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Application of Stochastic Models to Growth and Decline Episodes of Financial Data

A thesis submitted in partial fulfillment of the requirements of the degree of Master of Science in Mathematics

by

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Abstract

In this thesis work, we analyze fit of bivariate BEG and BTLG models to financial asset returns' episodes of growth and decline. Our data include foreign exchange rates, stock, and stock's indexes prices, and commoditites. We apply BEG and BTLG models to all data and decide if the models fit reasonably well based on univariate and bivariate fit methods. We also assess "stability" of the returns with respect to their geometric summation. Our results show BEG and BTLG models fitting best the foreign exchange rates.

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"If I have seen further, it is by standing on the shoulders of giants"-*Isaac Newton.*

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Contents

1. Introduction

Let $X_1, X_2, \ldots$ be independent, identically distributed (i.i.d.) exponential variables with parameter $\beta > 0$ and the probability density function (p.d.f.)

$$f(x) = \beta e^{-\beta x}, \;\; x > 0, \tag{1}$$

and let N be a geometric random variable with the p.d.f.

$$h(n) = \mathbb{P}(N = n) = p(1-p)^{n-1}, \;\; n = 1, 2, \ldots, \tag{2}$$

independent of the X_i's. The aforementioned distributions are denoted by $\mathcal{EXP}(\beta)$ and $\mathcal{GEO}(p)$, respectively. We are interested in the geometric maximum

$$Y = \max_{1 \le i \le N} X_i \tag{3}$$

and sum

$$X = \sum_{i=1}^{N} X_i \tag{4}$$

of the X_i's. The geometric maximum (3) of the X_i's has a *truncated logistic* distribution with parameters $\alpha = 1/p > 1$ and β and the p.d.f.

$$g(x) = \frac{\alpha \beta e^{-\beta x}}{[1 - (1-\alpha) e^{-\beta x}]^2}, \;\; x > 0, \tag{5}$$

studied by Marshall and Olkin [31]. Since the exponential distribution is closed under geometric compounding, the random sum (4) has an exponential distribution with parameter $p\beta$ (see, [10, 25]). Kozubowski and Panorska [29, 30] have showed that the resulting class of mixed (neither continuous nor discrete) bivariate distributions $(\sum_{i=1}^{N} X_i, N)$ and

$(\max_{1\le i\le N} X_i, N)$ are useful in stochastic modelling of financial data. These mixed distributions have been referred to as $\mathcal{BEG}$-bivariate distribution with *exponential* and *geometric* marginals; and $\mathcal{BTLG}$-bivariate distribution with *truncated logistic* and *geometric* marginals.

In fields such as water resources, climate research, and finance, maxima and geometric sums such as (3) and (4) respectively, appear quite often (see, [9, 25]). Notably, in many hydroclimatic problems which focus on water availability such as precitation or stream flow (see, [15]), the processes are characterized and studied in terms of *hydroclimatic episodes.* For a chosen reference threshold (eg., the mean, median etc. of all observations), hydroclimatic episodes are characterized by three random variables: *duration, maximum and magnitude.* The *duration* of an episode is the number of observations in that episode; the *maximum* is the largest (absolute) observation in that episode; the *magnitude* is the sum of the (absolute) observations in that episode. Studies of problems regarding water resource management, risk assessment and insurance claims have used the joint distributions (X, N) and (Y, N) for investigation and analysis (see, [10, 18, 37]).

In particular, in finance, suppose the X_i's are the log-returns $X_i = \log P_i - \log P_{i-1}$, $i = 1, 2, \dots$, corresponding to the values $P_0, P_1, \dots$ of an asset/process such as the daily price of a stock. Suppose that we use zero as the threshold. Then positive/negative values of X_i mean increase/decrease in the price of the asset. Analyzing the data, we divide it into positive/negative episodes, that is periods of growth/decline in price of that asset. In this framework, N represents the duration of a period of growth (or decline) of the asset, that is duration of an episode. Further, Y represents the *maximum*, and X represents the cumulative log-returns (*magnitude*), over this period.

In this thesis we analyze the following assets for fit of the $\mathcal{BEG}$ and $\mathcal{BTLG}$ models: currency exchange rates, stock prices, stock indexes and prices of commodities.

The thesis is organized as follows. First, we recall definitions and basic properties of $\mathcal{BEG}$ and $\mathcal{BTLG}$ in Section 2. Section 3 is devoted to data description. In Section 4, we focus on modelling of financial asset returns. In particular, we describe various methodologies of modelling financial returns prior to $\mathcal{BEG}$ and $\mathcal{BTLG}$. We also formulate a procedure of graphically assessing fit for both $\mathcal{BEG}$ and $\mathcal{BTLG}$ to the data. Sections 5 and 6 deal with fitting $\mathcal{BEG}$ to the growth and decline episodes respectively. Likewise, in Sections 7 and 8, we fit $\mathcal{BTLG}$ to the growth and decline episodes of our data sets. The last section is a summary of our findings.

2. Definitions and basic properties of **BEG** and **BTLG**

Let $X_1, X_2, \dots$ be independent, identically distributed (i.i.d.) exponential variables with parameter $\beta > 0$ and the probability density function (p.d.f.)

$$f(x) = \beta e^{-\beta x}, \ \ x > 0, \tag{6}$$

and let N be a geometric random variable with the p.d.f.

$$h(n) = \mathbb{P}(N = n) = p(1-p)^{n-1}, \ \ n = 1, 2, \dots, \tag{7}$$

independent of the X_i's. We denote these distributions by $\mathcal{EXP}(\beta)$ and $\mathcal{GEO}(p)$, respectively.

2.1. **The BEG model for (X, N).** The **BEG** distribution was defined and described by Kozubowski and Panorska [29]. We recall the definitions and some properties of this model here.

Definition 2.1. *A random vector (X, N) with the stochastic representation*

$$(X, N) \stackrel{d}{=} \left(\sum_{i=1}^{N} X_i, N \right), \tag{8}$$

where the X_i's are i.i.d. exponential variables (6) and N is a geometric variable (7), independent of the X_i's, is said to have a BEG distribution with parameters $\beta > 0$ and $p \in (0, 1)$. This distribution is denoted by $\mathcal{BEG}(\beta, p)$. The $\mathcal{BEG}(\beta, p)$ stands for bivariate distribution with exponential and geometric marginals.

The joint pdf of $(X, N) \sim \mathcal{BEG}(\beta, p)$ is given as

$$g(x,n) = \frac{p\beta^n}{(n-1)!}[x(1-p)]^{n-1}e^{-\beta x}, \;\; x > 0, \;\; n = 1, 2, \ldots. \tag{9}$$

The conditional pdf of X given $N = n$ has a gamma distribution given as

$$f_{X|N=n}(x) = \frac{\beta^n}{(n-1)!}x^{n-1}e^{-\beta x}, \;\; x > 0. \tag{10}$$

where X is the sum of n i.i.d exponential variables with parameter $\beta > 0$.

2.1.1. *Maximum likelihood estimators:* For a random sample $(X_1, N_1), \ldots, (X_n, N_n)$ from a $\mathcal{BEG}(\beta, p)$ distribution, the log-likelihood function takes the form

$$\log L(\beta, p) = n\overline{N}_n \log\beta + \sum_{i=1}^{n} \log\frac{X_i^{N_i-1}}{(N_i-1)!} - n\overline{X}_n\beta + n(\overline{N}_n - 1)\log(1-p) + n\log p, \tag{11}$$

where $\overline{X}_n$ and $\overline{N}_n$ are the sample means of the X_i's and the N_i's respectively. There exists a unique pair

$$\hat{\beta}_n = \frac{\overline{N}_n}{\overline{X}_n}, \;\; \hat{p}_n = \frac{1}{\overline{N}_n} \tag{12}$$

that maximizes the log-likelihood function (11). These maximum likelihood estimators (MLE's) are consistent, asymptotically normal, and efficient (see, [29]).

Proposition 2.1. *Let $(X_1, N_1), \ldots, (X_n, N_n)$ be i.i.d. variables from a $\mathcal{BEG}(\beta, p)$ distribution, such that the sample averages satisfy the relations $\overline{X}_n > 0$, $\overline{N}_n > 1$. Then there exist unique MLE's of β and p, given by (12). The vector MLE $(\hat{\beta}_n, \hat{p}_n)'$ is*

(i) consistent;

(ii) asymptotically normal, that is $\sqrt{n}[(\hat{\beta}_n, \hat{p}_n)' - (\beta, p)']$ *converges in distribution to a bivariate normal distribution with the (vector) mean zero and the covariance matrix*

$$\Sigma_{MLE} = \begin{bmatrix} \beta^2 p & 0 \\ 0 & p^2(1-p) \end{bmatrix}; \tag{13}$$

(iii) asymptotically efficient

Proposition 2.2. *If* $(X, N) \sim \mathcal{BEG}(\beta, p)$*, then* $\mathbb{E}X = (\beta p)^{-1}$, $\mathbb{E}N = p^{-1}$*, and* $Cov(X, N) = (1-p)(\beta p^2)^{-1}$*. The covariance matrix of* (X, N) *is*

$$\Sigma = \begin{bmatrix} \frac{1}{\beta^2 p^2} & \frac{1-p}{\beta p^2} \\ \frac{1-p}{\beta p^2} & \frac{1-p}{p^2} \end{bmatrix}, \tag{14}$$

and the correlation coefficient of X *and* N *is* $\rho = \sqrt{1-p}$.

Note that as $p \to 0$ (so that the mean of N converges to infinity), the correlation coefficient approaches 1 .

2.2. **The BTLG model for** (Y, N)**.** Here, we recall some of the properties of **BTLG** as defined by Kozubowski and Panorska [30].

Definition 2.2. *A random vector* (Y, N) *with the stochastic representation*

$$(Y, N) \stackrel{d}{=} \left(\max_{1 \le i \le N} X_i, N \right), \tag{15}$$

where the X_i*'s are i.i.d. exponential variables (6) and* N *is a geometric variable (7), independent of the* X_i*'s, is said to have a BTLG distribution with parameters* $\beta > 0$ *and* $p \in (0, 1)$*. This distribution is denoted by* $\mathcal{BTLG}(\beta, p)$*.*

The notation BTLG stands for *bivariate* distribution with *truncated logistic* and *geometric* marginals. By the above definition N has a geometric distribution. The cumulative distribution function (c.d.f.) of Y can be written as $F_Y(y) = G_N(F(x))$, where $F(x) = 1 - e^{-\beta x}$ is the c.d.f. of the X_i's and

$$G_N(z) = \mathbb{E}z^N = \frac{pz}{1-(1-p)z}$$

is the probability generating function (pgf) of N (see, e.g., [31]). Consequently, the cdf of Y is

$$F_Y(y) = \frac{p(1-e^{-\beta y})}{p+(1-p)e^{-\beta y}}, \;\; y \geq 0, \tag{16}$$

while the corresponding p.d.f. is given by (5) with $\alpha = 1/p > 1$. Y has a logistic distribution (see, e.g., [4, 24]) with parameters $\mu = \log[(1-p)/p]/\beta \in \mathbb{R}$ and $\sigma = 1/\beta > 0$, given by the c.d.f.

$$F(y) = \frac{1}{1+\exp\left(\frac{y-\mu}{\sigma}\right)}, \;\; y \in \mathbb{R}, \tag{17}$$

truncated below at zero. In the special case $\mu = 0$ ($p = 1/2$), Y has a (scaled) *half-logistic* (or *folded logistic*) distribution introduced in [3] and studied in [5, 6, 7, 8, 26]. The joint pdf of $(Y, N) \sim \mathcal{BTLG}(\beta, p)$ is

$$g(y, n) = n\beta p e^{-\beta y}[(1-e^{-\beta y})(1-p)]^{n-1}, \;\; y > 0, \;\; n = 1, 2, \ldots. \tag{18}$$

The conditional pdf of Y given $N = n$ is given as

$$f_{Y|N}(y|n) = n\beta e^{-\beta y}(1-e^{-\beta y})^{n-1}, \;\; y > 0, \tag{19}$$

where Y is the maximum of n i.i.d exponential variables with parameter $\beta > 0$. This is a special case with $\alpha = n$ of the *generalized exponential* distribution $\mathcal{GE}(\alpha, \beta)$ introduced in [19] (also called *exponentiated exponential* distribution in [21]) and studied in [20, 22, 23, 36, 35, 40].

2.2.1. *Maximum likelihood estimators:* The log-likelihood function

(20)
$$\log L(\beta, p) = n \left\{ \frac{1}{n} \sum_{j=1}^{n} \log \left[N_j (1 - e^{-\beta Y_j})^{N_j - 1} \right] + \log(\beta p) - \beta \overline{Y}_n + (\overline{N}_n - 1) \log(1 - p) \right\}$$

can be maximized separately with respect to β and p leading to a unique pair of maximum likelihood estimators (MLE's) given in the following result.

Proposition 2.3. *Let* $(Y_1, N_1), \ldots, (Y_n, N_n)$ *be i.i.d. observations from a* $\mathcal{BTLG}(\beta, p)$ *distribution. Then there exist unique MLE's of* β *and* p*, denoted by* $\hat{\beta}$ *and* $\hat{p}$*, respectively, where* $\hat{p} = 1/\overline{N}_n$ *and* $\hat{\beta} \in [1/\overline{Y}_n, \overline{N}_n/\overline{Y}_n]$ *is the unique solution of the equation*

$$\beta = \frac{1}{n\overline{Y}_n} \sum_{j=1}^{n} \left(1 + \frac{(N_j - 1) Y_j \beta}{e^{\beta Y_j} - 1} \right). \tag{21}$$

Remark 2.1. Since the function on the right-hand-side of (21) is decreasing in β, method of iterations, in particular, by using the method of moments estimate of β (which is always in the interval $[1/\overline{Y}_n, \overline{N}_n/\overline{Y}_n]$) as the starting value. (see, e.g., [29] for more details).

Proposition 2.4. *The vector MLE* $(\hat{\beta}_n, \hat{p}_n)'$ *given in Proposition 2.3 is*

(i) consistent;

(ii) asymptotically normal, that is $\sqrt{n}[(\hat{\beta}_n, \hat{p}_n)' - (\beta, p)']$ *converges in distribution to a bivariate normal distribution with the (vector) mean zero and the covariance matrix*

$$\Sigma_{MLE} = \begin{bmatrix} \frac{\beta^2}{1+d_p} & 0 \\ 0 & p^2(1-p) \end{bmatrix}; \tag{22}$$

(iii) asymptotically efficient

3. Data and Descriptive Statistics

In our analysis, we considered four different types of financial assets, namely *foreign exchange rates*, *commodities*, *stock indexes* and *stock prices*. For foreign exchange rates, we used daily exchange rates for the period of January 2, 1980 to May 21, 1996 between the British Pound and the following currencies: US dollar, Swiss Franc, Swedish Krona, Norwegian Krone, Canadian dollar, Australian dollar, Deutsche mark(deprecated). For commodities, we considered prices for Oil (WTI Cushing Spot prices) and Gas for our analysis. In both cases, we considered the daily and weekly prices. For prices of Oil, our data set covers the period from January 1986 to August 2010, while our data set for prices of Gold is from October 1996 to September 2010. Stock indexes used were S&P500 and NASDAQ-100, covering the periods of January 1980 to September 2010 and January 1988 to September 2010, respectively. In both cases, we considered the daily and weekly indexes. Lastly, we considered the following stock prices: daily and weekly prices for Chevron Corporation from January 02, 1970 to April 08, 2011; and daily and weekly stock prices for Bank of America Corporation (BOA) from May 29, 1986 to April 08, 2011. All the data sets were converted to log returns, i.e., log ratios of two consecutive data points.

Our data sets were sourced from www. finance. google. com, www. wikiposit. org, www. eia. gov and www.bloomberg.com

The descriptive statistics of our data sets are summarized in Table 1. Here MIN, 1ST QTR, 3RD QTR, AND MAX stand for MINIMUM, FIRST QUARTILE, THIRD QUARTILE AND MAXIMUM respectively.

Table 1. Descriptive Statistics

ASSETS	MIN	1ST QTR	MEDIAN	MEAN	3RD QTR	MAX
FOREIGN EXCHANGE RATES						
US Dollar	-3.866e-02	-3.572e-03	0.000e+00	-9.192e-05	3.419e-03	4.667e-02
Swiss Franc	-0.0394300	-0.0029070	0.0000000	-0.0001433	0.0029100	0.0282800
Swedish Krona	-0.0360500	-0.0024180	0.0000000	0.0000241	0.0023920	0.1830000
Norwegian Krone	-2.614e-02	-2.391e-03	0.000e+00	-2.253e-05	2.400e-03	7.177e-02
Canadian Dollar	-3.472e-02	-3.619e-03	0.000e+00	-5.359e-05	3.612e-03	3.462e-02
Australian Dollar	-6.164e-02	-4.043e-03	0.000e+00	-1.223e-05	3.741e-03	9.494e-02
Deutsche Mark	-0.0408200	-0.0022470	0.0000000	-0.0001166	0.0024000	0.0277700
COMMODITIES						
Daily Gold	-0.0994500	-0.0019000	0.0001619	0.0001769	0.0023480	0.0378200
Weekly Gold	-0.0563100	-0.0057090	0.0002999	0.0007502	0.0069790	0.0463400
Daily Oil	-1.765e-01	-5.355e-03	2.437e-04	7.222e-05	5.876e-03	8.317e-02
Weekly Oil	-0.0835300	-0.0106700	0.0013530	0.0003511	0.0118500	0.1091000
STOCK INDEXES						
Daily S&P500	-0.0994500	-0.0021240	0.0002330	0.0001332	0.0025030	0.0475900
Weekly S&P500	-0.0872200	-0.0050290	0.0013610	0.0006447	0.0062120	0.0493200
Daily NASDAQ-100	-1.765e-01	-5.355e-03	2.437e-04	7.222e-05	5.876e-03	8.317e-02
Weekly NASDAQ-100	-0.0835300	-0.0106700	0.0013530	0.0003511	0.0118500	0.1091000
STOCKS						
Daily BOA	-1.312e-01	-4.468e-03	0.000e+00	9.753e-05	4.332e-03	3.100e-01
Weekly BOA	-0.2635000	-0.0106600	-0.0009361	0.0004739	0.0092390	0.3099000
Daily Chevron	-8.226e-02	-4.036e-03	0.000e+00	-3.091e-05	3.754e-03	3.076e-01
Weekly Chevron	-0.0710200	-0.0099890	-0.0009953	-0.0001495	0.0084330	0.3194000

4. Modeling Financial Asset Returns using **BEG** and **BTLG**

4.1. **Literature Review.** Before the proposed joint distributions of (X, N) and (Y, N) by Kozubowski and Panorska [29, 30], there has been various models used to describe the marginal distribution of the daily log-returns of currency exchange rates including normal distribution, stable Paretian, scaled t-distribution, mixture of normals, double Weibull distribution, exponential power laws and asymmetric Laplace laws. Here, we describe briefly the various proposed models in turns.

4.1.1. *Normal Distribution.* At the early stages, the common assumption in building theoretical models and doing empirical research work, especially on returns of foreign exchange rates or common stock, was that the returns of exchange rate were normally distributed. Pioneered by Bachelier as a financial model [2], the normal distribution p.d.f.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} exp[-\frac{(x-\mu)^2}{2\sigma^2}], \text{ for all real } \quad x, \tag{23}$$

is symmetric and completely described by it's two parameters: the location (μ) and scale (σ). This distribution has proven useful because other test statistics are well established and easier to implement in analyzing financial data.

4.1.2. *Non-Normal Distributions.* Recent studies and distributional tests have revealed that the normal distribution, in most cases, does not adequately describe the foreign exchange data. Instead, non-normal distributions with fatter tails do a better job. The proposed non-normal distributions will be discussed as follows:

Stable Paretian Distribution: This was first applied to cotton futures by Mandelbrot [32] and further work has been done by Fama [16], Roll [17] and others. The family of stable Paretian distributions is defined by the log of their characteristic function. Namely, if X is a (symmetric) stable Paretian, then natural logarithm of its characteristic function $\ln \phi(t)$ is:

$$\ln \phi(t) = \ln E e^{itX} = i\delta t - \gamma \mid t \mid^{\alpha}, \tag{24}$$

where t is any real number, and i is $\sqrt{-1}$. The stable symmetric distributions have three parameters: δ, which is a location parameter, γ, which is a dispersion parameter, and α, which is the characteristic exponent. The characteristic exponent measures the "fatness" of the tails of the density function and varies from 0 to 2. The smaller the value α, the greater the probability contained in the extreme tails of the distribution, that is, the greater the chance of large positive and negative values. Thus, stable Paretian distributions can account for fat tailed empirical distributions via $\alpha < 2$. For special cases, the stable Paretian is a Cauchy Distribution when $\alpha = 1$; and a Normal distribution when $\alpha = 2$. The 'stable' feature of this distribution is that the characteristic exponent values do not change when observations are summed. That means that the distribution of the summed observations is of the same "type" as the original distribution of the data. For example, if the daily observations are normal, then the weekly and monthly are also normal ($\alpha = 2$), possibly with different dispersion and location. As Boothe and Glassman [11] observed, this stability property makes it a convenient distribution to use in rate of return calculations, where returns sampled at different frequencies are often of interest. A problem though with the stable Paretian in empirical applications is that second and higher order moments do not exist, except in the special case of the normal.

Scaled Student t-Distribution: The student density function is parametrized by m, a location parameter; H, a scale parameter; and v, the degrees of freedom parameter, and given by:

$$f(x) = \frac{\Gamma(\frac{1+v}{2})v^{v/2}\sqrt{H}}{\Gamma(1/2)\Gamma(v/2)}[v + H(x-m)^2]^{-(v+1)/2}, \tag{25}$$

for every real x. The Student distribution approaches the normal as the degrees of freedom aproach inifinity. When the Student random variable $(\tilde{x})$ with $v > 2$ is standardized by dividing $(\tilde{x} - E(\tilde{x}))$ by $\sqrt{VAR(\tilde{x})}$, then the density function of the resulting standardized variable has the following properties:

(1) It has fatter tails than the density function of conventional standardized normal random variable (i.e., normal with a mean equal to zero and variance equal to one), and

(2) It is higher than the standard normal density in the neighbourhood of their common mean, zero.

It is worth noting that the Student distribution is not a stable distribution because the parameters are not constant under addition of observations.

Mixture of Normals (Compound Normal Distribution): If the random variable x_t (the currency's return or stock return) occurs with probability $\alpha_i, i = 1, \ldots, k$ from a normal distribution $N(x_t|\mu_i, \sigma_i^2)$ where $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$, then it exhibits a compound normal distribution with probability density function $g(x)$, where

$$g(x) = \sum_{i=1}^{K} \alpha_i g_i(x \mid \mu_i, \sigma_i^2), \tag{26}$$

where $-\infty < x < \infty$ and $g_i(x \mid \mu_i, \sigma_i)$ is the p.d.f. of a normal random variable with mean μ_i and variance σ_i^2. As noted by Tucker and Pond [39], the compound normal distributions are likely to exhibit good fit for sample with populations characterized by (26). A good example of the use of this distribution has been documented by McFarland, Pettit and Sung [33]. Another use of (26), as observed by Boothe and Glassman [11], is mixing of two normal distributions with the same location (preserving symmetry) and different scales σ. This compound normal distribution has four parameters: one location (μ), two scales (σ_1, σ_2) and a mixing parameter (α_1, α_2). Such a distribution might arise if exchange rates from different days of the week or from tranquil and turbulent periods were normally distributed but had different variances. When the scales of the two distributions are equal, this distribution reduces to normal. The mixture of two normals is different from (and not a special case of) the Student's scale mixture.

Exponential Power Laws: A positive exponential power distribution has density

$$(27) \qquad f(x) = \frac{\alpha}{\Gamma(1/\alpha)} \frac{1}{\sigma} \begin{cases} \exp(-\frac{1}{\sigma^\alpha}) x^\alpha & for \quad x \geq 0 \\ 0 & for \quad x < 0, \end{cases}$$

where $\alpha > 0$ and σ are the shape and scale parameters, respectively. Further, a symmetrization of the (27) was proposed by Subbotin [38] and popularized by Box and Tiao [12, 13, 14], and Osiewalski and Steel [34]. Many others too have studied and shown the modelling potential of Exponential Power (EP) distributions to the log-returns of financial data.

More recently, Ayebo and Kozubowski [1] have proposed a skew exponential power model for modeling financial data arguing that exponential power laws, including their special cases of normal distributions ($\alpha = 2$) and Laplace distributions ($\alpha = 1$), which are symmetric, are not appropriate for modelling data with asymmetric empirical distributions. Having a symmetric p.d.f. g on $\mathbb{R}$, for any $\kappa > 0$, a skew density function is obtained as:

$$(28) \qquad f(x) = \frac{2\kappa}{1+\kappa^2} \begin{cases} g(x\kappa) & for \quad x \geq 0 \\ g(x/\kappa) & for \quad x < 0. \end{cases}$$

Similar to the symmetric EP laws, the proposed distribution (28) exhibits variety of tail behaviors including light tailed near-normal laws ($\alpha \approx 2$) and heavier tailed near-Laplace distributions ($\alpha \approx 1$). An example of a skew distribution that can be obtained in this way from a symmetric density is the skew Laplace law discussed in detail in [27]. If we take $g(x)$ to be symmetric Laplace density

$$g(x) = \frac{1}{2\sigma} \exp(-\frac{\mid x \mid}{\sigma}), \tag{29}$$

and apply the procedure given in equation (28), we obtain a skewed Laplace distribution.

Skewed Laplace distribution: The skew Laplace distribution has density (see e.g. [28])

$$f(x) = \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2} \begin{cases} \exp(-\frac{\kappa}{\sigma}) \mid x-\theta \mid) & for \quad x \geq \theta \\ \exp(-\frac{1}{\kappa\sigma}) \mid x-\theta \mid) & for \quad x < \theta, \end{cases} \tag{30}$$

and was successfully applied to modelling financial data returns in [28]. Their work showed a very good fit of the skew Laplace model to the distribution of currency exchange rates.

We give a description of the procedures used in fitting $\mathcal{BEG}$ and $\mathcal{BTLG}$ to episode data as follows:

4.2. Procedure for fitting $\mathcal{BEG}(\beta, p)$ distribution.

(1) Estimation of parameters $\hat{\beta}$ and $\hat{p}$.

We estimate the parameters $\hat{\beta}$ and $\hat{p}$ using (12), assumming the data are i.i.d $\mathcal{BEG}(\hat{\beta}, \hat{p})$.

(2) If the $\mathcal{BEG}$ distribution is correct, the positive log returns X_i are i.i.d exponential $\mathcal{EXP}(\hat{\beta})$ with the estimated mean of $1/(\hat{\beta})$. We check the fit in two ways:

(a) The histogram of the positive log returns X_i is overlaid with the estimated exponential $\mathcal{EXP}(\hat{\beta})$ distribution.

(b) We plot the probability plot of the positive log returns versus the quantiles of the estimated exponential $\mathcal{EXP}(\hat{\beta})$ distribution.

(3) If $\mathcal{BEG}$ distribution is correct, then the positive log-returns X_i should have the same distribution as the cumulative positive log returns of our original data. We check the fit in three ways:

(a) The histogram of the cumulative positive log returns is overlaid with the p.d.f. of exponential distribution with parameter $\hat{\beta}\hat{p}$.

(b) We plot the probability plot of the cumulative positve log returns versus the pdf of exponential distribution with parameter $\hat{\beta}\hat{p}$.

Remark 4.1. β is estimated from the positive cumulative log-returns (12) and not from the positive log returns.

(c) We plot the QQ plot of the positve log returns versus the cumulative positive log returns.

Remark 4.2. The positive log returns show "internal stability" with respect to the growth period if the positive cumulative log returns have the same distribution as the positive log returns. We observe stability if the QQ plot is close to straight line.

(4) From the $\mathcal{BEG}$ model, the duration N should have a geometric distribution with parameter $\hat{p}$. We first compute the theoretical relative frequencies for a geometric

random variable with parameter $\hat{p}$ and compare them with the empirical relative frequencies. Second, we compare the theoretical cumulative distribution function with the empirical cumulative distribution function.

(5) Goodness of fit for the Bivariate $\mathcal{BEG}$ model

(a) We compute the sample correlation (N_i, X_i) and compare with the theoretical correlation (2.2).

(b) According the $\mathcal{BEG}$ model, the conditional distributions of $X|N = k$ should have a gamma distribution $G(k, \hat{\beta})$ (10) with scale parameter k and shape parameter $\hat{\beta}$. We assess fit of the conditional distribution of X given $N = k$, where $k = 1, \ldots, n$ for those k when we have a reasonable amount of data. For each data set, the histogram is overlaid with the estimated gamma $G(k, \hat{\beta})$ p.d.f. We also plot the probability plot of the estimated gamma $G(k, \hat{\beta})$ p.d.f. versus cummulative positive log returns. We include least squares fit lines to aid visualization of their linearity.

4.3. Procedure for fitting $\mathcal{BTLG}(\beta, p)$ distribution.

(1) Estimation of the parameters $\hat{\beta}$ and $\hat{p}$. We estimate the parameters β and p by using their MLEs given in proposition 2.3.

(2) According to (15), if our $\mathcal{BTLG}$ model is correct, the maxima of the positive log returns per growth periods should have a truncated logistic distribution (17) with parameters $\hat{p}$ and $\hat{\beta}$. We assess the fit in two ways:

(a) We compare the theoretical c.d.f. curve with the empirical c.d.f. curve of a truncated logistic distribution.

(b) We plot the probability plot of the maxima versus theoretical quantiles of the truncated logistic distribution.

(3) For the $\mathcal{BTLG}$ model, the duration N should have a geometric distribution. Like the $\mathcal{BEG}$ model, we first compute the theoretical relative frequencies of a geometric distribution with parameter $\hat{p}$ and compare them with the empirical relative frequencies. Second, we compare the theoretical cumulative distribution function curve with the empirical cumulative distribution function curve. We include the least squares fit lines to aid visualization of their linearity.

(4) Goodness of fit for the Bivariate $\mathcal{BTLG}$ model

The conditional distributions of maxima given durations of the growth episodes should have a general exponential p.d.f. (19). We assess fit for the conditional distribution of X given $N = k$, where $k = 1, \ldots, n$, for a reasonable amount of data. For each data set, we plot the the theoretical generalized exponential c.d.f. versus the the empirical conditional c.d.f. We also plot the probability plot of the generalized exponential quantiles versus empirical quantiles of the conditional maxima .

Remark 4.3. The same procedures (4.3) and (4.2) are repeated for *decline* episodes, where we use absolute values of the log returns.

In the ensuing sections, we follow the procedures described in subsections 4.2 and 4.3 to fit $\mathcal{BEG}$ and $\mathcal{BTLG}$ to the log returns of financial assets in the following order: *foreign exchange rates, prices of commodities, stock prices, and stock indexes.*

5. Fitting **BEG** To Growth Episodes

5.1. Estimation of Parameters $\hat{\beta}$ and $\hat{p}$.

Table 2. Foreign Exchange Rates

Currency	$\hat{\beta}$	$\hat{p}$	$\hat{\beta}\hat{p}$
US Dollar	200.6742	0.501237	100.5854
Swiss Franc	249.6903	0.4869029	121.5749
Swedish Krona	272.0464	0.5420189	147.4543
Norwegian Krone	289.2154	0.5345419	154.5978
Canadian Dollar	201.3431	0.4934243	99.34757
Australian Dollar	172.7798	0.5029791	86.90463
Deutsche Mark	289.0577	0.4927461	142.4321

Table 3. Commodities

Commodity	$\hat{\beta}$	$\hat{p}$	$\hat{\beta}\hat{p}$
Daily Gold	327.2539	0.4700939	153.84017
Weekly Gold	114.5046	0.502809	57.57396
Daily Oil	127.9381	0.4912337	62.84749
Weekly Oil	71.67049	0.4279412	30.67075

Table 4. Stock Indexes

Stock Indices	$\hat{\beta}$	$\hat{p}$	$\hat{\beta}\hat{p}$
Daily S&P500	304.4922	0.4747868	144.5689
Weekly S&P500	139.2319	0.4605993	64.13011
Daily NASDAQ-100	127.9381	0.4912337	62.84749
Weekly NASDAQ-100	71.67049	0.4279412	30.67075

Table 5. Stock Prices

Stocks	$\hat{\beta}$	$\hat{p}$	$\hat{\beta}\hat{p}$
Daily BOA	132.8235	0.4991597	66.30011
Weekly BOA	55.72974	0.5259631	29.31179
Daily Chevron	173.6811	0.4994734	86.74906
Weekly Chevron	75.13167	0.5675403	42.64025

By way of using **BEG** model to fit growth episodes of *US dollar* as an example, we explain here the graphs included in this section and Section 6.

(1) **Figure 1**

   (a) Graph (A) provides a probability histogram (scaled total area of the bins equals to 1) of the positive log-returns overlaid with model exponential p.d.f. with estimated parameter $\hat{\beta} = 200.6742$.

   (b) Graph (B) shows probability plot of the same log returns data used in graph (A) with exponential model.

   (c) Graph (C) shows the probability histogram of *cumulative* positive log returns, i.e. magnitudes of the growth episodes overlaid with model exponential p.d.f. with estimated paramter $\hat{\beta}\hat{p} = 100.5854$.

   (d) Graph (D) shows empirical and model geometric c.d.f.'s for the duration of the growth episodes.

   (e) Table (6) is a summary table of the empirical frequencies, relative frequencies and theoretical probabilities (using geometric distribution with $\hat{p}$) of durations of the growth periods.

(2) **Figure 2**

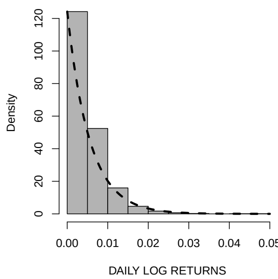
   (a) Graph (A) depicts *stability* of the log returns with respect to geometric summation. It is a QQ-plot of positive log returns versus cumulative positive log-returns (geometric sums of log returns). The two data sets exhibit stability by having the same type of distribution, i.e., having a staight line as depicted on the graph.

(b) Graph (B) shows a probability plot of the cumulative positive log-returns, i.e., the magnitudes of the growth (or decline) episodes versus theoretical quantiles of exponential distribution with parameter $\hat{\beta}\hat{p} = 100.5854$.

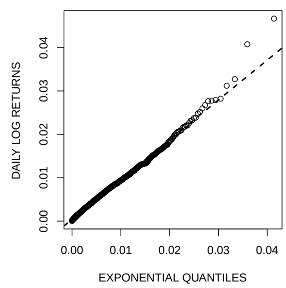
(c) Graph (C) through (F) are probability plots of $N_i = 1, 2, 3, 4 - day$ maxima of log returns versus model gamma distribution.

5.2. **Graphical fit:- Foreign exchange rates.**

5.2.1. *US Dollar (growth episodes).*



(A)



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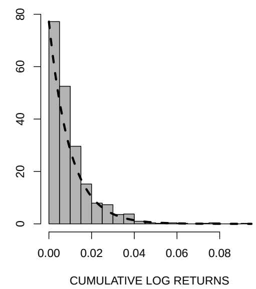

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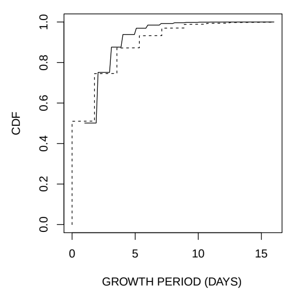


(D)

FIGURE 1. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 6. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	517	238	128	61	38	19	5	4
Relative Frequency	0.510	0.235	0.126	0.060	0.038	0.019	0.005	0.004
Model Probability	0.501	0.250	0.125	0.062	0.031	0.015	0.008	0.004

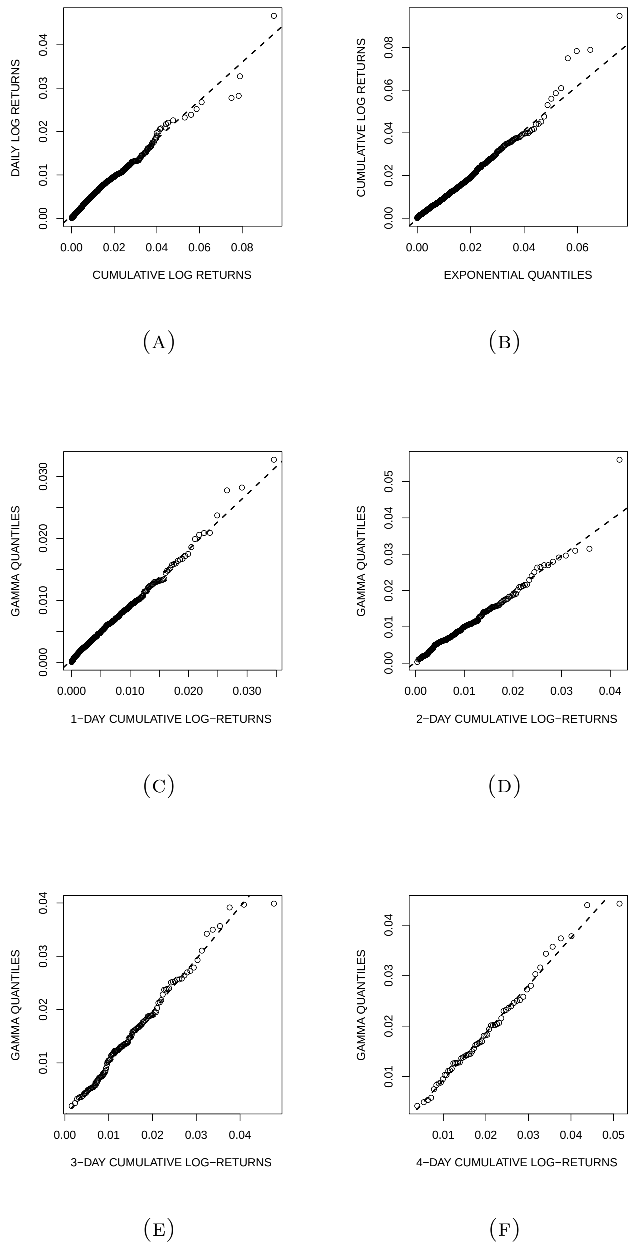
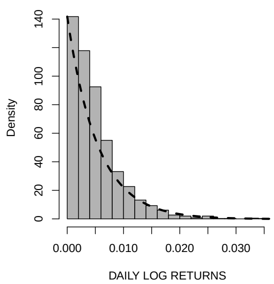


FIGURE 2. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.2.2. *Canadian Dollar (growth episodes).*



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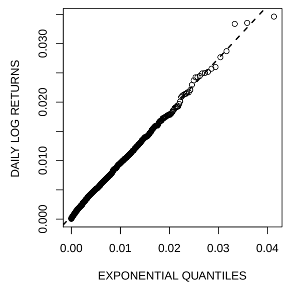
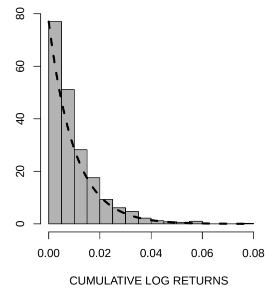


(B)



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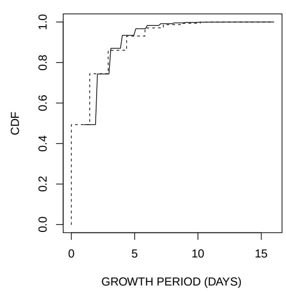

(D)

FIGURE 3. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 7. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	500	254	117	71	41	16	7	6
Relative Frequency	0.494	0.251	0.115	0.070	0.040	0.016	0.007	0.006
Model Probability	0.501	0.250	0.125	0.062	0.031	0.015	0.008	0.004

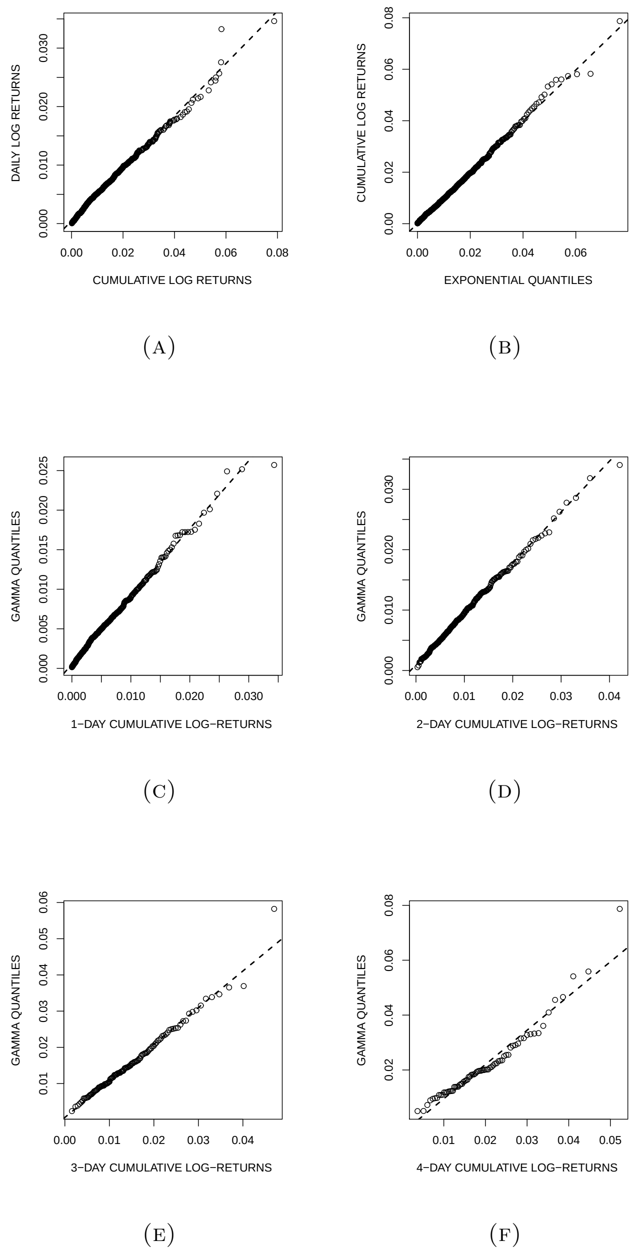

FIGURE 4. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.2.3. *Australian Dollar (growth episodes).*

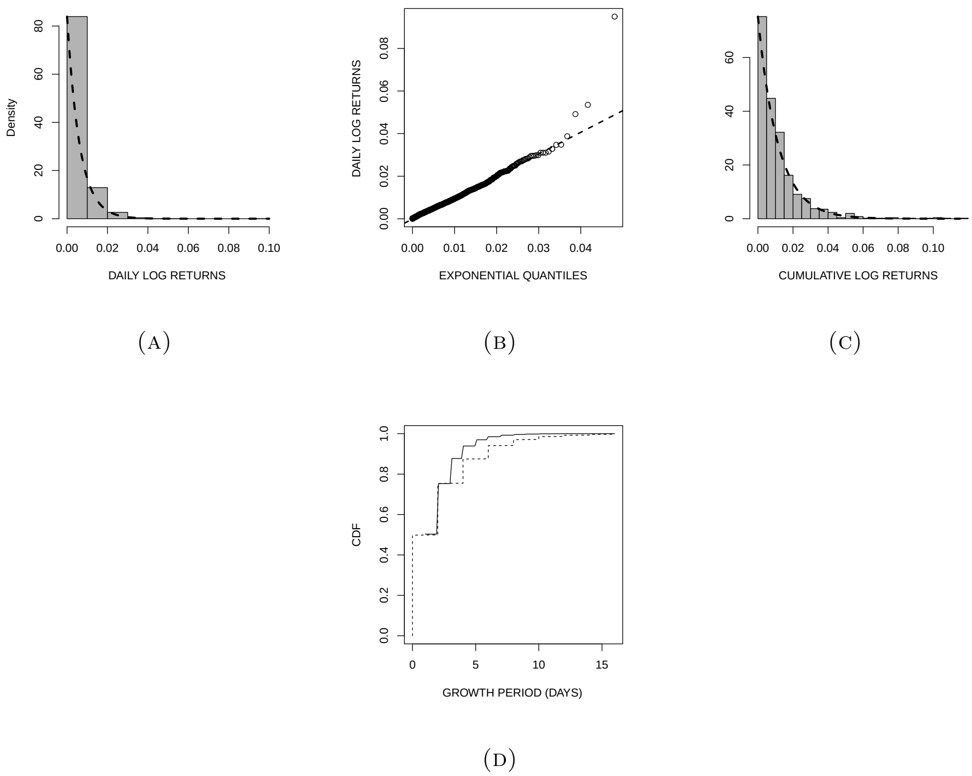

FIGURE 5. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 8. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	505	259	122	67	30	15	7	4
Relative Frequency	0.499	0.256	0.120	0.066	0.030	0.015	0.007	0.004
Model Probability	0.503	0.250	0.124	0.062	0.031	0.015	0.008	0.004

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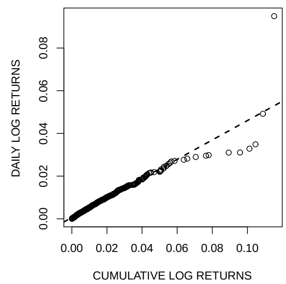
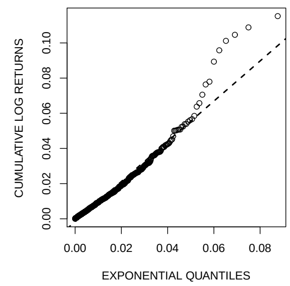


(A)



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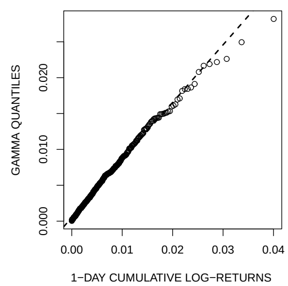

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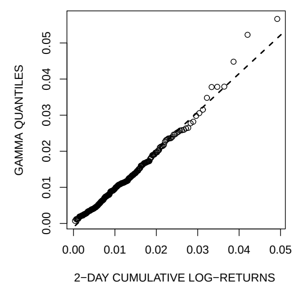

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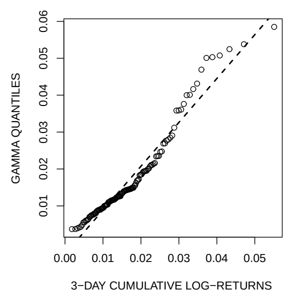

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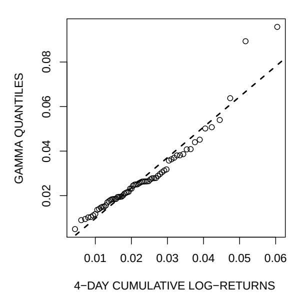

(F)

FIGURE 6. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.2.4. *Norwegian Krone (growth episodes).*

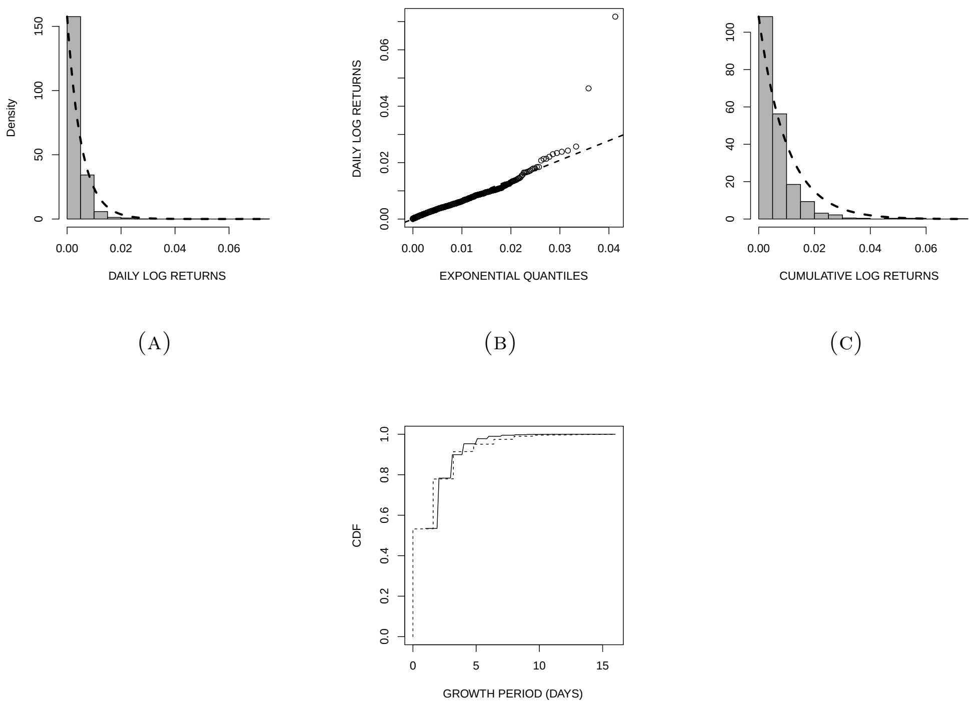

FIGURE 7. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 9. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	581	269	147	40	26	17	6	1
Relative Frequency	0.533	0.247	0.135	0.037	0.024	0.016	0.005	0.001
Model Probability	0.535	0.249	0.116	0.054	0.025	0.012	0.005	0.003

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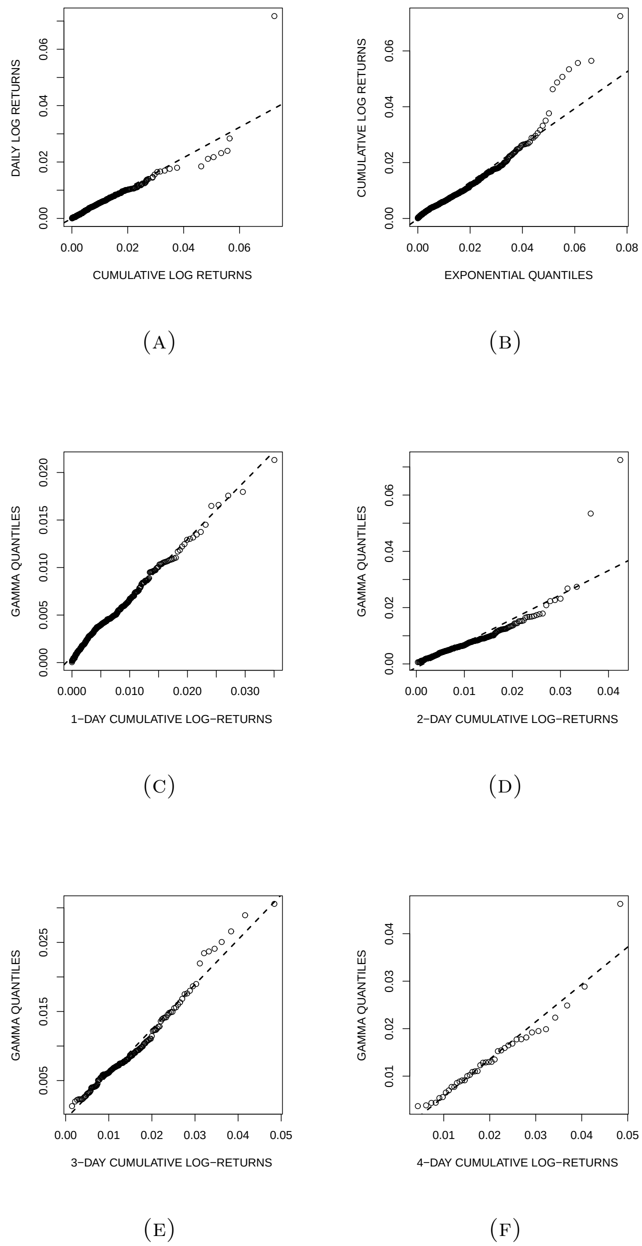
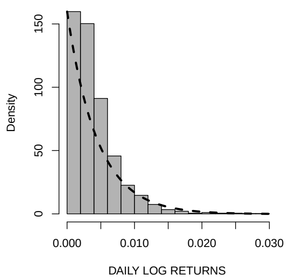


FIGURE 8. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.2.5. *Swiss Franc (growth episodes).*



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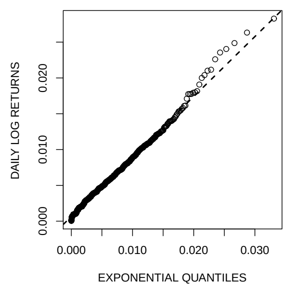

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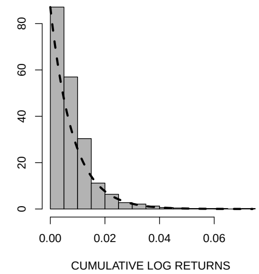

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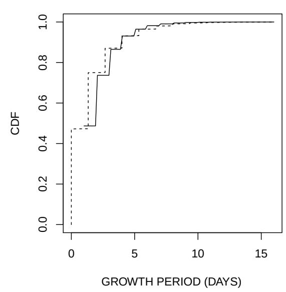
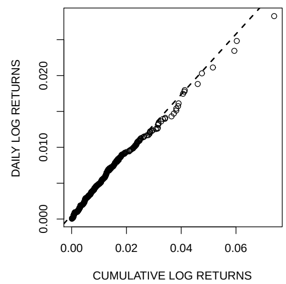


(D)

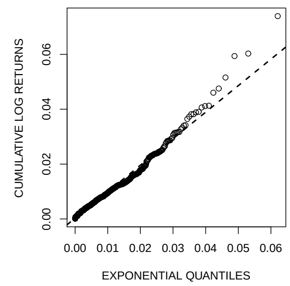
FIGURE 9. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 10. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	448	263	114	58	31	15	10	4
Relative Frequency	0.473	0.277	0.120	0.061	0.033	0.016	0.011	0.004
Model Probability	0.487	0.250	0.128	0.066	0.034	0.017	0.009	0.005



(A)



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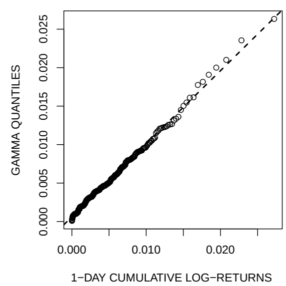

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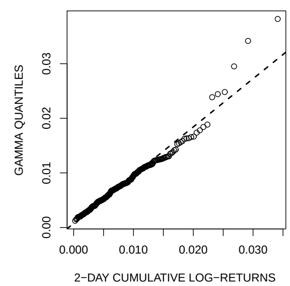

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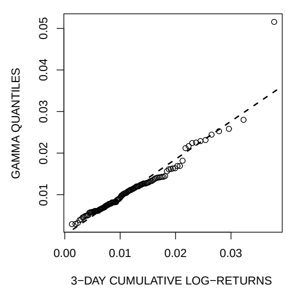

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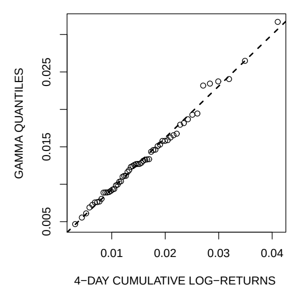
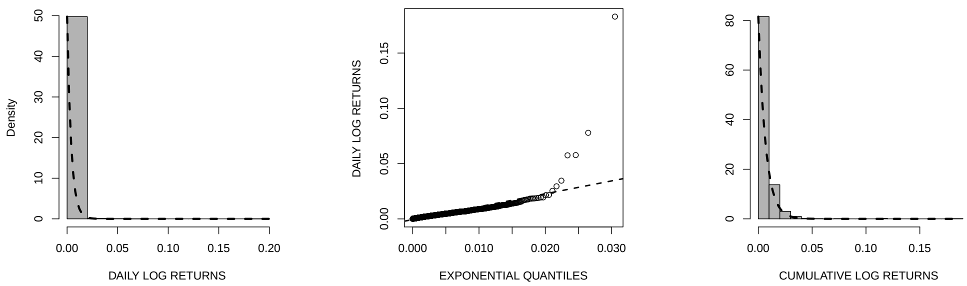


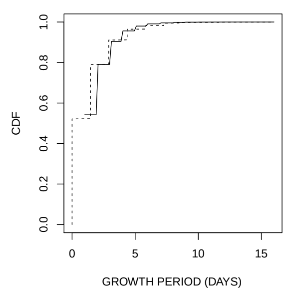
(F)

FIGURE 10. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.2.6. *Swedish Krona (growth episodes).*



(A) (B) (C)



(D)

FIGURE 11. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 11. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	569	292	132	58	19	14	3	0
Relative Frequency	0.522	0.268	0.121	0.053	0.017	0.013	0.003	0.000
Model Probability	0.542	0.248	0.114	0.052	0.024	0.011	0.005	0.002

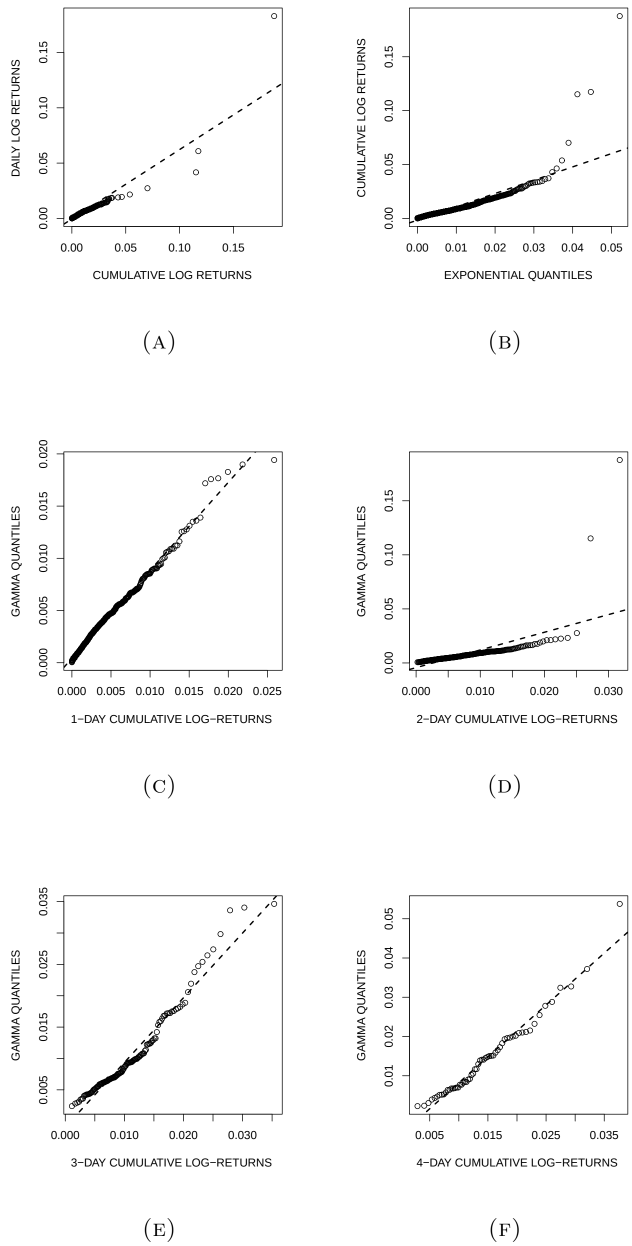


FIGURE 12. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.2.7. *Deutsche Mark (growth episodes).*

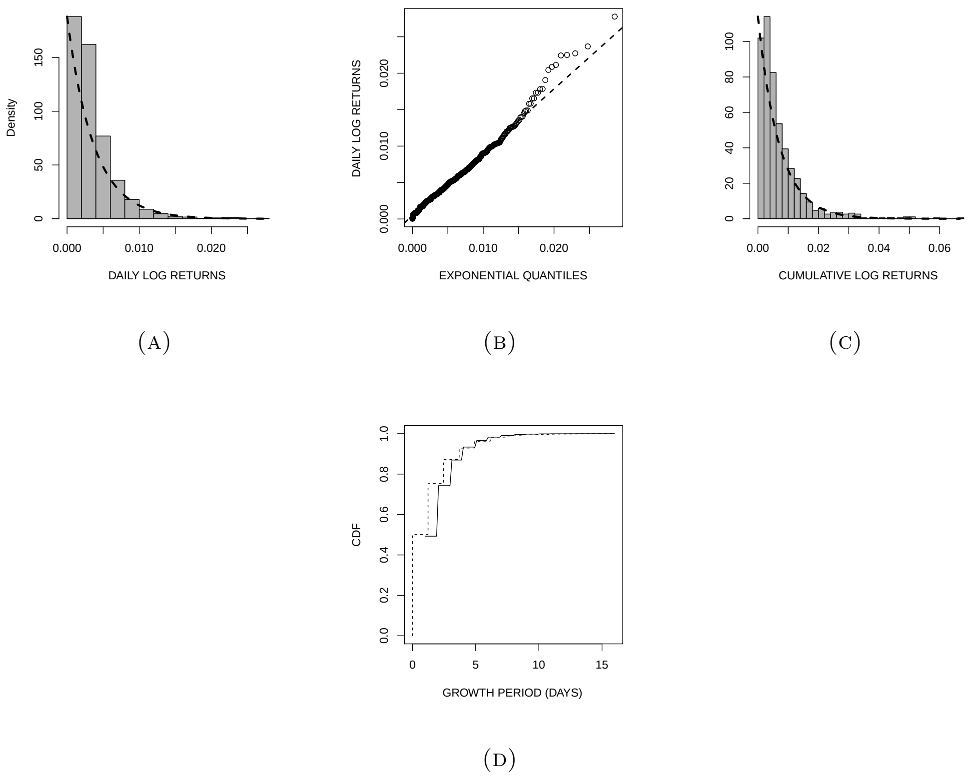


FIGURE 13. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 12. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	477	239	113	54	32	19	6	5
Relative Frequency	0.502	0.251	0.119	0.057	0.034	0.020	0.006	0.005
Model Probability	0.493	0.250	0.127	0.064	0.033	0.017	0.008	0.004

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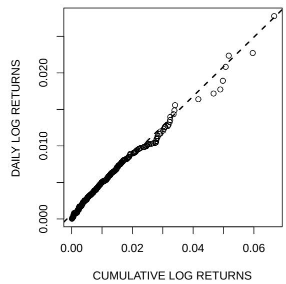
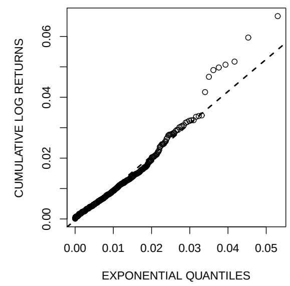
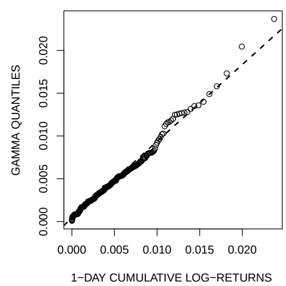


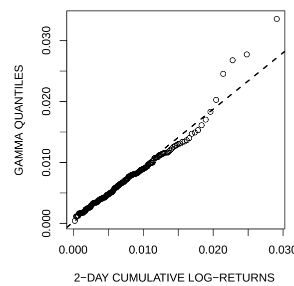
(A)



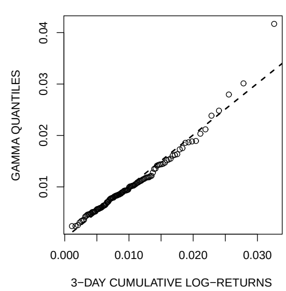
(B)



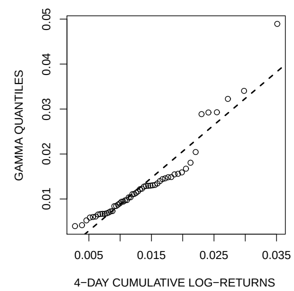
(C)



(D)



(E)

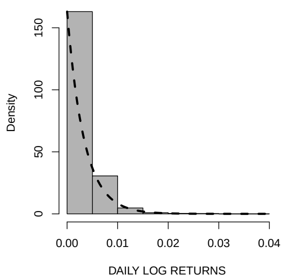


(F)

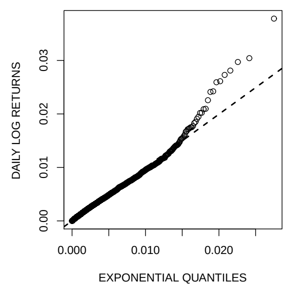
FIGURE 14. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.3. **Graphical fit:-Commodities.**

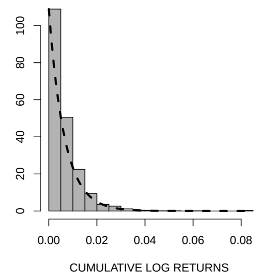
5.3.1. *Gold:-Daily log-returns (growth episodes).*



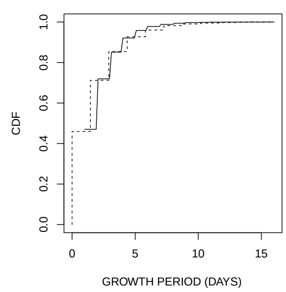
(A)



(B)



(C)



(D)

FIGURE 15. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 13. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	874	480	270	139	63	41	16	8
Relative Frequency	0.460	0.252	0.142	0.073	0.033	0.022	0.008	0.004
Model Probability	0.470	0.249	0.132	0.070	0.037	0.020	0.010	0.006

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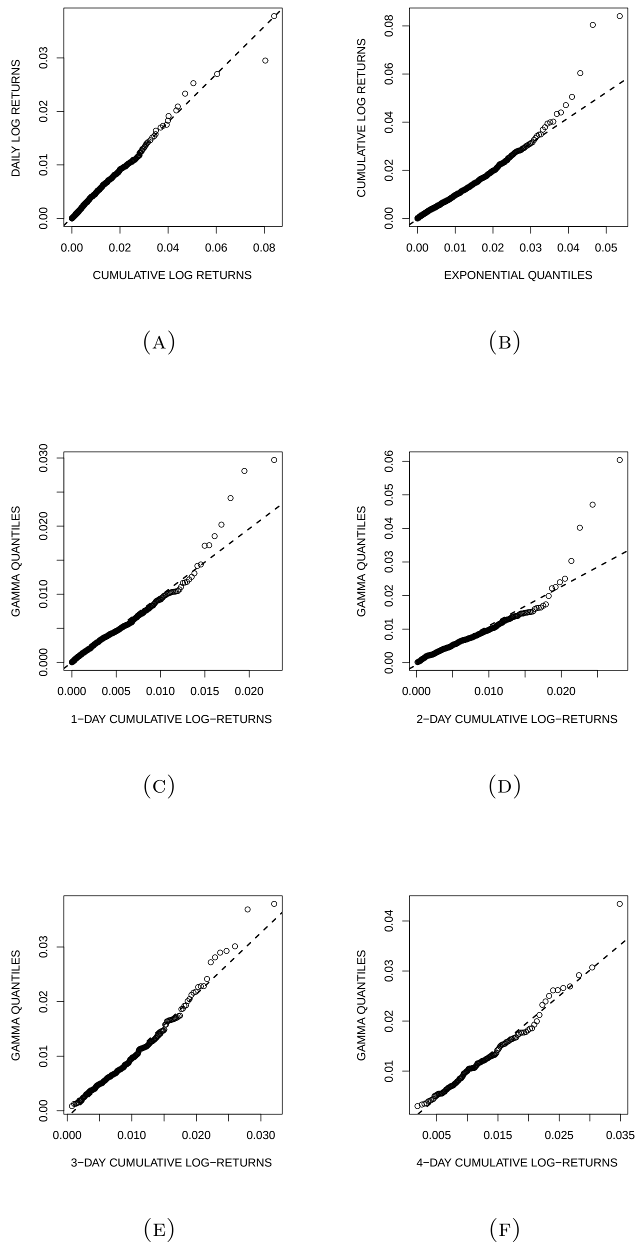
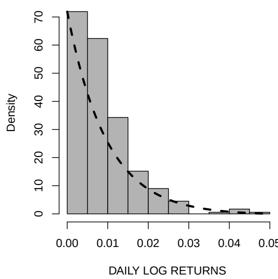
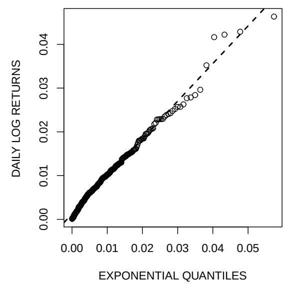


FIGURE 16. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

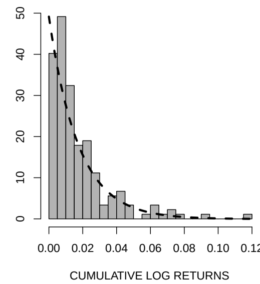
5.3.2. *Gold:-Weekly log-returns (growth episodes).*



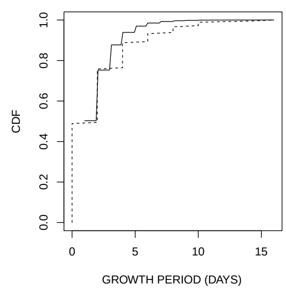
(A)



(B)



(C)



(D)

FIGURE 17. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 14. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	88	48	23	8	6	4	1	1
Relative Frequency	0.492	0.268	0.128	0.045	0.034	0.022	0.006	0.006
Model Probability	0.503	0.250	0.124	0.062	0.031	0.015	0.008	0.004

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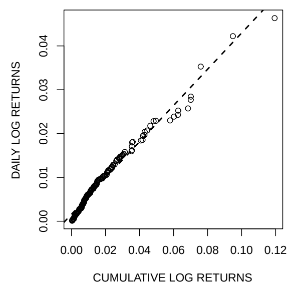
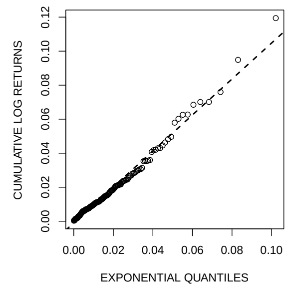
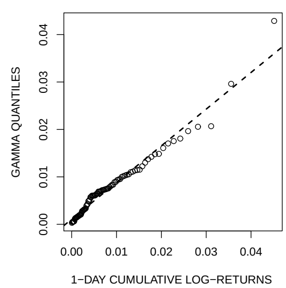


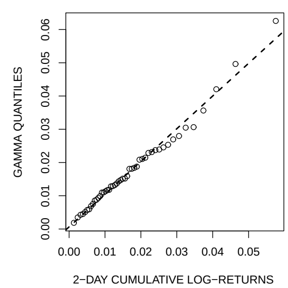
(A)



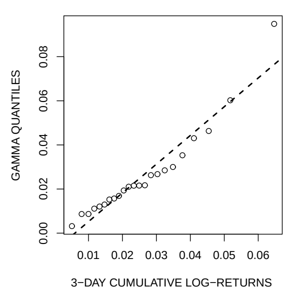
(B)



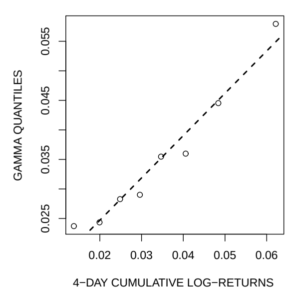
(C)



(D)



(E)



(F)

FIGURE 18. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.3.3. *Oil:-Daily log-returns (growth episodes).*

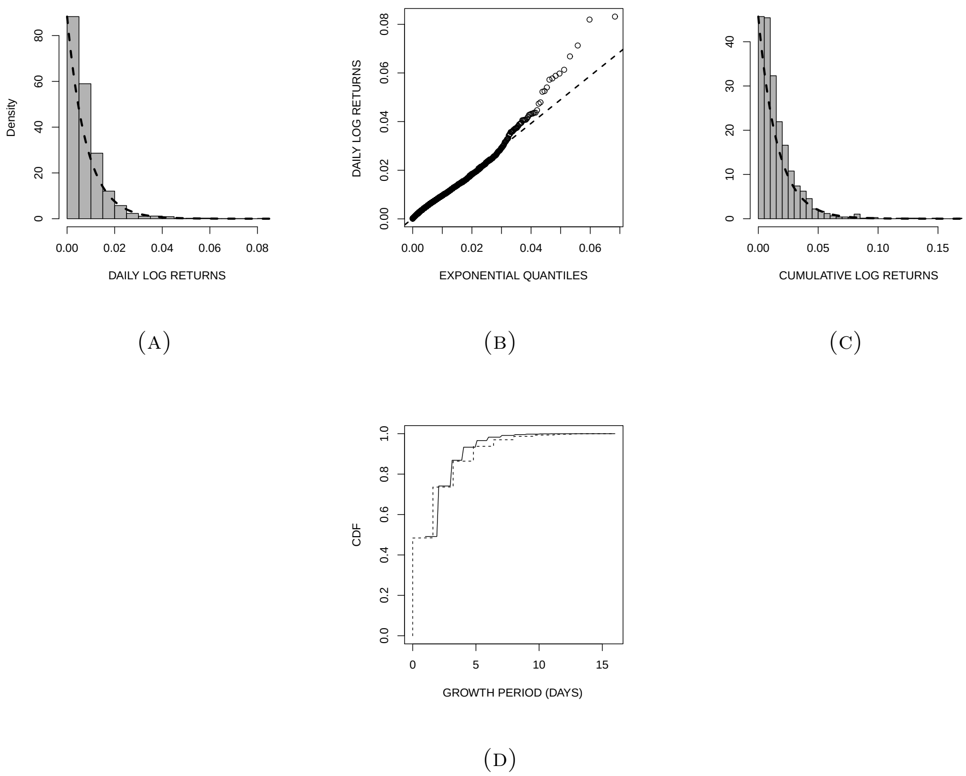


FIGURE 19. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 15. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	746	388	197	113	50	26	10	6
Relative Frequency	0.484	0.252	0.128	0.073	0.032	0.017	0.006	0.004
Model Probability	0.491	0.250	0.127	0.065	0.033	0.017	0.009	0.004

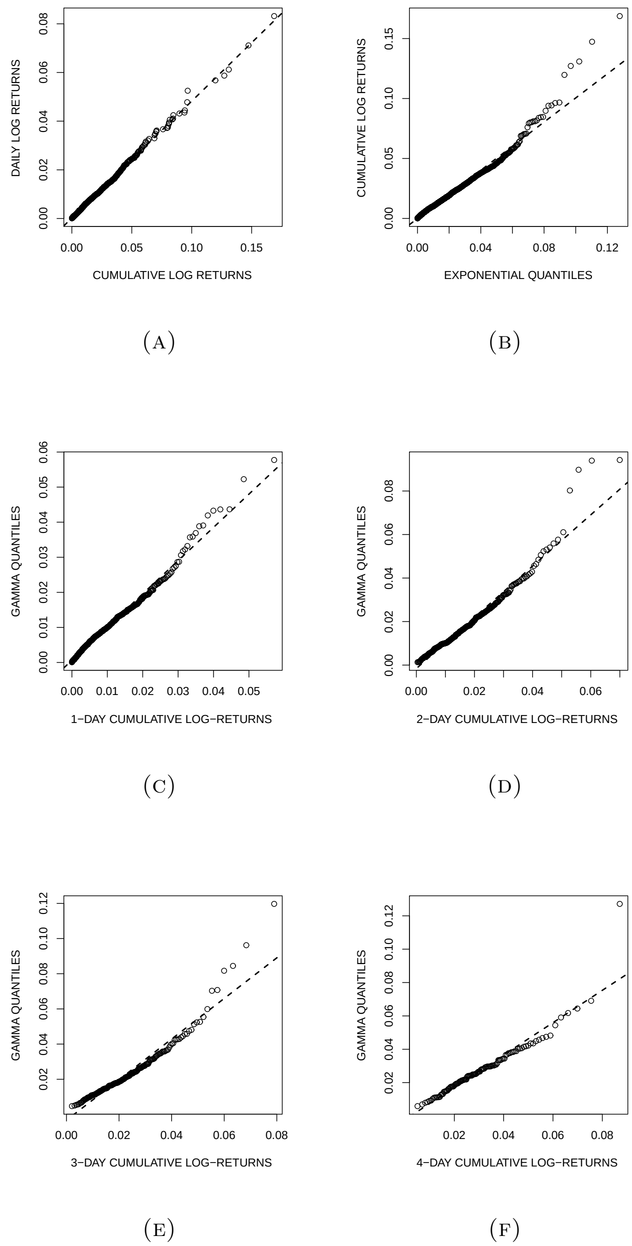


FIGURE 20. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.3.4. *Oil:-Weekly log-returns (growth episodes).*

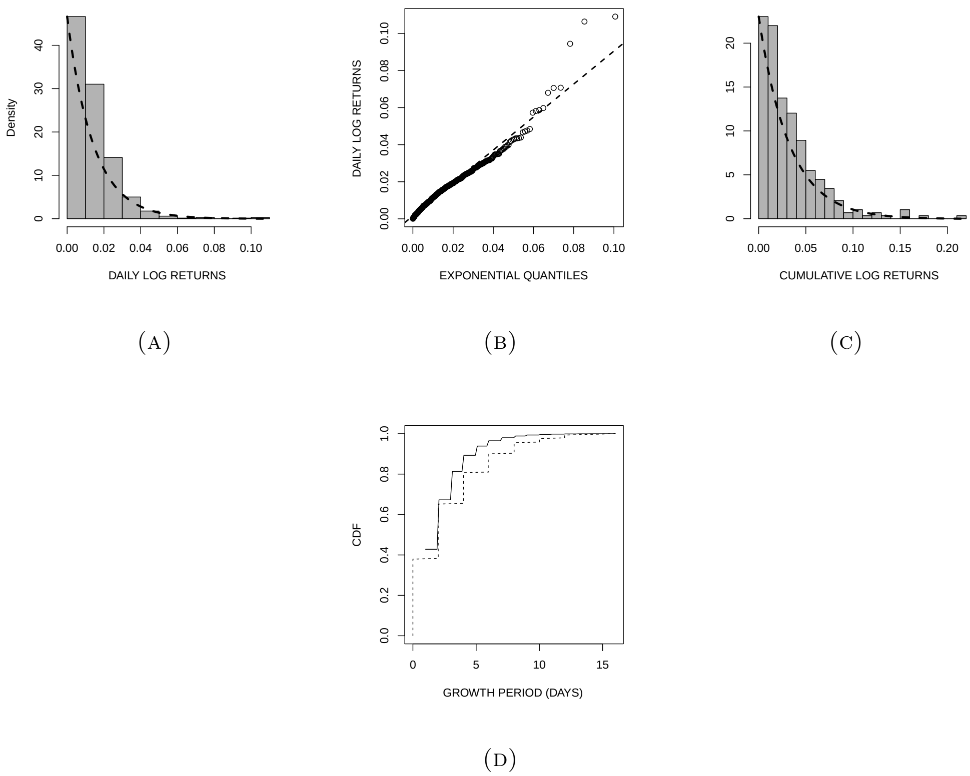


FIGURE 21. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 16. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	111	79	45	27	16	6	5	1
Relative Frequency	0.381	0.271	0.155	0.093	0.055	0.021	0.017	0.003
Model Probability	0.428	0.245	0.140	0.080	0.046	0.026	0.015	0.009

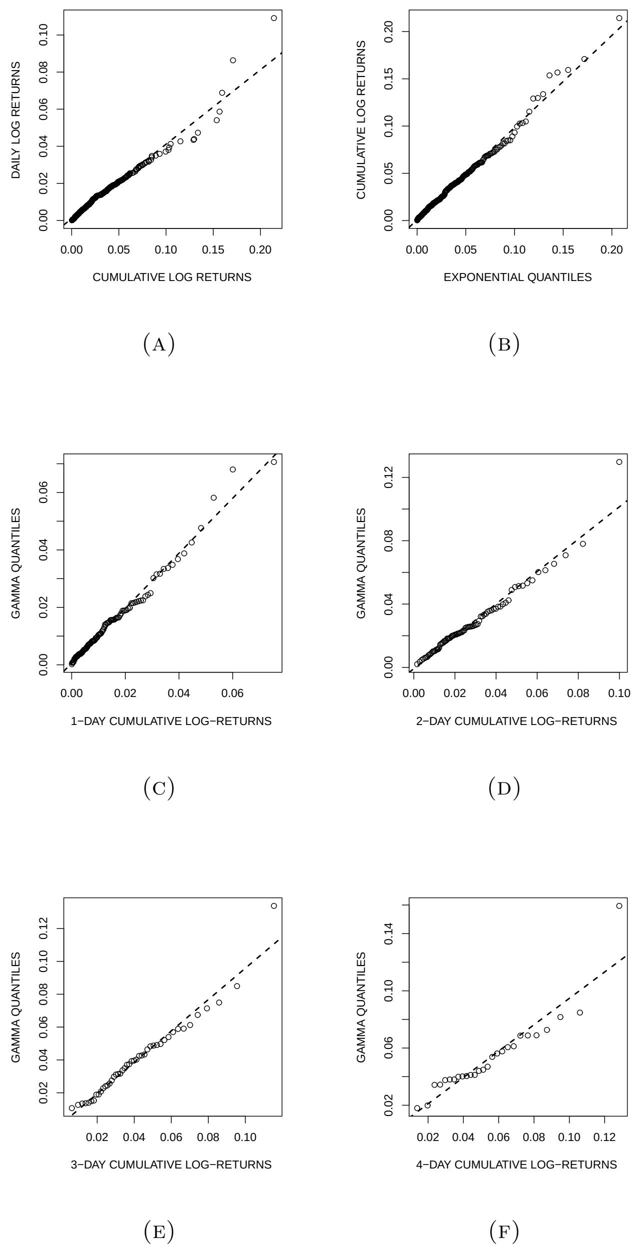


FIGURE 22. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.4. **Graphical fit:-Stock Indexes.**

5.4.1. *S&P500:-Daily log-returns (growth episodes).*

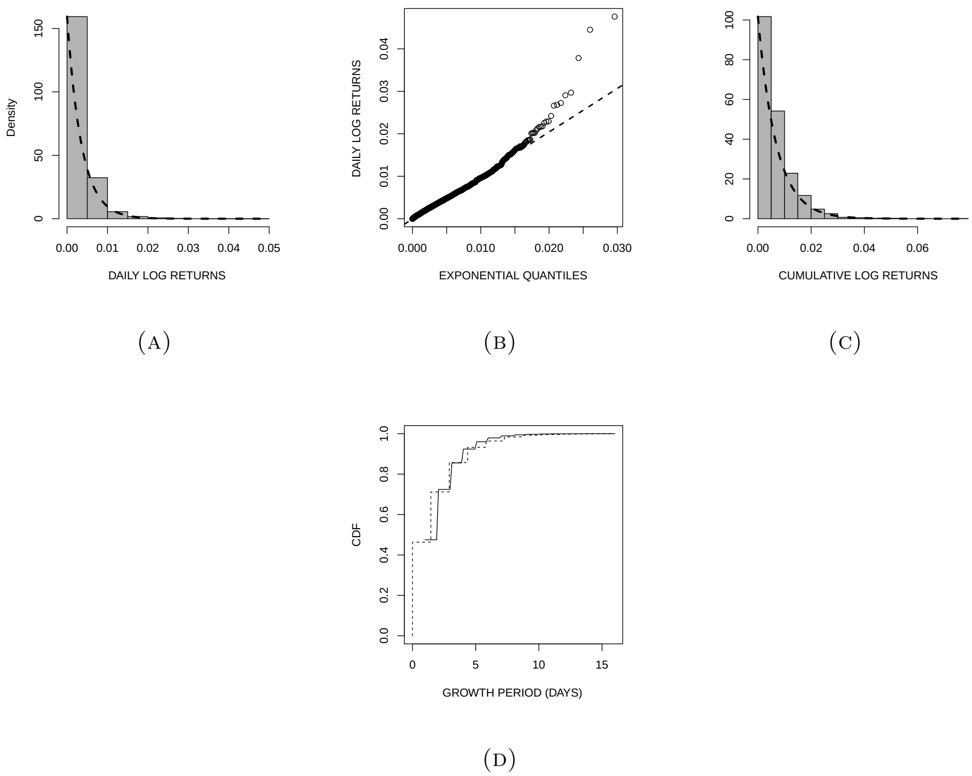


FIGURE 23. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 17. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	903	484	283	147	61	39	16	7
Relative Frequency	0.463	0.248	0.145	0.075	0.031	0.020	0.008	0.004
Model Probability	0.475	0.249	0.131	0.069	0.036	0.019	0.010	0.005

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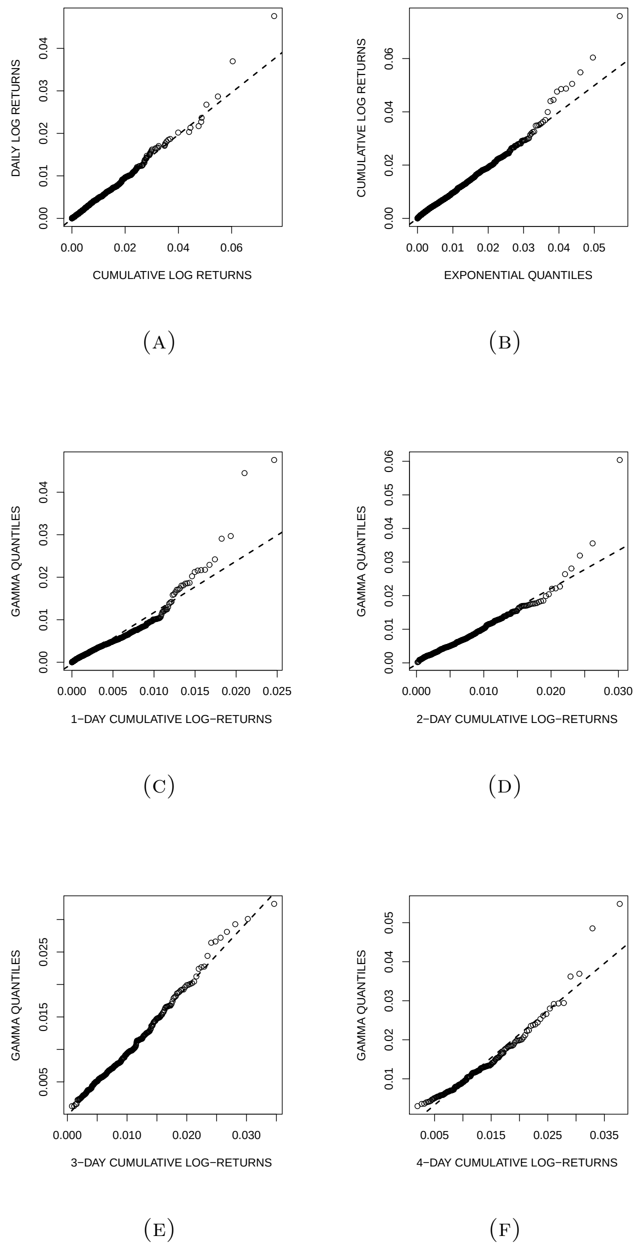
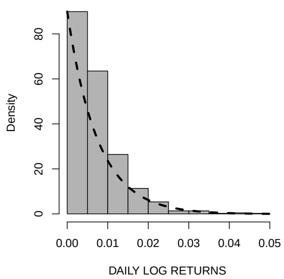
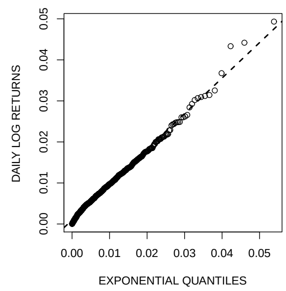


FIGURE 24. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

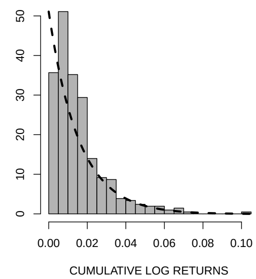
5.4.2. *S&P500:-Weekly log-returns (growth episodes).*



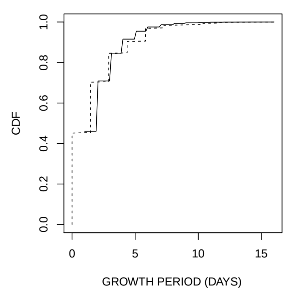
(A)



(B)



(C)



(D)

FIGURE 25. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 18. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	188	104	59	24	27	6	1	3
Relative Frequency	0.453	0.251	0.142	0.058	0.065	0.014	0.002	0.007
Model Probability	0.461	0.248	0.134	0.072	0.039	0.021	0.011	0.006

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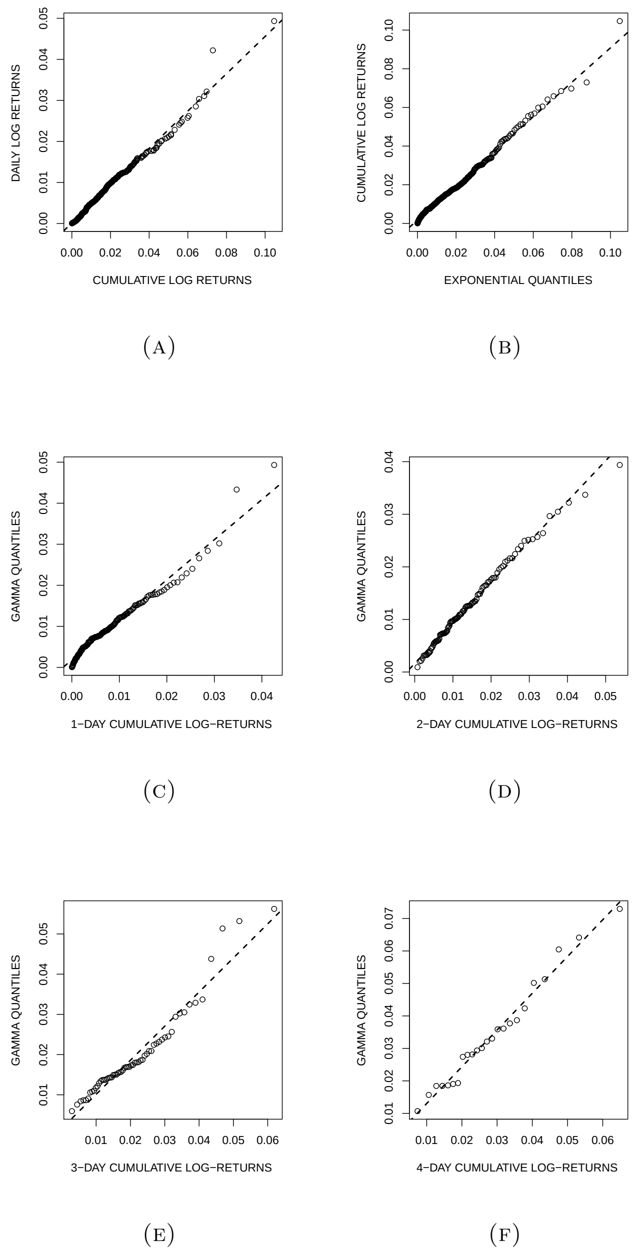

FIGURE 26. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.4.3. *NASDAQ-100:-Daily log-returns (growth episodes).*

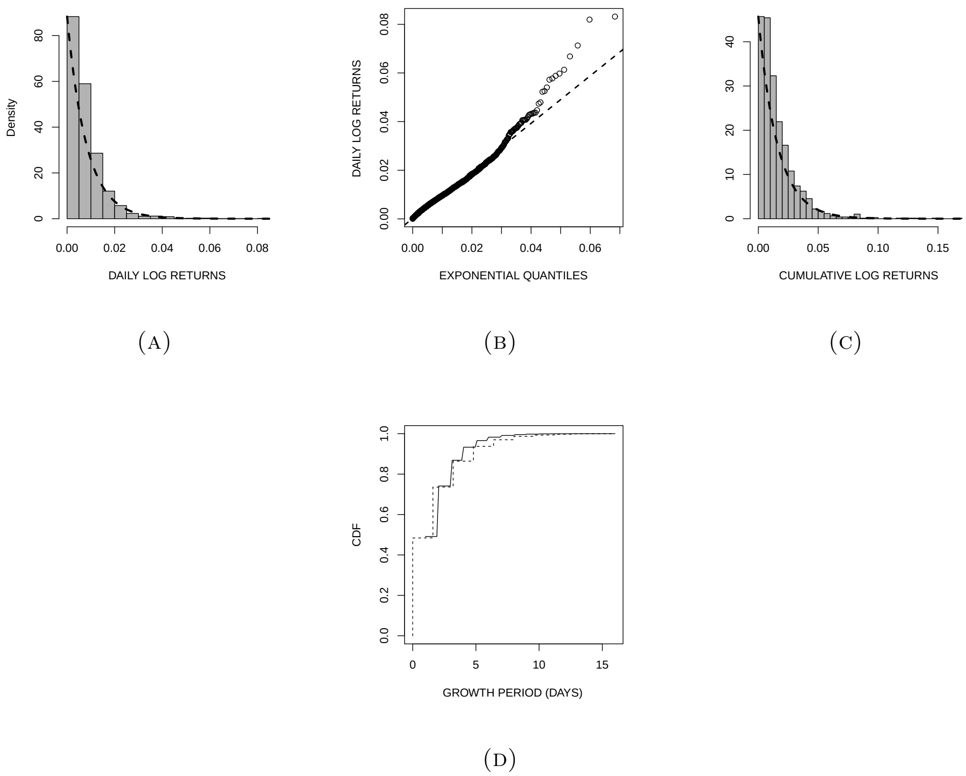

FIGURE 27. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 19. Frequency, relative frequency and geometric probability (model)

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Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	746	388	197	113	50	26	10	6
Relative Frequency	0.484	0.252	0.128	0.073	0.032	0.017	0.006	0.004
Model Probability	0.491	0.250	0.127	0.065	0.033	0.017	0.009	0.004

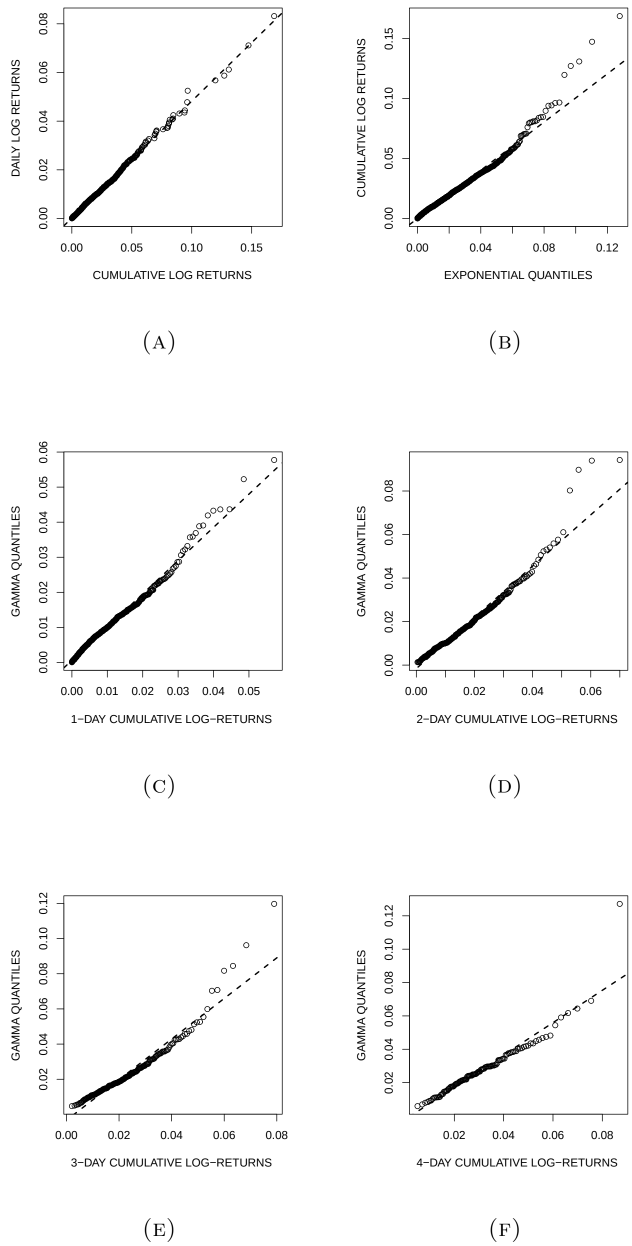
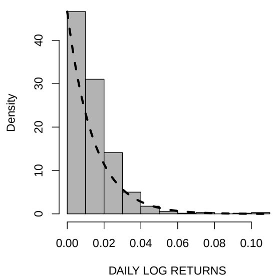
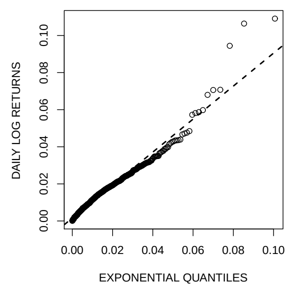


FIGURE 28. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

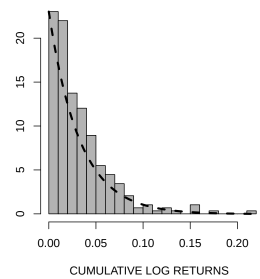
5.4.4. *NASDAQ-100:-Weekly log-returns (growth episodes).*



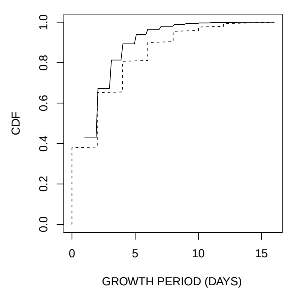
(A)



(B)



(C)



(D)

FIGURE 29. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 20. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	111	79	45	27	16	6	5	1
Relative Frequency	0.381	0.271	0.155	0.093	0.055	0.021	0.017	0.003
Model Probability	0.428	0.245	0.140	0.080	0.046	0.026	0.015	0.009

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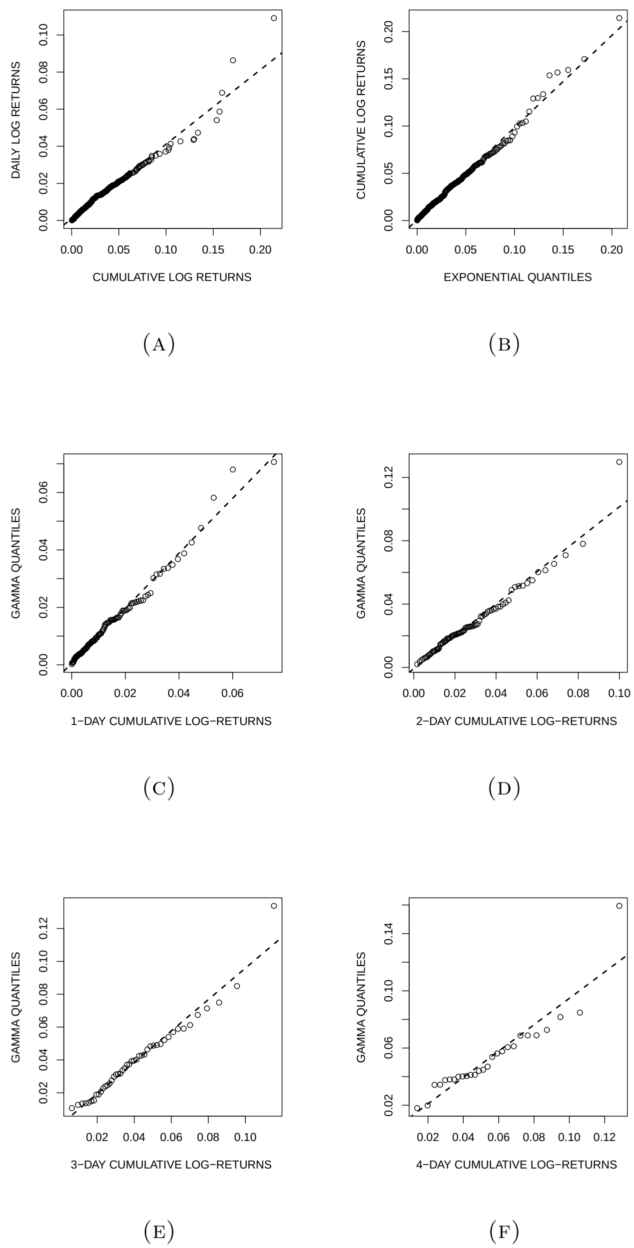
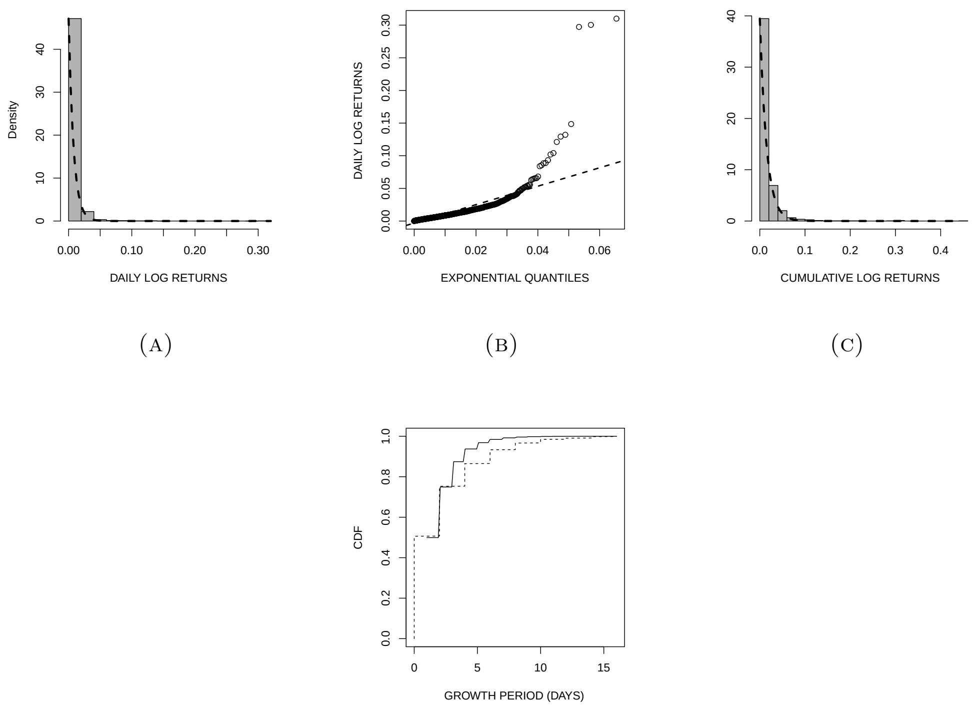


FIGURE 30. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.5. **Graphical fit:-Stocks.**

5.5.1. *BOA:-Daily log-returns (growth episodes).*



(A) (B) (C)

(D)

FIGURE 31. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 21. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	752	366	166	102	49	27	9	11
Relative Frequency	0.506	0.246	0.112	0.069	0.033	0.018	0.006	0.007
Model Probability	0.499	0.250	0.125	0.063	0.031	0.016	0.008	0.004

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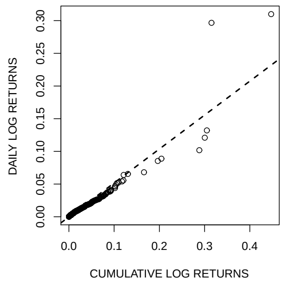
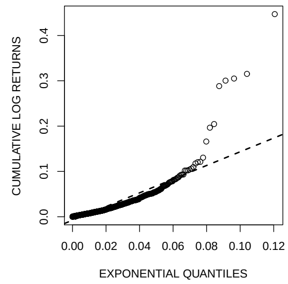
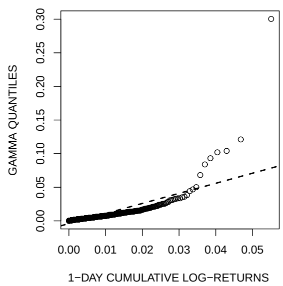


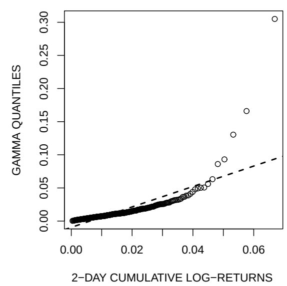
(A)



(B)



(C)



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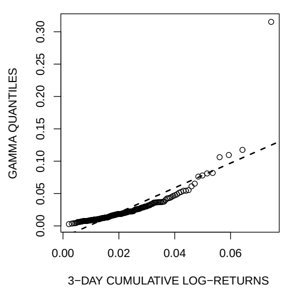

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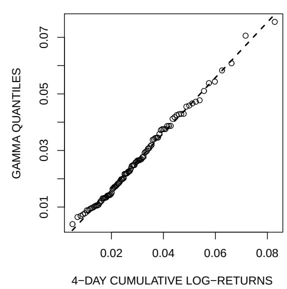
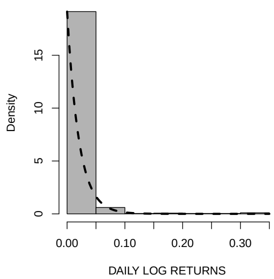


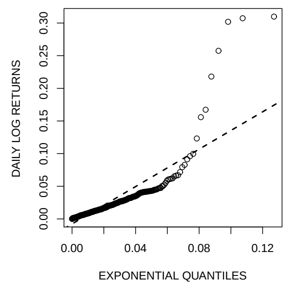
(F)

FIGURE 32. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.5.2. *BOA:-Weekly log-returns (growth episodes).*



(A)



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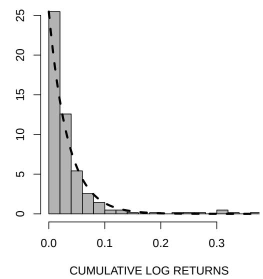
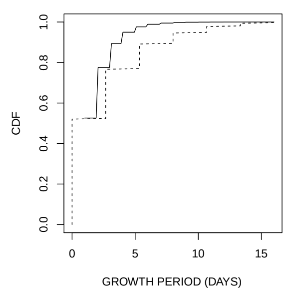


(C)

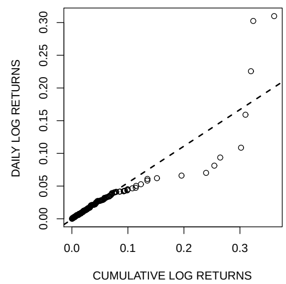


(D)

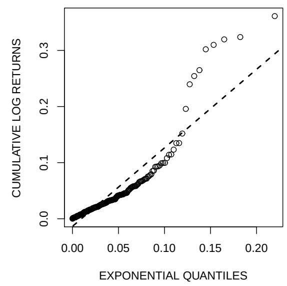
FIGURE 33. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 22. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	164	77	39	17	10	5	2	0
Relative Frequency	0.522	0.245	0.124	0.054	0.032	0.016	0.006	0.000
Model Probability	0.526	0.249	0.118	0.056	0.027	0.013	0.006	0.000



(A)



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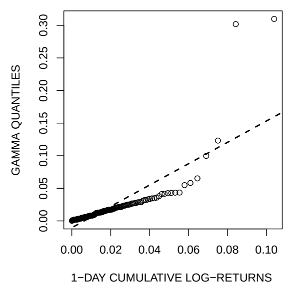
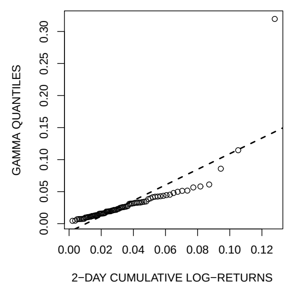


(C)



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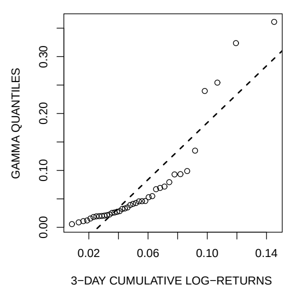
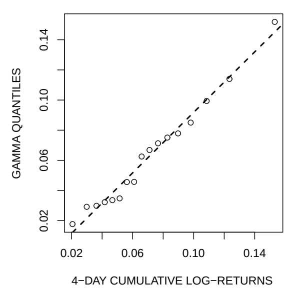


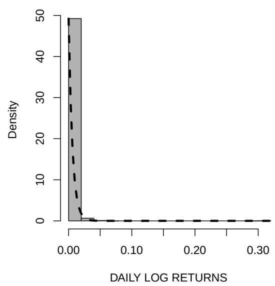
(E)



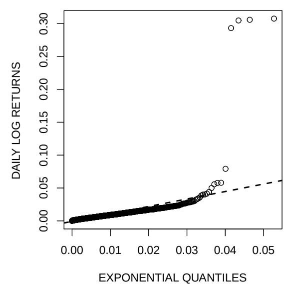
(F)

FIGURE 34. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

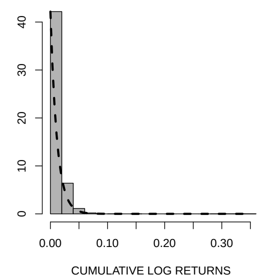
5.5.3. *Chevron:-Daily log-returns (growth episodes).*



(A)



(B)



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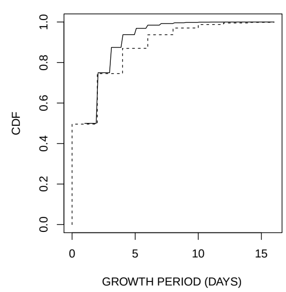

(D)

FIGURE 35. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 23. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	1176	591	295	159	79	41	18	7
Relative Frequency	0.496	0.249	0.124	0.067	0.033	0.017	0.008	0.003
Model Probability	0.499	0.250	0.125	0.063	0.031	0.016	0.008	0.004

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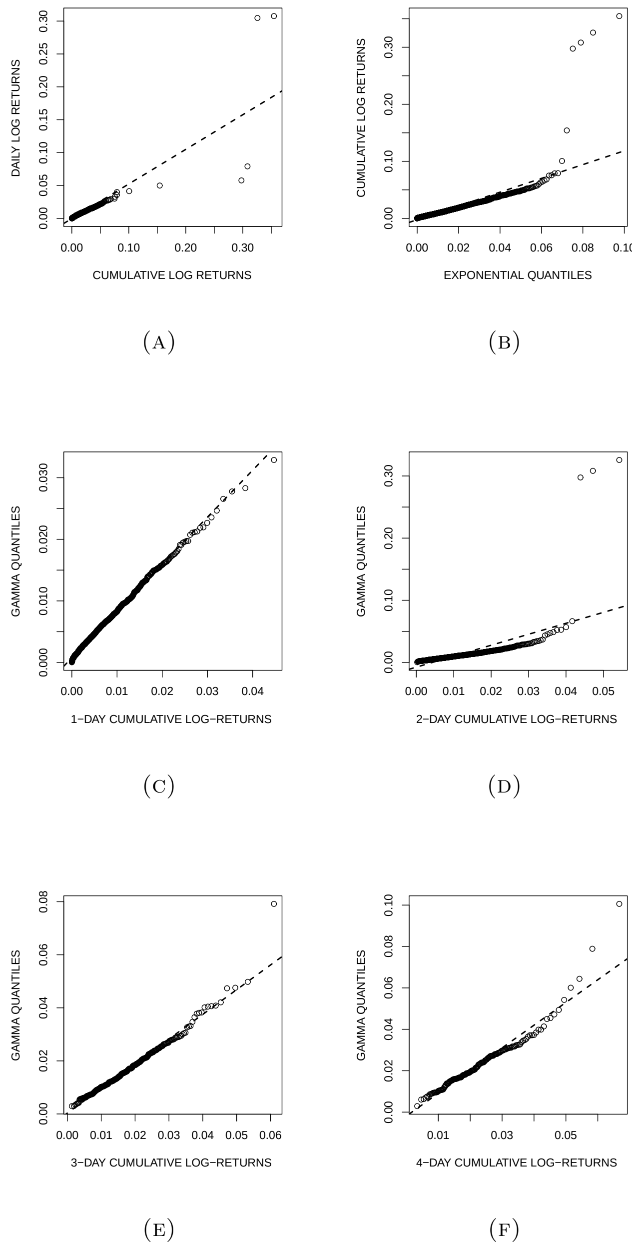


FIGURE 36. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

5.5.4. *Chevron:-Weekly log-returns (growth episodes).*

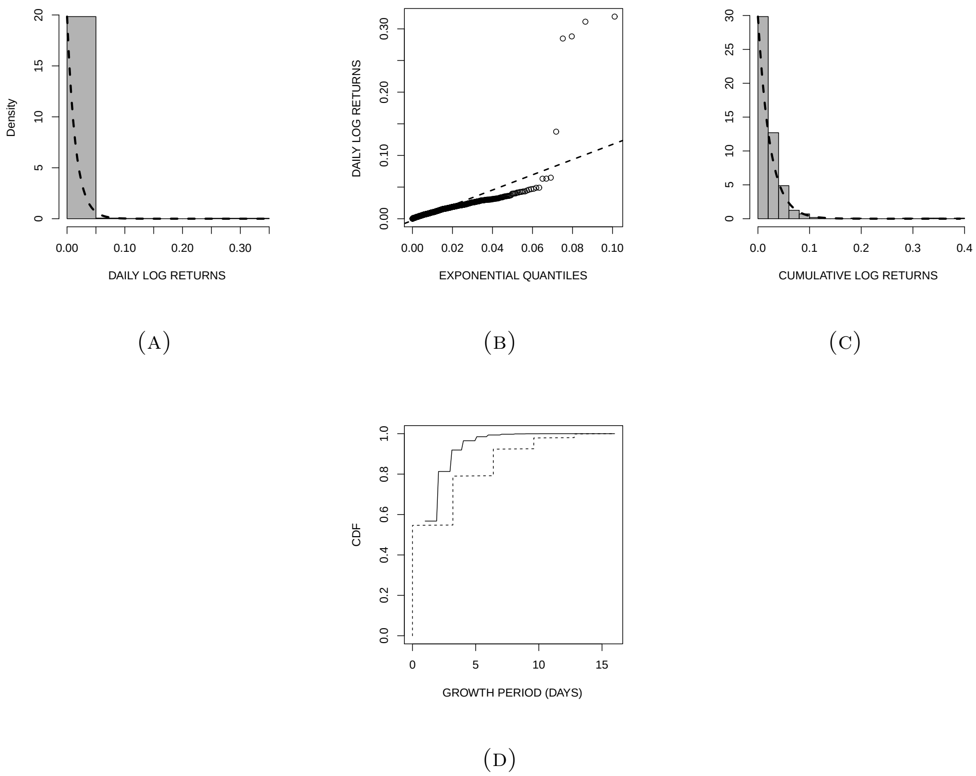
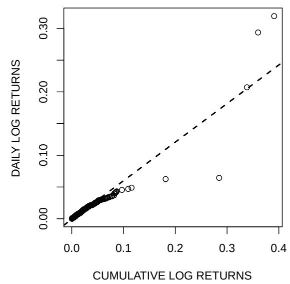


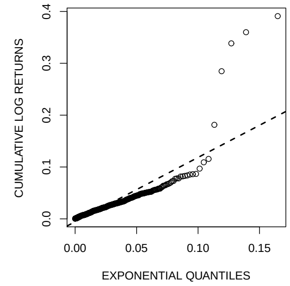
FIGURE 37. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 24. Frequency, relative frequency and geometric probability (model)

Growth period(N) in days	1	2	3	4	5	6	7	8
Frequency	0.547	0.243	0.133	0.055	0.020	0.002	0.000	0.000
Relative Frequency	0.547	0.243	0.133	0.055	0.020	0.002	0.000	0.000
Model Probability	0.568	0.245	0.106	0.046	0.020	0.009	0.000	0.000



(A)



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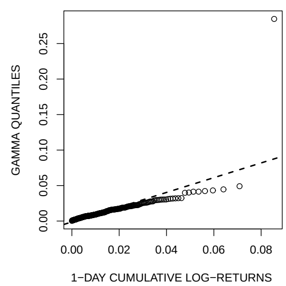
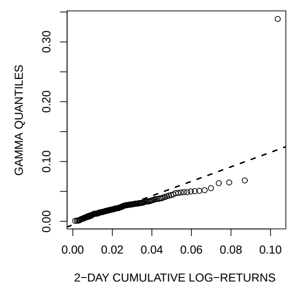


(C)



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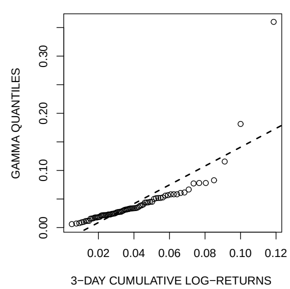

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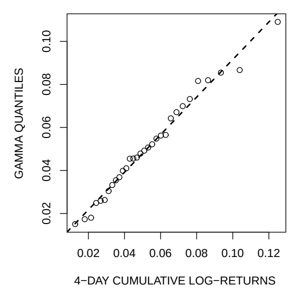


(F)

FIGURE 38. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6. Fitting **BEG** To Decline Episodes

6.1. Estimation of Parameters $\hat{\beta}$ and $\hat{p}$.

Table 25. Foreign Exchange Rates

Currency	$\hat{\beta}$	$\hat{p}$	$\hat{\beta}\hat{p}$
US Dollar	194.7644	0.4970559	96.8088
Swiss Franc	223.6643	0.5039872	112.7240
Swedish Krona	276.8503	0.5401388	149.5375
Norwegian Krone	281.1287	0.5425162	152.5169
Canadian Dollar	196.8236	0.4936647	97.16488
Australian Dollar	177.6454	0.4865385	86.43132
Deutsche Mark	259.0889	0.5115654	132.5409

Table 26. Commodities

Commodity	$\hat{\beta}$	$\hat{p}$	$\hat{\beta}\hat{p}$
Daily Gold	331.8171	0.5215245	173.0507
Weekly Gold	131.0821	0.5250737	68.82777
Daily Oil	123.0976	0.520081	64.02074
Weekly Oil	66.50906	0.484193	32.20322

Table 27. Stock Indexes

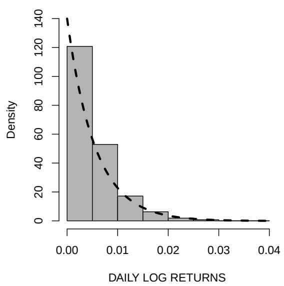
Stock Indices	$\hat{\beta}$	$\hat{p}$	$\hat{\beta}\hat{p}$
Daily S&P500	292.6476	0.5352731	156.6464
Weekly S&P500	128.8969	0.5905849	76.12456
Daily NASDAQ-100	123.0976	0.520081	64.02074
Weekly NASDAQ-100	66.50906	0.484193	32.20322

Table 28. Stock Prices

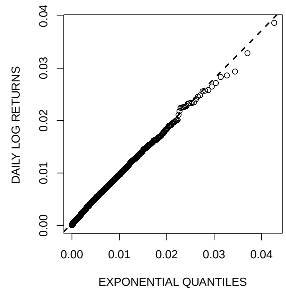
Stocks	$\hat{\beta}$	$\hat{p}$	$\hat{\beta}\hat{p}$
Daily BOA	138.0674	0.4936835	68.16162
Weekly BOA	66.45005	0.4679583	31.09585
Daily Chevron	180.7717	0.4744949	85.77523
Weekly Chevron	82.36326	0.5062837	41.69917

6.2. **Graphical fit:- Foreign exchange rates.**

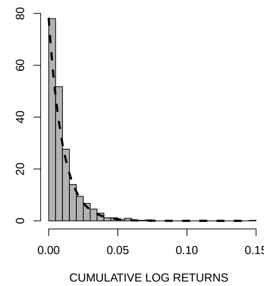
6.2.1. *US Dollar (decline episodes).*



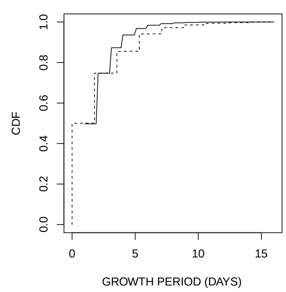
(A)



(B)



(C)

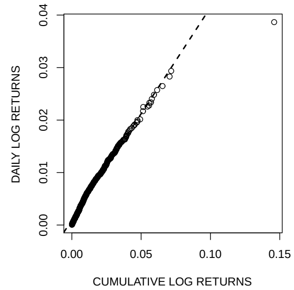


(D)

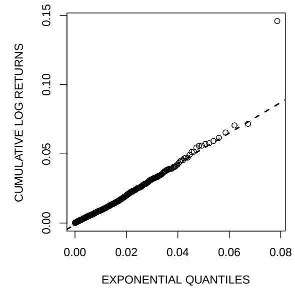
FIGURE 39. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 29. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	507	250	109	87	31	14	8	3
Relative Frequency	0.500	0.247	0.108	0.086	0.031	0.014	0.008	0.003
Model Probability	0.497	0.250	0.126	0.063	0.032	0.016	0.008	0.004



(A)



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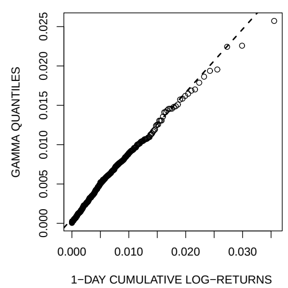
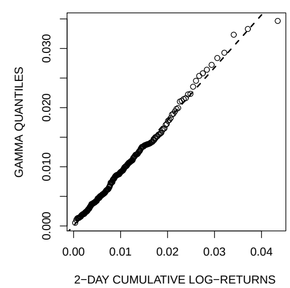


(C)



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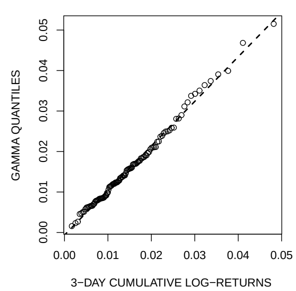

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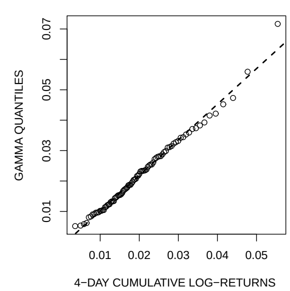
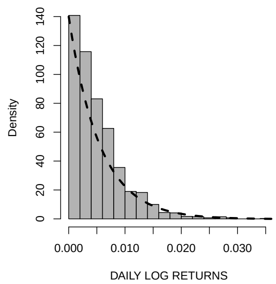


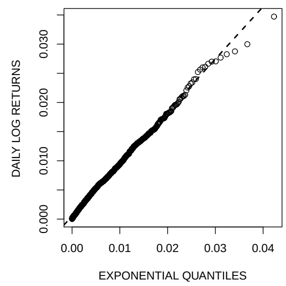
(F)

FIGURE 40. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

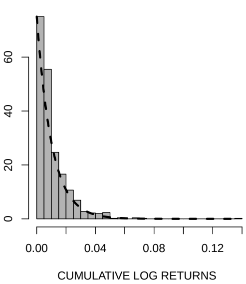
6.2.2. *Canadian Dollar (decline episodes).*



(A)



(B)



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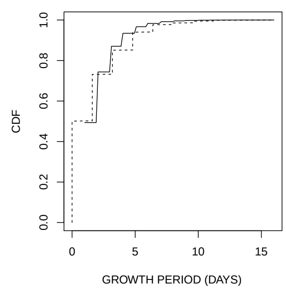
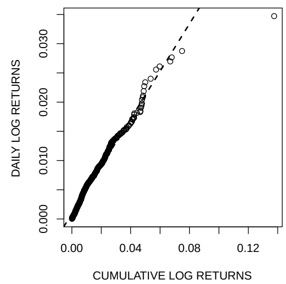


(D)

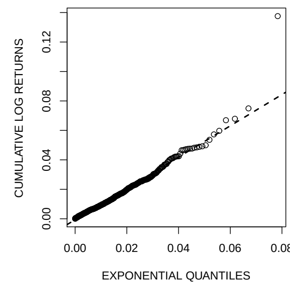
FIGURE 41. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 30. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	508	233	121	90	37	9	9	4
Relative Frequency	0.501	0.230	0.119	0.089	0.037	0.009	0.009	0.004
Model Probability	0.494	0.250	0.127	0.064	0.032	0.016	0.008	0.004



(A)



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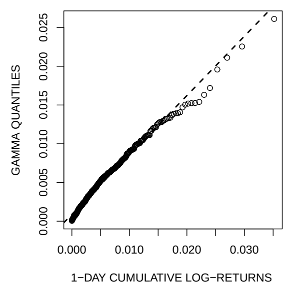

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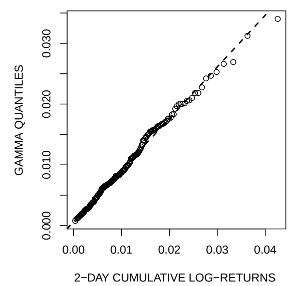

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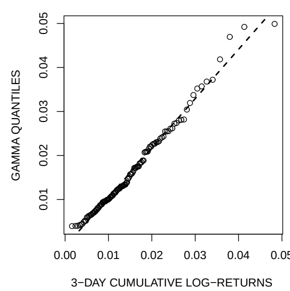
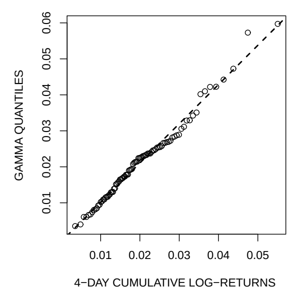


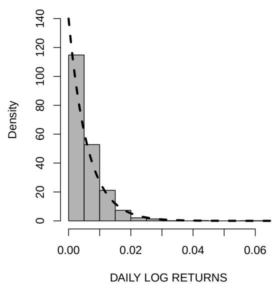
(E)



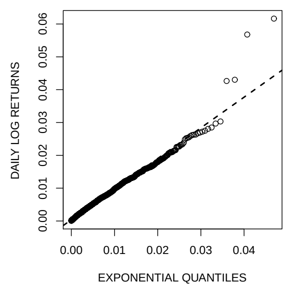
(F)

FIGURE 42. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

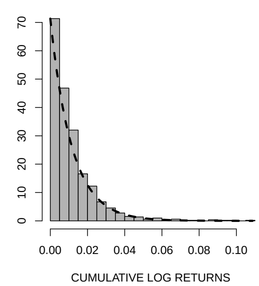
6.2.3. *Australian Dollar (decline episodes).*



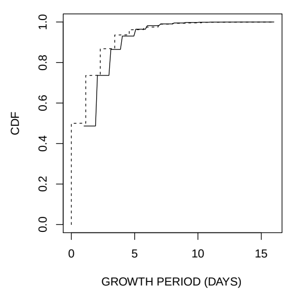
(A)



(B)



(C)



(D)

FIGURE 43. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 31. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	506	239	133	69	26	13	15	4
Relative Frequency	0.500	0.236	0.131	0.068	0.026	0.013	0.015	0.004
Model Probability	0.487	0.250	0.128	0.066	0.034	0.017	0.009	0.005

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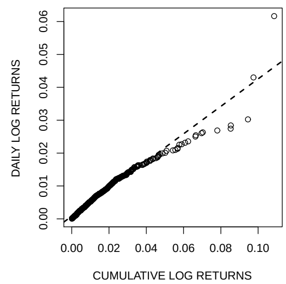
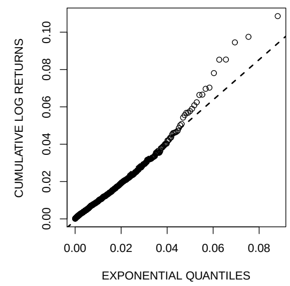
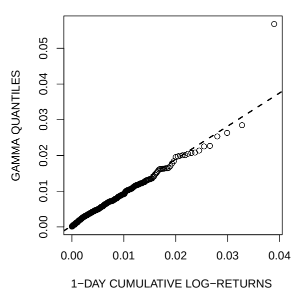


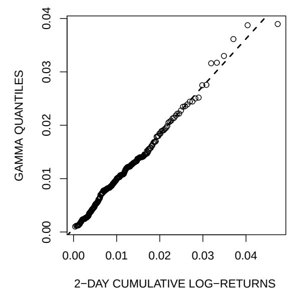
(A)



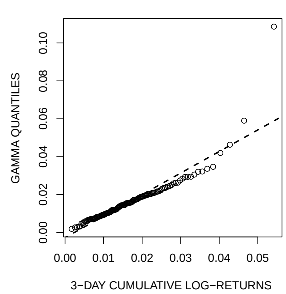
(B)



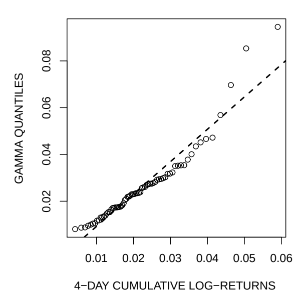
(C)



(D)



(E)



(F)

FIGURE 44. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.2.4. *Norwegian Krone (decline episodes).*

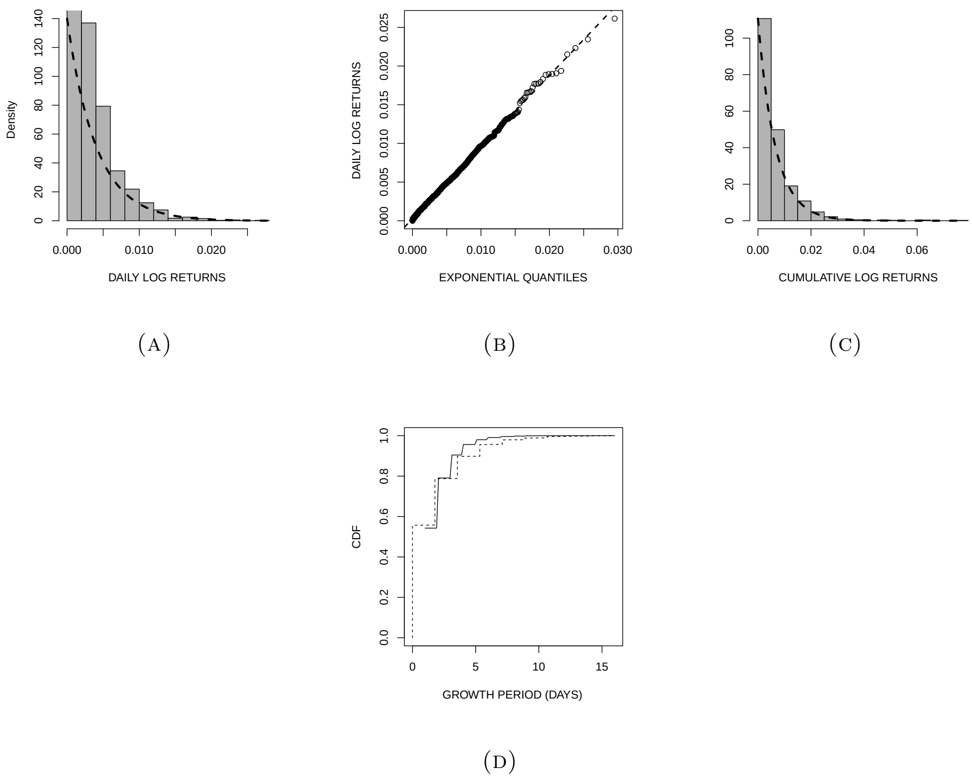


FIGURE 45. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 32. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	608	251	120	64	25	10	8	2
Relative Frequency	0.557	0.230	0.110	0.059	0.023	0.009	0.007	0.002
Model Probability	0.543	0.248	0.114	0.052	0.024	0.011	0.005	0.002

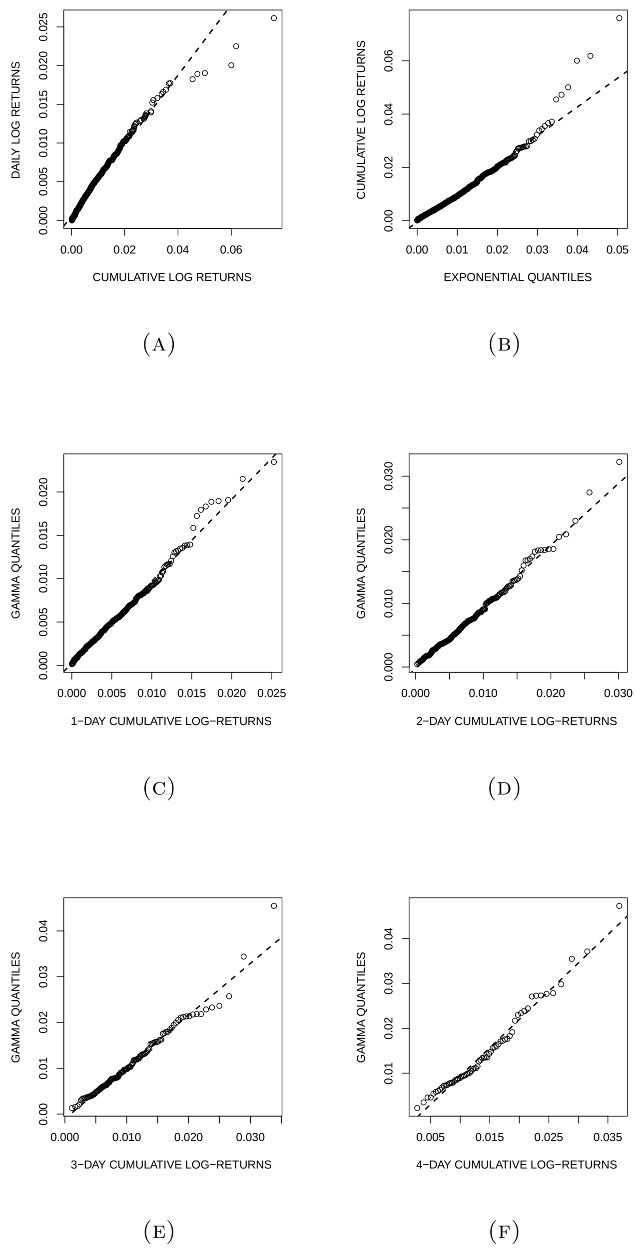
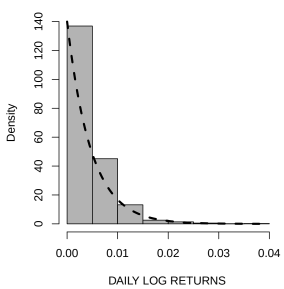
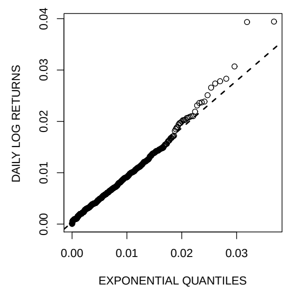


FIGURE 46. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

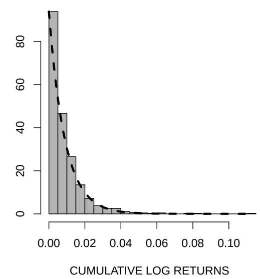
6.2.5. *Swiss Franc (decline episodes).*



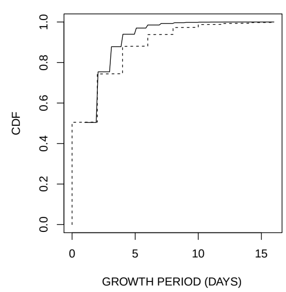
(A)



(B)



(C)



(D)

FIGURE 47. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 33. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	479	226	129	55	33	14	5	4
Relative Frequency	0.505	0.238	0.136	0.058	0.035	0.015	0.005	0.004
Model Probability	0.504	0.250	0.124	0.062	0.031	0.015	0.008	0.004

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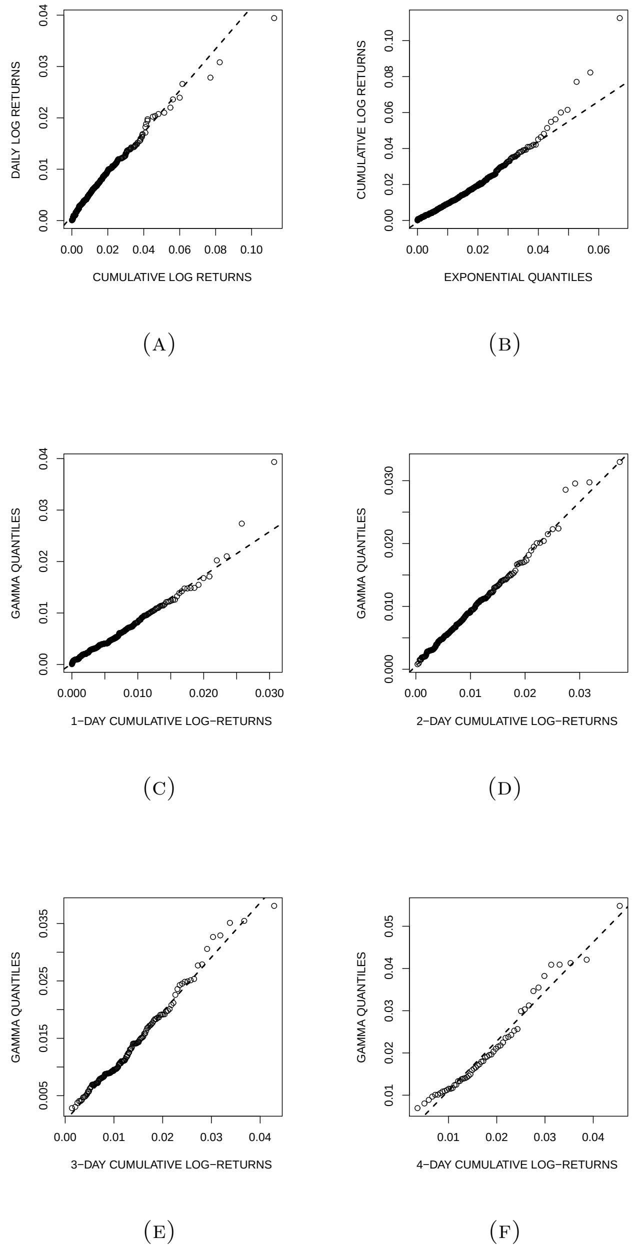
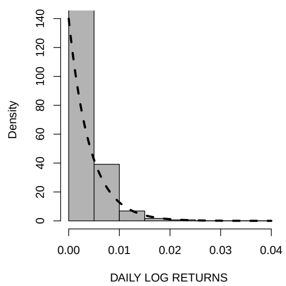
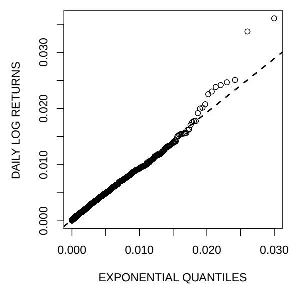


FIGURE 48. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

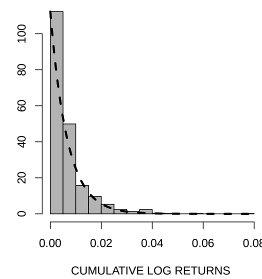
6.2.6. *Swedish Krona (decline episodes).*



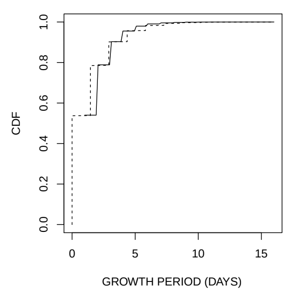
(A)



(B)



(C)



(D)

FIGURE 49. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 34. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	586	270	127	60	28	11	4	2
Relative Frequency	0.538	0.248	0.117	0.055	0.026	0.010	0.004	0.002
Model Probability	0.540	0.248	0.114	0.053	0.024	0.011	0.005	0.002

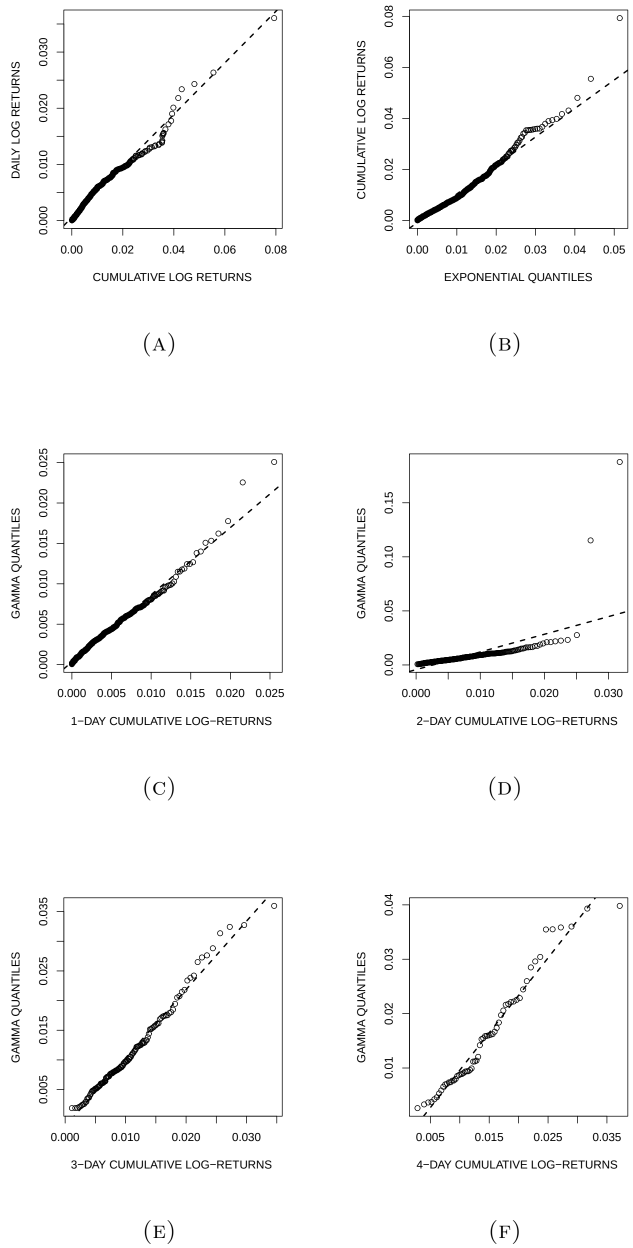
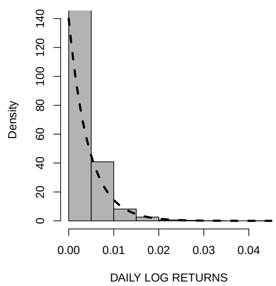
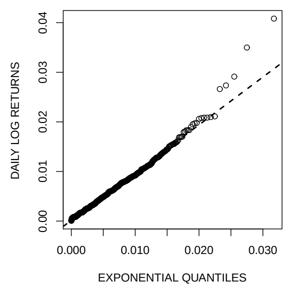


FIGURE 50. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

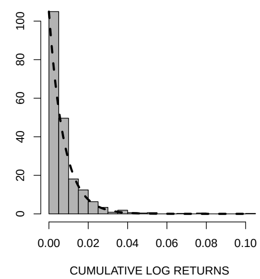
6.2.7. *Deusche Mark (decline episodes).*



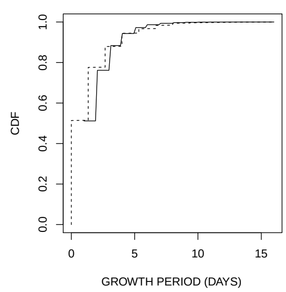
(A)



(B)



(C)



(D)

FIGURE 51. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 35. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	489	249	98	62	21	16	10	2
Relative Frequency	0.514	0.262	0.103	0.065	0.022	0.017	0.011	0.002
Model Probability	0.512	0.250	0.122	0.060	0.029	0.014	0.007	0.003

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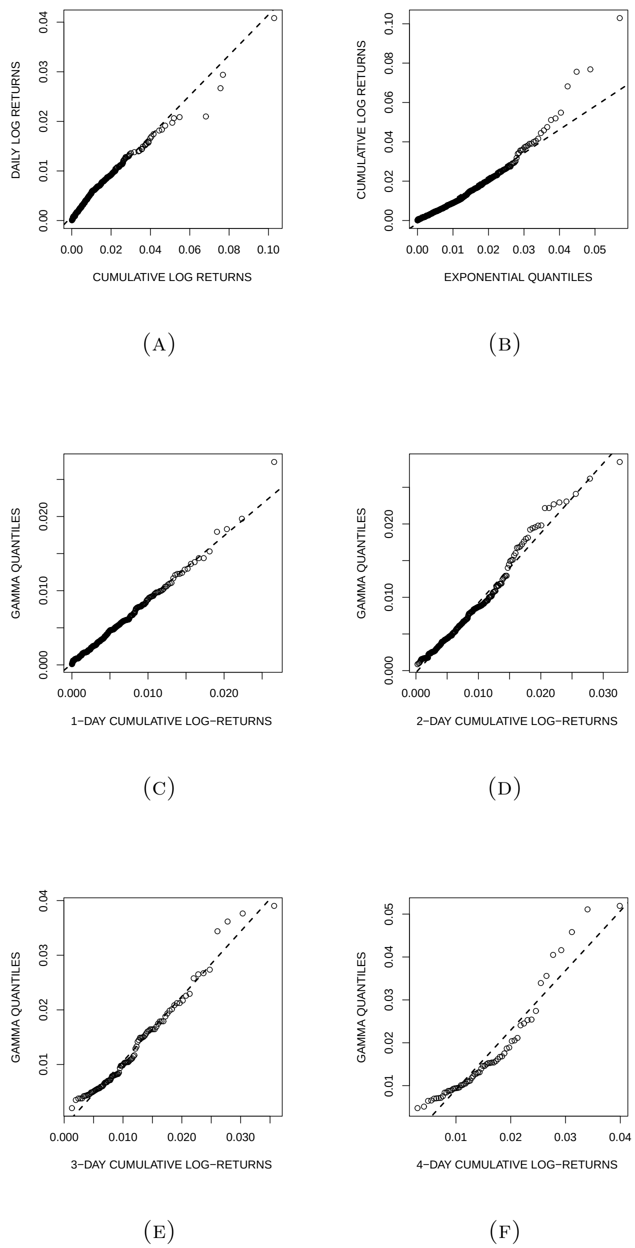
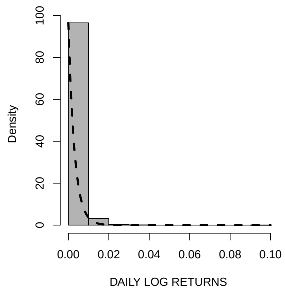


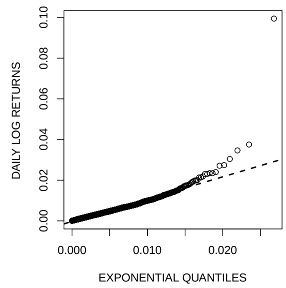
FIGURE 52. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.3. **Graphical fit:-Commodities.**

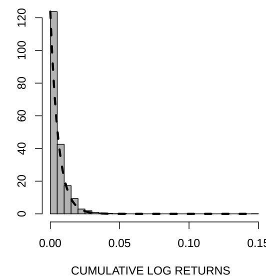
6.3.1. *Gold:-Daily log-returns (decline episodes).*



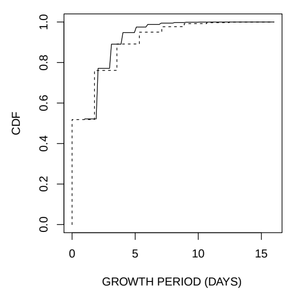
(A)



(B)



(C)



(D)

FIGURE 53. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 36. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	986	461	248	111	51	29	9	5
Relative Frequency	0.518	0.242	0.130	0.058	0.027	0.015	0.005	0.003
Model Probability	0.522	0.250	0.119	0.057	0.027	0.013	0.006	0.003

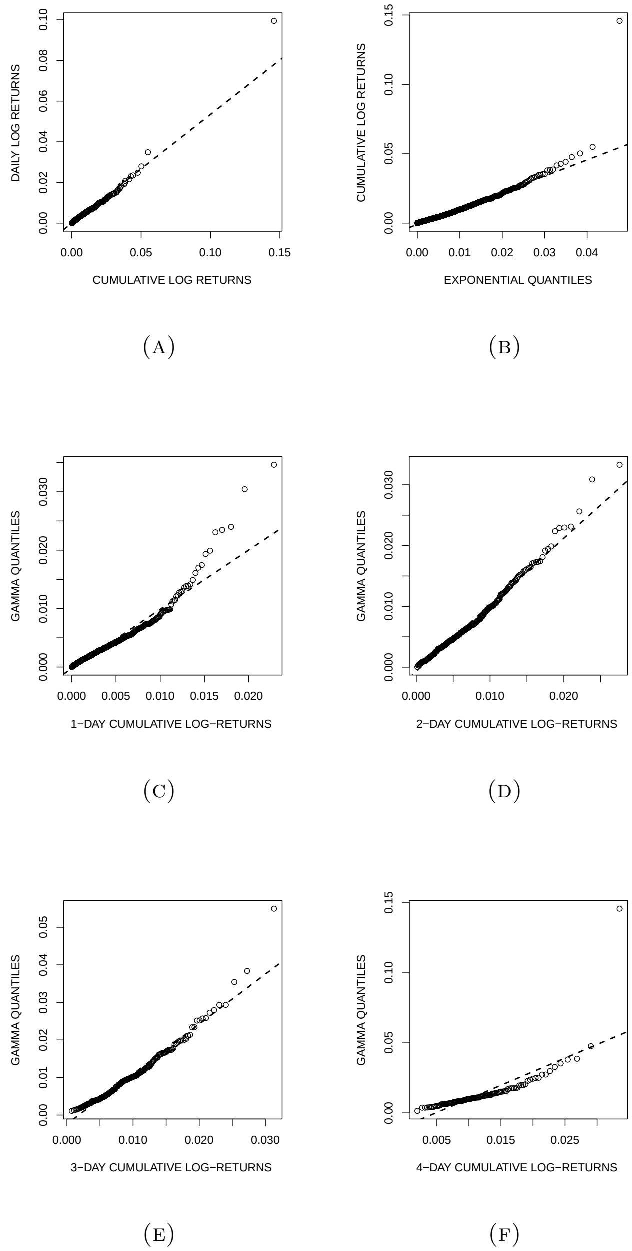


FIGURE 54. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.3.2. *Gold:-Weekly log-returns (decline episodes).*

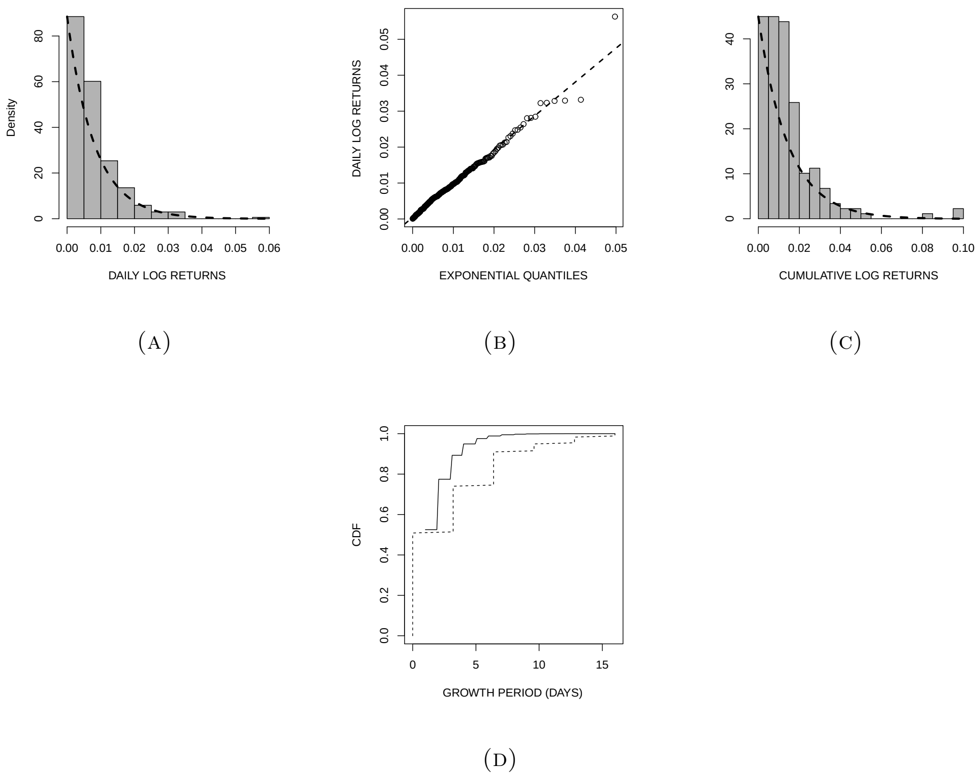
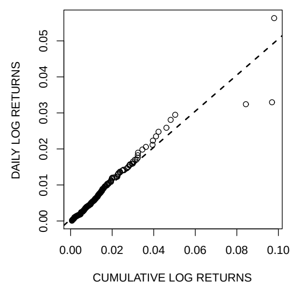


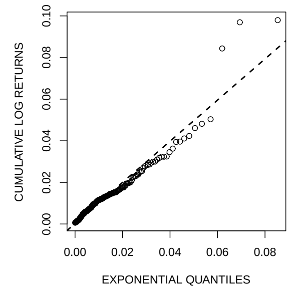
FIGURE 55. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 37. Frequency, relative frequency and geometric probability (model)

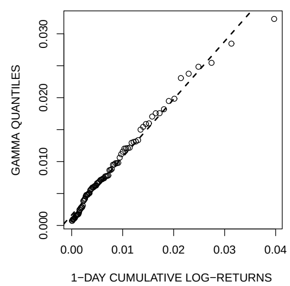
Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	91	41	30	7	6	3	–	–
Relative Frequency	0.511	0.230	0.169	0.039	0.034	0.017	–	–
Model Probability	0.525	0.249	0.118	0.056	0.027	0.013	–	–



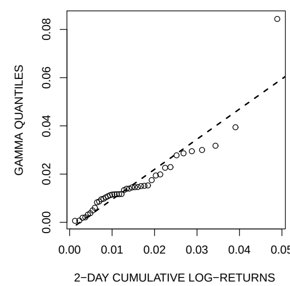
(A)



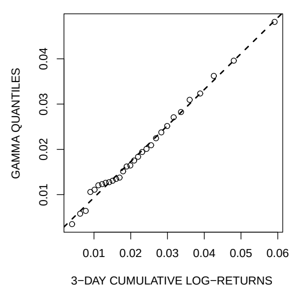
(B)



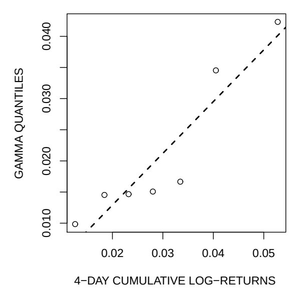
(C)



(D)



(E)



(F)

FIGURE 56. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.3.3. *Oil:-Daily log-returns (decline episodes).*

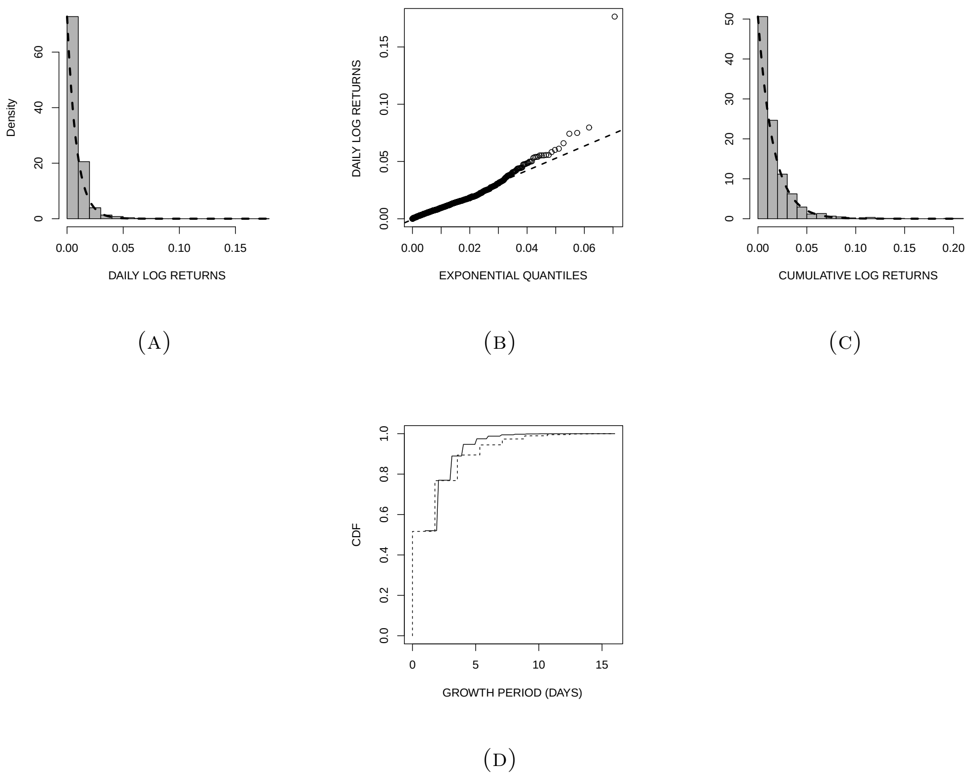


FIGURE 57. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 38. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	796	387	195	77	45	24	9	5
Relative Frequency	0.517	0.251	0.127	0.050	0.029	0.016	0.006	0.003
Model Probability	0.520	0.250	0.120	0.057	0.028	0.013	0.006	0.003

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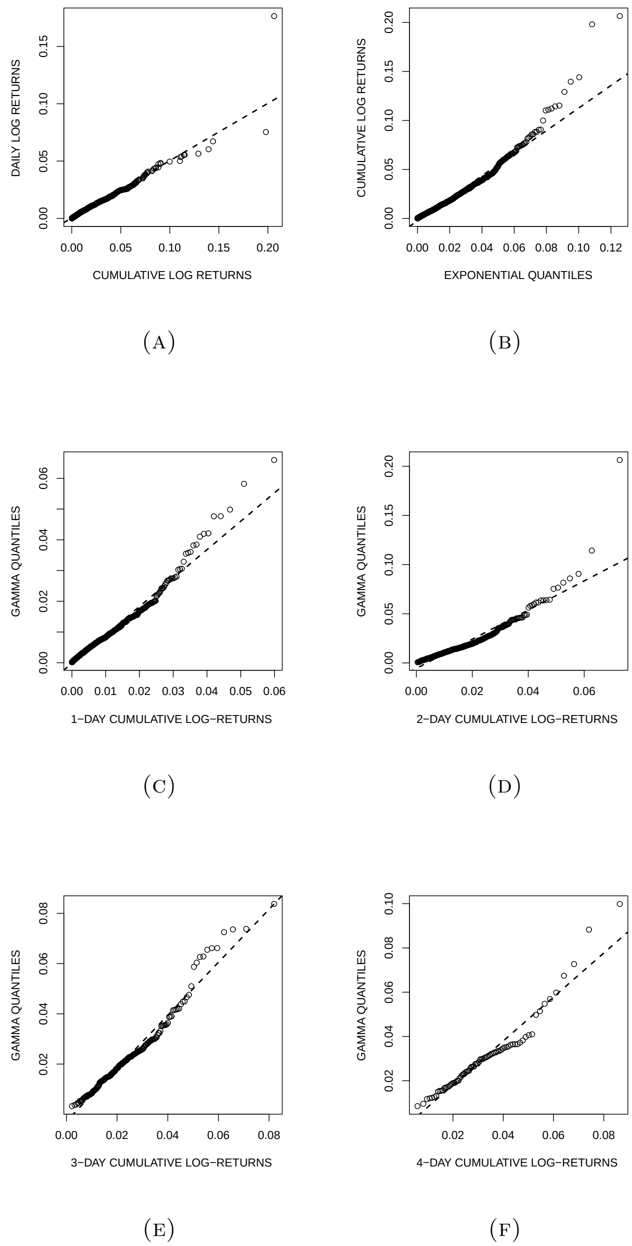
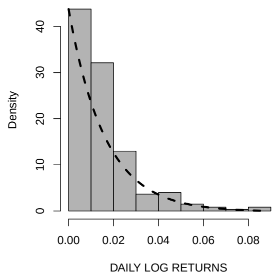
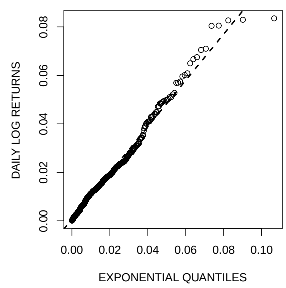


FIGURE 58. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

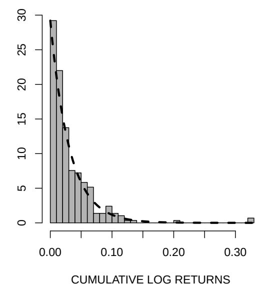
6.3.4. *Oil:-Weekly log-returns (decline episodes).*



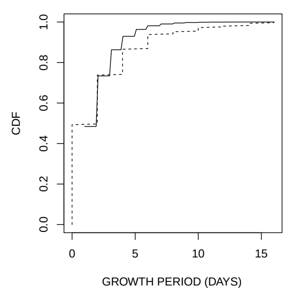
(A)



(B)



(C)



(D)

FIGURE 59. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 39. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	144	71	37	21	4	6	2	4
Relative Frequency	0.495	0.244	0.127	0.072	0.014	0.021	0.007	0.014
Model Probability	0.484	0.250	0.129	0.066	0.034	0.018	0.009	0.005

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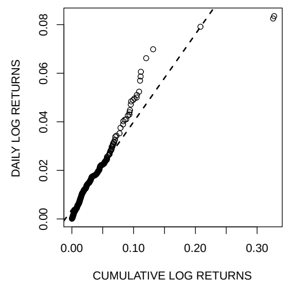
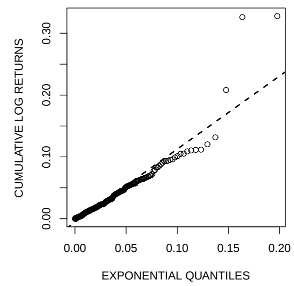
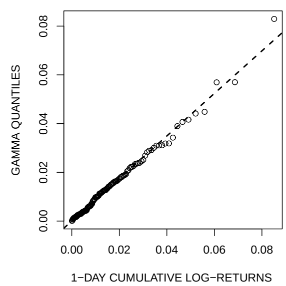


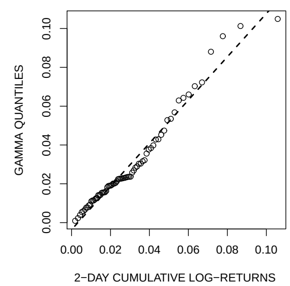
(A)



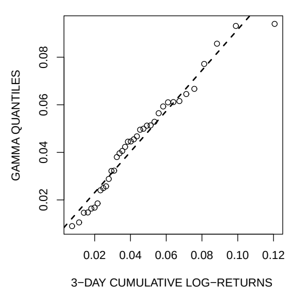
(B)



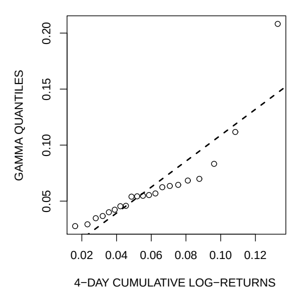
(C)



(D)



(E)



(F)

FIGURE 60. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.4. **Graphical fit:-Stock Indexes.**

6.4.1. *S&P500:-Daily log-returns (decline episodes).*

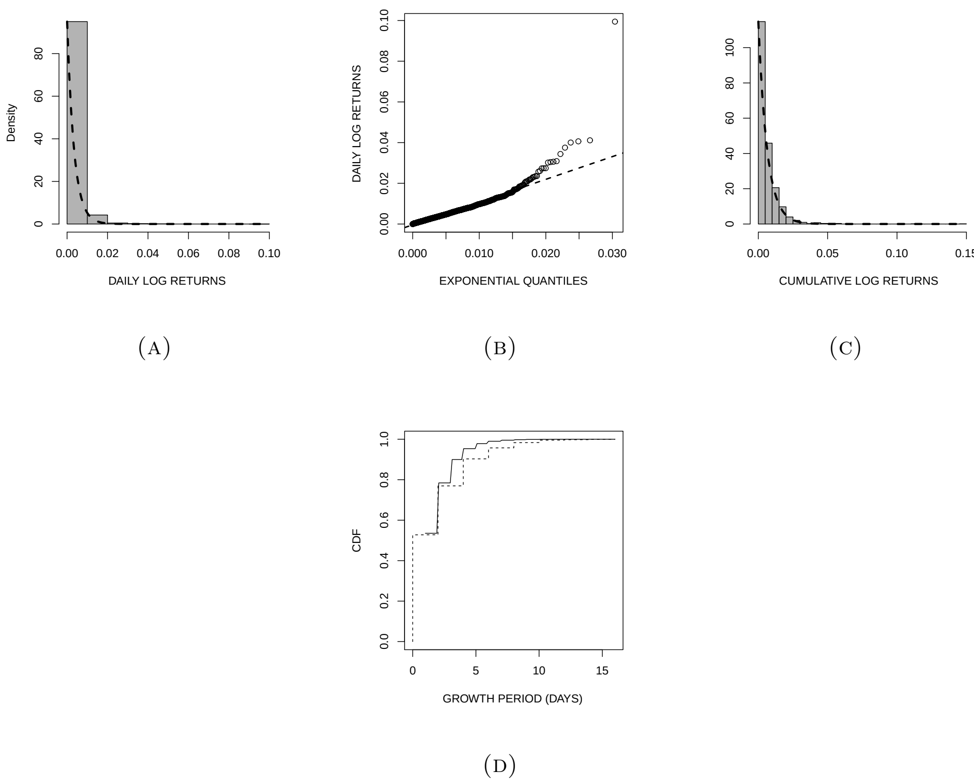


FIGURE 61. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 40. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	1029	471	260	106	51	24	4	4
Relative Frequency	0.528	0.242	0.133	0.054	0.026	0.012	0.002	0.002
Model Probability	0.535	0.249	0.116	0.054	0.025	0.012	0.005	0.003

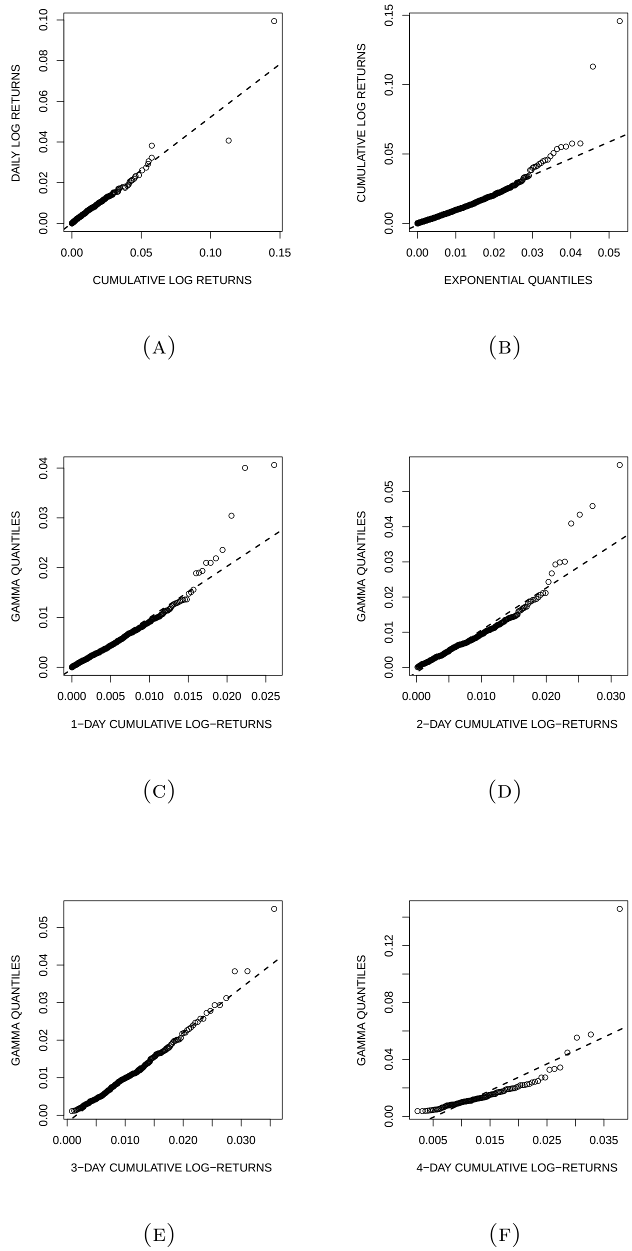


FIGURE 62. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.4.2. *S&P500:-Weekly log-returns (decline episodes).*

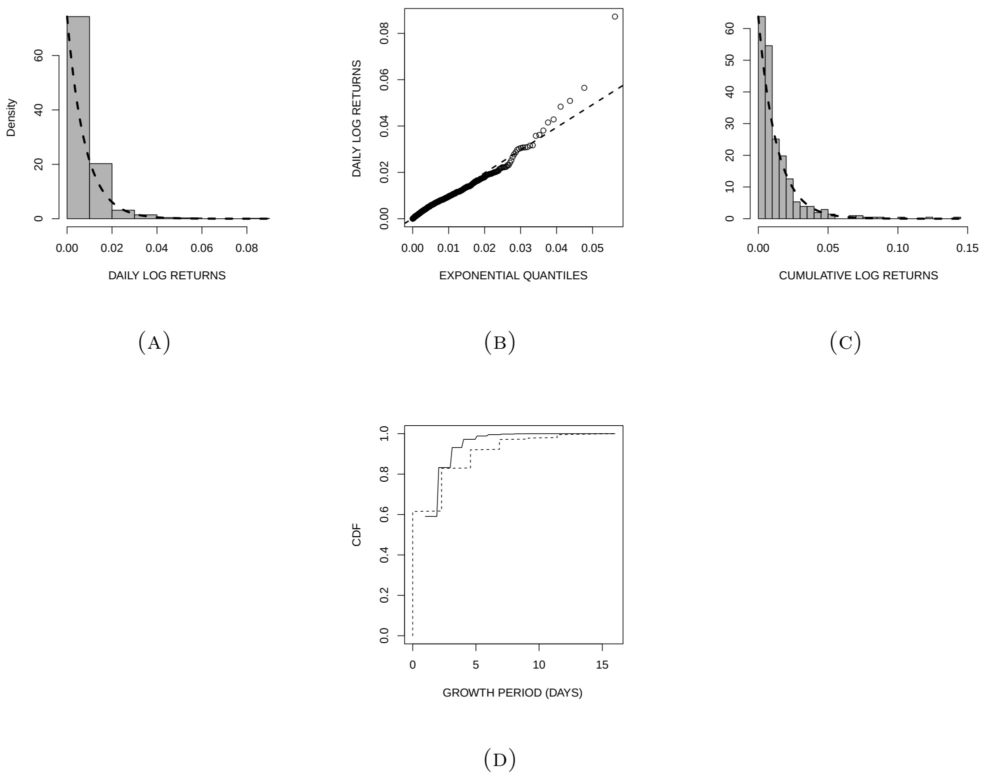


FIGURE 63. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 41. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	255	88	38	21	3	7	1	1
Relative Frequency	0.616	0.213	0.092	0.051	0.007	0.017	0.002	0.002
Model Probability	0.591	0.242	0.099	0.041	0.017	0.007	0.003	0.001

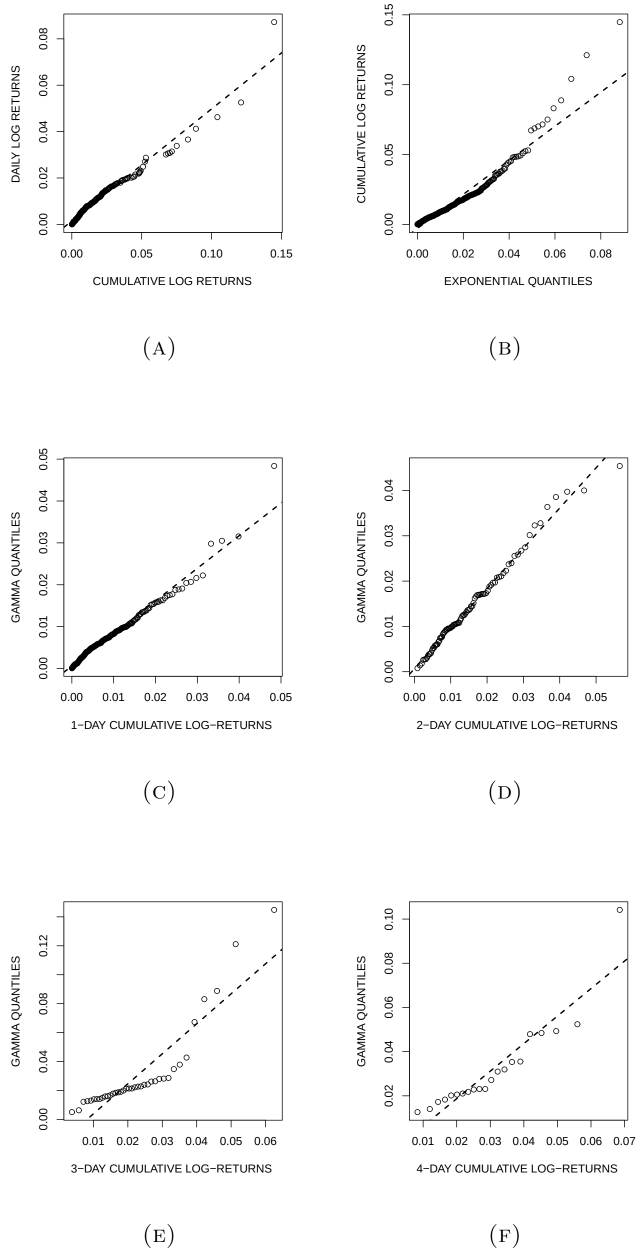
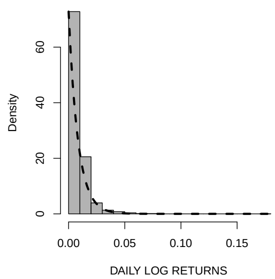
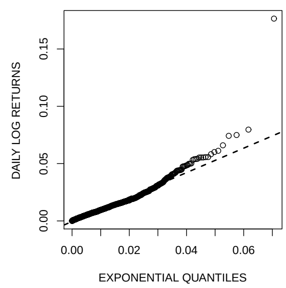


FIGURE 64. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

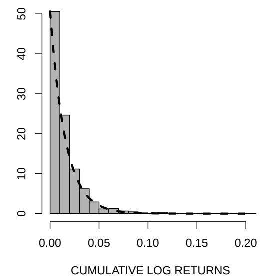
6.4.3. *NASDAQ-100:-Daily log-returns (decline episodes).*



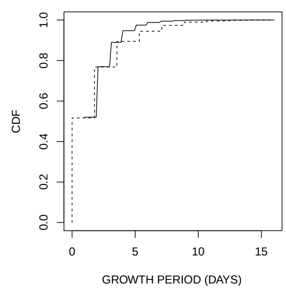
(A)



(B)



(C)

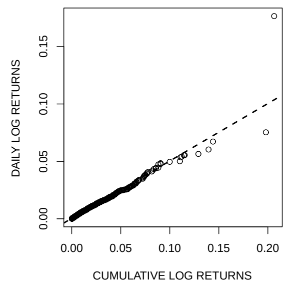


(D)

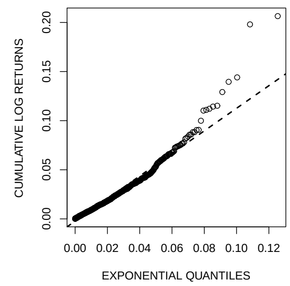
FIGURE 65. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 42. Frequency, relative frequency and geometric probability (model)

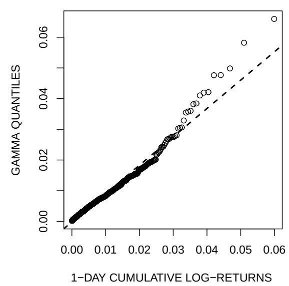
Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	144	71	37	21	4	6	2	4
Relative Frequency	0.517	0.251	0.127	0.050	0.029	0.016	0.006	0.003
Model Probability	0.520	0.250	0.120	0.057	0.028	0.013	0.006	0.003



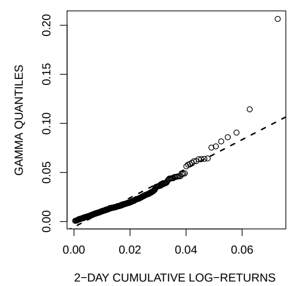
(A)



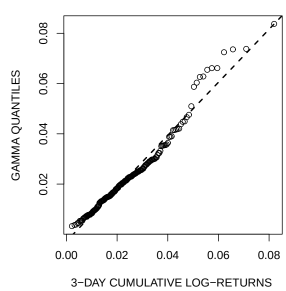
(B)



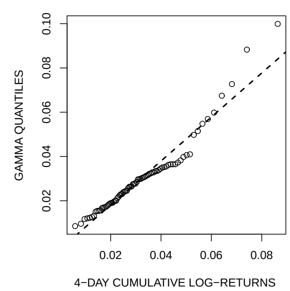
(C)



(D)



(E)



(F)

FIGURE 66. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.4.4. *NASDAQ-100:-Weekly log-returns (decline episodes).*

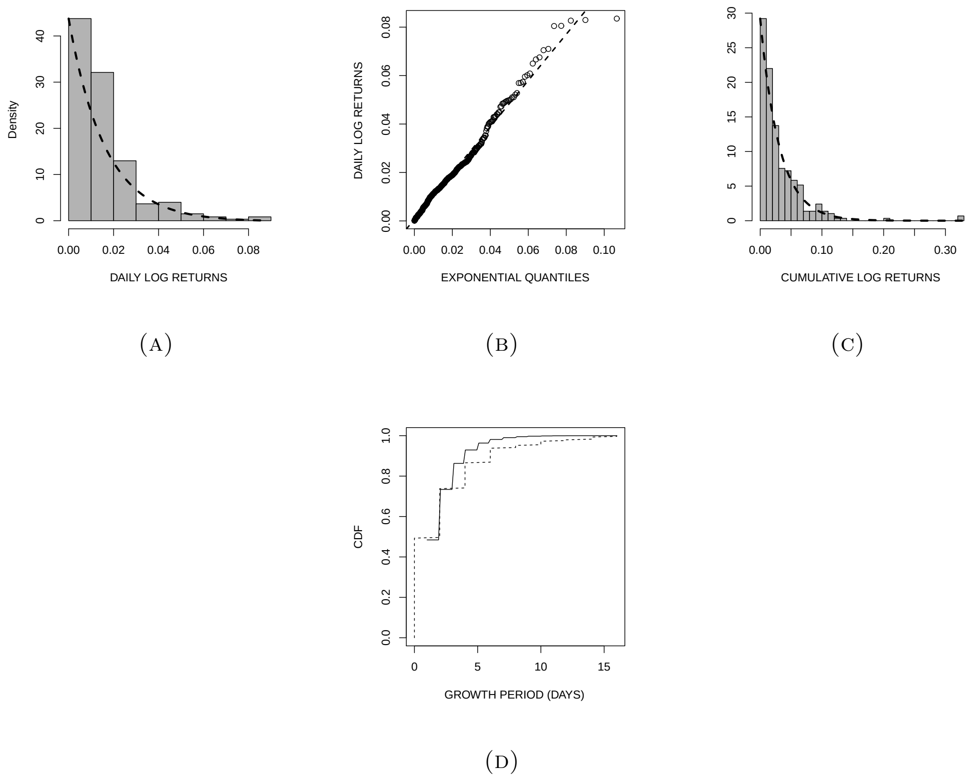


FIGURE 67. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 43. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	144	71	37	21	4	6	2	4
Relative Frequency	0.495	0.244	0.127	0.072	0.014	0.021	0.007	0.014
Model Probability	0.484	0.250	0.129	0.066	0.034	0.018	0.009	0.005

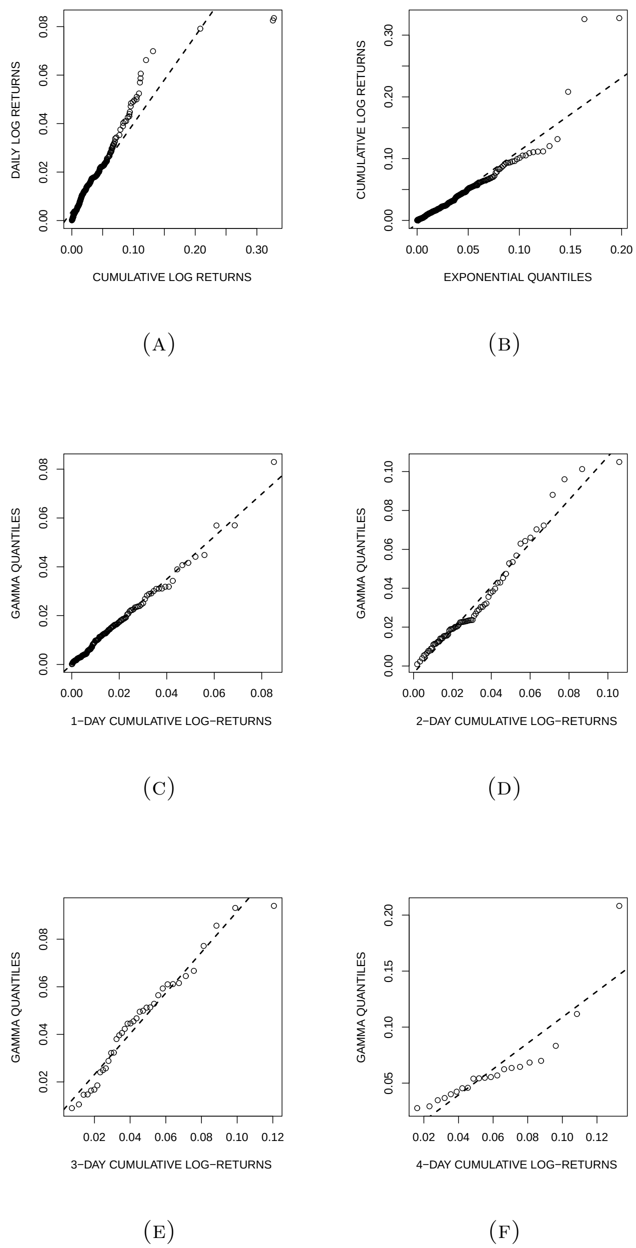
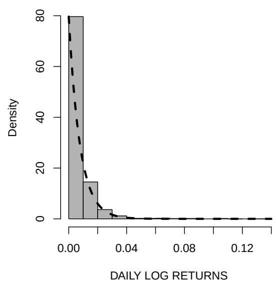


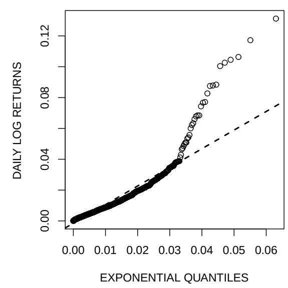
FIGURE 68. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.5. **Graphical fit:-Stocks.**

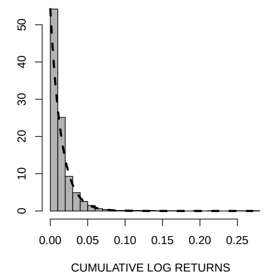
6.5.1. *BOA:-Daily log-returns (decline episodes).*



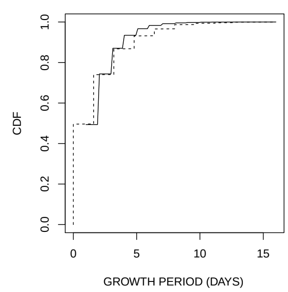
(A)



(B)



(C)



(D)

FIGURE 69. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 44. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	737	362	189	95	51	31	10	4
Relative Frequency	0.496	0.244	0.127	0.064	0.034	0.021	0.007	0.003
Model Probability	0.494	0.250	0.127	0.064	0.032	0.016	0.008	0.004

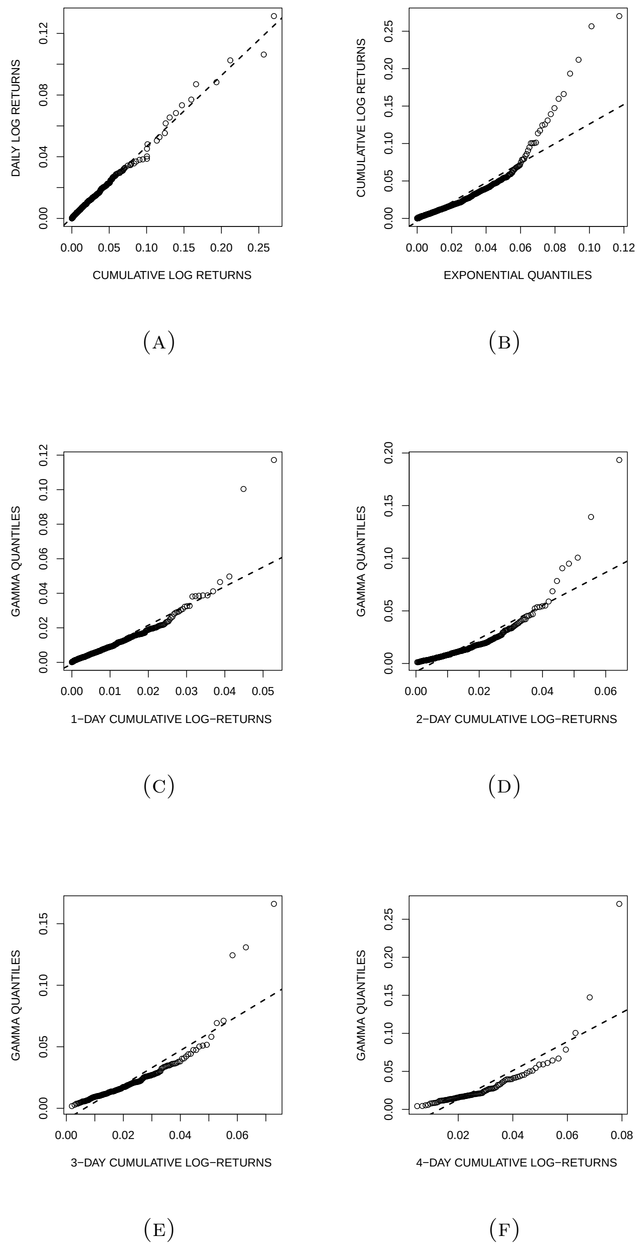
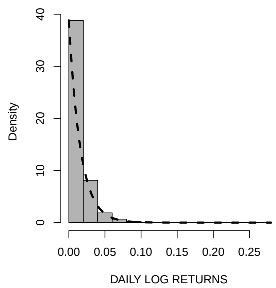
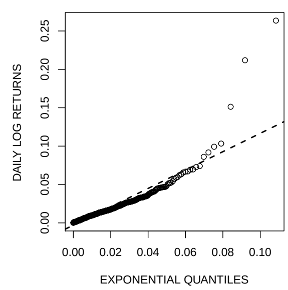


FIGURE 70. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

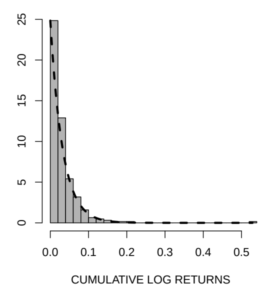
6.5.2. *BOA:-Weekly log-returns (decline episodes).*



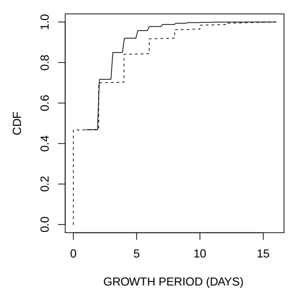
(A)



(B)



(C)



(D)

FIGURE 71. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 45. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	147	73	44	24	14	7	3	1
Relative Frequency	0.468	0.232	0.140	0.076	0.045	0.022	0.010	0.014
Model Probability	0.468	0.249	0.132	0.070	0.037	0.020	0.011	0.006

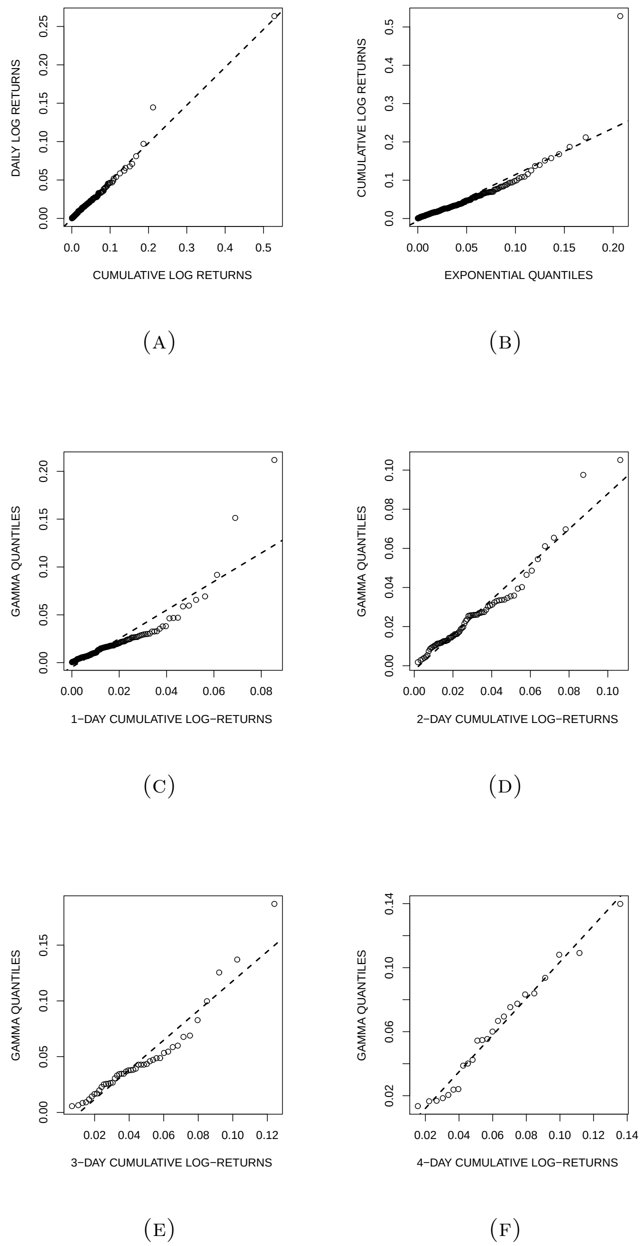


FIGURE 72. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.5.3. *Chevron:-Daily log-returns (decline episodes).*

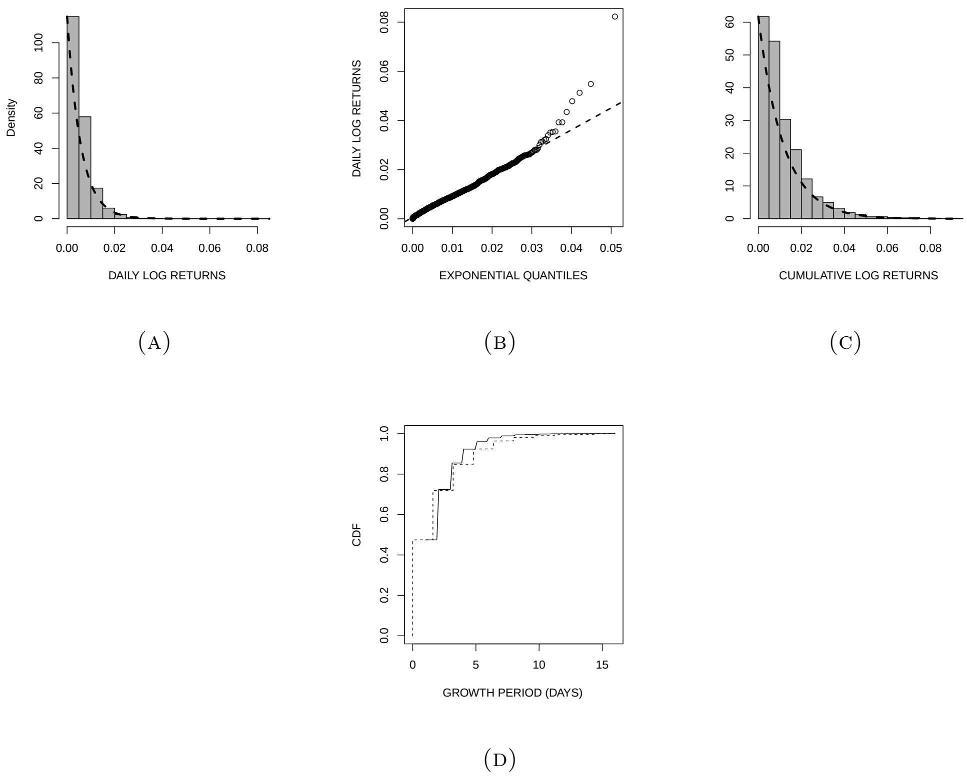


FIGURE 73. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

TABLE 46. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	1126	581	306	179	94	43	18	12
Relative Frequency	0.475	0.245	0.129	0.075	0.040	0.018	0.008	0.005
Model Probability	0.474	0.249	0.131	0.069	0.036	0.019	0.010	0.005

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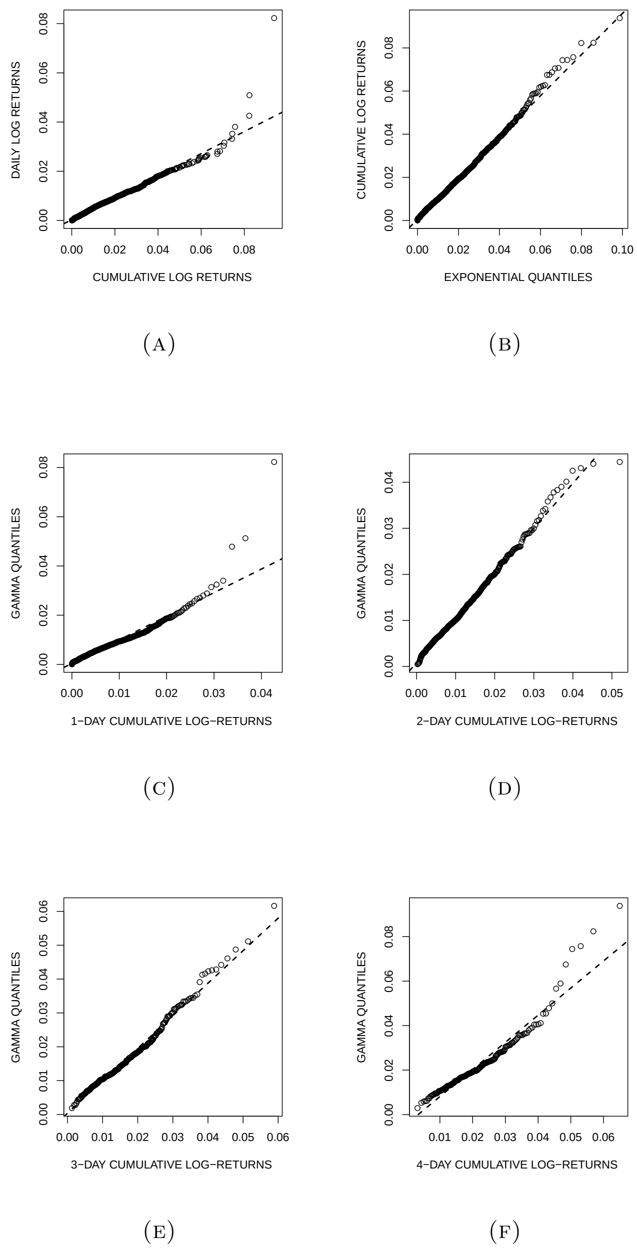


FIGURE 74. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

6.5.4. *Chevron:-Weekly log-returns (decline episodes).*

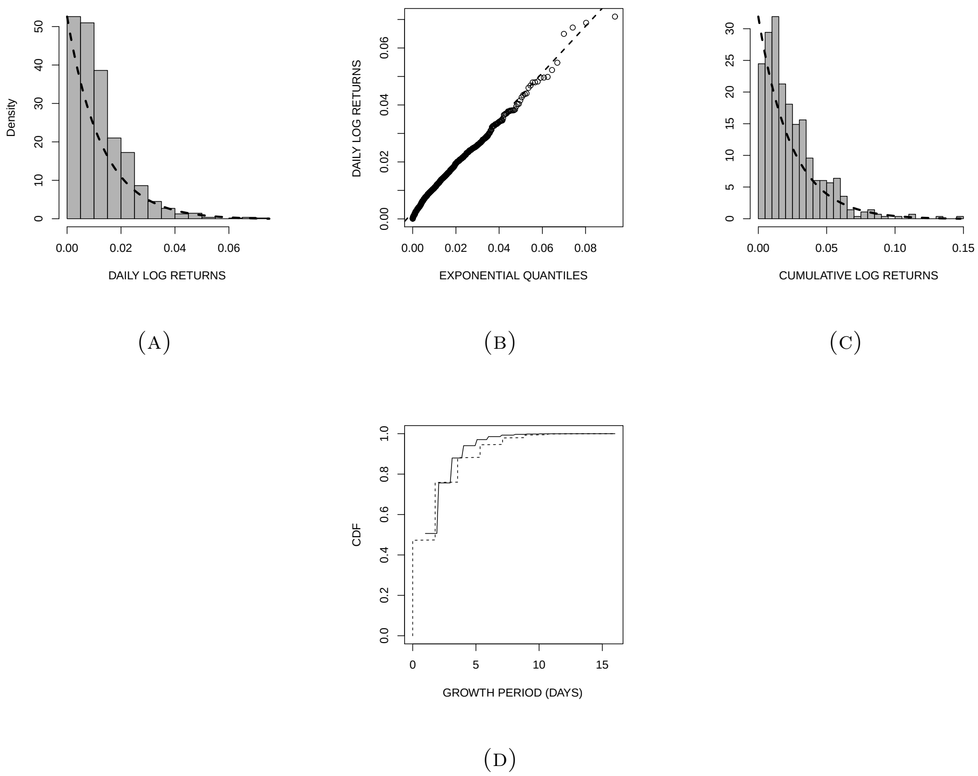


Figure 75. (A) Histogram of log-returns overlaid with exponential p.d.f., (B) Probability plot of log-returns vs. exponential p.d.f., (C) Histogram of cumulative log-returns overlaid with exponential p.d.f., (D) Empirical c.d.f. vs. geometric c.d.f.

Table 47. Frequency, relative frequency and geometric probability (model)

Decline period(N) in days	1	2	3	4	5	6	7	8
Frequency	267	161	69	36	19	8	3	1
Relative Frequency	0.473	0.285	0.122	0.064	0.034	0.014	0.005	0.002
Model Probability	0.506	0.250	0.123	0.061	0.030	0.015	0.007	0.004

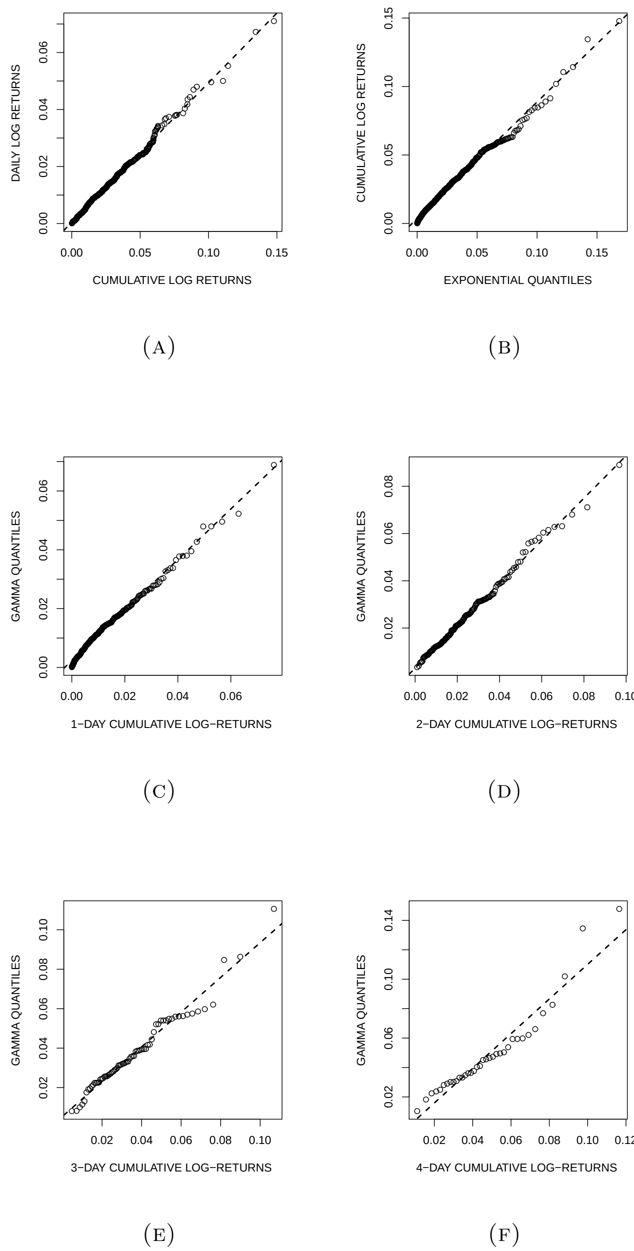


FIGURE 76. (A) QQ plot of cumulative log-returns vs. log-returns, (B) Probability plot of cumulative log-returns vs. exponential p.d.f., (C) Probability plot of log-returns with gamma model for $N_i = 1$, (D) Probability plot of log-returns with gamma model for $N_i = 2$, (E) Probability plot of log-returns with gamma model for $N_i = 3$, (F) Probability plot of log-returns with gamma model for $N_i = 4$.

7. Fitting **BTLG** To Growth Episodes

7.1. Estimation of Parameters ($\hat{\beta}$ and $\hat{p}$).

Table 48. Foreign Exchange Rates

Currency	$\hat{\beta}$	$\hat{p}$
US Dollar	218.206	0.5018201
Swiss Franc	268.4891	0.4918125
Swedish Krona	352.8282	0.5420189
Norwegian Krone	363.5448	0.5345419
Canadian Dollar	213.1977	0.4956962
Australian Dollar	187.4094	0.5041408
Deutsche Mark	384.9116	0.4927461

Table 49. Commodities

Commodity	$\hat{\beta}$	$\hat{p}$
Daily Gold	404.3835	0.4700939
Weekly Gold	145.6967	0.502809
Daily Oil	161.7645	0.4912337
Weekly Oil	91.14257	0.4279412

Table 50. Stock Indexes

Stock Indices	$\hat{\beta}$	$\hat{p}$
Daily S&P500	373.6386	0.4747868
Weekly S&P500	172.2238	0.4605993
Daily NASDAQ	161.7645	0.4912337
Weekly NASDAQ	91.14256	0.4279412

Table 51. Stock Prices

Stocks	$\hat{\beta}$	$\hat{p}$
Daily BOA	179.409	0.4991597
Weekly BOA	72.39661	0.5259631
Daily Chevron	221.7919	0.4994734
Weekly Chevron	95.80182	0.5675403

The graphical depiction of fitting **BTLG** model to growth and decline episodes of our financial data sets in this section and Section 8 is explained using the growth episodes of *US dollar* as an example:

(1) Graph (A) contains two (2) c.d.f.'s:

   (a) The empirical c.d.f. of the maxima of the growth episodes.

   (b) The fitted (theoretical model) c.d.f. of the truncated logistic distribution with estimators parameters $\hat{\beta} = 218.206$ and $\hat{p} = 0.5018201$.

(2) Graph (B) is a probability plot of maxima of log returns with quantiles of truncated logistic distribution.

(3) Graph (C) through (F) are probability plots of $N_i = 1, 2, 3, 4 - day$ maxima of log returns versus model generalized expnential distribution.

7.2. **Graphical fit:- Foreign exchange rates.**

7.2.1. *US Dollar (growth episodes).*

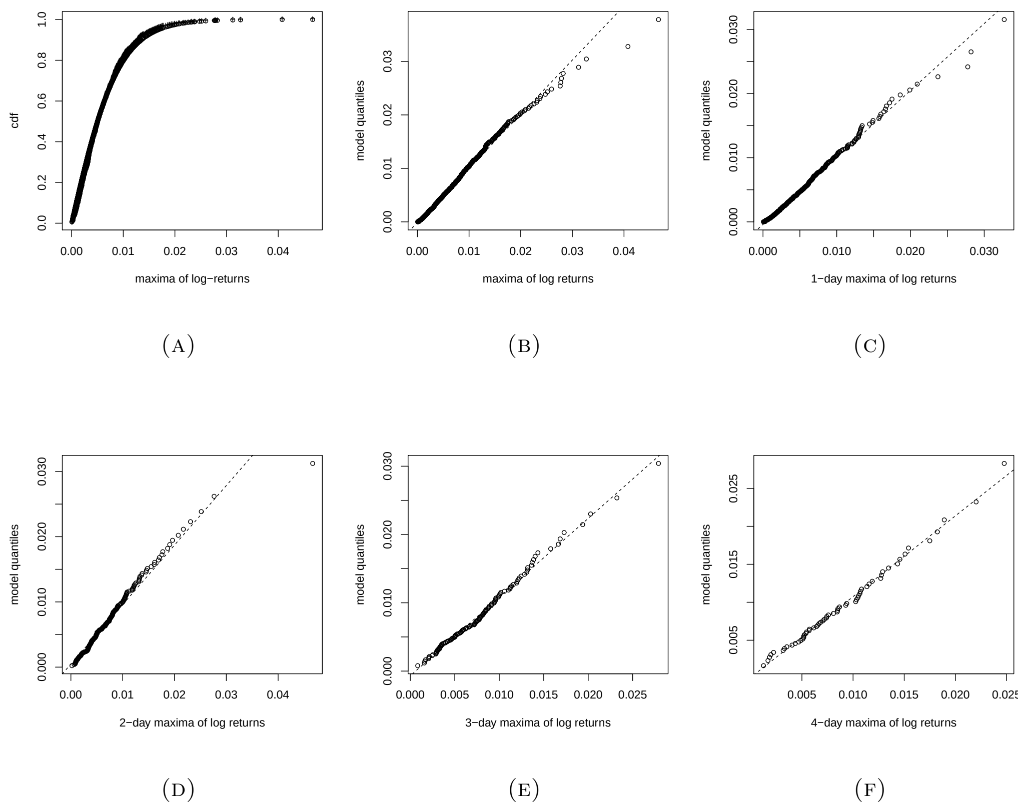


FIGURE 77. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.2.2. *Canada Dollar (growth episodes).*

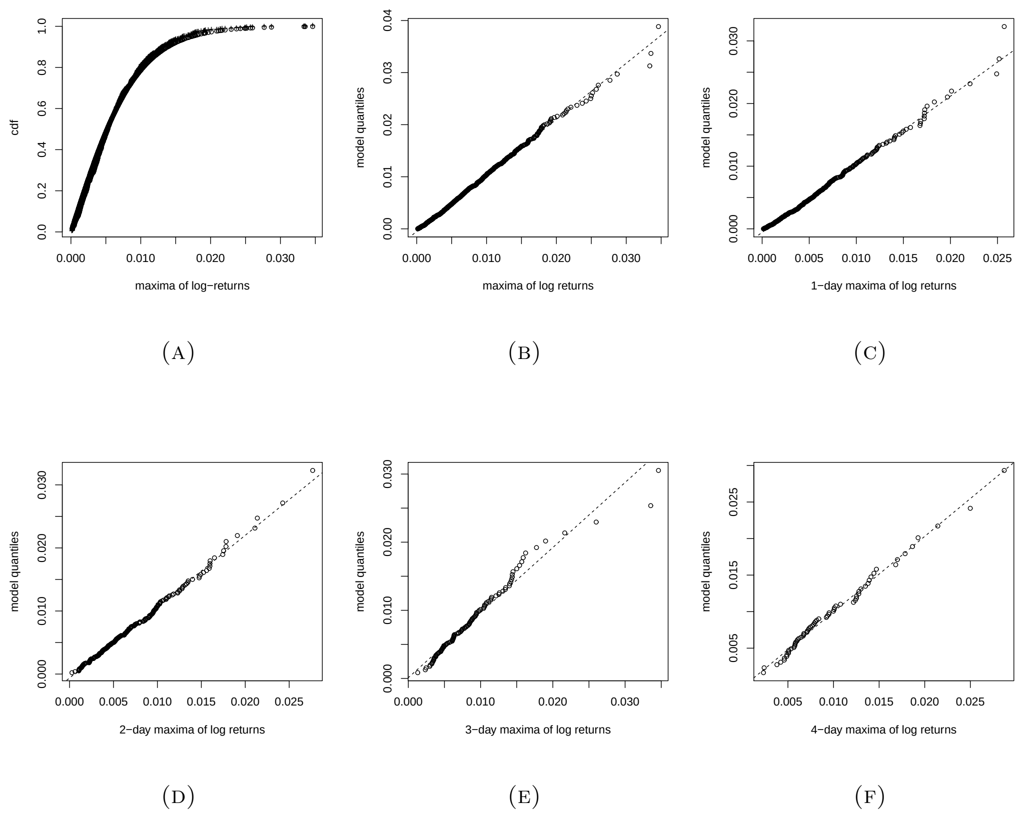


FIGURE 78. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.2.3. *Australian Dollar (growth episodes).*

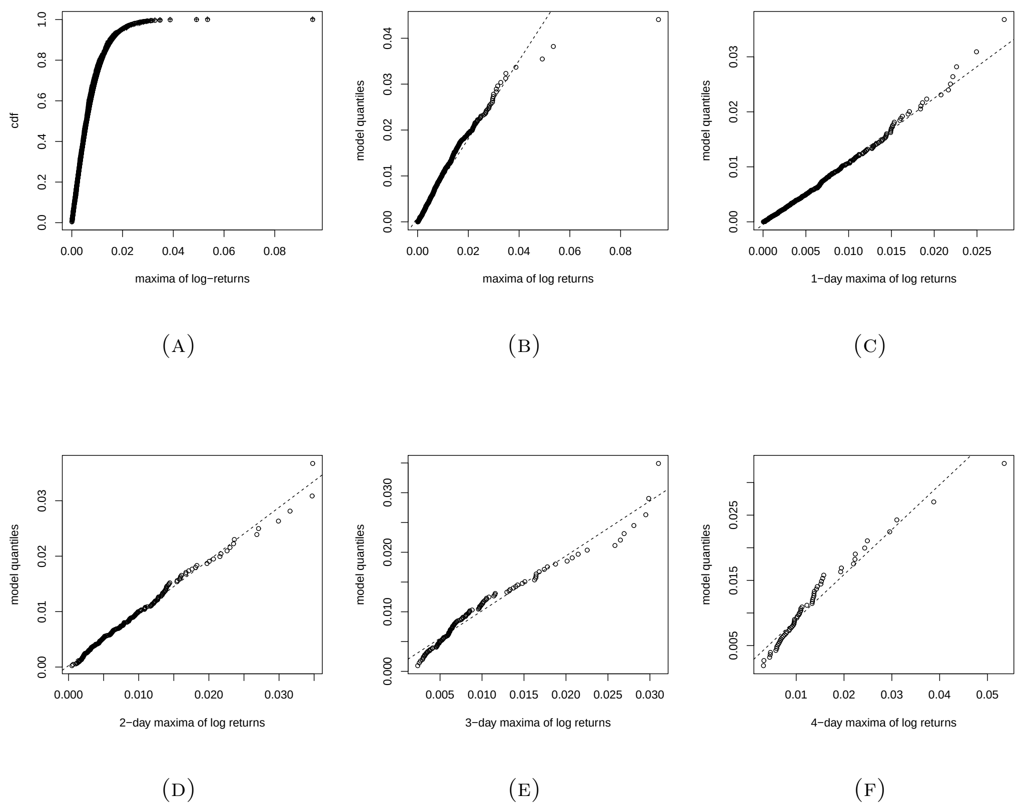


FIGURE 79. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.2.4. *Australian Dollar (growth episodes).*

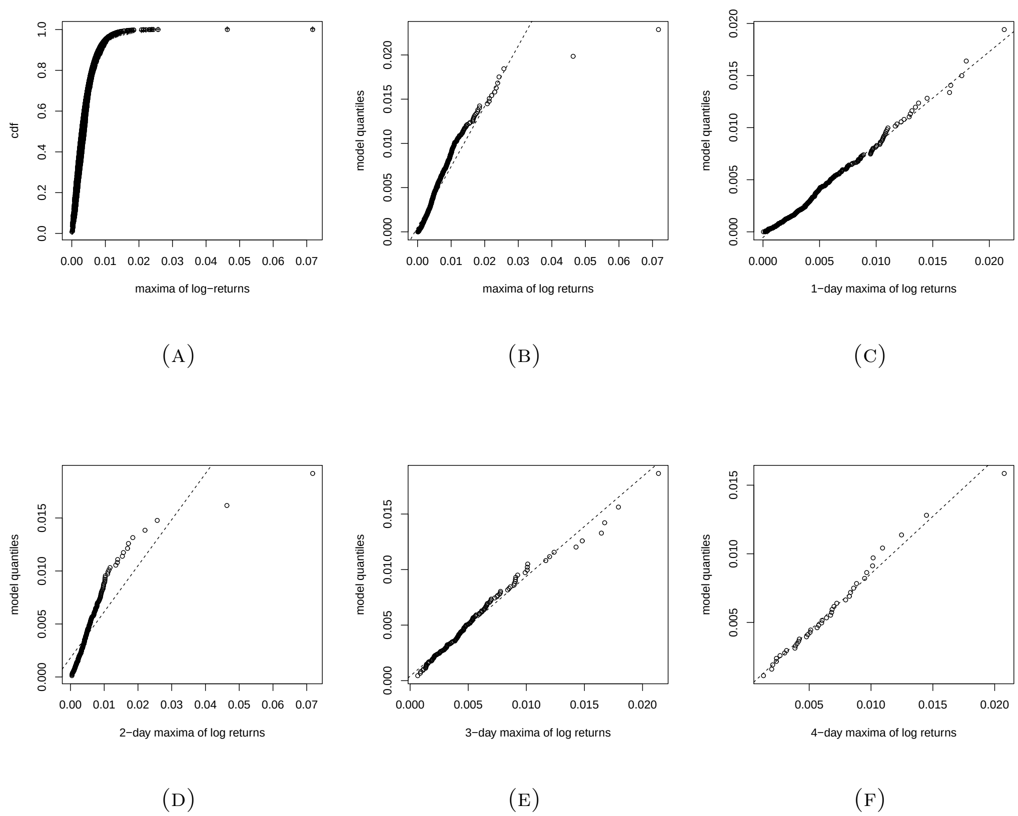


FIGURE 80. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.2.5. *Swiss Franc (growth episodes).*

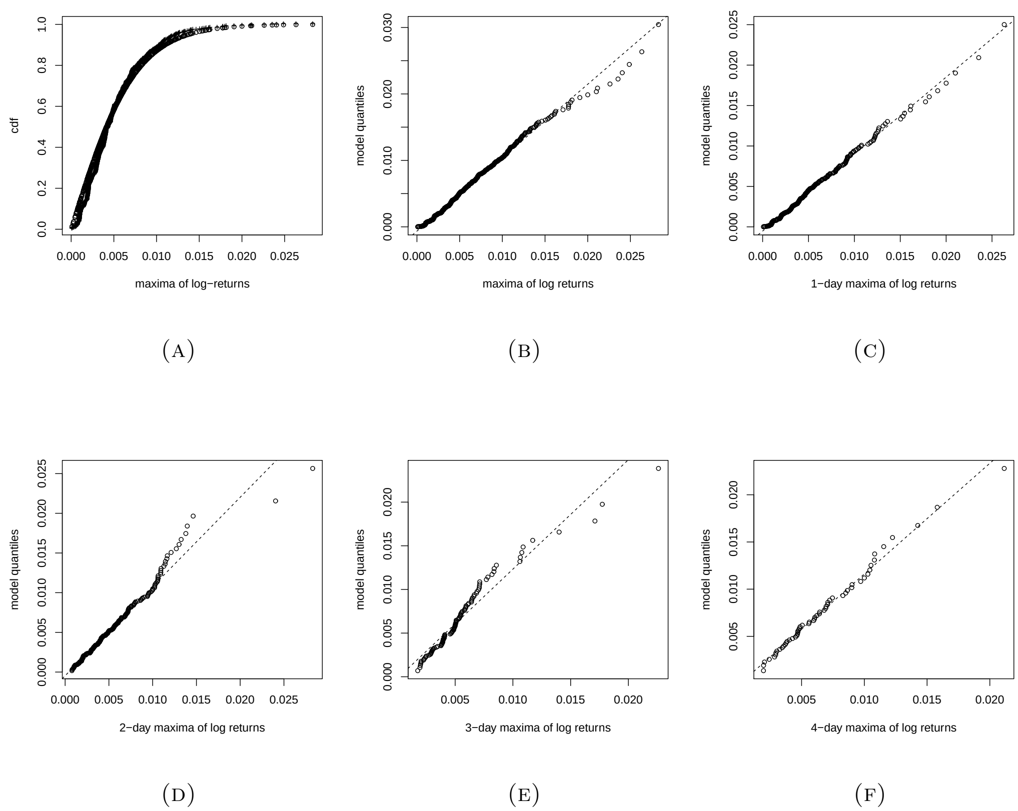


FIGURE 81. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.2.6. *Swedish Krona (growth episodes).*

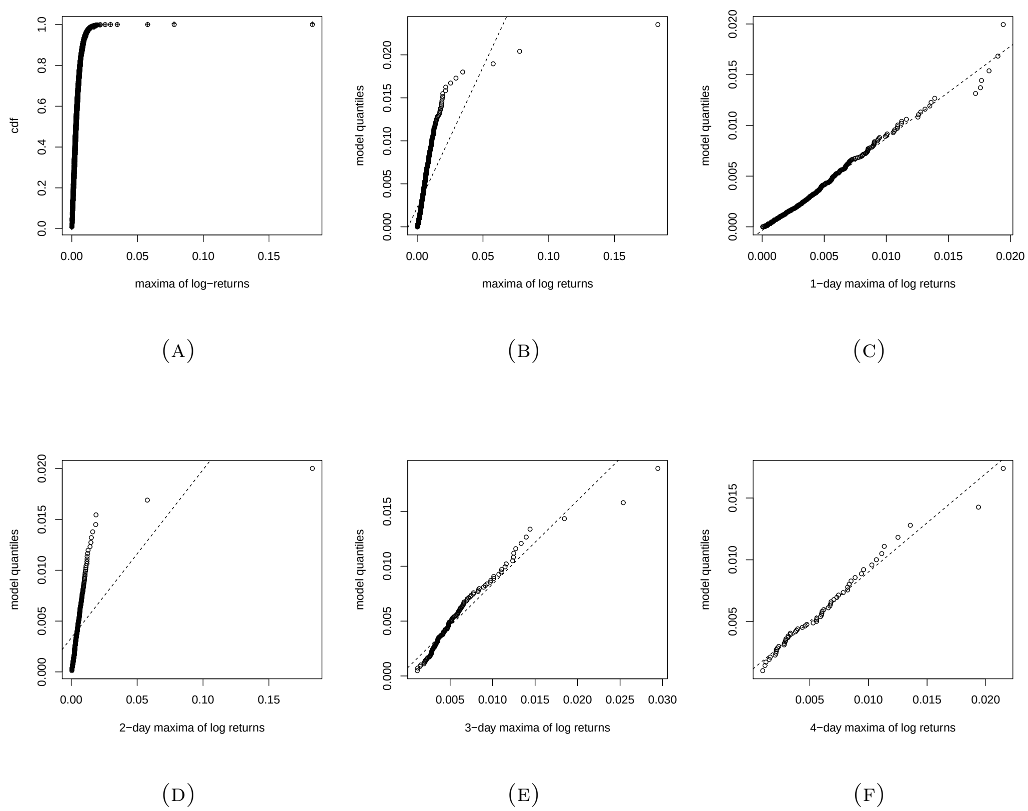
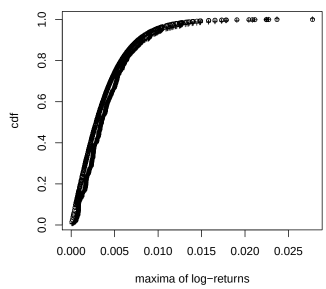
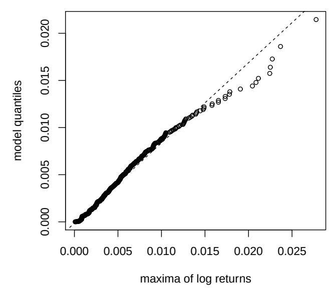


FIGURE 82. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

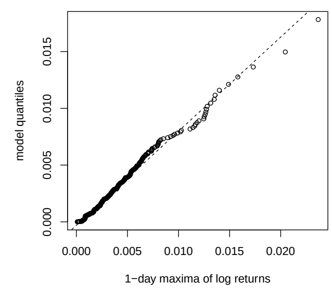
7.2.7. *Deutsche Mark (growth episodes).*



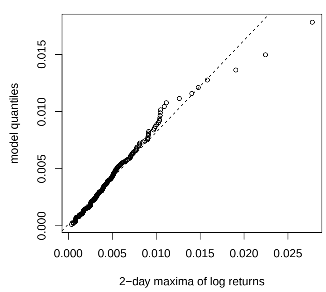
(A)



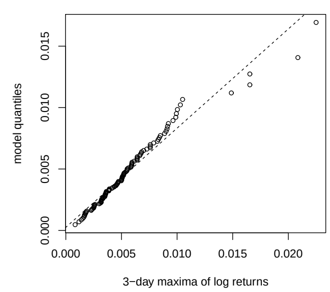
(B)



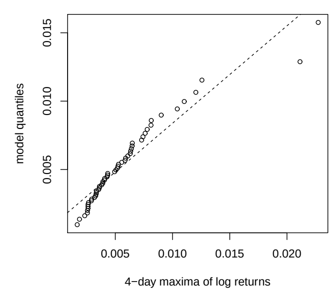
(C)



(D)



(E)



(F)

FIGURE 83. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.3. **Graphical fit:-Commodities.**

7.3.1. *Gold:-Daily log-returns (growth episodes).*

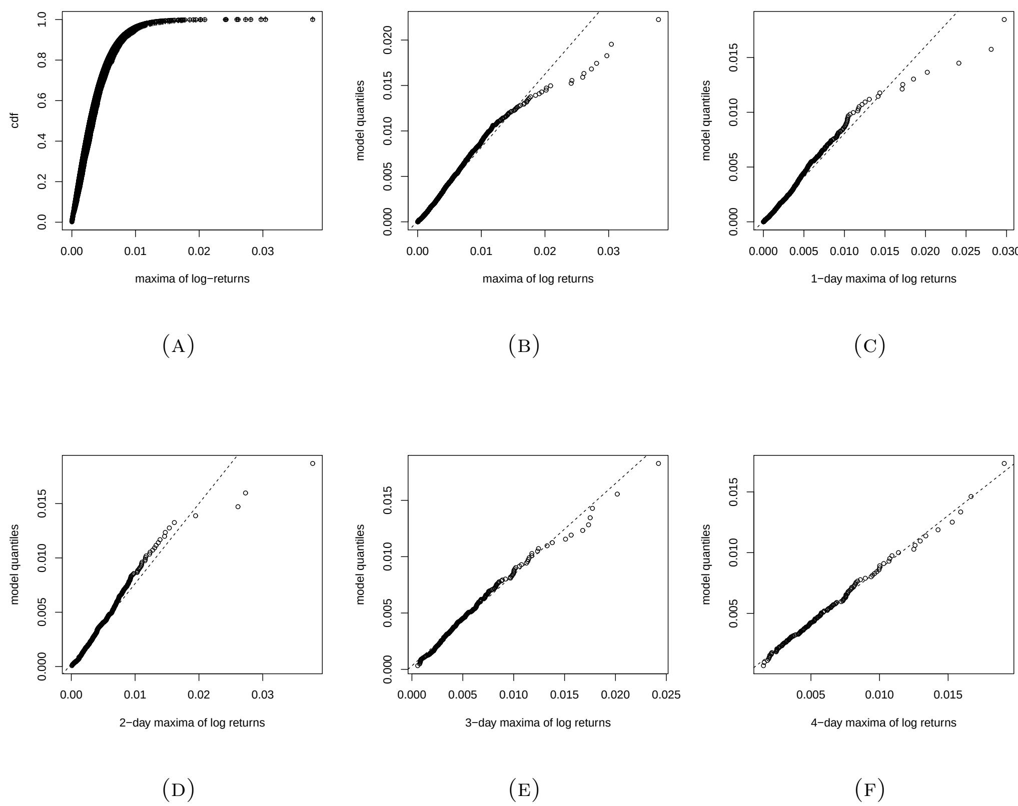


FIGURE 84. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.3.2. *Gold:-Weekly log-returns (growth episodes).*

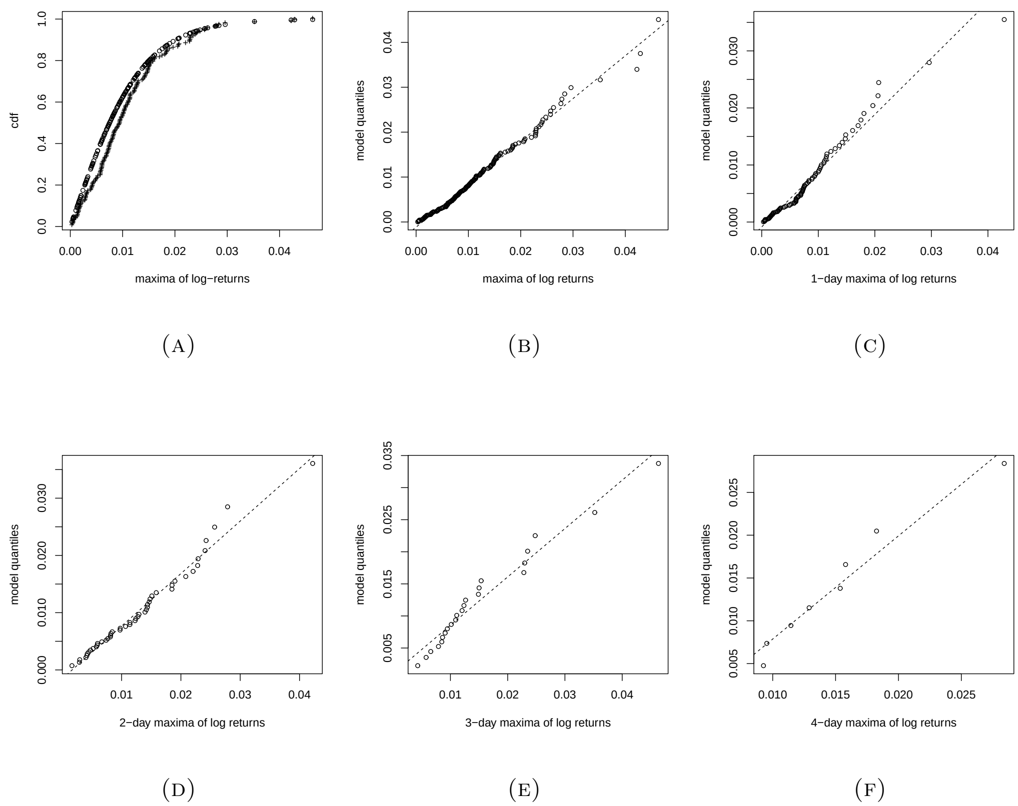


FIGURE 85. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.3.3. *Oil:-Daily log-returns (growth episodes).*

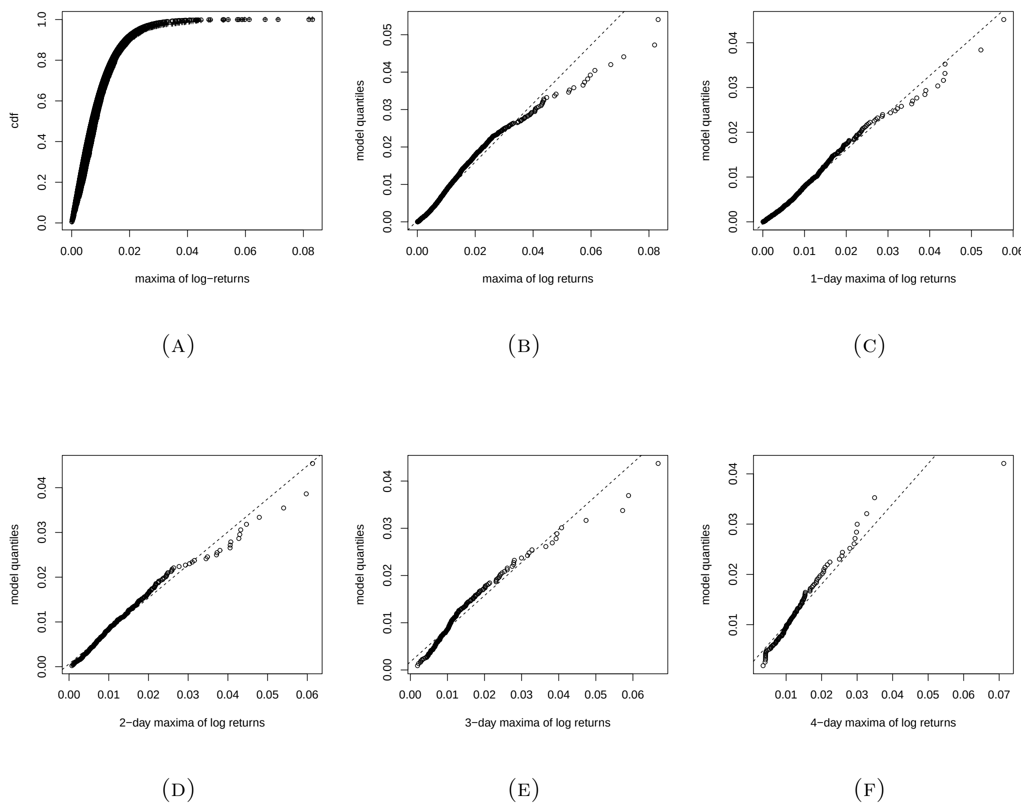
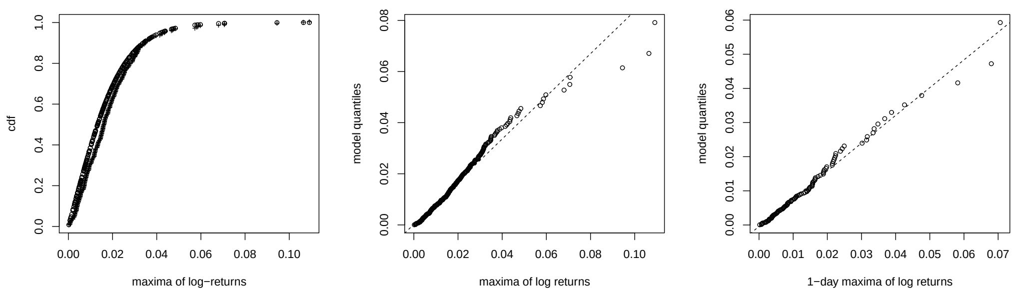
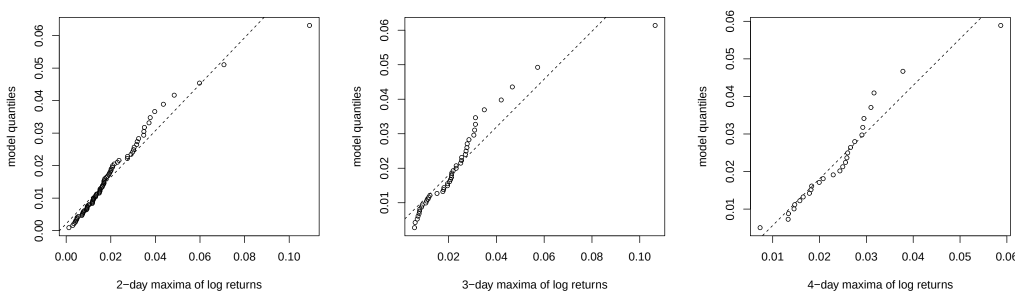


FIGURE 86. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.3.4. *Oil:-Weekly log-returns (growth episodes).*



(A) (B) (C)



(D) (E) (F)

FIGURE 87. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.4. **Graphical fit:-Stock Indexes.**

7.4.1. *S&P500:-Daily log-returns (growth episodes).*

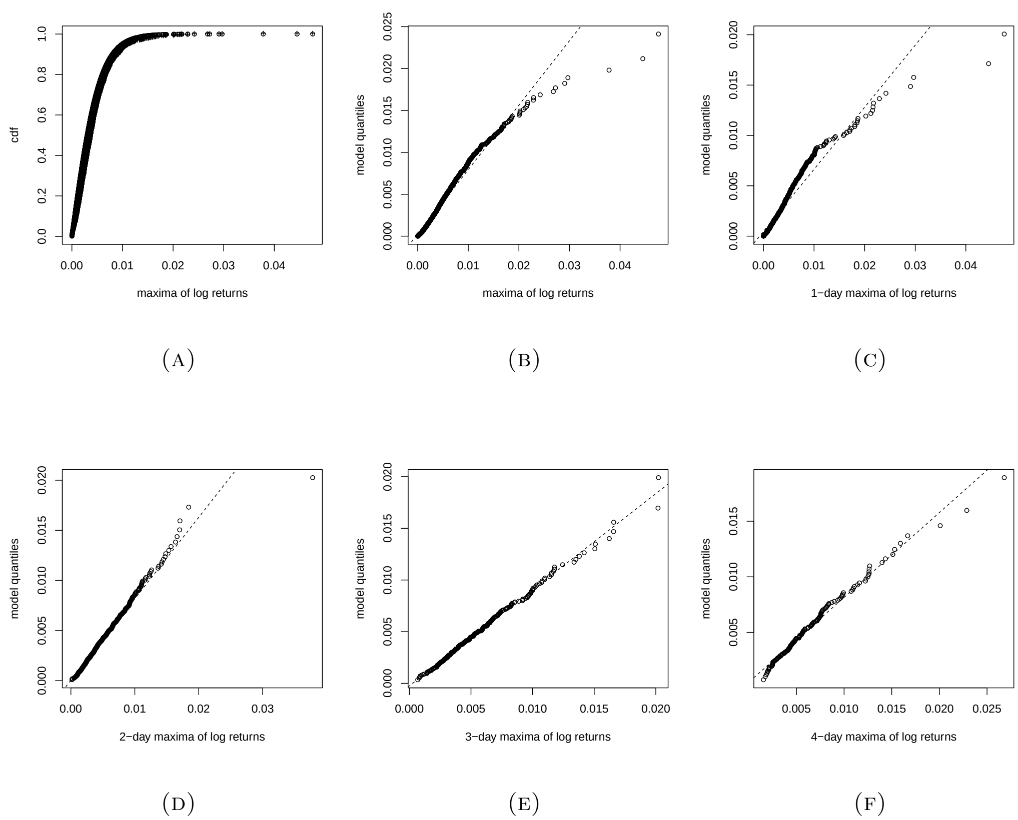


FIGURE 88. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.4.2. *S&P500:-Weekly log-returns (growth episodes).*

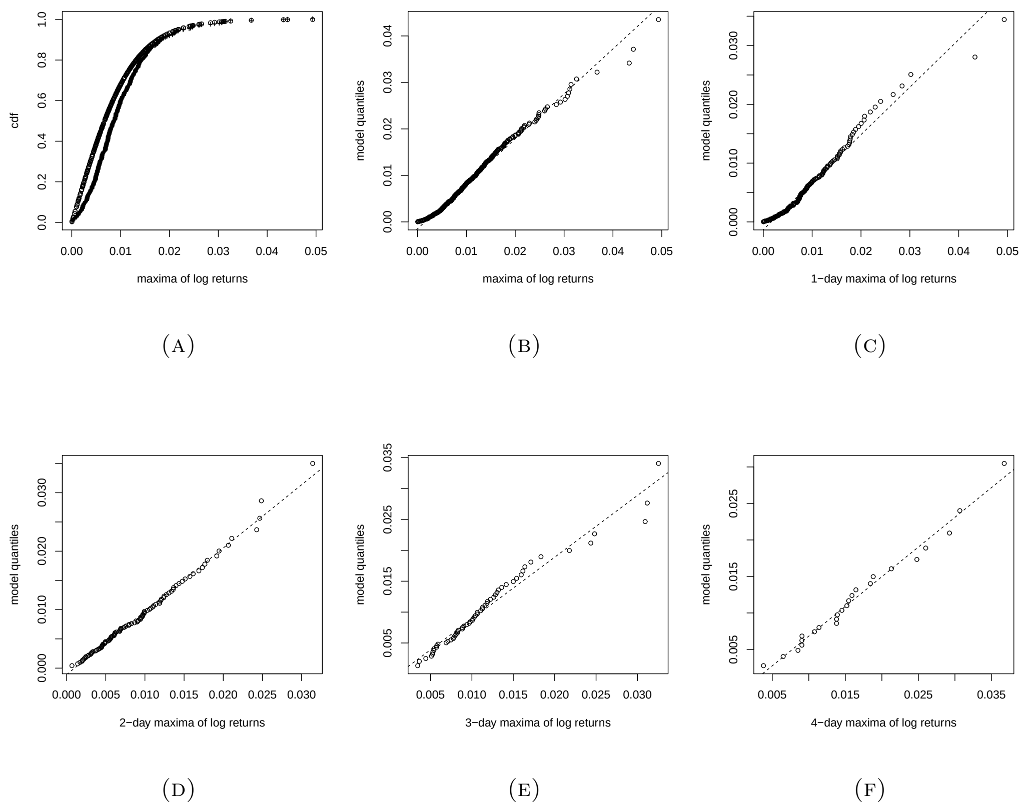


FIGURE 89. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.4.3. *NASDAQ-100:-Daily log-returns (growth episodes).*

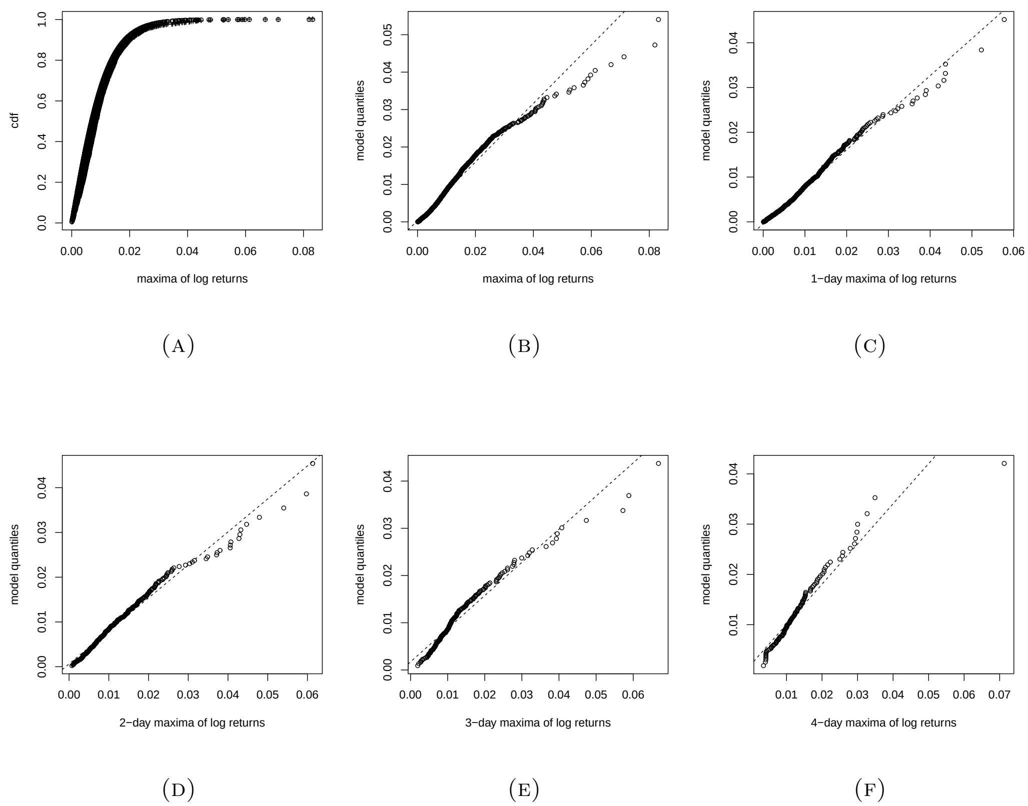


FIGURE 90. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.4.4. *NASDAQ-100:-Weekly log-returns (growth episodes).*

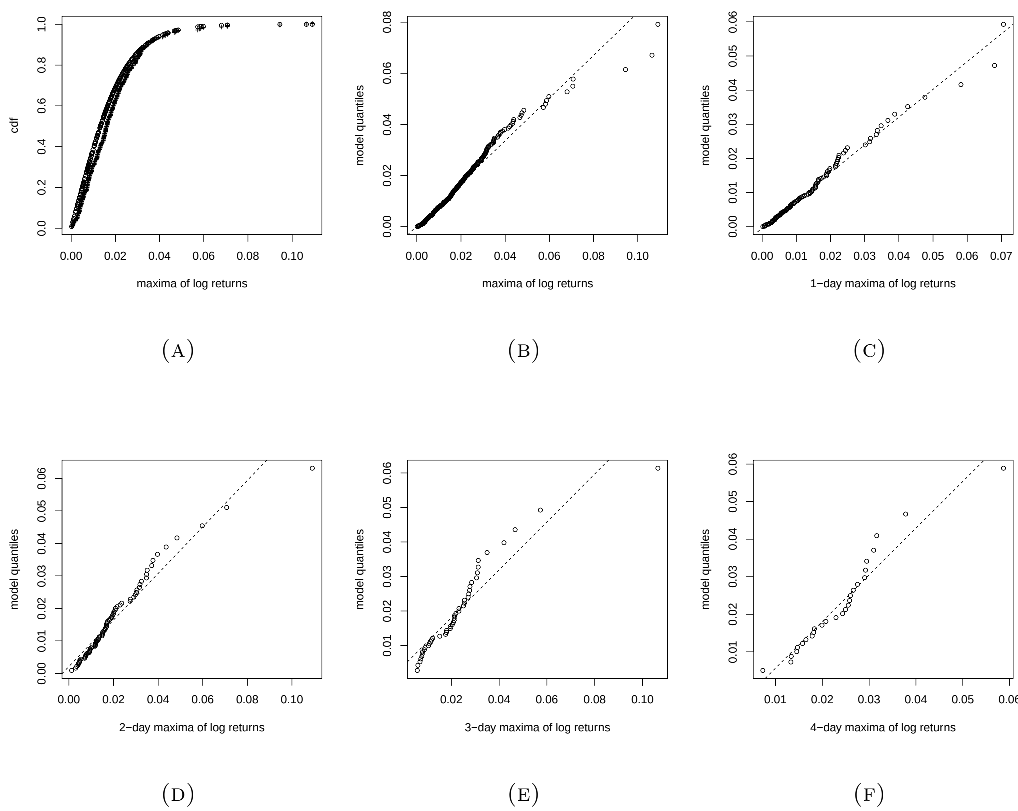


FIGURE 91. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.5. **Graphical fit:-Stocks.**

7.5.1. *BOA:-Daily log-returns (growth episodes).*

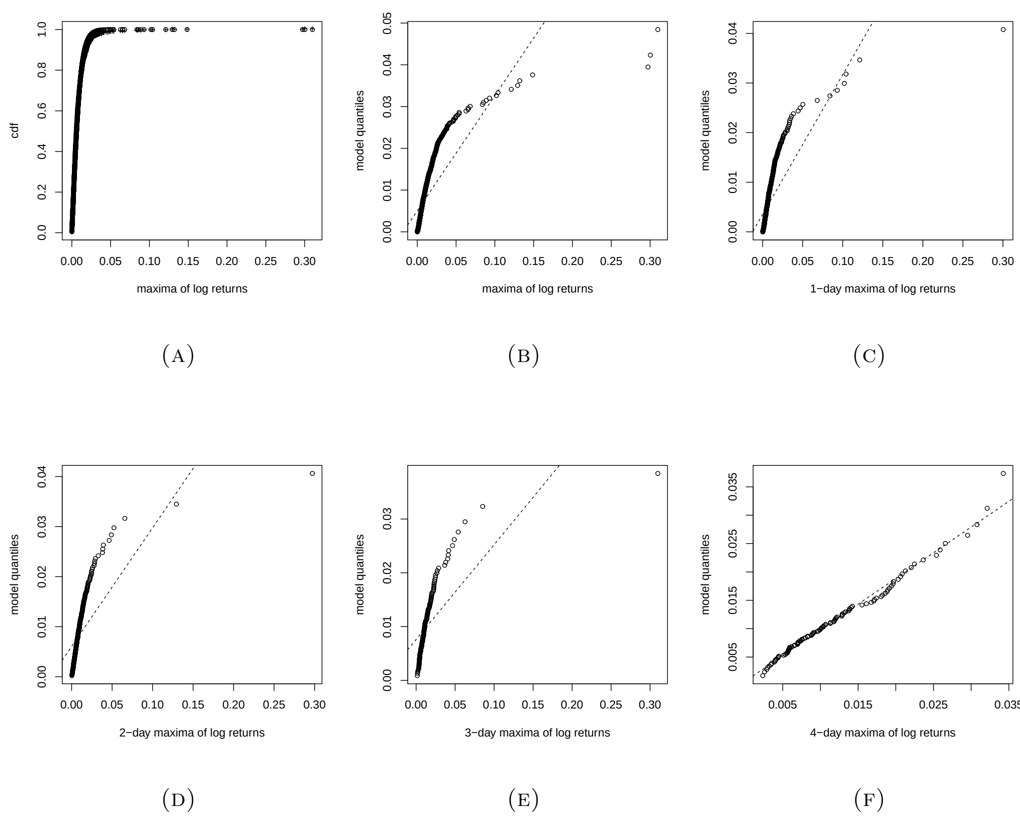


FIGURE 92. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

7.5.2. *BOA:-Weekly log-returns (growth episodes).*

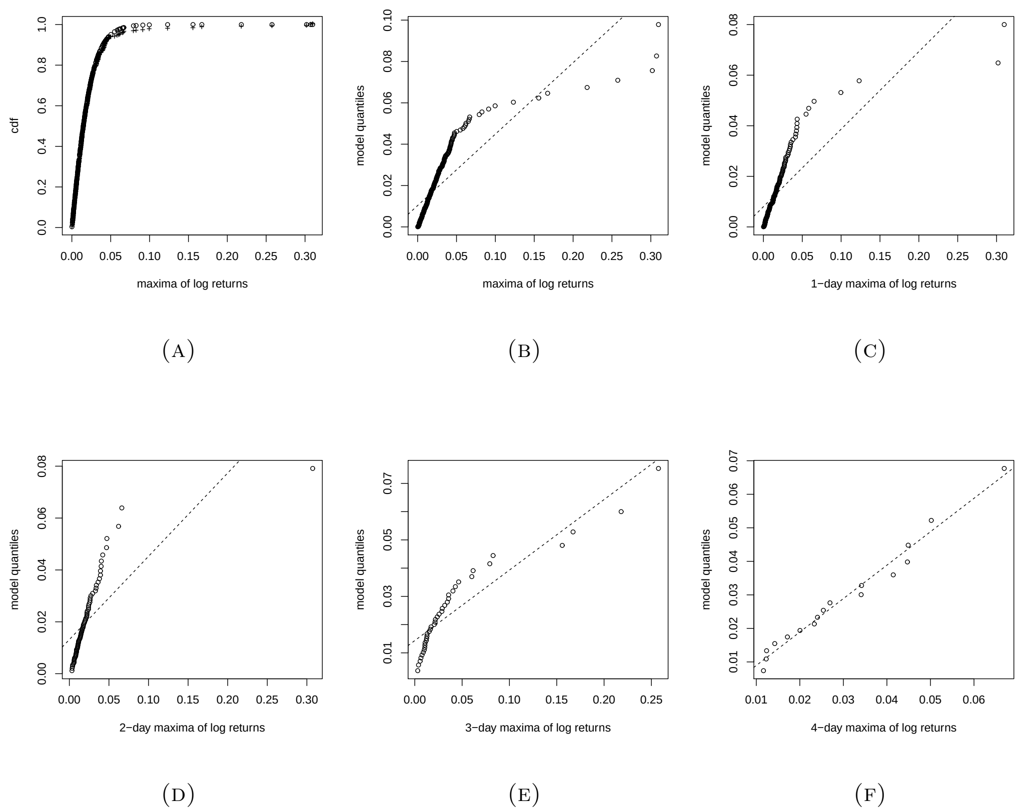


FIGURE 93. (A) Model (dotted) and empirical c.d.f.'s of maxima, (B) Probability plot of maxima versus theoretical quantiles of truncated logistic, (C) Probability plot of maxima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of maxima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of maxima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of maxima versus generalized exponential quantiles for $N = 4$.

8. Fitting **BTLG** To Decline Episodes

8.1. Estimation of Parameters ($\hat{\beta}$ and $\hat{p}$).

Table 52. Foreign Exchange Rates

Currency	$\hat{\beta}$	$\hat{p}$
US Dollar	212.8865	0.4951357
Swiss Franc	251.5544	0.5039872
Swedish Krona	364.1415	0.5401388
Norwegian Krone	372.6277	0.5425162
Canadian Dollar	212.4854	0.4914573
Australian Dollar	189.2294	0.4868356
Deutsche Mark	343.5336	0.5115654

Table 53. Commodities

Commodity	$\hat{\beta}$	$\hat{p}$
Daily Gold	424.3929	0.5215245
Weekly Gold	167.3977	0.5250737
Daily Oil	158.2818	0.520081
Weekly Oil	88.16383	0.484193

Table 54. Stock Indexes

Stock Indices	$\hat{\beta}$	$\hat{p}$
Daily S&P500	378.5208	0.5352731
Weekly S&P500	175.8548	0.5905849
Daily NASDAQ-100	161.7645	0.520081
Weekly NASDAQ-100	88.16385	0.484193

Table 55. Stock Prices

Stocks	$\hat{\beta}$	$\hat{p}$
Daily BOA	186.449	0.4936835
Weekly BOA	86.0469	0.4679583
Daily Chevron	231.524	0.4744949
Weekly Chevron	103.5215	0.5062837

8.2. **Graphical fit:- Foreign exchange rates.**

8.2.1. *US Dollar (decline episodes).*

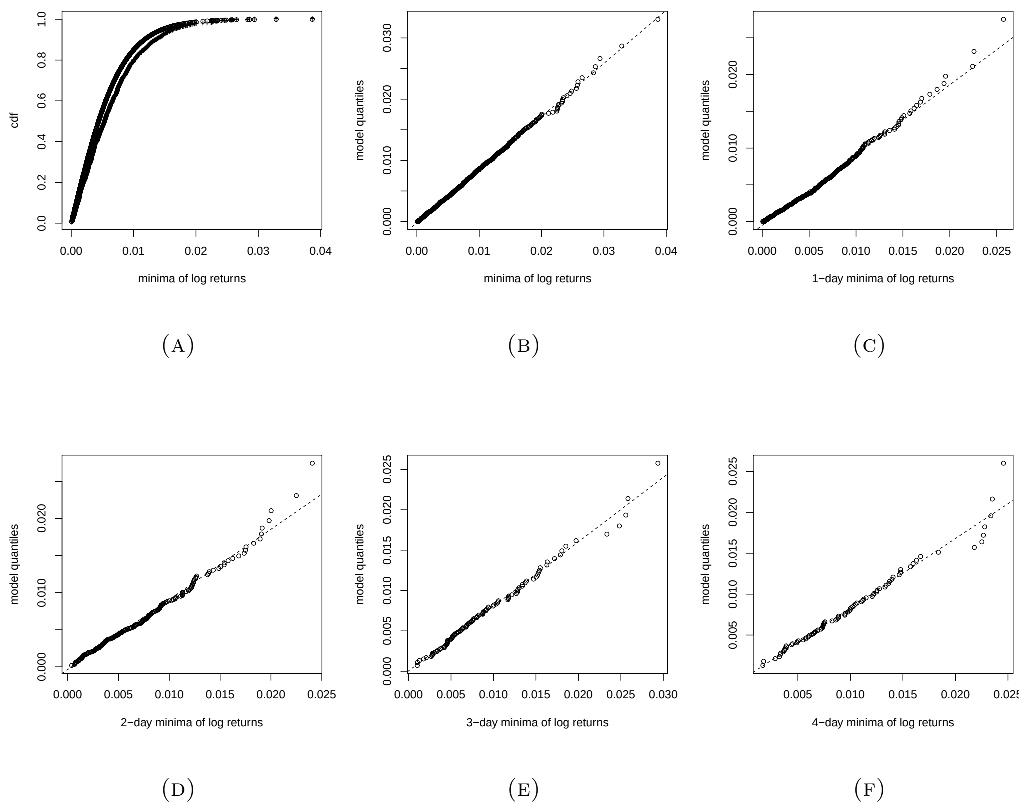


FIGURE 94. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.2.2. Canada Dollar (decline episodes).

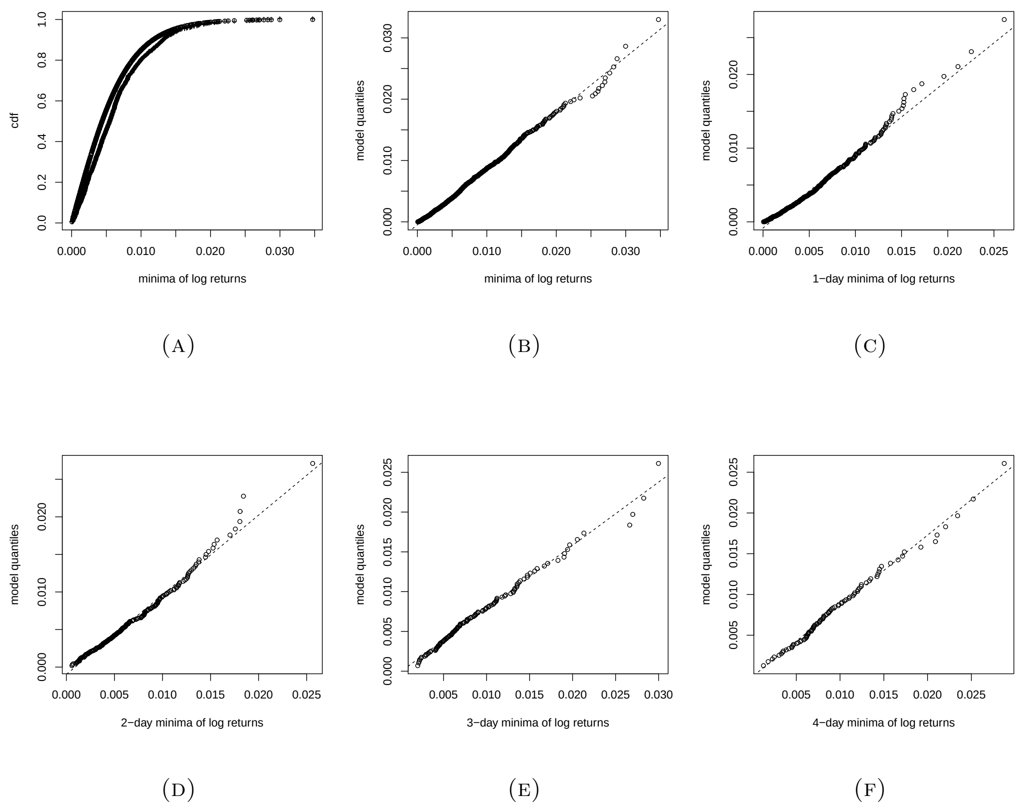


FIGURE 95. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.2.3. *Australian Dollar (decline episodes).*

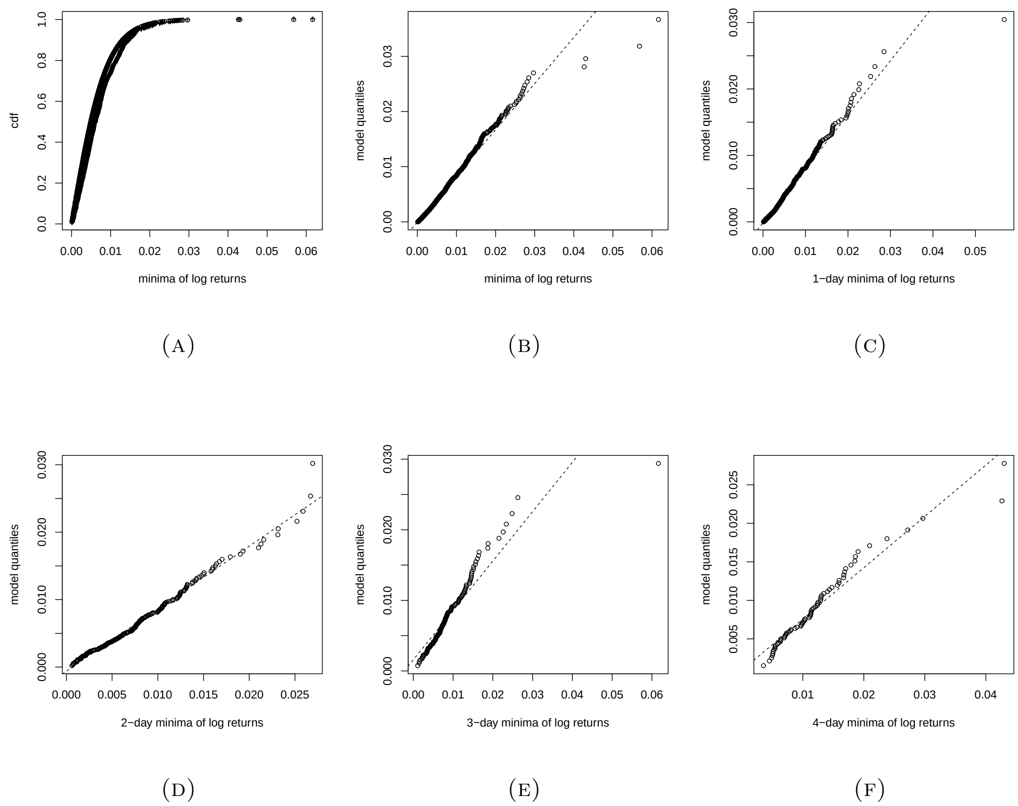


FIGURE 96. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.2.4. *Australian Dollar (decline episodes).*

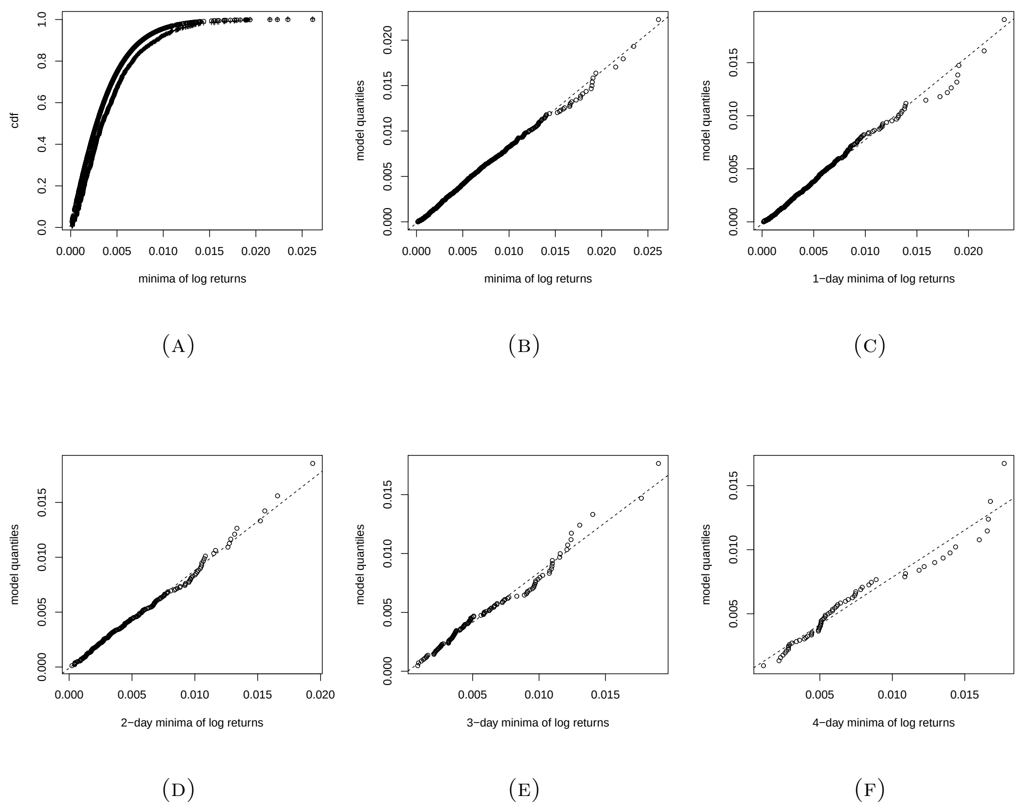


FIGURE 97. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.2.5. *Swiss Franc (decline episodes).*

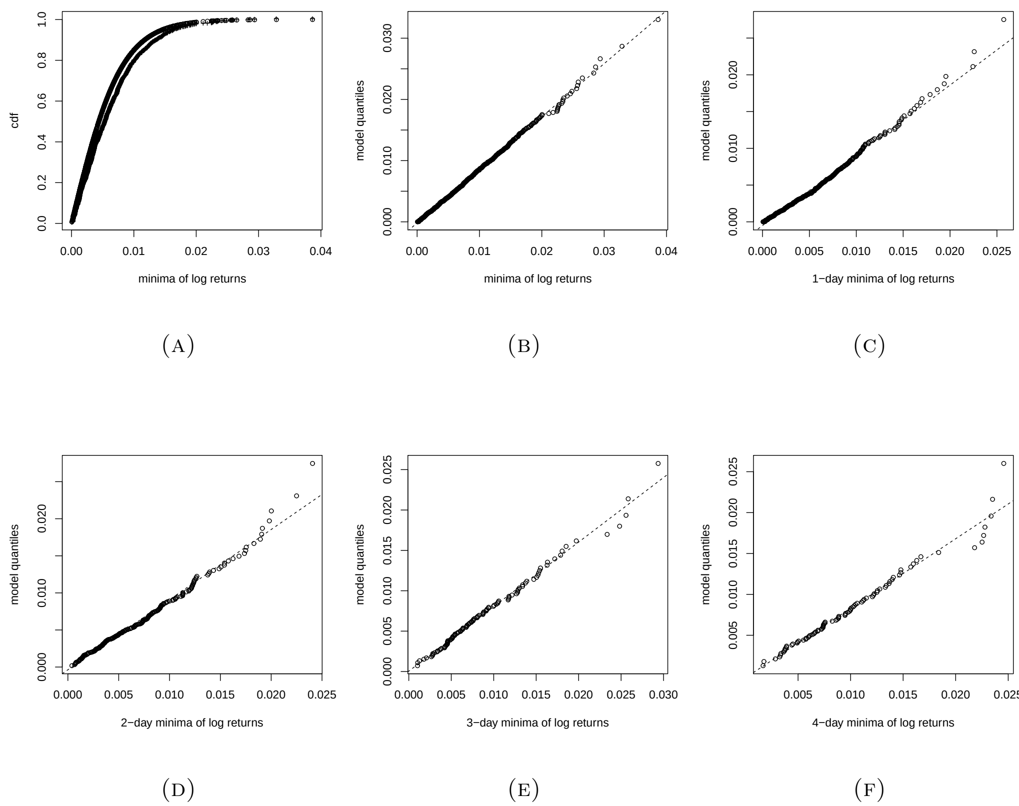


FIGURE 98. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.2.6. *Swedish Krona (decline episodes).*

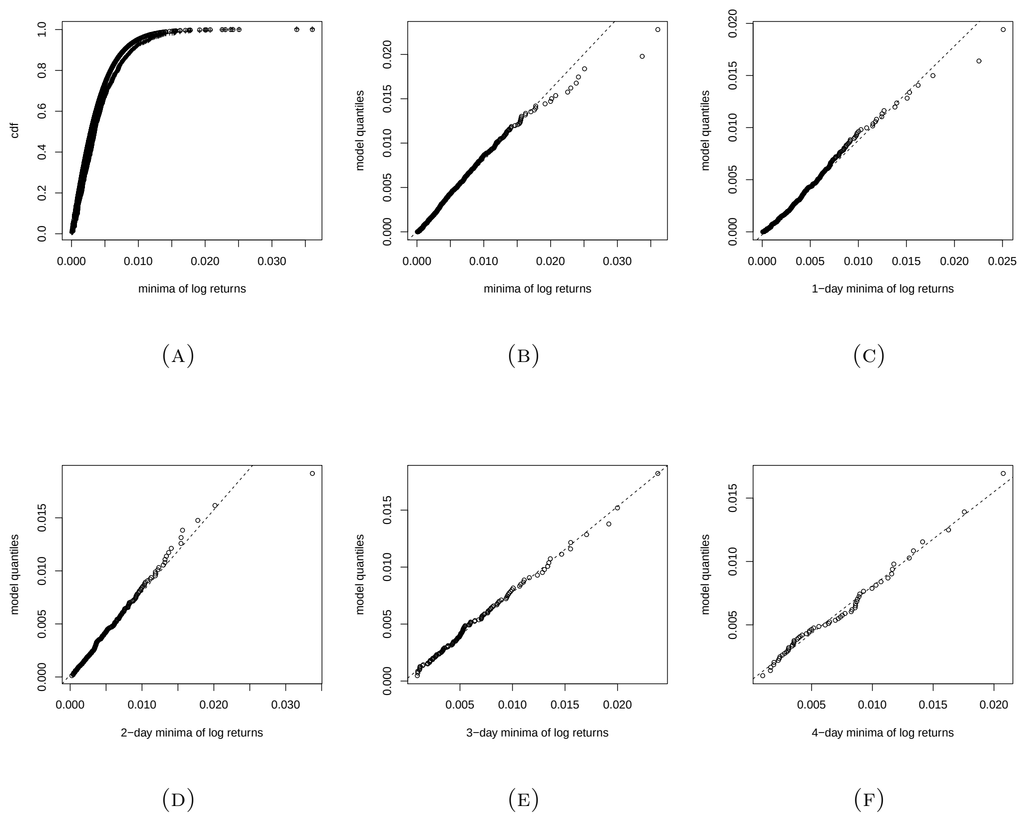


FIGURE 99. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.2.7. *Deutsche Mark (decline episodes).*

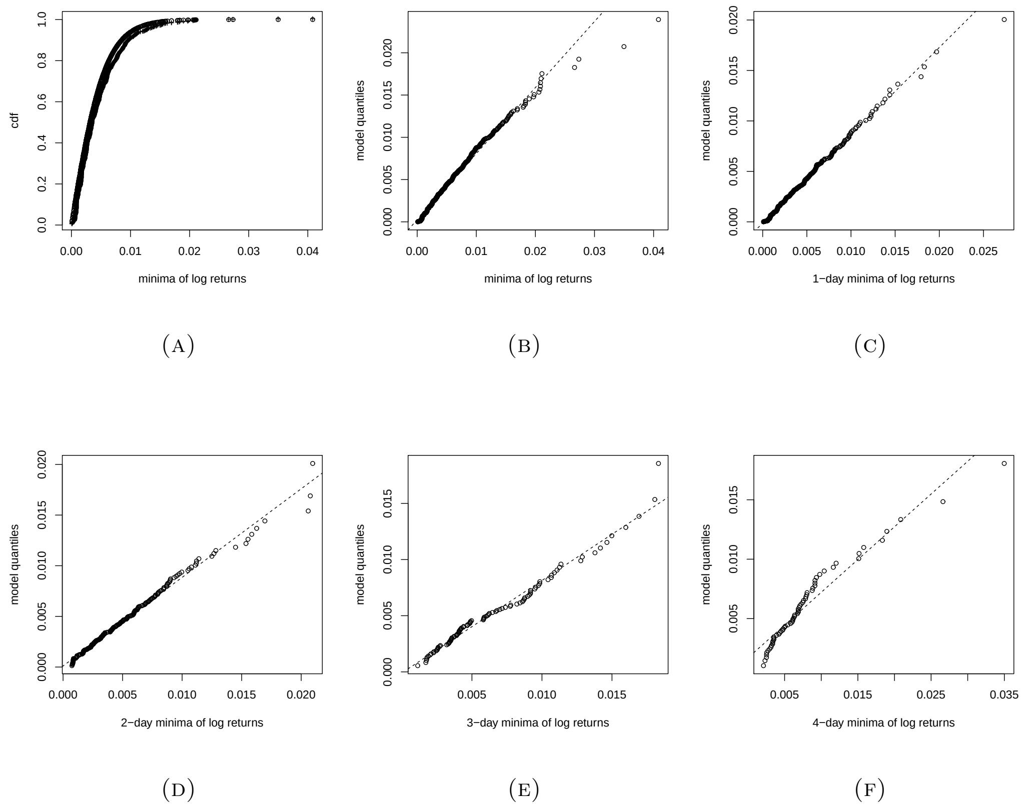


FIGURE 100. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.3. **Graphical fit:-Commodities.**

8.3.1. *Gold:-Daily log-returns (decline episodes).*

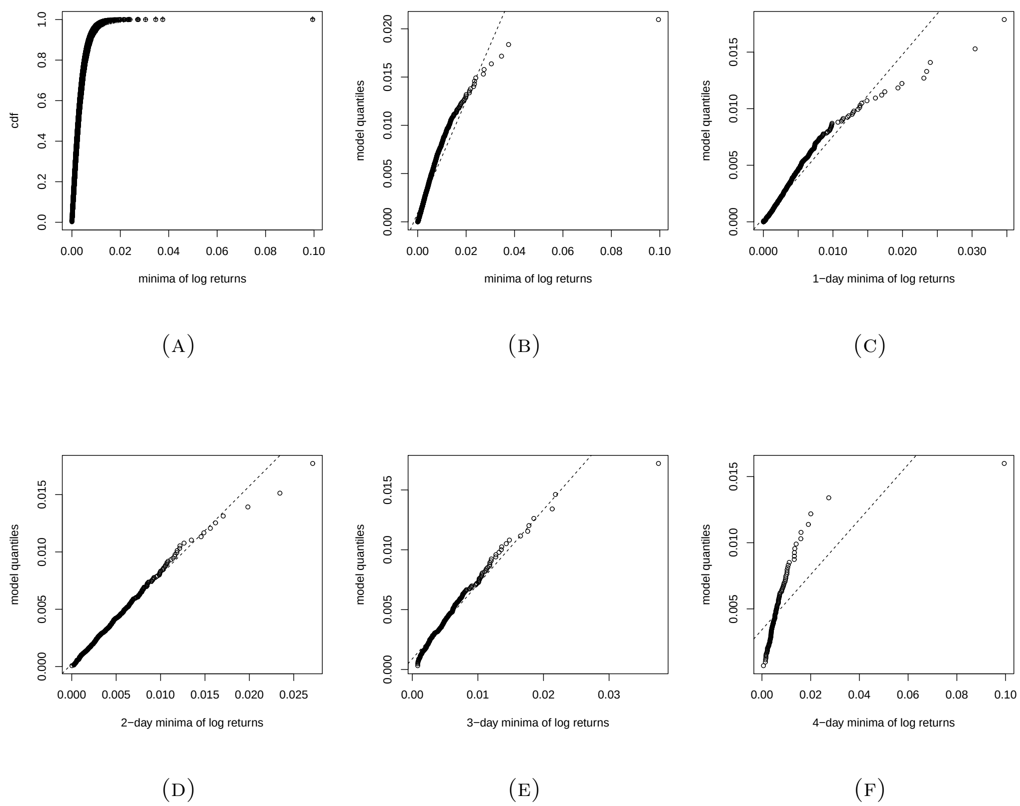


FIGURE 101. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.3.2. *Gold:-Weekly log-returns (decline episodes).*

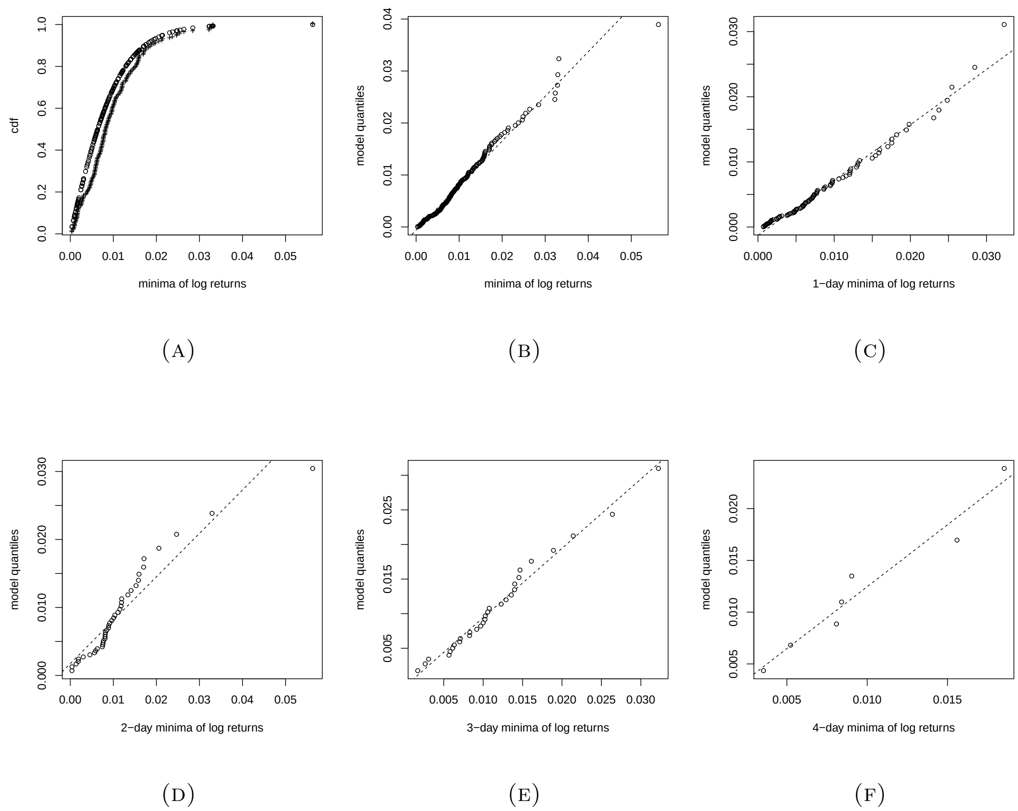


FIGURE 102. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.3.3. *Oil:-Daily log-returns (decline episodes).*

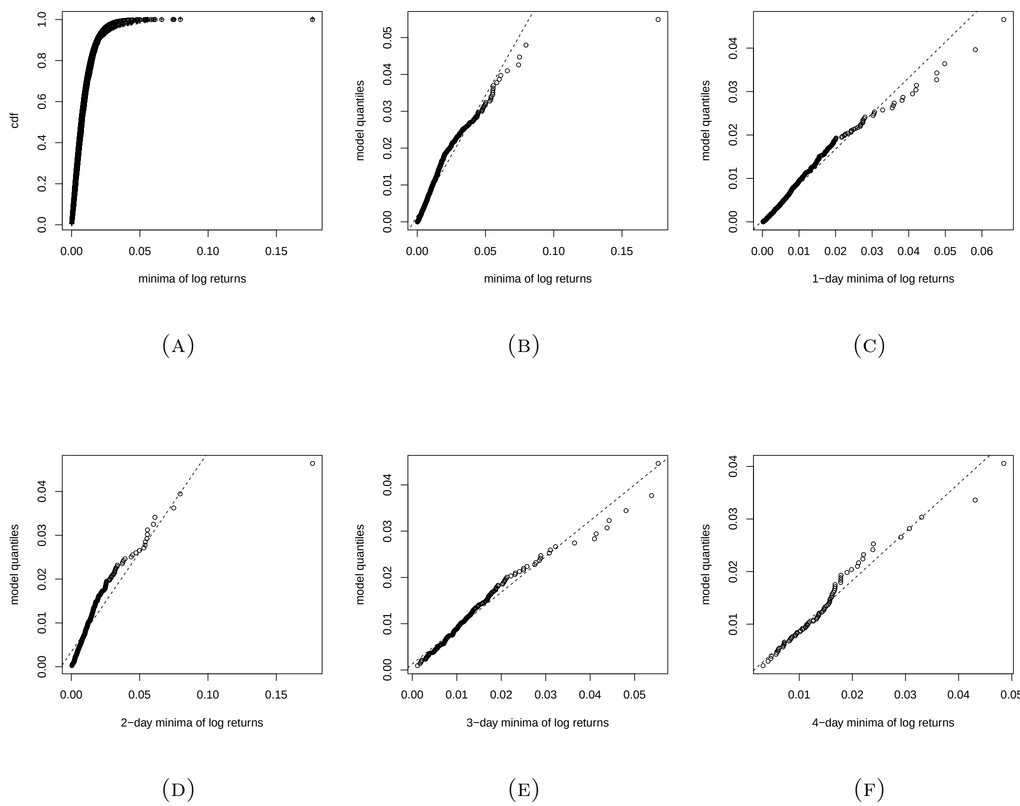


FIGURE 103. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.3.4. *Oil:-Weekly log-returns (decline episodes).*

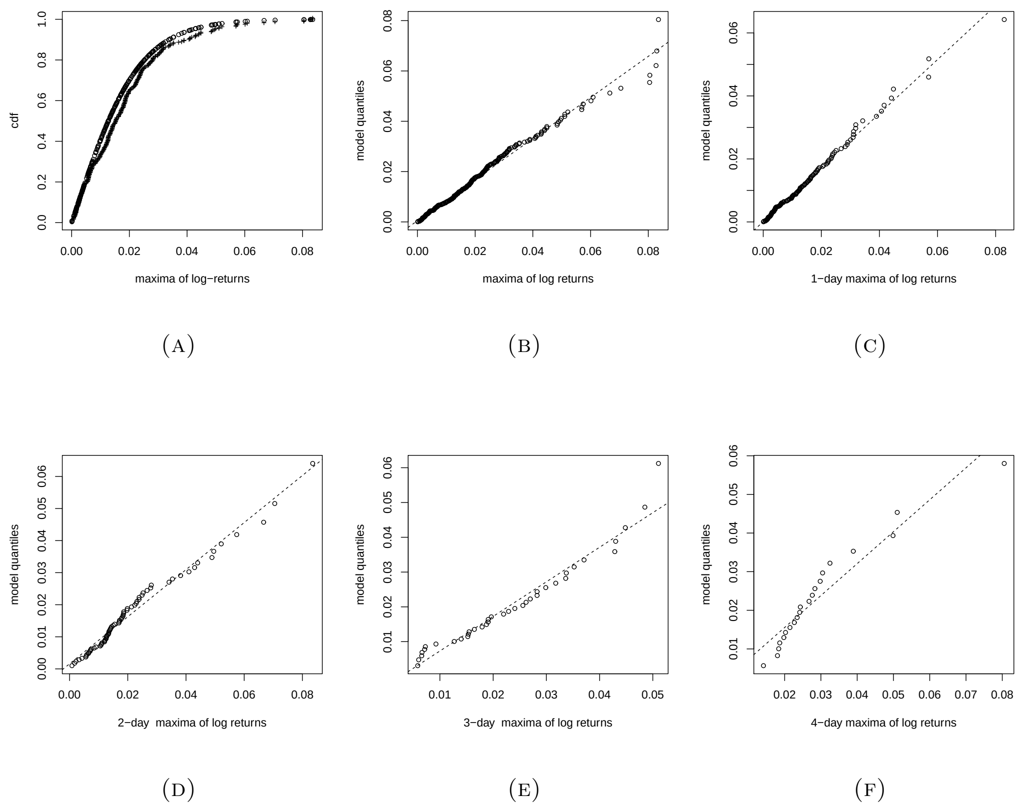


FIGURE 104. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.4. **Graphical fit:-Stock Indexes.**

8.4.1. *S&P500:-Daily log-returns (decline episodes).*

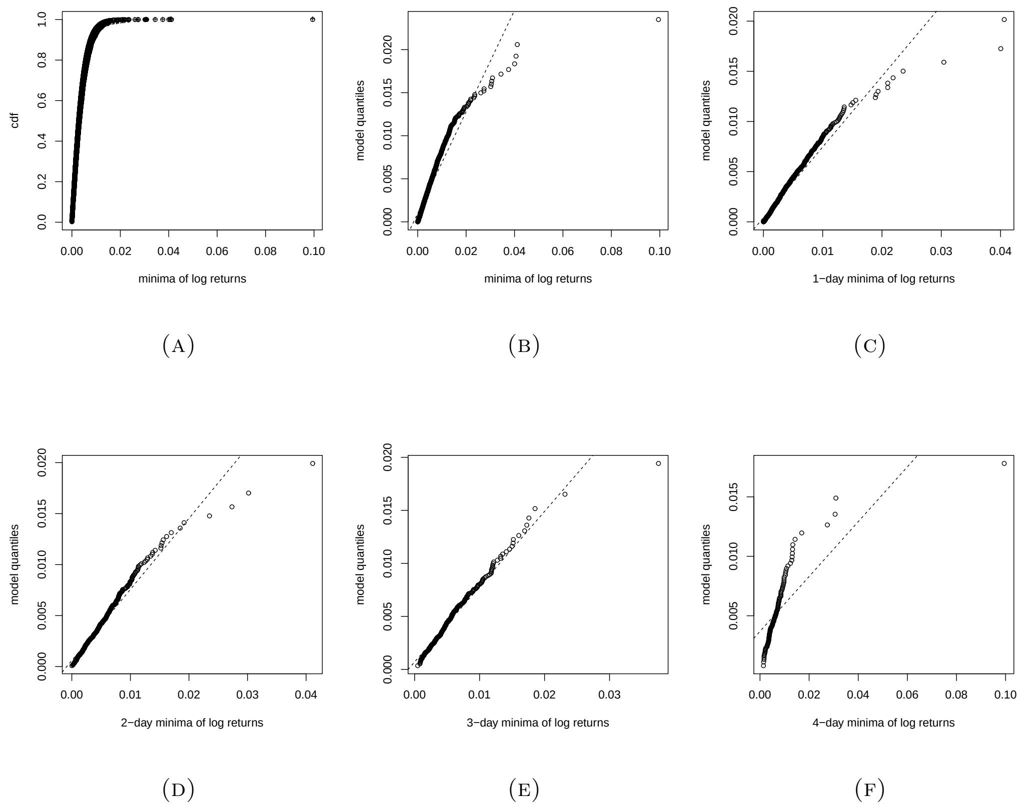


FIGURE 105. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.4.2. *S&P500:-Weekly log-returns (decline episodes).*

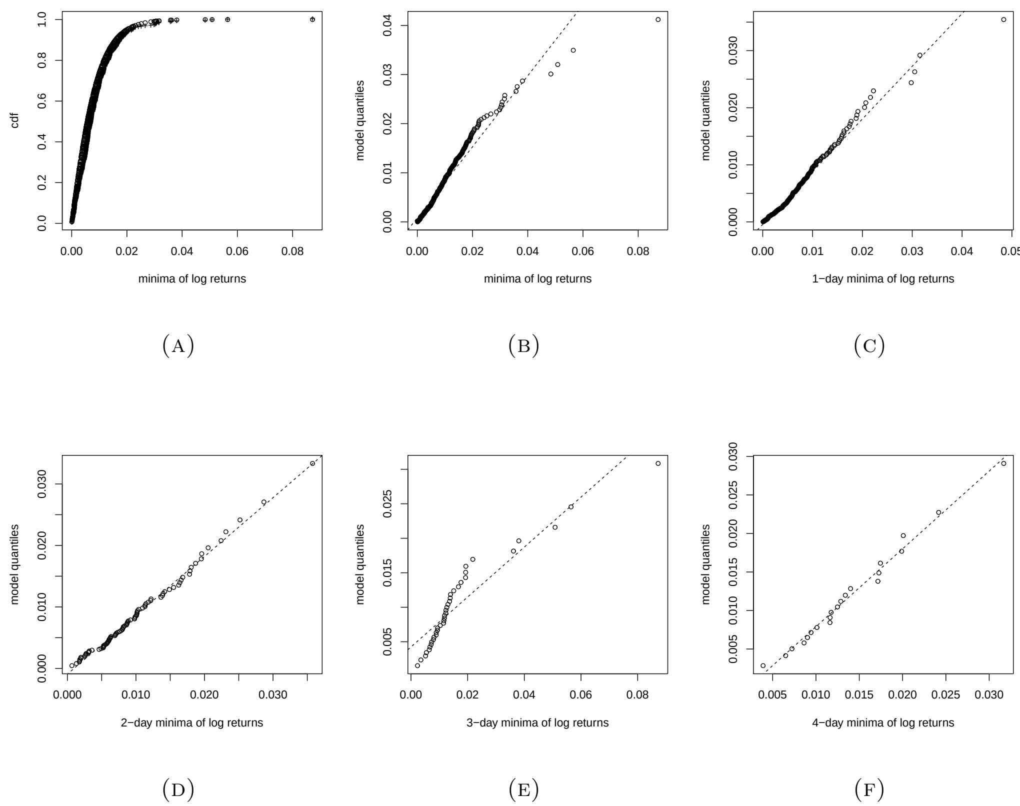


FIGURE 106. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.4.3. *NASDAQ-100:-Daily log-returns (decline episodes).*

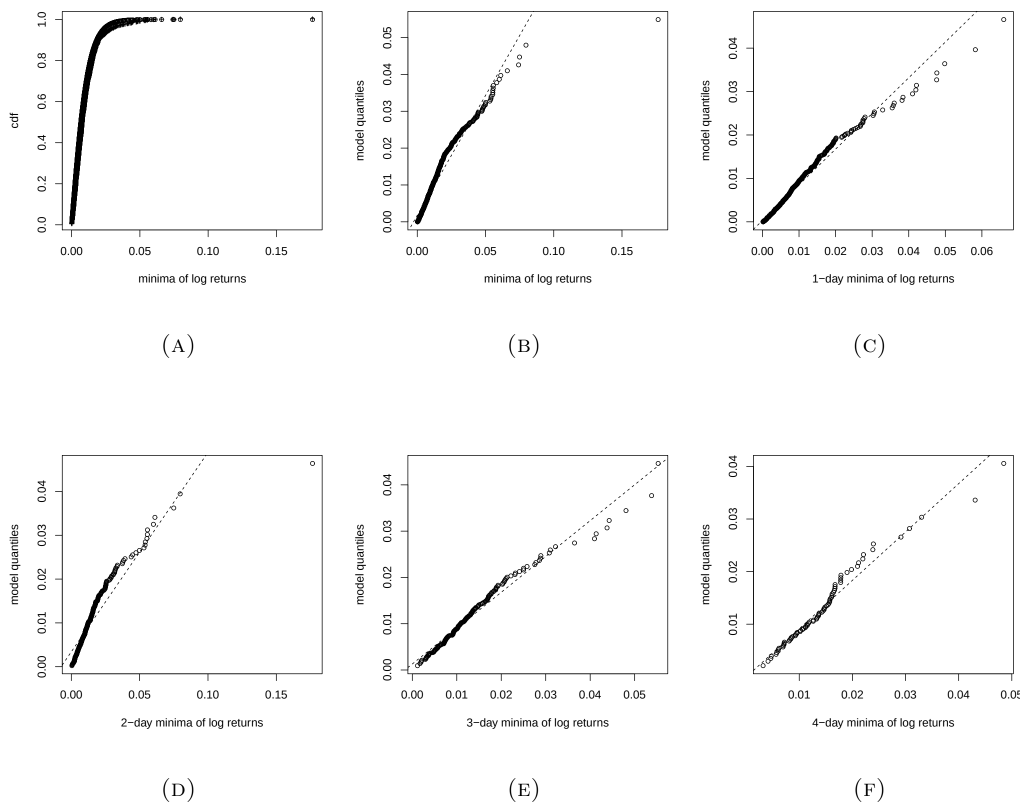


FIGURE 107. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.4.4. *NASDAQ-100:-Weekly log-returns (decline episodes).*

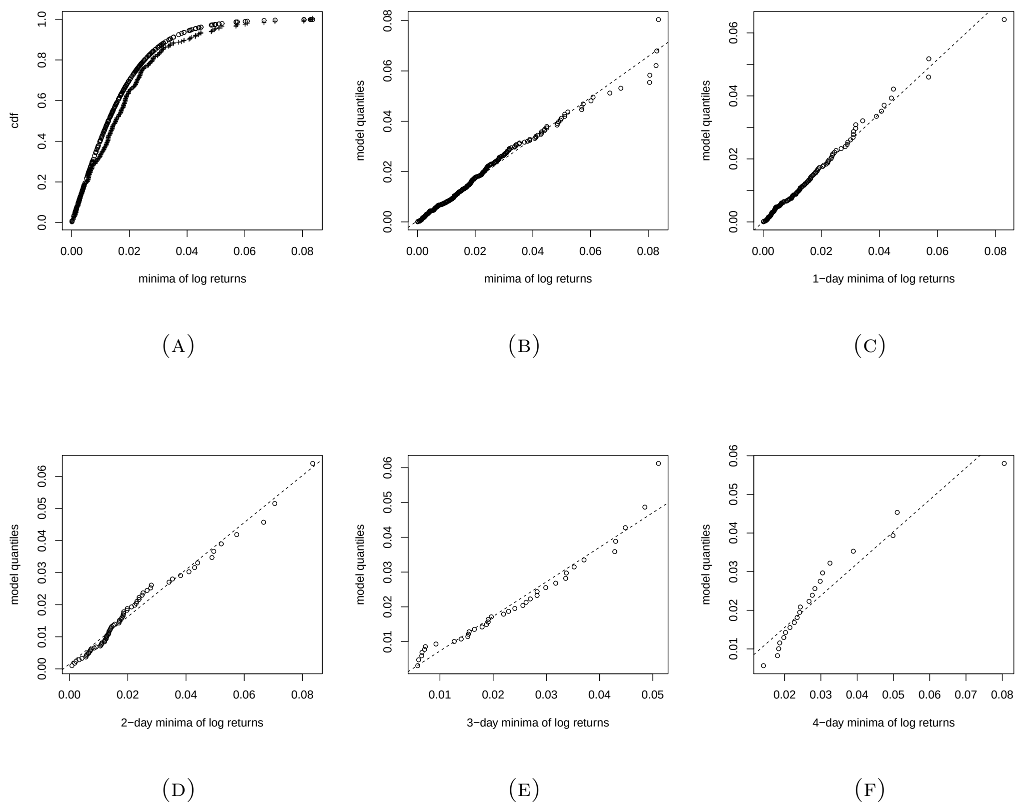


FIGURE 108. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.5. **Graphical fit:-Stocks.**

8.5.1. *BOA:-Daily log-returns (decline episodes).*

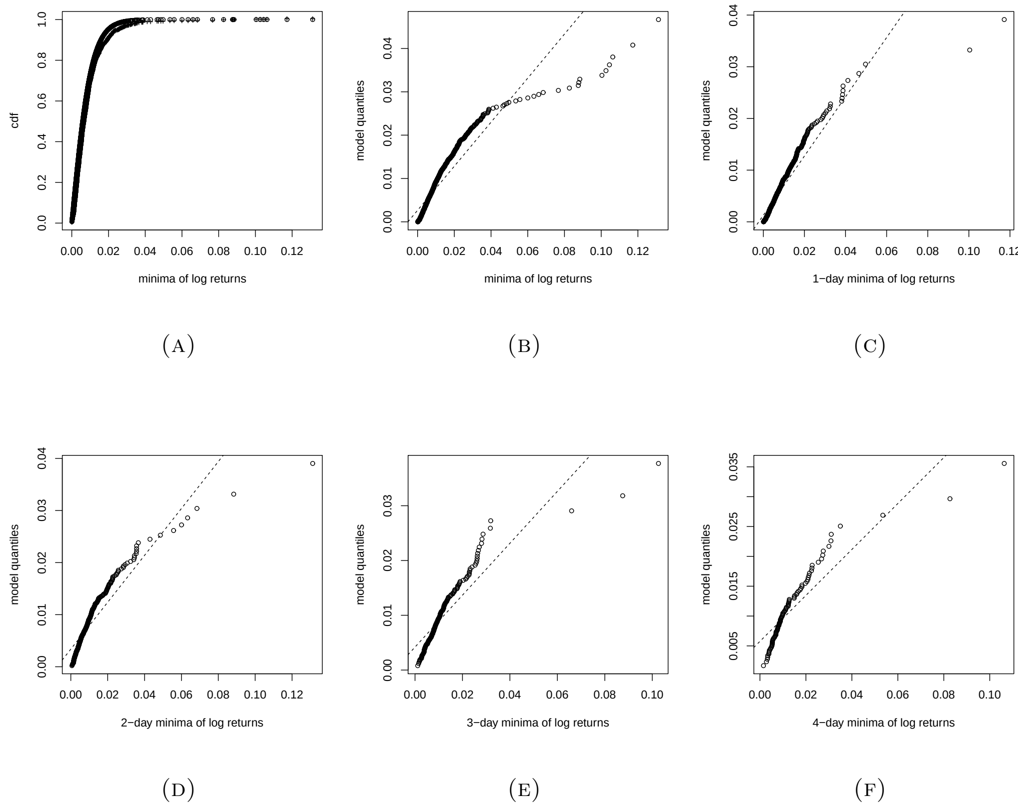


FIGURE 109. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

8.5.2. *BOA:-Weekly log-returns (decline episodes).*

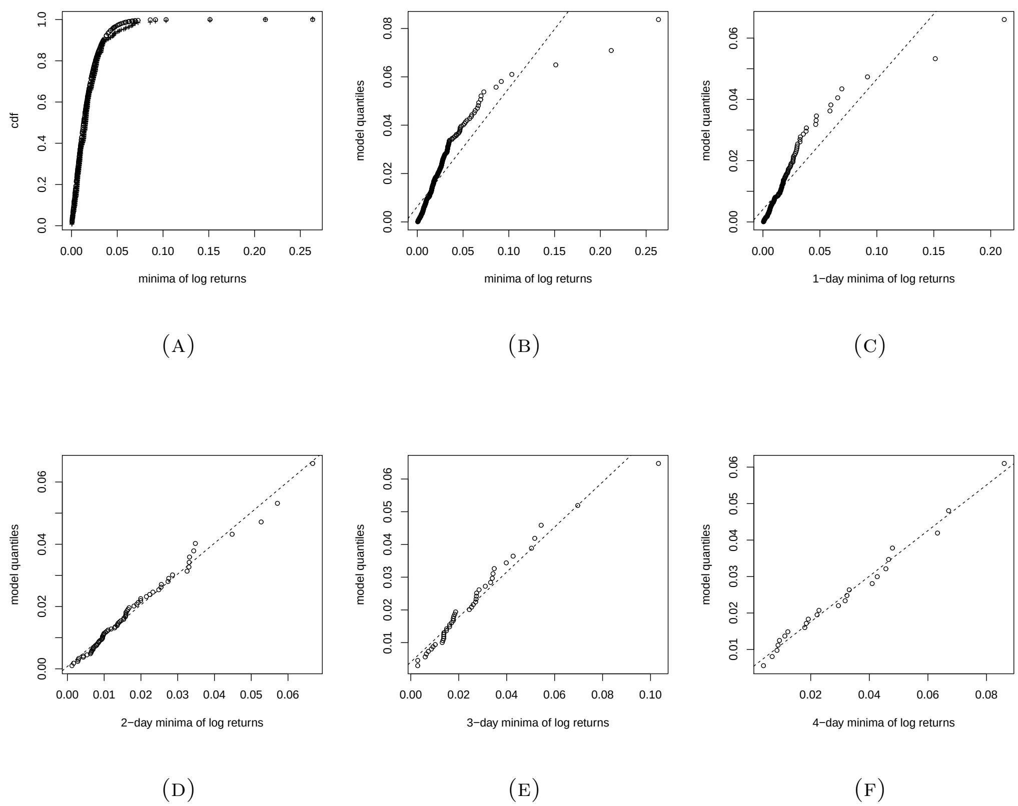


FIGURE 110. (A) Model (dotted) and empirical c.d.f.'s of minima, (B) Probability plot of minima versus theoretical quantiles of truncated logistics, (C) Probability plot of minima versus generalized exponential quantiles for $N = 1$, (D) Probability plot of minima versus generalized exponential quantiles for $N = 2$, (E) Probability plot of minima versus generalized exponential quantiles for $N = 3$, (F) Probability plot of minima versus generalized exponential quantiles for $N = 4$.

9. Summary

In this section we summarize the results of our data analysis for all assets considered. The results are preseted in Table (56). For each asset, we show whether *BEG* and *BTLG* models fit separately to growth and decline episodes. This is indicated by *Y (Yes)*, *N (No)* or *FY (Fairly Yes)*. We made the decision on the fit based on the graphical analysis presented in Sections 5, 6, 7 and 8. When the fit is very good, we categorize it as *Y*; whereas when the fit is very bad, it is categorized as *N*. In instances where the fit is reasonably good, we categorized it as *FY*. Stability property is evaluated based on probability plot of original log returns versus cumulative log returns (i.e., considering positive and negative log returns separately). If the probability plot yields a straight line, we categorized it as stable (*Y*), otherwise we categorized it as not stable (*N*). If the probability plot yields a reasonably straight line, we categorized it as fairly stable (*FY*).

The last column titled (*Overall*) shows our final assessement of fit of *BEG* and *BTLG* models to an asset.

Assets	Growth episodes BEG	Growth episodes BTLG	Growth episodes Stability	Decline episodes BEG	Decline episodes BTLG	Decline episodes Stability	Overall
FOREIGN EXCHANGE RATES							
US Dollar	Y	Y	Y	Y	Y	Y	Y
Swiss Franc	Y	FY	Y	FY	Y	Y	Y
Swedish Krona	FY	N	N	FY	FY	Y	FY
Norwegian Krone	FY	N	Y	FY	Y	FY	FY
Canadian Dollar	Y	Y	Y	Y	Y	Y	Y
Australian Dollar	FY	FY	FY	N	FY	FY	FY
Deutsche Mark	Y	FY	Y	N	FY	FY	FY
COMMODITIES							
Daily Gold	FY	FY	Y	FY	FY	Y	FY
Weekly Gold	Y	Y	Y	FY	Y	FY	Y
Daily Oil	FY	FY	Y	FY	FY	FY	FY
Weekly Oil	FY	FY	FY	FY	Y	N	FY
STOCK INDEXES							
Daily S&P500	FY	FY	Y	FY	FY	Y	FY
Weekly S&P500	Y	N	Y	FY	FY	Y	Y
Daily NASDAQ-100	FY	N	Y	FY	FY	Y	FY
Weekly NASDAQ-100	Y	N	Y	FY	Y	N	FY
STOCKS							
Daily BOA	N	N	N	N	N	Y	N
Weekly BOA	N	N	N	N	N	Y	N
Daily Chevron	N	N	N	FY	N	FY	N
Weekly Chevron	N	N	N	Y	N	Y	N

TABLE 56. Summary of results of fitting **BEG** and **BTLG** models to growth and decline episodes financial data sets. Here, **Y** stands for **YES**, **N** stands for **NO** and **FY** stands for **FAIRLY YES**.

In this thesis work, we generally observe that **BEG** and **BTLG** models are a good fit to the growth and decline episodes of foreign exchange rates. In particular, the **BEG** and **BTLG** fit remarkably well to the growth and decline episodes for *US Dollar, Canadian Dollar* and *Swiss Franc*. The **BEG** and **BTLG** models fairly fit to the growth and decline episodes of *Norwegian Krone, Swedish Krona, Australian dollar* and *Deutsche mark*. Additionally, these curriencies exhibit a remarkable stability property.

The **BEG** and **BTLG** models fairly fit to the growth and decline episodes of all the commodities and stock indexes considered for this thesis work except for weekly S&P500 indexes and weekly gold prices where we observe a remarkable fit.

On the other hand, the **BEG** and **BTLG** models do not fit to the growth and decline episodes of individual stock prices considered. The decline episodes, though, exhibit a remarkable stability property.

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