Systems of interacting diffusions with partial annihilation through membranes *

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Abstract

We introduce an interacting particle system in which two families of reflected diffusions interact in a singular manner near a deterministic interface I. This system can be used to model the transport of positive and negative charges in a solar cell or the population dynamics of two segregated species under competition. A related interacting random walk model with discrete state spaces has recently been introduced and studied in [9]. In this paper, we establish the functional law of large numbers for this new system, thereby extending the hydrodynamic limit in [9] to reflected diffusions in domains with mixed-type boundary conditions, which include absorption (harvest of electric charges). We employ a new and direct approach that avoids going through the delicate BBGKY hierarchy.

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1 Introduction

With motivation to model and analyze the transport of positive and negative charges in solar cells, an interacting random walk model in domains has recently been introduced in [9]. In that model, a bounded domain in $\mathbb{R}^d$ is divided into two adjacent sub-domains D_+ and D_- by an interface I. The subdomains D_+ and D_- represent the hybrid medias which confine the positive and the negative charges, respectively. At microscopic level, positive and negative charges are modeled by independent continuous time random walks on lattices inside D_+ and D_-. These two types of particles annihilate each other at a certain rate when they come close to each other near the interface I. This interaction models the annihilation, trapping, recombination and separation phenomena of the charges. Such a stochastic system can also model population dynamics of two segregated species under competition near their boarder. Under an appropriate scaling of the lattice size, the speed of the random walks and the annihilation rate, we proved in [9] that the hydrodynamic limit is described by a system of nonlinear heat equations that are

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coupled on the interface and satisfy Neumann boundary condition at the remaining part of the boundary.

While the random walk model in [9] is more amenable to computer simulation, it is subject to technical restrictions associated with the discrete approximations of both the diffusions performed by the particles and the underlying domains $D_{\pm}$. Furthermore, that model does not consider harvest of charges.

In this paper, a new continuous state stochastic model is introduced and investigated. This model is different from that of [9] in three ways: the particles perform reflected diffusions on continuous state spaces rather than random walks over discrete state spaces, particles are absorbed (harvested) at some regions (harvest sites) away from the interface I, and the annihilation mechanism near I is different. The model in this paper allows more flexibility in modeling the underlying spatial motions performed by the particles and in the study of their various properties. In particular, it is more convenient to work with when we study the fluctuation limit (or, functional central limit theorem) of the interacting diffusion system, which is the subject of an on-going project [10].

Here is a heuristic description of our new model (See figure 1): Let $D_{\pm}$ and I be as above.

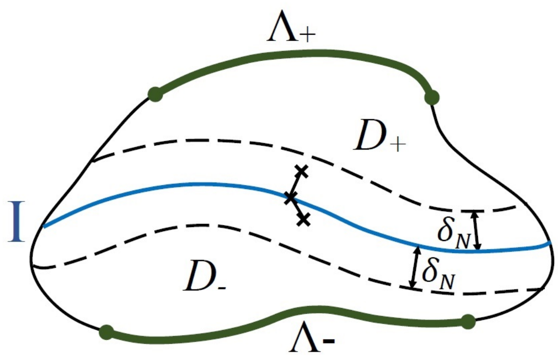

Figure 1: I = Interface, $\Lambda_{\pm}$ = Harvest sites

There is a harvest region $\Lambda_{\pm} \subset \partial D_{\pm} \setminus I$ that absorbs (harvests) $\pm$ charges, respectively, whenever it is being visited. Let N be the (common) initial number of particles in each of D_+ and D_-. For simplicity, we assume here that each particle in $D_{\pm}$ performs a Brownian motion with drift in the interior of $D_{\pm}$. These random motions model the transport of positive (respectively, negative) charges under an electric potential. When a particle hits the boundary, it is absorbed (harvested) on $\Lambda_{\pm}$, and is instantaneously reflected on $\partial D_{\pm} \setminus \Lambda_{\pm}$ along the inward normal direction of $D_{\pm}$. In other words, we assume that each particle in $D_{\pm}$ performs a reflected Brownian motions (RBM) with drift in $D_{\pm}$ that is killed upon hitting $\Lambda_{\pm}$. In addition, a pair of particles of opposite signs has a chance of being annihilated with each other when they are near I. Actually, when two particles of different types come within a small distance δ_N (which must occur near the interface I), they disappear with intensity $\frac{\lambda}{N\delta_N^{d+1}}$. Here $\lambda > 0$ is a given parameter modeling the rate of annihilation.

The choice of the scaling $\frac{\lambda}{N\delta_N^{d+1}}$ for the per-pair annihilation intensity is to guarantee that, in the limit $N \to \infty$, a non trivial proportion of particles is killed during the time interval $[0, t]$. Here is the heuristic reasoning. Since diffusive particles typically spread out in space, the number of pairs near the interface is of order $N^2\,\delta_N^{d+1}$ (because there are $N\delta_N$ number of particles in D_+ near I, and each of them is near to $N\delta_N^d$ number of particles in D_-). With the above choice of per-pair annihilation intensity, the expected number of pairs killed within t units of time is about $(N^2\,\delta_N^{d+1})\,(\frac{\lambda}{N\delta_N^{d+1}}\,t) = \lambda N t$ when $t>0$ is small. This implies that a non trivial proportion of particle is annihilated during $[0,t]$ and accounts for the boundary term in the hydrodynamic limit.

1.1 Main result and applications

We consider the normalized empirical measures

$$\mathfrak{X}_t^{N,+}(dx) := \frac{1}{N}\sum_{\alpha:\alpha\sim t} \mathbf{1}_{X_\alpha^+(t)}(dx) \quad\text{and}\quad \mathfrak{X}_t^{N,-}(dy) := \frac{1}{N}\sum_{\beta:\beta\sim t} \mathbf{1}_{X_\beta^-(t)}(dy).$$

Here $\mathbf{1}_y(dx)$ stands for the Dirac measure concentrated at the point y, while $\alpha\sim t$ (resp. $\beta\sim t$) denotes the condition that particle X_α^+ (resp. X_β^-) is alive at time t.

Our main result (Theorem 5.2) implies the following: Suppose each particle in $D_\pm$ is a RBM with gradient drift $\frac{1}{2}\nabla(\log\rho_\pm)$, where $\rho_\pm$ is a strictly positive function on $\overline{D}_\pm$. Suppose δ_N tends to zero and $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$. If $(\mathfrak{X}_0^{N,+},\,\mathfrak{X}_0^{N,-})$ converges in distribution, then the random measures $(\mathfrak{X}_t^{N,+},\,\mathfrak{X}_t^{N,-})$ converge in distribution to a deterministic limit $(u_+(t,x)\rho_+(x)dx,\,u_-(t,y)\rho_-(y)dy)$ for any $t>0$, where $(u_+,\,u_-)$ is the unique solution of the coupled heat equations

$$\begin{cases} \dfrac{\partial u_+}{\partial t} = \dfrac{1}{2}\Delta u_+ + \dfrac{1}{2}\nabla(\log\rho_+)\cdot\nabla u_+ & \text{on } (0,\infty)\times D_+ \\ u_+ = 0 & \text{on } (0,\infty)\times \Lambda_+ \\ \dfrac{\partial u_+}{\partial \vec{n}_+} = \dfrac{\lambda}{\rho_+}\, u_+u_-\,\mathbf{1}_{\{I\}} & \text{on } (0,\infty)\times \partial D_+\setminus\Lambda_+ \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dfrac{\partial u_-}{\partial t} = \dfrac{1}{2}\Delta u_- + \dfrac{1}{2}\nabla(\log\rho_-)\cdot\nabla u_- & \text{on } (0,\infty)\times D_- \\ u_- = 0 & \text{on } (0,\infty)\times \Lambda_- \\ \dfrac{\partial u_-}{\partial \vec{n}_-} = \dfrac{\lambda}{\rho_-}\, u_+u_-\,\mathbf{1}_{\{I\}} & \text{on } (0,\infty)\times \partial D_-\setminus\Lambda_-, \end{cases} \tag{1.2}$$

where $\vec{n_\pm}$ is the inward unit normal vector field on $\partial D_\pm$ of $D_\pm$ and $\mathbf{1}_{\{I\}}$ is the indicator function on I. Note that $\rho_\pm = 1$ corresponds to the particular case when there is no drift.

Remark 1.1. *Generalizations and Applications:* Actually, Theorem 5.2 is general enough to deal with any general symmetric reflected diffusions and covers the case when the constant λ is replaced by any continuous function $\lambda(x)$ on I. It is routine to generalize to any continuous time-dependent function $\lambda(t,x)$ and the details are left to the readers. Moreover, it is likely that a further generalization to tackle multiple deletion of particles near the interface (similar to that in [17]) can be done in an analogous way. As an immediate application of Theorem 5.2, we obtain an analytic formula for the asymptotic mass of positive charges harvested during the time interval $[0,T]$, which is

$$1-\int_{D_+} u_+(T,x)\rho_+(x)\,dx - \lambda\int_0^T\int_I u_+(s,z)u_-(s,z)\,d\sigma(z)\,ds.$$

Remark 1.2. The condition $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$ is an upper bound for the rate at which the annihilations distance δ_N tends to 0. Such kind of condition is necessary by the following reason: The dimension of I is $d+1$ lower than that of $D_+ \times D_-$. So we can choose δ_N small enough so that particles of different types cannot 'see' each other in the limit $N\to\infty$, resulting a decoupled linear system of PDEs with Dirichlet boundary condition on $\Lambda_\pm$ and Neumann boundary condition on $\partial D_\pm \setminus \Lambda_\pm$. See Example 5.3 for a rigorous statement and proof.

1.2 Key ideas

Theorem 3.2.39 of [22] from geometric theory asserts that

$$\lim_{\delta\to 0} \frac{\mathcal{H}^{2d}(I^\delta)}{c_{d+1}\,\delta^{d+1}} = \mathcal{H}^{d-1}(I), \tag{1.3}$$

where $I^\delta := \{(x,y)\in D_+\times D_- : |x-z|^2+|y-z|^2<\delta^2 \text{ for some } z\in I\}$, c_{d+1} is the volume of the unit ball in $\mathbb{R}^{d+1}$, and $\mathcal{H}^m$ is the m-dimensional Hausdorff measure. In Lemma 7.2, we strengthen it to

$$\lim_{\delta\to 0} \frac{1}{c_{d+1}\,\delta^{d+1}} \int_{I^\delta} f(x,y)\,dxdy = \int_I f(z,z)\,d\mathcal{H}^{d-1}(z) \tag{1.4}$$

uniformly in f from any equi-continuous family in $C(\overline{D}_+\times\overline{D}_-)$. Property (1.4) leads us to the following key observation that

$$\lim_{\delta\to 0}\lim_{N\to\infty} \frac{1}{c_{d+1}\,\delta^{d+1}}\mathbb{E}\int_0^T \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}(I^\delta)\,ds = \lim_{N\to\infty}\lim_{\delta\to 0} \frac{1}{c_{d+1}\,\delta^{d+1}}\mathbb{E}\int_0^T \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}(I^\delta)\,ds.$$

This interchange of limit in turn allows us to characterize the mean of any subsequential limit of $(\mathfrak{X}^{N,+},\,\mathfrak{X}^{N,-})$ by comparing the integral equations (4.1) satisfied by the hydrodynamic limit with its stochastic counterpart (7.6) . Using a similar argument, we can identify the second moment of any subsequential limit, and hence characterize any subsequential limit of $(\mathfrak{X}^{N,+},\,\mathfrak{X}^{N,-})$. We point out here that $\frac{1}{c_{d+1}\,\delta^{d+1}}\int_0^t \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}(I^\delta)\,ds$ quantifies the amount of interaction among the two types of particles, and is related (but different from) the *collision local time* introduced in [20]. The direct approach developed in this paper to establish the hydrodynamic limit avoids going through the delicate BBGKY hierarchy as was done in [9].

1.3 Literature

Interacting diffusion systems have been studied by many authors and they continue to be the subject of active research. See [30] and [32] for such a system on a circle whose hydrodynamic limit is established using the entropy method. We also mention [16] for a recent large deviation result for a system of diffusions in $\mathbb{R}$ interacting through their ranks. This large deviation principle implies convergence of the system to the hydrodynamic limit. However, the methods in these papers do not seem to work (at least not in a direct way) for our annihilating diffusion model due to the singular interaction on the interface.

An extensively studied class of stochastic particle systems is *reaction-diffusion systems* (R-D systems in short), whose hydrodynamic limits are described by R-D equations $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + R(u)$, where $R(u)$ is a function in u which represents the reaction. R-D systems is an important class of interacting particle systems arising from various contexts. They were investigated by many authors in both the discrete setting (particles perform random walks) and the continuous setting (particles perform continuous diffusions). For instance, for the case $R(u)$ is a polynomial in u, these systems were studied in [17, 18, 28, 29] on a cube with Neumann boundary conditions, and in [3, 4] on a periodic lattice. See also [7] for a survey of a class of discrete (lattice) models called the *Polynomial Model* which contains the *Schlögl's model*. Recently, perturbations of the voter models which contain the *Lotka-Volterra systems* are considered in [13]. The authors showed that the hydrodynamic limits are R-D equations and established general conditions for the existence of non-trivial stationary measures and for extinction of the particles. Another stochastic particle systems which are related to our annihilation-diffusion model is the *Fleming-Viot* type systems ([5, 6, 24]). In [6], Burdzy and Quastel studied an annihilating-branching system of two families of random walks on a domain. In their model, when a pair of particles of different types meet, they annihilate each other and they are immediately reborn at a site chosen randomly from the remaining particles of the same type. So the total number of particles of each type remains constant over the time, and thus this Fleming-Viot type system is different from the annihilating random walk model of [9]. The hydrodynamic limit of the model in [6] is described by a linear heat equation with zero average temperature. An elegant result obtained by P. Dittrich [17] is on a system of reflected Brownian motion on the unit interval $[0,1]$ with multiple deletion of particles. More precisely, any k-tuples $(2 \leq k \leq n)$ of particles with distances between them of order ε, say $(x^{i_1}, \cdots, x^{i_k})$, disappear with intensity $c_k(k-1)!\varepsilon^{k-1} \int_{[0,1]} p(\varepsilon^2, x^{i_1}, y) \cdots p(\varepsilon^2, x^{i_k}, y)\, dy$, where $c_k > 0$ are constants and $p(t,x,y)$ is the transition density of the reflected Brownian motion on $[0,1]$. The hydrodynamic limit is a R-D equation with reaction term $R(u) = -\sum_{k=2}^n c_k u^k$ and Neumann boundary condition. In contrast to [17], our model has two types of particles instead of one. Moreover, the interaction of our model is singular near the boundary and gives rise to a boundary integral term in the hydrodynamic limit.

The rest of the paper is organized as follows. Preliminary materials on setup, reflecting diffusions, and notations are given in Section 2. A rigorous description of the interacting stochastic particle system we are going to study in this paper is presented in Section 3. In section 4, we give an existence and uniqueness result of solution of a coupled heat equation with non-linear boundary condition, analogous to [9, Proposition 2.19]. The full statement of our main result (Theorem 5.2) of this paper is given in section 5. Section 6 is devoted to the proof of Theorem 5.2. The proof of a key proposition that identifies the first and second moments of subsequential limits of empirical distributions is given in Section 7.

2 Preliminaries

2.1 Reflected diffusions killed upon hitting a closed set $\Lambda \subset \overline{D}$

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and

$$W^{1,2}(D) = \{f \in L^2(D; dx) : \nabla f \in L^2(D; dx)\}.$$

Consider the bilinear form on $W^{1,2}(D)$ defined by

$$\mathcal{E}(f,g) := \frac{1}{2} \int_D \nabla f(x) \cdot \mathbf{a} \nabla g(x)\, \rho(x)\, dx,$$

where $\rho \in W^{1,2}(D)$ is a positive function on D which is bounded away from zero and infinity, $\mathbf{a} = (a^{ij})$ is a symmetric bounded uniformly elliptic $d \times d$ matrix-valued function such that $a^{ij} \in W^{1,2}(D)$ for each i, j. Since D is Lipschitz boundary, $(\mathcal{E}, W^{1,2}(D))$ is a regular symmetric Dirichlet form on $L^2(D; \rho(x)dx)$ and hence has a unique (in law) associated ρ-symmetric strong Markov process X (cf. [8]).

Definition 2.1. *Let $(\boldsymbol{a}, \rho)$ and X be as in the preceding paragraph. We call X an* ***$(\boldsymbol{a}, \rho)$-reflected diffusion****. A special but important case is when $\boldsymbol{a}$ is the identity matrix, in which X is called a reflected Brownian motion with drift $\frac{1}{2}\nabla(\log \rho)$. If in addition $\rho = 1$, then X is called a reflected Brownian motion (RBM).*

Denote by $\vec{n}$ the unit inward normal vector of D on ∂D. The Skorokhod representation of X tells us (see [8]) that X behaves like a diffusion process associated to the elliptic operator

$$\mathcal{A} := \frac{1}{2\,\rho}\,\nabla \cdot (\rho\,\mathbf{a}\nabla) \tag{2.1}$$

in the interior of D, and is instantaneously reflected at the boundary in the inward conormal direction $\vec{\nu} := \mathbf{a}\vec{n}$. It is well known (cf. [2, 25] and the references therein) that X has a transition density $p(t, x, y)$ with respect to the symmetrizing measure $\rho(x)dx$ (i.e., $\mathbb{P}_x(X_t \in dy) = p(t, x, y)\,\rho(y)dy$ and $p(t, x, y) = p(t, y, x)$), that p is locally Hölder continuous and hence $p \in C((0, \infty) \times \overline{D} \times \overline{D})$, and that we have the followings: for any $T > 0$, there are constants $c_1 \geq 1$ and $c_2 \geq 1$ such that

$$\frac{1}{c_1 t^{d/2}}\,\exp\left(\frac{-c_2|y-x|^2}{t}\right) \leq p(t, x, y) \leq \frac{c_1}{t^{d/2}}\,\exp\left(\frac{-|y-x|^2}{c_2\,t}\right) \tag{2.2}$$

for every $(t, x, y) \in (0, T] \times \overline{D} \times \overline{D}$. Using (2.2) and the Lipschitz assumption for the boundary, we can check that

$$\sup_{x\in\overline{D}}\ \sup_{0<\delta\leq\delta_0}\ \frac{1}{\delta}\int_{D^\delta} p(t, x, y)\,dy \quad \leq \quad \frac{C_1}{\sqrt{t}} + C_2 \quad \text{for } t \in (0, T] \quad \text{and} \tag{2.3}$$

$$\sup_{x\in\overline{D}} \int_{\partial D} p(t, x, y)\,\sigma(dy) \quad \leq \quad \frac{C_1}{\sqrt{t}} + C_2 \quad \text{for } t \in (0, T], \tag{2.4}$$

where C_1, C_2, $\delta_0 > 0$ are constants which depends only on d, T, the Lipschitz characteristics of D, the ellipticity of $\mathbf{a}$ and the lower and upper bound of ρ. Here $D^\delta := \{x \in D : \mathrm{dist}(x, \partial D) < \delta\}$. In fact (2.4) follows from (2.3) via Lemma 7.1.

Now we consider an $(\mathbf{a}, \rho)$-reflected diffusion *killed upon hitting a closed subset* Λ of $\overline{D}$. In particular, Λ can be subset of ∂D (this is the case for $\Lambda_\pm$ in figure 1). Define

$$X_t^{(\Lambda)} := \begin{cases} X_t, & t < T_\Lambda \\ \partial, & t \geq T_\Lambda, \end{cases} \tag{2.5}$$

where ∂ is a cemetery point and $T_\Lambda := \inf\{t > 0 : X_t \in \Lambda\}$ is the first hitting time of X on Λ. Since $\overline{D} \setminus \Lambda$ is open in $\overline{D}$, Theorem A.2.10 of [23] asserts that $X^{(\Lambda)}$ is a Hunt process on $(\overline{D} \setminus \Lambda) \cup \partial$ with transition function $P_t^\Lambda(x, A) = \mathbb{P}^x(X_t \in A,\, t < T_\Lambda)$. The characterization of the Dirichlet form of $X^{(\Lambda)}$ can be found in [11, Theorem 3.3.8] or [23, Theorem 4.4.2]; in particular, it implies that the semigroup $\{P_t^\Lambda\}_{t\geq 0}$ of $X^{(\Lambda)}$ is symmetric and strongly continuous on $L^2(\overline{D} \setminus \Lambda, \rho(x)dx)$. Clearly, $X^{(\Lambda)}$ has a transition density $p^{(\Lambda)}$ with respect to $\rho(x)dx$ (i.e. $P_t^\Lambda(x, dy) = p^{(\Lambda)}(t, x, y)\, \rho(y)\, dy$). Note that $p^{(\Lambda)}(t, x, y) \le p(t, x, y)$ for all $x, y \in D$ and $t > 0$.

So far Λ is only assumed to be closed in $\overline{D}$. We will also need the following regularity assumption.

Definition 2.2. *$\Lambda \subset \overline{D}$ is said to be **regular** with respect to X if $\mathbb{P}^x(T_\Lambda = 0) = 1$ for all $x \in \Lambda$, where $T_\Lambda := \inf\{t > 0 : X_t \in \Lambda\}$.*

This regularity assumption implies that $p^{(\Lambda)}(t, x, y)$ is jointly continuous in x and y up to the boundary. In particular, $p^{(\Lambda)}(t, x, y)$ is continuous for x and y in a neighborhood of I. We now gather some basic properties of $p^{(\Lambda)}(t, x, y)$ for later use.

Proposition 2.3. *Let X be an $(\boldsymbol{a}, \rho)$-reflected diffusion defined in Definition 2.1, and $p^{(\Lambda)}(t, x, y)$ be the transition density, with respect to $\rho(x)dx$, of X^Λ defined in (2.5). Suppose Λ is closed and regular with respect to X. Then $p^{(\Lambda)}(t, x, y) \ge 0$ and $p^{(\Lambda)}(t, x, y) = p^{(\Lambda)}(t, y, x)$ for all $x, y \in \overline{D}$ and $t > 0$. Moreover, $p^{(\Lambda)}(t, x, y)$ can be extended to be jointly continuous on $(0, \infty) \times \overline{D} \times \overline{D}$. The last property implies that the semigroup $\{P_t^\Lambda\}_{t \ge 0}$ of X^Λ is strongly continuous on the Banach space $C_\infty(\overline{D} \setminus \Lambda) := \{f \in C(\overline{D}) : f \text{ vanishes on } \Lambda\}$ equipped with the uniform norm on $\overline{D}$. The domain of the Feller generator of $\{P_t^{(\Lambda)}\}_{t \ge 0}$, denoted by $Dom(\mathcal{A}^{(\Lambda)})$, is dense in $C_\infty(\overline{D} \setminus \Lambda)$.*

Proof Define, for all $(t, x, y) \in (0, \infty) \times \overline{D} \times \overline{D}$,

$$q^{(\Lambda)}(t, x, y) := p(t, x, y) - r(t, x, y), \text{ where } r(t, x, y) := \mathbb{E}^x\left[p(t - T_\Lambda, X_{T_\Lambda}, y);\, t \ge T_\Lambda\right].$$

Using the fact that $x \mapsto \mathbb{P}^x(T_\Lambda < t)$ is lower semi-continuous (cf. Proposition 1.10 in Chapter II of [1]), it is easy to check that if Λ is closed and regular, then

$$\lim_{n \to \infty} \mathbb{P}^{x_n}(T_\Lambda < t) = 1 \tag{2.6}$$

whenever $t > 0$ and $x_n \in D$ converges to a point in Λ. Recall that $p(t, x, y)$ is symmetric in (x, y), has two-sided Gaussian estimates (2.2), and is jointly continuous on $(0, \infty) \times \overline{D} \times \overline{D}$. Using these properties of p together with (2.6), then applying the same argument of section 4 of Chapter II in [1], we have

(a) $q^{(\Lambda)}(t, x, y)$ is a density for the transition function X^Λ.

(b) $q^{(\Lambda)}(t, x, y) \ge 0$ and $q^{(\Lambda)}(t, x, y) = q^{(\Lambda)}(t, y, x)$ for all $x, y \in \overline{D}$ and $t > 0$.

(c) $q^{(\Lambda)}(t, x, y)$ is jointly continuous on $(0, \infty) \times \overline{D} \times \overline{D}$.

From (c), the semigroup $\{P_t^{(\Lambda)}\}$ of $X^{(\Lambda)}$ is strongly continuous by a standard argument. $C_\infty(\overline{D} \setminus \Lambda)$ is a Banach space since it is a closed subspace of $C(\overline{D})$. The Feller generator $Dom(\mathcal{A}^{(\Lambda)})$ of $\{P_t^{(\Lambda)}\}$ is dense in $C_\infty(\overline{D} \setminus \Lambda)$ because any $f \in C_\infty(\overline{D} \setminus \Lambda)$ is the strong limit $\lim_{t \downarrow 0} \frac{1}{t} \int_0^t P_s^{(\Lambda)} f ds$ in $C_\infty(\overline{D} \setminus \Lambda)$, and $\int_0^t P_s^{(\Lambda)} f ds \in Dom(\mathcal{A}^{(\Lambda)})$. $\square$

2.2 Assumptions and notations

We now return to our annihilating diffusion system. Recall that before being annihilated by a particle of the opposite kind near I, a particle in $D_\pm$ performs a reflected diffusion with absorption on $\Lambda_\pm \subset \partial D_\pm \setminus I$. If a particle is absorbed (in $\Lambda_\pm$) rather than annihilated (near I), it is considered to be harvested.

The following assumptions are in force throughout this paper.

Assumption 2.4. *(Geometric setting) Suppose D_+ and D_- are given adjacent bounded Lipschitz domains in $\mathbb{R}^d$ such that $I := \overline{D}_+ \cap \overline{D}_- = \partial D_+ \cap \partial D_-$ is $\mathcal{H}^{d-1}$-rectifiable. $\Lambda_\pm$ is a closed subset of $\overline{D}_\pm \setminus I$ which is regular with respect to the $(\boldsymbol{a}_\pm, \rho_\pm)$-reflected diffusion $X^\pm$, where $\rho_\pm \in W^{(1,2)}(D_\pm) \cap C(\overline{D}_\pm)$ is a given strictly positive function, $\boldsymbol{a}_\pm = (a^{ij}_\pm)$ is a symmetric, bounded, uniformly elliptic $d \times d$ matrix-valued function such that $a^{ij}_\pm \in W^{1,2}(D_\pm)$ for each i, j.*

Assumption 2.5. *(Parameter of annihilation) Suppose $\lambda \in C_+(I)$ is a given non-negative continuous function on I. Let $\widehat{\lambda} \in C(\overline{D}_+ \times \overline{D}_-)$ be an arbitrary but fixed extension of λ in the sense that $\widehat{\lambda}(z,z) = \lambda(z)$ for all $z \in I$. (Such $\widehat{\lambda}$ always exists.)*

Assumption 2.6. *(The annihilation distance) $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$, where $\{\delta_N\} \subset (0,\infty)$ converges to 0 as $N \to \infty$.*

Assumption 2.7. *(The annihilation potential) We choose* ***annihilation potential functions*** *$\{\ell_\delta : \delta > 0\} \subset C_+(\overline{D}_+ \times \overline{D}_-)$ in such a way that $\ell_\delta \le \frac{\widehat{\lambda}}{c_{d+1}\,\delta^{d+1}} \mathbf{1}_{I^\delta}$ on $D_+ \times D_-$ and*

$$\lim_{\delta \to 0} \left\| \ell_\delta - \frac{\widehat{\lambda}}{c_{d+1}\,\delta^{d+1}} \mathbf{1}_{I^\delta} \right\|_{L^2(D_+ \times D_-)} = 0 \tag{2.7}$$

Assumption 2.7 is natural in view of (1.3). Intuitively, if N is the initial number of particles, then δ_N is the annihilation distance and I^{δ_N} controls the frequency of interactions. As remarked in the introduction, we need to assume that the annihilation distance δ_N does not shrink too fast. This is formulated in Assumption 2.6.

Convention: To simplify notation, we suppress $\Lambda_\pm$ and write $X^\pm$ in place of $X^{\Lambda_\pm}$ for a $(\mathbf{a}_\pm, \rho_\pm)$-reflected diffusions on $D_\pm$ killed upon hitting $\Lambda_\pm$. We also use $p^\pm(t,x,y)$, $P^\pm_t$ and $\mathcal{A}^\pm$ to denote, respectively, the transition density w.r.t. $\rho_\pm$, the semigroup associated to $p^\pm(t,x,y)$ and the $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$-generator (called the Feller generator) of $X^\pm = X^{\Lambda_\pm}$. Under Assumption 2.4, $X^\pm$ is a Hunt (hence strong Markov) process on

$$D^\partial_\pm := \left(\overline{D}_\pm \setminus \Lambda^\pm\right) \cup \{\partial^\pm\},$$

where $\partial^\pm$ is the cemetery point for $X^\pm$ (see Proposition 2.3).

For reader's convenience, we list other notations that we will adopt here:

$\mathcal{B}(E)$	Borel measurable functions on E
$\mathcal{B}_b(E)$	bounded Borel measurable functions on E
$\mathcal{B}^+(E)$	non-negative Borel measurable functions on E
$C(E)$	continuous functions on E
$C_b(E)$	bounded continuous functions on E
$C^+(E)$	non-negative continuous functions on E
$C_c(E)$	continuous functions on E with compact support
$D([0,\infty),\,E)$	space of *càdlàg* paths from $[0,\infty)$ to E equipped with the Skorokhod metric

$C_\infty(\overline{D}\setminus\Lambda)$	$\{f\in C(\overline{D}):\, f$ vanishes on $\Lambda\}$
$C_\infty^{(n,m)}$	$\big\{\Phi\in C(\overline{D}_+^n\times\overline{D}_-^m):\, \Phi$ vanishes outside $(\overline{D}_+\setminus\Lambda_+)^n\times(\overline{D}_-\setminus\Lambda_-)^m\big\}$, see Subsection 7.2
$\mathcal{H}^m$	m-dimensional Hausdorff measure
I^δ	$\{(x,y)\in D_+\times D_-:\, \lvert x-z\rvert^2+\lvert y-z\rvert^2<\delta^2$ for some $z\in I\}$,
c_{d+1}	the volume of the unit ball in $\mathbb{R}^{d+1}$
ℓ_δ	the annihilating potential functions in Assumption 2.7
$\underline{\mathbf{X}}_t^{(N)}$	the configuration process defined in Subsection 3.1
S_N	$\cup_{m=1}^N\big(D_+^\partial(m)\times D_-^\partial(m)\big)\cup\{\partial\}$, the state space of $(\underline{\mathbf{X}}_t^{(N)})_{t\geq 0}$
$(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})$	the normalized empirical measure defined in Subsection 3.2
E_N	$\cup_{M=1}^N E_N^{(M)}\cup\{\mathbf{0}_*\}$, the state space of $(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})_{t\geq 0}$
$M_+(E)$	space of finite non-negative Borel measures on E, with weak topology
$M_{\leq 1}(E)$	$\{\mu\in M_+(E):\, \mu(E)\leq 1\}$
$\mathfrak{M}$	$M_{\leq 1}(\overline{D}_+\setminus\Lambda_+)\times M_{\leq 1}(\overline{D}_-\setminus\Lambda_-)$, see Section 5
$\{\mathcal{F}_t^X:\, t\geq 0\}$	filtration induced by the process (X_t), i.e. $\mathcal{F}_t^X=\sigma(X_s,\, s\leq t)$
$\mathbf{1}_x$	indicator function at x or the Dirac measure at x (depending on the context)
$\xrightarrow{\mathcal{L}}$	convergence in law of random variables (or processes)
$\langle f,\, \mu\rangle$	$\int f(x)\,\mu(dx)$
$x\vee y$	$\max\{x,\, y\}$
$x\wedge y$	$\min\{x,\, y\}$

3 Annihilating diffusion system

In this section, we fix $N\in\mathbb{N}$ and construct the normalized empirical measure process $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ and the configuration process $\underline{\mathbf{X}}^{(N)}$ for our annihilating particle system. In the construction, we will label (rather than annihilate) pairs of particles to keep track of the annihilated particles. This provides a *coupling* of our annihilating particle system and the corresponding system without annihilation.

Let $m\in\{1,2,\cdots,N\}$ (in fact, m can be any positive integer). Starting with m points in each of D_+^∂ and D_-^∂, we perform the following construction:

Let $\{X_i^\pm=X_i^{\Lambda_\pm}\}_{i=1}^m$ be $(\mathbf{a}_\pm,\, \rho_\pm)$-reflected diffusions on $D_\pm$ killed upon hitting $\Lambda_\pm$, starting from the given points in $D_\pm^\partial$. These $2m$ processes are constructed to be mutually independent. In case $X_i^\pm$ starts at the cemetery point $\partial^\pm$, we have $X_i^\pm(t)=\partial^\pm$ for all $t\geq 0$. Let $\{R_k\}_{k=1}^m$ be i.i.d. exponential random variables with parameter one which are independent of $\{X_i^+\}_{i=1}^m$ and $\{X_i^-\}_{j=1}^m$.

Define the first time of labeling (or annihilation) to be

$$\tau_1:=\inf\left\{t\geq 0:\frac{1}{2N}\int_0^t\sum_{i=1}^m\sum_{j=1}^m\ell_{\delta_N}(X_i^+(s),X_j^-(s))\,ds\geq R_1\right\}. \tag{3.1}$$

In the above, $\ell_{\delta_N}(x,y)=0$ if either $x=\partial^+$ or $y=\partial^-$. Hence particles absorbed at $\Lambda_\pm$ do not contribute to rate of labeling (or annihilation). At τ_1, we label exactly one pair in $\{(i,j)\}$ according to the probability distribution given by

$$\frac{\ell_{\delta_N}(X_i^+(\tau_1-),X_j^-(\tau_1-))}{\sum_{p=1}^m\sum_{q=1}^m \ell_{\delta_N}(X_p^+(\tau_1-),X_q^-(\tau_1-))} \quad \text{assigned to } (i,j).$$

Denote (i_1,j_1) to be the labeled pair at τ_1 (think of the labeled pair as begin removed due to annihilation of the corresponding particles).

We repeat this labeling procedure using the remaining unlabeled $2(m-1)$ particles. Precisely, for $k=2,3,\cdots,m$, we define

$$\tau_k := \inf\left\{t\geq 0 : \frac{1}{2N}\int_{\tau_1+\cdots+\tau_{k-1}}^{\tau_1+\cdots+\tau_{k-1}+t} \sum_{i\notin\{i_1,\cdots,i_{l-1}\}}\ \sum_{j\notin\{j_1,\cdots,j_{l-1}\}} \ell_{\delta_N}(X_i^+(s),X_j^-(s))\,ds \geq R_k\right\}.$$

At $\sigma_k := \tau_1+\tau_2+\cdots+\tau_{k-1}+\tau_k$, the k-th time of labeling (annihilation), we label exactly one pair (i_k,j_k) in $\{\,(i,j) : i\notin\{i_1,\cdots,i_{k-1}\},\, j\notin\{j_1,\cdots,j_{k-1}\}\,\}$ according to the probability distribution given by

$$\frac{\ell_{\delta_N}(X_i^+(\sigma_k-),X_i^-(\sigma_k-))}{\sum_{i\notin\{i_1,\cdots,i_{k-1}\}}\sum_{j\notin\{j_1,\cdots,j_{k-1}\}} \ell_{\delta_N}(X_i^+(\sigma_k-),X_i^-(\sigma_k-))} \quad \text{assigned to } (i,j).$$

We will study the evolution of the *unlabeled (or surviving* particles, which is described in detail below.

3.1 The configuration process $\underline{\mathbf{X}}^{(N)}$

We denote $D_\pm^\partial(m)$ the space of unordered m-tuples of elements in $D_\pm^\partial := \left(\overline{D}_\pm\setminus\Lambda^\pm\right)\cup\{\partial^\pm\}$. The **configuration space** for the particles is defined as

$$S_N := \cup_{m=1}^N \left(D_+^\partial(m)\times D_-^\partial(m)\right)\cup\{\partial\}, \tag{3.2}$$

where ∂ is a cemetery point (different from $\partial^\pm$).

We define $\underline{\mathbf{X}}_t^{(N)}\in S_N$ to be the following unordered list of (the position of) *unlabeled* (surviving) particles at time t. That is,

$$\underline{\mathbf{X}}_t^{(N)} := \begin{cases} \left(\,\{X_1^+(t),\cdots,X_m^+(t)\},\ \{X_1^-(t),\cdots,X_m^-(t)\}\,\right), & \text{if } t\in[0,\,\sigma_1=\tau_1); \\ \left(\,\{X_i^+(t)\}_{i\notin\{i_1,\cdots,i_{k-1}\}},\ \{X_j^-(t)\}_{j\notin\{j_1,\cdots,j_{k-1}\}}\,\right), & \text{if } t\in[\sigma_{k-1},\,\sigma_k),\ \text{for } k=2,3,\cdots,m; \\ \partial, & \text{if } t\in[\sigma_m,\,\infty). \end{cases}$$

By definition, $\underline{\mathbf{X}}_t^{(N)}\in D_+^\partial(m-k+1)\times D_-^\partial(m-k+1)$ when $t\in[\sigma_{k-1},\,\sigma_k)$, and $\underline{\mathbf{X}}_t^{(N)}=\partial$ if and only if all particles are labeled (annihilated) at time t (in particular, none of them is absorbed at $\Lambda^\pm$). We call $\underline{\mathbf{X}}^{(N)}=(\underline{\mathbf{X}}_t^{(N)})_{t\geq 0}$ the **configuration process**.

Denote $(\Omega,\mathcal{F},\wp)$ the ambient probability space on which the above random objects $\{X_i^+\}_{i=1}^m$, $\{X_i^-\}_{j=1}^m$, $\{R_i\}_{i=1}^m$ and $\{(i_1,j_1),\cdots,(i_m,j_m)\}$ are defined. For any $z\in S_N$, we define $\mathbb{P}^z$ to be the conditional measure $\wp\left(\,\cdot\mid\underline{\mathbf{X}}_0^{(N)}=z\right)$. From the construction, we have

Proposition 3.1. $\{\underline{\mathbf{X}}^{(N)}\}$ *is a strong Markov processes under* $\{\mathbb{P}^z :\ z \in S_N\}$.

The key is to note that the choice of (i_k, j_k) depends only on the value of $\underline{\mathbf{X}}^{(N)}_{\sigma_k-}$, and that

$$\tau_{k+1} = \inf\{t \geq 0 :\ A^{(k)}_t > R_{k+1}\}, \quad \text{where } A^{(k)}_t = \frac{1}{2N}\int_{\sigma_k}^{\sigma_k+t} \sum_{i=1}\sum_{j=1} \ell_{\delta_N}(X^+_i(s), X^-_j(s))\, ds.$$

Hence $\underline{\mathbf{X}}^{(N)}$ is obtained through a patching procedure reminiscent to that of Ikeda, Nagasawa and Watanabe [26]. The proof is standard and is left to the reader.

3.2 The normalized empirical process $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$

Next, we consider $E_N := \cup_{M=1}^N E_N^{(M)} \cup \{\mathbf{0}_*\}$, where

$$E_N^{(M)} := \left\{ \left(\frac{1}{N}\sum_{i=1}^M \mathbf{1}_{x_i},\ \frac{1}{N}\sum_{j=1}^M \mathbf{1}_{y_j} \right) :\ x_i \in D_+^\partial,\ y_j \in D_-^\partial \right\}$$

and $\mathbf{0}_*$ is an abstract point isolated from $\cup_{M=1}^N E_N^{(M)}$. We define the **normalized empirical measure** $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ by

$$(\mathfrak{X}^{N,+}_t,\, \mathfrak{X}^{N,-}_t) := U_N(\underline{\mathbf{X}}^{(N)}_t), \tag{3.3}$$

where $U_N :\ S_N \to E_N$ is the canonical map given by $U_N(\partial) := \mathbf{0}_*$ and

$$U_N :\ (\underline{x}, \underline{y}) = (x_1, \cdots, x_m,\, y_1, \cdots, y_m) \mapsto \left(\frac{1}{N}\sum_{i=1}^m \mathbf{1}_{x_i},\ \frac{1}{N}\sum_{j=1}^m \mathbf{1}_{y_j} \right)$$

For comparison, we also consider the empirical measure for the independent reflected diffusions *without* annihilation:

$$(\overline{\mathfrak{X}}^{N,+},\, \overline{\mathfrak{X}}^{N,-}) := \left(\frac{1}{N}\sum_{i=1}^m \mathbf{1}_{X^+_i(t)},\ \frac{1}{N}\sum_{j=1}^m \mathbf{1}_{X^-_j(t)} \right). \tag{3.4}$$

For any $\mu \in E_N$, we define $\mathbb{P}^\mu$ to be the conditional measure $\wp\big(\ \cdot \mid (\mathfrak{X}^{N,+}_0,\, \mathfrak{X}^{N,-}_0) = \mu\big)$. From Proposition 3.2, we have

Proposition 3.2. $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ *is a strong Markov processes under* $\{\mathbb{P}^\mu :\ \mu \in E_N\}$.

4 Coupled heat equation with non-linear boundary condition

Denote by $C_\infty([0,T]; \overline{D}\setminus\Lambda)$ the space of continuous functions on $[0,T]$ taking values in $C_\infty(\overline{D}\setminus\Lambda) := \{f \in C(\overline{D}) :\ f \text{ vanishes on } \Lambda\}$. We equip the Banach space $C_\infty([0,T]; \overline{D}_+\setminus\Lambda_+) \times C_\infty([0,T]; \overline{D}_-\setminus\Lambda_-)$ with norm $\|(u,v)\| := \|u\|_\infty + \|v\|_\infty$, where $\|\cdot\|_\infty$ is the uniform norm. Using a probabilistic representation and the Banach fixed point theorem in the same way as we did in the proof of the existence and uniqueness result for the PDE in [9, Propostion 2.19], we have the following:

Proposition 4.1. *Let $T > 0$ and $u_0^\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$. Then there is a unique element $(u_+, u_-) \in C_\infty([0,T]; \overline{D}_+ \setminus \Lambda_+) \times C_\infty([0,T]; \overline{D}_- \setminus \Lambda_-)$ that satisfies the coupled integral equation*

$$\begin{cases} u_+(t,x) = P_t^{\Lambda^+} u_0^+(x) - \frac{1}{2}\int_0^t \int_I p^{\Lambda^+}(t-r,x,z)[\lambda(z)u_+(r,z)u_-(r,z)]d\sigma(z)\,dr \\ u_-(t,y) = P_t^{\Lambda^-} u_0^-(y) - \frac{1}{2}\int_0^t \int_I p^{\Lambda^-}(t-r,y,z)[\lambda(z)u_+(r,z)u_-(r,z)]d\sigma(z)\,dr. \end{cases} \tag{4.1}$$

Moreover, (u_+, u_-) satisfies

$$\begin{cases} u_+(t,x) = \mathbb{E}^x\Big[u_0^+(X_t^{\Lambda^+}) \exp\Big(- \displaystyle\int_0^t (\lambda \cdot u_-)(t-s,\, X_s^{\Lambda^+})\, dL_s^{I,+} \Big) \Big] \\ u_-(t,y) = \mathbb{E}^y\Big[u_0^-(X_t^{\Lambda^-}) \exp\Big(- \displaystyle\int_0^t (\lambda \cdot u_+)(t-s,\, X_s^{\Lambda^-})\, dL_s^{I,-} \Big) \Big], \end{cases} \tag{4.2}$$

where $L^{I,\pm}$ is the boundary local time of $X^{\Lambda^\pm}$ on the interface I, i.e. the positive continuous additive functional having Revuz measure $\sigma|_I$, the surface measure σ restricted to I.

It can be shown that continuous functions (u_+, u_-) satisfying (4.1) is weakly differentiable and satisfies the following PDE (4.3)-(4.4) in the distributional sense.

Definition 4.2. *We call the unique solution $(u_+,\, u_-) \in C_\infty([0,T]; \overline{D}_+ \setminus \Lambda_+) \times C_\infty([0,T]; \overline{D}_- \setminus \Lambda_-)$ of (4.1) the* ***weak solution*** *to the following coupled PDEs starting from (u_0^+, u_0^-):*

$$\begin{cases} \dfrac{\partial u_+}{\partial t} = \mathcal{A}^+ u_+ & \text{on } (0,\infty) \times D_+ \\ u_+ = 0 & \text{on } (0,\infty) \times \Lambda_+ \\ \dfrac{\partial u_+}{\partial \vec{\nu_+}} = \dfrac{\lambda}{\rho_+}\, u_+ u_-\, \mathbf{1}_{\{I\}} & \text{on } (0,\infty) \times \partial D_+ \setminus \Lambda_+ \end{cases} \tag{4.3}$$

and

$$\begin{cases} \dfrac{\partial u_-}{\partial t} = \mathcal{A}^- u_- & \text{on } (0,\infty) \times D_- \\ u_- = 0 & \text{on } (0,\infty) \times \Lambda_- \\ \dfrac{\partial u_-}{\partial \vec{\nu_-}} = \dfrac{\lambda}{\rho_-}\, u_+ u_-\, \mathbf{1}_{\{I\}} & \text{on } (0,\infty) \times \partial D_- \setminus \Lambda_-, \end{cases} \tag{4.4}$$

where $\vec{\nu_\pm} := \boldsymbol{a}_\pm \vec{n}_\pm$ is the inward conormal vector field on $\partial D_\pm$. Here $\mathbf{1}_{\{I\}}$ is the indicator function of I.

5 Main result: rigorous statement

Denote by $M_{\le 1}(\overline{D}_\pm \setminus \Lambda_\pm)$ the space of non-negative Borel measures on $\overline{D}_\pm \setminus \Lambda_\pm$ with mass at most 1 and set

$$\mathfrak{M} := M_{\le 1}(\overline{D}_+ \setminus \Lambda_+) \times M_{\le 1}(\overline{D}_- \setminus \Lambda_-),$$

equipped with the topology of weak convergence.

Remark 5.1. $\mathfrak{M}$ is in fact a Polish space. Let $\{f_n; n \ge 1\}$ and $\{g_n; n \ge 1\}$ be sequences of continuous functions with $|f_n| \le 1$ and $|g_n| \le 1$ whose linear span are dense in $C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $C_\infty(\overline{D}_- \setminus \Lambda_-)$, respectively. For $\mu = (\mu_+, \mu_-)$ and $\nu = (\nu_+, \nu_-)$ in $\mathfrak{M}$, define

$$\varrho(\mu,\nu) := \sum_{n=1}^\infty 2^{-n} \left(\left| \int_{\overline{D}_+} f_n(x)(\mu_+ - \nu_+)(dx) \right| + \left| \int_{\overline{D}_-} g_n(y)(\mu_- - \nu_-)(dy) \right| \right).$$

It is well known that $\mathfrak{M}$ is a complete separable metric space under the metric ϱ. $\square$

Regard $\mathbf{1}_{\partial^\pm}$ as $\mathbf{0}^\pm$ and $\mathbf{0}_*$ as $(\mathbf{0}^+, \mathbf{0}^-)$, where $\mathbf{0}^\pm$ is the zero measure on $\overline{D}_\pm$, respectively. Clearly, $E_N \subset \mathfrak{M}$ for all N, and the processes $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$ have sample paths in $D([0,\infty),\, \mathfrak{M})$, the Skorokhod space of *càdlàg* paths in $\mathfrak{M}$.

We can now rigorously state our main result. In what follows, $\xrightarrow{\mathcal{L}}$ denotes convergence in law.

Theorem 5.2. *(Hydrodynamic Limit) Suppose that Assumptions 2.4 to 2.7 hold. If as $N \to \infty$, $(\mathfrak{X}_0^{N,+},\, \mathfrak{X}_0^{N,-}) \xrightarrow{\mathcal{L}} (u_+^0(x)\rho_+(x)dx,\, u_-^0(y)\rho_-(y)dy)$ in $\mathfrak{M}$, where $u_\pm^0 \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$, then*

$$(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-}) \xrightarrow{\mathcal{L}} (u_+(t,x)\rho_+(x)dx,\, u_-(t,y)\rho_-(y)dy) \quad \textit{in } D([0,T], \mathfrak{M})$$

for any $T > 0$, where $(u_+,\, u_-)$ is the unique weak solution of (4.3)-(4.4) *with initial value (u_+^0, u_-^0).*

As mentioned in Remark 1.2 in the introduction, an assumption on the rate at which δ_N tends to zero, such as Assumption 2.6, is *necessary* for Theorem 5.2 to hold. Below is a counter-example.

Example 5.3. Suppose that $\{X_i^+(t)\}_{i=1}^\infty$ and $\{X_j^-(t)\}_{j=1}^\infty$ are RBMs on $\overline{D}_+$ and $\overline{D}_-$, respectively, and they are all mutually independent. Note that X_i^+ and X_j^- never meet in the sense that

$$\mathbb{P}\left(X_i^+(t) = X_j^-(t) \text{ for some } t \in [0,\infty) \text{ and } i,\, j \in \{1,2,3,\cdots\} \right) = 0. \tag{5.1}$$

This implies that there exists $\{\delta_N\}$ so that $\sum_{N=1}^\infty \alpha_N < \infty$, where

$$\alpha_N := \mathbb{P}\left((X_i^+(t),\, X_j^-(t)) \in I^{\delta_N} \text{ for some } t \in [0,\infty) \text{ and } i,\, j \in \{1,2,\cdots,N\} \right). \tag{5.2}$$

Hence by Borel-Cantelli lemma, we know that with probability 1, there will be no annihilation for the particle system (which occurs only when a pair of particles are in I^{δ_N}) when N is sufficiently large. In this case, $(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})$ converges to $(P_t^+ u_0^+(x)dx,\, P_t^- u_0^-(y)dy)$ in distribution in $D([0,T], \mathfrak{M})$ instead, provided that $(\mathfrak{X}_0^{N,+},\, \mathfrak{X}_0^{N,-})$ converges to $(u_0^+(x)dx,\, u_0^-(y)dy)$ in distribution in $\mathfrak{M}$.

Question. We will see from Theorem 6.6 below that the tightness of $(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})$ holds without Assumption 2.6. Can we characterize all limit points of $(\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-})$ without Assumption 2.6? Is $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$ the sharpest condition for Theorem 5.2 to hold?

6 Hydrodynamic limit

Recall that Assumptions 2.4 to 2.7 are in force throughout this paper.

6.1 Martingales and tightness

In this subsection, we present some key martingales that are used to establish tightness of $(\mathfrak{X}^{N,+},\, \mathfrak{X}^{N,-})$. More martingales related to the time dependent process $(t,\, (\mathfrak{X}_t^{N,+},\, \mathfrak{X}_t^{N,-}))$ will be given in subsection 7.2.

6.1.1 Martingales for reflected diffusions

We will need the following collection of fundamental martingales, together with their quadratic variations, for reflected diffusions.

Lemma 6.1. *Suppose X^Λ is an $(\boldsymbol{a},\,\rho)$-reflected diffusion in a bounded Lipschitz domain D killed upon hitting Λ. Suppose all assumptions in Proposition 2.3 hold, and f is in the domain of the Feller generator $Dom(\mathcal{A}^{(\Lambda)})$. Then we have*

$$M(t) := f(X^\Lambda(t)) - f(X^\Lambda(0)) - \int_0^t \mathcal{A}^{(\Lambda)} f(X^\Lambda(s))\, ds \tag{6.1}$$

is an $\mathcal{F}_t^{X^\Lambda}$-martingale that is bounded on each compact time interval and has quadratic variation $\int_0^t (\boldsymbol{a}\nabla f\cdot\nabla f)\,(X^\Lambda(s))\,ds$ under $\mathbb{P}^x$ for any $x\in\overline{D}\setminus\Lambda$. Moreover, if X_1 and X_2 are independent copies of X^Λ, and if M_i is the above M with X^Λ replaced by X_i, then the cross variation $\langle M_1,\,M_2\rangle_t = 0$.

Proof For $f\in Dom(\mathcal{A}^{(\Lambda)})$, $M(t)$ defined in (6.1) is an $\mathcal{F}_t^{X^\Lambda}$-martingale that is bounded on each compact time interval. Since D is bounded, f is clearly in the domain of the L^2-generator of X^Λ. Hence it follows from the Fukushima decomposition of $f(X_t^\Lambda)$ (see [11, Theorems 4.2.6 and 4.3.11] that $M(t)$ is a martingale additive functional of X^Λ of finite energy having quadratic variation $\langle M(t)\rangle_t = \int_0^t(\mathbf{a}\nabla f\cdot\nabla f)(X^\Lambda(s))ds$. If X_1 and X_2 are independent copies of X^Λ, then M_1 and M_2 are independent and so $\langle M_1, M_2\rangle = 0$. □

An immediate consequence of Lemma 6.1 is

$$\int_0^t P_s^\Lambda(\mathbf{a}\nabla f\cdot\nabla f)(x)\,ds = \mathbb{E}^x[M(t)^2] \le 8(\|f\|^2 + \|\mathcal{A}^{(\Lambda)} f\|^2\, t^2) \quad \text{for } x\in\overline{D}, \tag{6.2}$$

where $\|g\|$ is the uniform norm of g on $\overline{D}$.

6.1.2 Martingales for annihilating diffusion system

Theorem 6.2. *Fix any positive integer N. Suppose $F\in C_b(E_N)$ is a bounded continuous function and $G\in\mathcal{B}(E_N)$ is a Borel measurable function on E_N such that*

$$\overline{M}_t := F(\overline{\mathfrak{X}}_t^{N,+}, \overline{\mathfrak{X}}_t^{N,-}) - \int_0^t G(\overline{\mathfrak{X}}_s^{N,+}, \overline{\mathfrak{X}}_s^{N,-})\, ds$$

is an $\mathcal{F}_t^{(\overline{\mathfrak{X}}^{N,+},\overline{\mathfrak{X}}^{N,-})}$-martingale under $\mathbb{P}^\mu$ for any $\mu\in E_N$. Then

$$M_t := F(\mathfrak{X}_t^{N,+}, \mathfrak{X}_t^{N,-}) - \int_0^t (G+KF)(\mathfrak{X}_s^{N,+}, \mathfrak{X}_s^{N,-})\, ds$$

is a $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$-martingale under $\mathbb{P}^\mu$ for any $\mu\in E_N$, where

$$KF(\nu) := -\frac{1}{2N}\sum_{i=1}^M\sum_{j=1}^M \ell_{\delta_N}(x_i,y_j)\left(F(\nu) \,-\, F\big(\nu^+ - N^{-1}\mathbf{1}_{\{x_i\}},\, \nu^- - N^{-1}\mathbf{1}_{\{y_j\}}\big)\right) \tag{6.3}$$

whenever $\nu = \left(\frac{1}{N}\sum_{i=1}^M \mathbf{1}_{\{x_i\}},\, \frac{1}{N}\sum_{j=1}^M \mathbf{1}_{\{y_j\}}\right)\in E_N^{(M)}$, and $KF(\boldsymbol{0}_*) := 0$.

Remark 6.3. (i) Theorem 6.2 indicates the infinitesimal generator of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ on $C_b(E_N)$ is given by $\overline{L} + K$, where $\overline{L}$ is the infinitesimal generator of $(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})$ on $C_b(E_N)$. Note that G is merely assumed to be Borel measurable, the above provides us with a broader class of martingales (such as $N_t^{(\phi_+,\phi_-)}$ in Corollary 6.4) than from using the $C_b(E_N)$-generator.

(ii) Theorem 6.2 can be generalized to deal with time-dependent functions $F_s \in C_b(E_N)$ $(s \geq 0)$. See Theorem 7.6 in subsection 7.2. $\square$

Proof of Theorem 6.2. We adopt the abbreviation $\mathfrak{X} := (\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ when there is no confusion. In particular, we write $\mathcal{F}_t^{\mathfrak{X}}$ in place of $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$. By Markov property for $\mathfrak{X}$, it suffices to show that for all $t \geq 0$ and $\nu \in E_N$,

$$\mathbb{E}^\nu\Big[F(\mathfrak{X}_t) - F(\mathfrak{X}_0) - \int_0^t (G + KF)(\mathfrak{X}_s)\,ds\Big] = 0. \tag{6.4}$$

The idea is to spit the time interval $[0,t]$ into pieces according to the jumping times of $F(\mathfrak{X}_s)$ ($s \in [0,t]$) caused by annihilation (excluding the jumps caused by absorbtion at the harvest sites $\Lambda^{\pm}$), then apply $\overline{M}$ in each piece and take into account the jump distributions.

Suppose $\nu = (\nu^+, \nu^-) = (\frac{1}{N}\sum_{i=1}^m \mathbf{1}_{x_i}, \frac{1}{N}\sum_{j=1}^m \mathbf{1}_{y_j}) \in E_N^{(m)}$. Recall that $\sigma_i := \tau_1 + \cdots \tau_i$ $(i = 1, 2, \cdots, m)$ is the time of the i-th labeling (annihilation) of particles. write

$$F(\mathfrak{X}_t) - F(\mathfrak{X}_0) = \sum_{i=0}^m \Big(F(\mathfrak{X}_{(t\wedge\sigma_{i+1})-}) - F(\mathfrak{X}_{t\wedge\sigma_i})\Big) + \sum_{j=1}^m \Big(F(\mathfrak{X}_{t\wedge\sigma_j}) - F(\mathfrak{X}_{(t\wedge\sigma_j)-})\Big), \tag{6.5}$$

where $\sigma_0 := 0$, $\sigma_{m+1} := \infty$ and $\mathfrak{X}_{s-} := \lim_{r\nearrow s}\mathfrak{X}_r$. Hence it suffices to show that

$$\mathbb{E}^\nu\Big[F(\mathfrak{X}_{(t\wedge\sigma_{i+1})-}) - F(\mathfrak{X}_{t\wedge\sigma_i}) - \int_{t\wedge\sigma_i}^{t\wedge\sigma_{i+1}} G(\mathfrak{X}_s)\,ds\Big] = 0 \quad \text{and} \tag{6.6}$$

$$\mathbb{E}^\nu\Big[F(\mathfrak{X}_{t\wedge\sigma_j}) - F(\mathfrak{X}_{(t\wedge\sigma_j)-}) - \int_{t\wedge\sigma_{j-1}}^{t\wedge\sigma_j} KF(\mathfrak{X}_s)\,ds\Big] = 0 \tag{6.7}$$

for $i \in \{0, 1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots, m\}$.

The left hand side of (6.6) equals

$$\begin{aligned}
&\mathbb{E}^\nu\left[\mathbb{E}^\nu\Big[F(\mathfrak{X}_{(t\wedge\sigma_{i+1})-}) - F(\mathfrak{X}_{t\wedge\sigma_i}) - \int_{t\wedge\sigma_i}^{t\wedge\sigma_{i+1}} G(\mathfrak{X}_s)\,ds\Big|\,\mathcal{F}^{\mathfrak{X}}_{t\wedge\sigma_i}\Big]\right] \\
= \;&\mathbb{E}^\nu\left[\mathbb{E}^{\mathfrak{X}_{\sigma_i}}\Big[F(\mathfrak{X}_{(t\wedge\sigma_{i+1}-\sigma_i)-}) - F(\mathfrak{X}_0) - \int_0^{t\wedge\sigma_{i+1}-\sigma_i} G(\mathfrak{X}_s)\,ds\Big]\,\mathbf{1}_{t>\sigma_i}\right] \\
= \;&\mathbb{E}^\nu\left[\mathbb{E}^{\mathfrak{X}_{\sigma_i}}\Big[F(\mathfrak{X}_{((t-\sigma_i)\wedge\tau_{i+1})-}) - F(\mathfrak{X}_0) - \int_0^{(t-\sigma_i)\wedge\tau_{i+1}} G(\mathfrak{X}_s)\,ds\Big]\,\mathbf{1}_{t>\sigma_i}\right].
\end{aligned}$$

The first equality follows from the strong Markov property of $\mathfrak{X}$ (applied to the stopping time σ_i) and the fact that the expression inside the expectation vanishes when $t \leq \sigma_i$. Note that σ_i is regarded as a constant w.r.t. the expectation $\mathbb{E}^{\mathfrak{X}_{\sigma_i}}$, because $\mathcal{F}^{\mathfrak{X}}_{\sigma_i}$ contains the sigma-algebra generated by σ_i. The second equality follows from the easy fact that $(t \wedge \sigma_{i+1}) - \sigma_i = (t - \sigma_i) \wedge (\sigma_{i+1} - \sigma_i) = (t - \sigma_i) \wedge \tau_{i+1}$ on $t > \sigma_i$. Therefore, to establish (6.6), it is enough to show that for any $\eta \in E_N$ and $w \geq 0$, we have

$$\mathbb{E}^\eta\Big[\,F(\mathfrak{X}_{(w\wedge\tau)-}) - F(\mathfrak{X}_0) - \int_0^{w\wedge\tau} G(\mathfrak{X}_s)\,ds\,\Big] = 0, \tag{6.8}$$

where τ is the time of the first annihilation for $\mathfrak{X}$ starting from η (i.e. $\tau = \tau_1$ under $\mathbb{P}^\eta$ where τ_1 is defined by (3.1)).

(6.8) obviously holds if η is the zero measure since both sides vanish. Suppose $\eta \in E_N^{(n)}$. Observe that τ is a stopping time for $\tilde{\mathcal{F}}_t^{\overline{\mathfrak{X}}} := \sigma(\mathcal{F}_t^{\overline{\mathfrak{X}}}, \{R_i;\, 1 \le i \le n\})$ and that $\overline{M}_t$ is a $\tilde{\mathcal{F}}_t^{\overline{\mathfrak{X}}}$-martingale under $\mathbb{P}^\eta$ since $\{R_i\}$ is independent of $\overline{\mathfrak{X}}$ under $\mathbb{P}^\eta$. Hence, by the optional sampling theorem, (6.8) is true, and so is (6.6).

Following the same arguments as above, the left hand side of (6.7) equals

$$\mathbb{E}^\nu\left[\mathbb{E}^{\mathfrak{X}_{\sigma_{j-1}}}\left[F(\mathfrak{X}_{(t-\sigma_{j-1})\wedge\tau_j}) - F(\mathfrak{X}_{((t-\sigma_{j-1})\wedge\tau_j)-}) + \int_0^{(t-\sigma_{j-1})\wedge\tau_j} KF(\mathfrak{X}_s)\,ds\right]\mathbf{1}_{t>\sigma_{j-1}}\right],$$

where σ_{j-1} is regarded as a constant w.r.t. the expectation $\mathbb{E}^{\mathfrak{X}_{\sigma_{j-1}}}$. Therefore, (6.7) holds if for any $\eta \in E_N$ and $\theta \ge 0$, we have

$$\mathbb{E}^\eta\Big[F(\mathfrak{X}_{\theta\wedge\tau}) - F(\mathfrak{X}_{(\theta\wedge\tau)-}) - \int_0^{\theta\wedge\tau} KF(\mathfrak{X}_s)\,ds\Big] = 0, \tag{6.9}$$

where τ is the time of the first killing for $\mathfrak{X}$ starting from η.

Suppose $\eta = (\frac{1}{N}\sum_{i=1}^n \mathbf{1}_{x_i},\, \frac{1}{N}\sum_{j=1}^n \mathbf{1}_{y_j}) \in E_N^{(n)}$ and $\mathfrak{X}_{\tau-} = (\frac{1}{N}\sum_{i=1}^n \mathbf{1}_{X_i^+(\tau-)},\, \frac{1}{N}\sum_{j=1}^n \mathbf{1}_{X_j^-(\tau-)})$, where $\{X_k^\pm : k = 1, \cdots, n\}$ are reflected diffusions killed upon hitting $\Lambda^\pm$ in the construction of $\mathfrak{X}$. At time τ, one pair of particles among $\{(X_i^+,\, \mathfrak{X}_j^-) : 1 \le i, j \le n\}$ is labeled (annihilated), where the pair $(X_i^+,\, \mathfrak{X}_j^-)$ is chosen to be labeled (annihilated) with probability $\frac{\ell_{\delta_N}(X_i^+(\tau-),\, X_j^-(\tau-))}{\sum_{p=1}^n\sum_{q=1}^n \ell_{\delta_N}(X_p^+(\tau-),\, X_q^-(\tau-))}$. Hence

$$\begin{aligned}
& \mathbb{E}^\eta\Big[F(\mathfrak{X}_{(\theta\wedge\tau)-}) - F(\mathfrak{X}_{\theta\wedge\tau})\Big] && (6.10)\\
= \; & \mathbb{E}^\eta\Big[\mathbb{E}^\eta\Big[F(\mathfrak{X}_{\tau-}) - F(\mathfrak{X}_\tau)\Big|\mathcal{F}^{\mathfrak{X}}_{\tau-}\Big]\,;\, \tau < \theta\Big]\\
= \; & \mathbb{E}^\eta\Big[\sum_{i=1}^n\sum_{j=1}^n \frac{\ell_{\delta_N}(X_i^+(\tau-),\, X_j^-(\tau-))}{\sum_{p=1}^n\sum_{q=1}^n \ell_{\delta_N}(X_p^+(\tau-),\, X_q^-(\tau-))} && (6.11)\\
& \qquad \Big(F(\mathfrak{X}_{\tau-}) \;-\; F\Big(\mathfrak{X}_{\tau-} - (\frac{1}{N}\mathbf{1}_{X_i^+(\tau-)},\, \frac{1}{N}\mathbf{1}_{X_j^-(\tau-)})\Big)\Big)\,;\, \tau < \theta\Big]\\
= \; & \mathbb{E}^\eta\Big[\frac{-(2N)\,KF(\mathfrak{X}_{\tau-})}{\sum_{p=1}^n\sum_{q=1}^n \ell_{\delta_N}(X_p^+(\tau-),\, X_q^-(\tau-))}\,;\, \tau < \theta\Big]\\
= \; & \mathbb{E}^\eta\Big[\int_0^\theta -KF(\mathfrak{X}_s)\,ds\Big].
\end{aligned}$$

The last equality follows from the fact that

$$\tau = \inf\Big\{t \ge 0 : \frac{1}{2N}\int_0^t \sum_{p=1}^n\sum_{q=1}^n \ell_{\delta_N}(X_p^+(s),\, X_q^-(s))\,ds \ge R\Big\},$$

where R is an independent exponential random variable of parameter 1 under $\mathbb{P}^\eta$ (see Proposition 2.2 of [12] for a rigorous proof). Hence (6.9) is established and the proof is complete. □

The following corollary is the key to the tightness of $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$. Recall that $\mathcal{A}^\pm$ is the Feller generator of the diffusion $X^\pm = X^{\Lambda_\pm}$ on $\overline{D}_\pm \setminus \Lambda_\pm$, respectively.

Corollary 6.4. *Fix any positive integer N. For any $\phi_\pm \in Dom(\mathcal{A}^\pm)$, we have*

$$\begin{aligned} M_t^{(\phi_+,\phi_-)} &:= \langle \phi_+, \mathfrak{X}_t^{N,+}\rangle + \langle \phi_-, \mathfrak{X}_t^{N,-}\rangle \\ &\quad - \int_0^t \langle \mathcal{A}^+\phi_+,\, \mathfrak{X}_s^{N,+}\rangle + \langle \mathcal{A}^-\phi_-,\, \mathfrak{X}_s^{N,-}\rangle - \frac{1}{2}\langle \ell_{\delta_N}(\phi_+ + \phi_-),\, \mathfrak{X}_s^{N,+}\otimes \mathfrak{X}_s^{N,-}\rangle\, ds \end{aligned}$$

is an $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$-martingale under $\mathbb{P}^\mu$ for any $\mu \in E_N$, where

$$\langle f(x,y),\, \mu^+(dx)\otimes\mu^-(dy)\rangle := \frac{1}{N^2}\sum_i\sum_j f(x_i,y_j) \quad \textit{whenever } \mu = (N^{-1}\sum_i \mathbf{1}_{x_i},\, N^{-1}\sum_j \mathbf{1}_{y_j}).$$

Moreover, $M_t^{(\phi_+,\phi_-)}$ has quadratic variation

$$\begin{aligned} [M^{(\phi_+,\phi_-)}]_t &= \frac{1}{N}\int_0^t \Big(\langle \boldsymbol{a}_+\nabla\phi_+\cdot\nabla\phi_+,\, \mathfrak{X}_s^{N,+}\rangle + \langle \boldsymbol{a}_-\nabla\phi_-\cdot\nabla\phi_-,\, \mathfrak{X}_s^{N,-}\rangle \\ &\qquad + \frac{1}{2}\langle \ell_{\delta_N}(\phi_+ + \phi_-)^2,\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\Big) ds \end{aligned} \tag{6.12}$$

and $\sup_{t\in[0,T]}\mathbb{E}^\mu[(M_t^{(\phi_+,\phi_-)})^2] \le \frac{C}{N}$ for some constant C that is independent of N and μ.

Proof From Lemma 6.1, we have the following two $\mathcal{F}_t^{(\overline{\mathfrak{X}}^{N,+},\overline{\mathfrak{X}}^{N,-})}$-martingales for $\phi_\pm \in Dom(\mathcal{A}^\pm)$:

$$\begin{aligned} \overline{M}_t^{(\phi_+,\phi_-)} &:= \langle \phi_+, \overline{\mathfrak{X}}_t^{N,+}\rangle + \langle \phi_-, \overline{\mathfrak{X}}_t^{N,-}\rangle - \int_0^t \langle \mathcal{A}^+\phi_+,\, \overline{\mathfrak{X}}_s^{N,+}\rangle + \langle \mathcal{A}^-\phi_-,\, \overline{\mathfrak{X}}_s^{N,-}\rangle\, ds \quad \text{and} \\ \overline{N}_t^{(\phi_+,\phi_-)} &:= (\langle \phi_+, \overline{\mathfrak{X}}_t^{N,+}\rangle + \langle \phi_-, \overline{\mathfrak{X}}_t^{N,-}\rangle)^2 \\ &\quad - \int_0^t 2\Big(\langle \phi_+, \overline{\mathfrak{X}}_s^{N,+}\rangle + \langle \phi_-, \overline{\mathfrak{X}}_s^{N,-}\rangle\Big)\Big(\langle \mathcal{A}^+\phi_+, \overline{\mathfrak{X}}_s^{N,+}\rangle + \langle \mathcal{A}^-\phi_-, \overline{\mathfrak{X}}_s^{N,-}\rangle\Big) \\ &\quad + \frac{1}{N}\Big(\langle \mathbf{a}_+\nabla\phi_+\cdot\nabla\phi_+,\, \overline{\mathfrak{X}}_s^{N,+}\rangle + \langle \mathbf{a}_-\nabla\phi_-\cdot\nabla\phi_-,\, \overline{\mathfrak{X}}_s^{N,-}\rangle\Big)\, ds. \end{aligned}$$

Note that $F_1(\mu) = F_1(\mu^+,\mu^-) := \langle \phi_+, \mu^+\rangle + \langle \phi_-, \mu^-\rangle$ is a function in $C(E_N)$, with the convention that $\phi_\pm(\partial^\pm) := 0$ and $F_1(\mathbf{0}_*) := 0$. A direct calculations shows that

$$KF_1(\mu) = \frac{-1}{2}\langle \ell_{\delta_N}(\phi_+ + \phi_-),\, \mu^+\otimes\mu^-\rangle$$

Therefore, by Theorem 6.2, $M_t^{(\phi_+,\phi_-)}$ is an $\mathcal{F}_t^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$-martingale. Similarly, $F_2(\mu) := (\langle \phi_+, \mu^+\rangle + \langle \phi_-, \mu^-\rangle)^2 \in C(E_N)$ and

$$KF_2(\mu) = -\left(\langle \phi_+, \mu^+\rangle + \langle \phi_-, \mu^-\rangle\right)\langle \ell_{\delta_N}(\phi_+ + \phi_-),\, \mu^+\otimes\mu^-\rangle + \frac{1}{2N}\langle \ell_{\delta_N}(\phi_+ + \phi_-)^2,\, \mu^+\otimes\mu^-\rangle.$$

Hence Theorem 6.2 asserts that

$$\begin{aligned} N_t^{(\phi_+,\phi_-)} &:= \Big(\langle \phi_+, \mathfrak{X}_t^{N,+}\rangle + \langle \phi_-, \mathfrak{X}_t^{N,-}\rangle\Big)^2 \\ &\quad - \int_0^t 2\Big(\langle \phi_+, \mathfrak{X}_s^{N,+}\rangle + \langle \phi_-, \mathfrak{X}_s^{N,-}\rangle\Big)\Big(\langle \mathcal{A}^+\phi_+, \mathfrak{X}_s^{N,+}\rangle + \langle \mathcal{A}^-\phi_-, \mathfrak{X}_s^{N,-}\rangle\Big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{N}\Big(\langle \mathbf{a}_+\nabla\phi_+\cdot\nabla\phi_+,\,\mathfrak{X}^{N,+}_s\rangle + \langle \mathbf{a}_-\nabla\phi_-\cdot\nabla\phi_-,\,\mathfrak{X}^{N,-}_s\rangle\Big)\\
&-(\langle\phi_+,\mathfrak{X}^{N,+}_s\rangle+\langle\phi_-,\mathfrak{X}^{N,-}_s\rangle)\langle\ell_{\delta_N}(\phi_++\phi_-),\,\mathfrak{X}^{N,+}_s\otimes\mathfrak{X}^{N,-}_s\rangle\\
&+\frac{1}{2N}\langle\ell_{\delta_N}(\phi_++\phi_-)^2,\,\mathfrak{X}^{N,+}_s\otimes\mathfrak{X}^{N,-}_s\rangle\,ds
\end{aligned}
$$

is an $\mathcal{F}^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}_t$-martingale. Since $\left(M^{(\phi_+,\phi_-)}_t\right)^2 - N^{(\phi_+,\phi_-)}_t$ is equal to the right hand side of (6.12), which is a continuous process of finite variation, it has to be $[M^{(\phi_+,\phi_-)}]_t$. This proves (6.12). Therefore,

$$
\begin{aligned}
&\mathbb{E}^\mu[(M^{(\phi_+,\phi_-)}_t)^2] = \mathbb{E}^\mu\Big[[M^{(\phi_+,\phi_-)}]_t\Big]\\
\le\;&\frac{1}{N}\bigg(\int_0^t\langle P^+_s\big(\mathbf{a}_+\nabla\phi_+\cdot\nabla\phi_+\big),\,\mathfrak{X}^{N,+}_0\rangle\,ds+\int_0^t\langle P^-_s\big(\mathbf{a}_-\nabla\phi_-\cdot\nabla\phi_-\big),\,\mathfrak{X}^{N,-}_0\rangle\,ds\\
&\qquad+\frac{1}{2}\,\|(\phi_++\phi_-)^2\|\int_0^t\langle\ell_{\delta_N},\,\mathfrak{X}^{N,+}_s\otimes\mathfrak{X}^{N,-}_s\rangle\bigg)\,ds\\
\le\;&\frac{1}{N}\bigg(8\big(\|\phi_+\|^2+\|\mathcal{A}^+\phi_+\|^2\,t^2\big)+8\big(\|\phi_-\|^2+\|\mathcal{A}^-\phi_-\|^2\,t^2\big)\\
&\qquad+\frac{1}{2}\,\|(\phi_++\phi_-)^2\|\int_0^t\langle\ell_{\delta_N},\,\mathfrak{X}^{N,+}_s\otimes\mathfrak{X}^{N,-}_s\rangle\bigg)\,ds,
\end{aligned}
$$

where we have used (6.2) in the last inequality. Finally, we show that

$$
\sup_{\mu\in E_N}\int_0^t\mathbb{E}^\mu[\langle\ell_{\delta_N},\,\mathfrak{X}^{N,+}_s\otimes\mathfrak{X}^{N,-}_s\rangle]\le 1. \tag{6.13}
$$

Let $(\tilde{\mathfrak{X}}^{N,+},\,\tilde{\mathfrak{X}}^{N,-})$ be the normalized empirical measure corresponding to the case $\Lambda_\pm$ being empty sets. By applying the martingale $M^{(\phi_+,\phi_-)}_t$ to the case $\Lambda_\pm$ being empty sets and $\phi_\pm=1$ (now 1 is in the domain of the Feller generator), we have

$$
\int_0^t\mathbb{E}[\langle\ell_{\delta_N},\tilde{\mathfrak{X}}^{N,+}_s\otimes\tilde{\mathfrak{X}}^{N,-}_s\rangle]\,ds=\Big(\langle 1,\,\tilde{\mathfrak{X}}^{N,+}_0\rangle-\mathbb{E}[\langle 1,\,\tilde{\mathfrak{X}}^{N,+}_t\rangle]\Big)\le 1.
$$

We then obtain (6.13) by a coupling of $(\mathfrak{X}^{N,+},\,\mathfrak{X}^{N,-})$ and $(\tilde{\mathfrak{X}}^{N,+},\,\tilde{\mathfrak{X}}^{N,-})$. The idea is that $(\tilde{\mathfrak{X}}^{N,+},\,\tilde{\mathfrak{X}}^{N,-})$ dominates $(\mathfrak{X}^{N,+},\,\mathfrak{X}^{N,-})$. This coupling can be constructed by labeling (rather than killing) particles which hit $\Lambda_\pm$, using the same method of subsection 3.1. Hence we obtain the desired bound for $\mathbb{E}^\mu[(M^{(\phi_+,\phi_-)}_t)^2]$. □

6.1.3 Tightness

The proof of tightness for $(\mathfrak{X}^{N,+},\,\mathfrak{X}^{N,-})$ is non-trivial because $\mathbb{E}\Big[\langle\ell_{\delta_N},\,\mathfrak{X}^{N,+}_s\otimes\mathfrak{X}^{N,-}_s\rangle^2\Big]$ blows up near $s=0$ in such a way that $\lim_{N\to\infty}\int_0^t\mathbb{E}\Big[\langle\ell_{\delta_N},\,\mathfrak{X}^{N,+}_s\otimes\mathfrak{X}^{N,-}_s\rangle^2\Big]\,ds=\infty$. To deal with this singularity at $s=0$, we will use the following lemma whose proof is based on the Prohorov's theorem. We omit the proof here since it is simple. A proof can be found in [21].

Lemma 6.5. *Let $\{Y_N\}$ be a sequence of real-valued processes such that $t\mapsto\int_0^t Y_N(r)\,dr$ is continuous on $[0,T]$ a.s., where $T\in[0,\infty)$. Suppose (i) and (ii) below holds.*

(i) *There exists* $q>1$ *such that* $\overline{\lim}_{N\to\infty}\mathbb{E}[\int_h^T |Y_N(r)|^q\,dr]<\infty$ *for any* $h>0$,

(ii) $\lim_{\alpha\searrow 0}\overline{\lim}_{N\to\infty}\mathbb{P}(\int_0^\alpha |Y_N(r)|\,dr>\varepsilon_0)=0$ *for any* $\varepsilon_0>0$.

Then $\{\int_0^t Y_N(r)\,dr\}_{N\in\mathbb{N}}$ *is tight in* $C([0,T],\mathbb{R})$.

Here is our tightness result for $(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})$. Note that it does not require Assumption 2.6.

Theorem 6.6. *(Tightness) Suppose* $\{\delta_N\}$ *tends to 0. Then* $\{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})\}$ *is tight in* $D([0,T],\mathfrak{M})$ *and any of subsequential limits is carried on* $C_{\mathfrak{M}}[0,T]$. *Moreover,* $\{J_N\}$ *is tight in* $C([0,T])$, *where* $J_N(t):=\int_0^t\langle \ell_{\delta_N},\,\mathfrak{X}^{N,+}_s\otimes\mathfrak{X}^{N,-}_s\rangle\,ds$.

Proof Recall from Remark 5.1 that $\mathfrak{M}$ is a complete separable metric space. Since $Dom(\mathcal{A}^{\pm})$ is dense in $C_\infty(\overline{D}_\pm\setminus\Lambda_\pm)$, we only need to check a "weak tightness criteria" (cf. Proposition 1.7 of [27]), i.e. it suffices to check that $\{(\langle\phi_+,\mathfrak{X}^{N,+}\rangle,\,\langle\phi_-,\mathfrak{X}^{N,-}\rangle)\}_N$ is tight in $D([0,T],\mathbb{R}^2)$ for any $\phi_\pm\in Dom(\mathcal{A}^{\pm})$. By Prohorov's theorem (see Theorem 1.3 and Remark 1.4 of [27]), $\{(\langle\phi_+,\mathfrak{X}^{N,+}\rangle,\,\langle\phi_-,\mathfrak{X}^{N,-}\rangle)\}_N$ is tight in $D([0,T],\mathbb{R}^2)$ if the following two properties (a) and (b) hold:

(a) For all $t\in[0,T]$ and $\varepsilon_0>0$, there exists a compact set $K(t,\varepsilon_0)\subset\mathbb{R}^2$ such that

$$\sup_N\mathbb{P}\Big((\langle\phi_+,\mathfrak{X}^{N,+}_t\rangle,\,\langle\phi_-,\mathfrak{X}^{N,-}_t\rangle)\notin K(t,\varepsilon_0)\Big)<\varepsilon_0.$$

(b) For all $\varepsilon_0>0$,

$$\lim_{\gamma\to 0}\overline{\lim_{N\to\infty}}\,\mathbb{P}\left(\sup_{\substack{|t-s|<\gamma\\0\le s,t\le T}}\Big|\Big(\langle\phi_+,\mathfrak{X}^{N,+}_t\rangle,\,\langle\phi_-,\mathfrak{X}^{N,-}_t\rangle\Big)-\Big(\langle\phi_+,\mathfrak{X}^{N,+}_s\rangle,\,\langle\ \phi_-,X^{N,-}_s\rangle\Big)\Big|_{\mathbb{R}^2}>\varepsilon_0\right)=0.$$

Property (a) is true since we can always take $K=[-\|\phi_+\|_\infty,\,\|\phi_+\|_\infty]\times[-\|\phi_-\|_\infty,\,\|\phi_-\|_\infty]$. To verify property (b), we only need to focus on $\mathfrak{X}^{N,+}$. Note that (writing $\phi=\phi_+$ for simplicity) by Corollary 6.4, we have

$$\langle\phi,\mathfrak{X}^{N,+}_t\rangle-\langle\phi,\mathfrak{X}^{N,+}_s\rangle=\int_s^t\langle\mathcal{A}^+\phi,\,\mathfrak{X}^{N,+}_r\rangle\,dr-\frac{1}{2}\int_s^t\langle\ell_{\delta_N}\,\phi,\,\mathfrak{X}^{N,+}_r\otimes\mathfrak{X}^{N,-}_r\rangle\,dr+(M_N(t)-M_N(s)),\tag{6.14}$$

where $M_N(t)$ is a martingale. So we only need to verify (b) with $\langle\phi,\mathfrak{X}^{N,+}_t\rangle-\langle\phi,\mathfrak{X}^{N,+}_s\rangle$ replaced by each of the three terms on the right hand side of (6.14).

The first term of (6.14) is obvious since $\langle\mathcal{A}^+\phi,\,\mathfrak{X}^{N,+}_r\rangle\le\|\mathcal{A}^+\phi\|$. For the third term of (6.14), recall that $\lim_{N\to\infty}\mathbb{E}\left[M_N(t)^2\right]=0$ by Corollary 6.4. Hence, by applying Chebyshev's inequality and then Doob's maximal inequality, we see that (b) is satisfied by the third term of (6.14).

For the second term of (6.14), we show that

$$\lim_{\gamma\to 0}\overline{\lim_{N\to\infty}}\,\mathbb{P}\left(\sup_{\substack{|t-s|<\gamma\\0\le s,t\le T}}\int_s^t\langle\ell_{\delta_N},\,\mathfrak{X}^{N,+}_r\otimes\mathfrak{X}^{N,-}_r\rangle\,dr>\varepsilon_0\right)=0.\tag{6.15}$$

Observe that, since $\langle \ell_{\delta_N}, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-}\rangle$ is non-negative, it suffices to prove (6.15) for the dominating case where $\Lambda_\pm$ are empty. We now prove this together with the tightness of $\{J_N\}$ at one stroke by applying Lemma 6.5 to the special case $q = 2$ and $Y_N(r) = \langle \ell_{\delta_N}, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-}\rangle$.

Using the Gaussian upper bound (2.2) for the heat kernel of the reflected diffusions, we have

$$\overline{\lim_{N\to\infty}} \int_h^T \mathbb{E}[\langle \ell_{\delta_N}, \overline{\mathfrak{X}}_s^{N,+} \otimes \overline{\mathfrak{X}}_s^{N,-}\rangle^2]\, ds \le C(d, D_+, D_-)\|\rho_+\|\,\|\rho_-\| \int_h^T s^{-2d}\, ds < \infty.$$

The hypothesis (i) of Lemma 6.5 is therefore satisfied, since $(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})$ dominates $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$.

It remains to verify hypothesis (ii) of Lemma 6.5, that is, to prove that for any $\varepsilon_0 > 0$, $\lim_{\alpha\to 0} \overline{\lim}_{N\to\infty} \mathbb{P}(J_n(\alpha) > \varepsilon_0) = 0$. By Corollary 6.4 again, for any $\phi \in Dom(\mathcal{A}^+)$, we have

$$\frac{1}{2}\int_0^t \langle \ell_{\delta_N}\phi, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}\rangle\, ds = \langle \phi, \mathfrak{X}_0^{N,+}\rangle - \langle \phi, \mathfrak{X}_t^{N,+}\rangle + \int_0^t \langle \mathcal{A}^+\phi, \mathfrak{X}_s^{N,+}\rangle\, ds + M_N(t), \quad (6.16)$$

where $M_N(t)$ is a martingale and $\lim_{N\to\infty} \mathbb{E}\left[(M_N(t))^2\right] = 0$ for all $t > 0$. Note that the left hand side of (6.16) is comparable to $J_N(t)$ whenever we pick $\phi \in Dom(\mathcal{A}^+)$ in such a way that $\ell_{\delta_N}\phi \approx \ell_{\delta_N}$. The idea is to pick $\phi \approx \mathbf{1}_{(D_+)_r}$, then let $r \to 0$ to bound $J_N(t)$ from above. Here $\mathbf{1}_{(D_+)_r}$ is the set of points in D_+ whose distance from the boundary is less than r. More specifically, for any $r > 0$, let $\psi_r \in C(\overline{D}_+)$ be such that $\psi_r = 1$ on $(D_+)_r$, $\psi_r = 0$ on $D_+ \setminus (D_+)_{2r}$ and $0 \le \psi \le 1$. Let $\phi_r \in Dom(\mathcal{A}^+) \cap C^+(\overline{D}_+)$ be such that $\|\phi_r - \psi_r\|_\infty = o(r)$. Such ϕ_r exists since $Dom(\mathcal{A}^+)$ is dense in $C(\overline{D}_+)$. Then (6.16) implies

$$\begin{aligned} 0 &\le J_N(\alpha) \\ &\le \left| \int_0^\alpha \langle \ell_{\delta_N} - \ell_{\delta_N}\phi_r, \mathfrak{X}_s^{N,+} \otimes \mathfrak{X}_s^{N,-}\rangle\, ds \right| + \langle \phi_r, \mathfrak{X}_0^{N,+}\rangle - \langle \phi_r, \mathfrak{X}_\alpha^{N,+}\rangle + \|\mathcal{A}^+\phi_r\|\,\alpha + |M_N(\alpha)| \\ &\le o(r)J_N(\alpha) + \langle \phi_r, \mathfrak{X}_0^{N,+}\rangle + \|\mathcal{A}^+\phi_r\|\,\alpha + |M_N(\alpha)| \quad \text{whenever } r > 2\delta_N. \end{aligned}$$

This is because when $r > 2\delta_N$, $\phi_r(x)$ is close to 1 on $(D_+)_{\delta_N}$. Hence we have, for $r > 2\delta_N$,

$$(1 - o(r))\, J_N(\alpha) \le \langle \phi_r, \mathfrak{X}_0^{N,+}\rangle + \|\mathcal{A}^+\phi_r\|\,\alpha + |M_N(\alpha)|.$$

From this, we have

$$\lim_{\alpha\to 0} \overline{\lim_{N\to\infty}} \mathbb{P}(J_N(\alpha) > 3\varepsilon_0) \le \overline{\lim_{N\to\infty}} \mathbb{P}\left(\langle \phi_r, \mathfrak{X}_0^{N,+}\rangle > \varepsilon_0(1 - o(r))\right).$$

Note that $0 \le \phi_r \le \mathbf{1}_{(D_+)_{2r}} + o(r)$. So for $r > 0$ small enough,

$$\mathbb{P}\Big(\langle \phi_r, \mathfrak{X}_0^{N,+}\rangle > \varepsilon_0(1 - o(r))\Big) \le \mathbb{P}(\langle \mathbf{1}_{(D_+)_{2r}}, \mathfrak{X}_0^{N,+}\rangle > \varepsilon_0/2).$$

Moreover, since $\mathfrak{X}_0^{N,+} \xrightarrow{\mathcal{L}} u_0^+(x)dx$ with $u_0^+ \in C(\overline{D})$, we have

$$\lim_{r\to 0} \overline{\lim_{N\to\infty}} \mathbb{P}(\langle \mathbf{1}_{(D_+)_{2r}}, \mathfrak{X}_0^{N,+}\rangle > \varepsilon_0/2) = 0.$$

Hence the second hypothesis of Lemma 6.5 is verified. We have shown that (ii) is true. Thus $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ is relatively compact. Property (ii) above also tells us that any subsequential limit has law concentrated on $C([0,\infty), \mathfrak{M})$ (detail can be found in [21]). □

6.2 Identifying subsequential limits

Recall that we have already established tightness of $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-}); N \geq 1\}$ in Theorem 6.6. Hence any subsequence has a further subsequence which converges in distribution in $D([0,T], \mathfrak{M})$. Let $\mathbb{P}^\infty$ be the law of an arbitrary subsequential limit $(\mathfrak{X}^{\infty,+},\, \mathfrak{X}^{\infty,-})$. Then $\mathbb{P}^\infty((\mathfrak{X}^{\infty,+},\, \mathfrak{X}^{\infty,-}) \in C([0,\infty), \mathfrak{M})) = 1$ by Theorem 6.6. Our goal is to show that

$$(\mathfrak{X}^{\infty,+},\, \mathfrak{X}^{\infty,-}) = (u_+(t,x)\rho_+(x)dx,\, u_-(t,y)\rho_-(y)dy) \quad \mathbb{P}^\infty - a.s.$$

An immediate question is whether $\mathfrak{X}^{\infty,+}$ and $\mathfrak{X}^{\infty,-}$ have densities with respect to the Lebesque measure. For this, we can compare $(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})$ with $(\overline{\mathfrak{X}}^{N,+},\, \overline{\mathfrak{X}}^{N,-})$ to get an affirmative answer. The construction in subsection 3.1 provides a natural coupling between $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ and $\{(\overline{\mathfrak{X}}^{N,+}, \overline{\mathfrak{X}}^{N,-})\}$. We summarize some preliminary information about $(\mathfrak{X}^{\infty,+},\, \mathfrak{X}^{\infty,-})$ in the following lemma. Its proof can be found in [21].

Lemma 6.7.

$$\mathbb{P}^\infty\Big(\langle \mathfrak{X}_t^{\infty,+}, \phi_+\rangle_{\rho_+} \leq \langle P_t^+ u_0^+, \phi_+\rangle_{\rho_+} \text{ and } \langle \mathfrak{X}_t^{\infty,-}, \phi_-\rangle_{\rho_-} \leq \langle P_t^- u_0^-, \phi_-\rangle_{\rho_-}$$
$$\text{for } t \geq 0 \text{ and } \phi_\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm) \Big) = 1.$$

In particular, both $\mathfrak{X}_t^{\infty,+}$ and $\mathfrak{X}_t^{\infty,-}$ are absolutely continuous with respect to the Lebesque measure for $t \geq 0$. Moreover, $(\mathfrak{X}_t^{\infty,+},\, \mathfrak{X}_t^{\infty,-}) = (v_+(t,x)\rho_+(x)dx,\, v_-(t,y)\rho_-(y)dy)$ for some $v_\pm(t) \in \mathcal{B}_b(D_\pm)$ with $v_+(t,x) \leq P_t^+ u_0^+(x)$ and $v_-(t,y) \leq P_t^- u_0^-(y)$ for a.e. $(x,y) \in D_+ \times D_-$.

The characterization $(\mathfrak{X}^{\infty,+},\, \mathfrak{X}^{\infty,-})$ will be accomplished by the following result of "mean-variance analysis":

Proposition 6.8. *For all $\phi_\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ and $t \geq 0$, we have*

$$\mathbb{E}^\infty[\langle v_\pm(t),\, \phi_\pm\rangle_{\rho_\pm}] = \langle u_\pm(t),\, \phi_\pm\rangle_{\rho_\pm}, \tag{6.17}$$
$$\mathbb{E}^\infty[\langle v_\pm(t),\, \phi_\pm\rangle_{\rho_\pm}^2] = \langle u_\pm(t),\, \phi_\pm\rangle_{\rho_\pm}^2. \tag{6.18}$$

where $v_\pm$ is the density of $\mathfrak{X}^{\infty,\pm}$, w.r.t. $\rho_\pm(x)dx$, stated in Lemma 6.7.

We postpone the proof of Proposition 6.8 to Section 7, and proceed to present the proof of Theorem 5.2.

6.3 Proof of Theorem 5.2

Proof Tightness of $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}$ was proved in Theorem 6.6. It remains to identify any subsequential limit. We conclude from (6.17) and (6.18) that

$$\langle \mathfrak{X}_t^{\infty,+},\, \phi_+\rangle = \langle u_+(t),\, \phi_+\rangle_{\rho_+} \quad \text{and} \quad \langle \mathfrak{X}_t^{\infty,-},\, \phi_-\rangle = \langle u_-(t),\, \phi_-\rangle_{\rho_-} \quad \mathbb{P}^\infty\text{-a.s.}$$

for any fixed $t > 0$ and $\phi_\pm \in C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$. Recall that $(\mathfrak{X}^{\infty,+},\, \mathfrak{X}^{\infty,-}) \in C([0,\infty), \mathfrak{M})$ by Theorem 6.6 and that $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ is separable. Hence through rational numbers and a countable dense subsets of $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$ to strengthen the previous statement to

$$\mathbb{P}^\infty\left((\mathfrak{X}_t^{\infty,+},\, \mathfrak{X}_t^{\infty,-}) = (u_+(t,x)\rho_+(x)dx,\, u_-(t,y)\rho_-(y)dy) \in \mathfrak{M} \quad \text{for every } t \geq 0 \right) = 1.$$

This completes the proof of Theorem 5.2. □

7 Characterization of the mean and the variance

The goal of this last section is to prove Proposition 6.8. We first strengthen a result from Geometric Measure Theory .

7.1 Minkowski content for $\{(z,z) : z \in I\}$

We first look at a single domain and prove a related result.

Lemma 7.1. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. If $\mathcal{F} \subset C(\overline{D})$ is an equi-continuous and uniformly bounded family of functions on $\overline{D}$, then*

$$\lim_{\varepsilon\to 0} \sup_{f\in\mathcal{F}} \Big| \frac{1}{\varepsilon} \int_{D_\varepsilon} f(x)dx - \int_{\partial D} f(x)\sigma(dx) \Big| = 0.$$

Proof The result holds trivially when $d = 1$, by the uniform continuity of f. We will only consider $d \geq 2$. The idea is to cut ∂D into small pieces so that f is almost constant in each piece, and then apply (1.3) in each piece.

Fix $\eta > 0$. There exists $\delta > 0$ such that $|f(x) - f(y)| < \eta$ whenever $|x - y| \leq \delta$. Since D is bounded and Lipschitz (or by a more general result by G. David in [14] or [15, Section 2]), we can reduce to local coordinates to obtain a partition $\{Q_i\}_{i=1}^N$ of ∂D in such a way that for any i, Q_i is the Lipschitz image of a bounded subset of $\mathbb{R}^{d-1}$ (hence it is $(\mathcal{H}^{d-1})$-rectifiable), $diam(Q_i) \leq \delta$ and ∂Q_i is $(\mathcal{H}^{d-2})$-rectifiable. Here ∂Q_i is the boundary of Q_i with respect to the topology induced by ∂D.

Let $(Q_i)_\varepsilon := \{x \in D : dist(x, Q_i) < \varepsilon\}$ and $(\partial Q_i)_\varepsilon := \{x \in D : dist(x, \partial Q_i) < \varepsilon\}$. Since $\{(Q_i)_\varepsilon \setminus (\partial Q_i)_\varepsilon\}_{i=1}^N$ are disjoint and $\cup_{i=1}^N (Q_i)_\varepsilon \setminus (\partial Q_i)_\varepsilon \subset D_\varepsilon \subset \cup_{i=1}^N (Q_i)_\varepsilon$, we have

$$\Big| \sum_{i=1}^N \int_{(Q_i)_\varepsilon} f\,dx - \int_{D_\varepsilon} f\,dx \Big| \leq \sum_{i=1}^N \int_{(\partial Q_i)_\varepsilon} |f|\,dx. \tag{7.1}$$

Therefore, we have

$$\begin{aligned}
& \Big| \frac{1}{\varepsilon} \int_{D_\varepsilon} f\,dx - \int_{\partial D} f\,d\sigma \Big| \\
\leq\ & \Big| \frac{1}{\varepsilon} \int_{D_\varepsilon} f\,dx - \frac{1}{\varepsilon} \sum_{i=1}^N \int_{(Q_i)_\varepsilon} f\,dx \Big| + \Big| \frac{1}{\varepsilon} \sum_{i=1}^N \int_{(Q_i)_\varepsilon} f\,dx - \int_{\partial D} f\,d\sigma \Big| \\
\leq\ & \frac{1}{\varepsilon} \sum_{i=1}^N \int_{(\partial Q_i)_\varepsilon} |f|\,dx + \sum_{i=1}^N \Big| \frac{1}{\varepsilon} \int_{(Q_i)_\varepsilon} f\,dx - \int_{Q_i} f\,d\sigma \Big| \quad \text{by (7.1)} \\
\leq\ & \sum_{i=1}^N \left(\|f\|_\infty \frac{|(\partial Q_i)_\varepsilon|}{\varepsilon} + \Big| \frac{1}{\varepsilon} \int_{(Q_i)_\varepsilon} f - f(\xi_i)\,dx \Big| + |f(\xi_i)| \Big| \frac{|(Q_i)_\varepsilon|}{\varepsilon} - \sigma(Q_i) \Big| + \Big| \int_{Q_i} f - f(\xi_i)\,d\sigma \Big| \right) \\
\leq\ & \eta \sum_{i=1}^N \left(\frac{|(Q_i)_\varepsilon|}{\varepsilon} + \sigma(Q_i) \right) + \|f\|_\infty \sum_{i=1}^N \left(\frac{|(\partial Q_i)_\varepsilon|}{\varepsilon} + \Big| \frac{|(Q_i)_\varepsilon|}{\varepsilon} - \sigma(Q_i) \Big| \right).
\end{aligned}$$

Since ∂Q_i and $(Q_i)_\varepsilon$ are $(\mathcal{H}^{d-2})$-rectifiable and $(\mathcal{H}^{d-1})$-rectifiable, respectively, [22, Theorem 3.2.39] tells us that

$$\lim_{\varepsilon\to 0}\frac{|(\partial Q_i)_\varepsilon|}{c_2\varepsilon^2}=\mathcal{H}^{d-2}(\partial Q_i)\quad\text{and}\quad \lim_{\varepsilon\to 0}\frac{|(Q_i)_\varepsilon|}{\varepsilon}=\mathcal{H}^{d-1}(Q_i),$$

where $c_m := |\{x\in\mathbb{R}^m : |x|<1\}|$. Thus,

$$\overline{\lim_{\varepsilon\to 0}}\Big|\frac{1}{\varepsilon}\int_{D_\varepsilon} f\,dx-\int_{\partial D} f\,d\sigma\Big|\le 2\eta\sum_i\sigma(Q_i)=2\sigma(\partial D)\,\eta.$$

Since $\eta>0$ is arbitrary and the above estimate is uniform over $f\in\mathcal{F}$, we get the desired result. □

Now we prove an analogous result for the interface I.

Lemma 7.2. *Under our geometric setting in Assumption 2.4, if $\mathcal{F}\subset C(\overline{D}_+\times\overline{D}_-)$ is an equi-continuous and uniformly bounded family of functions on $\overline{D}_+\times\overline{D}_-$, then*

$$\lim_{\delta\to 0}\sup_{f\in\mathcal{F}}\Big|(c_{d+1}\,\delta^{d+1})^{-1}\int_{I^\delta} f(x,y)\,dx\,dy-\int_I f(z,z)\,d\sigma(z)\Big|=0.$$

Proof By the same argument as in the proof of Lemma 7.1, we can construct a nice partition $\{Q_i\}_{i=1}^N$ of I and apply [22, Theorem 3.2.39 (p. 275)]. The only essential difference is that now we require $\partial Q_i\setminus\partial I$ to be $(\mathcal{H}^{d-2})-$rectifiable, where ∂I is the boundary of I with respect to the topology induced by ∂D_+, or equivalently by ∂D_-. Moreover, instead of (7.1), we now have

$$\Big|\sum_{i=1}^N\int_{(Q_i)_\delta} f\,dxdy-\int_{I^\delta} f\,dxdy\Big|\le\sum_{i=1}^N\int_{(\partial Q_i\setminus\partial I)_\delta}|f|\,dxdy. \tag{7.2}$$

Note that we do not need any assumption on ∂I. □

Corollary 7.3. *Suppose $\mathcal{F}\subset C(\overline{D}_+\times\overline{D}_-)$ is a family of equi-continuous and uniformly bounded functions on $\overline{D}_+\times\overline{D}_-$. Then*

$$\lim_{\varepsilon\to 0}\sup_{f\in\mathcal{F}}\Big|\int_{D_+}\int_{D_-}\ell_\varepsilon(x,y)f(x,y)\,dxdy-\int_I f(z,z)\,\sigma(dz)\Big|=0.$$

Remark 7.4. Following the same proof as above, clearly we can strengthen Lemma 7.2 and Corollary 7.3 by only requiring $\mathcal{F}$ to be equi-continuous and uniformly bounded on a neighborhood of the interface I. We can also generalize Lemma 7.1 to deal with $\int_J f(x)\,d\sigma(x)$ for any closed $\mathcal{H}^{d-1}$-rectifiable subset of J of ∂D, and by requiring $\mathcal{F}$ to be equi-continuous and uniformly bounded on a neighborhood of J.

7.2 Martingales for space-time processes

In this subsection, we collect some integral equations satisfied by $(\mathfrak{X}^{N,+},\,\mathfrak{X}^{N,-})$ that will be used later to identify the limit. These integral equations can be viewed as the Dynkins' formulae for our annihilating diffusion system, and will be proved rigorously by considering suitable martingales associated with the process $(t,\,(\mathfrak{X}^{N,+}_t,\,\mathfrak{X}^{N,-}_t))$.

Lemma 7.5. *Suppose X^Λ is an $(\boldsymbol{a}, \rho)$-reflected diffusion in a bounded Lipschitz domain D killed upon hitting a closed subset Λ of ∂D that is regular with respect to X. Then for any $T > 0$ and bounded measurable function ϕ on $\overline{D} \setminus \Lambda$, we have*

$$P^\Lambda_{T-s}\phi(X^\Lambda_s) \text{ is a } \mathcal{F}^{X^\Lambda}_s\text{-martingale for } s \in [0, T], \tag{7.3}$$

under $\mathbb{P}^x$ for any $x \in \overline{D}\backslash\Lambda$. Moreover, its quadratic variation is $\int_0^s \boldsymbol{a}\nabla P^\Lambda_{T-r}\phi\cdot\nabla P^\Lambda_{T-r}\phi(X^\Lambda(r))\,dr$.

Proof (7.3) follows from the Markov property of X^Λ. Denote by $\mathcal{L}^{(\Lambda)}$ the L^2-generator of $X^{(\Lambda)}$. Then for every $t \in [0, T)$, $P^\Lambda_{T-s}\phi \in Dom(\mathcal{L}^{(\Lambda)})$. It follows from the spectral representation of $\mathcal{L}^{(\Lambda)}$ that

$$\Big\|\frac{\partial P^\Lambda_{T-s}\phi}{\partial s}\Big\|_{L^2} = \| - \mathcal{L}^{(\Lambda)} P^\Lambda_{T-s}\phi\|_{L^2} \le \frac{\|\phi\|_{L^2}}{T-s}.$$

Thus $(s, x) \mapsto P^\Lambda_{T-s}\phi(x)$ for $s \in [0, T)$ and $x \in \overline{D} \setminus \Lambda$ is in the domain of the Dirichlet form for the space-time process $(s, X^{(\Lambda)}_s)$. By an application of the Fukushima decomposition in the context of time-dependent Dirichlet forms, one concludes that the quadratic variation of the martingale $s \mapsto P^\Lambda_{T-s}\phi(X^\Lambda_s)$ is $\int_0^s \mathbf{a}\nabla P^\Lambda_{T-r}\phi \cdot \nabla P^\Lambda_{T-r}\phi(X^\Lambda(r))\,dr$; see [31, Example 6.5.6]. □

As mentioned in Remark 6.3, a time-dependent version of Theorem 6.2 is valid. We now state it precisely. A proof can be obtained by following the same argument in the proof of Theorem 6.2, but now to the time dependent process $(t, (\mathfrak{X}^{N,+}_t, \mathfrak{X}^{N,-}_t))$. The detail is left to the reader.

Theorem 7.6. *Let $T > 0$, and $f_s \in C_b(E_N)$ and $g_s \in \mathcal{B}(E_N)$ for $s \in [0, T]$. Suppose*

$$\overline{M}_s := f_s(\overline{\mathfrak{X}}^{N,+}_s, \overline{\mathfrak{X}}^{N,-}_s) - \int_0^s g_r(\overline{\mathfrak{X}}^{N,+}_r, \overline{\mathfrak{X}}^{N,-}_r)\,dr$$

is a $\mathcal{F}^{(\overline{\mathfrak{X}}^{N,+},\overline{\mathfrak{X}}^{N,-})}_s$-martingale for $s \in [0, T]$, under $\mathbb{P}^\mu$ for any $\mu \in E_N$. Then

$$M_s := f_s(\mathfrak{X}^{N,+}_s, \mathfrak{X}^{N,-}_s) - \int_0^s (g_r + Kf_r)(\mathfrak{X}^{N,+}_r, \mathfrak{X}^{N,-}_r)\,dr$$

is a $\mathcal{F}^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}_r$-martingale for $s \in [0, T]$, under $\mathbb{P}^\mu$ for any $\mu \in E_N$, where the operator K is given by (6.3).

Consider $X_{(n,m)} := (X^+_1, \cdots, X^+_n,\ \mathfrak{X}^-_1, \cdots, X^-_m) \in (D^\partial_+)^n \times (D^\partial_-)^m$, which consists of independent copies of $X^\pm$'s. The transition density of $X_{(n,m)}$ w.r.t. $\rho_{(n,m)}$ is $p^{(n,m)}$, where

$$p^{(n,m)}(t, (\vec{x}, \vec{y}), (\vec{x'}, \vec{y'})) := \prod_{i=1}^n p^+(t, x_i, x'_i) \prod_{j=1}^m p^-(t, y_j, y'_j)$$
$$\rho_{(n,m)}(\vec{x}, \vec{y}) := \prod_{i=1}^n \rho_+(x_i) \prod_{j=1}^m \rho_-(y_j).$$

The semigroup of $X_{(n,m)}$, denoted by $P^{(n,m)}_t$, is strongly continuous on

$$C^{(n,m)}_\infty := \big\{\Phi \in C(\overline{D}^n_+ \times \overline{D}^m_-) :\ \Phi \text{ vanishes outside } (\overline{D}_+ \setminus \Lambda_+)^n \times (\overline{D}_- \setminus \Lambda_-)^m\big\} \tag{7.4}$$

Clearly, $C^{(1,0)}_\infty = C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $C^{(0,1)}_\infty = C_\infty(\overline{D}_- \setminus \Lambda_-)$.

Corollary 7.7. *Let n and m be any non-negative integers, $T>0$ be any positive number and $\Phi \in C_\infty^{(n,m)}$. Consider the function $f:[0,T]\times E_N \to \mathbb{R}$ defined as follows: $f(s,\mathbf{0}_*) := 0$ and for an arbitrary element $\mu \in E_N \setminus \{\mathbf{0}_*\}$, we can write $\mu = (\frac{1}{N}\sum_{i\in A_+}\mathbf{1}_{x_i},\, \frac{1}{N}\sum_{j\in A_-}\mathbf{1}_{y_j})$ for some index sets A_+ and A_-, then*

$$f(s,\,\mu) \;:=\; \sum_{\substack{i_1,\cdots,i_n\\ \text{distinct}}} \sum_{\substack{j_1,\cdots,j_m\\ \text{distinct}}} P_{T-s}^{(n,m)}\Phi(x^{i_1},\cdots,x^{i_n},\,y^{j_1},\cdots,y^{j_m}),$$

where the first summation is on the collection of all n-tuples $(i_1,\cdots,i_n)$ chosen from distinct elements of A_+, the second summation is on the collection of all m-tuples $(j_1,\cdots,j_m)$ chosen from distinct elements of A_-. Then we have

$$f\big(s,\,(\mathfrak{X}_s^{N,+},\mathfrak{X}_s^{N,-})\big) - \int_0^s Kf(r,\,\cdot)(\mathfrak{X}_r^{N,+},\mathfrak{X}_r^{N,-})\,dr$$

is a $\mathcal{F}_s^{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})}$-martingale for $s\in[0,T]$, under $\mathbb{P}^\nu$, for any $\nu\in E_N$.

Proof Clearly, $f(s,\cdot)\in C_b(E_N)$ for $s\in[0,T]$. By Lemma 7.5, we have $f(s,\overline{\mathfrak{X}}_s)$ is a $\mathcal{F}_s^{\overline{\mathfrak{X}}}$-martingale for $s\in[0,T]$ for all $T\geq 0$. Hence we can take g_r to be constant zero and f_r to be $f(r,\cdot)$ in Theorem 7.6 to finish the proof. □

As an immediate consequence, we obtain the Dynkin's formula for our system: For $0\le t\le T$, we have

$$\mathbb{E}\Big[f\big(T,\,(\mathfrak{X}_T^{N,+},\mathfrak{X}_T^{N,-})\big) - f\big(t,\,(\mathfrak{X}_t^{N,+},\mathfrak{X}_t^{N,-})\big) - \int_t^T Kf(r,\,\cdot)(\mathfrak{X}_r^{N,+},\mathfrak{X}_r^{N,-})\,dr\Big] = 0 \qquad (7.5)$$

Corollary 7.7 is the key to obtain the system of equations satisfied by the correlation functions of the particles in the annihilating diffusion system. This system of equations, usually called BBGKY hierarchy, will be formulated in the forthcoming paper [10]. The specific integral equations that we need to identify subsequential limits of $\{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})\}$ are stated in the following lemmas. These equations are a part of the BBGKY hierarchy.

Lemma 7.8. *For any $\phi_\pm \in C_\infty(\overline{D}_\pm\setminus\Lambda_\pm)$ and $0\le t\le T<\infty$, we have*

$$\begin{aligned} &\mathbb{E}\left[\langle\phi_+,\,\mathfrak{X}_T^{N,+}\rangle + \langle\phi_-,\,\mathfrak{X}_T^{N,-}\rangle\right] - \mathbb{E}\left[\langle P_{T-t}^+\phi_+,\,\mathfrak{X}_t^{N,+}\rangle + \langle P_{T-t}^-\phi_-,\,\mathfrak{X}_t^{N,-}\rangle\right] \qquad (7.6)\\ =\;& -\frac{1}{2}\int_t^T \mathbb{E}\left[\,\langle \ell_{\delta_N}\,(P_{T-r}^+\phi_+ + P_{T-r}^-\phi_-),\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle\,\right]\,dr \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\left[\langle\phi_+,\,\mathfrak{X}_T^{N,+}\rangle^2\right] - \mathbb{E}\left[\langle P_{T-t}^+\phi_+,\,\mathfrak{X}_t^{N,+}\rangle^2\right] \qquad (7.7)\\ =\;& -\int_t^T \mathbb{E}\left[\,\langle P_{T-r}^+\phi_+,\mathfrak{X}_r^{N,+}\rangle\,\langle \ell_{\delta_N}\,(P_{T-r}^+\phi_+),\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle\,\right]\,dr + +o(N), \end{aligned}$$

where $o(N)$ is a term which tends to zero as $N\to\infty$. A similar formula for (7.7) holds for $\mathfrak{X}^{N,-}$.

Proof Since $Dom(\mathcal{A}^{\pm})$ is dense in $C_\infty(\overline{D}_\pm \setminus \Lambda_\pm)$. Therefore, it suffices to check the lemma for $\phi_\pm \in Dom(\mathcal{A}^{\pm})$.

Identity (7.6) follows directly from Corollary 7.7 by taking $f(s,\mu) = \langle P^+_{T-s}\phi_+,\, \mu^+\rangle + \langle P^-_{T-s}\phi_-,\, \mu^-\rangle$.

For (7.7), we can apply Lemma 7.5 and Theorem 7.6, with $f_s(\mu) = \langle P^+_{T-s}\phi_+, \mu^+\rangle^2$ and $g_s(\mu) = \frac{1}{N}\langle \mathbf{a}_+\nabla P^+_{T-s}\phi_+ \cdot \nabla P^+_{T-s}\phi_+,\, \mu^+\rangle$, to obtain

$$
\begin{aligned}
& \mathbb{E}\Big[\langle \phi_+,\, \mathfrak{X}^{N,+}_T\rangle^2] - \mathbb{E}[\langle P^+_{T-t}\phi_+,\, \mathfrak{X}^{N,+}_t\rangle^2\Big] \\
= & -\int_t^T \mathbb{E}\left[\, \langle P^+_{T-r}\phi_+, \mathfrak{X}^{N,+}_r\rangle\, \langle \ell_{\delta_N}\,(P^+_{T-r}\phi_+),\, \mathfrak{X}^{N,+}_r\otimes\mathfrak{X}^{N,-}_r\rangle\right]\, dr \\
& +\frac{1}{2N}\int_t^T \mathbb{E}\left[2\mathbf{a}_+\nabla P^+_{T-s}\phi_+\cdot\nabla P^+_{T-s}\phi_+,\, \mathfrak{X}^{N,+}_r\rangle + \langle \ell_{\delta_N}\,(P^+_{T-r}\phi_+)^2,\, \mathfrak{X}^{N,+}_r\otimes\mathfrak{X}^{N,-}_r\rangle\right]\, dr.
\end{aligned}
$$

Note that the term with a factor $\frac{1}{N}$ converges to zero as $N\to\infty$. This can be proved by the same argument for the bound of the quadratic variation $\mathbb{E}^\mu[(M^{(\phi_+,\phi_-)}_t)^2]$ in Corollary 6.4. Hence we have (7.7). $\square$

We now derive the integral equations satisfied by the integrands (with respect to dr) on the right hand side of (7.6) and (7.7). The integrand (with respect to dr) of the right hand side of (7.7) is of the form

$$
\langle \phi,\, \mu^+\rangle\langle\varphi,\, \mu^+\otimes\mu^-\rangle = \frac{1}{N^3}\Big(\sum_i\sum_j \phi(x_i)\varphi(x_i,y_j) + \sum_\ell\sum_{i\neq\ell}\sum_j \phi(x_\ell)\varphi(x_i,y_j)\Big),
$$

where $\varphi\in\mathcal{B}(\overline{D}_+\times\overline{D}_-)$, $\phi=\phi_+\in\mathcal{B}(\overline{D}_+)$ and $\mu = (\frac{1}{N}\sum_i \mathbf{1}_{x_i},\, \frac{1}{N}\sum_j\mathbf{1}_{y_j})\in E_N$. We define

$$
\begin{aligned}
& P^{(*)}_t(\langle\phi,\, \mu^+\rangle\langle\varphi,\, \mu^+\otimes\mu^-\rangle) \qquad (7.8)\\
:= & \frac{1}{N^3}\Big(\sum_i\sum_j P^{(1,1)}_t(\phi\varphi)(x_i,y_j) + \sum_\ell\sum_{i\neq\ell}\sum_j P^{(2,1)}_t(\phi\varphi)(x_\ell,x_i,y_j)\Big) \\
= & \langle P^{(2,1)}_t(\phi\varphi)(x_1,x_2,y),\; \mu^+(dx_1)\otimes\mu^+(dx_2)\otimes\mu^-(dy)\rangle \\
& +\frac{1}{N}\,\langle\, P^{(1,1)}_t(\phi\varphi)(x,y) - P^{(2,1)}_t(\phi\varphi)(x,x,y),\; \mu^+(dx)\otimes\mu^-(dy)\,\rangle,
\end{aligned}
$$

In $P^{(1,1)}_t(\phi\varphi)$, we view $\phi\varphi$ as the function of two variables $(a,b)\mapsto\phi(a)\varphi(a,b)$; in $P^{(2,1)}_t(\phi\varphi)$, we view $\phi\varphi$ as the function of three variables $(a_1,a_2,b)\mapsto\phi(a_1)\varphi(a_2,b)$. The definition of $P^{(*)}_t$ is motivated by the fact that $f(s,\mu) := P^{(*)}_{T-s}\langle\phi_+\varphi,\, \mu^+\otimes\mu^+\otimes\mu^-\rangle$ is of the same form as the function in Corollary 6.4.

Lemma 7.9. *For any $\varphi\in C^{(1,1)}_\infty$, $\phi_\pm\in C_\infty(\overline{D}_\pm\setminus\Lambda_\pm)$ and $0\le t\le T<\infty$, we have*

$$
\begin{aligned}
& \mathbb{E}[\langle\varphi,\, \mathfrak{X}^{N,+}_T\otimes\mathfrak{X}^{N,-}_T\rangle] - \mathbb{E}[\langle P^{(1,1)}_{T-t}\varphi,\, \mathfrak{X}^{N,+}_t\otimes\mathfrak{X}^{N,-}_t\rangle] \\
= & -\frac{1}{2}\int_t^T \mathbb{E}\Big[\Big\langle\, \ell_{\delta_N}(x,y)\,\Big(\langle F_r(x,\cdot),\, \mathfrak{X}^{N,-}_r\rangle + \langle F_r(\cdot,y),\, \mathfrak{X}^{N,+}_r\rangle - \frac{1}{N}F_r(x,y)\Big), \\
& \qquad \mathfrak{X}^{N,+}_r(dx)\otimes\mathfrak{X}^{N,-}_r(dy)\Big\rangle\Big]\, dr \qquad (7.9)
\end{aligned}
$$

and

$$\begin{aligned}
&\mathbb{E}[\langle \phi_+,\, \mathfrak{X}_T^{N,+}\rangle\langle \varphi,\, \mathfrak{X}_T^{N,+}\otimes \mathfrak{X}_T^{N,-}\rangle] - \mathbb{E}\left[P^{(*)}_{T-t}\,\langle \phi_+\varphi,\, \mathfrak{X}_t^{N,+}\otimes \mathfrak{X}_t^{N,+}\otimes \mathfrak{X}_t^{N,-}\rangle\right] \\
= \; & -\frac{1}{2}\int_t^T \mathbb{E}\bigg[\Big\langle\, \ell_{\delta_N}(x,y)\,\Big(\langle H_r(x,\cdot,\cdot),\, \mathfrak{X}_r^{N,+}\otimes \mathfrak{X}_r^{N,-}\rangle \\
&\qquad +\langle H_r(\cdot,x,\cdot),\, \mathfrak{X}_r^{N,+}\otimes \mathfrak{X}_r^{N,-}\rangle + \langle H_r(\cdot,\cdot,y),\, \mathfrak{X}_r^{N,+}\otimes \mathfrak{X}_r^{N,+}\rangle \\
&\qquad -\frac{1}{N}\left[\langle 2H_r(x,x,\cdot),\, \mathfrak{X}_r^{N,-}\rangle + \langle H_r(\cdot,x,y),\, \mathfrak{X}_r^{N,+}\rangle + \langle H_r(x,\cdot,y),\, \mathfrak{X}_r^{N,+}\rangle\right] \\
&\qquad +\frac{1}{N}\left[\langle G_r(x,\cdot),\, \mathfrak{X}_r^{N,-}\rangle + \langle G_r(\cdot,y),\, \mathfrak{X}_r^{N,+}\rangle - \langle H_r(\cdot,\cdot,y),\, \mathfrak{X}_r^{N,+}\rangle\right] \\
&\qquad +\frac{1}{N^2}\left[2H_r(x,x,y) - G_r(x,y)\right]\Big),\, \mathfrak{X}_r^{N,+}(dx)\otimes \mathfrak{X}_r^{N,-}(dy)\Big\rangle\bigg]\,dr, \qquad (7.10)
\end{aligned}$$

where $F_r = P^{(1,1)}_{T-r}\varphi$, $G_r = P^{(1,1)}_{T-r}(\phi_+\varphi)$ *and* $H_r = P^{(2,1)}_{T-r}(\phi_+\varphi)$. *A similar formula for (7.10) holds for* $\mathbb{E}\left[\langle \phi_-,\, \mathfrak{X}_T^{N,-}\rangle\,\langle \varphi,\, \mathfrak{X}_T^{N,+}\otimes \mathfrak{X}_T^{N,-}\rangle\right]$.

Proof We first prove (7.9). Consider, for $s\in[0,T]$, $f_s(\mu) = f(s,\mu) := \langle P^{(1,1)}_{T-s}\varphi,\, \mu^+\otimes\mu^-\rangle$. Then (7.9) follows from Corollary 6.4 by directly calculating $\mathbb{E}[K(f_r)(\mathfrak{X}_r^{N,+},\mathfrak{X}_r^{N,-})]$ as follows: If $U_N(\vec{x},\vec{y}) = \mu$ where $(\vec{x},\vec{y})\in E_N^{(m)}$, then

$$\begin{aligned}
-Kf_r(\mu) &= \frac{1}{2N}\sum_{i=1}^m\sum_{j=1}^m \ell_{\delta_N}(x_i,y_j)\,\Big(f_r(\mu) - f_r(\mu^+ - \frac{1}{N}\mathbf{1}_{\{x_i\}},\, \mu^- - \frac{1}{N}\mathbf{1}_{\{y_j\}})\Big) \\
&= \frac{1}{2N}\sum_{i=1}^m\sum_{j=1}^m \ell_{\delta_N}(x_i,y_j)\,\Big(\frac{1}{N^2}\Big(\sum_l F_r(x_i,y_l) + \sum_k F_r(x_k,y_j) - F_r(x_i,y_j)\Big)\Big) \\
&= \frac{1}{2N}\sum_{i=1}^m\sum_{j=1}^m \ell_{\delta_N}(x_i,y_j)\,\Big(\frac{1}{N}\langle F_r(x_i),\mu^-\rangle + \frac{1}{N}\langle F_r(y_j),\mu^+\rangle - \frac{1}{N^2}F_r(x_i,y_j)\Big) \\
&= \frac{1}{2}\big\langle \ell_{\delta_N}(\langle F_r,\mu^-\rangle + \langle F_r,\mu^+\rangle - N^{-1}F_r),\, \mu^+\otimes\mu^-\big\rangle.
\end{aligned}$$

For (7.10), we choose $f_s(\mu) := P^{(*)}_{T-s}\langle \phi_+\varphi,\, \mu^+\otimes\mu^+\otimes\mu^-\rangle$ instead and follow the same argument as above. The expression on the right hand side of (7.10) follows from the observation that, for fixed (i,j), we have

$$\begin{aligned}
& N^3\Big(g_r(\mu) - g_r(\mu^+ - \frac{1}{N}\mathbf{1}_{\{x_i\}},\, \mu^- - \frac{1}{N}\mathbf{1}_{\{y_j\}})\Big) \\
= \; & \sum_q\sum_\ell H_r(x_i,x_q,y_\ell) + \sum_p\sum_\ell H_r(x_p,x_i,y_\ell) + \sum_p\sum_q H_r(x_p,x_q,y_j) \\
& -\sum_\ell H_r(x_i,x_i,y_\ell) - \sum_p H_r(x_p,x_i,y_j) - \sum_q H_r(x_i,x_q,y_j) + H_r(x_i,x_i,y_j) \\
& +\sum_\ell G_r(x_i,y_\ell) + \sum_p G_r(x_p,y_j) - G_r(x_i,y_j) \\
& -\sum_\ell H_r(x_i,x_i,y_\ell) - \sum_p H_r(x_p,x_p,y_j) + H_r(x_i,x_i,y_j).
\end{aligned}$$

The above expression can be obtained by using the Inclusion-Exclusion Principle. $\square$

The next two sections will be devoted to the proof of (6.17) and (6.18), respectively.

7.3 First moment

The goal of this subsection is to prove (6.17) in Proposition 6.8. The following key lemma allows us to interchange limits. This is a crucial step in our characterization of $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$, and is the step where Assumption 2.6 that $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$ is used.

Lemma 7.10. *Suppose Assumption 2.6 holds. Then for any $t>0$ and any $\phi\in C_\infty^{(1,1)}$, as $\varepsilon\to 0$, each of $\mathbb{E}^\infty\left[\langle \ell_\varepsilon\phi,\, v_+(t)\rho_+\otimes v_-(t)\rho_-\rangle\right]$ and $\mathbb{E}\left[\langle \ell_\varepsilon\phi,\, \mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle\right]$ converges uniformly in $N\in\mathbb{N}$ and in any initial distributions $\{(\mathfrak{X}_0^{N,+},\mathfrak{X}_0^{N,-})\}$. Moreover,*

$$A^\phi(t) := \lim_{\varepsilon\to 0}\mathbb{E}\left[\langle \ell_\varepsilon\phi,\, v_+(t)\rho_+\otimes v_-(t)\rho_-\rangle\right] = \lim_{N'\to\infty}\lim_{\varepsilon\to 0}\mathbb{E}\left[\langle \ell_\varepsilon\phi,\, \mathfrak{X}_t^{N',+}\otimes\mathfrak{X}_t^{N',-}\rangle\right]$$

for any subsequence $\{N'\}$ along which $\{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})\}_N$ converges to $(\mathfrak{X}^{\infty,+},\mathfrak{X}^{\infty,-})$ in distribution in $D([0,T],\mathfrak{M})$. Furthermore, $|A^\phi(t)|\le \|\phi\|\,\|P_t^+f\|\,\|P_t^-g\|\,\|\rho_+\|\,\|\rho_-\|\,\sigma(\partial I)$.

Proof Since $\rho_\pm\in C(\overline{D}_\pm)$ and is strictly positive, for notational simplicity, we assume without loss of generality that $\rho_\pm=1$. (The general case can be proved in the same way.) Recall from (7.9) that for any $\varphi\in C_\infty^{(1,1)}$, $\phi_\pm\in C_\infty(\overline{D}_\pm\setminus\Lambda_\pm)$ and $0\le s\le t<\infty$, we have

$$\begin{aligned}
&\mathbb{E}\left[\langle\varphi,\, \mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle\right]-\mathbb{E}\left[\langle P_{t-s}^{(1,1)}\varphi,\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\right] \qquad (7.11)\\
=\;& -\frac12\int_s^t \mathbb{E}\left[\Big\langle \ell_{\delta_N}\left(\langle P_{t-r}^{(1,1)}\varphi,\, \mathfrak{X}_r^{N,-}\rangle+\langle P_{t-r}^{(1,1)}\varphi,\, \mathfrak{X}_r^{N,+}\rangle-\frac1N P_{t-r}^{(1,1)}\varphi\right),\, \mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\Big\rangle\right]dr.
\end{aligned}$$

Note that $\ell_\varepsilon\phi\in C_\infty^{(1,1)}$ for ε small enough since I is disjoint from $\Lambda_\pm$. We fix $s\in(0,t)$. Putting $\ell_{\varepsilon_1}\phi$ and $\ell_{\varepsilon_2}\phi$, respectively, in the place of φ in (7.11) and then subtract, we have

$$\begin{aligned}
\Theta :=\;& \Big|\mathbb{E}[\langle \ell_{\varepsilon_1}\phi,\, \mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle]-\mathbb{E}[\langle \ell_{\varepsilon_2}\phi,\, \mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle]\Big| \qquad (7.12)\\
=\;& \Big|\mathbb{E}\left[\langle F_s,\, \mathfrak{X}_s^{N,+}\otimes\mathfrak{X}_s^{N,-}\rangle\right]-\frac12\int_s^t\mathbb{E}\Big[\Big\langle \ell_{\delta_N}\Big(\langle F_r,\, \mathfrak{X}_r^{N,-}\rangle+\langle F_r,\, \mathfrak{X}_r^{N,+}\rangle-\frac1N F_r\Big),\, \mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\Big\rangle\Big]dr\Big|\\
\le\;& \mathbb{E}\left[\langle |F_s|,\, \overline{\mathfrak{X}}_s^{N,+}\otimes\overline{\mathfrak{X}}_s^{N,-}\rangle\right]+\frac12\int_s^t\mathbb{E}\Big[\Big\langle \ell_{\delta_N}\langle|F_r|,\, \overline{\mathfrak{X}}_r^{N,-}\rangle,\, \overline{\mathfrak{X}}_r^{N,+}\otimes\overline{\mathfrak{X}}_r^{N,-}\Big\rangle\Big]\\
&+\frac12\mathbb{E}\Big[\Big\langle \ell_{\delta_N}\langle|F_r|,\, \overline{\mathfrak{X}}_r^{N,+}\rangle,\, \overline{\mathfrak{X}}_r^{N,+}\otimes\overline{\mathfrak{X}}_r^{N,-}\Big\rangle\Big]+\frac{1}{2N}\mathbb{E}\Big[\Big\langle |\ell_{\delta_N}F_r|,\, \overline{\mathfrak{X}}_r^{N,+}\otimes\overline{\mathfrak{X}}_r^{N,-}\Big\rangle\Big]dr\\
\le\;& \left\|P_s^{(1,1)}(|F_s|)\right\|+\frac12\int_s^t(A_1+A_2+A_3)\,dr,
\end{aligned}$$

where $F_r:=P_{t-r}^{(1,1)}(\ell_{\varepsilon_1}\phi-\ell_{\varepsilon_2}\phi)$, $A_1:=\left\|P_r^{(1,1)}\left(\ell_{\delta_N}P_r^{(0,1)}(|F_r|)\right)\right\|$, $A_2:=\left\|P_r^{(1,1)}\left(\ell_{\delta_N}P_r^{(1,0)}(|F_r|)\right)\right\|$, and $A_3:=\frac1N\left\|P_r^{(1,1)}\left(|\ell_{\delta_N}F_r|\right)\right\|$.

Clearly $\left\|P_s^{(1,1)}(|F_s|)\right\|\le\|F_s\|$. By applying Lemma 7.2 to the equi-continuous and uniformly bounded family

$$\{(x,y)\mapsto\phi(x)\,p(t-s,(a,b),(x,y)):\ (a,b)\in\overline{D}_+\times\overline{D}_-\}\subset C_\infty^{(1,1)}\subset C(\overline{D}_+\times\overline{D}_-),$$

we see that $\|F_s\|$ converges to zero uniformly for $N \in \mathbb{N}$ and for any initial configuration, as ε_1 and ε_2 both tend to zero.

By definition of A_1, (1.3), the Gaussian upper bound estimate (2.2) for the transition density p of the reflected diffusion, we have

$$\begin{aligned} A_1 &= \sup_{(a,b)} \int_{\overline{D}_+} \int_{\overline{D}_-} \ell_{\delta_N}(x,y) \left(\sup_y P_r^- \left(|F_r|\right)(x,y) \right) p(r,(a,b),(x,y))\, dxdy \\ &\le \left(\sup_{(x,y)} P_r^- \left(|F_r|\right)(x,y) \right) \frac{C(d,D_+,D_-)}{s^d} \quad \text{if } N \ge N(d,D_+,D_-). \end{aligned}$$

Using this bound, we have

$$\begin{aligned} &\int_s^t A_1 dr \\ \le\ & \frac{C}{s^d} \int_s^t \sup_{(x,y)} P_r^- \left(|P_{t-r}^{(1,1)} \left(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi\right)| \right)(x,y)\, dr \\ =\ & \frac{C}{s^d} \int_0^{t-s} \sup_{(x,y)} P_{t-w}^- \left(|P_w^{(1,1)} \left(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi\right)| \right)(x,y)\, dw \\ =\ & \frac{C}{s^d} \int_0^{t-s} \left(\sup_{(x,y)} \int_{D_-} \left| \int_{D_+} \int_{D_-} \left(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi\right)(\tilde{x},\tilde{y}) p(w,(x,b),(\tilde{x},\tilde{y}))\, d\tilde{x}\, d\tilde{y} \right| p^-(t-w,y,b)\, db \right) dw \\ \le\ & \frac{C}{s^d} \left(\int_0^\alpha \frac{2C}{\sqrt{w}}\, t^{-d/2}\, dw + \int_\alpha^{t-s} \left\| P_w^{(1,1)} \left(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi\right) \right\| dw \right). \end{aligned} \tag{7.13}$$

The last inequality holds for any $\alpha \in (0, t-s)$. This is because

$$\begin{aligned} & \sup_{(x,y)} \int_{D_-} \int_{D_+} \int_{D_-} \ell_\varepsilon(\tilde{x},\tilde{y})\, p(w,(x,b),(\tilde{x},\tilde{y}))\, d\tilde{x}\, d\tilde{y}\, p^-(t-w,y,b)\, db \\ =\ & \sup_{(x,y)} \int_{D_-} \int_{D_+} \ell_\varepsilon(\tilde{x},\tilde{y})\, p^+(w,x,\tilde{x})\, p^-(t,y,\tilde{y})\, d\tilde{x}\, d\tilde{y} \\ & \qquad \text{by Chapman-Kolmogorov equation for } p^- \\ \le\ & \frac{2C(d,D_+,D_-,T)}{\sqrt{w}}\, t^{-d/2} \quad \text{by applying the bound (2.3) on } D_+. \end{aligned}$$

Hence, from (7.13), by letting $\alpha \downarrow 0$ suitably and applying Lemma 7.2 to the equi-continuous and uniformly bounded family

$$\left\{ (x,y) \mapsto \phi(x)\, p(w,(a,b),(x,y)) :\ (a,b) \in \overline{D}_+ \times \overline{D}_-,\ w \in [\alpha, t-s] \right\} \subset C(\overline{D}_+ \times \overline{D}_-),$$

we see that $\int_s^t A_1\, dr$ converges to 0 as ε_1 and ε_2 tends to 0 uniformly for N large enough. The same conclusion hold for $\int_s^t A_2\, dr$ by the same argument.

So far we have not used the Assumption 2.6 of $\liminf_{N\to\infty} N\, \delta_N^d \in (0,\infty]$. We now use this assumption to show that $\int_s^t A_3\, dr$ tends to 0 uniformly for N large enough, as ε_1 and ε_2 tend to 0. By a change of variable $r \mapsto t-w$,

$$\int_s^t A_3 dr \ \le\ \int_0^{t-s} \sup_{(a,b)} \int_{D_+} \int_{D_-} p(t-w,(a,b),(x,y))\, \frac{1}{N} \ell_{\delta_N}(x,y) \left| P_w^{(1,1)} \left(\ell_{\varepsilon_1}\phi - \ell_{\varepsilon_2}\phi\right)(x,y) \right| dxdy\, dw$$

$$\leq \frac{2C_1}{s^{d/2}\, t^{d/2}} \int_0^\alpha \frac{1}{\sqrt{w}}\, dw + \frac{C_2}{N\, s^d} \int_\alpha^{t-s} \left\| P_w^{(1,1)} \left(\ell_{\varepsilon_1} \phi - \ell_{\varepsilon_2} \phi \right) \right\| dw.$$

The last inequality holds for any $\alpha \in (0, t-s)$, where $C_1 = C_1(d, D_+, D_-, T, \phi)$ and $C_2 = C_2(d, D_+, D_-)$. This is because for $\varepsilon > 0$,

$$\begin{aligned}
& \sup_{(a,b)} \int\int \left(\int\int p(w,(x,y),(\tilde{x},\tilde{y}))\, \ell_\varepsilon(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} \right) p(t-w,(a,b),(x,y))\, \frac{1}{N} \ell_{\delta_N}(x,y)\, dxdy \\
\leq\ & \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \sup_{(a,b)} \sup_{(\tilde{x},\tilde{y})} \frac{1}{c_{d+1} N \delta_N^{d+1}} \int_{D_+^{\delta_N}} \int_{D_- \cap B(x,\delta_N)} p(w,(x,y),(\tilde{x},\tilde{y})) p(t-w,(a,b),(x,y))\, dydx \\
\leq\ & \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \frac{C(d,D_-)}{t^{d/2}} \sup_a \sup_{\tilde{x}} \frac{1}{c_{d+1} N \delta_N^{d+1}} \int_{D_+^{\delta_N}} p^+(w,x,\tilde{x}) p^+(t-w,a,x)\, dx \\
\leq\ & \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \frac{C(d,D_-)}{t^{d/2}} \frac{C(d,D_+)}{s^{d/2}} \sup_{\tilde{x}} \frac{1}{c_{d+1} N \delta_N^{d+1}} \int_{D_+^{\delta_N}} p^+(w,x,\tilde{x})\, dx \\
& \qquad \text{by the Gaussian upper bound (2.2) for } p^+ \\
\leq\ & \frac{|I^\varepsilon|}{c_{d+1}\varepsilon^{d+1}} \frac{C(d,D_+,D_-)}{s^{d/2}\, t^{d/2}} \frac{1}{\sqrt{w}} \qquad \text{for } N \geq N(d, D_+), \\
& \quad \text{by the assumption } \liminf_{N\to\infty} N\, \delta_N^d \in (0,\infty] \text{ and the bound (2.3) on } D_+.
\end{aligned}$$

In conclusion, we have shown that $\left\{ \mathbb{E}[\langle \ell_\varepsilon \phi,\, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle] \right\}_{\varepsilon>0}$ is a Cauchy family and converges as $\varepsilon \to 0$ to a number in $[-\infty, \infty]$. Furthermore, the convergence is uniformly for N large enough and for any initial configuration. On other hand, since $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}_N$ converges in distribution to a continuous process to $(v_+(\cdot,x)dx,\, v_-(\cdot,y)dy)$ and $(\mu^+, \mu^-) \mapsto \langle \ell_\varepsilon \phi,\, \mu^+ \otimes \mu^- \rangle$ is a bounded continuous function on $\mathfrak{M}$, we have

$$\mathbb{E}^\infty \left[\langle \ell_\varepsilon \phi,\, v_+(t) \otimes v_-(t) \rangle \right] = \lim_{N'\to\infty} \mathbb{E} \left[\langle \ell_\varepsilon \phi,\, \mathfrak{X}_t^{N',+} \otimes \mathfrak{X}_t^{N',-} \rangle \right]$$

for all $t \geq 0$. Hence the proof for the convergence of $\lim_{\varepsilon\to 0} \mathbb{E}^\infty \left[\langle \ell_\varepsilon \phi,\, v_+(t) \otimes v_-(t) \rangle \right]$ is the same. Finally, the bound for $|A^\phi(t)|$ follows directly from Lemma 6.7 and Lemma 7.2. This bound also tells us that $A^\phi(t)$ actually lies in $\mathbb{R}$. □

From the above lemma, we immediately have

Corollary 7.11. *Suppose that Assumption 2.6 holds and $\{N'\}$ is any subsequence along which $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}_N$ converges to $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in distribution in $D([0,T], \mathfrak{M})$. Then for $\phi \in C_\infty(\overline{D}_+ \setminus \Lambda_+) \cup C_\infty(\overline{D}_- \setminus \Lambda_-)$, we have*

$$\lim_{N'\to\infty} \mathbb{E}[\langle \ell_{\delta_{N'}} \phi,\, \mathfrak{X}_r^{N',+} \otimes \mathfrak{X}_r^{N',-} \rangle] = A^\phi(r) \quad \textit{for } r > 0, \textit{ and}$$

$$\lim_{N'\to\infty} \int_s^t \mathbb{E} \left[\langle \ell_{\delta_{N'}} \phi,\, \mathfrak{X}_r^{N',+} \otimes \mathfrak{X}_r^{N',-} \rangle \right] dr = \int_s^t A^\phi(r)\, dr \quad \textit{for } 0 < s \leq t < \infty. \tag{7.14}$$

Question. It is an interesting question if one can strengthen (7.14) to include $s = 0$.

We can now present our proof for (6.17) by applying a Gronwall type argument to (7.18).

Proof of (6.17). Without loss of generality, we continue to assume $\rho_\pm = 1$. Recall from (7.6) that for $\phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $0 < s \le t < \infty$, we have

$$\mathbb{E}\left[\langle \phi_+, \mathfrak{X}^{N,+}_t \rangle\right] - \mathbb{E}\left[\langle P^+_{t-s}\phi_+, \mathfrak{X}^{N,+}_s \rangle\right] = -\frac{1}{2}\int_s^t \mathbb{E}\left[\langle \ell_{\delta_N}\, P^+_{t-r}\phi_+, \mathfrak{X}^{N,+}_r \otimes \mathfrak{X}^{N,-}_r \rangle\right] dr. \tag{7.15}$$

By (7.14), we can let $N \to \infty$ to obtain

$$\mathbb{E}^\infty[\langle \phi_+,\, v_+(t)\rangle] - \mathbb{E}^\infty[\langle P^+_{t-s}\phi_+,\, v_+(s)\rangle] = -\frac{1}{2}\mathbb{E}^\infty\left[\int_s^t A^{P^+_{t-r}\phi_+}(r)\, dr\right] \tag{7.16}$$

for $0 < s \le t < \infty$. Now let $s \to 0$. By the uniform bound for (v_+, v_-) given by Lemma 6.7, the continuity of $(v_+(s), v_-(s))$ in s and Lebesgue dominated convergence theorem, we obtain

$$\mathbb{E}^\infty[\langle \phi_+,\, v_+(t)\rangle] - \langle P^+_t\phi_+,\, u^+_0\rangle = -\frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\mathbb{E}^\infty\left[\langle \ell_\varepsilon\, P^+_{t-r}\phi_+,\, v_+(r)\otimes v_-(r)\rangle\right] dr. \tag{7.17}$$

Using the first equation in (4.1) in the definition of (u_+, u_-), the above equation (7.17) also holds if we replace (v_+, v_-) by (u_+, u_-). On subtraction, we get

$$\begin{aligned} &\big\langle \phi_+,\, u_+(t) - \mathbb{E}^\infty[v_+(t)]\big\rangle \\ = \;& -\frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\int_{D_-}\int_{D_+} \ell_\varepsilon(x,y)\, P^+_{t-r}\phi_+(x)\, \Big(u_+(r,x)u_-(r,y) - \mathbb{E}^\infty[v_+(r,x)v_-(r,y)]\Big)\, dxdy\, dr. \end{aligned} \tag{7.18}$$

The above equation holds for $\phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$ (and since ρ_+ has support in the entire domain $\overline{D}_+$), so we have

$$\begin{aligned} &u_+(t) - \mathbb{E}^\infty[v_+(t)] \\ = \;& -\frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\int_{D_-}\int_{D_+} \ell_\varepsilon(x,y)\, p^+(t-r,x,\cdot)\, \Big(u_+(r,x)u_-(r,y) - \mathbb{E}^\infty[v_+(r,x)v_-(r,y)]\Big)\, dxdy\, dr \end{aligned} \tag{7.19}$$

almost everywhere in D_+.

Let $w_\pm(t) := u_\pm(t) - \mathbb{E}^\infty[v_\pm(t)] \in \mathcal{B}_b(D_\pm)$ and $\|w_\pm(r)\|_\pm$ be the L^∞ norm in $D_\pm$. Then by the a.s. bound of $v_\pm$ in Lemma 6.7 and a simple use of triangle inequality, we have $\|u_+(r,x)u_-(r,y) - \mathbb{E}^\infty[v_+(r,x)v_-(r,y)]\| \le (\|u^+_0\|\,\|w_-(r)\| + \|u^-_0\|\,\|w_+(r)\|)$. On other hand,

$$\begin{aligned} &\int_{D_-}\int_{D_+} \ell_\varepsilon(x,y)\, p^+(t-r,x,a)\, dxdy \\ = \;& \frac{1}{c_{d+1}\varepsilon^{d+1}}\int_{I^\varepsilon} p^+(t-r,x,a)\, dxdy \\ \le \;& \frac{1}{c_{d+1}\varepsilon^{d+1}}\int_{D^\varepsilon_+}\int_{B(x,\varepsilon)\cap D^\varepsilon_-} p^+(t-r,x,a)\, dydx \\ \le \;& \frac{|B(x,\varepsilon)\cap D^\varepsilon_-|}{c_{d+1}\varepsilon^{d+1}}\int_{D^\varepsilon_+} p^+(t-r,x,a)\, dx \\ \le \;& \frac{C(d,D_+)}{\sqrt{t-r}} + \tilde{C}(d,D_+) \quad \text{uniformly for } a \in \overline{D}_+, \text{ for } \varepsilon < \varepsilon(d,D_+). \end{aligned} \tag{7.20}$$

Using these observations, it is easy to check that (7.19) implies

$$\|w_+(t)\|_+ \leq \int_0^t (\|u_0^+\| \, \|w_-(r)\| + \|u_0^-\| \, \|w_+(r)\|) \, \frac{C(d, D_+, T)}{\sqrt{t-r}} \, dr. \tag{7.21}$$

By the same argument, we have

$$\|w_-(t)\|_- \leq \int_0^t (\|u_0^+\| \, \|w_-(r)\| + \|u_0^-\| \, \|w_+(r)\|) \, \frac{C(d, D_-, T)}{\sqrt{t-r}} \, dr. \tag{7.22}$$

Adding (7.21) and (7.22), we have, for $C = C(\|u_0^+\|, \|u_0^-\|, d, D_+, D_-, T)$,

$$\|w_+(t)\|_+ + \|w_-(t)\|_- \leq C \int_0^t (\|w_-(r)\| + \|w_+(r)\|) \, \frac{1}{\sqrt{t-r}} \, dr. \tag{7.23}$$

By a "Gronwall type" argument (cf. [21]), we have $\|w_+(t)\|_+ + \|w_-(t)\|_- = 0$ for all $t \in [0, T]$. Since $T > 0$ is arbitrary, we have $\|w_+(t)\|_+ + \|w_-(t)\|_- = 0$ for all $t \geq 0$. This completes the proof for (6.17). □

7.4 Second moment

In this subsection, we give a proof for (6.18) in Proposition 6.8. We start with a key lemma that is analogous to Lemma 7.10.

Lemma 7.12. *Suppose Assumption 2.6 holds. Then for any $t > 0$ and any $\phi \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$, as $\varepsilon \to 0$, each of $\mathbb{E}^\infty \left[\langle \phi, \, v_+(t) \rangle_{\rho_+} \langle \ell_\varepsilon \phi, \, v_+(t)\rho_+ \otimes v_-(t)\rho_- \rangle \right]$ and $\mathbb{E} \left[\langle \phi, \, \mathfrak{X}_t^{N,+} \rangle \langle \ell_\varepsilon \phi, \, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle \right]$ converges uniformly for $N \in \mathbb{N}$ and for any initial distributions $\{(X_0^{N,+}, \mathfrak{X}_0^{N,-})\}$. Moreover, we have*

$$\begin{aligned} B^\phi(t) \;&:=\; \lim_{\varepsilon \to 0} \mathbb{E}^\infty \left[\langle \phi, \, v_+(t) \rangle_{\rho_+} \langle \ell_\varepsilon \phi, \, v_+(t)\rho_+ \otimes v_-(t)\rho_- \rangle \right] \\ &=\; \lim_{N' \to \infty} \lim_{\varepsilon \to 0} \mathbb{E} \left[\langle \phi, \, \mathfrak{X}_t^{N',+} \rangle \langle \ell_\varepsilon \phi, \, \mathfrak{X}_t^{N',+} \otimes \mathfrak{X}_t^{N',-} \rangle \right] \in \mathbb{R} \end{aligned}$$

for any subsequence $\{N'\}$ along which $\{(\mathfrak{X}^{N,+}, \mathfrak{X}^{N,-})\}_N$ converges to $(\mathfrak{X}^{\infty,+}, \mathfrak{X}^{\infty,-})$ in distribution in $D([0,T], \mathfrak{M})$. Similar results hold for $\phi \in C_\infty(\overline{D}_- \setminus \Lambda_-)$, but with $\langle \phi, \, v_-(t) \rangle_{\rho_-}$ and $\langle \phi, \, \mathfrak{X}_t^{N,-} \rangle$ in place of $\langle \phi, \, v_+(t) \rangle_{\rho_+}$ and $\langle \phi, \, \mathfrak{X}_t^{N,+} \rangle$ respectively.

Proof The proof follows the same strategy as that of Lemma 7.10, based on (7.10) rather than (7.9). We only provide the main steps. Without loss of generality, assume $\phi = \phi_+ \in C_\infty(\overline{D}_+ \setminus \Lambda_+)$ and $\rho_\pm = 1$.

Suppose $t > 0$ and $s \in (0, t)$ are fixed. Then (7.10) implies that

$$\begin{aligned} \Theta \;&:=\; \left| \mathbb{E} \left(\langle \phi, \, \mathfrak{X}_t^{N,+} \rangle \langle \ell_{\varepsilon_1} \phi, \, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle - \langle \phi, \, \mathfrak{X}_t^{N,+} \rangle \langle \ell_{\varepsilon_2} \phi, \, \mathfrak{X}_t^{N,+} \otimes \mathfrak{X}_t^{N,-} \rangle \right) \right| \qquad (7.24) \\ &\leq\; \left| \mathbb{E} \left(P_{t-s}^{(*)} \, \langle \, \phi(x_1)(\ell_{\varepsilon_1}(x_2, y) - \ell_{\varepsilon_2}(x_2, y))\phi(x_2), \; \mathfrak{X}_t^{N,+}(dx_1) \otimes \mathfrak{X}_t^{N,+}(dx_2) \otimes \mathfrak{X}_t^{N,-}(dy) \, \rangle \right) \right| \\ &\quad + \frac{1}{2} \int_s^t \mathbb{E} \Big[\Big\langle \, \ell_{\delta_N}(x, y) \Big(\langle H_r(x, \cdot, \cdot), \, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle \\ &\qquad\qquad + \langle H_r(\cdot, x, \cdot), \, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,-} \rangle + \langle H_r(\cdot, \cdot, y), \, \mathfrak{X}_r^{N,+} \otimes \mathfrak{X}_r^{N,+} \rangle \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{N}\left[\langle 2H_r(x,x,\cdot),\, \mathfrak{X}_r^{N,-}\rangle + \langle H_r(\cdot,x,y),\, \mathfrak{X}_r^{N,+}\rangle + \langle H_r(x,\cdot,y),\, \mathfrak{X}_r^{N,+}\rangle\right] \\
&+\frac{1}{N}\left[\langle G_r(x,\cdot),\, \mathfrak{X}_r^{N,-}\rangle + \langle G_r(\cdot,y),\, \mathfrak{X}_r^{N,+}\rangle + \langle H_r(\cdot,\cdot,y),\, \mathfrak{X}_r^{N,+}\rangle\right] \\
&+\frac{1}{N^2}\left[2H_r(x,x,y) - G_r(x,y)\right]\Big),\, \mathfrak{X}_r^{N,+}(dx)\otimes \mathfrak{X}_r^{N,-}(dy)\Big\rangle\Big]\, dr,
\end{aligned}
$$

where

$$
\begin{aligned}
G_r &:= \left|P_{t-r}^{(1,1)}\Big(\, \phi^2(x)\ (\ell_{\varepsilon_1}(x,y) - \ell_{\varepsilon_2}(x,y))\ \Big)\right| \in C_\infty^{(1,1)} \subset C(\overline{D}_+ \times \overline{D}_-) \quad \text{and} \\
H_r &:= \left|P_{t-r}^{(2,1)}\Big(\, \phi(x_1)\phi(x_2)\ (\ell_{\varepsilon_1}(x_2,y_1) - \ell_{\varepsilon_2}(x_2,y_1))\ \Big)\right| \in C_\infty^{(2,1)} \subset C(\overline{D}_+^2 \times \overline{D}_-).
\end{aligned}
$$

In the formula for G_r above, $P_{t-r}^{(1,1)}(\varphi(x,y)) \in C(\overline{D}_+ \times \overline{D}_-)$ is defined as

$$
(a,b) \mapsto \int_{\overline{D}_+\times\overline{D}_-} p^{(1,1)}(t-r,\,(a,b),\,(x,y))\, dxdy.
$$

The expression $P_{t-r}^{(2,1)}(\varphi(x,y))$ is defined in a similar way.

Comparison with $(\overline{\mathfrak{X}}^{N,+},\, \overline{\mathfrak{X}}^{N,-})$ then yields

$$
\Theta \le (1+\frac{1}{N})\,\|H_s\| + \frac{1}{N}\,\|G_s\| + \int_s^t \left(\sum_{i=1}^9 A_i + B_1 + B_2\right) dr, \tag{7.25}
$$

where, with abbreviations that will be explained,

$$
\begin{aligned}
A_1 &:= \left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)\big\|P_r^{(1,1)}H_r(x,\cdot,\cdot)\big\|\right)\right\| \\
A_2 &:= \left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)\big\|P_r^{(1,1)}H_r(\cdot,x,\cdot)\big\|\right)\right\| \\
A_3 &:= \left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)\big\|P_r^{(2,0)}H_r(\cdot,\cdot,y)\big\|\right)\right\| \\
A_4 &:= \frac{2}{N}\left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)\big\|P_r^{(0,1)}H_r(x,x,\cdot)\big\|\right)\right\| \\
A_5 &:= \frac{1}{N}\left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)\big\|P_r^{(1,0)}H_r(\cdot,x,y)\big\|\right)\right\| \\
A_6 &:= \frac{1}{N}\left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)\big\|P_r^{(1,0)}H_r(x,\cdot,y)\big\|\right)\right\| \\
A_7 &:= \frac{1}{N}\left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)\big\|P_r^{(0,1)}G_r(x,\cdot)\big\|\right)\right\| \\
A_8 &:= \frac{1}{N}\left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)\big\|P_r^{(1,0)}G_r(\cdot,y)\big\|\right)\right\| \\
A_9 &:= \frac{1}{N}\left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)\big\|P_r^{(1,0)}H_r(\cdot,\cdot,y)\big\|\right)\right\| \\
B_1 &:= \frac{2}{N^2}\left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)H_r(x,x,y)\right)\right\| \\
B_2 &:= \frac{1}{N^2}\left\| P_r^{(1,1)}\left(\ell_{\delta_N}(x,y)G_r(x,y)\right)\right\|
\end{aligned}
$$

In the above, the first $P_r^{(1,1)}$ acts on the (x,y) variable, while the second $P_r^{(i,j)}$ in each A_i acts on the ' $\cdot$ ' variable. Beware of the difference between $P_r^{(2,0)}H_r(\cdot,\cdot,y)$ and $P_r^{(1,0)}H_r(\cdot,\cdot,y)$ in

A_3 and A_9 respectively. In fact, $P_r^{(2,0)} H_r(\cdot,\cdot,y)$ is defined as the function on $\overline{D}_+^2$ which maps (a_1, a_2) to $\int_{D_+^2} p^{(2,0)}(r,\,(a_1,a_2),\,(x_1,x_2))\,H_r(x_1,x_2,y)\,d(x_1,x_2)$, while $P_r^{(1,0)} H_r(\cdot,\cdot,y)$ is defined as the function on $\overline{D}_+$ which maps a_1 to $\int_{D_+} p^{(1,0)}(r,\,a_1,\,x)\,H_r(x,x,y)\,dx$.

The rest of the proof goes in the same way as that for Lemma 7.10. For example, note that

$$\begin{aligned}\|H_s\| \;&=\; \sup_{(a_1,a_2,b_1)} \Big| \int_{D_+^2\times D_-} \phi(x_1)\phi(x_2)\ (\ell_{\varepsilon_1}(x_2,y_1)-\ell_{\varepsilon_2}(x_2,y_1)) \\ &\qquad\qquad p^{(2,1)}(t-s,(a_1,a_2,b_1),(x_1,x_2,y_1))\,d(x_1,x_2,y_1)\Big|.\end{aligned}$$

By applying Lemma 7.2 to the equi-continuous and uniformly bounded family

$$\big\{(x_1,x_2,y)\mapsto \phi(x_1)\phi(x_2)\,p^{(2,1)}(t-s,(a_1,a_2,b),(x_1,x_2,y)):\ (a_1,a_2,b)\in \overline{D}_+^2\times\overline{D}_-\big\}\subset C(\overline{D}_+^2\times\overline{D}_-),$$

we see that $\|H_s\|$ converges to zero uniformly for N large enough and for any initial configuration, as ε_1 and ε_2 both tend to zero. The integral term with respect to dr can be estimated as in the proof of Lemma 7.10, using the bound (2.3), Lemma 7.2 and Assumption 2.6 that $\liminf_{N\to\infty} N\,\delta_N^d \in (0,\infty]$.

We have shown that $\Big\{\mathbb{E}\Big[\langle\phi,\,\mathfrak{X}_t^{N,+}\rangle\langle\ell_\varepsilon\phi,\,\mathfrak{X}_t^{N,+}\otimes\mathfrak{X}_t^{N,-}\rangle\Big]\Big\}_{\varepsilon>0}$ is a Cauchy family which converges, as $\varepsilon\to 0$, uniformly for N large enough and for any initial configuration. Hence $B^\phi(t)$ in the statement of the lemma exists in $[-\infty,\infty]$. Finally, we have $B^\phi(t)\in\mathbb{R}$ since $|B^\phi(t)|<\infty$ by Lemma 6.7 and Lemma 7.2. □

From the above lemma, we immediately obtain

Corollary 7.13. *Suppose Assumption 2.6 holds and $\{N'\}$ is a subsequence along which $\{(\mathfrak{X}^{N,+},\mathfrak{X}^{N,-})\}$ converges to $(\mathfrak{X}^{\infty,+},\mathfrak{X}^{\infty,-})$ in distribution in $D([0,T],\mathfrak{M})$. Then for $\phi\in C_\infty(\overline{D}_+\setminus\Lambda_+)$,*

$$\lim_{N'\to\infty}\mathbb{E}\Big[\langle\phi,\,\mathfrak{X}_r^{N',+}\rangle\langle\ell_{\delta_{N'}}\phi,\,\mathfrak{X}_r^{N',+}\otimes\mathfrak{X}_r^{N',-}\rangle\Big]=B^\phi(r)\quad\textit{for } r>0,\quad\textit{and}$$

$$\lim_{N'\to\infty}\int_s^t \mathbb{E}\Big[\langle\phi,\,\mathfrak{X}_r^{N',+}\rangle\langle\ell_{\delta_{N'}}\phi,\,\mathfrak{X}_r^{N',+}\otimes\mathfrak{X}_r^{N',-}\rangle\Big]\,dr=\int_s^t B^\phi(r)\,dr\quad\textit{for } 0<s\le t<\infty. \tag{7.26}$$

We are now ready to give the

Proof of (6.18). As before, without loss of generality we assume $\rho_\pm=1$. Recall from (7.7) that for $\phi=\phi_+\in C_\infty(\overline{D}_+\setminus\Lambda_+)$ and $0<s\le t<\infty$, we have

$$\begin{aligned}&\mathbb{E}[\langle\phi,\mathfrak{X}_t^{N,+}\rangle^2]-\mathbb{E}[\langle P_{t-s}^+\phi,\mathfrak{X}_s^{N,+}\rangle^2] \qquad (7.27)\\ =\;& -\frac{1}{2}\int_s^t \mathbb{E}[\,\langle P_{t-r}^+\phi,\mathfrak{X}_r^{N,+}\rangle\,\langle\ell_{\delta_N}(P_{t-r}^+\phi),\,\mathfrak{X}_r^{N,+}\otimes\mathfrak{X}_r^{N,-}\rangle\,]\,dr+o(N).\end{aligned}$$

Letting $N'\to\infty$ in (7.26), we get

$$\mathbb{E}^\infty[\langle\phi,\,v_+(t)\rangle^2]-\mathbb{E}^\infty[\langle P_{t-s}^+\phi,\,v_+(s)\rangle^2=-\frac{1}{2}\int_s^t B^{P_{t-r}^+\phi}(r)\,dr$$

for $0 < s \le t < \infty$. Now let $s \to 0$. By the uniform bound for (v_+, v_-) given by Lemma 6.7, the continuity of $(v_+(s), v_-(s))$ in s and the Lebesgue dominated convergence theorem, we obtain

$$\mathbb{E}^\infty \left[\langle \phi_+,\, v_+(t)\rangle^2\right] - \langle P_t^+\phi,\, u_0^+\rangle^2 = -\frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\mathbb{E}^\infty \left[\langle P_{t-r}^+\phi,\, v_+(r)\rangle\langle \ell_\varepsilon P_{t-r}^+\phi,\, v_+(r)\otimes v_-(r)\rangle\right] dr. \tag{7.28}$$

Using the definition of (u_+, u_-), the above equation (7.28) also holds if we replace (v_+, v_-) by (u_+, u_-). On subtraction, we get

$$\begin{aligned}\mathbb{E}^\infty[\langle \phi,\, v_+(t)\rangle^2] - \langle \phi, u_+(t)\rangle^2 \quad &= \quad \frac{1}{2}\int_0^t \lim_{\varepsilon\to 0}\mathbb{E}^\infty \Big[\langle P_{t-r}^+\phi, u_+(r)\rangle\,\langle \ell_\varepsilon P_{t-r}^+\phi,\, u_+(r)\otimes u_-(r)\rangle \\ &\qquad -\langle P_{t-r}^+\phi,\, v_+(r)\rangle\,\langle \ell_\varepsilon P_{t-r}^+\phi,\, v_+(r)\otimes v_-(r)\rangle\Big]\, dr. \end{aligned} \tag{7.29}$$

The left hand side of (7.29) equals $\mathbb{E}^\infty[\langle \phi,\, v_+(t) - u_+(t)\rangle^2]$ because $\mathbb{E}^\infty[\langle \phi,\, v_+(t)\rangle] = \langle \phi, u_+(t)\rangle$. Since $\mathbb{E}^\infty[\langle \ell_\varepsilon P_{t-r}^+\phi,\, v_+(r)\otimes v_-(r)\rangle] = \langle \ell_\varepsilon P_{t-r}^+\phi,\, u_+(r)\otimes u_-(r)\rangle$, the integrand in the right hand side of (7.29) with respect to dr equals

$$\lim_{\varepsilon\to 0}\mathbb{E}^\infty \left[\langle \ell_\varepsilon P_{t-r}^+\phi,\, v_+(r)\otimes v_-(r)\rangle\,(\langle P_{t-r}^+\phi,\, u_+(r) - v_+(r)\rangle)\right] \le C\,\mathbb{E}^\infty\left[\left|\langle P_{t-r}^+\phi,\, u_+(r) - v_+(r)\rangle\right|\right].$$

The constant $C = C(\phi, f, g, D_+, D_-)$ above arises from the uniform bound for $v(r)$ in Lemma 6.7 and the bound (2.3). Hence we have

$$\mathbb{E}^\infty[\langle \phi,\, v_+(t) - u_+(t)\rangle^2] \le C\int_0^t \mathbb{E}^\infty\left[\left|\langle P_{t-r}^+\phi,\, u_+(r) - v_+(r)\rangle\right|\right] dr.$$

Letting $w_+(t) = u_+(t) - v_+(t)$, we obtain

$$\mathbb{E}^\infty[\langle \phi,\, w_+(t)\rangle^2] \le C\int_0^t \mathbb{E}^\infty[\langle P_{t-r}^+\phi,\, w_+(r)\rangle^2]\, dr. \tag{7.30}$$

We can then deduce by a "Gronwall-type" argument that $\mathbb{E}^\infty[\langle \phi,\, w_+(t)\rangle^2] = 0$ for all $t \ge 0$. In fact, by Fubinni's theorem, the left hand side of (7.30) equals

$$\int_{D_+}\int_{D_+} \phi(x_1)\phi(x_2)\,\mathbb{E}^\infty[w_+(t,x_1)w_+(t,x_2)]\,dx_1\,dx_2, \tag{7.31}$$

and the integrand with respect to dr of the right hand side of (7.30) is

$$\int_{D_+}\int_{D_+} \phi(a_1)\phi(a_2)\int_{D_+}\int_{D_+} p^+(t-r,x_1,a_1)p^+(t-r,x_2,a_2)\,\mathbb{E}^\infty[w_+(t,x_1)w_+(t,x_2)]\,dx_1dx_2da_1da_2.$$

Hence for a.e. $a_1, a_2 \in D_+$, we have

$$\begin{aligned} &\mathbb{E}^\infty[w_+(t,a_1)w_+(t,a_2)] \\ \le \quad & C\int_0^t\int_{D_+}\int_{D_+} p^+(t-r,x_1,a_1)p^+(t-r,x_2,a_2)\,\mathbb{E}^\infty[w_+(t,x_1)w_+(t,x_2)]\,dx_1\,dx_2\,dr. \end{aligned}$$

Let $\overline{f}(t) \triangleq \sup_{(a_1,a_2)\in\overline{D}_+^2}\left|\mathbb{E}^\infty[w_+(t,a_1)w_+(t,a_2)]\right|$, then the above equation asserts that $\overline{f}(t) \le C\int_0^t \overline{f}(r)\,dr$. Note that $\overline{f}(r) \in L^1[0,t]$ since it is bounded. Hence by Gronwall's lemma, we have $\overline{f}(t) = 0$ for all $t \ge 0$. This together with (7.31) yields $\mathbb{E}^\infty[\langle \phi,\, w_+(t)\rangle^2] = 0$. Hence $\mathbb{E}^\infty[\langle \phi,\, v_+(t)\rangle^2] = \langle \phi,\, u_+(t)\rangle^2$. The same holds for v_-. This completes the proof for (6.18). □

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