GLOBAL EXISTENCE FOR A COUPLED WAVE SYSTEM RELATED TO THE STRAUSS CONJECTURE

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ABSTRACT. A coupled system of semilinear wave equations is considered, and a small data global existence result related to the Strauss conjecture is proved. Previous results have shown that one of the powers may be reduced below the critical power for the Strauss conjecture provided the other power sufficiently exceeds such. The stability of such results under asymptotically flat perturbations of the space-time where an integrated local energy decay estimate is available is established.

1. INTRODUCTION

The purpose of this article is to establish global existence for a coupled system of wave equations, which is related to the Strauss conjecture, on asymptotically flat space-times that permit a localized energy estimate. It is now well-known that nonlinear wave equations

$$\Box u := (\partial_t^2 - \Delta)u = F_p(u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

where

$$\sum_{0\le j\le 2} |u|^j |\partial_u^j F_p(u)| \lesssim |u|^p \quad \text{for } u \text{ small} \tag{1.1}$$

have global solutions for sufficiently small initial data provided $p > p_c$ where $p_c > 1$ solves

$$(n-1)p_c^2 - (n+1)p_c - 2 = 0. \tag{1.2}$$

Moreover, blow up is known to occur for $p < p_c$. These results originated in [23] for $n = 3$ where $p_c = 1 + \sqrt{2}$, and following [44] the problem became known as the Strauss conjecture. Global existence in general dimension was eventually established in [20], [45]; see the references therein for many intermediate results. Blow up below the critical exponent was proved in [40]. See also [39], [51] for further results at the critical exponent.

In the current work, we shall examine a system of the form

$$\Box u = |v|^p, \quad \Box v = |u|^q.$$

In the flat case, the coupled system was examined in [16], and it was shown that global existence may be established for powers in the nonlinearity below the critical

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exponent provided the power on the coupled equation exceeds the same. Indeed, setting

$$C(p,q) = \max\Big\{\frac{q+2+p^{-1}}{pq-1}, \frac{p+2+q^{-1}}{pq-1}\Big\} - \frac{n-1}{2}, \tag{1.3}$$

it was shown that small data global existence holds for $C(p,q) < 0$ and that such fails for $C(p,q) > 0$. Notice that $C(p,p) = 0$ corresponds precisely to (1.2). In particular, note that small data global existence may be established for powers $p < p_c$ provided that the other power q sufficiently exceeds p_c. In addition to [16], see [17], [14] for treatments of $C(p,q) > 0$ and [1], [15], [25], and [21] for analysis of the critical curve $C(p,q) = 0$. Moreover, see the overview [26] of this and related problems.

Here we seek to establish the same using techniques that are sufficiently robust so as to allow background geometries. Specifically, we shall use a variant of the weighted Strichartz estimates of [22], [19], which were further developed in [27], [35], and the localized energy estimate to prove such global existence.

We shall examine operators of the form

$$Pu = \partial_\alpha g^{\alpha\beta}\partial_\beta u + b^\alpha \partial_\alpha u + cu \tag{1.4}$$

on space-times M where $M = \mathbb{R}_+ \times \mathbb{R}^3$ or $M = \mathbb{R}_+ \times (\mathbb{R}^3 \backslash \mathcal{K})$ where $\mathcal{K}$ has a smooth boundary and $\mathcal{K} \subset \{x : |x| < R_0\}$. Here g is a Lorentzian metric, and we make the assumption that g can be written as

$$g(t,x) = m + g_0(t,r) + g_1(t,x) \tag{1.5}$$

where $m = \operatorname{diag}(-1,1,1,1)$ is the Minkowski metric. The components g_0 and g_1 will represent long-range and short-range perturbations respectively. They are asymptotically flat in the sense that

$$\|\partial^\mu_{t,x} g_{i,\alpha\beta}\|_{\ell_1^{i+|\mu|} L^\infty_{t,x}} = \mathcal{O}(1), \quad i = 0,1, \quad |\mu| \le 3.\text{[1]} \tag{1.6}$$

Due to the need to commute with spatial rotations, the long-range perturbation g_0 is assumed to be spherically symmetric in the sense that the coefficients only (spatially) depend on $r = |x|$ and

$$g - g_1 = (-1+\tilde g_{00}(t,r))dt^2 + 2\tilde g_{01}(t,r)\,dt\,dr + (1+\tilde g_{11}(t,r))dr^2 + (1+\tilde g_{22}(t,r))r^2\,d\omega^2_{\mathbb{S}^2}. \tag{1.7}$$

and, by (1.6), $\|\partial^\mu_{t,x}\tilde g_{\alpha\beta}\|_{\ell_1^{|\mu|}L^\infty_{t,x}} = \mathcal{O}(1)$ for $|\mu| \le 3$. The coefficients of the lower-order perturbations decay are assumed to decay as follows:

$$\|\partial^\mu_{t,x} b\|_{\ell_1^{1+|\mu|}L^\infty_{t,x}} + \|\partial^\mu_{t,x} c\|_{\ell_1^{2+|\mu|}L^\infty_{t,x}} = \mathcal{O}(1), \quad |\mu| \le 2. \tag{1.8}$$

We shall also assume that the perturbations admit a (weak) localized energy decay. More specifically, we assume that there is R_1 (with $R_1 > R_0$ in the case

[1]Here, as in [35], for a norm A, we set

$$\|u\|_{\ell^s_q A} = \Big\| 2^{js}\|\phi_j(x)u(t,x)\|_A \Big\|_{\ell^q_{j\ge 0}}, \quad \sum_{j\ge 0}\phi_j^2(x) = 1, \quad \operatorname{supp}\phi_j \subset \{\langle x\rangle \approx 2^j\}.$$

that $M = \mathbb{R}_+ \times (\mathbb{R}^3 \backslash \mathcal{K}))$ so that if u solves $Pu = F$ then

$$\|\partial\partial^\mu u\|_{L^\infty_t L^2_x} + \|(1-\chi)\partial\partial^\mu u\|_{\ell^{-1/2}_\infty L^2_{t,x}} + \|\partial^\mu u\|_{\ell^{-3/2}_\infty L^2_{t,x}} \lesssim \|u(0,\cdot)\|_{H^{|\mu|+1}} + \|\partial_t u(0,\cdot)\|_{H^{|\mu|}} + \sum_{|\nu|\le|\mu|} \|\partial^\nu F\|_{L^1_t L^2_x} \tag{1.9}$$

for all $|\mu| \le 2$. Here χ is a smooth function that is identically 1 on $B_{R_1/2} := \{|x| \le R_1/2\}$ and is supported on B_{R_1}.

On $(1+3)$-dimensional Minkowski space (i.e. when $g_0 \equiv g_1 \equiv 0$), it is known that

$$\|\partial u\|_{L^\infty_t L^2_x} + \|\partial u\|_{\ell^{-1/2}_\infty L^2_{t,x}} + \|u\|_{\ell^{-3/2}_\infty L^2_{t,x}} \lesssim \|\partial u(0,\cdot)\|_{L^2} + \|\Box u\|_{L^1_t L^2_x},$$

which is a stronger version of the $\mu = 0$ estimate above. And as the flat d'Alembertian commutes with $\partial_{t,x}$, the higher order variants readily follow. Such estimates originated in [36]. They follow, e.g., by multiplying $\Box u$ by a multiplier of the form $\frac{r}{r+2^j}\partial_r u + \frac{n-1}{2}\frac{1}{r+2^j}u$, integrating over $[0,T]\times\mathbb{R}^3$, and integrating by parts. See, e.g., [43], [29]. And see, e.g., [34] for a more complete history. These estimates are known to be rather robust in the asymptotically flat regime. Even without the cutoff, they are known to hold for small, possibly time-dependent perturbations of Minkowski space [29, 30], [33, 32], [2] and for time-independent nontrapping perturbations in the product manifold setting due to, e.g., [8], [6], [42]. See [31] for the most general results in the nontrapping regime.

The presence of trapped rays is a known obstruction to the localized energy estimate [37], [38]. The asymptotic flatness restricts the possibility of trapped rays to a compact set, and when the trapping is sufficiently weak, a localized energy estimate where one, say, cuts off away from the trapping may sometimes be recovered. Allowing for this is the reason for the cutoff in assumption (1.9). Previous results have then verified (1.9) in a number of settings where trapping occurs, including on the Schwarzschild space-time [4, 5], [10, 11], [28], on Kerr space-times with $a \ll M$ [47] (see also [3], [9, 12] for some closely related results and [13] for a related result that holds for the full subextremal range $|a| < M$), and on certain warped product manifolds that contain degenerate trapping [7].

We now introduce the specific problem at hand. With two possibly different operators P_1 and P_2 subject to hypotheses (1.5)-(1.9), we examine the coupled system

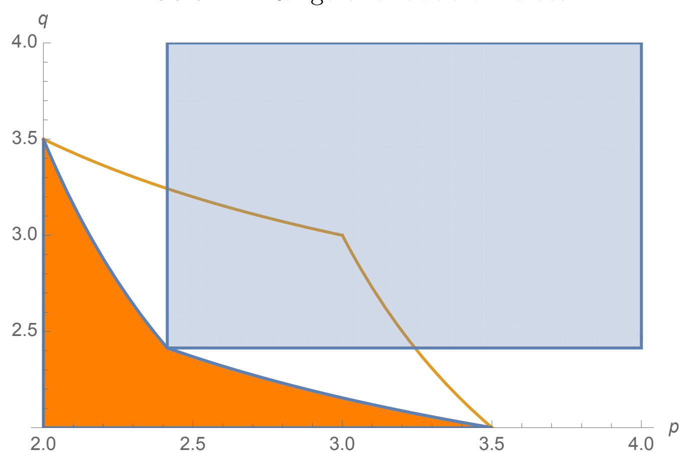
$$\begin{aligned} P_1 u &= F_p(v), & P_2 v &= F_q(u),\\ u(0,x) &= f_1(x), & v(0,x) &= f_2(x),\\ \partial_t u(0,x) &= g_1(x), & \partial_t v(0,x) &= g_2(x). \end{aligned} \tag{1.10}$$

Here F_p and F_q are two functions satisfying (1.1).

For the system (1.10), we shall establish the following small data global existence result:

Theorem 1.1. *Suppose that P_1 and P_2 are operators of the form (1.4) so that (1.5), (1.6), (1.7), (1.8), and (1.9) hold. Moreover assume that $2 < p, q$ and $C(p,q) < 0$.*

FIGURE 1. Range of allowable indices



Then if $f_1, g_1, f_2, g_2 \in C_c^\infty$[2] *and*

$$\|(f_1, f_2)\|_{H^3} + \|(g_1, g_2)\|_{H^2} \le \varepsilon \tag{1.11}$$

with ε *sufficiently small, there exists a global solution* (u, v) *to* (1.10).

We note that the techniques to prove Theorem 1.1 also work in four spatial dimensions, but as they require $p, q \ge 2$ and $p_c = 2$ in this case, nothing new is gained over [35].

This result follows a number of studies that established various existence results related to the Strauss conjecture in the presence of background geometry. In exterior domains, these included [18], [22], [41], and [52]. And on asymptotically flat backgrounds, see [42], [49], [27], [35], and [48]. See, also, the expository article [50]. A key component of many of these results is the weighted Strichartz estimate of [22], [19]. Here, in particular, we rely on the variant of that developed in [35], which is based on the local energy estimates of [33].

Figure 1 demonstrates the range of allowable indices. Theorem 1.1 allows for any pair of indices (p, q) that land outside of the shaded region in the lower left corner. The techniques of, e.g., [35], however, only trivially apply in the rectangular region where $p, q > p_c = 1 + \sqrt{2}$. The other curve will be discussed further in Section 3.

The method that we shall employ is similar in spirit to that of [35], which in turn is based on a number of preceding works. Near infinity, where the asymptotic flatness allows us to think of the geometry as a small perturbation of Minkowski space, variants of the weighted Strichartz estimates of [22], [19] are employed. The assumed localized energy estimate handles the remaining compact region, where the geometry has the most significant role. It also allows the analyses done in the two regions to be glued together. Such a strategy has become common. See, e.g., [27], [28], [33, 32], [34], [46].

[2]For simplicity of exposition, we have taken the data here to be compactly supported, but this may be replaced by a condition such as [35, (5.3)].

2. Main Estimates

Before we proceed to the main estimates for the linear equation, we first introduce some notation. We shall be using a restricted set of the classical invariant vector fields. We let $\Omega_{ij} = x_i\partial_j - x_j\partial_i$ denote the generators of spatial rotations. And we let $Y = \{\nabla_x, \Omega\}$ and $Z = \{\partial, \Omega\}$, where $\partial = (\partial_t, \nabla_x)$ denotes the space-time gradient. We also introduce the shorthand $|Z^{\le m}u| = \sum_{|\mu|\le m} |Z^\mu u|$ for summing over multi-indices of order $\le m$. A similar notation where the absolute values are replaced by a norm shall also be used. Finally, for the mixed norms that appear in the weighted Strichartz estimates below, we fix the convention

$$\|f\|_{L^p_tL^q_rL^s_\omega} = \Big[\int\Big(\int\Big[\int |f(t,r\omega)|^s\,d\omega_{\mathbb{S}^2}\Big]^{q/s} r^2\,dr\Big)^{p/q}\,dt\Big]^{1/p},$$

with the obvious changes for Lebesgue indices of ∞.

The main linear estimate that will be applied near infinity, where the operators may be viewed as small perturbations of the flat d'Alembertian, is the following weighted Strichartz estimate:

Theorem 2.1 ([35])**.** *Suppose that P is an operator of the form (1.4) satisfying the hypotheses (1.5), (1.6), (1.7), (1.8), and (1.9). Suppose that $w(0,\cdot)$ and $\partial_t w(0,\cdot)$ are compactly supported and satisfy (1.11). Then there exists $R_2 > R_1$ so that for any $R > R_2$ if ψ_R is identically 1 on $\{|x| \ge 2R\}$ and vanishes on $\{|x| < R\}$, then we have*

$$\|\psi_R Z^{\le 2}w\|_{\ell_p^{\frac{3}{2}-\frac{4}{p}-s}L^p_tL^p_rL^2_\omega} \le C_1\varepsilon + C_1\|\psi_R Y^{\le 1}Pw(0,\cdot)\|_{\dot H^{s-1}} + C_1\|\psi_R^{\tilde p}Z^{\le 2}Pw\|_{\ell_1^{-\frac{1}{2}-s}L^1_tL^1_rL^2_\omega} + C_1\|\partial^{\le 2}Pw\|_{L^1_tL^2_rL^2_\omega} \tag{2.1}$$

and

$$\|\psi_R w\|_{\ell_p^{\frac{3}{2}-\frac{4}{p}-s}L^p_tL^p_rL^2_\omega} \lesssim \varepsilon + \|\psi_R^{\tilde p}Pw\|_{\ell_1^{-\frac{1}{2}-s}L^1_tL^1_rL^2_\omega} + \|Pw\|_{L^1_tL^2_rL^2_\omega} \tag{2.2}$$

for any $p \in (2,\infty)$, $s \in (1/2 - 1/p, 1/2)$, and $\tilde p > 0$.

These estimates originated in [22] and [19] for the flat d'Alembertian. The above version is essentially from [35], which in particular allows for asymptotically flat operators and does not necessitate compactly supported initial data though we assume that here for simplicity. These estimates follow by interpolating a variant of the localized energy estimate with a trace theorem on the sphere. We note a few minor modifications from the version in [35]: (1) We have allowed a different power of the cutoff function in the right sides; (2) We have more carefully stated the requirements on R so that in the sequel we are able to use the same R for both u and v; (3) We have stated separately the case where no vector fields are applied since this is used when showing that our iteration is Cauchy. These all follow from trivial modifications of the proof of [35].

A key to tying the region near infinity to the remaining compact region where (1.9) is the primary tool is the following weighted Sobolev inequalities. These are variants of the original estimates of [24] and follow by localizing, applying Sobolev embeddings on $\mathbb{R}\times\mathbb{S}^2$, and adjusting the volume elements to match those of $\mathbb{R}^3$ in polar coordinates. See, e.g., [27], [35] for proofs.

Lemma 2.2. *On $\mathbb{R}^3$, for $R \geq 1$, $\beta \in \mathbb{R}$, and $2 \leq p \leq q \leq \infty$, we have*

$$\|r^\beta u\|_{L^q_r L^\infty_\omega(r\geq R+1)} \lesssim \sum_{|\mu|\leq 2} \|r^{\beta-\frac{2}{p}+\frac{2}{q}} Y^\mu u\|_{L^p_r L^2_\omega(r\geq R)}. \tag{2.3}$$

If $2 \leq p \leq q \leq 4$ and $\beta \in \mathbb{R}$, we also have

$$\|r^\beta u\|_{L^q_r L^4_\omega(r\geq R+1)} \lesssim \sum_{|\mu|\leq 1} \|r^{\beta-\frac{2}{p}+\frac{2}{q}} Y^\mu u\|_{L^p_r L^2_\omega(r\geq R)}. \tag{2.4}$$

3. Small data global existence

We now prove Theorem 1.1. We shall apply (2.1) to u with $(p,s) = \left(q, \frac{7+4p-3pq}{2-2pq}\right)$ and to v with $(p,s) = \left(p, \frac{7+4q-3pq}{2-2pq}\right)$. The requirement from Theorem 2.1 that $s > \frac{1}{2} - \frac{1}{p}$ then corresponds to

$$\frac{2+p+q^{-1}}{pq-1} < 1, \quad \frac{2+q+p^{-1}}{pq-1} < 1$$

respectively. In the case of $n = 3$, this produces the requirement that $C(p,q) < 0$.

We shall assume, without loss of generality, that (p,q) satisfy

$$p(q-2) < 3, \quad q(p-2) < 3, \tag{3.1}$$

which corresponds to the condition that $s < 1/2$ in Theorem 2.1. It is these conditions that are represented by the curve in Figure 1, below which they are satisfied. In the unshaded region to the left of the rectangle, if (p,q) satisfy $C(p,q) < 0$ but one of the above conditions is violated, one may simply choose any $\tilde{q} < q$ so that $C(p,\tilde{q}) < 0$ and so that both conditions in (3.1) hold. One simply imagines $|u|^q = |u|^{q-\tilde{q}}|u|^{\tilde{q}}$ and argues as below with the exponent $\tilde{q}$ replacing q. Simple Sobolev embeddings control the remaining $q - \tilde{q}$ powers. In the unshaded region below the rectangle, one argues similarly by reducing the power of p. In the shaded rectangle, the methods of [35] apply directly and no further argument is needed.

Let $s_1 = \frac{7+4p-3pq}{2-2pq}$, $s_2 = \frac{7+4q-3pq}{2-2pq}$, $\alpha_1 = \frac{3}{2} - \frac{4}{q} - s_1$, and $\alpha_2 = \frac{3}{2} - \frac{4}{p} - s_2$. We note that the power of the weight in the right side of (2.1) then satisfies:

$$-\frac{1}{2} - s_1 = p\alpha_2, \quad -\frac{1}{2} - s_2 = q\alpha_1.$$

We solve (1.10) via an iteration, and at this point, the arguments that are used are akin to those of [22], [27], and [35]. Setting $u_{-1} \equiv 0$ and $v_{-1} \equiv 0$, we recursively define u_j, v_j, $j \geq 0$ to solve

$$\begin{aligned} P_1 u_j &= F_p(v_{j-1}), & P_2 v_j &= F_q(u_{j-1}), \\ u_j(0,x) &= f_1(x), & v_j(0,x) &= f_2(x), \\ \partial_t u_j(0,x) &= g_1(x), & \partial_t v_j(0,x) &= g_2(x). \end{aligned} \tag{3.2}$$

We introduce the quantity

$$\begin{aligned} M_k(u,v) = \|\psi_R Z^{\leq k} u\|_{\ell^{\alpha_1}_q L^q_t L^q_r L^2_\omega} + \|\psi_R Z^{\leq k} v\|_{\ell^{\alpha_2}_p L^p_t L^p_r L^2_\omega} \\ + \|\partial^{\leq k}(u,v)\|_{\ell^{-3/2}_\infty L^2_t L^2_r L^2_\omega} + \|\partial^{\leq k}\partial(u,v)\|_{L^\infty_t L^2_r L^2_\omega}. \end{aligned} \tag{3.3}$$

Our first goal is to inductively show that $M_2(u_j, v_j) \le 4C_2\varepsilon$ for some uniform constant C_2. We first note that from (1.11) (and the assumption that the data are compactly supported) we easily obtain

$$\|\psi_R Y^{\le 1} P_1 u_j(0,\cdot)\|_{\dot H^{s_1-1}} + \|\psi_R Y^{\le 1} P_2 v_j(0,\cdot)\|_{\dot H^{s_2-1}} \le C\varepsilon.$$

And thus, for C_2 chosen large enough, by (2.1),

$$\begin{aligned}\|\psi_R Z^{\le 2} u_j\|_{\ell^{\alpha_1}_q L^q_t L^q_r L^2_\omega} + \|\psi_R Z^{\le 2} v_j\|_{\ell^{\alpha_2}_p L^p_t L^p_r L^2_\omega} \le C_2\varepsilon \\ + C\|\psi_R^p Z^{\le 2} F_p(v_{j-1})\|_{\ell^{p\alpha_2}_1 L^1_t L^1_r L^2_\omega} + C\|\psi_R^q Z^{\le 2} F_q(u_{j-1})\|_{\ell^{q\alpha_1}_1 L^1_t L^1_r L^2_\omega} \\ + C\|\partial^{\le 2} F_p(v_{j-1})\|_{L^1_t L^2_r L^2_\omega} + C\|\partial^{\le 2} F_q(u_{j-1})\|_{L^1_t L^2_r L^2_\omega}.\end{aligned} \tag{3.4}$$

And from (1.11) and (1.9), we have

$$\begin{aligned}\|\partial^{\le 2}(u_j, v_j)\|_{\ell^{-3/2} L^2_t L^2_r L^2_\omega} + \|\partial^{\le 2}\partial(u_j, v_j)\|_{L^\infty_t L^2_r L^2_\omega} \le C_2\varepsilon \\ + C\|\partial^{\le 2} F_p(v_{j-1})\|_{L^1_t L^2_r L^2_\omega} + C\|\partial^{\le 2} F_q(u_{j-1})\|_{L^1_t L^2_r L^2_\omega}.\end{aligned} \tag{3.5}$$

From these, we first notice that $M_2(u_0, v_0) \le 2C_2\varepsilon$, which provides the base case for the induction.

We then assume that $M_2(u_{j-1}, v_{j-1}) \le 4C_2\varepsilon$ and show that $M_2(u_j, v_j) \le 4C_2\varepsilon$. We first notice that (1.1) gives

$$|Z^{\le 2} F_p(u)| \lesssim |u|^{p-1}|Z^{\le 2} u| + |u|^{p-2}|Z^{\le 1} u|^2. \tag{3.6}$$

By the Sobolev embeddings $H^2_\omega \subset L^\infty_\omega$ and $H^1_\omega \subset L^4_\omega$ on $\mathbb{S}^2$ and Hölder's inequality, this gives

$$\begin{aligned}\|Z^{\le 2} F_p(v_{j-1})\|_{L^2_\omega} &\lesssim \|v_{j-1}\|^{p-1}_{L^\infty_\omega}\|Z^{\le 2} v_{j-1}\|_{L^2_\omega} + \|v_{j-1}\|^{p-2}_{L^\infty_\omega}\|Z^{\le 1} v_{j-1}\|^2_{L^4_\omega} \\ &\lesssim \|Z^{\le 2} v_{j-1}\|^p_{L^2_\omega}.\end{aligned}$$

It then immediately follows that

$$\|\psi_R^p Z^{\le 2} F_p(v_{j-1})\|_{\ell^{p\alpha_2}_1 L^1_t L^1_r L^2_\omega} \lesssim \|\psi_R Z^{\le 2} v_{j-1}\|^p_{\ell^{\alpha_2}_p L^p_t L^p_r L^2_\omega} \lesssim (M_2(u_{j-1}, v_{j-1}))^p \lesssim \varepsilon^p. \tag{3.7}$$

The last inequality results from the inductive hypothesis. A similar argument shows that

$$\|\psi_R^q Z^{\le 2} F_q(u_{j-1})\|_{\ell^{q\alpha_1}_1 L^1_t L^1_r L^2_\omega} \lesssim \varepsilon^q. \tag{3.8}$$

To finish the proof of the boundedness of $M_2(u_j, v_j)$, it remains to examine the $L^1_t L^2_r L^2_\omega$ pieces in the right sides of (3.4) and (3.5). The analyses will be done separately outside of a ball of radius $2R+1$, where the resulting terms will be compared to the weighted Strichartz portions of $M_2(u_{j-1}, v_{j-1})$, and inside the remaining compact set where the localized energy portions will be used.

We begin with the former using the weighted Sobolev inequalities of Lemma 2.2. Starting again at (3.6), we have

$$\begin{aligned}\|Z^{\le 2} F_p(v_{j-1})\|_{L^1_t L^2_{r\ge 2R+1} L^2_\omega} \lesssim \|r^{-\frac{\alpha_2}{p-1}} v_{j-1}\|^{p-1}_{L^p_t L^{\frac{2p(p-1)}{p-2}}_{r\ge 2R+1} L^\infty_\omega} \|r^{\alpha_2}\psi_R Z^{\le 2} v_{j-1}\|_{L^p_t L^p_r L^2_\omega} \\ + \left\|r^{\frac{2}{p-2}\left(-\alpha_2 - \frac{2}{p} + \frac12\right)} v_{j-1}\right\|^{p-2}_{L^p_t L^\infty_{r\ge 2R+1} L^\infty_\omega} \|r^{\alpha_2 + \frac{2}{p} - \frac12} Z^{\le 1} v_{j-1}\|^2_{L^p_t L^4_{r\ge 2R+1} L^4_\omega}.\end{aligned} \tag{3.9}$$

Since (3.1) guarantees that $s_1 < 1/2$, it follows that $p\alpha_2 = -\frac{1}{2} - s_1 \geq -1$, which is equivalent to
$$-\frac{\alpha_2}{p-1} - \frac{2}{p} + \frac{p-2}{p(p-1)} \leq \alpha_2.$$
Thus, (2.3) yields
$$\|r^{-\frac{\alpha_2}{p-1}} v_{j-1}\|_{L^p_t L^{\frac{2p(p-1)}{p-2}}_{r\geq 2R+1} L^\infty_\omega} \lesssim \|r^{\alpha_2} Z^{\leq 2} v_{j-1}\|_{L^p_t L^p_{r\geq 2R} L^2_\omega}.$$
The estimate (2.4) directly applies to yield
$$\|r^{\alpha_2+\frac{2}{p}-\frac{1}{2}} Z^{\leq 1} v_{j-1}\|_{L^p_t L^4_{r\geq 2R+1} L^4_\omega} \lesssim \|r^{\alpha_2} Z^{\leq 2} v_{j-1}\|_{L^p_t L^p_{r\geq 2R} L^2_\omega}.$$
As above, $p\alpha_2 \geq -1$, which implies that
$$\frac{2}{p-2}\Big(-\alpha_2 - \frac{2}{p} + \frac{1}{2}\Big) - \frac{2}{p} \leq \alpha_2.$$
Hence (2.3) gives
$$\Big\|r^{\frac{2}{p-2}\left(-\alpha_2-\frac{2}{p}+\frac{1}{2}\right)} v_{j-1}\Big\|_{L^p_t L^\infty_{r\geq 2R+1} L^\infty_\omega} \lesssim \|r^{\alpha_2} Z^{\leq 2} v_{j-1}\|_{L^p_t L^p_{r\geq 2R} L^2_\omega}.$$
Plugging each of these bounds into (3.9) and using that ψ_R is identically 1 on $r \geq 2R$, it follows that

(3.10)
$$\|Z^{\leq 2} F_p(v_{j-1})\|_{L^1_t L^2_{r\geq 2R+1} L^2_\omega} \lesssim \|\psi_R Z^{\leq 2} v_{j-1}\|^p_{\ell^{\alpha_2}_p L^p_t L^p_r L^2_\omega} \lesssim (M_2(u_{j-1}, v_{j-1}))^p \lesssim \varepsilon^p.$$
And an analogous argument in the symmetric variable q shows that
$$\|Z^{\leq 2} F_q(u_{j-1})\|_{L^1_t L^2_{r\geq 2R+1} L^2_\omega} \lesssim \varepsilon^q, \tag{3.11}$$
which leaves the analysis of the $L^1_t L^2_r L^2_\omega$ pieces over the region $r \leq 2R+1$ where the coefficients of Z are bounded.

We again start with (3.6) and apply Hölder's inequality to obtain
$$\begin{aligned} \|Z^{\leq 2} F_p(v_{j-1})\|_{L^1_t L^2_{r\leq 2R+1} L^2_\omega} &\lesssim \|v_{j-1}\|^{p-2}_{L^\infty_t L^\infty_r L^\infty_\omega} \|v_{j-1}\|_{L^2_t L^\infty_{r\leq 2R+1} L^\infty_\omega} \|\partial^{\leq 2} v_{j-1}\|_{L^2_t L^2_{r\leq 2R+1} L^2_\omega} \\ &\quad + \|v_{j-1}\|^{p-2}_{L^\infty_t L^\infty_r L^\infty_\omega} \|\partial^{\leq 1} v_{j-1}\|^2_{L^2_t L^4_{r\leq 2R+1} L^4_\omega}. \end{aligned} \tag{3.12}$$
By Sobolev embeddings, we have
$$\|v_{j-1}\|_{L^\infty_t L^\infty_r L^\infty_\omega} \lesssim \|\partial^{\leq 1} v_{j-1}\|_{L^\infty_t L^6_r L^6_\omega} \lesssim \|\partial^{\leq 1} \partial v_{j-1}\|_{L^\infty_t L^2_r L^2_\omega}.$$
Similarly, Sobolev embeddings (with a localizing factor) give
$$\|v_{j-1}\|_{L^2_t L^\infty_{r\leq 2R+1} L^\infty_\omega} \lesssim \|\partial^{\leq 2} v_{j-1}\|_{L^2_t L^2_{r\leq 2R+2} L^2_\omega} \lesssim \|\partial^{\leq 2} v_{j-1}\|_{\ell^{-3/2}_\infty L^2_t L^2_r L^2_\omega}$$
and
$$\|\partial^{\leq 1} v_{j-1}\|_{L^2_t L^4_{r\leq 2R+1} L^4_\omega} \lesssim \|\partial^{\leq 2} v_{j-1}\|_{L^2_t L^2_{r\leq 2R+2} L^2_\omega} \lesssim \|\partial^{\leq 2} v_{j-1}\|_{\ell^{-3/2}_\infty L^2_t L^2_r L^2_\omega}.$$
These bounds in (3.12) show that
$$\begin{aligned} \|Z^{\leq 2} F_p(v_{j-1})\|_{L^1_t L^2_{r\leq 2R+1} L^2_\omega} &\lesssim \|\partial^{\leq 1} \partial v_{j-1}\|^{p-2}_{L^\infty_t L^2_r L^2_\omega} \|\partial^{\leq 2} v_{j-1}\|^2_{\ell^{-3/2}_\infty L^2_t L^2_r L^2_\omega} \\ &\lesssim (M_2(u_{j-1}, v_{j-1}))^p \lesssim \varepsilon^p. \end{aligned} \tag{3.13}$$

And an analogous argument gives

$$\|Z^{\le 2} F_q(u_{j-1})\|_{L^1_t L^2_{r\le 2R+1} L^2_\omega} \lesssim \varepsilon^q. \tag{3.14}$$

Using (3.7), (3.8), (3.10), (3.11), (3.13), and (3.14) in (3.4) and (3.5), we obtain

$$M_2(u_j, v_j) \le 2C_2\varepsilon + C_3\varepsilon^{\min(p,q)}$$

for some constant C_3 that is independent of j. Since $p, q \ge 2$, if ε is sufficiently small, the desired bound $M_2(u_j, v_j) \le 4C_2\varepsilon$ follows.

It remains to show that the sequence converges. To do so, we shall show

$$M_0(u_j - u_{j-1}, v_j - v_{j-1}) \le \frac{1}{2} M_0(u_{j-1} - u_{j-2}, v_{j-1} - v_{j-2}), \tag{3.15}$$

and this will complete the proof. Applying (2.2) and (1.9), we have

$$\begin{aligned} &\|\psi_R(u_j - u_{j-1})\|_{\ell^{\alpha_1}_q L^q_t L^q_r L^2_\omega} + \|\psi_R(v_j - v_{j-1})\|_{\ell^{\alpha_2}_p L^p_t L^p_r L^2_\omega} \\ &\lesssim \|\psi_R^p(F_p(v_{j-1}) - F_p(v_{j-2})\|_{\ell_1^{p\alpha_2} L^1_t L^1_r L^2_\omega} + \|\psi_R^q(F_q(u_{j-1}) - F_q(u_{j-2}))\|_{\ell_1^{q\alpha_1} L^1_t L^1_r L^2_\omega} \\ &\qquad + \|F_p(v_{j-1}) - F_p(v_{j-2})\|_{L^1_t L^2_r L^2_\omega} + \|F_q(u_{j-1}) - F_q(u_{j-2})\|_{L^1_t L^2_r L^2_\omega}, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &\|(u_j - u_{j-1}, v_j - v_{j-1})\|_{\ell^{-3/2} L^2_t L^2_r L^2_\omega} + \|\partial(u_j - u_{j-1}, v_j - v_{j-1})\|_{L^\infty_t L^2_r L^2_\omega} \\ &\qquad \lesssim \|F_p(v_{j-1}) - F_p(v_{j-2})\|_{L^1_t L^2_r L^2_\omega} + \|F_q(u_{j-1}) - F_q(u_{j-2})\|_{L^1_t L^2_r L^2_\omega}. \end{aligned} \tag{3.17}$$

Noting that

$$|F_p(v_{j-1}) - F_p(v_{j-2})| \lesssim (|v_{j-1}|^{p-1} + |v_{j-2}|^{p-1})|v_{j-1} - v_{j-2}|, \tag{3.18}$$

we can quickly observe that

$$\begin{aligned} &\|\psi_R^p(F_p(v_{j-1}) - F_p(v_{j-2})\|_{\ell_1^{p\alpha_2} L^1_t L^1_r L^2_\omega} \\ &\quad \lesssim \Big(\|\psi_R v_{j-1}\|^{p-1}_{\ell^{\alpha_2}_p L^p_t L^p_r L^\infty_\omega} + \|\psi_R v_{j-2}\|^{p-1}_{\ell^{\alpha_2}_p L^p_t L^p_r L^\infty_\omega}\Big)\|\psi_R(v_{j-1} - v_{j-2})\|_{\ell^{\alpha_2}_p L^p_t L^p_r L^2_\omega} \\ &\quad \lesssim \Big(\|\psi_R Z^{\le 2} v_{j-1}\|^{p-1}_{\ell^{\alpha_2}_p L^p_t L^p_r L^2_\omega} + \|\psi_R Z^{\le 2} v_{j-2}\|^{p-1}_{\ell^{\alpha_2}_p L^p_t L^p_r L^2_\omega}\Big) \\ &\qquad\qquad \times \|\psi_R(v_{j-1} - v_{j-2})\|_{\ell^{\alpha_2}_p L^p_t L^p_r L^2_\omega} \\ &\quad \lesssim (4C_2\varepsilon)^{p-1} M_0(u_{j-1} - u_{j-2}, v_{j-1} - v_{j-2}). \end{aligned} \tag{3.19}$$

A similar argument yields

$$\|\psi_R^q(F_q(u_{j-1}) - F_q(u_{j-2})\|_{\ell_1^{q\alpha_1} L^1_t L^1_r L^2_\omega} \lesssim (4C_2\varepsilon)^{q-1} M_0(u_{j-1} - u_{j-2}, v_{j-1} - v_{j-2}). \tag{3.20}$$

We also start at (3.18) to control the $L^1_t L^2_r L^2_\omega$ terms. There we get, using (2.3) as above,

$$\begin{aligned} &\|F_p(v_{j-1}) - F_p(v_{j-2})\|_{L^1_t L^2_{r\ge 2R+1} L^2_\omega} \\ &\quad \lesssim (\|r^{-\frac{\alpha_2}{p-1}} v_{j-1}\|^{p-1}_{L^p_t L^{\frac{2p(p-1)}{p-2}}_{r\ge 2R+1} L^\infty_\omega} + \|r^{-\frac{\alpha_2}{p-1}} v_{j-2}\|^{p-1}_{L^p_t L^{\frac{2p(p-1)}{p-2}}_{r\ge 2R+1} L^\infty_\omega})\|r^{\alpha_2}\psi_R(v_{j-1} - v_{j-2})\|_{L^p_t L^p_r L^2_\omega} \\ &\quad \lesssim [(M_2(u_{j-1}, v_{j-1}))^{p-1} + (M_2(u_{j-2}, v_{j-2}))^{p-1}] M_0(u_{j-1} - u_{j-2}, v_{j-1} - v_{j-2}) \\ &\quad \lesssim \varepsilon^{p-1} M_0(u_{j-1} - u_{j-2}, v_{j-1} - v_{j-2}). \end{aligned} \tag{3.21}$$

And similarly

$$\|F_q(u_{j-1}) - F_q(u_{j-2})\|_{L^1_t L^2_{r\geq 2R+1} L^2_\omega} \lesssim \varepsilon^{q-1} M_0(u_{j-1}-u_{j-2}, v_{j-1}-v_{j-2}). \tag{3.22}$$

Finally, using Sobolev embeddings as in (3.13),

$$\begin{aligned}
&\|F_p(v_{j-1}) - F_p(v_{j-2})\|_{L^1_t L^2_{r\leq 2R+1} L^2_\omega} \\
&\quad \lesssim \big[\|v_{j-1}\|^{p-2}_{L^\infty_t L^\infty_r L^\infty_\omega} \|v_{j-1}\|_{L^2_t L^\infty_{r\leq 2R+1} L^\infty_\omega} + \|v_{j-2}\|^{p-2}_{L^\infty_t L^\infty_r L^\infty_\omega} \|v_{j-2}\|_{L^2_t L^\infty_{r\leq 2R+1} L^\infty_\omega}\big] \\
&\qquad\qquad\qquad\qquad \times \|v_{j-1}-v_{j-2}\|_{L^2_t L^2_{r\leq 2R+1} L^2_\omega} \\
&\quad \lesssim \big[(M_2(u_{j-1}, v_{j-1}))^{p-1} + (M_2(u_{j-2}, v_{j-2}))^{p-1}\big] M_0(u_{j-1}-u_{j-2}, v_{j-1}-v_{j-2}) \\
&\quad \lesssim \varepsilon^{p-1} M_0(u_{j-1}-u_{j-2}, v_{j-1}-v_{j-2}),
\end{aligned} \tag{3.23}$$

and by the same procedures

$$\|F_q(u_{j-1}) - F_q(u_{j-2})\|_{L^1_t L^2_{r\leq 2R+1} L^2_\omega} \lesssim \varepsilon^{q-1} M_0(u_{j-1}-u_{j-2}, v_{j-1}-v_{j-2}). \tag{3.24}$$

Plugging (3.19)-(3.24) into (3.16) and (3.17) immediately yields (3.15) provided that ε is sufficiently small, and this completes the proof.

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