed algebraic subgroup of G, and it is a basic fact, see e.g. Borel [Bo2], that both G and H have maximal compact subgroups K, resp. L, with $L \subset K$, which are strong deformation retracts of G, resp. H, and of the same ranks as G and H.

For example this will include the cases where $V = M$ is one of the spaces of complex matrices $M = Sym_m$ or $M = Sk_m$ (for $m = 2k$) acted on by $\mathrm{GL}_m(\mathbb{C})$ by $B \cdot A = BAB^T$, or , $M = M_{m,m}$ and $\mathrm{GL}_m(\mathbb{C})$ acts by left multiplication. Each of these representations have open orbits and the resulting prehomogeneous space has an exceptional orbit variety $\mathcal{E}$ which is a hypersurface.

Definition 1.1. The *determinantal hypersurface* for the space of $m \times m$ symmetric or general matrices, denoted by $M = Sym_m$ or $M = M_{m,m}$ is the hypersurface of singular matrices defined by $\det : M \to \mathbb{C}$ and denoted by $\mathcal{D}_m^{sy}$ for $M = Sym_m$, or $\mathcal{D}_m$ for $M = M_{m,m}$. For the space of $m \times m$ skew-symmetric matrices $M = Sk_m$ (for $m = 2k$) the determinantal hypersurface of singular matrices is defined by the Pfaffian $\mathrm{Pf} : Sk_m \to \mathbb{C}$, and is denoted by $\mathcal{D}_m^{sk}$.

A second class of examples consists of "equidimensional representations" where $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} G$ with an open orbit $\mathcal{U}$. Then, necessarily $\mathcal{E}$ is a hypersurface and the isotropy subgroup H of a point in the open orbit is finite. If an appropriate defining equation for $\mathcal{E}$, obtained from the coefficient determinant of the associated vector fields for the action, is reduced then $\mathcal{E}$ is a "linear free divisor", introduced by Mond and Buchweitz [BM]; and otherwise it is a slightly weaker linear free* divisor with nonreduced defining equation. Partial results on the topology of the complement were obtained if G is reductive, see e.g. [GMNS]) and also for the complement and Milnor fiber for "block representations"for G solvable (see [DP], [DP2]). We shall determine the topology in the general case.

For any of these cases, the action of G commutes with the usual $\mathbb{C}^*$-action on V; hence, the exceptional orbit variety $\mathcal{E}$ is also invariant, and hence has a homogeneous reduced defining equation f of degree n. In the case of equidimensional representations there is a defining equation given by the coefficient determinant of the associated vector fields of degree $N = \dim_{C} V = \dim_{C} G$. If $\mathcal{E}$ is a linear free divisor, this is a reduced definng equation, which we may choose for f; if not then $\deg f = n < N$.

We consider the Milnor fibration of the hypersurface germ $(\mathcal{E}, 0)$ in the standard form as $f^{-1}(S^1_\delta) \cap D^{2N}_\varepsilon$ with D^{2N}_ε the disk about 0 of sufficiently small radius ε, S^1_δ the boundary of the disk about 0 in $\mathbb{C}$ of radius δ, for $0 < \delta << \varepsilon$. Because of the homogeneity of f we may adapt a standard argument for isolated singularities to obtain the global description of the Milnor fibration.

Lemma 1.2. *The Milnor fibration of $(\mathcal{E}, 0)$ is diffeomorphic to the fibration $f|E : E \to S^1$, where $E = f^{-1}(S^1)$, with fiber $F = f^{-1}(1)$. This is the restriction of the fibration $f : V\backslash\mathcal{E} \to \mathbb{C}^*$ to $S^1 \subset \mathbb{C}^*$, and the inclusion $E \subset V\backslash\mathcal{E}$ is a homotopy equivalence.*

Proof. The proof is a slight modification of that for the case of isolated singularities. It uses the induced homogeneous $\mathbb{R}_+$-action to establish a diffeomorphism of fibrations between $E = f^{-1}(S^1)$ and $f^{-1}(S^1_\delta)$. If $0 < \delta << \varepsilon$ are sufficiently small, it is proven that the restriction of the distance-squared function $\| \cdot \|^2$ to $f^{-1}(S^1_\delta)\backslash D^{2N}_\varepsilon$ has no critical points, and by Morse theory we deduce that the Milnor fibration is diffeomorphic as a fibration to $f^{-1}(S^1_\delta)$. For the case of equidimensional representations of solvable linear algebraic groups the details are given in the proof of [DP, Thm 3.2]. The general case follows the same line of reasoning. □

We refer to the fibration $F \hookrightarrow E \to S^1$ as the *global Milnor fibration* and F as the *global Milnor fiber*. We note that by e.g. Kato-Masumoto [KM], provided $N \geq 2$, F is path-connected.

Second, the cohomology of the complement $V\backslash\mathcal{E}$ (and hence E), can be expressed in terms of the maximal compact subgroups.

Lemma 1.3. *If G and the isotropy subgroup H of $v_0 \in \mathcal{U}$ have maximal compact subgroups K, resp. L, then $V\backslash\mathcal{E}$ is homotopy equivalent to K/L. Hence,*

$$H^*(V\backslash\mathcal{E}) \simeq H^*(K/L) .$$

Proof. First, the action of G on $v_0 \in \mathcal{U}$, gives the open orbit $V\backslash\mathcal{E} = \mathcal{U} \simeq G/H$ as a homogenenous space. It is sufficient to show the inclusion $K/L \hookrightarrow G/H$ is a homotopy equivalence. As pointed out by Shrawan Kumar, this is a simple consequence of the long exact homotopy sequence for a fibration. The groups G, resp. H, are homotopy equivalent to their maximal compact subgroups, K, resp.L.

We consider the long exact homotopy sequence of the fibrations (see e.g. [Sp, Chap 7, §2, Thm 10]) in the horizontal rows of (1.1).

$$\begin{array}{ccccc} L & \longrightarrow & K & \longrightarrow & K/L \\ {\scriptstyle i'}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle i_0}\downarrow \\ H & \longrightarrow & G & \longrightarrow & G/H \end{array} \tag{1.1}$$

which has the form

$$\begin{array}{ccccccc} \longrightarrow & \pi_j(L, Id) & \longrightarrow & \pi_j(K, Id) & \longrightarrow & \pi_j(K/L, \{L\}) & \longrightarrow \\ & {\scriptstyle i'_*}\downarrow & & {\scriptstyle i_*}\downarrow & & {\scriptstyle i_{0\,*}}\downarrow & \\ \longrightarrow & \pi_j(H, Id) & \longrightarrow & \pi_j(G, Id) & \longrightarrow & \pi_j(G/H, \{H\}) & \longrightarrow \end{array} \tag{1.2}$$

We note that when $j = 0$, both i_* and i'_* are bijections of pointed sets so K is connected and H and L have the same finite number of connected components. The homotopy equivalence $i' : L \hookrightarrow H$ restricts to a homotopy equivalence $i' : L_0 \hookrightarrow H_0$ of the connected components of the identity. Thus, $i'_* : \pi_j(L, Id) = \pi_j(L_0, Id) \simeq \pi_j(H_0, Id) = \pi_j(H, Id)$ is an isomorphism for all $j \geq 0$. Hence, we can apply the five Lemma to conclude first that $\pi_i(K/L, \{L\}) \simeq \pi_i(G/H, \{H\})$ for all $i \geq 0$. As both are path-connected CW-complexes, it follows by Whitehead's theorem that K/L is homotopy equivalent to G/H, giving the result. □

Next, we consider the global Milnor fibration $F \hookrightarrow E \to S^1$. First, from the long exact homotopy sequence for the fibration, we obtain $\pi_i(F) \simeq \pi_i(E)$ for $i \geq 2$ and, as F is path-connected, the short exact sequence

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \xrightarrow{\ p\ } \pi_1(S^1) \longrightarrow 1\,. \tag{1.3}$$

Hence, by the Hurewicz theorem and the universal coefficient theorem, $p_* : H_1(E) \to H_1(S^1)$ is surjective, and $p^* : H^1(S^1; \mathbf{k}) \to H^1(E; \mathbf{k})$ is injective for any field $\mathbf{k}$ of characteristic 0, so the image of a generator of $H^1(S^1; \mathbf{k})$ is a nonzero class $s_1 \in H_1(E)$. Furthermore, this class restricts to 0 in $H^1(F; \mathbf{k})$, so then does the ideal $\langle s_1 \rangle \cdot H^*(E; \mathbf{k})$.

Example 1.4. For the case of complex cohomology, if γ is a generator of $\pi_1(S^1)$, then there is a $\beta \in \pi_1(E)$ such that $f_*(\beta) = \gamma$. Hence, $\frac{df}{f} = f^*(\frac{dz}{z})$ is a closed 1-form which satisfies

$$\int_\beta \frac{df}{f} \;=\; \int_{f_*(\beta)} \frac{dz}{z} \;=\; 2\pi i\,.$$

Hence ,$\omega_1 = \frac{df}{f}$ defines a nontrivial cohomology class s_1 in $H^1(E; \mathbb{C})$. Moreover, its restriction to F is 0 as $df = 0$ as $f \equiv 1$ on F. Thus, the ideal $\mathbb{C}\langle s_1 \rangle \cdot H^*(E; \mathbb{C})$ restricts to 0 in $H^*(F; \mathbb{C})$.

Model Complex Geometry for the Milnor Fiber. We note that although by Lemma 1.2 the Milnor fiber and global Milnor fiber are diffeomorphic, they are not holomorphically diffeomorphic. The different complex structure on the global Milnor fiber allows us to introduce an alternate way to view the topology on the global Milnor fiber resulting from it having a form of "complex model geometry" in the sense of Thurston (see e.g. [Th, Chap. 3]). By this we mean a smooth affine submanifold $X \subset \mathbb{C}^N$ together with an algebraic action of a connected linear algebraic group G on X, which is transitive with H an isotropy subgroup of a point.

We denote it by the triple (X, G, H). It provides a method for beginning to understand the geometry of the global Milnor fiber. Actually, Thurston's definition of geometry over the reals required X be simply connected, which we refer to it as a *simple model geometry*. However, without this restriction, the following is true.

Proposition 1.5. *For a prehomogeneous space defined by the representation $\rho : G \to \mathrm{GL}(V)$ with exceptional orbit variety a hypersurface, there is a connected codimension one algebraic subgroup G'_0 of G which acts transitively on the global Milnor fiber F, with isotropy subgroup H' so that (F, G', H') defines a model complex geometry on F. This model is simple in the case of determinantal hypersurfaces.*

The proof of this will follow from results in § 2. This model geometry type representation of the global Milnor fiber is the essential ingredient, in place of Morse theory, for the analysis of the topology. To carry out this analysis, we will make considerable use of the Wang cohomology sequence to relate the cohomology of E with that of F.

The Wang Sequence. We consider a fibration $F \hookrightarrow E \stackrel{p}{\to} S^1$. We let $\sigma = \sigma_1$ be the monodromy map where σ_t denotes the lift of the curve $\varphi(t) = e^{2\pi i t}$ on S^1. to the family of homeomorphisms $\sigma_t : F \to F_t = p^{-1}(\varphi(t))$.

Definition 1.6. We say that the fibration $F \hookrightarrow E \to S^1$ is *(rationally) cohomologically trivial* if $\sigma^* : H^*(F; \mathbb{Q}) \to H^*(F; \mathbb{Q})$ is the identity.

Remark 1.7. By the universal coefficient theorem it likewise follows that for any field $\mathbf{k}$ of characteristic 0, $\sigma^* = id$ on $H^*(F; \mathbf{k})$ and conversely if it holds for $\mathbf{k}$ of characteristic 0 then it also holds for $\mathbb{Q}$. Also, if $H^*(F; \mathbb{Z})$ is a free abelian group then it also follows that the Milnor fibration is cohomologically trivial for integer coefficients.

We also might ask whether the monodromy is geometrically trivial in that it is homotopy equivalent to the identity. David Mond has been able to show directly for several examples in the equidimensional case, including certain quiver representations, that the monodromy is geometrically trivial; however, for certain other equidimensional cases we discussed neither of us were unable to verify this. For now there is no general result for geometric triviality of monodromy, even for certain special classes such as quiver representations, so it is still an open question. However, we will see that being cohomologically trivial will suffice for many questions.

To begin, there is the following result.

Proposition 1.8. *For the fibration $F \hookrightarrow E \to S^1$, with F connected and $\mathbf{k}$ a field of characteristic 0, the following are equivalent.*

i) *The fibration is cohomologically trivial.*

ii) *there is an isomorphism of graded vector spaces.*

$$\begin{aligned} H^*(E; \mathbf{k}) &\simeq \Lambda^* \mathbf{k}\langle s_1 \rangle \otimes H^*(F; \mathbf{k}) \\ &\simeq H^*(F; \mathbf{k}) \;\oplus\; \mathbf{k}\langle s_1 \rangle \otimes H^*(F; \mathbf{k}) \,. \end{aligned} \tag{1.4}$$

iii)

$$\dim_{\mathbf{k}} H^*(E; \mathbf{k}) \;=\; 2 \dim_{\mathbf{k}} H^*(F; \mathbf{k}) \,. \tag{1.5}$$

Moreover, if the preceding hold then (1.4) is an isomorphism of graded $\Lambda^* \mathbf{k}\langle s_1\rangle$*-modules, where the exterior algebra* $\Lambda^* \mathbf{k}\langle s_1\rangle$ *is on one generator* s_1*.*

Proof. First, we immediately observe that ii) imples iii). Second, suppose i) holds. We consider the Wang sequence in cohomology (see e.g. [Sp, Chap. 8 §5 Cor. 6]) applied to our situation.

$$(1.6) \qquad \longrightarrow H^q(F;\mathbf{k}) \xrightarrow{\theta} H^q(F;\mathbf{k}) \longrightarrow H^{q+1}(E;\mathbf{k}) \xrightarrow{i^*} H^{q+1}(F;\mathbf{k}) \xrightarrow{\theta} H^{q+1}(F;\mathbf{k}) \longrightarrow$$

where $\theta = id - \sigma^*$ for σ the monodromy map.

If the fibration is cohomologically trivial, then $\sigma^* = id$, and (1.6) reduces to the short exact sequences for $q \geq -1$,

$$(1.7) \qquad 0 \longrightarrow H^q(F;\mathbf{k}) \longrightarrow H^{q+1}(E;\mathbf{k}) \xrightarrow{i^*} H^{q+1}(F;\mathbf{k}) \longrightarrow 0$$

It then follows that the Betti numbers, which equal $\beta_q = \dim_{\mathbf{k}} H^q(\,\cdot\,;\mathbf{k})$, satisfy $\beta_{q+1}(E) = \beta_q(F) + \beta_{q+1}(F)$ for $q \geq -1$ (where $\beta_{-1} = 0$). This says that there is a graded vector space isomorphism (1.4) and also that (1.5) holds.

Third, if iii) holds we show that i) holds. For any $q \geq 0$, let $K_q = \ker(I - \sigma^*) : H^q(F;\mathbf{k}) \to H^q(F;\mathbf{k})$ and $C_q = \operatorname{coker}(I - \sigma^*)$. Then, $\dim_{\mathbf{k}} K_q = \dim_{\mathbf{k}} C_q$ and from (1.6)

$$(1.8) \qquad \beta_{q+1}(E) \;=\; \dim_{\mathbf{k}} K_{q+1} + \dim_{\mathbf{k}} C_q \;=\; \dim_{\mathbf{k}} K_{q+1} + \dim_{\mathbf{k}} K_q\,.$$

Hence, if we choose an $r > \dim F$ so that $H^j(F;\mathbf{k}) = 0$ for $j \geq r$, then summing (1.8) over q yields

$$\begin{aligned} \dim_{\mathbf{k}} H^*(E;\mathbf{k}) \;&=\; \sum_{q=-1}^{r-1} \dim_{\mathbf{k}} K_{q+1} \;+\; \sum_{q=0}^{r} \dim_{\mathbf{k}} K_q \\ &=\; 2\sum_{q=0}^{r} \dim_{\mathbf{k}} K_q \qquad (1.9) \end{aligned}$$

Hence, if the fibration is not cohomologically trivial then for some q, $\dim_{\mathbf{k}} K_q < \dim_{\mathbf{k}} H^q(F;\mathbf{k})$. It follows that the RHS of (1.9) is less than $2\dim_{\mathbf{k}} H^*(F;\mathbf{k})$, so iii) doesn't hold.

Lastly, if the above hold, then (1.4) is an isomorphism of graded vector spaces, and (1.6) reduces to the short exact sequences (1.7). Thus, for each q, we can find a set of elements $\{\varphi_1^{(q)}, \dots, \varphi_{m_q}^{(q)}\}$ in $H^q(E;\mathbf{k})$ which restrict to a basis for $H^q(F;\mathbf{k})$. Then, we can apply the Leray-Hirsch Theorem, see e.g. [Sp, Chap. 5, §7,Thm 9], to conclude that $H^*(E;\mathbf{k})$ is a free $H^*(S^1;\mathbf{k})$-module on this set of generators. As $H^*(S^1;\mathbf{k}) \simeq \Lambda^*\mathbf{k}\langle s_1\rangle$, we conclude that (1.4) is an isomorphism of graded $\Lambda^*\mathbf{k}\langle s_1\rangle$-modules. □

Topology of the Complement, Milnor Fiber, and Link. We consider a representation $\rho : G \to \mathrm{GL}(V)$ which belongs to the special class of prehomogeneous spaces: with open orbit $\mathcal{U}$, isotropy subgroup H of $v_0 \in \mathcal{U}$, with maximal compact subgroups K, resp. L, exceptional orbit variety $\mathcal{E}$ which is a hypersurface, and whose global Milnor fiber is cohomologically trivial. We apply the preceding results to draw conclusions about the cohomology of the complement, the Milnor fiber and link of $\mathcal{E}$ using the preceding notation.

We have already considered both the cohomology of the complement and Milnor fiber. Since $\mathcal{E}$ is invariant under the usual $\mathbb{C}^*$-action, we may use the restricted $\mathbb{R}_+$-action to conclude that $L(\mathcal{E}) = \mathcal{E} \cap S^{2N-1}$ is diffeomorphic to the link of $\mathcal{E}$, where S^{2N-1} denotes the unit sphere about the origin in V.

To describe the cohomology of the link we use the following notation. For a compact connected orientable manifold X, we introduce the graded vector space $\widetilde{H^*(X; \mathbf{k})}\,[r]$ which will denote the vector space $H^*(X; \mathbf{k})$, truncated at the top degree and shifted upward by degree r. Then we can summarize the topology of the exceptional orbit variety $\mathcal{E}$ by the following.

Proposition 1.9. *Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation which belongs to the special class of prehomogeneous spaces.*

i) Cohomology of Milnor fiber F:
If the global Milnor fibration is cohomologically trivial, then

$$H^*(F; \mathbf{k}) \simeq H^*(E; \mathbf{k}) / \left(\mathbf{k}\langle s_1\rangle \smile H^*(E; \mathbf{k})\right)$$

where as graded vector spaces,

$$H^*(E; \mathbf{k}) \simeq H^*(F; \mathbf{k}) \oplus \left(\mathbf{k}\langle s_1\rangle \otimes H^*(F; \mathbf{k})\right) ;$$

ii) Cohomology of the Complement $V\backslash\mathcal{E}$:

$$H^*(V\backslash\mathcal{E}; \mathbf{k}) \simeq H^*(K/L; \mathbf{k}) ;$$

iii) Cohomology of the Link $L(\mathcal{E})$:
If K/L is orientable, then

$$\widetilde{H}^*(L(\mathcal{E}); \mathbf{k}) \simeq \widetilde{H^*(K/L; \mathbf{k})}\,[2N - 2 - \dim_{\mathbb{R}} K/L] ;$$

Proof. We have already discussed i) in Proposition 1.8, and ii) in Lemma 1.3.

For the link, as $\mathcal{E}$ is homogeneous (of degree n), the complement $V\backslash\mathcal{E}$ is also invariant under the usual $\mathbb{C}^*$ action. Thus, we may contract using the $\mathbb{R}_+$-action to conclude that $\mathcal{E}\backslash\{0\}$ is homotopy equivalent to the link $L(\mathcal{E})$ and $V\backslash\mathcal{E}$ is homotopy equivalent to the complement of the link $S^{2N-1}\backslash L(\mathcal{E})$. Thus,

$$H^*(S^{2N-1}\backslash L(\mathcal{E}); \mathbf{k}) = H^*(V\backslash\mathcal{E}; \mathbf{k}) . \tag{1.10}$$

Second, we apply Alexander duality for subspaces of spheres (see e.g. [Ma, Chap. XIV, Thm 6.6]) to conclude for all j

$$\widetilde{H}^j(L(\mathcal{E}); \mathbf{k}) \simeq \widetilde{H}_{2N-2-j}(S^{2N-1}\backslash L(\mathcal{E}); \mathbf{k}) . \tag{1.11}$$

Hence, combined with (1.10), we obtain for all j

$$\widetilde{H}^j(L(\mathcal{E}); \mathbf{k}) \simeq \widetilde{H}_{2N-2-j}(V\backslash\mathcal{E}; \mathbf{k}) . \tag{1.12}$$

By Lemma 1.3

$$H^*(V\backslash\mathcal{E}; \mathbf{k}) \simeq H^*(K/L; \mathbf{k}) \tag{1.13}$$

If $q = \dim_{\mathbb{R}} K/L$, then, as K/L is compact and orientable, by Poincaré duality $H^{q-r}(K/L; \mathbf{k}) \simeq H^r(K/L; \mathbf{k})$ for all $0 \leq r \leq q$. Thus, for $j = 2N - 2 - (q - r)$, we obtain from (1.12) for $0 < r < q$

(1.14)

$$\widetilde{H}^{2N-2-q+r}(L(\mathcal{E}); \mathbf{k}) \simeq \widetilde{H}_{q-r}(K/L; \mathbf{k}) \simeq \widetilde{H}^{q-r}(K/L; \mathbf{k}) \simeq \widetilde{H}^r(K/L; \mathbf{k}) .$$

If $r = q$ we obtain

$$\widetilde{H}^{2N-2)}(L(\mathcal{E});\mathbf{k}) \;\simeq\; \widetilde{H}^{0}(K/L;\mathbf{k}) \;=\; 0\,.$$

This last equation also follows as $\dim_{\mathbb{R}} L(\mathcal{E}) = 2N-3$. If $j = 0$ in (1.12) and $N > 2$ then as $2N-2 > N = \dim_{\mathbb{R}} K/L$,

$$\widetilde{H}^{0}(L(\mathcal{E});\mathbf{k}) \;\simeq\; \widetilde{H}^{2N-2}(K/L;\mathbf{k}) \;=\; 0\,.$$

Combining (1.14) and the comment following it with (1.13), yields the result if $N > 2$.

The only cases when $N \leq 2$ are Sym_1, $M_{1,1}$, and Sk_2, each of which is dimension 1 with $\mathrm{GL}_1(\mathbb{C}) = C^*$, $K = S^1$, $H = \{1\}$, and $\mathcal{E} = \{0\}$. In each case the Milnor fiber is a single point, the link is empty, and the complement is homotopy equivalent to S^1. Then, iii) has to be understood as correct for $\widetilde{H}^j(L(\mathcal{E});\mathbf{k})$ for $j \geq 0$ and then the result is true. □

Then, we will use these results to determine when the global Milnor fibrations arising from two classes of exceptional orbit hypersurfaces are cohomologically trivial, and show that the cohomology $H^*(K/L;\mathbf{k})$ is either an exterior algebra or a module over an exterior algebra.

2. Modeling Global Milnor Fibers for Exceptional Orbit Hypersurfaces

Let $\rho : G \to GL(V)$ be a representation of a connected linear algebraic group G defining a prehomogeneous space with exceptional orbit variety $\mathcal{E}$ a hypersurface. We will identify a subgroup of G which defines a model complex geometry on the global Milnor fiber F. Using this we relate the corresponding maximal connected compact subgroups via an exact sequence that gives a fibration which we show is always cohomologically trivial.

For now we consider the general case with $\mathcal{E}$ a hypersurface with reduced homogenous defining equation $h = 0$. We first define a representation χ of G on $\mathbb{C}\langle h\rangle$ as follows. If $g \in G$, then we let $g^*h(v) = h(g^{-1}\cdot v)$ for $v \in V$. As the action of g^{-1} preserves the orbits of G, g^*h still vanishes on $\mathcal{E}$. Thus, by the Nullstellensatz, g^*h is a multiple of h. Also, as the action of g is given by a linear transformation of V, g^*h is again a polynomial of the same degree as h; hence, $g^*h = c_g \cdot h$, for $c_g \in \mathbb{C}$. This defines a representation $\chi_0 : G \to \mathrm{GL}(\mathbb{C}\langle h\rangle) \simeq \mathbb{C}^*$. We let $G' = \ker(\chi_0)$ and G'_0 denote the connected component of the identity of G'. We begin with a simple Lemma.

Lemma 2.1. *In the preceding situation, χ_0 is non-trivial and induces by a lifting an exact sequence*

$$1 \longrightarrow G'_0 \longrightarrow G \xrightarrow{\;\chi\;} \mathbb{C}^* \longrightarrow 1 \tag{2.1}$$

where G' and G'_0 are linear algebraic groups with $\dim_{\mathbb{C}} G'_0 = \dim_{\mathbb{C}} G - 1$ and $\mathrm{rank}\,(G'_0) = \mathrm{rank}\,(G) - 1$. Moreover, if G is reductive, respectively solvable, then so is G'_0 reductive, respectively solvable.

Proof. First, if χ_0 were trivial, then G acts trivially on h; hence, it will preserve the fibers $h^{-1}(w)$ for any $w \in \mathbb{C}$. This says that the orbits of G are all of positive codimension, so there is no open orbit, a contradiction. Thus, χ_0 is surjective and $G/G' \simeq \mathbb{C}^*$.

Since G' is a closed subgroup of G, it is an algebraic subgroup, as is the subgroup G'_0. Also, G' and G'_0 have the same dimension and rank. From $G/G' \simeq \mathbb{C}^*$ we see $\dim_{\mathbb{C}} G'_0 = \dim_{\mathbb{C}} G' = \dim_{\mathbb{C}} G - 1$.

As G' is algebraic, it has only finitely many connected components. Thus, the quotient map $G/G'_0 \to G/G' \simeq \mathbb{C}^*$ is a finite covering map of $\mathbb{C}^*$ and hence $G/G'_0 \simeq \mathbb{C}^*$. Thus, we have the exact sequence given in (2.1), where χ denotes the lifting of χ_0 for the covering map.

Next, if T denotes a maximal algebraic torus of G of dimension $k = \operatorname{rank}(G)$, then $\chi_0|T$ is still onto $\mathbb{C}^*$, otherwise its image would be $\{1\}$. Since every element in G is conjugate to an element of T, χ would be trivial on G, which as we just saw is impossible. Thus, $\ker(\chi_0|T)$ is a torus of dimension $k-1$. Thus, $\operatorname{rank}(G') \geq k-1$. If $\operatorname{rank}(G') = k$, then there is a maximal algebraic torus $T' \subset G'$ of rank k. Then, T' is also a maximal algebraic torus of G and $\chi_0|T'$ is trivial, a contradiction. Thus, $\operatorname{rank}(G') = k-1$.

If G is solvable, then so is its subgroup G'_0. If G is reductive, and K is the maximal compact subgroup with Lie algebra $\mathfrak{k}$, then $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$. If $\tilde{\chi} : \mathfrak{g} \to \mathbb{C}$ is the Lie algebra homomorphism associated to χ, then as it is $\mathbb{C}$ linear, it is the complexification of $\tilde{\chi}|\mathfrak{k}$. Hence, the Lie algebra of G', $\mathfrak{g}' = \ker(\tilde{\chi})$ is the complexification of $\ker(\tilde{\chi}) = \mathfrak{k}'$, the Lie algebra of K'. Thus, if G'_0 and K'_0 denote the corresponding connected components of the identity, then as K'_0 is compact, and $\mathfrak{k}'$ a real form for $\mathfrak{g}'$, then by [GW, Thm 2.4.7, Prop. 2.4.2], G'_0 and then G' are reductive.

This completes the proof.

□

Second, we consider the associated exact sequence of maximal compact subgroups. For the maximal compact subgroup $K \subset G$, $\chi(K)$ is a connected compact subgroup of $\mathbb{C}^*$, and hence is either $\{1\}$ or S^1. Again if $\chi(K) = \{1\}$, let T_0 be a maximal torus of K contained in a maximal algebraic torus T of G and homotopy equivalent to it. Since $\chi_0|T_0$ is trivial, it follows that $\chi|T$ is trivial, which is a contradiction.

We let $\chi' = \chi|K$ and $K' = \ker(\chi')$, $G' \cap K$. We also let K'_0 be the connected component of the identity of K', so $K'_0 \subset G'_0$.

Then, we have the following diagram with rows exact, and the vertical arrows are inclusions.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G'_0 & \longrightarrow & G & \xrightarrow{\chi} & \mathbb{C}^* & \longrightarrow & 1 \\ & & {\scriptstyle i'}\uparrow & & {\scriptstyle i}\uparrow & & {\scriptstyle i_0}\uparrow & & \\ 1 & \longrightarrow & K'_0 & \longrightarrow & K & \xrightarrow{\chi'} & S^1 & \longrightarrow & 1 \end{array} \tag{2.2}$$

We remark that the bottom row is exact, because the covering $K/K'_0 \to K/K' \simeq S^1$ is the restriction of the covering for $G/G'_0 \to G/G'$.

There are three things that we want to show in this situation:

i) i' is a homotopy equivalence;
ii) G'_0 acts transitively on the global Milnor fiber F; and
iii) the fibration $K'_0 \hookrightarrow K \to S^1$ is cohomologically trivial.

Lemma 2.2. *In the above situation in the diagram (2.2), i' is a homotopy equivalence.*

Proof. Each of the rows of (2.2) gives fibrations: $G'_0 \hookrightarrow G \to \mathbb{C}^*$ and $K'_0 \hookrightarrow K \to S^1$. Since K is the maximal compact subgroup of G, $i : K \hookrightarrow G$ is a homotopy equivalence, as is the inclusion $i_0 : S^1 \hookrightarrow \mathbb{C}^*$. Then, we may consider the induced maps for the long exact sequences in homotopy for the two fibrations, and again by the 5-lemma, since both i and i_0 are homotopy equivalences, it follows that $i'_* : \pi_j(K'_0) \simeq \pi_j(G'_0)$ for all $j \geq 0$. Since they are both path-connected CW-complexes, it follows by Whitehead's theorem that i' is a homotopy equivalence. □

Next, as G'_0 acts trivially on h, it acts on the fibers of h, and in particular on the global Milnor fiber F. Let H' be the isotropy subgroup of G'_0 for a point v_0 in F, and let $L' \subset K'_0$ be the maximal compact subgroup of H'.

Lemma 2.3. *In the preceding situation, G'_0 acts transitively on the global Milnor fiber and $K'_0/L' \subset F$ is a deformation retract. In the equidimensional case, H' is finite and hence equals L' and G'_0 is a finite regular covering space of F with group of covering transformations H'.*

Proof. By Lemma 2.1, $\dim_{\mathbb{C}} G'_0 = \dim_{\mathbb{C}} G - 1$ and it acts on the global Milnor fiber F of $\dim_{\mathbb{C}} F = N - 1$, where $\dim_{\mathbb{C}} V = N$. If $v_0 \in F$, then as $v_0 \in \mathcal{U}$, the orbit map $\Phi : G \to V$ sending $g \mapsto g \cdot v_0$ is a local submersion at Id. Hence, if $d\Phi(Id)|T_{Id}G'_0$ has rank $< N-1$, then $d\Phi(Id)$ has rank $< N$, contradicting it being a submersion. Thus, $d\Phi(Id)|T_{Id}G'_0$ has rank $N-1$, and as G'_0 preserves the fibers of h, $\Phi(G'_0) \subset F$, so its image is a neighborhood of v_0. As this is true at each point $v_0 \in F$, The orbit of v_0 under G'_0 is open in F. Thus, any orbit of G' in F is open in F. As F is connected (by e.g. Kato-Matsumoto [KM]), there can only be one orbit and G'_0 acts transitively on F. Hence, if H' is the isotropy subgroup in G'_0 of v_0, F is diffeomorphic to G'_0/H'. Also, by the same argument given in Lemma 1.3, $K'_0/L' \subset G'_0/H'$ is a deformation retract, establishing the first result.

In the equidimensional case, $\dim_{\mathbb{C}} G'_0 = \dim_{\mathbb{C}} F = N - 1$. As they have the same dimension, the isotropy subgroup of a point is a zero dimensional algebraic subgroup, and hence finite. As G'_0 is connected and $F \simeq G'_0/H'$, G'_0 is a covering space as stated.

□

Remark 2.4. We see now that Proposition 1.5 follows from Lemma 2.3, since (F, G'_0, H') gives the model complex geometry for F. For the determinantal hypersurfaces, the singular sets for $\mathcal{D}_m^{sy}$, $\mathcal{D}_m$, and $\mathcal{D}_m^{sk}$ (m even) have codimension ≥ 2, so by Kato-Matsumoto [KM] the Milnor fibers are simply connected, so the model is simple.

We also remark that Lemma 2.3 gives an alternate approach to the topological structure of the Milnor fiber. Rather than trying to represent it as being homotopy equivalent to a bouquet of spheres, it gives a compact submanifold which is a deformation retract.

We complete this section by establishing for a fibration of connected compact Lie groups that cohomological triviality always hold.

Lemma 2.5. *For an exact sequence of compact connected Lie groups*

$$1 \longrightarrow K'_0 \longrightarrow K \xrightarrow{\chi'} S^1 \longrightarrow 1 \tag{2.3}$$

The associated fibration $K'_0 \hookrightarrow K \to S^1$ is cohomologically trivial.

Proof. First, by the same arguments as in Lemma 2.1 but for compact groups, $\dim_{\mathbb{R}} K'_0 = \dim_{\mathbb{R}} K - 1$ and $\operatorname{rank} K'_0 = \operatorname{rank} K - 1 = k - 1$.

Thus, by the basic Hopf structure theorem, the cohomology with coefficients in a field $\mathbf{k}$ of characteristic 0 of a compact connected Lie group of rank k is a Hopf algebra which is isomorphic to an exterior algebra on k generators of odd degrees (see e.g. Borel [Bo] which treats the more general Leray Structure theorem for Hopf algebras, where $\mathbf{k}$ may be a perfect field). Thus, $H^*(K; \mathbf{k})$ is isomorphic to an exterior algebra on k generators and $H^*(K'_0; \mathbf{k})$ is isomorphic to an exterior algebra on $k-1$ generators. Thus, $\dim_{\mathbf{k}} H^*(K; \mathbf{k}) = 2^k$, and $\dim_{\mathbf{k}} H^*(K'_0; \mathbf{k}) = 2^{k-1}$. Hence, $\dim_{\mathbf{k}} H^*(K; \mathbf{k}) = 2 \dim_{\mathbf{k}} H^*(K'_0; \mathbf{k})$, so by Proposition 1.8 the fibration is cohomologically trivial. □

This last result by itself will not imply that the global Milnor fibration is cohomologically trivial. In fact, we see in the next section that it fails for a family of determinantal hypersurfaces. However, the preceding will provide the crucial pieces in §5 to prove that for any prehomogeneous space defined from an equidimensional representation $\rho : G \to \mathrm{GL}(V)$ of a connected linear algebraic group G, the global Milnor fibration is cohomologically trivial.

3. Topology of Determinantal Hypersurfaces

We first consider the determinantal hypersurfaces which arise as the exceptional orbit varieties of the representations of $\mathrm{GL}_m(\mathbb{C})$ on $M = Sym_m$, $M_{m,m}$, or Sk_m (for $m = 2k$), given before Definition 1.1. We separately state the results for the topology of the Milnor fibers, and the complements and links.

Theorem 3.1. *The Milnor fibers of the determinantal hypersurfaces are homogeneous spaces homotopy equivalent to compact symmetric spaces of "classical type" (classified by Cartan) given in Table 1. The cohomology with coefficients in a field* $\mathbf{k}$ *of characteristic* 0 *of the Milnor fiber is either an exterior algebra, or a free module on two genersators over an exterior algebra, as given in Table 1. Also, the Milnor fibration is cohomologically trivial for all cases except the case of* $m \times m$ *symmetric matrices with* m *even.*

Moreover, for the regular and skew-symmetric cases the formulas remain true when $\mathbf{k}$ *is replaced by* $\mathbb{Z}$*, while for the symmetric case there is* 2*-torsion in the Milnor fiber and the mod-*2 *cohomology of the Milnor fiber is given by the* $\mathbb{Z}_2$*-exterior algebra*

$$H^*(SU_m/SO_m(\mathbb{R}); \mathbb{Z}_2) \;=\; \Lambda^* \, \mathbb{Z}_2\langle s_2, s_3, \ldots, s_m\rangle$$

with s_j *of degree* j.

Along with the topology of the Milnor fiber, we can also give the topology of the complement and the (co)homology of the link in the next theorem.

Theorem 3.2. *The complements* $M \backslash \mathcal{D}$ *of the determinantal hypersurfaces* $\mathcal{D}$ *are homogeneous spaces* G/H *which are homotopy equivalent to quotients of maximal compact subgroups* K/L*, given in column 2 of Table 2. The cohomology with coefficients in a field* $\mathbf{k}$ *of characteristic* 0 *of the complement is an exterior algebra as given in column 3 of Table 2. Moreover, for the regular and skew-symmetric cases the formulas remain true when* $\mathbb{C}$ *is replaced by* $\mathbb{Z}$.

Determinantal Hypersurface	Milnor Fiber F	Symmetric Space	$H^*(F, \mathbf{k})$
$\mathcal{D}_m^{sy}$ (m = 2k+1)	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$\Lambda^*\mathbf{k}\langle e_5, e_9, \ldots, e_{2m-1}\rangle$
$\mathcal{D}_m^{sy}$ (m = 2k)	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$\Lambda^*\mathbf{k}\langle e_5, e_9, \ldots, e_{2m-3}\rangle \cdot \{1, e_m\}$
$\mathcal{D}_m$	$SL_m(\mathbb{C})$	SU_m	$\Lambda^*\mathbf{k}\langle e_3, e_5, \ldots, e_{2m-1}\rangle$
$\mathcal{D}_m^{sk}, m = 2k$	$SL_{2k}(\mathbb{C})/Sp_k(\mathbb{C})$	SU_{2k}/Sp_k	$\Lambda^*\mathbf{k}\langle e_5, e_9, \ldots, e_{2m-3}\rangle$

TABLE 1. The generators of the cohomology e_k are in degree k; and the structure is an exterior algebra for all cases except for $\mathcal{D}_{2k}^{sy}$, for which the structure is a free module on 1 and e_m over the exterior algebra.

Determinantal Hypersurface	Complement $M \backslash \mathcal{D}$	$H^*(M\backslash\mathcal{D}, \mathbf{k}) \simeq H^*(K/L, \mathbf{k})$	Shift
$\mathcal{D}_m^{sy}$ (m = 2k+1)	$GL_m(\mathbb{C})/O_m(\mathbb{C}) \sim U_m/O_m(\mathbb{R})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-1}\rangle$	$\binom{m+1}{2} - 2$
$\mathcal{D}_m^{sy}$ (m = 2k)	$GL_m(\mathbb{C})/O_m(\mathbb{C})$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-3}\rangle$	$\binom{m+1}{2} + m - 2$
$\mathcal{D}_m$	$GL_m(\mathbb{C}) \sim U_m$	$\Lambda^*\mathbf{k}\langle e_1, e_3, \ldots, e_{2m-1}\rangle$	$m^2 - 2$
$\mathcal{D}_m^{sk}$ (m = 2k)	$GL_{2k}(\mathbb{C})/Sp_k(\mathbb{C}) \sim U_{2k}/Sp_k$	$\Lambda^*\mathbf{k}\langle e_1, e_5, \ldots, e_{2m-3}\rangle$	$\binom{m}{2} - 2$

TABLE 2. The cohomology of the complements $M\backslash\mathcal{D}$ and links $L(\mathcal{D})$ for each determinantal hypersurface $\mathcal{D}$. The complements, are homotopy equivalent to the quotients of maximal compact subgroups K/L with cohomology given in the third column, where the generators of the cohomology e_k are in degree k; and the structure is an exterior algebra. For the links $L(\mathcal{D})$, the cohomology is isomorphic as a vector space to the cohomology of the complement truncated in the top degree and shifted by the degree indicated in the last column.

The cohomology of the link $L(\mathcal{D})$ is isomorphic as a graded vector space to the cohomology of the complement truncated in the top degree and shifted in degree. The shift is $N-2$ where $N = \dim_{\mathbb{C}} M$ for all cases except the symmetric case with m even; and in that case the shift is $N - 2 + m$. Then the products in the reduced cohomology $\widetilde{H}^*(L(\mathcal{D}); \mathbb{C})$ are 0 except possibly for the single product $e_1^* \smile 1^*$, where the e_1^* and 1^* are the images in degrees $N-1$, resp. $N-2$, of $e_1, 1 \in H^*(M\backslash\mathcal{D}; \mathbb{C})$ in all cases except for the symmetric case with m even.

Example 3.3 (Simplest Determinantal Hypersurfaces which are Morse Singularities)**.** We consider the complex cohomology for the simplest examples in Table 1 of dimension > 1 and clarify the notation. First, for $\mathcal{D}_2^{sy}$ we have $m = 2$ with $k = 1$. As $4k - 3 = 1 < 5$, the exterior algebra is on 0 generators. By this we mean it is just $\mathbb{C} \cdot 1$, with 1 the identity in the cohomology algebra. The Milnor fiber then has complex cohomology $\mathbb{C}\langle 1, e_2\rangle$. As $\mathcal{D}_2^{sy}$ is defined by an equation of the form

$xz - y^2 = 0$, we see it is a Morse singularity with cohomology as stated. In addition, in Table 2, the complement has cohomology $\mathbb{C}\langle 1, e_1\rangle$; and after truncating e_1 and shifting by 3, we obtain that the reduced complex cohomology of the link $L(\mathcal{D}_2^{sy})$, which is $\mathbb{R}P^3$, is nonzero only in degree 3, where it is $\mathbb{C}$.

Next $\mathcal{D}_2$ is defined by an equation of the form $xw - yz = 0$. Again it is a Morse singularity and the table with $m = 2$ gives for the complex cohomology of the Milnor fiber $\Lambda^*\mathbb{C}\langle e_3\rangle = \mathbb{C}\langle 1, e_3\rangle$. The complement has cohomology $\Lambda^*\mathbb{C}\langle e_1, e_3\rangle = \mathbb{C}\langle 1, e_1, e_3, e_1e_3\rangle$, which when truncated and shifted gives the reduced complex cohomology of the link $L(\mathcal{D}_2)$ as being $\mathbb{C}$ in degrees 2, 3 and 5 and zero otherwise. Similarly, $\mathcal{D}_4^{sy}$ is defined by the Pfaffian of the form $xw - yv + zu = 0$, which is again a Morse singularity and the Table with $m = 2$ gives as the complex cohomology of the Milnor fiber $\Lambda^*\mathbb{C}\langle e_5\rangle = \mathbb{C}\langle 1, e_5\rangle$; and the reduced complex cohomology of the link $L(\mathcal{D}_2^{sk})$ is $\mathbb{C}$ in degrees 4, 5 and 9 and zero otherwise. Because in the later two cases the link is a compact oriented manifold, we see that the one possible nonzero product $e_1^* \smile 1^*$ is indeed nontrivial.

Example 3.4. Beyond the cases in Example 3.3, the classical approach of representing the Milnor fiber as a bouquet of spheres no longer applies as the cohomology algebra would be trivial with no torsion. For example, for $\mathcal{D}_3^{sy}$, the Table gives the complex cohomology as $\Lambda^*\mathbb{C}\langle e_5\rangle = \mathbb{C}\langle 1, e_5\rangle$; however, we also have for $\mathbb{Z}_2$-cohomology, $\Lambda^*\mathbb{Z}_2\langle s_2, s_3\rangle = \mathbb{Z}_2\langle 1, s_2, s_3, s_2s_3\rangle$, which gives 2-torsion in degrees 2 and 3 and a non-trivial product structure. Likewise, other higher determinantal hypersurfaces have Milnor fibers not homotopy equivalent to a bouquet of spheres.

The cohomology of the complement $Sym_3 \backslash \mathcal{D}_3^{sy}$ is the exterior algebra $\Lambda^*\mathbb{C}\langle e_1, e_5\rangle = \mathbb{C}\langle 1, e_1, e_5, e_1e_5\rangle$. Also, $N - 2 = 6 - 2 = 4$, so the link $L(\mathcal{D}_3^{sy})$ has reduced complex cohomology $\mathbb{C}\langle 1^*, e_1^*, e_5^*\rangle$ with degrees $4, 5, 9$. where the product $e_1^* \smile 1^*$ still has to be determined.

Stable Homotopy Groups of the Milnor Fibers. Third, using the periodicity Theorems of Bott, we can also give the stable homotopy groups of the Milnor fibers up to the appropriate stable range. Again this is because we have already identified them as being homotopy equivalent to the classical symmetric spaces, so the results of Bott, as applied to these spaces, yield the calculations, with it only being necessary to observe where the stable range begins in each case. For the three cases, we the corresponding infinite dimensional symmetric spaces are given by:

$$\mathbb{SU} = \cup_{n=1}^{\infty} SU_n, \quad \mathbb{SU}/\mathbb{SO} = \cup_{n=1}^{\infty} SU_n/SO_n, \quad \text{and} \quad \mathbb{SU}/\mathbb{S}p = \cup_{n=1}^{\infty} SU_{2n}/Sp_n\,.$$

We also let the Milnor fibers of the determinantal hypersurfaces be denoted by: F_m, F_m^{sy}, and F_m^{sk}. Then, the homotopy groups of the Milnor fibers can be given up to the end of the stable range in terms of the stable homotopy groups of the infinite dimensional symmetric spaces as follows.

Theorem 3.5. *The homotopy groups of the Milnor fibers up to the end of the stable range are as follows.*

i)
$$\pi_j(F_m) \simeq \pi_j(SU_m) \simeq \pi_j(\mathbb{SU}) \qquad \textit{for } j < 2m$$

ii)
$$\pi_j(F_m^{sy}) \simeq \pi_j(SU_m/SO_m) \simeq \pi_j(\mathbb{SU}/\mathbb{SO}) \qquad \textit{for } j < m - 1$$

ii) *for* $m = 2k$

$$\pi_j(F_m^{sk}) \;\simeq\; \pi_j(SU_{2k}/Sp_k) \;\simeq\; \pi_j(\mathbb{SU}/\mathbb{S}p) \qquad \text{for } j < 4k-2$$

where the stable homotopy groups are given in Table 3.

$\pi_j(\mathbb{G}/\mathbb{H})$ $j =$	0	1	2	3	4	5	6	7	8	9
$\mathbb{SU}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\mathbb{SU}/\mathbb{SO}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\mathbb{SU}/\mathbb{S}p$	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

TABLE 3. The stable homotopy groups of the infinite dimensional symmetric spaces $\mathbb{G}/\mathbb{H}$. They are periodic of period dividing 8, with the periodicity beginning at $j = 2$.

We note that for the Milnor fibers F_m, $m \geq 6$, F_m^{sy}, $m \geq 11$, and F_m^{sk}, $m \geq 6$, exhibit all of the homotopy groups appearing in Table 3 in their stable ranges.

Proofs of the Theorems.

Proof of Theorem 3.1. We first observe that in all three cases the determinantal hypersurface is homogeneous defined by either $\det^{-1}(0)$ for $M = Sym_m$ or $M = M_{m,m}$ or $\mathrm{Pf}^{-1}(0)$ for $M = Sk_m$ (for $m = 2k$). By Lemma 1.2, we may consider the Milnor fibers of the global Milnor fibration.

The simplest case is for the Milnor fiber for the determinantal hypersurface $\mathcal{D}_m$ for $m \times m$ matrices $M_{m,m}$, which is just $F = \det^{-1}(1) = SL_m(\mathbb{C})$. It is homotopy equivalent to its maximal compact subgroup SU_m, and so, for example, by [MT, Chap. 3, Thm 6.5],its cohomology is a Hopf algebra given by the exterior algebra

$$H^*(SU_m; \mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}\langle e_3, \dots, e_{2m-1}\rangle\,.$$

where on the e_i are of degree i (which correspond by transgression homomorphisms to the Chern classes). Furthermore, by replacing $\mathbb{Z}$ by $\mathbf{k}$, a field of characteristic 0, we obtain the corresponding result for cohomology with coefficients in $\mathbf{k}$.

The second case $\mathcal{D}_m^{sy}$ is for the $m \times m$ symmetric case $M = Sym_m$. We claim the Milnor fiber $F = \det^{-1}(1)$, is diffeomorphic to $SL_m(\mathbb{C})/SO_m(\mathbb{C})$. First, the action of $SL_m(\mathbb{C})$ on F by $A \mapsto BAB^T$ is transitive. We know by the classification of symmetric matrices under the equivalence $A \mapsto BAB^T$, that if $\det(A) \neq 0$, then there is a $B \in GL_m(\mathbb{C})$ such that $BAB^T = I_m$, the $m \times m$ identity matrix. If $\det(B) = b \neq 0$ and $\det(A) = 1$, then $\det(B)\det(A)\det(B^T) = b^2 = 1 = \det(I_m)$. Hence, if we replace B by $B' = b^{-1}B$, then $B'AB'^T = b^{-2}\cdot I_m = I_m$, and $\det(B') = 1$. Thus, the orbit of I_m under $SL_m(\mathbb{C})$ is F.

Moreover, the isotropy subgroup of I_m under the action of $SL_m(\mathbb{C})$ is $\{B \in SL_m(\mathbb{C}) : B\, I_m\, B^T = I_m\}$, which is $SO_m(\mathbb{C})$. Hence, $F \simeq SL_m(\mathbb{C})/SO_m(\mathbb{C})$. Lastly, the groups $SL_m(\mathbb{C})$, resp. $SO_m(\mathbb{C})$, are homotopy equivalent to their maximal compact subgroups, which are SU_m, resp. $SO_m(\mathbb{R})$. Hence, by the argument given in Lemma 1.3, $SL_m(\mathbb{C})/SO_m(\mathbb{C})$ is homotopy equivalent to $SU_m/SO_m(\mathbb{R})$, which is one of the classical symmetric spaces (see [MT, Chap. 3, §6]). Thus, $H^*(SL_m(\mathbb{C})/SO_m(\mathbb{C})) \simeq H^*(SU_m/SO_m(\mathbb{R}))$. The calculation of $H^*(SU_m/SO_m(\mathbb{R}))$

is given in e.g. [MT, Chap. 3, Thm 6.7] depends on whether m is even or odd. If m is odd $= 2k+1$, then for a field $\mathbf{k}$ of characteristic 0, it is an exterior algebra

$$H^*(SU_m/SO_m(\mathbb{R}); \mathbf{k}) \; = \; \Lambda^*\mathbf{k}\langle e_5, e_9, ..., e_{2m-1}\rangle \, .$$

If m is even $= 2k$, then it is a free module of rank 2 over an exterior algebra

$$H^*(SU_m/SO_m(\mathbb{R}); \mathbf{k}) \; = \; \Lambda^*\mathbf{k}\langle e_5, e_9, ..., e_{2m-3}\rangle\{1, e_m\}$$

In the last case $e_m^2 \neq 0$, but instead equals an expression involving products of the odd degree generators..

The remaining case is for the $m \times m$ skew-symmetric case $M = Sk_m$ with $m = 2k$. In this case the global Milnor fiber $F = \mathrm{Pf}^{-1}(1)$ is diffeomorphic to $SL_m(\mathbb{C})/Sp_m(\mathbb{C})$.

First, the action of $SL_m(\mathbb{C})$ on F by $A \mapsto BAB^T$ is transitive. We know by the classification of skew-symmetric matrices under the equivalence $A \mapsto BAB^T$, that if $\det(A) \neq 0$, then there is a $B \in GL_m(\mathbb{C})$ such that $BAB^T = J_m$, where J_m is the 2×2 block diagonal matrix with 2×2 diagonal blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Again if $\det(B) = b \neq 0$ and $\mathrm{Pf}(A) = 1$, then $\mathrm{Pf}(BAB^T) = \det(B)\mathrm{Pf}(A) = b = 1 = \mathrm{Pf}(J_m)$. Hence, $B \in SL_m(\mathbb{C})$; thus, the orbit of J_m under $SL_m(\mathbb{C})$ is F. The isotropy group of J_m in $SL_m(\mathbb{C})$ is $\{B \in SL_m(\mathbb{C}) : BJ_mB^T = J_m\} = Sp_k(\mathbb{C})$. Thus, as before, $F \simeq SL_{2k}(\mathbb{C})/Sp_k(\mathbb{C})$.

Lastly, the groups $SL_{2k}(\mathbb{C})$, resp. $Sp_k(\mathbb{C})$, are homotopy equivalent to their maximal compact subgroups, which are $SU_{2k}(\mathbb{C})$, resp. Sp_k. We conclude $SL_{2k}(\mathbb{C})/Sp_k(\mathbb{C})$ is homotopy equivalent to $SU_{2k}(\mathbb{C})/Sp_k$, which is again one of the classical symmetric spaces of compact type (see [MT, Chap. 3, §6]).

Then, by [MT, Chap. 3, Thm 6.7]

$$H^*(SU_{2k}(\mathbb{C})/Sp_k; \mathbb{Z}) = \Lambda^*\mathbb{Z}\langle e_5, e_9, ..., e_{2m-3}\rangle$$

where again e_i has degree i and $m = 2k$. We obtain the corresponding result with $\mathbb{Z}$ replaced by $\mathbf{k}$.

For the symmetric case, we again apply [MT, Chap. 3, Thm 6.7] to conclude that the mod-2 cohomology is as claimed.

The only remaining claim is that the Milnor fibrations are cohomologically trivial except in the case of $m \times m$ symmetric matrices with m even. This will follow from the results in Theorem 3.2 together with Proposition 1.8. □

Proof of Theorem 3.2. The proof will follow in each case from Proposition 1.9.

In the case of $M_{m,m}$, the complement is $\mathrm{GL}_m(\mathbb{C})$, which has maximal compact subgroup U_m with cohomology a Hopf algebra given by the exterior algebra

$$H^*(U_m; \mathbb{Z}) \; \simeq \; \Lambda^*\mathbb{Z}\langle e_1, e_3, \dots, e_{2m-1}\rangle \, ,$$

and replacing $\mathbb{Z}$ by $\mathbf{k}$ we obtain the corresponding result for cohomology with $\mathbf{k}$ coefficients.

Second, in the skew-symmetric case $M = Sk_m$ with $m = 2k$, the isotropy subgroup of $\mathrm{GL}_m(\mathbb{C})$ for J_m is $\{B \in \mathrm{GL}_m(\mathbb{C}) : BJ_mB^T = J_m\} = Sp_m(\mathbb{C})$. They have maximal compact subgroups U_m, resp. $Sp_k(\mathbb{R})$. Second by [MT, Chap. 3, Thm 6.7],

$$H^*(U_{2k}(\mathbb{C})/Sp_k; \mathbb{Z}) = \Lambda^*\mathbb{Z}\langle e_1, e_5, e_9, ..., e_{2m-3}\rangle \, ,$$

and we can again replace $\mathbb{Z}$ by $\mathbf{k}$ to obtain the corresponding result for cohomology with coefficients in $\mathbf{k}$.

Third, the symmetric case is the most subtle. We will verify it for complex cohomology and then the result will follow for any field $\mathbf{k}$ of characteristic 0. For $M = Sym_m$, the isotropy subgroup of $\mathrm{GL}_m(\mathbb{C})$ for I_m is $\{B \in \mathrm{GL}_m(\mathbb{C}) : BI_mB^T = I_m\} = O_m(\mathbb{C})$. They have maximal compact subgroups U_m, resp. $O_m(\mathbb{R})$; and $O_m(\mathbb{R})$ has two connected components with $SO_m(\mathbb{R})$ being the connected component of the identity. Thus, first by Proposition 1.9, $H^*(M\backslash\mathcal{E};\mathbb{C}) \simeq H^*(U_m/O_m(\mathbb{R});\mathbb{C})$. Second by [MT, Chap. 3, Thm 6.7], we may compute the cohomology $H^*(U_m/SO_m(\mathbb{R});\mathbb{C})$, which depends on whether m is even or odd. If m is odd $= 2k+1$, then it is an exterior algebra

$$H^*(U_m/SO_m(\mathbb{R});\mathbb{C}) \;=\; \Lambda^*\mathbb{C}\langle e_1, e_5, e_9, ..., e_{2m-1}\rangle \,. \tag{3.1}$$

If m is even $= 2k$, then it is a free module of rank 2 over an exterior algebra

$$H^*(U_m/SO_m(\mathbb{R});\mathbb{C}) \;=\; \Lambda^*\mathbb{C}\langle e_1, e_5, e_9, ..., e_{2m-3}\rangle\{1, e_m\} \,. \tag{3.2}$$

To obtain the results for $H^*(U_m/O_m(\mathbb{R});\mathbb{C})$ from $H^*(U_m/SO_m(\mathbb{R});\mathbb{C})$ we use that $p : U_m/SO_m(\mathbb{R}) \to U_m/O_m(\mathbb{R})$ is a double covering obtained from $U_m/SO_m(\mathbb{R})$ as a quotient of the action of $O_m(\mathbb{R})/SO_m(\mathbb{R}) \simeq \mathbb{Z}_2$. This action is given by a covering transformation τ of $U_m/SO_m(\mathbb{R})$ defined as follows. We let C_m denote the diagonal matrix which equals 1 except for the last entry, which is -1. Then, $\tau(A \cdot SO_m(\mathbb{R})) = A \cdot C_m \cdot SO_m(\mathbb{R})$. This is well-defined as $SO_m(\mathbb{R})$ is normal in $O_m(\mathbb{R})$ so if $A' = A \cdot D$ with $D \in SO_m(\mathbb{R})$, then $A' \cdot C_m = A \cdot D \cdot C_m = A \cdot C_m \cdot D'$ with $D' = \cdot C_m^{-1} \cdot D \cdot C_m \in SO_m(\mathbb{R})$. Then, $U_m/O_m(\mathbb{R})$ is the resulting quotient.

We obtain by a standard result, see Lemma 4.2,

$$H^*(U_m/O_m(\mathbb{R});\mathbb{C}) \;\simeq\; H^*(U_m/SO_m(\mathbb{R});\mathbb{C})^{\mathbb{Z}_2} \,. \tag{3.3}$$

The subtle point of the calculation is to compute the RHS of (3.3). Then, the result for the symmetric case follows from the following Lemma.

Lemma 3.6. *The action of τ^* on $H^*(U_m/O_m(\mathbb{R});\mathbb{C})$ is given by $\tau^*(e_{4i+1}) = e_{4i+1}$ for all i, and if m is even, $\tau^*(e_m) = -e_m$*

It follows from the Lemma that in the odd dimensional case $m = 2k+1$ the entire cohomology is invariant under τ^*. By contrast, in the even dimensional case $m = 2k$, the exterior algebra generated by all of the odd degree generators e_{4i+1} is invariant under τ^*; however the elements in the module on e_m over the exterior algebra are all sent to their negatives. Thus, the invariant cohomology under τ^* is exactly the exterior algebra as stated. Before proving the Lemma, which gives the result for the symmetric case, we give the remaining argument for the link.

By Proposition 1.9 we obtain the cohomology as the truncated and shifted cohomology of the complement. If K/L is orientable then the degree is shifted by $2N - 2 - \dim_{\mathbb{R}} K/L$. With the sole exception of the $m \times m$ symmetric matrices with m even, for the determinantal hypersurfaces, K/L is orientable. For example, we can see this because the cohomology computed in Table 2 is nonzero in the degree $= \dim_{\mathbb{R}} K/L$. However, for $m \times m$ symmetric matrices with m even, this is not true, so K/L is not orientable so we will consider this case separately.

Then, for the other cases, the complement is a quotient of reductive groups G/H with $N = \dim_{\mathbb{C}}(G/H) = \dim_{\mathbb{R}} K/L$, so the shift is given by $N-2$. For the symmetric case with m even, the cohomology is still an exterior algebra, but top nonzero cohomology occurs in degree m below the top degree ($N = \dim_{\mathbb{R}} K/L$), if we follow the argument in Proposition 1.9, we see that the shift must be altered to $N - 2 + m$, giving the result as claimed.

The properties of the cohomology product follow by considering degrees. We have $\dim_{\mathbb{R}} L(\mathcal{D}) = 2N-3$, $H^{2N-4}(L(\mathcal{D}); \mathbb{C}) = 0$ (as $H^2(K/L; \mathbb{C}) = 0$) in each case. Excluding the symmetric case with m even, the lowest positive degree classes have degrees $N-2$ for 1^* and $N-1$ for e_1^*, then the lowest dimensional products in the reduced cohomology are given by: $1^* \cup 1^*$ of degree $2N-4$ and hence 0, and $e_1^* \cup 1^*$ of degree $2N-3$. All other products have higher degree and hence are 0. For the symmetric case with m even, the lowest degree term 1^* has degree $N-2+m$, so the lowest degree product will have degree $2(N-2+m) = 2N-3+(2m-1) > 2N-3$ so all products are zero in this case.

Lastly, we return to the question of the Milnor fibration being cohomologically trivial. Since $H^*(E; \mathbb{C}) \simeq H^*(M\backslash\mathcal{E}; \mathbb{C})$, the calculations of the cohomology given by Theorems 3.1 and 3.2 (and the universal coefficient theorem) imply in each case except for the symmetric case with m even, that (1.5) is satisfied, so by Proposition 1.8, the Milnor fibration is cohomologically trivial. In the symmetric case with $m = 2k$ even, from Table 1 we see the cohomology of the Milnor fiber is a free module on two generators over an exterior algebra on $k-1$ generators, and has total dimension 2^k; while from Table 2 the cohomology of the total space is an exterior algebra on k generators and has the same total dimension 2^k. Thus, (1.5) does not hold, so the Milnor fibration for these cases is not cohomologically trivial. □

Proof of Lemma 3.6. We let τ' denote the diffeomorphism of U_n defined by $\tau'(A) = A \cdot C_m$. Then, τ' descends in the quotient map $p : U_n \to U_n/O_n(\mathbb{R})$ to yield τ. Furthermore, by [MT, Chap. 3, §6, Thm 6.7], the map p^* in complex cohomology uniquely sends $p^*(e_{4j+1}) = \tilde{e}_{4j+1}$ in $H^*(U_n; \mathbb{C})$ for all j with e_{4j+1} nonzero in $H^*(U_n/O_n(\mathbb{R}); \mathbb{C})$. Here we use $\tilde{e}_{2j+1}$ to denote the exterior algebra generators of $H^*(U_n; \mathbb{C})$. Since U_n is connected, we can choose a path from C_m to Id, and this defines a homotopy between τ' and the identity map on U_n. Thus, ${\tau'}^*(\tilde{e}_{4j+1}) = \tilde{e}_{4j+1}$ for all j. Then,

$$p^*(\tau^*(e_{4j+1})) \;=\; {\tau'}^*(p^*(e_{4j+1})) \;=\; \tau' * (\tilde{e}_{4j+1}) \;=\; \tilde{e}_{4j+1} \;=\; p^*(e_{4j-3})\,.$$

By the uniqueness of $p^*(e_{4j+1})$, we conclude $\tau^*(e_{4j+1}) = e_{4j+1}$ as claimed.

It remains to consider in the case $m = 2k$, $\tau^*(e_m)$ which is the Euler class $e(E)$ of the m dimensional bundle E over $U_m/SO_m(\mathbb{R})$ defined from the standard representation of $SO_m(\mathbb{R})$ on $\mathbb{R}^m$. We consider the fibration $p' : U_m/T^k \to U_m/SO_m(\mathbb{R})$ where T^k is the k-torus $SO_2(\mathbb{R}) \times \cdots \times SO_2(\mathbb{R})$ (with k factors). Then ${p'}^* : H^*(U_m/SO_m(\mathbb{R}); \mathbb{C}) \to H^*(U_m/T^k; \mathbb{C})$ is injective. Also, the pull-back ${p'}^*(E)$ splits into oriented real 2-plane bundles (or using $SO_2(\mathbb{R}) \simeq U_1$, complex line bundles) $L_1 \oplus L_2 \oplus \cdots \oplus L_k$. Then $\tau^*(e(E)) = e(\tau^* E)$. Since C_m is in the normalizer of T^k, we can define a diffeomorphism τ'' of U_m/T^k by $\tau''(A \cdot T^k) = (A \cdot C_m \cdot T^k)$. This is seen to be well-define just as was τ'.

Then, ${p'}^*(\tau^*(e_m)) = {p'}^*(\tau^*(e(E))) = {\tau''}^*({p'}^*(e(E)))$. Also, by the splitting, ${p'}^*(e(E)) = \prod_{j=1}^k e(L_j)$. Thus, ${\tau''}^*({p'}^*(e(E))) = \prod_{j=1}^k {\tau''}^* e(L_j)$. Now, the effect of τ'' on the first $k-1$ factors of T^k is as the identity so ${\tau''}^* e(L_j) = e(L_j)$ for $j < k$. On the last factor τ'' acts by multiplication by the reflection matrix $C_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This changes the orientation of L_k and hence changes the sign of $e(L_k)$.

Thus, $\tau''^*e(L_k) = -e(L_k)$. Hence, $\tau''^*(p'^*(e(E))) = -e(E)$. Finally, from the above we conclude

$$p'^*(\tau^*(e(E))) \;=\; \tau''^*(p'^*(e(E))) \;=\; -p'^*(e(E))\,.$$

Since p'^* is injective, we finally conclude $\tau^*(e_m) = \tau^*(e(E)) = -e(E) = -e_m$, as claimed. □

Proof of Theorem 3.5. Lastly, for Theorem 3.5, we begin by noting that the groups in Table 3 are obtained from Bott periodicity, see [MT, Chap. 4, §6,Table 4.1]. Then, as we already know that the Milnor fibers are homotopy equivalent to the specific classical symmetric spaces, it is only necessary to see that the stable ranges are as stated. This follows by standard type arguments applying the homotopy long exact sequence to the fibrations $SU_n \hookrightarrow SU_{n+1} \to S^{2n+1}$, resp. $SO(n)(\mathbb{R}) \hookrightarrow SO_{n+1}(\mathbb{R}) \to S^n$, and $Sp_n \hookrightarrow Sp_{n+1} \to S^{4n-1}$, together with $SO_n(\mathbb{R}) \hookrightarrow SU_n \to SU_n/SO_n(\mathbb{R})$, and $Sp_n \hookrightarrow SU_{2n} \to SU_{2n}/Sp_n(\mathbb{R})$ to obtain

i)
$$\pi_j(SU_n) \;\simeq\; \pi_j(SU_{n+1}) \qquad \text{for} \quad j \leq 2n-1\,;$$

ii)
$$\pi_j(SU_n/SO_n(\mathbb{R})) \;\simeq\; \pi_j(SU_{n+1}/SO_{n+1}(\mathbb{R})) \qquad \text{for} \quad j \leq n-1\,; and$$

iii)
$$\pi_j(SU_{2n}/Sp_n) \;\simeq\; \pi_j(SU_{2(n+1)}/Sp_{n+1}) \qquad \text{for} \quad j \leq 4n-2\,.$$

This gives the stable range. □

4. Cohomology of the Complement and Link for the Equidimensional Case

We next return to the class of special prehomogeneous spaces defined by equidimensional representations $\rho : G \to GL(V)$ of a connected linear algebraic group. In this section, we compute the topology of the complement and link of the exceptional orbit variety $\mathcal{E}$.

Topology of the Complement.

Theorem 4.1. *Consider a prehomogeneous space defined by an equidimensional representation of a connected linear algebraic group $\rho : G \to \mathrm{GL}(V)$ with exceptional orbit variety $\mathcal{E}$ and maximal compact subgroup K. Then, for a field $\mathbf{k}$ of characteristic* 0*,*

$$H^*(V\backslash\mathcal{E};\mathbf{k}) \;\simeq\; H^*(K;\mathbf{k}) \;=\; \Lambda^*\mathbf{k}\langle s_1, s_2, \ldots, s_k\rangle\ . \tag{4.1}$$

where s_j are classes of odd degree q_j.

In addition, $\pi_i(V\backslash\mathcal{E}) \simeq \pi_i(K)$ for $i > 1$; and there is a short exact sequence

$$1 \longrightarrow \pi_1(K) \longrightarrow \pi_1(V\backslash\mathcal{E}) \longrightarrow H \longrightarrow 1 \tag{4.2}$$

where H is the isotropy subgroup of G at a point $v_0 \in V\backslash\mathcal{E}$.

Proof. First, for (4.1) we may again apply the Hopf structure theorem for a compact connected Lie group K and a field $\mathbf{k}$ of characteristic 0, $H^*(K;\mathbf{k})$, to conclude it is a Hopf algebra which is isomorphic to an exterior algebra on classes of odd degree. This gives the structure for $H^*(K;\mathbf{k})$ in (4.1)

As K is homotopy equivalent to G we conclude that $H^*(G;\mathbf{k})$ is also given by the RHS of (4.1). However, let $v_0 \in \mathcal{U}$, the open orbit of G, and let H be the isotropy subgroup of v_0, which is finite as $\dim_{\mathbb{C}} G = \dim_{\mathbb{C}} V$. Then, $G/H \simeq \mathcal{U} = V\backslash\mathcal{E}$. Thus, we may replace $V\backslash\mathcal{E}$ by G/H. Lastly, it is sufficient to show that

$$H^*(G/H;\mathbf{k}) \;=\; H^*(G;\mathbf{k})\ . \tag{4.3}$$

This follows in two steps. First, there is the standard Lemma (see e.g. [Bn, Chap III, Thm2.4]).

Lemma 4.2. *If the finite group H acts freely on a manifold M, then for a field $\mathbf{k}$ of characteristic 0 or relatively prime to $|H|$,*

$$H^*(M;\mathbf{k})^H \;=\; H^*(M/H;\mathbf{k})$$

where $H^*(M;\mathbf{k})^H$ denotes the subspace invariant under the induced action of H on cohomology

Hence,

$$H^*(G;\mathbf{k})^H \;=\; H^*(G/H;\mathbf{k}) \tag{4.4}$$

Second, we have the averaging map

$$\begin{aligned} avg_H : H^*(G;\mathbf{k}) &\to H^*(G;\mathbf{k})^H \\ [\tau] &\mapsto \frac{1}{|H|}\sum_{\sigma\in H}\sigma^*([\tau]) \end{aligned} \tag{4.5}$$

which is an isomorphism. This follows since as G is connected, for any cocycle τ on G, the cohomology classes $[\sigma^*(\tau)] = [\tau]$ for all $\sigma \in G$; so $[avg_H(\tau)] = [\tau]$.

For the second part, we use the long exact sequence in homotopy for the fibration $H \hookrightarrow G \to G/H$ (just as in [DP, §3]) to obtain the desired isomorphisms and the exact sequence. □

We state an immediate consequence for the case of an equidimensional representation $\rho : G \to \mathrm{GL}(V)$ of a connected solvable linear algebraic group with an open orbit. As G has a maximal compact subgroup T^k, where $k = \mathrm{rank}\,(G)$, we obtain from Theorem 4.1.

Corollary 4.3. *Suppose $\rho : G \to \mathrm{GL}(V)$ is an equidimensional representation of a connected solvable linear algebraic group which defines a prehomogeneous space with exceptional orbit variety $\mathcal{E}$. If $\mathrm{rank}\,(G) = k$, then $V\backslash\mathcal{E}$ is a $K(\pi,1)$ with π a finite extension of $\mathbb{Z}^k$ by the finite isotropy group H of a point in $\mathcal{U}$; and*

$$H^*(V\backslash\mathcal{E};\mathbf{k}) \;=\; \Lambda^*\mathbf{k}\langle s_1, s_2, \dots, s_k\rangle\ .$$

where each s_j is of degree one.

Remark 4.4. Since an equidimensional representation of a connected solvable linear algebraic group with open orbit $\mathcal{U}$ is a (possibly nonreduced) block representation, the first part of Corollary 4.3 was given in [DP]. The second part extends for cohomology the results in [DP] which for the special cases of representations corresponding to (modified) Cholesky-type factorizations furthermore showed that the complement and Milnor fibers were homotopy tori.

Topology of the Link. We can immediately deduce the cohomology of the link of the exceptional orbit variety $\mathcal{E}$. For an exterior algebra $A = \Lambda^*\mathbf{k}\langle s_1, s_2, \dots, s_k\rangle$, with generators s_i of odd degrees q_i,we let

$$\widetilde{\Lambda^*\mathbf{k}}\langle s_1, s_2, \dots, s_k\rangle\,[r]$$

denote the algebra obtained from A by removing the top degree term, and shifting degrees upward by degree r. Then, there is the following result for the link $L(\mathcal{E})$ of $\mathcal{E}$.

Theorem 4.5. *Let $\rho : G \to \mathrm{GL}(V)$ be an equidimensional representation of a linear algebraic group defining a prehomogeneous space with exceptional orbit variety $\mathcal{E}$, and a maximal compact subgroup K. Then, for a field $\mathbf{k}$ of characteristic 0, there is an isomorphism of graded vector spaces*

$$\widetilde{H}_*(L(\mathcal{E}); \mathbf{k}) \;\simeq\; \widetilde{H}^*(L(\mathcal{E}); \mathbf{k}) \;\simeq\; \widetilde{\Lambda^*\mathbf{k}}\langle s_1, s_2, \dots, s_k\rangle\,[2N - 2 - \dim_{\mathbb{R}} K]$$

where $N = \dim_{\mathbb{C}} V = \dim_{\mathbb{C}} G$ and $k = \mathrm{rank}\,(G)$.

Proof. This result follows from Proposition 1.9 using Theorem 4.1. □

Corollary 4.6. *In the situation of Theorem 4.5,*

i) *If G is solvable and $\not\simeq (\mathbb{C}^*)^N$, then the degree shift $> N-2$; hence products in the reduced cohomology $\widetilde{H}^*(L(\mathcal{D}); \mathbf{k})$ are 0.*
ii) *When G is reductive, the degree shift equals $N-2$, so products in the reduced cohomology $\widetilde{H}^*(L(\mathcal{D}); \mathbb{C})$ are 0 except possibly for the single product $e_1^* \cup 1^*$, where the e_1^* and 1^* are the images in degrees $N-1$, resp. $N-2$, of $e_1, 1 \in H^*(M\backslash\mathcal{D}; \mathbf{k})$.*

Proof of Corollary 4.6. If G is solvable and $\not\simeq (\mathbb{C}^*)^N$ then $\dim_{\mathbb{R}} K < \dim_{\mathbb{C}} G = N$; hence, $2N - 2 - \dim_{\mathbb{R}} K > N - 2$. Thus the product of two classes in $\widetilde{H}^*(L(\mathcal{D}); \mathbf{k})$ have degree $\geq 2N-2$ and hence is 0.

For the equidimensional case when G is reductive, $\dim_{\mathbb{C}} G = \dim_{\mathbb{R}} K$ and $H = L$ is finite, so $N = \dim_{\mathbb{C}} G = \dim_{\mathbb{R}} K/L$. Hence, the degree shift is simply $N-2$.

Then, the argument follows as for the analogous property in Theorem 3.2. □

As a second corollary we consider the number of irreducible components of $\mathcal{E}$. Each component W_i such will contribute a generator from the fundamental class $\left[W_i \cap S^{2N-1}\right]$ to $H_{2N-3}(L(\mathcal{E}); \mathbf{k})$. Applying Theorem 4.5 we obtain as a corollary

Corollary 4.7. *For a representation $\rho : G \to \mathrm{GL}(V)$ as in Theorem 4.5, the number of irreducible components of $\mathcal{E}$ equals $\dim H_1(K; \mathbf{k})$. In particular, when G is solvable, the number equals $\mathrm{rank}\,(G)$.*

Proof. The generators of

$$H_{2N-3}(L(\mathcal{E}); \mathbf{k}) \;\simeq\; H^{2N-3}(L(\mathcal{E}); \mathbf{k})\,.$$

are given by the fundamental classes $\left[W_i \cap S^{2N-1}\right]$ for the components W_i of $\mathcal{E}$. Thus, by Theorem 4.5, the number of components equals $\dim_{\mathbf{k}} H_1(K; \mathbf{k})$.

If G is solvable of rank k, then $K = T^k$ so $\dim_{\mathbf{k}} H_1(T^k; \mathbf{k}) = k$ □

Example 4.8. We compare the topology of the links for the exceptional orbit hypersurfaces in $M_{2,2}$ defined for 2×2 matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$: $\mathcal{D}_2$, the determinantal hypersurface of singular matrices (defined by $xw - yz$), for the representation of left multiplication by $\mathrm{GL}_2(\mathbb{C})$; the linear free* divisor $\mathcal{E}_2'$ (defined by $x(xw-yz)$) for the representation for Cholesky factorization using the solvable group $B_2 \times N_2^T$ (with B_2 the Borel subgroup of lower triangular matrices and N_2^T the upper triangular unipotent matrices); and the linear free divisor $\mathcal{E}_2$ (defined by $xy(xw-yz)$) for the representation for modified Cholesky factorization using the solvable group $B_2 \times C_2^T$ (with B_2 the Borel subgroup of lower triangular matrices and $C_2^T \simeq \mathbb{C}^*$ the invertible diagonal matrices with top entry 1) (also see [DP2] for the second and third). Table 4 gives the maximal compact subgroups K, compact model for the Milnor fiber, and the reduced cohomology of the links in the nonvanishing dimensions. These exhibit the increased complexity of the cohomology of the link, and changes in the topology of the Milnor fiber resulting from successively adding two hyperplanes to obtain the linear free divisor $\mathcal{E}_2$.

$\mathcal{E}$	h	K	F model	$\widetilde{H}^J(L(\mathcal{E}), \mathbf{k})$ $j=2$	3	4	5
$\mathcal{D}_2$	$xw-yz$	U_2	$SU_2 \simeq S^3$	$\mathbf{k}$	$\mathbf{k}$	0	$\mathbf{k}$
$\mathcal{E}_2'$	$x(xw-yz)$	T^2	S^1	0	0	$\mathbf{k}$	$\mathbf{k}^2$
$\mathcal{E}_2$	$xy(xw-yz)$	T^3	T^2	0	$\mathbf{k}$	$\mathbf{k}^3$	$\mathbf{k}^3$

TABLE 4. Three exceptional orbit hypersurfaces arising from equidimensional representations in Example 4.8, with defining equation $h = 0$, together with the maximal compact subgroup K of G, the compact model for the global Milnor fiber F, and the nonzero reduced cohomology groups $\widetilde{H}^J(L(\mathcal{E}), \mathbf{k})$.

5. Topology of the Milnor Fiber for Equidimensional Representations

We next apply the preceding results on the cohomology of the complement together with properties of the the Wang sequence in §1 and the results for equidimensional representations in §2 to compute the topology of the Milnor fiber of $\mathcal{E}$. We use the notation of §§1 and 2.

Theorem 5.1. *Consider a prehomogeneous space defined by an equidimensional representation $\rho : G \to \mathrm{GL}(V)$ of a connected linear algebraic group G of rank k with maximal compact subgroup K and exceptional orbit variety $\mathcal{E}$. Then, the global Milnor fibration of $\mathcal{E}$ is cohomologically trivial. Furthermore, for a field $\mathbf{k}$ of characteristic* 0,

$$H^*(F; \mathbf{k}) \;\simeq\; \Lambda^*\mathbf{k} < e_2, \dots, e_k > \;.$$

Here $e_j = i^*(s_j)$, where $i : F \hookrightarrow V \backslash \mathcal{E}$ is the inclusion and

$$H^*(V\backslash\mathcal{E}; \mathbf{k}) \;\simeq\; H^*(K; \mathbf{k}) \;\simeq\; \Lambda^*\mathbf{k} < s_1, s_2, \dots, s_k >$$

with $\deg s_1 = 1$.

Moreover, the homotopy groups of F are given by $\pi_j(F) \simeq \pi_j(G)$ for $j \geq 2$; and there is the exact sequence

$$(5.1) \qquad 1 \longrightarrow \pi_1(G_0') \longrightarrow \pi_1(F) \longrightarrow H \longrightarrow 1$$

where H is the isotropy group of G'_0 for a point in F and $\pi_1(G'_0)$ is in the exact sequence

$$1 \longrightarrow \pi_1(G'_0) \longrightarrow \pi_1(G) \longrightarrow \mathbb{Z} \longrightarrow 1 \tag{5.2}$$

Proof. We can apply apply Proposition 1.9 and Theorem 4.1, to compute for the global Milnor fibration, the cohomology

$$H^*(E;\mathbf{k}) \;\simeq\; H^*(V\backslash\mathcal{E};\mathbf{k}) \;\simeq\; H^*(K;\mathbf{k})\,, \tag{5.3}$$

where

$$H^*(K;\mathbf{k}) \;\simeq\; \Lambda^*\mathbf{k}\langle s_1,\ldots,s_k\rangle \tag{5.4}$$

with $k = \operatorname{rank} K$.

Second, by Lemma 2.3, G'_0 is a finite covering space of F, so by Lemma 4.2 and then Lemma 2.2,

$$H^*(F;\mathbf{k}) \;\simeq\; H^*(G'_0;\mathbf{k}) \;\simeq\; H^*(K'_0;\mathbf{k}) \tag{5.5}$$

Finally by Lemma 2.5, since $K'_0 \hookrightarrow K \to S^1$ is cohomologically trivial, we have by Proposition 1.8 that $\dim_{\mathbf{k}} H^*(K;\mathbf{k}) = 2\dim_{\mathbf{k}} H^*(K'_0;\mathbf{k})$. Combining the preceding we obtain

$$\dim_{\mathbf{k}} H^*(E;\mathbf{k}) \;=\; 2\dim_{\mathbf{k}} H^*(F;\mathbf{k})\,. \tag{5.6}$$

Thus, by Proposition 1.8 the global Milnor fibration is cohomologically trivial. Also, by the discussion for (1.3) in §1, there is a nontrivial class $s_1 \in H^1(E;\mathbf{k})$, and we may choose s_1 to be one of the generators of the exterior algebra. Thus, combining (5.3) and (5.4) we have an isomorphism of algebras

$$H^*(E;\mathbf{k}) \;\simeq\; \Lambda^*\mathbf{k}\langle s_2,\ldots,s_k\rangle \;\oplus\; (\langle s_1\rangle \cdot \Lambda^*\mathbf{k}\langle s_2,\ldots,s_k\rangle)\;. \tag{5.7}$$

where the product in the second summand is cup prouct.

As the cohomology of the global Milnor fibration is cohomologically trivial, the second summand on the RHS of (5.7) maps by $i^* : H^*(E;\mathbf{k}) \to H^*(F;\mathbf{k})$ to be 0. It follows by consideration of dimensions and Proposition 1.8 that the first summand maps by i^* isomorphically to $H^*(F;\mathbb{C})$. This gives the desired conclusion.

For the homotopy groups $\pi_j(F)$, we use the finite covering space $G'_0 \to F$ with covering group H'. The exact sequence (5.1) follows from the basic relation between fundamental groups for a regular covering, and the higher homotopy groups follow from e.g. the long exact homotopy sequence for a fibration. Lastly the exact sequence (5.2) follows from the long exact homotopy sequence of the fibration $G'_0 \hookrightarrow G \to \mathbb{C}^*$ since both G'_0 and G are connected. □

Linear Free and Free* Divisors for Solvable Linear Algebraic Groups. We consider the case of an equidimensional representation $\rho : G \to \mathrm{GL}(V)$ defining a prehomogeneous space when G is solvable, with exceptional orbit variety $\mathcal{E}$. As G has a maximal compact subgroup T^k, where $k = \operatorname{rank}(G)$, we already know by Theorems 4.5 and 5.1 that the cohomology of both the complement $V\backslash\mathcal{E}$ and Milnor fiber F are an exterior algebras on k, resp. $k-1$ generators, all of which are of degree 1.

As in Theorem 4.1 of [DP], we can explicitly construct the generators from the basic relative invariants p_i of the representation. ρ. By Corollary 4.7, there are $k = \operatorname{rank} G$ irreducible components W_i of $\mathcal{E}$. By [SK], the homogeneous defining equations p_i for W_i are *basic relative invariants* and are independent. We let

$\omega_i = \frac{dp_i}{p_i} = p_i^*(\frac{dz}{z})$, which provide k closed one-forms on $\mathcal{U} = V\backslash\mathcal{E}$. The reduced homogeneous defining equation for $\mathcal{E}$ is given by $f = \prod_{i=1}^{k} p_i$. Then, $\tilde{\omega} = \frac{df}{f}$ defines a cohomology class in $H^1(V\backslash\mathcal{E};\mathbb{C})$ whose restriction $\tilde{\omega}|F = 0$ as $df|F = 0$ (since $f \equiv 1$ on F).

We deduce an extension of the result of Damon-Pike [DP].

Theorem 5.2. *Let $\rho : G \to \mathrm{GL}(V)$ be an equidimensional representation of a solvable linear algebraic group G of rank k, defining a prehomogeneous space with exceptional orbit variety $\mathcal{E}$. Then,*

i) *$H^1(V\backslash\mathcal{E},\mathbb{C})$ is the exterior algebra on the set of generators ω_i for $i = 1,\dots k$.*
ii) *$H^1(F,\mathbb{C})$ is generated by the $\{\omega_i, i = 1,\dots,k\}$ with a single relation $\sum_{i=1}^{k}\omega_i = 0$. Hence, $H^*(F,\mathbb{C})$ is the exterior algebra on any subset of $k-1$ of the ω_i.*

Proof. We know by Theorem 4.1 that $H^*(V\backslash\mathcal{E};\mathbb{C})$ is an exterior algebra on k generators of degree 1. Then, we can apply same the methods in [DP, §4] to show that the k closed 1-forms ω_i defined from the basic relative invariants p_i are a set of generators for $H^1(V\backslash\mathcal{E};\mathbb{C})$; and hence, are exterior algebra generators for $H^*(V\backslash\mathcal{E};\mathbb{C})$. By Theorem 5.1, the complex cohomology of the global Milnor fiber is an exterior algebra on the pull-backs of $k-1$ of the generators $\{\omega_i\}$ for $H^1(V\backslash\mathcal{E};\mathbb{C})$. Since the pullback of $\sum_{i=1}^{k}\omega_i = \tilde{\omega} = 0$, and $k-1$ of the generators will suffice.

□

Equidimensional Representations of Reductive Linear Algebraic Groups. In the case where G is a connected reductive linear algebraic group with maximal compact subgroup K, and $\rho : G \to \mathrm{GL}(V)$ defines a prehomogeneous space, then the results apply even if the exceptional orbit variety $\mathcal{E}$ is not a linear free divisor. However, we give one consequence for an important class of such representations for quivers of finite type considered by Buchweitz-Mond which do give linear free divisors.

Quivers of Finite Representation Type. A quiver is a connected finite directed graph Γ with edges $e(\Gamma) = \{\ell_j\}$, vertices $v(\Gamma) = \{v_i\}$, where we denote the initial vertex for ℓ_j by $i(\ell_j)$ and end point by $e(\ell_j)$. To define a *representation of the quiver* Γ, we associate to each vertex v_i a finite dimensional complex vector space V_i of dimension d_i and for each edge ℓ_j a linear transformation $\varphi_j : V_{i(\ell_j)} \to V_{e(\ell_j)}$. Then, the tuple $\{\varphi_j\}_{\ell_j\in e(\Gamma)}$ of linear transformations $\varphi_j : V_{i(\ell_j)} \to V_{e(\ell_j)}$ is a quiver representation. With an ordering on the vertices $\{v_i \in v(\Gamma)\}$, we let $\mathbf{d} = (d_i)$ denote the *dimension vector* for the quiver representation.

Together a set of such transformations $\{\varphi_j\}$ forms a quiver representation space $V \simeq \prod_{\ell_j\in\mathcal{L}} \mathrm{Hom}(V_{i(\ell_j)}, V_{e(\ell_j)})$. The group $\tilde{G} = \prod_{v_i\in v(\Gamma)} \mathrm{GL}(V_i)$ acts on V by

$$\{\psi_i\}\cdot\{\varphi_j\} \;=\; \{\psi_{e(\ell_j)}\circ\varphi_j\circ\psi_{i(\ell_j)}^{-1}\} \qquad \textit{for} \quad \{\psi_i\}\in\tilde{G},\ \{\varphi_j\}\in V$$

The group $\mathbb{C}^*$ embeds in each $\mathrm{GL}(V_i)$ as the standard $\mathbb{C}^*$-action. The diagonal embedding of $\mathbb{C}^*$ in $\tilde{G}$ defines a subgroup of $\tilde{G}$ which acts trivially on V. We let $G = \tilde{G}/\mathbb{C}^*$. Then, the quiver is said to be of *finite representation type* if G has only a finite number of isomorphism classes of *indecomposable quiver representations.*

The classification of quivers of finite representation type was done by Gabriel [G, G2] and they correspond to the Dynkin diagrams of type A, D, or E with the indecomposable quiver representations for appropriate dimension vectors $\mathbf{d}$ which are the positive Schur roots corresponding to the Dynkin diagram. The quiver arrows can go in either direction. Then, it is a fact that for these dimension vectors the representation of G on V is an equidimensional representation with an open orbit, which hence defines a prehomogeneous space whose exceptional orbit variety, denoted $\mathcal{D}_{(\Gamma,\mathbf{d})}$, is called the *discriminant of the quiver*. Buchweitz-Mond prove in [BM] that for quivers of finite representation type the discriminant $\mathcal{D}_{(\Gamma,\mathbf{d})}$ is a linear free divisor.

As a result of Theorems 5.1 and 4.5, we can compute the cohomology in characteristic 0 of both the link and Milnor fiber of the quiver discriminant.

To do so we need a simple lemma.

Let G be a connected linear algebraic group with $j : \mathbb{C}^* \hookrightarrow G$ a subgroup, such that for $S^1 \subset \mathbb{C}^*$, $j' : S^1 \subset K$ for K a maximal compact subgroup of G, of rank k. By Hopf's structure theorem,

$$H^*(K;\mathbf{k}) \;\simeq\; \Lambda^*\mathbf{k} < s_1, \dots, s_k > \; . \tag{5.8}$$

with each s_i of odd degree q_i and s_1 of degree 1.

Lemma 5.3. *In the preceding situation, suppose that ${j'}^*(s_1)$ generates $H^1(S^1;\mathbf{k})$. Then,*

$$H^*(G/\mathbb{C}^*;\mathbf{k}) \;\simeq\; \Lambda^*\mathbf{k} < s_2, \dots, s_k > \; . \tag{5.9}$$

Proof. By the same argument given in Lemma 1.3, $K/S^1 \hookrightarrow G/\mathbb{C}^*$ is a homotopy equivalence. Hence it is sufficient to prove the result for K/S^1. Then, $\{1, s_1\}$ restrict to a basis for $H^*(S^1;\mathbf{k})$. Hence, the Leray-Hirsch theorem applied to the fibration $S^1 \hookrightarrow K \xrightarrow{p} K/S^1$ yields that $H^*(K;\mathbf{k})$ is a free $H^*(K/S^1;\mathbf{k})$-module on $\{1, s_1\}$, where the module structure is via p^* and p^* is injective. Also by (5.8), $H^*(K;\mathbf{k})$ is a free $\Lambda^*\mathbf{k} < s_2, \dots, s_k >$-module on $\{1, s_1\}$. Thus,

$$H^*(K;\mathbf{k})/(s_1 \smile H^*(K;\mathbf{k})) \;\simeq\; \Lambda^*\mathbf{k} < s_2, \dots, s_k > ;$$

and also by the Leray-Hirsch theorem,

$$H^*(K;\mathbf{k})/(s_1 \smile H^*(K;\mathbf{k})) \;\simeq\; p^*H^*(K/S^1;\mathbf{k}) \cdot \{1\} \, .$$

Thus, if we compose p^* with projection onto the quotient by cup product with s_1. We obtain an isomorphism

$$H^*(K/S^1;\mathbf{k}) \;\simeq\; \Lambda^*\mathbf{k} < s_2, \dots, s_k > \; .$$

□

Now in the case of quivers for a graph Γ and quiver representation space V with dimension vector $\mathbf{d}$, we have the subgroup $\mathbb{C}^* \hookrightarrow \tilde{G} = \prod \mathrm{GL}(V_i)$. The maximal compact subgroup of $\mathrm{GL}(V_i) \simeq \mathrm{GL}_{d_i}(\mathbb{C})$ is the unitary group U_{d_i}, which has cohomology

$$H^*(U_{d_i};\mathbf{k}) \;\simeq\; \Lambda^*\mathbf{k}\langle s_1^{(i)}, \dots, s_{d_i}^{(i)} \rangle \, .$$

with $s_j^{(i)}$ of degree $2j-1$. Then, $\tilde{K} = \prod_{v_i \in v(\Gamma)} \in U_{d_i}$ is the maximal compact subgroup of $\tilde{G}$ and its cohomology is given by

$$H^*(\tilde{K};\mathbf{k}) \;\simeq\; \otimes_{v_i \in v(\Gamma)} \Lambda^*\mathbf{k}\langle s_1^{(i)}, \dots, s_{d_i}^{(i)} \rangle \tag{5.10}$$

which is again an exterior algebra. We denote the RHS of (5.10) by $\Lambda^*(\Gamma, \mathbf{d})$. We let $K = \tilde{K}/S^1$ which is the maximal compact subgroup of G.

To compute the cohomology of K we use Lemma 5.3. First, the degree 1 generator $s_1^{(i)}$ arises at the pull-back of the generator of $H^1(S^1; \mathbf{k})$ via $\det : U_{d_i} \to S^1$. The composition of $S^1 \hookrightarrow U_{d_i}$ with the determinant map sends $z \to z^{d_i}$, which is a covering map but still induces an isomorphism on cohomology with coefficients in $\mathbf{k}$. Thus, the pull-back of $s_1^{(i)}$ to $H^1(S^1; \mathbf{k})$ via the inclusion is a generator.

Hence, the pull-back of any degree 1 generator $\sum_{v_i \in v(\Gamma)} a_i s_1^{(i)}$ not belonging to a certain codimension one subspace of $H^*(U_{d_i}; \mathbf{k})$ will generate $H^1(S^1; \mathbf{k})$. We choose such a generator s_1, which can be chosen to be one of the generators of the exterior algebra. Then, we have by Lemma 5.3

$$H^*(K; \mathbf{k}) \;\simeq\; \Lambda^*(\Gamma, \mathbf{d})/(s_1 \cdot \Lambda^*(\Gamma, \mathbf{d})) \tag{5.11}$$

which is an exterior algebra obtained by removing a generator of degree 1 from (5.10).

Second, by the discussion for (1.3) in §1 and Lemma 2.5 in §2, there is a nonzero class $s_2 \in H^1(K; \mathbf{k})$ obtained as the pull-back of the generator of $H^1(S^1; \mathbf{k})$ via the composition of the map $\chi' : K \to S^1$ with the projection $\tilde{K} \to \tilde{K}/S^1 = K$. This will be different from s_1. Then, we have the following structure theorem for the cohomology of the Milnor fiber and link of the quiver discriminant.

Theorem 5.4. *For the quiver representation space V for a quiver Γ of finite type with dimension vector $\mathbf{d}$, having an open orbit of indecomposable quiver representations, let $F_{(\Gamma, \mathbf{d})}$ denote the Milnor fiber of the discriminant $\mathcal{D}_{(\Gamma, \mathbf{d})}$, and $L(\mathcal{D}_{(\Gamma, \mathbf{d})})$ the link. Then,*

$$H^*(F_{(\Gamma, \mathbf{d})}; \mathbf{k}) \;\simeq\; \Lambda^*(\Gamma, \mathbf{d})/(s_1, s_2) \cdot \Lambda^*(\Gamma, \mathbf{d}) \tag{5.12}$$

which is the exterior algebra on the generators of (5.10) but with two degree 1 generators removed. Also,

$$\widetilde{H}^*(L(\mathcal{D}_{(\Gamma, \mathbf{d})}); \mathbf{k}) \;\simeq\; \widetilde{\Lambda^*(\Gamma, \mathbf{d})}/(s_1 \cdot \widetilde{\Lambda^*(\Gamma, \mathbf{d})})[\dim_{\mathbb{C}} V - 2] \tag{5.13}$$

which is the exterior algebra on the generators of (5.10) with one degree 1 generator removed, then truncated in the top degree, and then shifted by degree $\dim_{\mathbb{C}} V - 2$.

Remark 5.5. For an odd integer $k > 1$, the number of generators in degree k of the exterior algebras in (5.12) and (5.13) is given by $|\{v_j \in v(\Gamma) : d_j \geq \frac{k+1}{2}\}|$. While the number of degree 1 generators is either 2, resp. 1, less than $|v(\Gamma)|$.

Proof. This follows from from the preceding discussion, Lemma 5.3, together with Theorems 5.1 and 4.5. □

Example 5.6. We consider the quiver representation corresponding to D_4 which has 4 vertices with a central one v_1 connected to the other three by edges, with the direction toward the central vertex. The dimension vector $\mathbf{d}$ has dimensions $\dim_{\mathbb{C}} V_1 = 2$, and for the other vertices $\dim_{\mathbb{C}} V_i = 1$. By Mond and Buchweitz, we can view the linear transformations as determined by vectors $(x_i, y_i) \in \mathbb{C}^2$, for $i = 1, 2, 3$. The quiver discriminant consists of the triple of vectors for which at least two of them lie in a common line. This is defined by the equation $(x_1y_2 - x_2y_1)(x_1y_3 - x_3y_1)(x_2y_3 - x_3y_2) = 0$ in $\mathbb{C}^6$. The group $G = \left(\mathrm{GL}(\mathbb{C}^2) \times (\mathbb{C}^*)^3\right)/\mathbb{C}^*$ has maximal compact subgroup $K = (U_2 \times T^3)/S^1$. Then, by Theorem 5.4 we have the cohomology of the Milnor fiber

$$H^*(F_{(D_4, \mathbf{d})}; \mathbf{k}) \;\simeq\; \Lambda^*\mathbf{k}\langle s_1, s_2, s_3\rangle \tag{5.14}$$

and

$$\widetilde{H}^*(L(\mathcal{D}_{(D_4,\mathbf{d})});\mathbf{k}) \simeq \widetilde{\Lambda^*\mathbf{k}}\langle s_1, s_2, s_3, s_4\rangle[4] \tag{5.15}$$

where we let s_1 have degree 3 and the other s_i have degree 1. Thus, $H^j(F_{D_4};\mathbf{k})$ is nonzero in degrees $0 \leq j \leq 5$ and has dimensions $1, 2, 1, 1, 2, 1$; while $\widetilde{H}^j(L(\mathcal{D}_{(D_4,\mathbf{d})});\mathbf{k})$ is nonzero in degrees $4 \leq j \leq 9$ and has dimensions $1, 3, 3, 2, 3, 3$. The top degree 9 of dimension 3 corresponds to the 3 irreducible components of $\mathcal{D}_{(D_4,\mathbf{d})}$.

Remark 5.7. Mond and Buchweitz have examined the effect of reversing various quiver arrows in a quiver representation of finite type. David Mond has indicated they have found that for certain quiver representations, the discriminant changes when the directions of certain quiver arrows are changed. However, it follows from the results obtained here that the rational cohomology of the Milnor fiber, the complement, and the cohomology of the link (as a graded vector space) will not change. A natural question is which topological invariants will detect this change.

6. Topology of Formal Sums of Exceptional Orbit Hypersurfaces

We conclude by combining the preceding results with a result of Mutsuo Oka [Ok], together with the results from Siersma et al to exhibit a large collection of highly nonisolated hypersurface singularities whose Milnor fibers are either joins of compact manifolds or bouquets of suspensions of such a join, and whose topology we can explicitly compute.

Topology of Formal Sums of Exceptional Orbit Hypersurfaces. We use the notion of a *formal linear combination of hypersurface singularities.* We consider $f_i : \mathbb{C}^{n_i}, 0 \to \mathbb{C}, 0$ for $i = 1, \dots, r$, defining hypersurfaces $X_i, 0 \subset \mathbb{C}^{n_i}, 0$. We regard the $\mathbb{C}^{n_i}$ as distinct spaces and let $\pi_i : \prod_{i=1}^{r} \mathbb{C}^{n_i} \to \mathbb{C}^{n_i}$ denote projection on the i-th factor. Then, for $a_i \in \mathbb{C}^*$ we let $f = \sum_{i=1}^{r} a_i f_i \circ \pi_i$. Then, f defines a hypersurface in $\prod_{i=1}^{r} \mathbb{C}^{n_i}$ which we will refer to as the *formal linear combination of the hypersurfaces defined by the* f_i. We will denote it by $f = \oplus_{i=1}^{r} a_i f_i$.

We are interested in the special case where the f_i define exceptional orbit hypersurfaces $\mathcal{E}_i$ of the types we considered earlier. Then, we will first combine the earlier results we obtained for them with the following result of Mutsuo Oka [Ok, Thm 1] which considerably extends a classical result of Thom-Sébastiani.

Theorem 6.1 (Oka)**.** *Let $g : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ and $h : \mathbb{C}^m, 0 \to \mathbb{C}, 0$ be weighted homogeneous germs with Milnor fibers Y, resp. Z, then $f = g \oplus h : \mathbb{C}^{n+m}, 0 \to \mathbb{C}, 0$ has Milnor fiber X homotopy equivalent to the join $Y * Z$. Moreover, the monodromy of f is the join of the monodromies of g and h.*

To use this result we recall the consequences for cohomology with coefficients in a field $\mathbf{k}$ of characteristic 0 (see e.g. [Ok] or [CF, Chap. 5]).

$$\tilde{H}^{\ell}(Y * Z;\mathbf{k}) \simeq \oplus_{i=0}^{\ell-1} \left(\tilde{H}^i(Y;\mathbf{k}) \otimes \tilde{H}^{\ell-i-1}(Z;\mathbf{k})\right) \tag{6.1}$$

Hence, we have the following isomorphism of cohomology viewed as graded vector spaces

$$\tilde{H}^*(Y * Z;\mathbf{k}) \simeq (\tilde{H}^*(Y;\mathbf{k}) \otimes \tilde{H}^*(Z;\mathbf{k}))[1] \tag{6.2}$$

where as earlier "[1]"will denote shift increasing degrees by 1. Then, if the monodromy on $H^*(Y;\mathbf{k})$ is denoted by σ_g and that on $H^*(Z;\mathbf{k})$, by σ_h, then the join of the monodromies, denoted $\sigma_g * \sigma_h$, which by Theorem 6.1 gives the monodromy σ_f, is given by the tensor product $\sigma_g \otimes \sigma_h$ on each summand.

We may combine Oka's theorem with our earlier results. For $i = 1, \dots, r$, let $f_i : \mathbb{C}^{n_i}, 0 \to \mathbb{C}, 0$ define an exceptional orbit hypersurface $\mathcal{E}_i, 0$ with global Milnor fiber F_i with model compact submanifold M_i, global Milnor fibration $p_i : E_i \to S^1$, and monodromy σ_i on $\tilde{H}^*(F_i; \mathbf{k})$.

Theorem 6.2. *Let $f : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ be given by the formal linear combination $\oplus_{i=1}^r a_i f_i$ with $a_i \in \mathbb{C}^*$, and f_i as above. It defines a hypersurface $\mathcal{E}$, with global Milnor fiber F, global Milnor fibration $p : E \to S^1$, and monodromy σ. Then*

i) *F is homotopy equivalent to the join of compact manifolds $M_1 * M_2 * \cdots * M_r$;*
ii) *hence,*

$$\tilde{H}^*(F; \mathbf{k}) \;\simeq\; \left(\otimes_{i=1}^r \; \tilde{H}^*(M_i; \mathbf{k})\right)[r-1]. \tag{6.3}$$

iii) *Also, the monodromy $\sigma = \sigma_1 * \sigma_2 * \cdots * \sigma_r$; so*
iv) *if the Milnor fibration of each f_i is cohomologically trivial, then the Milnor fibration of f is cohomologically trivial.*
v) *In the case of iv),*

(6.4)

$$H^*(\mathbb{C}^n \backslash \mathcal{E}; \mathbf{k}) \;\simeq\; H^*(E; \mathbf{k}) \;\simeq\; \Lambda^* \mathbf{k}\{s_1\} \otimes \Big(\mathbf{k}\{1\} \oplus (\otimes_{i=1}^r \; \tilde{H}^*(M_i; \mathbf{k}))[r-1]\Big)$$

where $\mathbf{k}\{1\}$ has degree 0 and $\deg(s_1) = 1$.

Proof. For i) ii) and and iii), we use induction on the number r of terms in the formal linear combination $f = \oplus_{i=1}^r a_i f_i$. If the result is true when $m < r$ then $f = g \oplus h$ where $g = \oplus_{i=1}^{r-1} a_i f_i$ and $h = a_r f_r$. As each f_i is homogeneous on different coordinates, both g and h are weighted homogeneous for a common set of weights on $\mathbb{C}^n$, so Oka's theorem applies and the Milnor fiber $X = F$ of f is homotopy equivalent to $Y * Z$ with Y and Z denoting the global Milnor fibers of g, resp. h (and the Milnor fiber of h is diffeomorphic to that of f_r, i.e. F_r). Also, the monodromy $\sigma_f = \sigma_g * \sigma_h$. By the inductive assumption Y is homotopy equivalent to $F_1 * F_2 * \cdots * F_{r-1}$, with monodromy $\sigma_g = \sigma_1 * \cdots * \sigma_{r-1}$. so by Oka's theorem, X is homotopy equivalent to $(F_1 * F_2 * \cdots * F_{r-1}) * F_r$, with monodromy $\sigma_f = \sigma_1 * \cdots * \sigma_{r-1} * \sigma_r$. Hence, by repeated application of (6.2) we see that $\tilde{H}^*(F; \mathbf{k})$ has the indicated form as a graded vector space.

Furthermore, for iv) if each global Milnor fibration $p_i : E_i \to S^1$ is cohomologically trivial, then $\sigma_i \equiv id$ so by iii) $\sigma_f \equiv id$ and the Milnor fiber of f is cohomologically trivial. Thus, by Proposition 1.8 $H^*(E; \mathbf{k})$ is given by (6.4). As f is weighted homogeneous, by an analogous argument as in Lemma 1.2 it follows that $\mathbb{C}^n \backslash \mathcal{E}$ has E as a deformation retract; thus, the remainder of (6.4) follows. □

We next separately give the form of the (co)homology of the link $L(\mathcal{E})$. To do so we will introduce some notation. Given a (finite dimensional) graded vector space $W = \oplus W_j$ over the field $\mathbf{k}$, we let $\mathbf{k}[m]$ denote $\mathbf{k}$ with a grading of degree m; and apply a grading to $\hom(W, \mathbf{k}[m]) \simeq \oplus \hom(W_j, \mathbf{k}[m])$ so that if W_j has graded degree ℓ_j, then $\hom(W_j, \mathbf{k}[m])$ has graded degree $m - \ell_j$.

Corollary 6.3. *If f is given as a formal sum as above and the Milnor fiber of each f_i is cohomologically trivial, then*

$$\widetilde{H}^*(L(\mathcal{E});\mathbf{k}) \;\simeq\; \mathbf{k}[2n-3] \oplus \mathrm{hom}(\otimes_{i=1}^r \tilde{H}^*(M_i;\mathbf{k}), \mathbf{k}[2n-2])[-(r-1)] \oplus \mathrm{hom}(\otimes_{i=1}^r \tilde{H}^*(M_i;\mathbf{k}), \mathbf{k}[2n-2])[-r]\,. \tag{6.5}$$

Proof. By v) of Theorem 6.2, $H^*(\mathbb{C}^n\backslash\mathcal{E};\mathbf{k})$ is given by (6.4). Then, by the same argument given in Proposition 1.9, $S^{2n-1}\backslash L(\mathcal{E})$ is homotopy equivalent to $\mathbb{C}^n\backslash\mathcal{E}$. Hence, we may again apply Alexander duality as (1.11) in the proof of Proposition 1.9 to obtain the isomorphism of graded vector spaces

$$\widetilde{H}^{2n-2-j}(L(\mathcal{E});\mathbf{k}) \;\simeq \mathrm{hom}(\widetilde{H}_j(S^{2n-1}\backslash L(\mathcal{E});\mathbf{k}), \mathbf{k}[2n-2])\,. \tag{6.6}$$

Summing (6.6) over j yields an isomorphism of graded vector spaces

$$\widetilde{H}^*(L(\mathcal{E});\mathbf{k}) \;\simeq \mathrm{hom}(\widetilde{H}_*(S^{2n-1}\backslash L(\mathcal{E});\mathbf{k}), \mathbf{k}[2n-2]) \tag{6.7}$$

By (6.4), we may decompose the RHS of (6.7) as the direct sum of graded vector spaces

$$\mathrm{hom}(\mathbf{k}\{s_1\}, \mathbf{k}[2n-2]) \oplus \mathrm{hom}\Big((\otimes_{i=1}^r \tilde{H}^*(M_i;\mathbf{k}))[(r-1)], \mathbf{k}[2n-2]\Big) \oplus \mathrm{hom}\Big(\mathbf{k}\{s_1\} \otimes (\otimes_{i=1}^r \tilde{H}^*(M_i;\mathbf{k}))[r-1], \mathbf{k}[2n-2]\Big)\,. \tag{6.8}$$

As graded vector spaces, the first summand is $\mathbf{k}[2n-3]$, and

$$\mathbf{k}\{s_1\} \otimes (\otimes_{i=1}^r \tilde{H}^*(M_i;\mathbf{k}))[r-1] \;\simeq\; (\otimes_{i=1}^r \tilde{H}^*(M_i;\mathbf{k}))[r]\,.$$

Then, for the second and third summands in (6.8), the positive shift inside "hom" can be moved to a negative shift outside. Hence, when we add $\mathbf{k}\{1\}$ to both sides of (6.7), we obtain from (6.6) and (6.7) the isomorphism of graded vector spaces (6.5). ☐

Unfortunately $H^*(L(\mathcal{E});\mathbf{k})$ in this more general case cannot be given in such a simple form as a shifted and upper truncated cohomology of a compact orientable manifold or an exterior algebra as in iii) of Proposition 1.9 and Theorems 3.2 and 4.5. However, we can still view the cohomology as a direct sum of the shifted upper truncated exterior algebras which appear in the theorems.

Example 6.4. As a simple example we consider the formal sum

$$f \;=\; a_1 \det\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} + a_2 y_1 y_2 \det\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

which defines a hypersurface singularity in $\mathbb{C}^{13}$. We denote the first term by $f_1(x)$ and the second by $f_2(y)$. Then, by Theorem 3.1, the Milnor fiber of f_1 is homotopy equivalent to SU_3 and has cohomology isomorphic to an exterior algebra $\Lambda^*\mathbf{k}(e_3, e_5)$. Also, by [DP2] (see also Example 4.8), f_2 defines a linear free divisor resulting from the action of a solvable linear algebraic group and by [DP] it has Milnor fiber homotopy equivalent to torus, with cohomology an exterior algebra $\Lambda^*\mathbf{k}(e_1, e_1')$. Thus, by Theorem 6.2 the Milnor fiber of f is homotopy equivalent to $SU_3 * T^2$ with reduced cohomology $\left(\widetilde{\Lambda}^*\mathbf{k}(e_3, e_5) \otimes \widetilde{\Lambda}^*\mathbf{k}(e_1, e_1')\right)[1]$, which is 0 in degrees < 5 and in degrees $5 \leq \ell \leq 11$ has dimensions $2, 1, 2, 1, 0, 2, 1$. We see from the dimensions that it is a direct sum of three shifted copies of $\widetilde{\Lambda}^*\mathbf{k}(e_1, e_1')$

with nonzero dimensions $2, 1$ or three shifted copies of $\widetilde{\Lambda}^*\mathbf{k}(e_3, e_5)$ which has its dimensions $1, 0, 1, 0, 0, 1$ between degrees 3 and 8.

Also, both Milnor fibrations are cohomologically trivial by Theorems 3.1 and 5.1; thus, by iii) of Theorem 6.2, the Milnor fibration of f is cohomologically trivial. Hence, by Corollary 6.3, the link $L(\mathcal{E})$ has reduced cohomology which is 0 in degrees < 12 and in degrees $12 \leq \ell \leq 23$ has dimensions $1, 3, 2, 1, 3, 3, 3, 2, 0, 0, 0, 1$. Here we see the group of lower nonzero dimensions obtained from those for the Milnor fiber written in reverse order $1, 2, 0, 1, 2, 1, 2$ and added to another copy shifted by 1.

Milnor Fibers Homotopy Equivalent to a Bouquet of Suspensions of Joins of Compact Manifolds.

As a last step, we consider a formal sum $h = f \oplus g$, where f is a formal sum of f_i which define exceptional orbit hypersurfaces as considered earlier, and g is weighted homogeneous and has a Milnor fiber which is homotopy equivalent to a bouquet of spheres $\vee_{i=1}^k S^{n_i}$. Then, we may again apply Oka's Theorem to conclude

Proposition 6.5. *The Milnor fiber of $h = f \oplus g$ as above has the homotopy type of a bouquet of spaces, each of which is an iterated $n_i + 1$ suspension $S^{n_i+1}(*_{j=1}^r M_j)$ for $i = 1, \dots, k$.*

Remark 6.6. Thus, within the class of nonisolated hypersurface singularities obtained from formal sums, the Milnor fibers which are a bouquet of spheres are replaced more generally by bouquets of spaces each of which are suspensions of joins of compact manifolds.

Proof. Let $X = *_{j=1}^r M_j$,which is homotopy equivalent to the Milnor fiber of f. By Oka's Theorem, the Milnor fiber of h is homotopy equivalent to the join $X * (\vee_{i=1}^k S^{n_i})$. Let p denote the common point of the bouquet. Then, this join is the union of the joins $X * S^{n_i}$, and any two intersect in the common subspace $X * \{p\}$. This is a cone on X, and is hence contractible. Moreover, let $U_i \subset S^{n_i}$ be a contractible neighborhood of p, with $\varphi_i : U_i \to \{p\}$ a strong deformation retraction. Then, the join of φ_i with the identity on X gives a strong deformation retraction $\tilde{\varphi}_i$ of $X * U_i$ to $X * \{p\}$. Thus, together these give a strong deformation retraction of $\cup_{i=1}^k X * U_i$ to $X * \{p\}$. As $X * \{p\}$ has $\{p\}$ as a strong deformation retract, we conclude that $\cup_{i=1}^k X * U_i$ has $\{p\}$ as a strong deformation retract. Hence, we may collapse $\cup_{i=1}^k X * U_i$ to $\{p\}$ and retain the same homotopy type. It follows that $X * (\vee_{i=1}^k S^{n_i})$ is homotopy equivalent to $\vee_{i=1}^k (X * S^{n_i})$, and each $X * S^{n_i}$ is homeomorphic to the $n_i + 1$ iterated suspension $S^{n_i+1}(X)$. Together these give the result. □

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