Good Approximate Quantum LDPC Codes from Spacetime Circuit Hamiltonians

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ABSTRACT

We study *approximate* quantum low-density parity-check (QLDPC) codes, which are approximate quantum error-correcting codes specified as the ground space of a frustration-free local Hamiltonian, whose terms do not necessarily commute.

Such codes generalize stabilizer QLDPC codes, which are exact quantum error-correcting codes with sparse, low-weight stabilizer generators (i.e. each stabilizer generator acts on a few qubits, and each qubit participates in a few stabilizer generators). Our investigation is motivated by an important question in Hamiltonian complexity and quantum coding theory: do stabilizer QLDPC codes with constant rate, linear distance, and constant-weight stabilizers exist?

We show that obtaining such optimal scaling of parameters (modulo polylogarithmic corrections) is possible if we go beyond stabilizer codes: we prove the existence of a family of $[[N, k, d, \varepsilon]]$ approximate QLDPC codes that encode $k = \widetilde{\Omega}(N)$ logical qubits into N physical qubits with distance $d = \widetilde{\Omega}(N)$ and approximation infidelity $\varepsilon = 1/\text{polylog}(N)$. The code space is stabilized by a set of 10-local *noncommuting* projectors, with each physical qubit only participating in polylog N projectors. We prove the existence of an efficient encoding map and show that the spectral gap of the code Hamiltonian scales as $\widetilde{\Omega}(N^{-3.09})$. We also show that arbitrary Pauli errors can be locally detected by circuits of polylogarithmic depth.

Our family of approximate QLDPC codes is based on applying a recent connection between circuit Hamiltonians and approximate quantum codes (Nirkhe, et al., ICALP 2018) to a result showing that random Clifford circuits of polylogarithmic depth yield asymptotically good quantum codes (Brown and Fawzi, ISIT 2013). Then, in order to obtain a code with sparse checks and strong detection of local errors, we use a *spacetime* circuit-to-Hamiltonian construction in order to take advantage of the parallelism of the Brown-Fawzi circuits. Because of this, we call our codes *spacetime codes*.

The analysis of the spectral gap of the code Hamiltonian is the main technical contribution of this work. We show that for any depth D quantum circuit on n qubits there is an associated spacetime circuit-to-Hamiltonian construction with spectral gap $\Omega(n^{-3.09} D^{-2} \log^{-6}(n))$. To lower bound this gap we use a Markov chain decomposition method to divide the state space of partially completed circuit configurations into overlapping subsets corresponding to uniform circuit segments of depth $\log n$, which are based on bitonic sorting circuits. We use the combinatorial properties of these circuit configurations to show rapid mixing between the subsets, and within the subsets we develop a novel isomorphism between the local update Markov chain on bitonic circuit configurations and the edge-flip Markov chain on equal-area dyadic tilings, whose mixing time was recently shown to be polynomial (Cannon, Levin, and Stauffer, RANDOM 2017). Previous lower bounds on the spectral gap of spacetime circuit Hamiltonians have all been based on a connection to exactly solvable quantum spin chains and applied only to 1+1 dimensional nearest-neighbor quantum circuits with at least linear depth.

CCS CONCEPTS

• **Theory of computation → Quantum complexity theory**.

KEYWORDS

quantum computing, error-correcting codes, low-density parity check, Hamiltonian complexity

A full version of this paper is available on the arXiv [9].

1 INTRODUCTION

A central result in the theory of classical error correcting codes is that there exist families of *good* linear $[N, k, d]$ codes, which have linear dimension $k = \Omega(N)$, linear distance $d = \Omega(N)$, constant

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sparsity parity checks, and linear time encoding and decoding algorithms. These *low-density parity check* (LDPC) codes [22] have many theoretical as well as practical applications.

A grand challenge in quantum information theory is to construct a *quantum* counterpart to classical LDPC codes with similarly optimal parameters. Traditionally this effort has focused on CSS stabilizer codes[1], where the notion of sparse parity checks corresponds to stabilizer generators that each act on $O(1)$ physical qubits, with each qubit participating in only $O(1)$ of such checks. The existence of QLDPC codes with good parameters and fast encoding/decoding algorithms would have significant practical impact; for example, Gottesman has shown these would imply schemes for fault tolerant quantum computation with constant overhead [26].

Despite many years of investigation, we do not yet know of QLDPC codes that simultaneously achieve constant rate and relative distance while maintaining constant locality and sparsity. The QLDPC codes of [35, 41] have a constant rate, but the minimum distance does not exceed $O(\sqrt{N})$ where N is the number of physical qubits. So far the QLDPC code with the best distance scaling is the construction of Freedman, Meyers and Luo [21] which achieves minimum distance distance $\Theta(\sqrt{N}\log^{1/4} N)$, but only encodes a single qubit. Bravyi and Hastings gave a probabilistic construction of a code with constant rate and linear distance, but the stabilizer generators each act on $\sqrt{N}$ physical qubits [11]. Hastings proved that, assuming a conjecture about high dimensional geometry, there exist QLDPC codes encoding a constant number of qubits (i.e. have vanishing rate) with distance scaling as $\Omega(N^{1-\xi})$ for any $\xi > 0$ [28, 29].

The question of whether good QLDPC codes exist also has importance for Hamiltonian complexity and the construction of exotic models in physics. This connection arises because any QECC code space that can be enforced by a set of constant-weight check operators can also be identified as the ground space of a local Hamiltonian. A central goal in these areas is to identify classes of local Hamiltonians with robust entanglement properties, and QLDPC codes provide a fruitful source of candidates. However, if the local terms are stabilizers then H is always a commuting Hamiltonian, and despite the richness of these systems they only capture a subset of local Hamiltonians and the properties they can exhibit.

Here we explore the QLDPC Conjecture (which posits that there exist asymptotically good QLDPC codes) through the correspondence between QLDPC codes and local Hamiltonians. This leads us to relax the requirement of being a CSS stabilizer code in two ways:

(1) The code satisfies an *approximate* error-correction property: after an error channel is applied the decoding procedure recovers encoded states up to some $1 - \varepsilon$ fidelity, where $\varepsilon = o(1)$.
(2) The codespace is specified as the groundspace of a frustration-free local Hamiltonian $H = \Pi_1 + \cdots + \Pi_m$, where the local projectors Π_i don't necessarily commute.

Codes satisfying the approximate reovery condition are known as *approximate quantum error correcting codes* (AQECC), and codes with noncommuting frustration-free local check terms have been

considered as a generalization of QLDPC in Hamiltonian complexity, therefore we call codes satisfying satisfying these conditions *approximate QLDPC codes*.

1.1 Our Results

Our main result is a construction of approximate QLDPC codes with nearly-optimal parameters.

THEOREM 1.1. *For infinitely many N there exists N-qubit subspaces $\{C_N\}$ with the following properties:*

(1) *C_N is an AQECC that encodes $k = \tilde{\Omega}(N)$ logical qubits in N physical qubits, has distance $d = \tilde{\Omega}(N)$, approximation error $\varepsilon = O(1/\text{polylog } N)$, and a $\text{poly}(N)$ time encoding algorithm.*
(2) *C_N is the ground space of a frustration-free local Hamiltonian $H^{(N)} = \sum H_i^{(N)}$ such that each term $H_i^{(N)}$ acts on $O(1)$ qubits, and each physical qubit participates in at most $\text{polylog } N$ terms.*
(3) *The Hamiltonian $H^{(N)}$ has spectral gap $\tilde{\Omega}(N^{-3.09})$ and it is spatially local in $\text{polylog}(N)$ dimensions (i.e. it can be embedded in $\mathbb{R}^{\text{polylog } N}$ with finite qubit density and geometrically local interactions).*

Here, the notation $\tilde{\Omega}(\cdot)$ suppresses factors of $\text{polylog } N$.

The fact that the local check terms do not commute means that it is impossible to measure them all simultaneously. However, in Section 4 we show that any Pauli error will increase the energy of at least one local check term by at least $1/\text{polylog}(N)$, and we use this to show that this family of codes is capable of *locally* detecting arbitrary Pauli errors with $\text{polylog}(N)$ depth circuits.

THEOREM 1.2. *For each code C_N, there exists a collection $\mathcal{D}_N$ of $\text{polylog}(N)$-local projectors satisfying the following properties:*

(1) *Each projector $\Pi \in \mathcal{D}_N$ acts on 10 physical qubits in the code and $s = \text{polylog}(N)$ ancilla qubits initialized in the $|0\rangle$ state, and $\Pi |\psi\rangle |0^s\rangle = 0$ for all $\Pi \in \mathcal{D}$ if and only if $|\psi\rangle \in C^N$.*
(2) *For all Pauli channels $\mathcal{E}$ acting on N qubits, for all codewords $|\psi\rangle \in C_N$, there exists a projector $\Pi \in \mathcal{D}_N$ such that*

$$\text{Tr}\left(\Pi\left(\mathcal{E}(\psi) \otimes |0^s\rangle\langle 0^s|\right)\right) \geq (1 - \alpha)(1 - 2^{-\text{polylog}(N)})$$

where $\psi = |\psi\rangle\langle\psi|$ and α is the total weight of the channel $\mathcal{E}$ on the set of Pauli operators that stabilize C.

Furthermore, there exists a measurement M, implementable by a circuit of $\text{polylog}(N)$ depth acting on $O(N \text{polylog}(N))$ qubits, such that for all Pauli channels $\mathcal{E}$ and for all codewords $|\psi\rangle \in C_N$

$$\text{Tr}\left(M\left(\mathcal{E}(\psi) \otimes |0^{Ns}\rangle\langle 0^{Ns}|\right)\right) \geq (1 - \alpha)(1 - 2^{-\text{polylog}(N)}).$$

Our construction of this family of codes is based on a recently discovered connection between AQECC and Feynman-Kitaev (FK) Hamiltonians [39]. FK Hamiltonians have ground states of the form $\frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle |\psi_t\rangle$, where $|\psi_t\rangle = U_t...U_1 |0^n\rangle$[2] is the state of a quantum circuit at time t, and are used to prove the quantum version of the Cook-Levin theorem. The connection to AQECC is based on mapping the encoding circuit of a QECC to the ground space

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of a local Hamiltonian. To construct the family of codes in Theorem 1.1 we apply the connection formed in [39] to a randomized construction of good quantum codes with polylogarithmic depth encoding circuits [14]. The polylogarithmic factors in our construction arise from the additional "clock" qubits that are used in this mapping from circuits to ground states. However, the standard FK construction uses a single global clock variable and does not allow for gates to be applied in parallel; to take full advantage of these parallel encoding circuits we present a substantial new technical analysis of the many-clock "spacetime" [13, 38] version of the FK construction that assigns an independent clock variable t_i to each qubit i in the circuit[3].

The spacetime circuit Hamiltonian enforces a ground state that is a uniform superposition over all valid configurations of these clocks (where validity is determined by the pattern of gates in the circuit), and it is unitarily equivalent to the normalized Laplacian of a random walk on the high-dimensional space of partially completed circuit configurations. Spacetime circuit Hamiltonians have been used previously for universal adiabatic computation and QMA-completeness constructions that are spatially local on a square lattice and do not require perturbative gadgets [13, 25, 36]. The analysis of the spectral gap in these previous works has always relied on the exact solutions to certain $1 + 1$ dimensional quantum spin chains [32]. Here we develop a nearly tight lower bound on the spectral gap of the spacetime circuit Hamiltonian for a particular uniform class of circuits based on bitonic sorting networks. These sorting networks are used to transform a depth D circuit with arbitrary connectivity and n qubits into a depth $D \log(n)^2$ circuit with spatially local connectivity in $\log(n)$ dimensions. By analyzing these sorting networks we prove the following general theorem.

Theorem 1.3. *For any depth D quantum circuit of 2-local gates on n qubits, where n is a power of 2, there is an associated spacetime circuit-to-Hamiltonian construction which is spatially local in polylog(n) dimensions and has a spectral gap that is*

$$\Omega(n^{-3.09} D^{-2} \log^{-6}(n)).[4]$$

The spectral gap of a code Hamiltonian lower bounds the soundness of the code, since it determines the minimum energy of states outside of the code space. In our code construction we take $D = \text{polylog}(n)$, and since the circuit Hamiltonian acts on a total of $N = n \, \text{polylog}(n)$ qubits this accounts for the bound on the spectral gap in Theorem 1.1. Since our proof holds for any circuit with arbitrary connectivity we state the general result here for future potential applications to QMA and universal adiabatic computation.

Because the code Hamiltonians of Theorem 1.1 are based on a spacetime circuit-to-Hamiltonian construction, we call our QECCs *spacetime codes*.

1.2 Discussion

We believe that our approximate QLDPC codes, beyond being an attempt to address the QLDPC Conjecture via a different perspective, also illustrate a compelling synthesis of various intriguing concepts

in quantum information theory, and furthermore, highlight several connections that deserve closer investigation.

Approximate quantum error correction. AQECCs generalize QECCs by only requiring that the quantum information stored in the code, after the action of an error channel, be recoverable with fidelity at least $1 - \varepsilon$. AQECCs have long been known to be capable of achieving better parameters than standard QECCs [17, 34], though the necessary and sufficient conditions for approximate recovery were only established within the last decade [8]. AQECCs have found applications to fault-tolerant quantum computation [12, 33] through the analysis of realistic perturbations to exact QECCs, and have recently experienced a resurgence in popularity in physics due to connections made with the holographic correspondence in quantum gravity [4]. Recently [20] have considered a version of local AQECCs which also include the possibility of locally approximate correction of errors in order to investigate the ultimate limits of the storage of quantum information in space. One can interpret our approximate QLDPC codes as providing another demonstration that the AQECC condition is a useful relaxation that facilitates the construction of codes with superior parameters than what is (known to be) achievable in the standard QECC framework.

Codes from local Hamiltonians. As previously mentioned, QLDPC codes have been a fruitful source of local Hamiltonians with robust entanglement properties, which are central objects of study in quantum Hamiltonian complexity and condensed matter theory. The first example of a QLDPC code was Kitaev's toric code, which is also a canonical example of a topologically ordered phase of matter [31]. Most research on QECC has been focused on stabilizer codes, like the toric code, for which the associated code Hamiltonians are commuting and frustration-free. In this paper we proceed in the opposite direction by asking: what kinds of quantum codes can we construct from local Hamiltonians whose terms don't necessarily commute? With this perspective, the extensive toolbox of techniques for constructing and analyzing Hamiltonians in quantum computing and quantum physics becomes immediately useful. This approach is inspired by several recent papers:

(1) In [19], Eldar, et al. defined general QLDPC codes to be subspaces S that are stabilized by a collection of local projectors $\{\Pi_i\}$; in other words, $\Pi_i |\psi\rangle = 0$ for all i if and only if $|\psi\rangle \in S$. They call the Π_i projectors "parity checks" in analogy to the parity check terms of CSS codes; however, the projectors $\{\Pi_i\}$ need not be parity checks in the traditional sense.

(2) In [20], Flammia, et al. formalized a notion of local AQECCs that includes an additional condition of approximate local correctability. This notion was applied to derive bounds on the ultimate limits of the storage of quantum information in spatially local codes.

(3) In [10], Brandao et al. show that qutrit systems on a line with nearest-neighbor interactions can form approximate QLDPC that encode $\log(N)$ qubits with distance $\log(N)$, and also show that AQECC can appear generically in energy subspaces of local Hamiltonians.

(4) In [39], Nirkhe, et al. shows that by using the Feynman-Kitaev circuit-to-Hamiltonian construction and a *non-local* CSS code, one can obtain a *local* approximate QECC where

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[3]The term "spacetime" comes from relativistic physics, in which time is necessarily measured by local clocks.

[4]The full proof has been ommitted and can be found in the full version [9]. A sketch is found in Section 1.4.

the corresponding Hamiltonian's ground space is approximately the original CSS code.

Although there are still many hurdles to overcome before codes with noncommuting checks can be realistically applied to fault-tolerance protocols, these recent developments form an exciting frontier in the study of local Hamiltonians. Another example of this connection is that the approximate codes developed in [39] and extended here can be seen as an instance of the recently formalized notion of Hamiltonian sparsification [3].

Comparison with the sparse subsystem codes of [5]. In [5] Bacon et al. construct subsystem codes with distance $\Omega(N^{1-\xi})$ for $\xi = O(1/\sqrt{\log N})$ and constant weight gauge generators, and these were termed "sparse subsystem codes." These are the best parameters achieved to date for any exact QECC in the ground space of a local Hamiltonian. Even more remarkable, in relation to the present work, is the fact that the codes of Bacon et al. have local checks that arise in a completely different way from quantum circuits. The difference is that [5] considers fault-tolerant circuit gadgets (instead of encoding circuits as in [39] and this work) and enforces the correct operation of these Clifford circuits according to the Gottesman-Knill theorem (rather than FK circuit Hamiltonians).

Another difference between these code constructions is that the code Hamiltonians of Bacon et al. are necessarily frustrated due to the fact that the noncommuting gauge generators are all Pauli operators, which therefore anticommute and share no simultaneous eigenstates. Although frustration does not always preclude the possibility of local error correction [20], there is no lower bound established on the spectral gap of the codes in [5] (and so there may be states outside the codespace with exponentially small energy), and detecting an error on a single qubit requires measuring poly(N) gauge generators in order to ascertain the syndromes of nonlocal stabilizers. With this understanding we summarize past results on QECC with strong parameters in Table 1.

Connections with QPCP. One of the most significant open problems in Hamiltonian complexity is to resolve the quantum PCP (QPCP) conjecture [1], which posits that quantum proofs can be made probabilistically checkable. Since local Hamiltonians and the complexity class QMA are the respective quantum generalizations of constraint satisfaction problems and NP, the QPCP conjecture is equivalent to the statement that it is QMA-complete to decide whether the ground state energy of a Hamiltonian $H = \sum_{i=1}^{m} H_i$ is less than a or greater than b (under the promise that one of these is the case), where $b - a > \frac{c}{(m \cdot \max_i \|H_i\|)}$ for some $c = \Omega(1)$ corresponds to constant relative precision. One reason this question is difficult is any trivial state which is output by a constant-depth quantum circuit acting on a product state can be given as an NP witness, and many of the commonly studied classes of local Hamiltonians necessarily have low-energy trivial states. Therefore in order for the QPCP conjecture to hold there must be some Hamiltonian with no low-energy trivial states, and even this weaker NLTS conjecture [27] remains an open problem.

One approach to resolving the NLTS and QPCP conjectures is to develop the quantum analogue of locally testable codes, which are defined in [2] as codes with frustration-free but not necessarily commuting local checks, good parameters, and a *soundness* property

which states that the energy of a state with respect to the constraints grows linearly with its distance from the code space. Therefore constructing good QLDPC is necessary for constructing QLTC, but it is not sufficient since in general QLDPC may have low energy states outside the code space. This collection of open challenges that are stimulating innovations in Hamiltonian complexity is known as the robust entanglement zoo [18], since they all involve generalizing known properties of quantum ground states to states with constant relative distance above the code space.

Just as the classical PCP Theorem indirectly transforms a Cook-Levin computational tableau into a probabilistically checkable CSP, a QPCP construction could be seen as transforming the FK circuit-to-Hamiltonian construction into a local Hamiltonian with robust entanglement. While known limitations on generalized FK constructions make such a direct approach unlikey [7, 23, 24], our Theorem 4.1 on local error detection in polylog(N) depth is the first result to quantitatively substantiate the belief that the spacetime Hamiltonian construction is more robust than the standard global-clock FK Hamiltonian. Specifically, we show that the energy of a state after the application of a Pauli error channel is inversely proportional to the *depth* of the circuit in the spacetime construction, whereas it is proportional to the *size* of the circuit in the standard FK construction. In the full version [9] we describe an alternate version of our approximate QLDPC construction that is based on global-clock FK and a modified distribution over time steps of the quantum circuit, and this version can achieve any scaling of the approximation error $\varepsilon(N) > 0$ at the expense of decreasing the spectral gap to $\tilde{\Omega}(\varepsilon N^{-3})$, but this substantially weakens the corresponding version of Theorem 4.1 and forces the local error detection circuits to have superlinear depth. This results suggest that continued investigation into alternative circuit-to-Hamiltonian constructions might be a fruitful direction of research, and might possibly make headway towards the mystery of the QPCP conjecture.

1.3 Description of the Code Hamiltonian

In [39] it was recognized that the FK Hamiltonian which maps circuits to ground states could be used to develop a set of local checks for AQECCs for which only an efficient encoding circuit had previously been found. For a circuit with local gates $U_1, ..., U_T$ the FK Hamiltonian's ground states are

$$|\Psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle_C \otimes (U_t U_{t-1} \cdots U_1) |\psi, 0 \ldots 0\rangle_S .$$

Such states are called *history states*. The register C, called the *clock register*, indicates how many gates have been applied to the all zeroes state, which is stored in register S (called the *state register*) containing an initial state $|\psi\rangle$ and ancillas.

Although this state has only a $1/(T+1)$ fidelity with the output of the circuit, the standard technique for increasing the overlap to be inverse polynomially close to 1 is to pad the end of the circuit with identity gates (for recent work on more efficient methods for biasing the history state towards its endpoints, see [7, 15]). This technique allows history states to capture approximate versions of QECC that have efficient encoding circuits. The approximation error of the code is directly related to history state overlap with the output of the encoding circuit.

Table 1: Past results on QECC with strong parameters

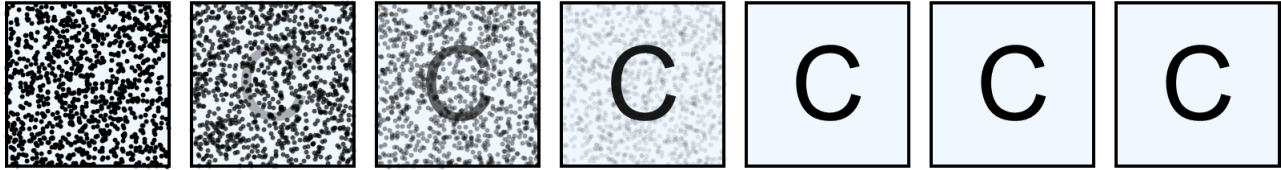| Reference	# of logical qubits	Distance	Locality	Notes
[41]	$\Theta(N)$	$\Theta(\sqrt{N})$	$O(1)$	CSS Stabilizer code
[21]	$O(1)$	$O(\sqrt{N \log N})$	$O(1)$	CSS Stabilizer code
[11]	$\Theta(N)$	$\Theta(N)$	$\Omega(\sqrt{N})$	CSS Stabilizer code
[28, 29]	$O(1)$	$\Omega(N^{1-\xi})$ for all $\xi > 0$	$O(\log N)$	CSS code, assumes conjecture in high dimensional geometry
[5]	$O(N)$	$\Omega(N^{1-\xi})$ for all $\xi > 0$	$O(1)$	Subsystem Stabilizer code, frustrated Hamiltonian
This paper	$\Omega(N/\text{polylog } N)$	$\Omega(N/\text{polylog } N)$	$O(1)$	approximate QLDPC code

Figure 1: The approximate nature of the codes introduced in [39] arises from the fact that part of the history state superposition corresponding to early time steps, which do not match the output of the encoding circuit and are treated as noise in our analysis. Once a sufficient depth to form a codeword is reached, the computation can be padded with identity gates in order to increase the overlap of this approximate codeword with the original codeword it is approximating.

The Hamiltonian which enforces the ground space spanned by states of the form (1.3) is formed by projectors that check the input state of the computation, as well as *propagation terms* that check that the branch of the superposition corresponding to time t and the branch corresponding to time $t + 1$ differ by the application of the gate U_{t+1} to the state register. The linear ordering of the computation $U_1, \ldots, U_T$ is enforced via the sum of these propagation terms. The propagation Hamiltonian is unitarily equivalent to a normalized Laplacian on the path graph with vertices $\{0, \ldots, T\}$ and therefore has a spectral gap that is $\Theta(T^{-2})$. For the purpose of lower bounding the energy of excitations that leave the code space, it is important to check the spectral gap of the full Hamiltonian including the input check terms, see Section 1.4 for further discussion.

In this work we use the spacetime version of the FK circuit Hamiltonian [13], which assigns a clock register to each computational qubit, and has a ground space spanned by uniform superpositions over all valid time configurations $\tau = (t_1, ..., t_n)$ of the state of the computation after the gates prior to τ have been performed,

$$|\psi\rangle = \frac{1}{|\mathcal{T}|^{1/2}} \sum_{\tau \in \mathcal{T}} |\tau\rangle_C \otimes U(\tau \leftarrow 0)|0 \cdots 0\rangle_S .$$

Here $\mathcal{T}$ is the set of all valid time configurations τ, which is any vector $(t_1, \ldots, t_n)$ that the clock registers could hold if a subset of gates that respected causal dependence were applied. To avoid boundary effects at the beginning and end of the computation we use circular (periodic) time, which involves reversing the gates in the second half of the circuit so that the computation returns to its initial state. In Section 2.3 we implement these periodic clocks using qubits.

The necessity of including these causal constraints is one of the complications introduced by the use of spacetime circuit Hamiltonians, but a far more significant challenge is lower bounding the spectral gap of the spacetime propagation Hamiltonian. In contrast with single-clock circuit Hamiltonians, the geometric arrangement of the gates in the circuit now has a significant effect on the spectrum of the spacetime circuit Hamiltonian due to the causal constraints. All lower bounds in previous works apply to spacetime Hamiltonians in 2 spatial dimensions, which represent 1 (space) + 1 (time) dimensional quantum circuits. This is not only due to the importance of planar connectivity for practical applications, but it is also a symptom of the general fact that exactly solvable models in mathematical physics are hardly known beyond 1 + 1 dimensions. The 1 + 1 dimensional circuit propagation Hamiltonian is unitarily equivalent to a stochastic model describing the evolution of a string in the plane. For higher dimensional circuits it corresponds to the dynamics of membranes or crystal surface growth, where no known solutions are available. To overcome this in the present work we use sorting networks to turn arbitrary random circuits into circuits with uniform connectivity, and then we apply powerful techniques and past results from the theory of Markov chains to analyze the resulting high-dimensional spacetime circuit Hamiltonians.

1.4 Proof Sketch for the Spectral Gap Analysis

Our analysis of the spectral gap Δ_{prop} of the spacetime circuit propagation Hamiltonian begins with the standard mapping from H_{prop} to a a Markov chain transition matrix P. [5] To analyze the latter,

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[5]The re-scaled Hamiltonian $H_{\text{prop}}/\|H_{\text{prop}}\|$ is unitarily equivalent to a normalized graph Laplacian $\mathcal{L}$ for the graph with vertices corresponding to valid time configurations and edges corresponding to local gate updates on those time configurations. P is the transition matrix for the random walk on this graph, which is obtained from $I - \mathcal{L}$ by a similarity transformation. The point is that these mappings provide an algebraic relation between Δ_{prop} and Δ_P.

we apply a Markov chain decomposition method due to Madras and Randall [37], which is used to split the Markov chain and its state space into pieces that are easier to analyze individually. For our decomposition of choice these pieces come in several closely related variants, which all essentially correspond to the set of time configurations contained within the final phase of a bitonic sorting circuit (as shown in Figure 3 for 8 inputs) which we call a bitonic block. An arbitrary circuit consisting of 2-local gates can be transformed into a sequence of consecutive bitonic blocks, with at most a polylogarithmic factor of blow up in the depth[6].
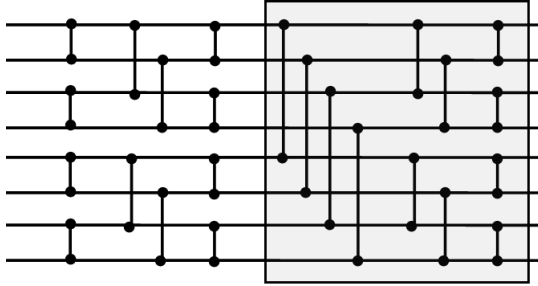

Figure 2: A bitonic sorting architecture on $n = 8$ bits. We refer to the final phase of the architecture, corresponding to the last $\log(n) = 3$ layers enclosed in a gray box, as a bitonic block. Note that the gates in each layer are executed simultaneously, but are drawn as non-overlapping for visual clarity. An arbitrary circuit consisting of 2-local gates can be transformed to have the architecture of consecutive repetitions of bitonic blocks at the cost of increasing the depth by a factor of $\log(N)^2$.

After dividing the set of valid time configurations Ω (the state space of the Markov chain) into subsets Ω_i of configurations confined to bitonic blocks of the form illustrated in Figure 3, the subsets will form a quasi-linear chain in the sense that Ω_i and Ω_j have nonempty intersections when $|i - j| \leq \log n$. To apply the decomposition method we need to analyze (1) the spectral gap of the restricted Markov chains P_i that are confined to stay within each of the subsets Ω_i, and (2) the spectral gap of an aggregate Markov chain $\overline{P}$ that moves between the blocks based on transition probabilities related to the size of the inetrsections of the blocks.

As suggested by its quasi-linear connectivity, the spectral gap of the aggregate chain can be lower bounded using Cheeger's inequality in a manner similar to how it is done for the path graph Laplacian. The main technical challenge is to accurately compute the transition probabilities $\overline{P}(i, j) = \pi(\Omega_i \cap \Omega_j)/(\Theta\pi(\Omega_i))$, which involve the ratio of the number of configurations within each of the blocks to the number within the pairwise intersections, $|\Omega_i \cap \Omega_j|/|\Omega_i|$, as well as the maximum number of blocks Θ that can contain any particular time configuration. In the full version [9], we develop a recurrence relation to exactly count these configurations and show that the former is constant for consecutive blocks (and decays doubly exponentially with $|i - j|$ for longer distance transitions), and the latter is logarithmic in n. Using asymptotic properties of

[6]See full version [9] for proof.

the recurrence relation we show that the transition probabilities between $i, i + 1$ are equal to $(\phi \log n)^{-1}$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. If there are m blocks in total so that the length of the path is m, we can use Cheeger's inequality to show that the spectral gap $\Delta_{\overline{P}}$ of the aggregate chain satisfies

$$\Delta_{\overline{P}} \geq (\phi m \log n)^{-2}. \tag{1}$$

Turning to the analysis of the restricted chains P_i, we present the discovery of a surprising and beautiful connection between valid time configurations of architectures of the form shown in Figure 2 with combinatorial structures known as dyadic tilings [30]. Dyadic tilings are tilings of the unit square by equal-area dyadic rectangles, which are rectangles of the form $[a2^{-s}, (a+1)2^{-s}] \times [b2^{-t}, (b+1)2^{-t}]$, where a, b, s, t are nonnegative integers. These tilings have a natural recursive characterization: beginning from the unit square, draw a line that is either a horizontal or vertical bisector. This divides the square into two rectangles, and in each of these one chooses a horizontal or vertical bisector, and so on. After $\ell = \log(n)$ such recursive steps one obtains a dyadic tiling of rank ℓ with a total of n dyadic rectangles, each with area $1/n$. Some examples are given in Figure 5.

For a spacetime circuit with n qubits, we choose the blocks Ω_i in the decomposition so that for each block there is an exact bijection between the time configurations within the block and the set of equal-area dyadic tilings of rank $\ell = \log n$. Moreover, it turns out that the natural Markov chain on time configurations can also be mapped onto a previously known Markov chain for dyadic tilings called the edge-flip chain. This Markov chain selects a rectangle of area $1/n$ in the current dyadic tiling and one of its four edges at random, and flips this edge if the result would be another dyadic tiling. The correspondence is described in Figure 6.

The mixing time of this edge flip chain was an open problem for over a decade, but has recently been the subject of a tour de force analysis that establishes an upper bound on the mixing time that is polynomial in n. Adapting these results using our bijection between these Markov chains yields

$$\Delta_{P_i} = \Omega\left(n^{-4.09}\right), \quad \text{for all} \quad i = 1, ..., m, \tag{2}$$

where the value of the exponent can be taken to be $\log(17) = 4.087...$. Once (1) and (2) are established, we combine them according to the decomposition result,

$$\Delta_P \geq \frac{1}{2}\Delta_{\overline{P}} \min_{i=1,...,m} \Delta_{P_i} = \Omega\left(n^{-4.09}m^{-2}\text{polylog}(n)^{-1}\right),$$

which is an inverse polynomial lower bound on the gap. The circuit propagation Hamiltonian is equivalent to the Markov chain P scaled by a factor of n, and so we obtain $\Delta_{\text{prop}} = \widetilde{\Omega}(n^{-3.09})$. Finally, using the version of the spacetime Hamiltonian with circular time we show that every state in the code space has overlap $1/\text{polylog}(n)$ with the input terms and so the geometrical lemma yields a gap of $\widetilde{\Omega}(n^{-3.09})$ for the full code Hamiltonian.

2 PRELIMINARIES

2.1 Approximate QLDPC Codes

Definition 2.1 (Approximate QLDPC code). A 2^k-dimensional subspace C of $(\mathbb{C}^2)^{\otimes N}$ is a $[[N, k, d, \varepsilon, \ell, s]]$ *approximate QLDPC code*
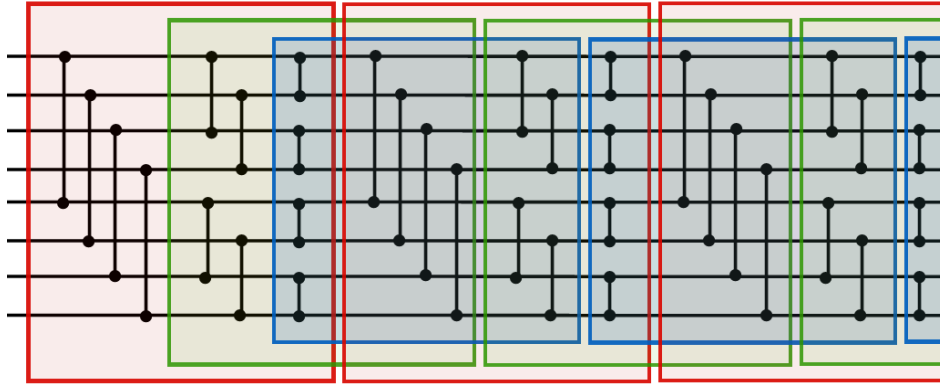
Figure 3: (Color Figure) The Markov chain block decomposition for a sequence of padded bitonic sorting architectures on 8 bits. The set of valid time configurations contained entirely within the i-th colored rectangle constitutes the block Ω_i. The set of time configurations in two rectangles of different colors are related by a permutation of the qubit wires. The aggregate chain $\overline{P}$ has a nonzero transition probability $\overline{P}(i,j)$ iff the rectangles corresponding to the blocks Ω_i and Ω_j are overlapping. Each block Ω_i has a nonzero transition probability to $\log N$ other blocks Ω_j. Every valid time configuration is contained in at least one of the blocks, and no time configuration is contained in more than $\log N$ blocks.
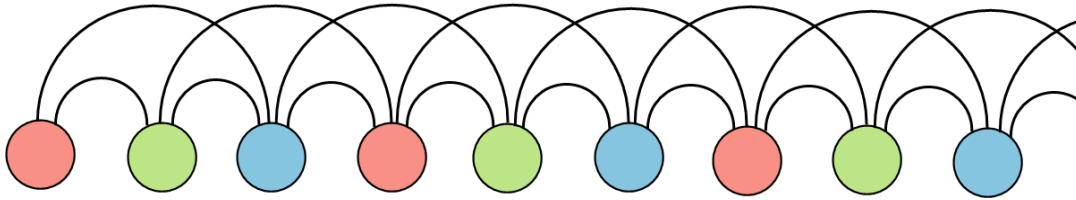

Figure 4: (Color Figure) An illustration of the states and transitions in the aggregate chain corresponding to the subsets of time configurations contained with the blocks in Figure 3.
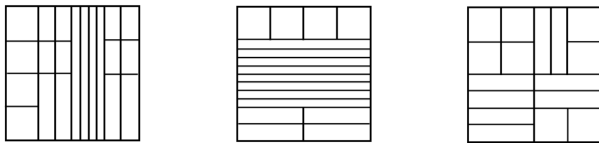

Figure 5: Examples of dyadic tilings of rank 4.

iff there exists a (not necessarily commuting) set of projectors $\{H_1, \ldots, H_m\}$ acting on N qubits such that

(1) Each term H_i acts on at most ℓ qubits (i.e. *locality*) and each qubit participates in at most s terms (i.e. *sparsity*).
(2) For all $|\psi\rangle$, we have that $|\psi\rangle \in C$ if and only if $\langle\psi| H |\psi\rangle = 0$, where $H = H_1 + \cdots + H_m$.
(3) There exist encoding and recovery maps Enc, Rec such that for all $|\phi\rangle \in (\mathbb{C}^2)^{\otimes k} \otimes \mathcal{R}$ where $\mathcal{R}$ is some purifying register, for all completely positive trace preserving maps $\mathcal{E}$ acting on at most $(d-1)/2$ qubits, we have that the image of Enc is exactly the code C and

$$F(\text{Rec} \circ \mathcal{E} \circ \text{Enc}(|\phi\rangle\langle\phi|), |\phi\rangle\langle\phi|) \geq 1 - \varepsilon$$

where $F(\cdot, \cdot)$ denotes the fidelity function. Here, the maps Enc, $\mathcal{E}$, and Rec do not act on register $\mathcal{R}$.

2.2 Parallel Quantum Circuits

Our model for random depth D Clifford circuits is to choose, for each layer L_t, a random partition $\{(p, q)\}$ of the n qubits, and then for each pair (p, q) let $U_t[p, q]$ be a uniformly chosen gate from the two-qubit Clifford group (i.e., the set of all unitaries that preserve the Pauli group under conjugation).

Brown and Fawzi showed that for $D = O(\log^3 n)$, the circuit C is an encoding circuit for a good error-correcting code with high probability [14]:

THEOREM 2.2 ([14]). *For all $\delta > 0$, for all integers $n, k, d > 0$ satisfying*

$$\frac{k}{n} \leq 1 - h(d/n) - \log(3)d/n - 4\delta,$$

with $h(\cdot)$ as the binary entropy function, the circuit C described in the paragraph above is an encoding circuit for a $[[n, k, d]]$ stabilizer code with probability at least $1 - \Omega(n^{-8})$. In other words, with high probability the subspace $C = \{C |\psi\rangle |0\rangle^{\otimes(n-k)} : |\psi\rangle$ is a k-qubit state$\}$ is a $[[n, k, d]]$ stabilizer code.

Since the circuits are Clifford circuits, the resulting code is a stabilizer code.
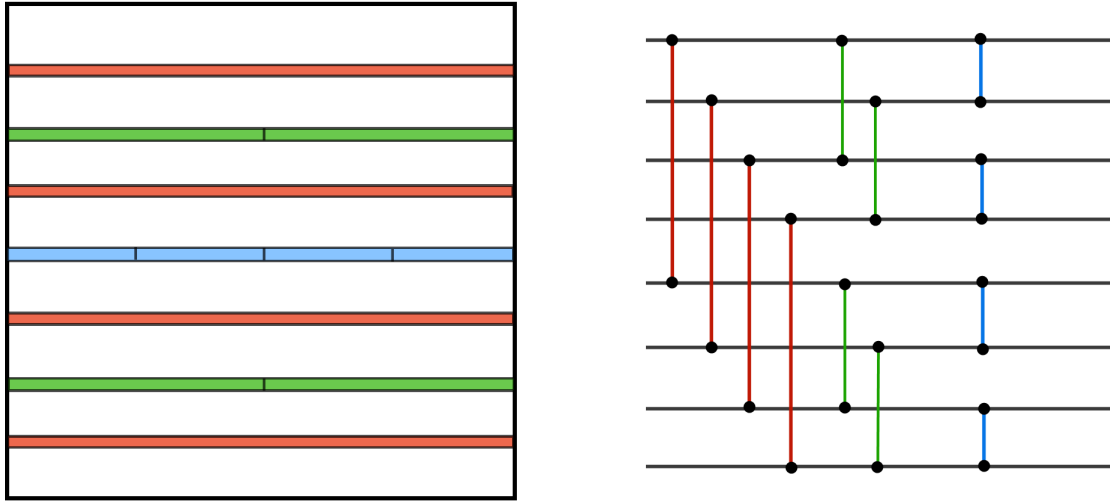
Figure 6: (Color Figure) A color-coding of the correspondence between dyadic tilings and valid time configurations of a bitonic sorting circuit. The colored line segments in (a) correspond to sub-edges which when rotated by $\pi/2$ about their midpoint will be sub-edges of a vertical edge in some dyadic tiling. These edges are placed in correspondence with the gates of the bitonic sorting circuit in (b), with the convention that colored line segments in (a) are ordered from left to right and from top to bottom, and the gates in a given commuting layer in (b) are enumerated from top to bottom. Given an arbitrary dyadic tiling, one checks which of the colored line segments in (a) correspond to vertical sub-edges in the tiling, and these correspond to gates that are in the past causal cone of the bitonic time configuration associated with that tiling.

2.3 The Spacetime Circuit Hamiltonian Construction

As mentioned in the introduction, we use a small variant of the spacetime circuit Hamiltonian of Breuckmann and Terhal [13] to create our code Hamiltonian. A description of the construction can be found in the full version [9].

2.4 Bitonic Sorting Networks

In this section, we describe a class of circuits called *bitonic sorting networks*. These are parallel circuits, devised by Batcher [6], that are used to efficiently sort data arrays. Specifically, these are circuits acting on n elements with depth $O(\log^2 n)$. In each layer of the circuit, pairs of elements are compared and swapped. Equivalently, for every permutation π on n elements, there is a bitonic sorting network consisting of SWAP and identity gates that implements π.

Bitonic sorting networks will be a crucial component of our code construction, as we use them to "uniformize" the random Brown-Fawzi encoding circuits before applying the spacetime circuit Hamiltonian construction. The uniformity of the resulting circuits will be the key ingredient that allows us to analyze the spectral gap of the Hamiltonian.

Definition 2.3 (Bitonic block [6]). For a positive integer ℓ, the bitonic block of *rank* ℓ, $\mathcal{B}_\ell$, is a circuit architecture acting on 2^ℓ qubits. $\mathcal{B}_\ell$ is recursively defined with the architecture $\mathcal{B}_1$ being an architecture consisting of a single layer, $\mathcal{L}_1$, with a gate between qubits 1 and 2 (see part (a) of Figure 7).

For $\ell > 1$, the bitonic block $\mathcal{B}_\ell$ is a depth ℓ architecture with the first layer, $\mathcal{L}_1$ being $2^{\ell-1}$ gates connecting qubit i to $i + 2^{\ell-1}$

for $i = 1, 2, \ldots, 2^{\ell-1}$. The following $\ell - 1$ layers, $\mathcal{L}_\ell, \mathcal{L}_{\ell-1}, \ldots, \mathcal{L}_2$ are defined recursively as $\mathcal{B}_{\ell-1}^{\otimes 2}$ where one of the two blocks acts on the qubits $\{1, 2, \ldots, 2^{\ell-1}\}$ and the other on the qubits $\{2^{\ell-1} + 1, 2^{\ell-1} + 2, \ldots, 2^\ell\}$.

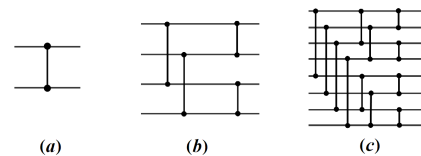See Figure 7 for illustrations of blocks $\mathcal{B}_2$, and $\mathcal{B}_3$.

Figure 7: (a) Bitonic block $\mathcal{B}_1$. (b) Bitonic block $\mathcal{B}_2$. (c) Bitonic block $\mathcal{B}_3$.

2.5 Uniformizing Circuits for Spacetime Hamiltonians

We now present a general method for encoding depth D circuits C into a spacetime circuit Hamiltonian, in a way that allows us to give a good lower bound on the spectral gap. Let C denote a circuit of depth D consisting of layers $L_1, \ldots, L_D$, where each L_t is a set of $n/2$ two-qubit gates. We preprocess the circuit C in multiple steps to obtain a slightly larger-depth circuit C'. We "uniformize" the circuit using the bitonic sorting networks described in the previous section. The circuit C will not, in general, correspond to nearest-neighbor interactions in small dimension. We add bitonic sorting networks in between each layer L_t of C to ensure that all the Clifford

gates act on adjacent qubits. Because of the regular structure of the sorting networks, the resulting circuit will consist of nearest-neighbor interactions on a hypercube of dimension $\ell = \log n$. A more formal description can be found in the full version [9].

3 CONSTRUCTION OF THE CODE HAMILTONIAN

Here we describe our code construction in detail. Let $\varepsilon > 0$ be the desired target approximation error. Let n, k, d be integers satisfying Theorem 2.2 where $k, d = \Omega(n)$. Let C_0 denote a Clifford circuit of depth $D_0 = O(\log^3 n)$ that is an encoding circuit of an $[[n, k, d]]$ code C_{BF}, as promised by Theorem 2.2. Let $L_1, \ldots, L_{D_0}$ be the D_0 layers of C_0, where each L_t is a set of $n/2$ two-qubit Clifford gates.

The first preprocessing step is to replace all the Clifford gates by gates from the set $\{I, H, S, CNOT\}$. This is possible because the gate set generates the Clifford group; thus every two-qubit Clifford gate can be written as a $O(1)$-length product of I, H, S, and $CNOT$ gates. The depth of this circuit is $D_1 = O(D_0)$. Let C_1 denote this circuit.

Next, we *pad* the circuit to have depth $3D_1/\varepsilon$ where the last $1-(\varepsilon/3)$ fraction of the layers are simply applications of the identity gate on consecutive pairs of qubits. Call this padded circuit C_1'; its depth is $D_1' = 3D_1/\varepsilon$.

Now, let C_2 be the circuit obtained by preprocessing C_1' as described in Section 2.5. This has depth $D = O(\log^5 n)$. Let H denote the corresponding spacetime circuit Hamiltonian, acting on $N = O(nD)$ qubits. Let C denote the ground space of H. This will be our code.

THEOREM 3.1. *For all $\varepsilon > 0$, the subspace C is a $[[N, k, d, \varepsilon, \ell, s]]$ approximate QLDPC code, for $k = \Omega(N/\log^5 N)$, $d = \Omega(N/\log^5 N)$, $\ell = 9$, and $s = \text{polylog}(N)$.*[7]

Additionally, we demonstrate that there is an efficient circuit generating a ground-state of the Hamiltonian.

THEOREM 3.2. *There exists an encoding circuit of polynomial size in n which on input $|\psi\rangle$ generates the state $\text{Enc}(\psi)$. In particular, the polynomial size circuit generating the state is $\log(n) + 2$ spatially local.*[8]

4 LOCAL DETECTION OF PAULI ERRORS

The described code is capable of local detection of errors on space-time codewords with probability $1 - 2^{-\text{polylog}(N)}$ with polylog(N)-depth circuits. The class of errors that we handle is the set of tensor products of Pauli operators on the physical qubits (which includes data and time qubits). Interestingly, we can detect Pauli errors even if the weight of the error (the number of qubits affected) exceeds the distance of the spacetime code!

THEOREM 4.1. *With probability*[9] *$1 - 2^{-\Omega(\log^2 n)}$, there exists a collection $\mathcal{D}$ of polylog(N)-local projectors satisfying the following properties:*

(1) *Each projector $\Pi \in \mathcal{D}$ acts on 10 physical qubits of the codeword, and acts on $s = \text{polylog}(N)$ ancilla qubits initialized in the $|0\rangle$ state.*

(2) *For all n-qubit states $|\psi\rangle$, we have that $\Pi |\psi\rangle |0^s\rangle = 0$ for all $\Pi \in \mathcal{D}$ if and only if $|\psi\rangle$ is a codeword in the spacetime code C.*

(3) *For all Pauli channels $\mathcal{E}$, for all codewords $|\psi\rangle \in C$, there exists a projector $\Pi \in \mathcal{D}$ such that*

$$\text{Tr}\left(\Pi\left(\mathcal{E}(\psi) \otimes |0^s\rangle\langle 0^s|\right)\right) \geq (1-\alpha)(1 - 2^{-\text{polylog}(N)})$$

where $\psi = |\psi\rangle\langle\psi|$ and $\alpha = \sum_{P \in \mathcal{S}} c_P$ is the weights of the channel $\mathcal{E}$ on the Pauli stabilizers in $\mathcal{S}$.

Furthermore, there exists a measurement M, implementable by a circuit of $\text{polylog}(N)$ depth acting on $O(N \text{polylog}(N))$ qubits, such that for all Pauli channels $\mathcal{E}$ and for all codewords $|\psi\rangle \in C$

$$\text{Tr}\left(M\left(\mathcal{E}(\psi) \otimes |0^{Ns}\rangle\langle 0^{Ns}|\right)\right) \geq (1-\alpha)(1 - 2^{-\text{polylog}(N)}).[10]$$

REFERENCES

[1] Dorit Aharonov, Itai Arad, and Thomas Vidick. 2013. Guest column: the quantum PCP conjecture. *ACM SIGACT news* 44, 2 (2013), 47–79.

[2] Dorit Aharonov and Lior Eldar. 2015. Quantum locally testable codes. *SIAM J. Comput.* 44, 5 (2015), 1230–1262.

[3] Dorit Aharonov and Leo Zhou. 2018. On Gap-Simulation of Hamiltonians and the Impossibility of Quantum Degree-Reduction. *arXiv preprint arXiv:1804.11084* (2018).

[4] Ahmed Almheiri, Xi Dong, and Daniel Harlow. 2015. Bulk locality and quantum error correction in AdS/CFT. *Journal of High Energy Physics* 2015, 4 (2015), 163.

[5] Dave Bacon, Steven T Flammia, Aram W Harrow, and Jonathan Shi. 2017. Sparse quantum codes from quantum circuits. *IEEE Transactions on Information Theory* 63, 4 (2017), 2464–2479.

[6] K. E. Batcher. 1968. Sorting Networks and Their Applications. In *Proceedings of the April 30–May 2, 1968, Spring Joint Computer Conference (AFIPS '68 (Spring))*. ACM, New York, NY, USA, 307–314. https://doi.org/10.1145/1468075.1468121

[7] Johannes Bausch and Elizabeth Crosson. 2018. Analysis and limitations of modified circuit-to-Hamiltonian constructions. *Quantum* 2 (Sept. 2018), 94. https://doi.org/10.22331/q-2018-09-19-94

[8] Cédric Bény and Ognyan Oreshkov. 2010. General conditions for approximate quantum error correction and near-optimal recovery channels. *Physical review letters* 104, 12 (2010), 120501.

[9] Thomas C. Bohdanowicz, Elizabeth Crosson, Chinmay Nirkhe, and Henry Yuen. 2018. Good approximate quantum LDPC codes from spacetime circuit Hamiltonians. (2018). arXiv:arXiv:1811.00277

[10] Fernando GSL Brandao, Elizabeth Crosson, M Burak Şahinoğlu, and John Bowen. 2017. Quantum Error Correcting Codes in Eigenstates of Translation-Invariant Spin Chains. *arXiv preprint arXiv:1710.04631* (2017).

[11] Sergey Bravyi and Matthew B Hastings. 2014. Homological product codes. In *Proceedings of the forty-sixth annual ACM symposium on Theory of computing*. ACM, 273–282.

[12] Sergey Bravyi, Matthew B Hastings, and Spyridon Michalakis. 2010. Topological quantum order: stability under local perturbations. *Journal of mathematical physics* 51, 9 (2010), 093512.

[13] Nikolas P Breuckmann and Barbara M Terhal. 2014. Space-time circuit-to-Hamiltonian construction and its applications. *Journal of Physics A: Mathematical and Theoretical* 47, 19 (2014), 195304.

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[7]The proof has been ommitted and can be found in the full version [9].

[8]The proof has been ommitted and can be found in the full version [9].

[9]Here the probability is taken over the randomized construction of the code.

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[10]The proof has been ommitted and can be found in the full version [9].

[14] Winton Brown and Omar Fawzi. 2013. Short random circuits define good quantum error correcting codes. In *Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on*. IEEE, 346–350.

[15] Libor Caha, Zeph Landau, and Daniel Nagaj. 2018. Clocks in Feynman's computer and Kitaev's local Hamiltonian: Bias, gaps, idling, and pulse tuning. *Physical Review A* 97, 6 (2018), 062306.

[16] A Robert Calderbank and Peter W Shor. 1996. Good quantum error-correcting codes exist. *Physical Review A* 54, 2 (1996), 1098.

[17] Claude Crépeau, Daniel Gottesman, and Adam Smith. 2005. Approximate quantum error-correcting codes and secret sharing schemes. In *Proceedings of the 24th annual international conference on Theory and Applications of Cryptographic Techniques*. Springer-Verlag, 285–301.

[18] Lior Eldar and Aram W Harrow. 2017. Local Hamiltonians whose ground states are hard to approximate. In *Foundations of Computer Science (FOCS), 2017 IEEE 58th Annual Symposium on*. IEEE, 427–438.

[19] Lior Eldar, Maris Ozols, and Kevin F Thompson. 2016. The Need for Structure in Quantum LDPC Codes. *arXiv preprint arXiv:1610.07478* (2016).

[20] Steven T Flammia, Jeongwan Haah, Michael J Kastoryano, and Isaac H Kim. 2017. Limits on the storage of quantum information in a volume of space. *Quantum* 1 (2017), 4.

[21] Michael H Freedman, David A Meyer, and Feng Luo. 2002. Z2-systolic freedom and quantum codes. *Mathematics of quantum computation, Chapman & Hall/CRC* (2002), 287–320.

[22] Robert G. Gallager. 1963. Low-Density Parity-Check Codes. (1963).

[23] Anand Ganti and Rolando Somma. 2013. On the gap of Hamiltonians for the adiabatic simulation of quantum circuits. *International Journal of Quantum Information* 11, 07 (2013), 1350063.

[24] Carlos E González-Guillén and Toby S Cubitt. 2018. History-state Hamiltonians are critical. *arXiv preprint arXiv:1810.06528* (2018).

[25] David Gosset, Barbara M. Terhal, and Anna Vershynina. 2015. Universal Adiabatic Quantum Computation via the Space-Time Circuit-to-Hamiltonian Construction. *Phys. Rev. Lett.* 114 (Apr 2015), 140501. Issue 14. https://doi.org/10.1103/PhysRevLett.114.140501

[26] Daniel Gottesman. 2013. Fault-tolerant quantum computation with constant overhead. *arXiv preprint arXiv:1310.2984* (2013).

[27] Matthew B Hastings. 2013. Trivial low energy states for commuting Hamiltonians, and the quantum PCP conjecture. *Quantum Information & Computation* 13, 5-6 (2013), 393–429.

[28] Matthew B Hastings. 2017. Quantum Codes from High-Dimensional Manifolds. In *LIPIcs-Leibniz International Proceedings in Informatics*, Vol. 67.

[29] Mathew B Hastings. 2017. Weight reduction for quantum codes. *Quantum Information & Computation* 17, 15-16 (2017), 1307–1334.

[30] Svante Janson, Dana Randall, and Joel Spencer. 2002. Random Dyadic Tilings of the Unit Square. 21 (10 2002), 225–251.

[31] A Yu Kitaev. 2003. Fault-tolerant quantum computation by anyons. *Annals of Physics* 303, 1 (2003), 2–30.

[32] Tohru Koma and Bruno Nachtergaele. 1997. The spectral gap of the ferromagnetic XXZ-chain. *Letters in Mathematical Physics* 40, 1 (1997), 1–16.

[33] Yi-Chan Lee, Courtney G Brell, and Steven T Flammia. 2017. Topological quantum error correction in the Kitaev honeycomb model. *Journal of Statistical Mechanics: Theory and Experiment* 2017, 8 (2017), 083106.

[34] Debbie W Leung, Michael A Nielsen, Isaac L Chuang, and Yoshihisa Yamamoto. 1997. Approximate quantum error correction can lead to better codes. *Physical Review A* 56, 4 (1997), 2567.

[35] Seth Lloyd, Peter Shor, and Kevin Thompson. 2017. polylog-LDPC Capacity Achieving Codes for the Noisy Quantum Erasure Channel. *arXiv preprint arXiv:1703.00382* (2017).

[36] Seth Lloyd and Barbara M Terhal. 2016. Adiabatic and Hamiltonian computing on a 2D lattice with simple two-qubit interactions. *New Journal of Physics* 18, 2 (2016), 023042.

[37] Neal Madras and Dana Randall. 2002. Markov chain decomposition for convergence rate analysis. *Ann. Appl. Probab.* 12, 2 (05 2002), 581–606. https://doi.org/10.1214/aoap/1026915617

[38] Ari Mizel, Daniel A Lidar, and Morgan Mitchell. 2007. Simple proof of equivalence between adiabatic quantum computation and the circuit model. *Physical review letters* 99, 7 (2007), 070502.

[39] Chinmay Nirkhe, Umesh Vazirani, and Henry Yuen. 2018. Approximate Low-Weight Check Codes and Circuit Lower Bounds for Noisy Ground States. In *45th International Colloquium on Automata, Languages, and Programming (ICALP 2018) (Leibniz International Proceedings in Informatics (LIPIcs))*, Vol. 107. 91:1–91:11. https://doi.org/10.4230/LIPIcs.ICALP.2018.91

[40] Andrew Steane. 1996. Multiple-particle interference and quantum error correction. *Proc. R. Soc. Lond. A* 452, 1954 (1996), 2551–2577.

[41] Jean-Pierre Tillich and Gilles Zémor. 2014. Quantum LDPC codes with positive rate and minimum distance proportional to the square root of the blocklength. *IEEE Transactions on Information Theory* 60, 2 (2014), 1193–1202.