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# An Investigation of Student Learning in Beginning Algebra Using Classroom Teaching Experiment Methodology and Design Research

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## Abstract

Students in grades 6-8 often struggle with learning expressions, equations, and functions (NCTM, 2011). This study investigated what sense sixth-grade students make when solving algebra tasks presented in a whole class teaching experiment (Lamberg & Middleton, 2009; Middleton, Gorard, Taylor, & Bannan-Ritland, 2008; Steffe & Thompson, 2000) using design research (Cobb, 2000; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Cobb & Yackel, 1996; Gravemeijer, 1994).

The teaching episodes were video recorded and all student work was documented and analyzed. The teaching experiment was an iterative process conducted in three phases. Data collection and analysis was a parallel process with prospective analysis occurring throughout the teaching experiment and retrospective analysis occurring after the teaching experiment.

This research developed theories about the students' learning process in algebra, as well as techniques designed to support their learning. The instructional unit that was used in this study was developed with the goal of promoting student learning of the algebra tasks and was modified to further student understanding of the tasks. The realized learning trajectory for extending arithmetic to algebraic expressions, solving one-variable equations, and representing functions was also documented.

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# CHAPTER I: INTRODUCTION AND STATEMENT OF THE PROBLEM

Algebra, in its most simple form, is a branch of mathematics that generalizes arithmetic by using a symbolic language, substituting letters for numbers. This area of mathematics has been of interest in human problem solving research for several decades (Kieran, 1989). Traditionally, in the United States, the teaching of algebra consists of two or more courses that students begin in secondary school and often repeat in a remedial math course in post-secondary school (Kaput, 1995). This repetition occurs for some post-secondary students because they do not have a deep understanding of algebra. Moreover, only a small percentage of students that repeat algebra attain proficiency when they take the class again (Finkelstein, Fong, Tiffany-Morales, Shields, & Huang, 2012). The preliminary topics in school algebra usually consist of variables, simplification of algebraic expressions, equations in one unknown, and equation solving (Kieran, 1989).

In general, traditional approaches to teaching algebra begin by devoting time to learning skills and procedures before applying them to problems (French, 2002). Procedural algebra, or the "drill" approach, involves incremental steps and trains students using redundant practice of similar problems. This stems from a prevailing tradition present in algebra where the teacher introduces students to a new topic by demonstrating it using an example, and then students practice similar problems using the same procedure introduced by the worked example (French, 2002). As a result, students in grades 6-8 often struggle with learning expressions, equations, and functions (NCTM, 2011) because students are learning a procedure for solving problems, and not conceptually understanding why the procedure works.

A position statement from the National Council of Teachers of Mathematics (NCTM, 2008) asserted that algebra is more than a set of procedures for manipulating symbols. According to Kirshner (1993), the drill approach "trains students in non-reflective competence" (p. 3). Alternatively, the other approach is to begin with the problem, identify and learn the necessary skills and procedures needed to solve it, then think through and make sense of the problem (Hiebert & Lefevre, 1986; Star, 2000). Although procedural algebra drills provide students with a systematic way to solve mathematical problems, a broader conceptual understanding of algebra is necessary to develop algebraic thinking (Kamol & Har, 2010). It is possible to thoroughly integrate rules, procedures, algorithms, sense making, and meaningful problems into the algebra learning process (Friedlander & Arcavi, 2012). This broader conception of algebra, rather than skillfully using algebraic procedures, helps students make connections, generalize, and represent relationships.

## Problem Statement and Research Question

Learning mathematics with understanding means that students have conceptual, factual, and procedural knowledge (Bransford, Brown, & Cocking, 2000). Conceptual, factual, and procedural knowledge are not independent; they are interwoven. The National Research Council (NRC, 2001) contends that mathematical proficiency is achieved through the interdependent components: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (p. 117). Mathematical thinking and learning requires that students understand not only what they are doing, but also why they are doing it. The Common Core State Standards (CCSS) for Mathematics (National Governor's Association/Council of Chief State School Officers

[NGA/CCSSO], 2010) emphasize that learning in mathematics should take into account a balanced combination of procedure and understanding. Students that understand mathematics should not rely on only procedural knowledge, but conceptual knowledge as well. Moreover, students' learning about the context of the mathematics, as well as the relationships between operations, numbers, and symbols, is crucial.

**Rationale and Significance**

The Common Core State Standards for Expressions and Equations state that students in grade 6 should learn to (a) apply and extend previous understandings of arithmetic to algebraic expressions, (b) reason about and solve one-variable equations and inequalities, and (c) represent and analyze quantitative relationships between dependent and independent variables (NGA/CCSSO, 2010). Kaput (1999) described the understanding of algebra as a "rich web of connections" (p. 4) that involve five interrelated forms of thinking: (a) generalizing arithmetic to algebra, (b) using symbols in a meaningful way, (c) study of structure, (d) study of patterns and functions, and (e) mathematical modeling and combining the first four forms. Clearly, students need more than a procedural understanding of algebra in order to participate in these forms of algebraic thinking. However, the five interrelated forms of algebraic thinking are not well represented in classrooms where traditional algebra is taught (Banerjee & Subramaniam, 2011; Drijvers, Goddijn, & Kindt, 2011; Kaput, 2000). Recognizing that students taught traditional algebra face many difficulties with learning algebra due to lack of exposure to the five interrelated forms of algebraic thinking, this study aimed to develop a teaching approach that helps beginning students conceptually understand algebra.

A whole class teaching experiment (Lamberg & Middleton, 2009; Middleton, Gorard, Taylor, & Bannan-Ritland, 2008; Steffe & Thompson, 2000) on algebra using design research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Cobb, 2000; Cobb & Yackel, 1996; Gravemeijer, 1994) was conducted. The teaching experiment occurred in a sixth-grade classroom. The idea for this teaching experiment came out of discussions with the classroom teacher, the teacher-researcher, and a mathematics education professor. This study was motivated by the researcher's desire to help students gain an in-depth understanding of beginning algebra, taking into account that increased understanding of how students make sense of beginning algebra might not only generate theory on how students learn algebra, but also advance curriculum development, pre-service teacher preparation, and in-service teacher professional development. The students that participated in this study had never formally been introduced to variables or algebra. The instructional unit that was used in this study was researcher-developed with the goal of promoting student learning of the algebra tasks. It was modified to further student understanding of the tasks, based on how students interacted with the tasks in the unit.

The purpose of this study was to document and understand how sixth-grade students think mathematically when presented with algebra tasks in a teaching experiment. The following questions guided this study: how do sixth-grade students think mathematically when solving arithmetic and algebra tasks in a whole class teaching experiment and what are the means of supporting and organizing student learning of algebra?

## Summary

Chapter one is an introduction to the study that includes an overview of the background of the study and also the need to better understand how beginning algebra students make sense of algebra tasks. The five interrelated forms of algebraic thinking are presented as a framework that was used in this study for development of an instructional unit. Motivation and rationale for the exploration into how students make sense of algebra tasks are also discussed.

Chapter two is a review of literature that examines prior research in the learning and teaching of mathematics. This review informed the curriculum and tasks in the teaching experiment, designed to explore how to support sixth-grade students' understanding of algebra as outlined in the Common Core State Standards for Mathematics (NGA/CCSSO, 2010). Specifically, the researcher created a hypothetical learning trajectory (Simon, 1995) and investigated the realized learning trajectory (Lamberg & Middleton, 2009) that emerged as a result of the instructional unit. Additionally, the researcher intended to understand what strategies and mathematical reasoning practices students used to make sense of the algebra tasks.

The first section in chapter two is a review of the research on how students learn mathematics and how to support learning in mathematics. The next section summarizes what beginning students should know and be able to do in algebra and how students learn algebra. Then, a review of hypothetical learning trajectories and curriculum development is discussed. This review of literature also includes the theories of social constructivism and community of learning as they relate to learning in mathematics. Chapter two concludes with a discussion on the need for continued research on how students make sense of beginning algebra tasks and includes the research goals for this study.

Chapter three provides the methods for data collection and analysis in the teaching experiment. It also describes the frameworks that were used to analyze and interpret the data. Chapter three concludes with a discussion of objectivity, reliability, and validity in design research.

Chapter four presents the research findings. This chapter begins with an introduction of the context of the classroom community. Next, the realized whole class learning trajectory is described in detail, including students' understanding of the algebra tasks, examples of student work, and the role of the tasks in supporting student learning. The results of the pretest and posttest analysis are also provided. Chapter four continues with findings that illustrate how students made connections from arithmetic to algebra and how students viewed the relationship between a variable as a known value and a variable as an unknown value. Chapter four concludes with a visualization of how the lesson plans in the instrucitonal unit were modified during and after the teaching experiment.

Chapter five presents the conclusions on learning and teaching mathematics from the teaching experiment using design research. This chapter also discusses the practical and theoretical relevance of the results of the teaching experiment. Additionally, chapter five suggests that the realized learning trajectory be the revised hypothetical learning trajectory for future research.

# CHAPTER II: REVIEW OF LITERATURE

This section describes the current research on learning and teaching mathematics from the perspective of a social situation in the classroom or the classroom microculture (Cobb, 1999). The purpose of this review is not only to examine how students learn to think mathematically, but also to examine how students engage in algebraic thinking. This review synthesizes the research on how students learn mathematics within a classroom setting and was used to inform the instructional unit. This unit was used in the teaching experiment and was designed to support student learning in algebra.

First, a general review of the research on how students learn mathematics and how to support learning in mathematics is provided. The next section considers what beginning students should know and be able to do in algebra and how students learn algebra. Additionally, a review of hypothetical learning trajectories and curriculum development is discussed. This review of literature also discusses the theory of collective mathematical development defined as classroom mathematical practices that are established by the classroom community (Cobb, 1999). This chapter concludes with a discussion on the need for continued research on how students make sense of beginning algebra tasks and includes the research goals for this study.

## Learning Mathematics

It is important to understand mathematical thinking and learning in order to develop a curriculum that supports student learning in algebra. Learning takes place in the context of the classroom and the student experiences learning mathematics by interactions with the teacher, students, and content (Gee, 2007). The way in which students experience mathematics within the classroom setting, the content that they learn, and the curriculum that is used shapes the learning that takes place (Elsaleh, 2010). Traditionally, in mathematics, the curriculum materials are viewed as the main policy foundation that improves instruction and learning on a large scale (Stein & Kaufman, 2010). However, the nature of teachers' instruction combined with how the curriculum is implemented in a mathematics classroom is actually the factor that influences what student learning takes place (Brown, 2009). Stein and Kaufman (2010) provided evidence that no direct causal relationship exists between curriculum and student learning in mathematics; rather, there is a connection between student learning and how a teacher instructs students using the curriculum.

A curriculum contains tasks that provide opportunities for learning. Curriculum influences student learning based on how the teacher chooses to incorporate the curriculum and the opportunities presented within the curriculum. These interactions with teachers, students, and content occur within the classroom community that is made up of students and the teacher (Cobb, 1999). The factors that influence mathematical learning are embedded within the mathematical practices established by the classroom community in which members of the community explain and justify solutions, focusing "on the taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas" (Cobb, 1999, p. 9). Therefore, classroom mathematical practices depict changes in whole class mathematical thinking, but also consider individual students' understanding.

Students in the classroom are part of the "culture of mathematical learning," or "semiotic domain" (Gee, 2007, p. 24). A semiotic domain is "grounded in a group of people who have cognitive and social interests and help uphold a set of standards and norms" (Gee, 2008, p. 137). In other words, individuals within a semiotic domain have their own associated way of thinking and interacting. Figure 1 presents a framework for a community of learning where mathematical thinking takes place. This framework takes into account that mathematical thinking occurs through interactions among teachers, students, and content. A community of learning must exist in a mathematics classroom for students to engage in mathematical thinking. The next section explains this framework in detail.

**Figure 1.** Framework for a Community of Learning

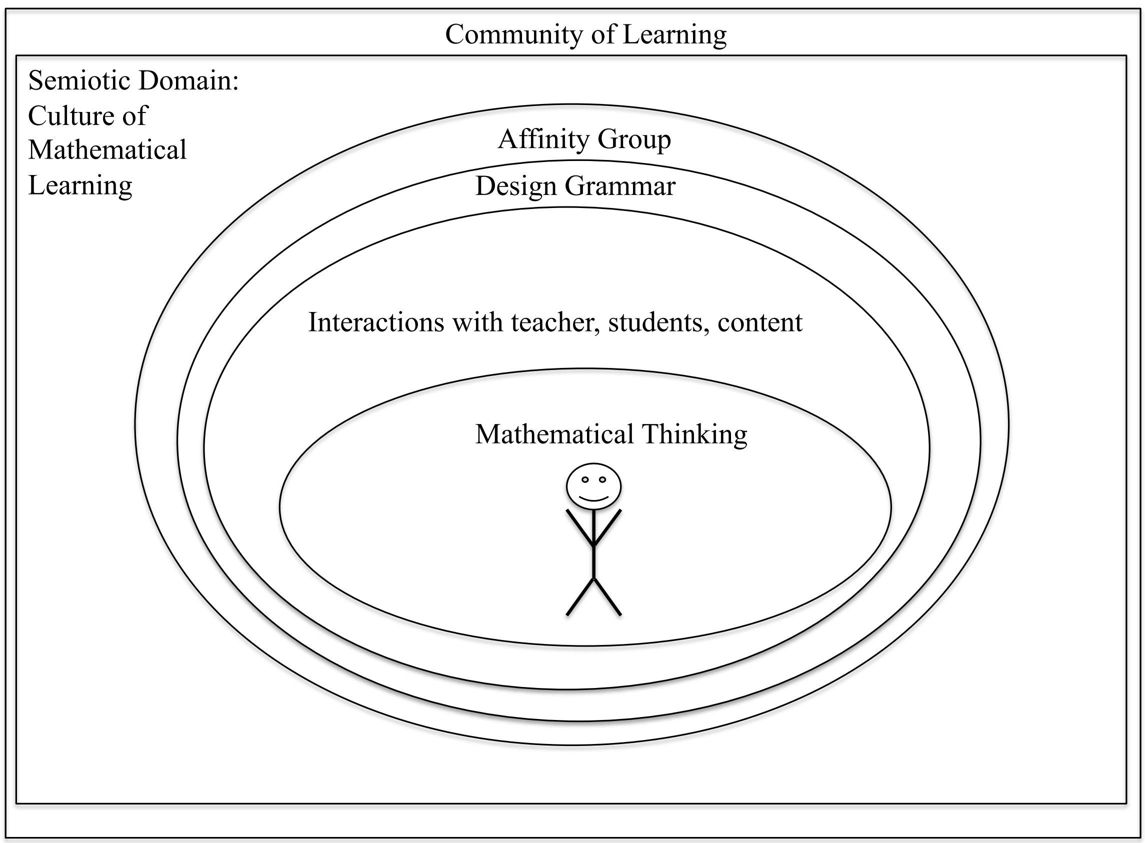



*Figure 1.* Mathematical thinking occurs when interactions and discussion, using the design grammar of the affinity group, between teachers, students, and content exist. The affinity group is within the semiotic domain of mathematical learning (adapted from Gee, 2004).

**Learning and Thinking in Mathematics**

Mathematical thinking is a state of mind in which the thinker has a conceptual understanding of the mathematics and has become part of the community of mathematics. The authors of *Adding It Up* contend that all students have the ability to learn to think mathematically (NRC, 2001). Devlin (2012) defines mathematical thinking as:

> Mathematical thinking is more than being able to do arithmetic or solve algebra problems. In fact, it is possible to think like a mathematician and do fairly poorly when it comes to balancing your checkbook. Mathematical thinking is a whole way of looking at things, of stripping them down to their numerical, structural, or logical essentials, and of analyzing the underlying patterns. Moreover, it involves *adopting the identity* of a mathematical thinker. (pp. 59-61, emphasis in the original)

A "semiotic domain" (Gee, 2004) within the community of learning mathematics is the culture that is established through interactions among community members. In order to be completely a part of a semiotic domain, a person must join an "affinity group" and know the "design grammar" used in the domain that allows communication about information, values, and ideas that are specific to the domain (Gee, 2004, 2007). In a classroom setting, the students that belong to the community of mathematical learning have their own way of communicating and interacting with each other, the teacher, and the content**.** Members of the semiotic domain that build a relationship based on a shared interest form an affinity group. In other words, affinity groups are the small groups that make up a larger community (Gee, 2004). For instance, consider the community of snowboarding in which separate affinity groups exist. Three snowboarding affinity groups are free ride snowboarders, free style snowboarders, and free carve snowboarders. Each group of snowboarders requires a specific type of equipment and gear and has its own set of fans and competitions. Although many people snowboard, they are not part of an affinity group until they have the equipment and style and can communicate using vocabulary specific to snowboarding with the other members of the group. This communication requires knowledge of the design grammar of the affinity group. In particular, the affinity group of free style snowboarders uses specific design grammar that includes distinct terminology such as half pipes, tabletop jumps, rail slides, spines, hips, and quarter pipes (terrain park attractions). Free ride snowboarders and free carve snowboarders may not know the design grammar of free style snowboarders and vice versa.

Semiotic domains in the community of mathematical learning are composed of content, academic level, or interest-specific affinity groups that communicate through the language specific to the mathematics topic area. It is problematic when students are not mastering the semiotic domain in school mathematics. Instead, they are learning a series of skills and details without learning the design grammar, and thus, are not able to join an affinity group associated with the domain. Mathematical thinkers must understand how meaning is constructed by the design grammar of mathematics. In other words, to think mathematically, a person must be able to understand and communicate in the domain of mathematics, and not just memorize facts or repeat details without knowing the meaning of the design grammar.

**Communication.** Gee's (2004) theory of semiotic domains, the Common Core State Standards (CCSS) for Mathematics (NGA/CCSSO, 2010), and the National Council of Teachers of Mathematics Standards (NCTM, 2000) emphasized communication as essential for how students should learn mathematics (Lamberg, 2013). Communication occurs through student-student interactions, student-teacher interactions, and student-content interactions. Eisenhart (1988) suggested that mathematical learning is both a process of constructing meaning as an individual as well as a process of acculturation of mathematical meanings in society. This implies that students learn mathematics individually but also by interacting with others. Cobb, Yackel, and Wood (1992) described communication as a process where learning occurs during social interaction, which is a circular sequence of events. In other words, learning as social interaction is not a linear process; it is messy and complex. Moreover, the theory of social constructivism (Vygotsky, 1978) reflects this perspective where the central tenet is that learners construct their own understanding by participating in meaningful shared discourse. Vygotsky (1978) wrote, “Every function in the child’s cultural development appears twice: first, on the social level, and later, on the individual level”. This theory takes into account the critical role of the social nature of language and culture in student learning.

**Figure 2.** Continuum of the Zone of Proximal Development

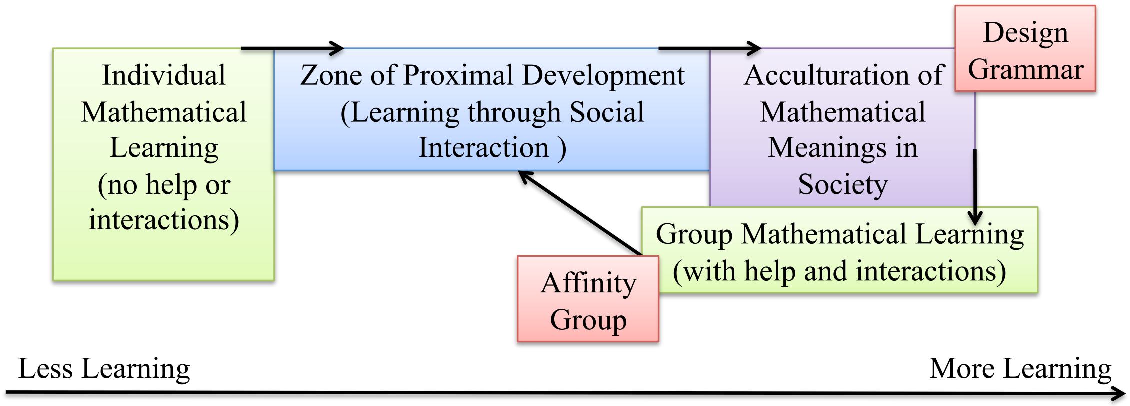



*Figure 2*. The continuum of the zone of proximal development shows that students learn more through social interaction. Social interaction leads to the acculturation of mathematical meanings through the use of design grammar (Gee, 2004; 2007). Group mathematical learning forms an affinity group that leads back to the Zone of Proximal Development (Vygotsky, 1978).

Additionally, Vygotsky's (1978) Zone of Proximal Development (ZPD) is based on the idea that students are limited in what they can learn independently and that more can be learned with guidance of teachers or collaboration with others (Carlile & Jordan, 2005). In a community of mathematical learning, students learn the design grammar of an affinity group through interactions with teachers, students, and content. The design grammar in a mathematics classroom is the shared meaning of mathematical conventions and vocabulary that students and teachers develop through their experiences and discussions. Knowledge of the design grammar allows for mathematical discourse and the opportunity for learning to occur. These interactions, which are largely accomplished through discussion, make up the ZPD. Figure 2 depicts the ZPD and how it relates to the process of constructing meaning by becoming part of an affinity group and using design grammar.

Upon entering the ZPD, students learn through social interaction. This interaction leads to acculturation of mathematical meanings in society through the knowledge of the design grammar associated with the semiotic domain. Once students engage in group mathematical learning, by participating in the culture and communicating using design grammar, they are part of the affinity group. In an ideal learning community, this cycle continues and contributes to the construction of new knowledge (Steinbring, 2005).

In education, the culture of mathematical learning consists of an affinity group that practices the language and symbols, or design grammar, of this community. A person will not be able to become a competent mathematical thinker without joining the semiotic domain, or culture, of mathematical learning. In addition, Gee (2004) and Devlin (2012) concurred that entering a semiotic domain does not imply becoming an expert in that particular discipline, rather it implies achieving competency by interacting in the discipline associated with that domain.

**Teaching to Support Learning in Mathematics**

Teaching mathematics is a complex endeavor because mathematics involves a language, a way of thinking, a specific skill set, and a multitude of knowledge. Effective math teachers must be able to support the learning of their students by helping them reflect on their thinking, identify mistakes, and improve strategies. Horizon Research, Inc. (Weiss, Pasley, Smith, Banilower, & Heck, 2003) conducted a study that examined mathematics and science education in the United States. This study considered the main objectives of math education as helping students learn important mathematical concepts and deepening students' abilities to successfully engage in the mathematical process. After an in-depth analysis of lessons, Weiss, Pasley, Smith, Banilower, and Heck (2003) identified five factors that distinguish effective lessons from ineffective lessons. Effective lessons: (a) engage students with the mathematics content, (b) create an environment conducive to learning, (c) ensure access for all students, (d) use questioning to monitor and promote understanding, and (e) help students make sense of the mathematics content (p. 39). Within the semiotic domain of mathematical learning, the supportive learning community begins with teachers that cultivate this environment that facilitates effective mathematics lessons.

How students learn mathematics is directly affected by experiences in mathematics classrooms, and these experiences are created by and depend on the teaching of mathematics. Pedagogy influences the scope and nature of the mathematics that the students learn (Schoenfeld, 1992). Therefore, the teacher needs to select worthwhile mathematical tasks and involve students in discussion (NCTM, 2000). Furthermore, teachers must be able to understand what their students already know in order to connect new mathematical knowledge to their previous knowledge (NRC, 2001). This requires teachers to have a strong foundation in mathematics content, curriculum, and pedagogy (Ball & Bass, 2000). To teach mathematics effectively, teachers need to know the big mathematical themes and be able to present these as interconnected topics (Ma, 2010). Additionally, teachers should realize that many ways to teach and learn mathematics exist and they should incorporate a variety of methods into their pedagogy (Sheffield & Cruikshank, 2001). Teachers should augment their teaching beyond the textbook, continually reflect and improve, and be aware of how students are learning mathematics (Ma, 2010; NCTM, 2000; Sheffield & Cruikshank, 2001). Moreover, teachers should recognize that curriculum is not a static document to be implemented;

rather, it is a toolbox with tasks and lessons to be selected and adjusted by the teacher to support student learning.

Teaching mathematics is a complex process that should be viewed as more than an extensive list of topics. The *Principles and Standards for School Mathematics* (NCTM, 2000) articulated six principles that should be "deeply intertwined with school mathematics" (NCTM, 2000, p. 12). The six principles for school mathematics are: (a) Equity, (b) Curriculum, (c) Teaching, (d) Learning, (e) Assessment, and (f) Technology. Thus, the teaching of mathematics is an intricate system that must consider multiple details linking content, student learning, environment, and equity. Figure 3 displays the complex network of teaching mathematics.

**Figure 3.** Network of Teaching Mathematics

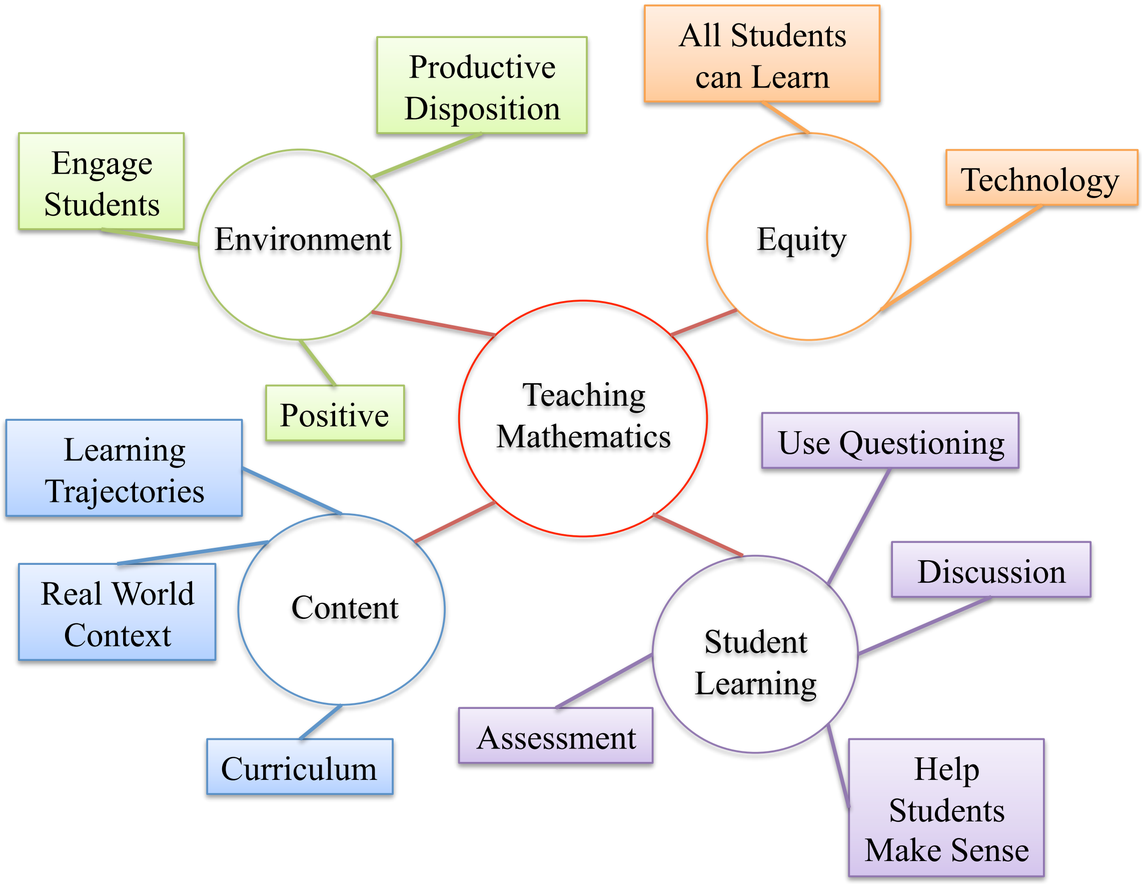



*Figure 3.* Teaching mathematics is a complex network that involves student learning, equity, environment, and content.

**Content.** Mathematically proficient students are able to skillfully compute, apply, understand, reason, and engage in mathematics (NRC, 2001). The NCTM (2000) identified the content strands that students at each grade level should learn: numbers and operations, algebra, geometry, measurement, and data analysis and probability. The CCSS in Mathematics contains content standards similar to the NCTM Standards; however, the CCSS also provides progressions, specific to each grade level, in which the content should be learned (NGA/CCSSO, 2010).

***Learning trajectories.*** A hypothetical learning trajectory (HLT) (Simon, 1995) is a model of student learning that consists of the goal for students' learning, the tasks that will be used to promote students' learning, and hypotheses about the process of this learning. The National Research Council (NRC, 2007) defined learning progressions as "successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (p. 214). Moreover, the CCSS in Mathematics (NGA/CCSSO, 2010) stated that the standards emerged from, "research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (p. 4). Reconceptualizing a mathematics concept to lead to student learning and understanding is challenging (Simon & Tzur, 2004). For instance, development of a HLT requires the teacher or teacher-researcher to have a solid understanding of the mathematics content and the current knowledge of the students in order to make hypotheses about the process of student learning and to select learning tasks based on these hypotheses.

Middleton et al. (2008) described the notion that research in education is similar to the process of design in engineering except that the products in education research are learning environments (Scardamalia & Bereiter, 2006). These enterprises are similar in that engineers and education researchers must continually keep the outcomes of a prototype product in mind (Lamberg & Middleton, 2009). These outcomes in education are the hypothetical learning trajectories that specify crucial understandings of the content and map out tasks that hypothetically move students to deeper understandings of the content. Moreover, the "function structure" of a system is a hypothesized model that designates the important components of a system and how they relate and interact

(Lamberg & Middleton, 2009, p. 243). Thus, the primary function structure in education research is the system that aims to advance the goal of learning (Scardamalia & Bereiter, 2006).

A hypothetical learning trajectory must be developed, tested, and refined to produce a realized learning trajectory (RLT) (Lamberg & Middleton, 2009). The RLT results from the HLT and shows what actually happened in the education research. The RLT explains what occurred in the research with respect to the goal of promoting learning, why it happened, and how it happened. Once a RLT is established, it can then become the HLT and be retested and redeveloped to find a new and improved RLT. In other words, the realized learning trajectory represents a replacement of the original hypothetical model (Steffe & Thompson, 2000).

***Curriculum.*** A quality curriculum is essential to support student learning in mathematics because it determines what mathematical ideas and content students will have the opportunity to learn. Curriculum needs to encompass both content and opportunities for reflection and assessment. A curriculum is coherent if it leads to the development of big mathematical ideas by linking content, focusing on important mathematics, and articulating across grade levels (NCTM, 2000). For example, students should learn that mathematics is more than unrelated and disconnected topics; rather, it is a highly organized body of knowledge. Moreover, the NRC (2001, 2005) established that the teaching and learning of mathematics should emphasize problem solving and sense making rather than isolated procedures and skills. Thus, curriculum must contain challenging content and assessment opportunities, as well as connections among mathematical ideas throughout grade levels.

Curriculum is developed around essential mathematical content goals, linked throughout each grade level. Ernest (1998) argued that complete understanding of the objectives of mathematics education is not feasible without taking into consideration the philosophy of mathematics. In order to answer the question, “what is the purpose of teaching and learning mathematics” (Ernest, 2004, p. 17), teachers, researchers, and curriculum planners must explore the role that mathematics plays in society (White-Fredette, 2010). Standards-based curricula reflect the views of the NCTM (2000) standards and embody a constructivist philosophy of student learning.

Constructivism advocates for students to construct their own knowledge by reorganizing, integrating, and assimilating new procedures and concepts into their existing knowledge (Chomsky, 1968; Piaget, 1954), and social constructivism asserts that learning mathematics is something that people do, instead of something that people attain (Forman, 2003). Social constructivism in mathematics education emphasizes communication and participation in a mathematical learning community (Cobb, Yackel, & Wood, 1992). In order to reflect the role of mathematics in society, curriculum must take into account how mathematics is used in the real world and why it is important to learn mathematics.

According to The NCTM (2000), mathematics education consists of two types of curricula: traditional curricula and standards-based curricula. Traditional curricula, which are teacher-centered, emphasize mathematical procedures more than conceptual understanding. Standards-based curricula, which are student-centered, emphasize conceptual understanding, problem solving, and connections among mathematical topics. Textbooks are central in current mathematics education and, therefore, The NCTM

(2000) calls for the use of textbooks that incorporate meaningful and interesting tasks, while the CCSS (NGA/CCSSO, 2010) asserts that curriculum designers should strive to connect mathematical practices to mathematical content in order to teach students how to be mathematically proficient. Integration of mathematical content with mathematical practices in curriculum will produce a foundation for mathematical thinking and learning.

**Environment.** In an environment conducive to learning mathematics, the teacher plays a vital role. The teacher not only needs to use interesting and engaging mathematics problems, but also encourage discussion and provide representations of multiple methods, support conceptual understanding, and encourage mathematical thinking (Picone-Zocchia & Martin-Kniep, 2008). The NRC (2001) contended that students must develop a "productive disposition" toward mathematics and believe that they are capable of learning and using mathematics (p. 131). Teachers that have a productive disposition are confident doers of mathematics and encourage and support their students. Students that are learning in a positive environment should feel comfortable expressing their mathematical approaches and engaging in problem solving. Moreover, mathematics teaching should not focus entirely on the content, but should consider the interactions that occur among teachers, students, and content as well (Cohen & Ball, 1999). These valuable interactions set the stage for productive mathematical thinking and learning, as well as expose mathematical misconceptions.

Teachers can create an environment conducive to learning by providing students with problems that can generate many solutions. Students' justifications for their responses can be used as a basis for the content of the lesson. Teachers that encourage students to volunteer and explain their solutions create a positive environment in which students feel comfortable describing their mathematical thinking, even if the discussion exposes errors (Czarnocha & Maj, 2008).

**Equity.** Ensuring access to mathematical learning for all students requires high expectations and strong support for all students (NCTM, 2000). High expectations mean that teachers provide mathematics problems that are challenging and unfamiliar. The problems should slightly exceed a student's skill level, so that they do not become routine (Picone-Zocchia & Martin-Kniep, 2008). Equity requires high expectations, accommodation of differences, and high quality instructional programs for all students. The NCTM (2000) clarified that equity does not mean that all students should receive the same instruction; rather, it calls for "reasonable and appropriate accommodations as needed" to support the learning of each student (p. 12). In other words, all students should be provided access to mathematics teaching that is responsive to their prior knowledge, intellectual strengths, and personal interests (NCTM, 2000). Moreover, the equity principle makes clear that all students can learn mathematics, have the opportunity to use technology to enhance their learning of mathematics, and receive support from the teacher as well as the curriculum so that they have the opportunity to excel.

***Technology.*** In learning mathematics, calculators and computers are essential because they allow students to explore and solve problems that might otherwise be inaccessible or would be overly tedious. In addition, with the use of technology, students can examine visual representations of abstract mathematics. These representations aid student learning by generating visualizations of mathematics that students are unable or unwilling to create (NCTM, 2000). Moreover, technology can help students learn mathematics by providing access to more examples and dynamic representations.

Geogebra is an example of an interactive algebra, geometry, and calculus tool that students use to construct geometric figures and explore conjectures through the use of algebra (Hohenwarter & Borcherds, 2012).

The CCSS (NGA/CCSSO, 2010) expected mathematically proficient students to use appropriate tools strategically. These tools include technology such as calculators, spreadsheets, computer algebra systems, statistical packages, and dynamic software. Students should learn that technology can help them visualize and explore different types of mathematics and they should know which technology is appropriate. The NCTM (2000) standards and the CCSS (NGA/CCSSO, 2010) contended that technological tools support student learning by deepening their understanding of mathematical concepts. Additionally, touch screen technology might potentially allow students to engage in relational thinking because mathematical expressions can be moved as virtual physical objects (Ottmar & Landy, 2012).

**Student learning.** It is important for students to have a balance of factual, procedural, and conceptual knowledge to learn mathematics (NGA/CCSSO, 2010). Moreover, learning mathematics with understanding requires that students have knowledge of the relationships among numbers, symbols, and operations, as well as the context. Students can be given the opportunity to learn mathematics in context in a variety of ways. Inoue and Buczynski (2011) promoted the use of open-ended questions because they can incorporate contextualized, interesting situations that require both conceptual and procedural mathematics. Open-ended questions can have multiple approaches and solutions that require students to engage in mathematical thinking. Contextual approaches to learning mathematics fall between problem solving and procedural approaches. Picone-Zocchia and Martin-Kniep (2008) recommended that a contextual problem be introduced after students have learned the skills and procedures necessary to solve the problem. In the problem solving approach, a problem is introduced before skills and procedures to create a rationale for the mathematical content.

***Help students make sense of the mathematics content.*** Within the NCTM Standards are the process standards that consist of problem solving, reasoning and proof, communication, connections, and representation (NCTM, 2000). The purpose of the standards for mathematical practice are to make sense of problems and persevere in solving them, reason abstractly and quantitatively, construct viable arguments and critique the reasoning of others, model with mathematics, use appropriate tools strategically, attend to precision, look for and make use of structure, and look for and express regularity in repeated reasoning (NCTM, 2000; NRC, 2001). Although there are various systems for classifying and organizing the body of mathematics, a consensus among the mathematical teaching community exists on the attributes associated with successful mathematical thinkers (Picone-Zocchia & Martin-Kniep, 2008). Students that can effectively think mathematically know the meaning of mathematical symbols, representations, and procedures, and can comprehend mathematical concepts, operations, and relations (Picone-Zocchia & Martin-Kniep, 2008).

Helping students make sense of the mathematics content means that students are learning with understanding. Learning with understanding is more powerful than merely memorizing (NRC, 2001). Memorizing algorithms and rules does not foster understanding of mathematics because memorization does not produce connections or meaning. Rowan and Bourne (2001) recommended that students should be encouraged to build their "math power" (p. 3). To build math power, students create and use procedures that they can relate to and understand. In other words, students construct their own meaning of mathematical problems and the teacher guides this learning through insightful questions and discussion.

Students that have learned mathematics through constructing their own knowledge develop a conceptual understanding of mathematics. Students with a conceptual understanding of mathematics value the importance and context in which mathematics is useful and have the ability to connect new ideas to previous knowledge (NRC, 2001). Conceptual knowledge can be achieved by constructing connections among pieces of information or by creating relationships between existing knowledge and new information (Hiebert & Lefevre, 1986). Although students need conceptual understanding to make sense of mathematics content, they should also have a procedural understanding of mathematics. Procedural understanding includes knowing the formal language or system of mathematics. Procedural knowledge is associated with having a sequential nature as opposed to a relational nature (Hiebert & Lefevre, 1986). According to The NCTM (2000), in order for students to learn mathematics with understanding, they not only need to acquire procedural, computational skills, but also the ability to build new knowledge from prior knowledge. Moreover, students should understand that mathematical ideas are important, interrelated, and useful. In addition, students should be taught to connect procedures with conceptual knowledge so that the procedures are learned with meaning and that mathematics is more than a list of objectives.

Furthermore, students learn mathematics by attending to structure and thinking relationally. Although thinking about mathematical relations is implicit in correct computations, relational thinking allows thinking about these relations to become more explicit. Relational thinking occurs when a student studies two or more mathematical ideas and analyzes these ideas by the examination of connections among them (Molina, Castro, & Ambrose, 2005). This type of thinking is developed by guiding students to attend to how operations and numbers are related in a mathematical statement. Understanding relations among mathematical concepts is essential when learning mathematics (Hiebert & Carpenter, 1992). In relational thinking, understanding relationships among symbols is more important than arriving at the correct answer (Molina et al., 2005). Additionally, students can engage in relational thinking by treating mathematical equations and expressions as physical objects that can be moved according to mathematical rules (Ottmar & Landy, 2012).

***Use questioning to monitor and promote understanding.*** In learning mathematics, teachers and students should be asking questions. Although teachers should be asking questions to support different types of learning, they should also be listening to students' questions to understand students' thinking. Questions can be categorized as convergent or divergent. Convergent questions seek specific and correct answers, while divergent questions are generally open-ended and encourage critical thinking (Duron, Limbach, & Waugh, 2005). In Bloom's Taxonomy, convergent questions align with lower levels of knowledge, comprehension, and application, while divergent questions align with higher levels of analysis, synthesis, and evaluation (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956). For example, a convergent mathematics question might ask for the definition or formula of slope of a line, and a divergent question might ask for a justification or description of why that is the definition or formula of slope of a line. Divergent questions foster student-centered discussion and support critical thinking (Duron et al., 2005). By using questioning, teachers and students engage in discussion that provides them with the opportunity to participate in the community of mathematical learning.

Questioning also allows the teacher to discover the students' knowledge base and extend their knowledge by fostering learning of new ideas and understandings. Research posits that higher order, divergent, questions increase learning among students (Cotton, 2001; Walsh & Sattes, 2005). Fusco (2012) contends that teacher questioning has the greatest impact on student learning because it urges students to extend their thinking.

Moreover, teachers can support student learning in mathematics by listening to students' questions and helping students become better questioners (Boaler, 2002). Students' questions not only reveal how they are thinking about a problem, but also uncover any misconceptions or insights. Students should have plenty of opportunities and be prompted to formulate and ask different types of questions. Students will become familiar with different types of questions as long as the teacher is incorporating a variety of questioning techniques and encouraging discussion (Duron et al., 2005).

***Assessment.*** Assessment in mathematics refers to students' understanding, skill, and disposition toward mathematics. The NCTM (1995) presented six assessment standards for assessing students' understanding of concepts and procedures, enhancing mathematics learning, promoting equity, being an open process, promoting valid inferences, and being a coherent process (NCTM, 2000). These standards addressed the need for assessment to be intertwined in the routine of a classroom. Moreover, assessment should be ongoing and influence instructional decisions (Even, 2005).

Assessment informs the daily instructional decisions made by the teacher by monitoring students' progress, evaluating students' achievement, and making misconceptions known (Davis, 1996; NCTM, 1995). Student learning is assessed not only to inform the teacher (and others), but also to guide the students' learning (Even, 2005). Assessment supports teaching and learning by providing information about students' growth and development in their understanding of mathematics.

### Conclusions on Teaching and Learning Mathematics

It is essential that students learn how to think mathematically by being engaged in interactions with the teacher, other students, and the mathematical content. Thinking mathematically embraces more than just doing mathematics; it is a way of thinking. This skill is necessary in the modern world and can be practiced by participating in Gee's (2004) notion of a semiotic domain that contains an affinity group that applies a specific design grammar for communication among the group. Mathematical thinking can be thought of as a state of mind that is realized by joining a mathematical thinking community.

To support student learning in mathematics, teachers must focus on mathematical thinking and reasoning (NCTM, 2000). This focus begins with writing effective lessons that engage students, create a learning environment, are equitable, promote understanding, and help students understand the content. Moreover, teachers must adjust and utilize curriculum to promote student learning in mathematics.

## Learning Algebra

A wide variety of perspectives on the goals and approaches to algebra education exist (Drijvers, Goddijn, & Kindt, 2011). This worldwide discussion includes questions about how algebra should be taught, the integration of technology into algebra education, and the nature of school algebra (e.g., abstract, procedural, empirical, etc.). Debates, called “Math Wars”, are taking place in many countries and are centered on the goals, approaches, and achievements in mathematics education (Klein, 2007; Schoenfeld, 2004), including the relationship between conceptual understanding and procedural skills in the teaching and learning of algebra (Drijvers et al., 2011). According to the National Math Panel, students’ conceptual understanding and procedural understanding of algebra are intertwined (U.S. Department of Education, 2007), thus eliminating a central argument in the Math Wars: the importance of conceptual understanding versus procedural understanding in algebra.

**A Brief History of Algebra**

Algebra can be characterized simply; it has to do with numbers and structures (Drijvers et al., 2011). For example, $x + 2 = 5$ is an algebraic activity, whereas $3 + 2 = 5$ is an arithmetic activity. The former example has structure in that subtracting 2 from both sides of the equal sign can find the unknown quantity. In fact, the Persian mathematician, Al-Khwarizmi (830), considered to be one of the fathers of algebra, defined “al-jabr” as a method in which we can eliminate subtractions by adding the same quantity to each side of the equation (Pickover, 2009). His algebra book, *Kitab al-mukhtasar fi hisab al-jabr wa’l-muqabala* (*The Compendious Book on Calculation by Completion and Balancing*, 830) was the first book written on the systematic solutions to linear and quadratic equations (Pickover, 2009). For instance, $x^2 = 4x - 3x^2$ can be simplified to $4x^2 = 4x$ by adding $3x^2$ to both sides of the equation. Al-Khwarizmi (830) also described the method of “al-muqabala”, gathering like terms on the same side of the equation (Pickover, 2009). These techniques allowed people to simplify, or break down, complicated mathematical problems into smaller steps. Students continue to learn these ancient techniques, as well as extensions of these methods when working with algebraic expressions and equations.

Years before Al-Khwarizmi introduced his systematic solution of algebraic equations, Diophantus (250), a Greek mathematician (the other father of algebra), introduced the syncopated stage of algebra in his book *Arithmetica* (Pickover, 2009). In the syncopated style of writing equations, abbreviations are used instead of the full words. Diophantus is credited with introducing specific and consistent algebraic notations as well as treating fractions as numbers (Pickover, 2009). Therefore, he did not require that there be a whole number solution to an algebraic equation; he also accepted rational number solutions.

Generally, the historical development of algebra occurred in three overlapping stages: (a) the rhetorical stage, (b) the syncopated stage, and (c) the symbolic stage (Katz, 2006). During the rhetorical stage of algebra, all statements and arguments were verbal or written without any symbols. For instance, rhetorical algebra problems were similar to present day word problems. This stage began with the ancient Egyptians in 1650 BC and continued through 1500 AD European Algebra (Boyer, 1991; Kline, 1972). During the rhetorical stage, teaching algebra consisted of teaching by example and provided little reason or explanations for the given outcomes (Baumgart, 1989).

As already noted, Diophantus, a Greek mathematician, introduced some symbolism to algebra during the syncopated stage. Although this stage had been introduced, the rhetorical stage was more common for many more centuries. Traces of symbolic algebra began in the ancient Hindu civilization (800 BC), but became fully sophisticated in René Descartes' *La Géométrie* in 1637 (Pickover, 2009). In *La Géométrie*, Descartes demonstrated how geometric shapes and figures could be studied using algebraic representations. Moreover, Descartes proposed teaching algebra and geometry as one subject, a debate that is ongoing among mathematics educators. In the symbolic stage, mathematicians were deliberately using symbols to write algebraic sentences, as opposed to their incidental use, and before the formal manipulation of the symbols had been established according to algebraic rules (Katz, 2006).

**The new math.** A major period in the history of algebra in the United States involved the "New Math" era, which began in the 1950s and continued through the 1960s (Klein, 2003). The New Math period emerged as a result of disagreements between mathematicians and psychologists about procedural instruction and conceptual understanding in mathematics education (Bossé, 1995). Algebra in the New Math curricula focused on mathematical structure, formalisms, and abstract proof with minimal regard to basic skills or real-world applications. This curricula was subject to public criticism due in part to the reality that many teachers did not have the mathematical knowledge required to teach the rigorous content; moreover, the curricula were so theoretical in nature that it was deemed unnecessary because it did not relate to students' experiences (Klein, 2003). By the early 1970s, the National Science Foundation discontinued funding for these types of curricula; thus terminating the New Math.

## Meanings of Algebra

Presently, a distinction exists between what algebra means for mathematicians and what algebra means in school (Drijvers et al., 2011). For mathematicians, algebra is considered abstract and based on mathematical proof, and includes such vocabulary as groups, rings, and fields. However, algebra as taught in school is associated with symbolic rules and repetition, along with real-world applications.

**Algebraic Thinking**

The NCTM (2000) *Principles and Standards for School Mathematics* asserts that all students should learn algebra. The NCTM (2000) views algebra as a curriculum strand beginning in prekindergarten and the Common Core State Standards (CCSS) for Mathematics places emphasis on algebra and algebraic thinking beginning in kindergarten (NGA/CCSSO, 2010). By including algebra and algebraic thinking throughout the school curriculum, students have a solid understanding and foundation as they prepare for more advanced mathematics. Although The NCTM (2000) and the NGA/CCSSO (2010) recommend introducing algebraic topics in the early grades, the standards for grades 6-8 focus concentrated attention on algebra. The NCTM (2000) Algebra Standard states that students should be able to (a) understand patterns, relations, and functions, (b) represent and analyze mathematical situations and structures using algebraic symbols, (c) use mathematical models to represent and understand quantitative relationships, and (d) analyze change in various contexts (p. 222). These standards are found throughout the continuum from Kindergarten through twelfth-grade.

**Forms of algebraic thinking.** Algebraic thinking begins in Prekindergarten with recognizing and duplicating patterns with numbers (NCTM, 2006). It continues throughout high school with the central topics of generalizing and understanding patterns, studying change, and the concept of function (Van de Walle, Karp, & Bay-Williams, 2010). Therefore, many aspects of algebraic thinking are present. Kaput's (1999) five interrelated forms of thinking form a network that contains: (a) Generalizing Arithmetic to Algebra, (b) using symbols in a meaningful way, (c) study of structure, (d) study of patterns and functions, and (e) mathematical modeling and combining the first four forms. Figure 4 is adapted from Kaput's (1999) Five Aspects of Algebra and demonstrates that algebraic thinking has two levels within the five domains of algebra.

**Figure 4.** Framework for Five Forms of Algebraic Thinking

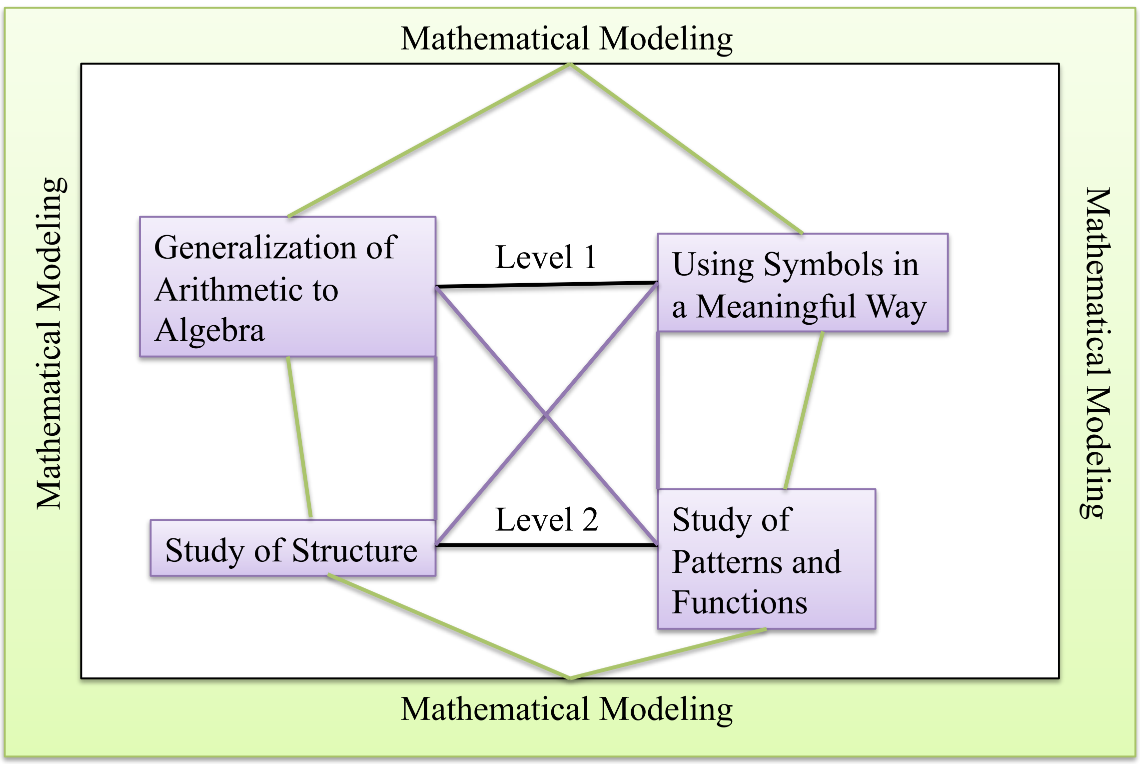



*Figure 4.* Network that displays the two levels of algebraic thinking and their relatedness to Kaput's (1999) five forms of algebraic thinking.

Level 1 occurs when students generalize arithmetic to algebra and develop the ability to use symbols in a meaningful way. Once students have become proficient in Level 1, they naturally move to Level 2, where they begin to study the structure of algebraic expressions and equations and study patterns and functions in algebra. Mathematical modeling permeates Levels 1 and 2 (Kaput, 1999). The five forms are related and interconnected and should be learned throughout elementary, middle, and high school. Although algebra is a separate subject that is taught during the middle grades, algebraic thinking is embedded in all areas of mathematics (Van de Walle et al., 2011). In the next sections, the Framework for Five Forms of Algebraic Thinking (Kaput, 1999) will be used to discuss how students learn algebra and what students need to know and be able to do in the middle grades.

**Framework for Five Forms of Algebraic Thinking**

**Level 1: Generalization of arithmetic to algebra.** Arithmetic involves computations with specific numbers, whereas algebra generalizes mathematical concepts and introduces abstraction. Students of arithmetic are concerned with the numerical value of the computation and students of algebra are concerned with applying mathematical laws and operations correctly to find all solutions. Algebra generalizes to arithmetic to find *all* solutions to a particular equation. Generalizing about the behaviors and properties of operations involves noticing patterns and regularities that are fundamental to arithmetic and algebra (Russell, Schifter, & Bastable, 2011). The example below provides an arithmetic problem that can be generalized to an algebraic problem.

**Figure 5.** The Generalization of Arithmetic to Algebra

| Arithmetic | Algebra |
|---|---|
| Check that: $(1\text{-}2)\ (1+2+2^2+2^3+2^4) = 1-2^5$ | Check that: $(1\text{-}x)\ (1+x+x^2+\ldots+x^n) = 1-x^{n+1}$ |

*Figure 5*. An example from Wu (2009) demonstrating the generalization of arithmetic to algebra.

To generalize from arithmetic to algebra, students must be able to think algebraically or abstractly. Kaput (2008) defined algebraic thinking as building, expressing, and justifying mathematical relationships. Algebraic thinking promotes awareness of implicit knowledge and this awareness reinforces understanding of the operations and patterns that can later be generalized to algebra.

Historically, the transition from arithmetic to algebra is difficult for students (Kieran & Chalouh, 1993; Knuth, 2000; Lee & Wheeler, 1989). One reason for this could be that their knowledge of the two subjects is not connected. For example, students learning algebra must move from arithmetic problems to symbolic representations of relationships with variables (Moseley & Brenner, 2009), requiring students to adapt their prior knowledge to new experiences. This requires the integration of symbols used in arithmetic (e.g., +, -, x, •, ÷, and =) to be used in the transition to algebra with variable expressions. Moseley and Brenner (2009) define this integration process using “the arithmetic schema” and “the algebraic schema” (p. 5). The arithmetic schema is defined as knowledge that is centered on procedures and relations among known quantities and algebraic schema is knowledge that is focused on operations and strategies applied to known and unknown quantities (Moseley & Brenner, 2009).

***An example of the arithmetic schema.*** The arithmetic schema requires the quantities to be known and operations are applied to find a numerical answer. For instance, a bag contains 30 blue marbles and 10 green marbles. Find the total number of marbles in the bag. This example requires the computation of 30 + 10 (Figure 6). Students solve this by addition and solely focus on the answer as opposed to the process (Booth, 1989).

***An example of the algebraic schema.*** The algebraic schema requires awareness of the processes of operations involving numerals and variables. For example, a bag contains a total of 40 marbles with 12 more blue marbles than green marbles. Find the number of green marbles and the number of blue marbles in the bag. In this problem, the number of each color of marbles is not numerically known and must be represented using variables (Figure 6). Solving this problem requires students to have a conceptual understanding of the problem, use variable equations and expressions, and apply processes of operations.

**Figure 6.** Arithmetic and Algebraic Schema Marble Examples

| Arithmetic Schema | Algebraic Schema |
|---|---|
| A bag contains 30 blue marbles and 10 green marbles. Find the total number of marbles in the bag. | A bag contains a total of 40 marbles with 12 more blue marbles than green marbles. Find the number of green marbles and the number of blue marbles in the bag. |
| 30 blue marbles<br>10 green marbles<br>total marbles = 30 + 10 = 40 | $b$ = blue marbles<br>$g$ = green marbles<br>total marbles = 40<br>$b = 12 + g$<br>$b + g = 40$<br>Substitute $b = 12 + g$ into $b + g = 40$ for $b$<br>$12 + g + g = 40$<br>Solve for $g$<br>$12 + 2g = 40$<br>$2g = 28$<br>$g = 14$<br>Substitute $g = 14$ into $b = 12 + g$ to find $b$<br>$b = 12 + 14 = 26$<br>$b = 26$<br>$g = 14$ |
| *Figure 6*. The arithmetic schema focuses on computation, which this example demonstrates is simply addition or counting. The algebraic schema focuses on processes and conceptual understanding. Notice the use of variables to represent quantities in the algebraic schema; without variables, students would have to guess and check repeatedly to find the solution. The algebraic schema allows for solving the problem, even if the total number of marbles was unknown. | |

***From arithmetic to algebra.*** Herscovics and Linchevski (1994) contended that beginning algebra students proceed from thinking arithmetically to thinking algebraically by thinking about and discussing numerical relations using everyday language and eventually representing these relations with variables. This transition involves using letters to represent numbers and knowledge of the mathematical operations and method symbolized by numbers and letters or variables (Kieran & Chalouh, 1993). In other words, algebraic thinking considers the method and process, whereas arithmetic thinking is focused on computation. Blanton (2008) illustrated a variety of ways arithmetic can be generalized to algebra that include (a) designing meaningful computation problems that lead to generalizations about operations and properties, (b) making known quantities unknown, (c) varying known quantities, and (d) building an algebraic view of equality (p. 29).

*Computation problems that lead to algebraic generalizations.* All properties in arithmetic can extend to algebra. These properties are true for all real numbers, and thus, students should explore the properties using natural numbers, whole numbers, integers, rational numbers, and eventually variables. Algebraic thinking occurs when students describe arithmetic properties using everyday language and eventually represent these properties using symbolic expressions. Figure 7 (adapted from Blanton, 2008, p. 15) describes how to design meaningful computation problems that lead to generalizations of operations and properties (note that $a$, $b$, and $c$ represent any real number).

**Figure 7.** Meaningful Computation Problems That Lead to Algebraic Generalizations

| Property | Arithmetic | In Language | Algebra |
|---|---|---|---|
| Commutative Property of Addition | $2 + 3 = 3 + 2$ | "We can add numbers in any order." | $a + b = b + a$ |
| Commutative Property of Multiplication | $2 \bullet 3 = 3 \bullet 2$ | "We can multiply numbers in any order." | $ab = ba$ |
| Association Property of Addition | $(2 + 3) + 4 = 2 + (3 + 4)$ | "We can group numbers in a sum any way we want and still get the same answer."<br><br>"We can add the first two numbers first and then the last number, or we can add the last two numbers first and then the first number, and get the same answer." | $(a + b) + c = a + (b + c)$ |
| Associative Property of Multiplication | $(2 \bullet 3) \bullet 4 = 2 \bullet (3 \bullet 4)$ | "We can group numbers in a product any way we want and still get the same answer."<br><br>"We can multiply the first two numbers first and then the last number, or we can multiply the last two | $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ |

| | | | |
|---|---|---|---|
| | | numbers first and then the first number, and get the same answer.” | |
| Distributive Property | $2(4 + 3) = 2 \bullet 4 + 2 \bullet 3$<br><br>$2(4 - 3) = 2 \bullet 4 - 2 \bullet 3$ | “Multiplying a number is the same as multiplying its addends by the number, then adding the products.” | $a(b + c) = ab + ac$<br><br>$a(b – c) = ab - ac$ |
| Additive Identity Property | $2 + 0 = 2$ | “If we add 0 to any number, we will get the same number.” | $a + 0 = a$ |
| Multiplicative Identity | $2 \bullet 1 = 2$ | “If we multiply 1 to any number, we will get the same number.” | $a \bullet 1 = a$ |
| Additive Inverse Property | $2 + (-2) = 0$ | “If we add inverses/opposites of the same number then we get 0.” | $a + (-a) = 0$ |
| Multiplicative Inverse Property | $2 \bullet (1/2) = 1$ | “If we multiply inverses then we get 1.” | $a \bullet (1/a) = 1$ |
| Zero Property | $3 \bullet 0 = 0$ | “Anytime we multiply a number by 0, we get 0.” | $a \bullet 0 = 0$ |

*Figure 7*. Properties of mathematics on the set of real numbers in arithmetic, language, and algebra.

*Making known quantities unknown.* Blanton (2008) recommended that students solve problems that contain unknown values. For instance, an arithmetic problem with known quantities can be rewritten to include unknown quantities. In the next example (Figure 8), an arithmetic problem is rewritten as an algebra problem.

**Figure 8.** Algebraic Representation of an Arithmetic Problem

| Arithmetic | Algebra |
|---|---|
| "If Ann has \$5 and Joe has \$8 more than Ann, how much money does Joe have?" | "If Ann has some money and Joe has \$8 more than Ann, how much money does Joe have?" |
| *Figure 8.* Arithmetic to Algebra Example | |

In the arithmetic problem, students simply find that they must add 8 + 5 to determine the amount of money that Joe has. In the algebra problem, students must be able to represent the unknown amount of money using symbols. To guide student thinking, teachers might ask questions about the information provided in the problem and the value they need to find. Students should also be prompted to compare the amount of money that Ann has to the amount of money that Joe has. With guidance, students will be able to represent the money that Joe has as $a + 8$, where $a$ represents the amount of money that Ann has. By introducing symbols in this way, students naturally develop algebraic thinking.

*Varying known quantities.* Variables have many different possible definitions. May and Van Engen (1959) stated, "Roughly speaking, a variable is a symbol for which one substitutes names for some objects, usually a number in algebra" (p. 70). By varying a quantity in an arithmetic problem, students begin to learn the notion of "variable". The next example (Figure 9, adapted from Blanton & Kaput, 2003) shows how to vary a quantity in arithmetic to allow students to explore patterns, relationships, and variables.

**Figure 9.** Varying a Quantity in Arithmetic to Explore Algebra

| **Arithmetic** | **Algebra** |
|---|---|
| "I want to buy a tee shirt that costs \$14. I have \$8 saved already. How much more money do I need to earn to buy the shirt?" | "Suppose the tee shirt costs \$15. If I have \$8 saved already, write a number sentence that describes how much more money I need to buy the tee shirt. What if the shirt costs \$16? \$17? Write number sentences for each of these cases. If *P* stands for the price of any tee shirt I want to buy, write a number sentence using *P* that describes how much more money I need to buy the tee shirt." |

*Figure 9.* An example that shows how to vary a quantity in arithmetic to allow students to explore patterns, relationships, and variables.

In the arithmetic problem, students can write the solution using an unknown quantity as $8 + x = 14$ or $14 – 8 = x$ where $x$ is the amount of money needed to buy the tee shirt. However, the algebra problem produces a series of number sentences as follows:

$$14 – 8 = 6$$

$$15 – 8 = 7$$

$$16 – 8 = 8$$

$$17 – 8 = 9$$

By examining these number sentences, students can identify which part varies (the total cost of the tee shirt). Then, students can translate to an algebraic equation that expresses how much more money is needed to buy the shirt, $P – 8 = x$, where $P$ is the price of any shirt and $x$ is the amount of money needed to buy the shirt. Any arithmetic problem can be rewritten to include unknown quantities. This type of thinking is much more powerful than solving a single number sentence (Blanton, 2008).

*Building an algebraic view of equality.* Students view the equal sign in arithmetic as an action symbol, meaning that they should compute something on the left of the equal sign and place the answer on the right of the equal sign (Carpenter, Franke, & Levi, 2003). Students obtain this misconception of the equal sign through repeatedly performing computations. Thus, many students fail to see the role of the equal sign as indication of a relationship between two quantities (Blanton, 2008). The next example shows how to help students learn the algebraic role of the equal sign (Figure 10).

**Figure 10.** Algebraic Role of the Equal Sign

| **Arithmetic** | **Algebra** |
|---|---|
| Find the sum of 10 and 5.<br>10 + 5 = ____ | Express the number 15 as a sum of two numbers.<br>15 = *a* + *b* |

*Figure 10*. An example that shows how to help students see the role of the equal sign as a relationship between two quantities.

In the arithmetic example, students are computing a sum. However, in the algebraic example, students will find a variety of pairs of numbers that add to 15. From this list of sums, students will learn that 1 + 14 = 8 + 7 means that the equal sign is expressing a relationship of equality between both sides of the equation. Moreover, this type of problem can be extended to include the commutative property (e.g., 5 + 6 = 6 + 5).

**Level 1: Using symbols in a meaningful way.** Algebra is composed of its own standardized body of symbols, signs, and rules that govern the language of algebra. In other words, algebra has its own grammar and syntax that allows one to formulate algebraic ideas clearly and compactly (Drijvers et al., 2011). Although this symbolic language is very powerful, it is also detached and formal in relation to the actual context of a problem. Algebraic language represents algebraic ideas using an abstract system.

***Decontextualizing a problem.*** The CCSS for Mathematical Practice state that mathematically proficient students should develop skills to reason abstractly and quantitatively (NGA/CCSSO, 2010). Furthermore, students need to not only have the ability to decontextualize a problem and illustrate it symbolically, but also contextualize a symbolic representation and understand its referents. The NCTM (2000) recommended that students in grades 6-8 explore relationships among symbolic expressions and graphs and also use symbolic algebra to represent situations and solve problems. Consider the following example:

> The area of a rectangular garden is 20 $ft^2$ and the length of the garden is 8 ft more than the width. Find the dimensions of the garden.

To decontextualize this problem, students need to know that the area of a rectangle is given by $A = lw$, and that the area of the garden is 20 $ft^2$, so $20 = l$w. Furthermore, the length $l$ is 4 ft more than the width $w$. Thus, $l = w + 8$. So, the symbolic representation of this algebra problem is:

$$20 = (w + 8)(w).$$

To contextualize this algebraic depiction, students should be able to describe each part of the equation, and also consider the units involved. The symbolic language of algebra is more than the memorization of rules; it involves the ability to model mathematical situations with symbols, understand the manipulation of these symbols, and have a fundamental understanding of the concept of variables and algebraic structures (Kieran, 1996).

***Learning variables with meaning.*** Learning the meaning behind symbols and variables is essential for students to become proficient in algebra. Students cannot understand how to solve an algebra equation without knowing the meaning of the equal sign and variables (Van de Walle et al., 2011). Kaput (1995) found that many students view algebra as "little more than many different types of rules about how to write and rewrite strings of letters and numerals, rules that must be remembered for the next quiz or test" (p. 4). Thus, students must find meaning in algebra not only to understand why they are solving algebraic equations, but also what situations these equations represent.

Research on algebra shows that algebra students have difficulty interpreting letters as variables and studies have focused on how students learn to represent values using variables (National Research Council [NRC], 2001). Once students learn to work with variables without thinking about the numbers that the variable might represent, they have achieved manipulation of "opaque formalisms" (Kaput, 1995, p. 8). Variables can be thought of as unknown values or as changing, varying quantities. Students need a deep understanding of both variables as unknown values and changing quantities in order to learn algebra.

*Learning variables as unknown values.* By the middle grades, students should be familiar with finding missing values and open mathematical sentences where a blank square or underline represents the unknown value, but they may not have connected the missing value with the word "variable" (Van de Walle, et al., 2011). Students can begin to learn about variables by using a variety of symbols and letters to represent unknown quantities. According to Carpenter, Franke, and Levi (2003), students must learn the mathematician's rule where the same symbol or letter in an equation must represent the same number every place it occurs. For example, in the number sentence:

$$\_\_\_ + \_\_\_ + 2 = \_\_\_ + 5 + 1$$

The ___ must be the number 4 every time.

In addition to learning variables as placeholders, story problems can introduce variables by representing a specific unknown. A simple example could be:

Jane's bookcase contains 17 books. She takes some books to school and the bookcase is left with 12 books. How many books did Jane take?

Although this story problem can be solved without using variables or algebra, students can express it using symbols as: $b + 12 = 17$. By placing the variables in context, students develop a better understanding of their meaning (Van de Walle et al., 2011).

*Learning variables as changing quantities.* In many cases, variables can represent more than one value. However, students have difficulty with this notion because they have only learned that a variable represents a particular number, as in the previous examples. Usiskin (1988) identified the differences between two conceptions of variables: (a) variables as unknowns or constants and (b) variables as varying quantities (p. 10). Variables as unknowns or constants are used in algebraic equations where the main goal is to simplify or solve. Variables as varying quantities are seen in equations where variables are arguments. For instance, in a linear equation $y = 3x + 1$, both $x$ and $y$ are the arguments and can represent any values that make the equality true. In other words, the ordered pair $(x, y)$ can be any $x$ and $y$ values that that lie on the line and, thus, are a solution to $y = 3x + 1$. Some mathematics educators think that algebra should be introduced through the use of variables as changing quantities instead of variables as unknowns (Fey & Good, 1985; Usiskin, 1988).

***Learning the equal sign with meaning.*** In arithmetic, the equal sign is used to indicate the process of computation, whereas in algebra, the equal sign represents equivalence between two expressions (Carpenter et al., 2003; Molina et al., 2005). In general, all the computation in addition takes place on the left of the equal sign, for example, 15 + 5 = ___. This leads students to think that the equal sign is a signal to compute and does not require a broad understanding of the meaning of equality (Molina et al., 2005). Davis (1984) and Falkner, Levi, and Carpenter (1999) recommended using open number sentences and true/false number sentences to promote understanding of the equal sign and relational thinking. Mann (2004) suggested teaching the equal sign as a balance between two sides of the equation.

*Relational thinking and the equal sign.* Carpenter, Levi, Franke, and Zeringue (2005) defined relational thinking as focusing on the properties and ways of thinking about operations, rather than focusing exclusively on the procedure to calculate the correct answer. In algebra, students need to learn that the equal sign is a relation (Kieran et al., 1993; Matz, 1982) and this can be achieved by using open number sentences. For example, Van de Walle et al. (2011) explained that relational thinking could be nurtured by exploring true/false and open sentences similar to the examples in Figure 11.

**Figure 11.** Relational Thinking Examples

| True/False | Open Sentences |
|---|---|
| 24 – 6 = 20 – 2<br>5 • 12 = 30 • 2<br>9 + 14 = 31 – 8 | 32 + ___ = 41 + 5<br>6 + 6 + 7 = ___ - 5 |
| *Figure 11*. True/false and open sentence examples that demonstrate the equal sign as a relationship between quantities, not as an operand. | |

Relational thinking is implicit when students can solve the previous examples; however, discussion about why a math sentence is true or false or how students found the answer to an open number sentence makes relational thinking more explicit (Molina et al., 2005). Moreover, Molina, Castro, and Ambrose (2005) found that asking students to write their own open sentences greatly contributed to students' understanding of the equal sign in algebra.

*Balance and the equal sign.* Conceptualizing the equal sign as a balance scale, where the two expressions of either side of the equal sign have the same value, attaches meaning to the equal sign. Mann (2004) contended that students need to transition from thinking that the equal sign indicates, "the answer is", to the equal sign indicates, "is the same as", before they begin learning algebra. Students in elementary school should be helped to understand that the equal sign symbolizes balance and equivalence. A balance scale will assist students with developing the meaning of the equal sign by using objects, numbers, and, eventually, can be extended to variables (Figure 12). By having students check their answers, they will verify that both sides of the scale have an equal value. In the next example, both sides of the scale are equal to 9. The scale reinforces that two sides of an equation are equivalent, which means that they must remain balanced.

**Figure 12.** Balance Scale Example

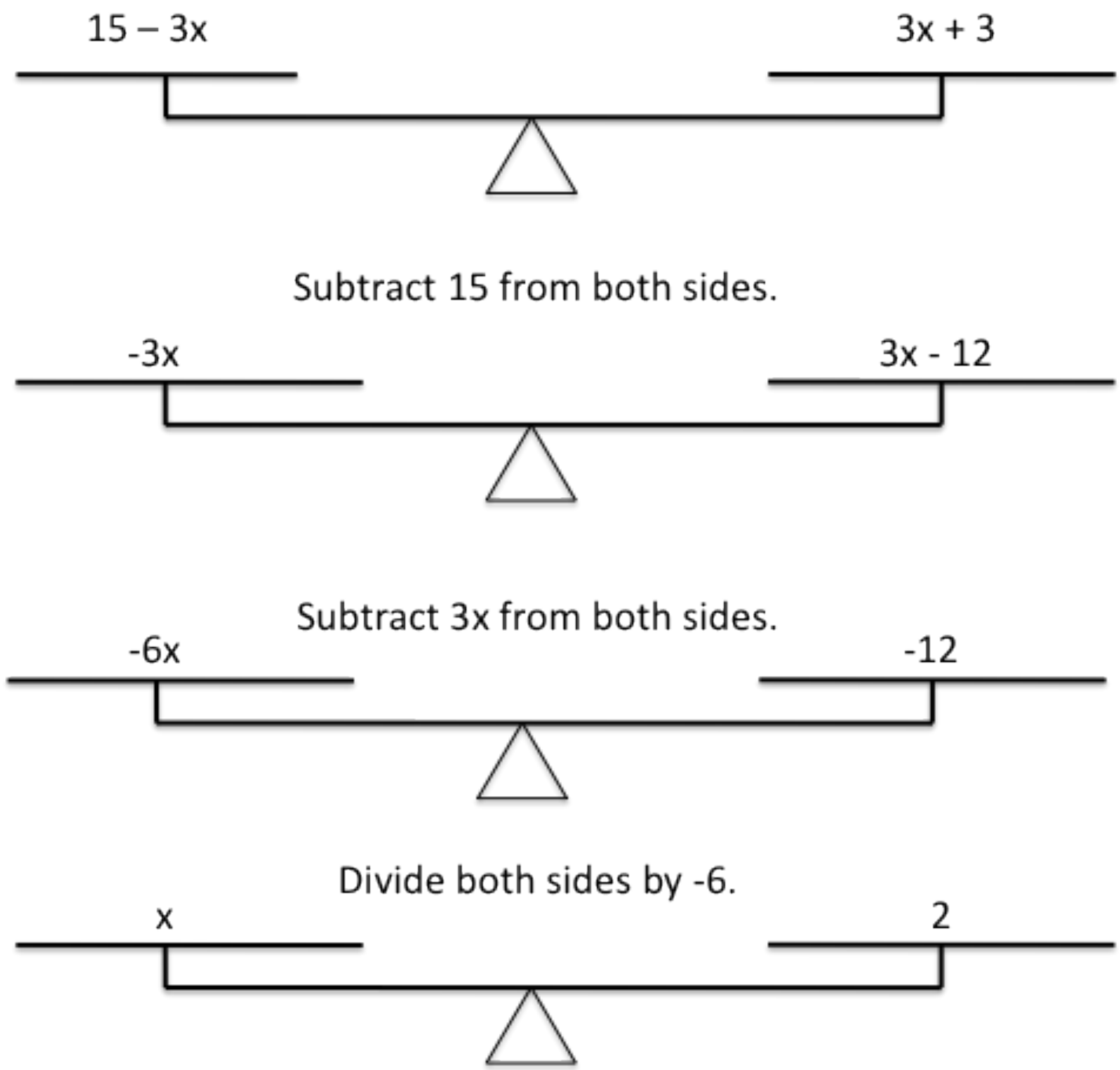



*Figure 12*. The balance scale example gives meaning to the equal sign in that both sides must be the same value. In order for a scale to remain balanced, both sides must be equal.

**Level 2: Study of structure.** The study of algebraic structure is an extension of the study of the structure of the number system (Greeno, 1991). Learning the number system requires knowledge of the properties of real numbers (previously discussed in the "From arithmetic to algebra section" of this paper p. 10) and this structure can be extended to variables, and thus, algebra. Furthermore, a powerful way to promote algebraic thinking involves attempting to justify conjectures derived from the real number system (Van de Walle et al., 2011). At the elementary level, justification of conjectures usually involves the use of examples; however, students benefit from the challenge to prove that a conjecture is always true because this exercise engages students in mathematical thinking (Carpenter et al., 2003).

Successful students of algebra should be able to describe the structure of equations and expressions, but are generally unable to do so (Davis, 1984; Kirshner, 1989). Algebraic structure can be considered in terms of the shape or order of an expression or equation (Novotná & Hoch, 2008). The logical structures in algebra can be classified as (a) the surface structure, (b) the systemic structure, and (c) the visual structure. According to Kaput (2000), learning the structures of algebra enriches understanding, provides frameworks for computations, and contributes to the mathematical foundation necessary for higher-level mathematics.

***Meanings of structural algebra.*** Researchers define the term structure differently and it can mean different things to different people (e.g., Dreyfus & Eisenberg, 1996; Hoch & Dreyfus, 2004). For example, algebraic structure can be considered in terms of the shape and order of an expression or equation (Novotná & Hoch, 2008), whereas structure sense can be considered to be an extension of number sense, an intuition for numbers such as seeing when an answer is incorrect or instinctively choosing a correct answer (Greeno, 1991). Moreover, algebraic structure can be classified into three categories: (a) surface structure, (b) systemic structure, and (c) visual structure.

*Surface structure.* The surface structure of an algebraic expression or equation has to do with its visual structure and not the symbols representing the unknown quantity. For instance, the surface structure of $2(m + 1) + 3 = 6$ is an alternate representation of the surface structure of $2x + 3 = 6$. The Algebra Learning Project (Wagner, Rachlin, & Jensen, 1984) found that students have difficulty with manipulating multi-term expressions as a single unit and thus do not recognize the surface structure (Kieran, 1989). In order to learn algebra proficiently, students should learn the similarities and differences in the surface structures of equations or expressions. Furthermore, studies that have focused on students' knowledge of "parsing", the syntactic analysis of expressions or equations, have found that students have difficulty identifying the surface structure on expressions involving several combinations of operations on numbers and variables (Davis, 1979; Davis, Jockusch, & McKnight, 1978; Greeno, 1982; Matz, 1980).

Syntactic analysis in algebra involves breaking down expressions or equations to understand each part (e.g., term, operation, property, etc.) and their relatedness. Thompson and Thompson's (1987) study incorporated expression trees on a computer screen that helped students recognize the surface structure of algebraic expressions and equations. By using a computer program called "EXPRESSIONS", students were able to explicitly model the intrinsic structure of expressions, which pushed them to think about the equation's structure. Results of this study indicated that attention to structure is important for student learning in algebra (Thompson & Thompson, 1987).

*Systemic structure.* The systemic structure of an algebraic equation refers to the equivalence of the left-hand and right-hand sides of an algebraic equation (Kieran, 1989). In arithmetic, students are expected to generalize that $5 + 3 = 8$ can be expressed as $8 – 3$

= 5. A study conducted with 12-year old algebra students concluded that they had difficulty recognizing the systemic structure of algebraic equations (Kieran, 1984). For example, they could not generalize that $x + 3 = 8$ can be expressed as $3 = 8 - x$. This misconception leads back to developing the meaning of the equal sign. Learning the systemic structure of algebraic equations involves knowledge of properties that are used to "balance" both sides of the equation such as the addition property of equality and the multiplication property of equality (i.e., if $a = b$, then $a + c = b + c$ and $ac = bc$). The process of solving equations relies on knowledge of systemic structure and that the equal sign is a relation between the two sides of an equation.

*Visual structure.* Kirshner and Awtry (2004) elaborated on the systemic structure of equations by introducing "visual salience in algebra" (p. 229). Many rules in algebra are visually salient, meaning that the right-hand and left-hand sides of equations are naturally related (Kirshner & Awtry, 2004). For instance, a visually salient algebraic rule is $x(y + z) = xy + xz$. This rule is visually salient because the right-hand side of the equation visually follows from the left-hand side of the equation. A non-visually salient rule is $x^2 - y^2 = (x - y)(x + y)$ because the right-hand side of the equation is not visually obvious from the left-hand side of the equation.

A visually salient rule has an immediate connection between both sides of the equation (Kirshner & Awtry, 2004). Students learning algebra become confused because some rules are visually salient and some are not. Learning the visual structure of algebra engages students in thinking about the structure, or organization, when simplifying an algebraic expression or solving an equation (Kirshner, 1993). One way to learn the visual structure is to provide students with structural models of expressions, equations, and solution processes. This approach to learning algebra begins with undefined symbols and explicit rules, similar to axioms in geometry, and meaning is constructed logically by the methods used (Kirshner, 1993).

**Level 2: Study of patterns and functions.**

***Patterns and formulas in algebra.*** Algebra can be used as a tool to predict mathematical patterns. These patterns are frequently introduced through the use of activities that involve geometric and numerical patterns (NRC, 2001). An example from *Adding it Up* (2001) follows:

> Triangular numbers can be built with dots. The first four triangular numbers are 1, 3, 6, and 10. Predict the number of dots in the 20th triangle and give a rule for predicting the number of dots in any triangle. (p. 277)

In this example, the algebra of patterns is involved to predict the number of dots in subsequent triangles. After finding this pattern, it is possible to generalize the pattern into a formula. Therefore, algebra not only concerns investigating, identifying, and formulating patterns, but also examines the underlying algebraic structures (Drijvers et al., 2011). Moreover, formulas represent these patterns and structures in algebra.

The *Principles and Standards* (NCTM, 2000) and the CCSS (NGA/CCSSO, 2010) support that students should be able to identify and explain patterns in algebra and be able to generalize such patterns with tables, graphs, and symbolic rules. Once students identify a pattern, they should be able to make use of the structure to form an expression or equation that models the pattern. By predicting patterns and emphasizing generalization, students can develop the skills not only to recognize and identify structure, but also to appreciate algebraic expressions as common statements (NRC, 2001).

*Literal equations.* In addition to using formulas to model situations and patterns in algebra, they can also be used to manipulate literal equations. Literal equations contain more than one variable and are introduced in algebra to study the deductive process of solving for a particular variable (NGA/CCSSO, 2010). For example, the perimeter of a rectangle is given by the formula $P = 2l + 2w$, where $l$ is the length and $w$ is the width of the rectangle. This formula can be rearranged to solve for either $l$ or $w$. For example, the $l = (P - 2w)/2$. This process involves the logical analysis of the original formula and the application of algebraic techniques to rearrange the formula. Scholars contend that formulas are a powerful technique to describe structure and explain patterns (Drijvers et al., 2011; Ernest, 2004; Van Amerom, 2003).

***Functions.*** The algebra of functions is primarily concerned with the dependent relationships between variables (Drijvers et al., 2011). In the middle grades, students should understand that a function is a rule that assigns to each input exactly one output (NGA/CCSSO, 2010). For example, the slope-intercept equation $y = 2x + 5$ is a linear function whose graph is a straight line. This describes a line with slope 2 and $y$-intercept 5. Moreover, for each input value $x$, exactly one output exists for the value $y$. In addition to knowing the definition of a function, students should also be able to use graphs of functions and analyze the nature of changes (NCTM, 2000). The CCSS (NGA/CCSSO, 2010) also states that students should know how to qualitatively describe the functional relationship between two quantities through the analysis of a graph. For instance, students should know how to describe where the function is increasing and decreasing, whether the function is linear or nonlinear, and the values of the function at certain points using function tables and graphs.

*Function notation.* Students in the middle grades should also know how to use function notation (NGA/CCSSO, 2010). For example, if $f$ is a function and $x$ is in the domain of $f$, then $f(x)$ denotes the output of $f$ with $x$ as the input. Then, the graph of $f$ is given by the equation $y = f(x)$. In first year algebra, function notation is new to most students and can be confusing since $f$ has become the parameter, instead of the argument $x$ (Usiskin, 1988). However, students must learn function notation since it is used extensively in advanced mathematics and computer science.

*Patterns as functions.* Patterns are visible in every area of mathematics and part of thinking algebraically is identifying and describing patterns. In algebra, students study patterns of relationships and functions. The algebra of patterns is about investigating, identifying and formulating similarities relating to general patterns (Drijvers et al., 2011). Thompson's (1994) perspective of learning mathematics maintains that concepts emerge over time and are related to previous concepts. Thus, to learn functions, students must already have knowledge of expressions, variables, arithmetic operations, and quantity. Students learn the concept of function in phases that contain different representations and components of functions (Markovits, Eylon, & Bruckheimer, 1986). Figure 13 (adapted from Markovits et al., 1986) displays the components of a function and how functions are presented in different forms.

**Figure 13.** Function Representation and Function Components

| | Function Components | | |
|---|---|---|---|
| **Function Representation** | **Domain** | **Range** | **Rules** |
| **Verbal** | Verbal or mathematical notation | Verbal or mathematical notation | Verbal |
| **Arrow Diagram** | A circle around the elements of the domain | A circle around the elements of the range | Arrows |
| **Algebraic** | Verbal or mathematical notation | Verbal or mathematical notation | Formula |
| **Graphical** | The horizontal or x-axis of a coordinate plane | The vertical or y-axis of a coordinate plane | A set of points on the coordinate plane |

*Figure 13*. The function representations can be verbal, arrow diagrams, algebraic, or graphical. The function components include domain, range, and rules (adapted from Markovits et al., 1986).

**Figure 14.** Function Representation Examples

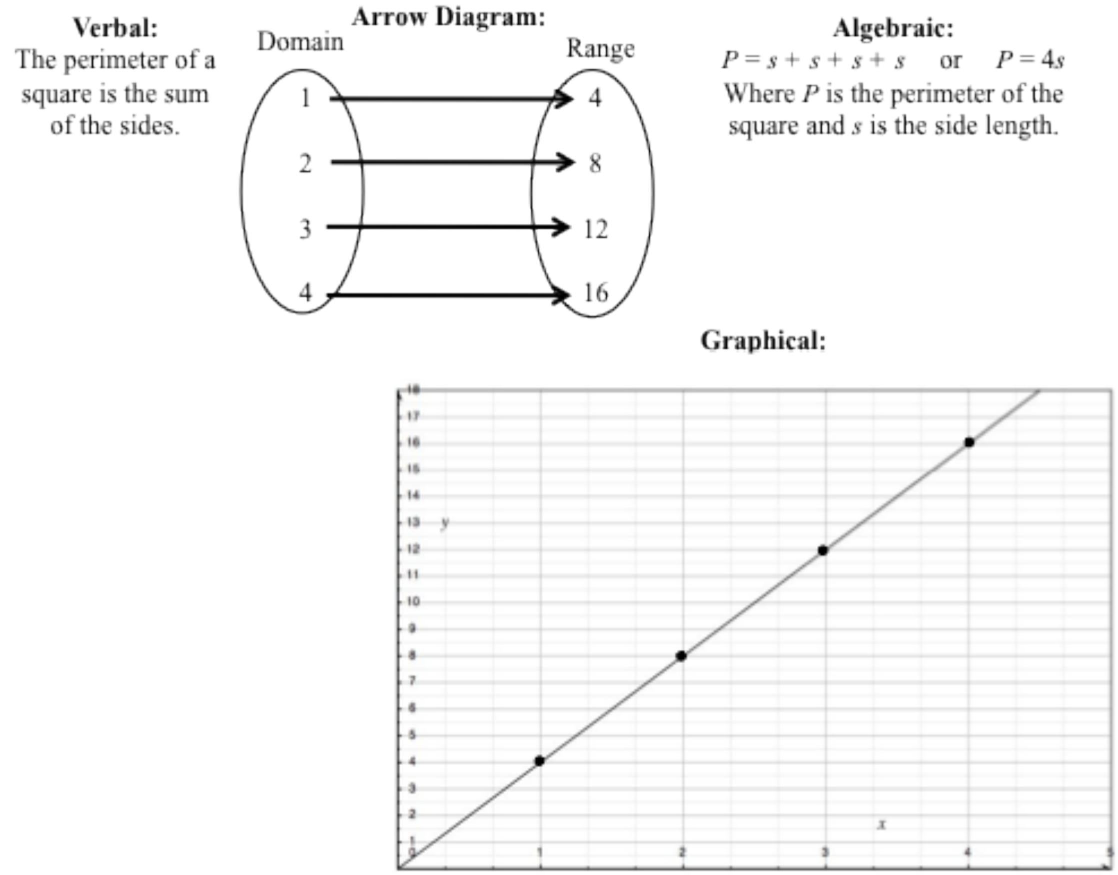



*Figure 14*. These examples show four representations of the function for the perimeter of a square.

When learning functions, students learn that functions have three components: the domain, the range, and the rule. Next, students learn that a function can be represented in different ways such as verbally, in an arrow diagram, algebraically, and graphically (see Figure 14). Once students learn these forms, they learn how to translate among the different representations. For example, students learn that a linear equation is a function and a line on a coordinate plane can also represent this linear function. Essentially, a student learns functions by understanding the representations and components of a function and how they are related (Markovits et al., 1986).

**Mathematical modeling.**

***Empirical algebra.*** Empirical approaches to algebra include relating the symbolic systems to real-world situations, graphs and tables, or arithmetic patterns (Kirshner, 2001). This approach to simplifying algebraic expressions and solving equations places the emphasis on learning the referential meanings of algebra, not merely on the manipulation of symbols with no purpose (Booth, 1989). Therefore, mathematical modeling (see Figure 15) involves the conversion of a word problem, written or verbal, to an equation, inequality, or system (NGA/CCSSO, 2010).

**Figure 15.** Example for Modeling with Mathematics

| Words | Equation |
|---|---|
| Susan is two years older than her brother, Bob. The sum of their ages is 8. How old is Bob and how old is Susan. | Susan's age $S$ and Bob's age $B$<br>Susan is two years older than Bob: $S = B + 2$<br>Sum of their ages is 8: $S + B = 8$ or $B + 2 + B = 8$<br>Solve for $B$: $B + 2 + B = 8$<br>$2B + 2 = 8$<br>$2B = 8$<br>$B = 4$<br>Find $S$: $S = B + 2 = 4 + 2 = 6$<br>Bob is 4 years old and Susan is 6 years old. |

*Figure 15*. An example that shows how mathematical modeling involves the conversion of the word problem to algebra.

Linear programming provides many examples that use equations and inequalities to model real-world situations and their restrictions. A typical linear programming problem determines the maximum profit for some situation, taking into account the limitations of materials and labor. In middle school algebra, students solve simple linear programming problems that generally involve modeling the situation with inequalities in two variables and graphing the inequalities on the $(x, y)$ plane to find the feasible region, or the area on the graph where the shading intersects (see Figure 16).

**Figure 16.** Feasible Region

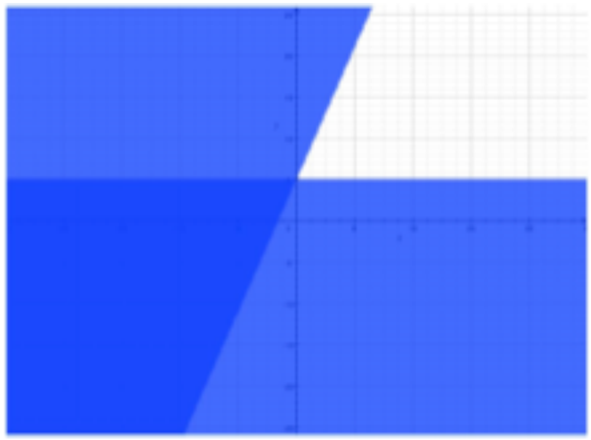


*Figure 16*. The feasible region is in dark blue for the system of inequalities $\begin{cases} y > 2x + 5 \\ y \le 5 \end{cases}$.

Although linear programming problems offer applications for inequalities, mathematical modeling also occurs through other types of practical applications that are useful in students' everyday lives. The NCTM (2000) suggests that modeling in the middle grades involve representing real-world situations symbolically and through the use of linear functions, while also beginning the exploration of some nonlinear functions. Moreover, students in middle school should know how to use algebraic models to understand quantitative relationships. Translating word problems into algebra often involves creating equations and inequalities and then finding the value of one or more variables. Thus, learning how to solve equations and inequalities becomes an essential component of algebra.

Mathematical modeling is present throughout Level 1 and Level 2 of the algebraic thinking framework and many view mathematical modeling as the main objective of algebra (Izsák, 2003; Kaput, 1999; Schoenfeld, 1992; Usiskin, 1988). Mathematical modeling is the process of taking a real world situation and attempting to represent it with mathematics or "mathematize it" (Kaput, 1999, p. 17). Mathematical modeling is not referring to using manipulatives as an example of the mathematics, but rather, simulating real phenomena with equations. Students learn to model real situations using mathematics by thinking about situations that contain multiple and connected representations. Placing an algebra problem in context helps students make sense of the mathematics and supports conceptual understanding of these abstract representations (Earnest & Baiti, 2008). Mathematical modeling is learned through the generalization of arithmetic to algebra, using symbols in a meaningful way, studying structure, and studying patterns and functions.

**The Organization of the Common Core Standards for Algebra**

In the CCSS for Mathematics Content (NGA/CCSSO, 2010), the strand for operations and algebraic thinking is present from grades K-5. In grades 6-8, algebraic thinking exists throughout the strand for expressions and equations as well as the strand for functions (NGA/CCSSO, 2010). Figure 17 (adapted from Illustrative Mathematics, 2011) shows the progression of the algebra content standards from grades K-8.

**Figure 17.** Progression of Algebra Common Core Content Standards Grades K-8

| | Operations and Algebraic Thinking (K, 1, 2, 3, 4, 5) | Expressions and Equations (6, 7, 8) | Functions (8) |
|---|---|---|---|
| K | • Understand addition as putting together and adding to, and understanding subtraction as taking apart and taking from. | | |
| 1 | • Represent and solve problems involving addition and subtraction.<br>• Understand and apply properties of operations and the relationship between addition and subtraction.<br>• Add and subtract within 20.<br>• Work with addition and subtraction equations. | | |
| 2 | • Represent and solve problems involving addition and subtraction.<br>• Add and subtract within 20.<br>• Work with addition and subtraction equations. | | |
| 3 | • Represent and solve problems involving multiplication and division.<br>• Understand properties of multiplication and the relationship between multiplication and division.<br>• Multiply and divide within 100.<br>• Solve problems involving the four operations, and identify and explain patterns in arithmetic. | | |
| 4 | • Use the four operations with whole numbers to solve problems.<br>• Gain familiarity with factors and multiples.<br>• Generate and analyze patterns. | | |
| 5 | • Write and interpret numerical expressions.<br>• Analyze patterns and relationships. | | |
| 6 | | • Apply and extend previous understandings of arithmetic to algebraic expressions.<br>• Reason about and solve one-variable equations and inequalities.<br>• Represent and analyze quantitative relationships between dependent and independent variables. | |
| 7 | | • Use properties of operations to generate equivalent expressions.<br>• Solve real-life and mathematical problems using numerical and algebraic expressions and equations. | |
| 8 | | • Work with radicals and integer exponents.<br>• Understand the connections between proportional relationships, lines, and linear equations.<br>• Analyze and solve linear equations and pairs of simultaneous linear equations. | • Define, evaluate, and compare functions.<br>• Use functions to model relationships between quantities. |

*Figure 17*. In the progression of the Algebra Common Core Content Standards for Grades K-8, operations and algebraic thinking are taught in grades K-5, expressions and equations are taught in grades 6-8, and functions are introduced in grade 8.

Specific algebra content standards are outlined in the CCSS for eighth grade algebra and high school algebra (NGA/CCSSO, 2010). Algebra content can be divided into five strands as shown in Figure 18 (adapted from Illustrative Mathematics, 2011). The function strand contains three additional sub-strands that are found throughout algebra. Moreover, Figure 18 is a summary of the CCSS for eighth grade and high school level algebra content.

**Figure 18.** Common Core Content Standards Summary for Algebra

| Algebra | | | |
|---|---|---|---|
| **Seeing Structure in Expressions** | **Arithmetic with Polynomials and Rational Expressions** | **Creating Equations** | **Reasoning with Equations and Inequalities** |
| • Interpret the structure of expressions.<br>• Write expressions in equivalent forms to solve problems. | • Perform arithmetic operations on polynomials.<br>• Understand the relationship between zeros and factors of polynomials.<br>• Use polynomial identities to solve problems.<br>• Rewrite rational expressions. | • Create equations that describe numbers or relationships. | • Understand solving equations as a process of reasoning and explain the reasoning.<br>• Solve equations and inequalities in one variable.<br>• Solve systems of equations.<br>• Represent and solve equations and inequalities graphically. |

| Functions | | |
|---|---|---|
| **Interpreting Functions** | **Building Functions** | **Linear, Quadratic, and Exponential Models** |
| • Understand the concept of a function and use function notation.<br>•Interpret functions that arise in applications in terms of the context.<br>• Analyze functions using different representations. | • Build a function that models a relationship between two quantities.<br>• Build new functions from existing functions. | • Construct and compare linear, quadratic, and exponential models and solve problems.<br>• Interpret expressions for functions in terms of the situation they model. |

*Figure 18*. Algebra contains five strands: seeing structure in expressions, arithmetic with polynomials and rational expressions, creating equations, reasoning with equations and inequalities, and functions. The function strand has three subsections: interpreting functions, building functions, and linear, quadratic, and exponential models.

## Conclusions on Learning Algebra

Algebra does not have a simple definition and algebra in elementary school differs considerably from algebra in college (Usiskin, 1988). Algebra in an elementary classroom involves examples and questions that support the development of students' algebraic thinking. Algebraic thinking is developed through good questions that press students to articulate their mathematical understanding (Blanton, 2008). Moreover, French (2002) contends that, "Algebra is a very economical language: making sense of an expression, a step in an argument or a complete argument is very dependent on a deep understanding of all the component parts as well as the overall logic of the argument" (p. 190). This thinking can be expanded through different approaches to teaching algebra such as connecting algebra to arithmetic, procedural algebra, relational algebra, empirical algebra, and structural algebra.

Learning algebra occurs simultaneously with learning to think mathematically. Learning algebra involves generalization, the use of symbols, the study of algebraic structure, the study of patterns and functions, and mathematical modeling. These five themes form a framework for algebraic thinking in which understanding is intensified through the use of real world context. Algebra is not just symbol manipulation—it is a broad, rich mathematical topic that all students can learn through the framework for algebraic thinking.

The early stages of learning algebra are critical in that students build relational understanding and the ability to apply ideas to a variety of new situations (French, 2002). Algebraic thinking incorporates fluency with procedures and skills, as well as understanding and applying ideas to the real world. Moreover, students who are engaging in algebraic thinking are able to see meaning and purpose in the algebra that they are doing.

## Summary and Critique of Research

Understanding how to better support the learning of beginning students in algebra involves having knowledge on how students learn mathematics, especially algebra, as well as the implications for curriculum development and teaching mathematics. Algebra is inherently complex; however, algebra contains many formal, structural aspects (Kaput, 1995). Gaining the ability to identify the building blocks of a structure is a useful technique in algebra (Cuoco, Goldenberg, & Mark, 1996). An abstract algebraic expression can be made simple by breaking it into a combination of simple objects.

Thus, the literature implies that successful students are generally able to describe the structure of the symbolic language of algebra (Kirshner, 1993). However, a need exists for further research on how beginning algebra students develop a structural understanding of algebra through learning the structure of arithmetic expressions (Banerjee & Subramaniam, 2005).

The review of the literature indicated that there was no clear and concise definition of algebraic thinking. Specifically, little is known about children's ability to advance from arithmetic to algebra and use algebraic notation (Carraher, Schliemann, Brizuela, & Earnest, 2006). Research suggests that algebra can be integrated into arithmetic and arithmetic can be modified to include algebra, through generalizing the properties of real numbers (Moseley & Brenner, 2009). Researchers contend (Darley, 2009; Ketterlin-Geller, Jungjohann, Chard, & Baker, 2007; Usiskin & Bell, 1983) that it is impossible to learn arithmetic without dealing implicitly or explicitly with variables. Students' knowledge of how to translate and generalize from arithmetic to algebra helps support mathematical thinking. It is possible to integrate algebra and arithmetic, but early algebra education is not thoroughly described in the literature.

Students must have a strong understanding of the symbols that they are using in order to meaningfully engage in algebra. In addition, students need to understand that variables can be quantities, and that the quantities can vary, rather than simply knowing that variables are unknown values. Moreover, students need a solid understanding of the equal sign to be successful in algebra. They must learn that an equation is an equal relation between two expressions. Having a relational understanding of the equal sign is important for success in algebra; however, limited research exists on how to explicitly develop students' knowledge of the equal sign (Knuth, Stevens, McNeil, & Alibali, 2006).

The study investigated how sixth-grade students understand extending arithmetic to algebraic expressions and solving one-variable equations through tasks presented in a teaching experiment. The research study informs the development of an instructional unit to support sixth-grade students' understanding of algebra as outlined in the Common Core State Standards for Mathematics (NGA/CCSSO, 2010). The following research questions guided this study:

1. How do sixth-grade students think mathematically when solving arithmetic and algebra tasks in the classroom teaching experiment?
    - What is the whole class learning trajectory that emerges?
2. What are the means of supporting and organizing student learning of algebra?
    - What is the role of the tasks in supporting learning?
    - What design decisions are made to modify the tasks in the instructional unit and why?

The next chapter describes the research methodology, including a rationale for the research approach, a description of the research sample, the methods of data collection, the analysis and synthesis of data, issues of trustworthiness, and limitations of the study. It concludes with a brief overview of the study.

# CHAPTER III: METHOD

## Rationale for Design Research

This study investigated how students learn algebra through a whole class teaching experiment using design research. The design research approach (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Collins, Joseph, & Bielaczyc, 2004; Kelly, 2003) was used to investigate how students learn in a whole class setting and documented the realized learning trajectory and the mechanisms for supporting learning. The framework of design research allows researchers to observe as well as intervene throughout the study process. It involves engineering learning environments; systematically studying what takes place, and making adjustments to the curriculum (Cobb et al., 2003; Collins et al., 2004; Kelly, 2003). The overarching objective of design research is to develop a better understanding of the learning ecology (Gravemeijer & Cobb, 2006). A learning ecology in mathematics involves the interactions among teachers, students, content, and curriculum and how these relations affect teaching and learning. Furthermore, design researchers develop theories about the learning process, as well as techniques designed to support learning (Cobb et al., 2003). Design research methodology was employed to study and understand the means of supporting and organizing student learning of algebra through tasks presented in a teaching experiment.

## The Research Sample

Students from a sixth-grade classroom in an urban elementary school in a western state participated in the study. The sample included a total of 22 predominately Latino(a) students, ages 11 to 12. There were 11 female students and 11 male students. The majority of the students were from lower to middle socioeconomic backgrounds. Students sat in groups of four assigned by the teacher.

**Table 1.** Research Participant Summary (N = 22)

| Student | Ethnicity | Gender | Group |
|---|---|---|---|
| MH1F | Latino | Male | F |
| MH2F | Latino | Male | F |
| FH3F | Latina | Female | F |
| FC4F | Caucasian | Female | F |
| MH11 | Latino | Male | 1 |
| MH21 | Latino | Male | 1 |
| MH12 | Latino | Male | 2 |
| MH22 | Latino | Male | 2 |
| MH32 | Latino | Male | 2 |
| FH42 | Latina | Female | 2 |
| MH13 | Latino | Male | 3 |
| MH23 | Latino | Male | 3 |
| FH33 | Latina | Female | 3 |
| FH43 | Latina | Female | 3 |
| MT14 | Tongan | Male | 4 |
| MH24 | Latino | Male | 4 |
| FH34 | Latina | Female | 4 |
| FH44 | Latina | Female | 4 |
| FH15 | Latina | Female | 5 |
| FH25 | Latina | Female | 5 |
| FH35 | Latina | Female | 5 |
| FH45 | Latina | Female | 5 |

*Note*. Student, ethnicity, gender, and group

The teacher that participated in this study had a master's degree in education and had taught fifth and sixth grades for two years. She regularly participates in professional development sessions in the school district and is trained in the Common Core State Standards for Mathematical Content and Practices. The teacher was chosen because she has a close working relationship with the lead researcher and wanted to learn new methods of teaching algebra. The research team consisted of the lead researcher, Diana Moss (Mathematics Education Doctoral Student and Former Middle Grades Teacher) and the faculty researcher, Teruni Lamberg (Mathematics Education Professor and Former Primary School Teacher). There were two others who operated the video cameras.

## Treatments

The lead researcher developed the instructional unit (Appendix A) based on a review of research on how students learn algebra and mathematics to address the sixth-grade CCSS for Expressions and Equations (NGA/CCSSO, 2010). The researcher received feedback from the classroom teacher and the faculty researcher as the unit was being developed. This instructional unit (Appendix A) initially served as the hypothetical learning trajectory (Simon, 1995). The context of *Soccer* was used to help students think about the meaning of expressions, equations, and functions. Soccer was chosen to relate mathematical concepts to a real-world context that interested the students in this class. The teaching experiment was conducted over four weeks through a design research approach. The lesson times ranged from an hour to an hour and a half and took place during regular math instruction time. Each day, the lesson was modified based on teacher feedback and analysis of data and student work. The lead researcher communicated with the research team and the classroom teacher before, during, and after each lesson in order to make modifications for subsequent lessons.

## Design of this Study

This section provides the methodological framework and a detailed description of the five phases of data collection and analysis in this teaching experiment. Additionally, the interpretive framework used to analyze the data is presented.

**Methodological Framework**

A teaching experiment is a type of design research. The primary purpose of a teaching experiment is for researchers to personally observe students' learning and reasoning (Steffe & Thompson, 2000). Moreover, a teaching experiment forms the core of classroom design research (Gravemeijer, 2004) and is conducted in a small or large learning setting over a variable amount of time. For example, this type of study can occur over just a few hours or an entire academic year. The participants usually include a teacher-researcher, a group of students, and an observer-researcher (Steffe & Thompson, 2000). Unless the official teacher becomes completely involved in the research process, the teacher-researcher assumes the role of the teacher during the experiment (Molina, Castro, & Castro, 2007). The researchers interact with and become part of the environment in which they are studying, permitting relationships to form among researchers, teachers, and students. This situation is distinct to many other forms of research because very few boundaries exist between the researchers and other participants.

Teaching experiments analyze student learning by repeating the process of developing and testing instructional activities. According to Steffe and Thompson (2000), the purpose of a teaching experiment in mathematics is to explore and explain students' learning process and mathematical activity throughout this process. Confrey and Lachance (2000) describe a teaching experiment as a planned teaching intervention that takes place in a classroom during a period of academic instruction. The framework of a teaching experiment has not been established formally because its actual structure varies within each context and student response guides the researchers (Steffe &

Thompson, 2000). However, a teaching experiment that uses design research must involve an iterative process (see Figure 20). Moreover, throughout the experiment, researchers must constantly be evaluating and questioning the underlying meaning of students' learning and behavior, which informs the next modification of the intervention and, thus, the next iteration in the design process. Figure 19 (adapted from Middleton et al., 2008) shows a logic model of a teaching experiment using design research that framed this study.

**Figure 19.** Model of a Teaching Experiment Using Design Research

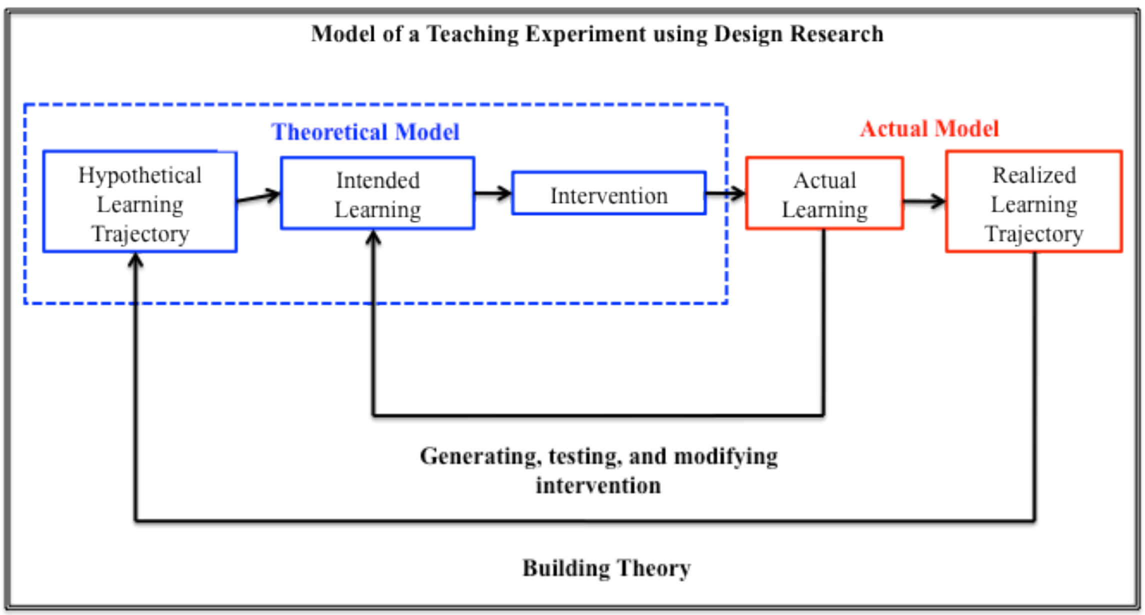



*Figure 19.* A theoretical model and an actual model are embedded in the model of a teaching experiment using design research. The theoretical model begins with the HLT, which leads to the intended learning, which informs the intervention. The actual model begins with the actual learning that takes place and the realized learning trajectory is based on the actual learning. These sub-models are connected through generating, testing, and modifying the intervention and through building theory. (adapted from Middleton et al., 2008)

**Iterative design.** Iterative design is an essential component of design research, and therefore, of teaching experiments that employ design research. In a teaching experiment, the recurrent cycle of developing and testing instructional activities, designing teaching interventions, collecting and analyzing data, and making adjustments in the design is the iterative process. A study that aims to analyze student learning begins with making an experimental model that leads to a HLT. The preliminary model is based on the researchers' theoretical assumptions and previous experience (Molina et al., 2007). In this teaching experiment, the hypothetical learning trajectory was based on the CCSS for Expressions and Equations (NGA/CCSSO, 2010), which is the progression from expressions to equations to functions (see Appendix A). Moreover, in a teaching experiment using design research, the teacher-researcher interacts with the students, as a teacher normally would in a classroom, and the researchers collect data. Then, these data inform the research by either confirming or rejecting the initial hypotheses about student learning. The process continues and eventually leads to a final model that has been adjusted and refined to obtain an optimal product. During this cycle, the researchers not only intend to investigate if the model performs as expected, but also are willing to redesign and fine-tune the model based on their observations (Confrey, 2006; Steffe & Thompson, 2000). Appendix A contains the initial model of the instructional unit and Appendix B contains the fine-tuned and redesigned model of the instructional unit. The iterative process of design research is shown in Figure 20 (adapted from Middleton et al., 2008).

**Figure 20.** The Iterative Process of Design Research

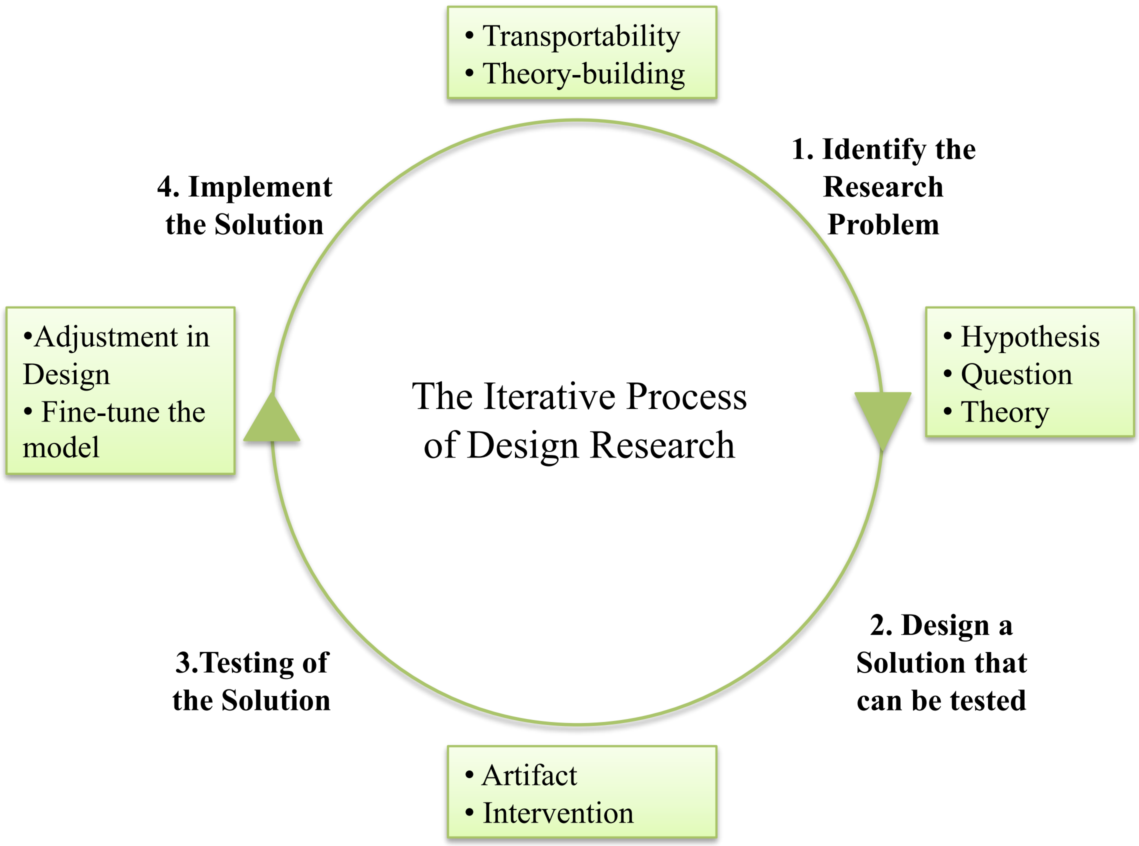



*Figure 20*. The iterative process of design research is shown as a circular system that begins with the identification of the research problem. A hypothesis, question, or theory is developed that addresses the research problem. This leads to the design of a testable solution that informs the artifact or intervention. The solution is tested, and adjustments are made in the design and model. Lastly, the solution is implemented, which leads to transportability of the model and theory-building. (adapted from Middleton et al., 2008)

The methodological framework of a teaching experiment using design research (Middleton et al., 2008; Lamberg & Middleton, 2009) was used in this study to inform the development of an instructional unit to support sixth grade students understanding of algebra. This model begins with a theoretical model, expressions to equations to functions, of a hypothetical learning trajectory and ends with a realized model that emerges as a result of the instructional unit.

**Interpretive Framework**

Cobb, Stephan, McClain, and Gravemeijer (2001) offer a methodology for the analysis of collective learning of a classroom community. This methodology takes into account the evolution of classroom mathematical practices. Cobb et al. (2001) described classroom mathematical practices as "taken-as-shared ways of reasoning, arguing, and symbolizing established while discussing particular mathematical ideas" (p. 126). This approach to analyzing data is interpretive in that it takes into account both the social perspective (the meanings the class is making as a whole) and the psychological perspective (the meanings and interpretations of individual students as they participate in the classroom community). The following figure (Figure 21) provides the interpretive framework that is adapted from the Lamberg (2001) framework. The framework outlines the type of data that was collected and how the data was collected, triangulated, and analyzed. The framework takes into account sense making that takes place through activity within the context. The arrows on the framework represent the affordances and constraints that take place.

**Figure 21.** Interpretive Framework

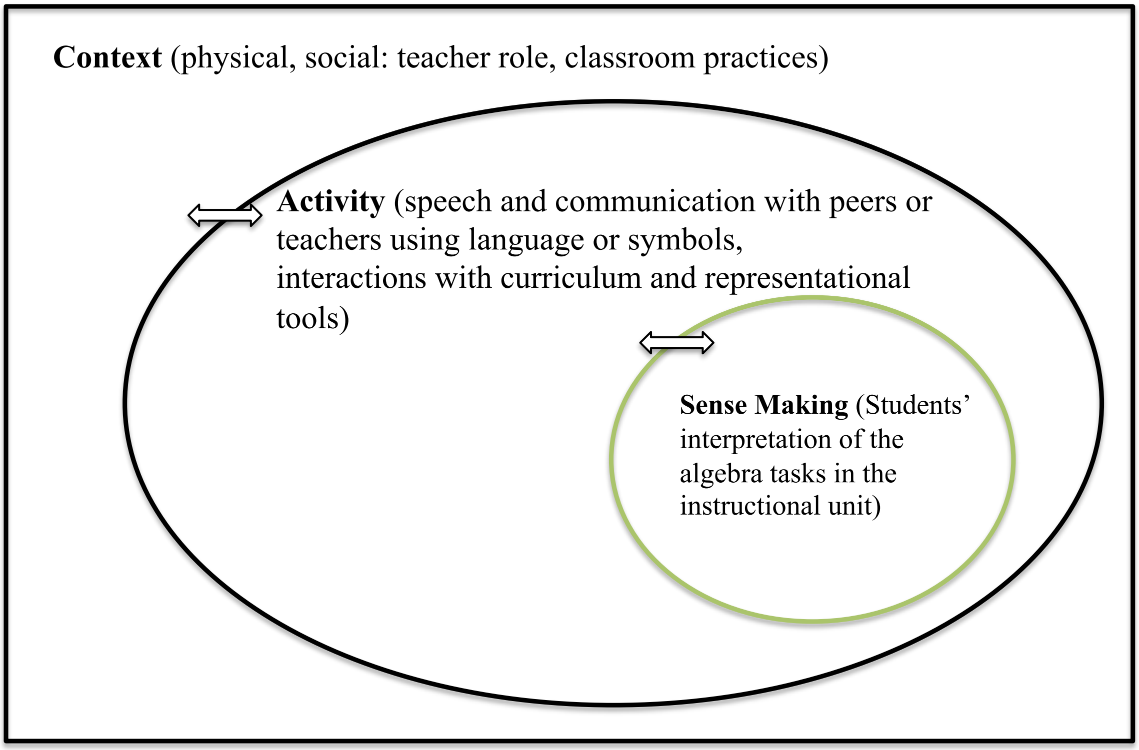



*Figure 21.* An Interpretive Framework (adapted from Lamberg, 2001) for the analysis of sense making of algebra tasks by examining the layers of activity within the classroom context with respect to students' interpretation of the algebra tasks in the instructional unit.

The following section summarizes in detail the methods and procedures used in each phase of the data collection and analysis.

**Data Collection and Analysis**

The researcher used a variety of data sources in order to analyze this teaching experiment using design research. Any source that related to the broader phenomena being studied in the experiment was collected (Cobb et al., 2003). Multiple sources of data allowed for retrospective analysis once the study was complete, as well as iterative analysis throughout the study. Data sources included field notes (Maxwell, 2005), video recordings, and documentation of anything that occurred in the classroom such as students' work and researchers' reflections. The researchers and teacher had regular and ongoing discussions about the in-class interventions and interpretations of the data. These discussions were important because they enhanced the quality of the study and research process. This teaching experiment was an iterative process conducted in three phases. The first phase consisted of observation and pretest. The second phase consisted of teach/reflect/plan, interview, and prospective analysis. The third phase consisted of posttest and retrospective analysis. Data collection and analysis was a parallel process with prospective analysis occurring throughout the teaching experiment and retrospective analysis occurring after the teaching experiment (see Figure 22).

**Figure 22.** Phases of Data Collection and Analysis

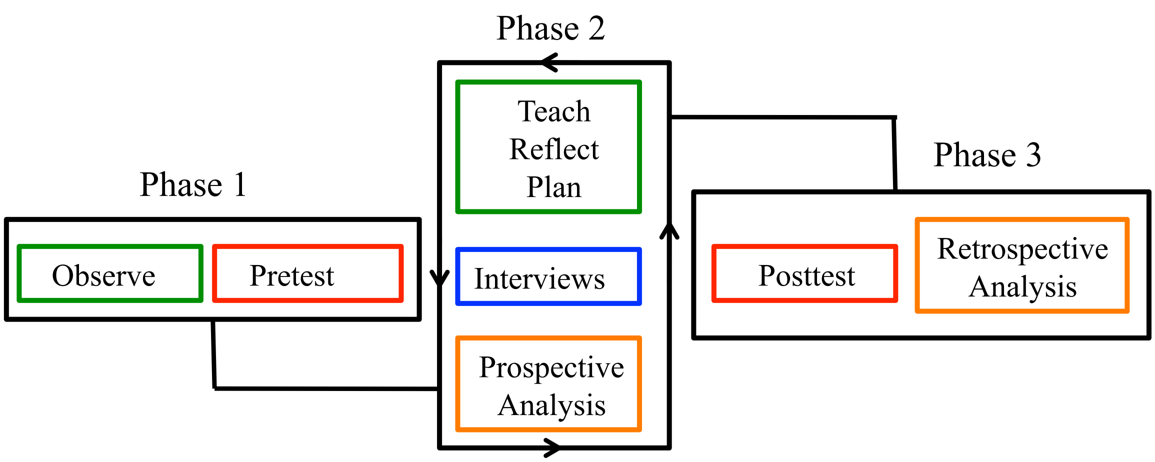



*Figure 22.* The iterative process of data collection and analysis in the teaching experiment.

**Phase 1.** Phase 1 of the data collection and analysis consisted of two parts: observe and pretest. These parts were intended to help the researcher design the hypothetical learning trajectory, interventions, and instructional unit.

*Observe.* Exploratory observations occurred in the weeks prior to the teaching experiment. The researcher took field notes (Maxwell, 2005) of what occurred in the classroom and anything that she thought informed the teaching experiment. Moreover, this phase allowed the researcher to observe the classroom social norms during whole class discussions and small group activity (Cobb et al., 1996). The researcher also was able to observe the teacher's role in the classroom and the classroom layout. These observations were intended to provide the researcher with a context in which to situate the hypothetical learning trajectory with regards to students' level of algebraic thinking.

*Pretest.* A pretest was given to all the students on the first day of the teaching experiment. The pretest was designed to determine students' prior knowledge of extending arithmetic to algebraic expressions and solving one-variable equations. The analysis occurred directly after the students took the pretest. The tests were dichotomously scored based on correctness. If the answer to a problem on the pretest was correct, then the student received one point and if the answer to a problem on the pretest was incorrect, then the student received zero points. The total points possible on the pretest was 20. Thus, if a student missed five points, then his or her score was 15/20 or 75%. The objective of the pretest was to examine the problem types that students were able to solve and to inform the researcher of students' existing mathematical knowledge, misconceptions, and strategies, which were taken into account when creating the hypothetical learning trajectory and instructional unit.

**Phase 2.** Phase 2 of the data collection and analysis involved three parts: teach/reflect/plan, interviews/discussion, and prospective analysis. During phase 2, the actual teaching experiment took place. The research team planned each lesson in the instructional unit, the teacher taught the lessons, and interviews and prospective analysis informed succeeding lessons. The researcher followed a detailed protocol for data collection and analysis (Appendix C) and this cycle continued throughout the teaching of the instructional unit and lasted four weeks. The data collected in phase 2 included video of the teaching sessions, video of the student discussions and comments, video and field notes of the teacher/researcher debriefing interviews, lesson plans, and student work (See Figure 22).

*Teach/Reflect/Plan.* The teach/reflect/plan part of phase 2 included making decisions about the teaching approaches, the types of activities and tasks, and the order and ways that would address the objectives of the research (Molina et al., 2007). In this teaching experiment, the objective was to investigate the means of supporting and organizing student learning of algebra. These decisions about planning the teaching episodes occurred only after completing a thorough review of literature in mathematics education related to algebra.

*Interviews.* The teacher and researcher had debriefing interview sessions after each teaching episode. The teacher and researcher also had a final debriefing interview session after the instructional unit was complete. During these interviews, the teacher and researcher used the Lesson Reflection Protocol (see Appendix D) to guide their discussions. The researcher took field notes and the sessions were recorded on video.

*Prospective analysis.* Data analysis was performed during two periods of the research process: throughout the study and after the teaching experiment had been completed. The early analyses of data were performed after each teaching episode. These prospective analyses informed the modifications of the interventions during future teaching episodes and facilitated creation and revision of hypotheses and conjectures (Molina et al., 2007).

Data sources collected during this phase included video recordings of the teaching sessions, video of student discussions and comments, video recordings and field notes of student interviews and discussions, video recordings and field notes of teacher/researcher debriefing interview sessions, lesson plans, and student work. The following figure describes in detail how each source was collected.

**Figure 23.** Data Sources Collected in Phase 2

| **Video – Teaching Sessions** |
|---|
| ▪ All teaching sessions were recorded using two video cameras.<br>▪ Camera 1 was in the front of the room and focused on the teacher and the board at the front of the room to capture what was being written or modeled on the board. Camera 1 was zoomed in.<br>▪ Camera 2 was in the back of the group and captured the conversation that was occurring. Camera 2 was zoomed out. |
| **Video – Student Discussions and Comments** |
| ▪ During whole class discussions, Camera 1 was in the front of the room and was zoomed in on the written work. This camera focused on the teacher and students as they wrote and discussed. Camera 2 was zoomed out and in the back of the group and captured the conversation that occurred.<br>▪ During small group discussions, Camera 1 focused on the group interaction. Camera 2 focused on capturing a cross section of conversations taking place with other small groups who consented to be video recorded. The purpose was to capture different ways students thought about a particular problem. |
| **Video/ Field Notes – Teacher/Researcher Debriefing Interview Sessions** |
| ▪ The researcher and teacher debriefed after each teaching session and debriefed during a final session after the instructional unit was complete.<br>▪ The researcher and teacher used the Lesson Reflection Protocol Questions (Appendix D) to guide their discussions.<br>▪ Camera 1 recorded these sessions and the researcher took field notes. |
| **Lesson Plans** |
| ▪ Lesson plans were collected and modified throughout the teaching experiment. Field notes were taken on the lesson plans. |
| **Student Work** |
| ▪ All written student work was collected each day after the teaching session. |

*Figure 23*. A description of the data sources collected in phase 2 of the teaching experiment.

The prospective analysis of the data occurred every day after each teaching episode. During this analysis, the lead researcher made a checklist to document student learning and made typed logs in separate documents of class learning, misconceptions, the key mechanisms for shifts in student thinking, changes made to the lesson plans, and teacher interviews. Then, these logs were analyzed and condensed, using the interpretive framework (Figure 21), which created snapshots of each day that documented what occurred during each teaching episode. The following figure shows an example of the process of analysis for a single teaching episode that occurred on Day 4.

**Figure 24.** Day 4 Example of Analysis

**Checklist based on Student Work**

| Day 4<br>Symbolizing with Variable | Combining Like Terms |
|---|---|
| [redacted] | |
| x | x |
| x | x |
| x | x |
| a | |
| [redacted] | |
| x | |
| x | |
| [redacted] | |
| x | x |
| x | |
| x | x |
| x | |
| [redacted] | |
| | |
| x | |
| x | |
| x | |
| [redacted] | |
| x | x |
| x | x |
| x | x |
| x | x |
| [redacted] | |
| | x |
| x | x |
| x | x |
| | x |
| [redacted] | |

**Logs based on Data Sources**

| **Class Learning Log:** 9/12/13 |
|---|
| ▪ In small groups, students were able to count the number of hexagons and pentagons on a soccer ball and represent hexagon with *h* and pentagon with *p*. Some students were able to write an equation representing the number of hexagons on a soccer ball. They needed more time to do the rest of the lesson.<br>▪ Some students do not understand that *h* is just one hexagon and not all the hexagons. The students are beginning to see that equation means the same on both sides. They understand the difference between expression and equation. |

- To reinforce like terms, we will finish this lesson tomorrow and do the worksheet. We will also work on multiplication by teaching the cost of soccer ball lesson to show that 2x means 2 • x. Students that come up to the board and show their work need to ask classmates if they have any questions.
- Need to clarify that the variable is the number of objects. (not number of sides of hexagon)

**Misconceptions:** 9/12/13

| | **Misconception** | **Remedy** |
|---|---|---|
| **Like Terms** | 10h + 12p = 32 | Teach like terms again using fruit. |
| | h + p = hp | Again, more on like terms tomorrow. |
| | h = hexagons instead of h = 1 hexagon | Some understood that h is representing one hexagon. Clarify tomorrow |
| | 10h • 2h = 20h | Teach multiplication |

**Key Mechanisms for Shifts in Student Thinking:** 9/12/13

- What does the variable represent? 1 hexagon or all the hexagons?
- Does the variable represent the number of sides of a hexagon?
- One student thought that the variable represented 6 since a hexagon has 6 sides.

- What do both sides of the equation represent? Are they the same?
- Keep reiterating that the variable is not just a random letter. It means something.

- How can you show 20h in other ways? (Such as 5h + 15h = 20h)
- Can we write this as 20h = 5h + 15h

- It is important to keep showing the equal relationship between both sides of the equation.

**Lesson Log:** 9/12/13

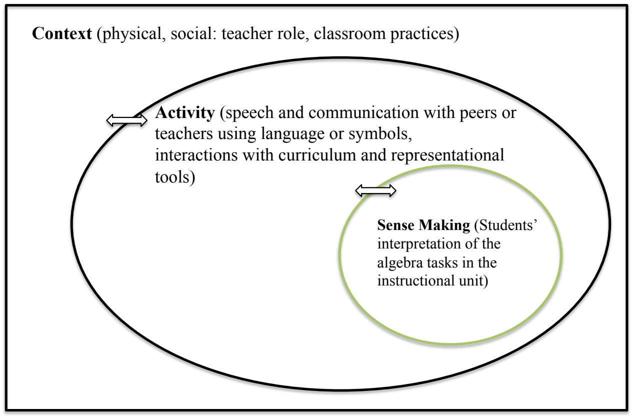
- The lesson went longer than expected. We did not get to defining like and unlike terms.
- Keeping the lesson the same, but splitting it over two classes.
- It took the students a long time to figure out how to represent hexagon and pentagon with a variable.
- If there is time, then we will also do the price of a soccer ball lesson tomorrow, but most likely we will just do like and unlike terms.

**Teacher Log:** 9/12/13

- The lesson went well.
- She thinks the unit is going to take double the time. (I think pacing is okay)
- Teacher thinks it is going to take a lot longer.
- Took coaxing to get students to figure out answer
- Misconceptions: idea of 20 + h means 20h (penguins + hexagons)
- Teacher thinks students understood hexagon means 1 hexagon
- Do the pentagons as a review
- Students knew multiplication better than addition and subtraction
- Seems like they get multiplication
- We will define like and unlike terms tomorrow.

⇩

**Interpretive Framework Used for Analysis**




**Snapshot of the Teaching Episode on Day 4**

| Date/ Activity | Mathematical Meaning | Errors/ Misconceptions | Activity that led to misconception | Context of small groups | Context of whole group | Role of teacher/ teacher conceptions | What did we change and why? |
|---|---|---|---|---|---|---|---|
| **Day 4 9/12/13 Adding and Subtracting Like terms** Students will develop an understanding of variables in mathematics and will learn that like terms can be added and subtracted. Students will also learn to model patterns with algebra. | Variable<br><br>Equality<br><br>Like Terms<br><br>Unlike Terms | Like Terms:<br><br>$10h + 12p = 32$<br><br>$h + p = hp$<br><br>$h$ = hexagons instead of $h = 1$ hexagon<br><br>$10h \cdot 2h = 20h$ | Students counted the number of hexagons and the number of pentagons on a soccer ball and were then asked to represent the total number using variables. | Small groups came up with the total amount of hexagons and pentagons and symbolized using variables as $h + p = T$ | Small groups shared with whole group and began to get to like terms such as $h + h =$ 2h | Took coaxing to get students to figure out the answer.<br><br>Students seemed like they knew multiplication better than addition and subtraction.<br><br>Thinks that we should formally define like and unlike terms.<br><br>Teacher led whole group discussion and facilitated small group discussions. | Took too long and was too messy for students to count the number of pentagons and hexagons.<br><br>Should have told them the total amount of each.<br><br>This lesson took a lot longer than expected so was split over three days.<br><br>This lesson could be renamed "Symbolizing with Variables." |

*Figure 24.* This is an example of the student learning checklist, class learning log, misconceptions, key mechanisms for shifts in student thinking, the lesson log, and the teacher log for Day 4. These logs were condensed using the Interpretive Framework and resulted in the Snapshot of the Teaching Episode on Day 4.

**Phase 3.** Phase 3 of the data collection and analysis consists of the posttest and retrospective analysis.

*Posttest.* The posttest was administered to the whole class and was exactly the same as the pretest. The analysis occurred directly after the students took the posttest. The tests were dichotomously scored based on correctness. The objective of the posttest was to determine individual growth of students in the whole class with regard to extending arithmetic to algebraic expressions and solving one-variable equations.

*Retrospective analysis.* The final analysis was retrospective in that it included analysis of all the data collected during the teaching experiment. The product of this analysis was a historical explanation that details the pattern that emerged from the teaching experiment (Cobb et al., 2003). This pattern might be reproduced in other teaching experiments. Additionally, by conducting retrospective analysis of the data, a theory developed that is informed by the observations of student learning and instruction during the experiment. Moreover, this final analysis is an honest account of what occurred during the teaching experiment (Cobb et al., 2003).

The qualitative data generated by this study was voluminous (Patton, 1980). The retrospective analysis consisted of formal analysis of the video recordings, student work, and field notes. An adaptation of the Data Analysis Spiral (Creswell, 2007) was used to analyze the qualitative data. Creswell (2007) describes this process as “moving in analytic circles rather than using a fixed linear approach” (p. 150). Figure 25 shows an adapted version of Creswell’s (2007) Data Analysis Spiral for a teaching experiment using design research.

**Figure 25.** The Retrospective Data Analysis Loops

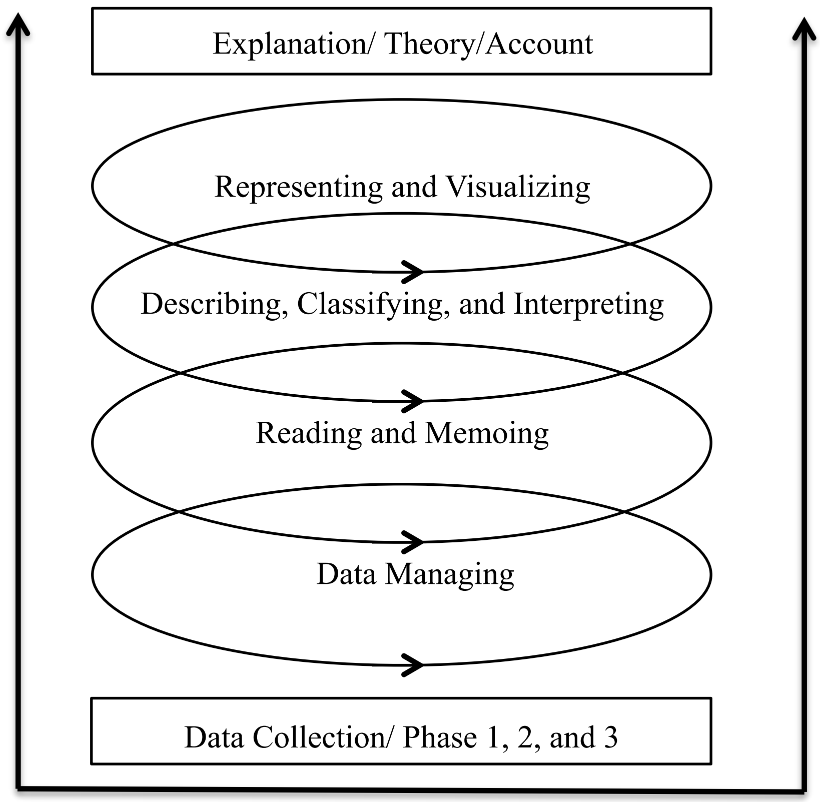



*Figure 25.* The Data Analysis Loops (adapted from Creswell, 2007) for retrospective analysis of qualitative data from a teaching experiment using design research.

After the instructional unit was complete, the data collection during phases 1, 2, and 3 is also complete. The Data Managing is the first ring in the spiral analysis. The researcher organized the data (see Figure 23) after each teaching episode into file folders and computer files. During this loop, the researcher also converted the data to plain text, the correct format for computer analysis. For instance, the researcher transcribed the video recordings using the computer program HyperTRANSCRIBE, which automatically exported the text to plain text. Additionally, during this cycle, the researcher organized all student work into folders that are labeled by date and lesson plan topic.

The Reading and Memoing ring allowed the researcher to get a sense of the data as a whole, before breaking it into themes (Agar, 1980). Creswell (2007) recommends writing memos, key ideas or concepts that occur to the reader, in the margins of field notes or transcripts. Writing memos with the research questions in mind helped the researcher to reflect on the study as a whole. After reading and memoing, the natural process of the spiral continues to the Describing, Classifying, and Interpreting ring. This loop involves coding and categorizing the data into themes as outlined by Creswell (2007). Themes in this teaching experiment emerged from the data where sense making by the students was afforded or constrained by the activity or the context. Because this analysis was purely qualitative, the researcher was engaged in interpreting the data and learned from the data (Lincoln & Guba, 1985).

The last loop of the analysis is the presentation of the data. Creswell (2007) describes this final phase as “a packaging of what was found in text, tabular, or figure form” (p. 154). In this design research study, the product of the analysis is an honest account of what occurred during the teaching experiment and a presentation of a theory of learning that emerged from the data (Cobb et al., 2003). Moreover, the refined and tested hypothetical learning trajectory became the realized learning trajectory.

The quantitative data in this teaching experiment consisted of the pretest and the posttest. The pretest and posttest each contained twenty questions and were scored dichotomously (1/0) based on exactness. One point was given for the correct answer and zero points for an incorrect answer. The means of the pretest and posttest were analyzed using a one-tailed dependent *t*-test. This test compared the two means to see if one mean was significantly greater than the other mean (Field, 2009). Additionally, statistics are reported to show the pre and posttest scores for each individual student. Results are included in a chart.

## Objectivity, Reliability, and Validity in Design Research

Objectivity, reliability, and validity are required for design research to be a scientifically accepted research method. However, these qualities in design research are controlled differently than in experimental research design (Barab & Kirshner, 2001). Unlike traditional empirical research, the goal of design research is to generate new and useful theories (Edelson, 2002). Therefore, objectivity, reliability, and validity of the results in design research cannot be described in the same way as in the traditional empirical approach.

### Objectivity

Objectivity is concerned with the trustworthiness of the analyses (Cobb & Gravemeijer, 2008). In order for this credibility to be achieved, the collection and analysis of data will be systematic and open to criticism by other researchers. Design researchers are in a different position than experimental researchers in that they try to promote objectivity while simultaneously facilitating the interventions. The Design-Based Research Collective (DBRC, 2003) noted that “design-based researchers regularly find themselves in the dual intellectual roles of advocate and critic” (p. 7). Thus, the design researchers will triangulate (Maxwell, 2005) multiple sources and types of data to reduce the risk that the conclusions of the retrospective analysis reflect systemic biases. The data set generated in a design study will include video interviews, video recordings of classroom discussions, copies of students’ work, lesson plans, and field notes. Triangulating the data and using a constant comparison method of analysis (Glaser &

Strauss, 1967) will serve as justification for the final results. Additionally, claims are justified by reviewing the phases of analysis, including backtracking to the original sources of data (Cobb & Gravemeijer, 2008).

**Reliability**

A teaching experiment using design research involves many decisions by the teacher-researcher, observer-researcher, and other members of the design team. Because each learning setting is unique, exact replication of an entire design research study is almost impossible. Instead, to promote reliability of the findings, repetition of the analyses occurred within a single design experiment across cycles (DBRC, 2003). Additionally, the researchers triangulated multiple data sources.

**Validity**

The iterative cycle and collaborative nature of design research over time ensured that the results are valid. In other words, the nature of a design-based research approach allows for increased alignment of theory, design, practice, and measurement (DBRC, 2003). Messick (1992) argued that the consequential validity of a claim is based on the changes that it produces in a given system. In this design research, these changes are considered to be evidence in support of validity (Barab & Squire, 2004). Thus, the instances of student learning triggered by a model in a teaching experiment are considered evidence of the validity of the results.

## Summary

A teaching experiment using design research was conducted in a sixth-grade class. This chapter described the data collection and analysis that was used to inform the development of an instructional unit to support sixth-grade students understanding of algebra as outlined in the CCSS (NGA/CCSSO, 2010). Moreover, this chapter presented the frameworks that were used in this study. The findings from this teaching experiment are presented in the next chapter.

# CHAPTER IV: RESEARCH FINDINGS

This chapter describes the results from the whole class teaching experiment with regards to the realized learning trajectory that emerged. This chapter addresses the following research questions:

1. How do sixth grade students think mathematically when solving arithmetic and algebra tasks in the classroom teaching experiment?
    - What is the whole class learning trajectory that emerges?
2. What are the means of supporting and organizing student learning of algebra?
    - What is the role of the tasks in supporting learning?
    - What design decisions are made to modify the tasks in the instructional unit and why?

In the first section of this chapter, a description of the context of the classroom community is provided. The realized learning trajectory is presented and briefly discussed in the second section. The chapter also contains a theory of learning with an example of how student thinking developed through making mathematical connections and the results of the one-tailed dependent *t*-test. Next, a detailed explanation of how the realized whole class learning trajectory developed is given. Examples of the tasks and how the tasks supported student learning and mathematical thinking are provided throughout the chapter. The conclusion of the chapter is a presentation of how the lesson plans in the instructional unit were modified to support and organize student learning of algebra.

**Context of the Classroom Community**

The teaching experiment took place in a sixth grade classroom where mathematics lessons occurred once a day for four weeks. The lesson times were consistent, occurring on Monday and Thursday from 10:15 – 11:15 a.m. and Tuesday, Wednesday, and Friday from 12:00 to 1:30 p.m. For the majority of the lessons, students worked at their desks, arranged in groups of four. Each group, including the focus group, was assigned a country name that represented a team participating in the 2014 World Cup.

The CCSS for Expressions and Equations (NGA/CCSSO, 2010) was the foundation for the original instructional unit (Version 1 in Appendix A) designed for this teaching experiment. This hypothetical learning trajectory was based on the review of research and includes the following lesson sequence: expressions, equations, and functions. The final version (Final Version in Appendix B) of the instructional unit consists of tasks for small group discussion, whole group discussion, and individual work that occurred at the beginning or the end of the lesson. During whole class discussions, students volunteered to go to the front of the classroom and explain their work. The students and teacher used both the document camera and the board at the front of the classroom to present their work. While volunteers presented, the other students asked questions and agreed or disagreed with the presenter. This generated whole class discussion. The lessons also allowed for small group discussions in which students shared their thinking with their groups and tried to clarify misconceptions. The lesson modifications and the transformation of the instructional unit are presented in detail at the end of this chapter.

The role of the teacher consisted of facilitating discussion and clarifying misconceptions. When students worked in small groups or individually, the teacher walked around the room and interacted with them, asking questions to help students clarify their own thinking. The teacher understood that the instructional unit was a tool for teaching and that she could adapt it as needed based on student understanding of the tasks. During the teaching experiment, the researcher and teacher collaborated daily and weekly to evaluate and revise the sequence of topics in the unit. These revisions were based on the analysis of student learning that took place during the teaching experiment.

## Realized Learning Trajectory

The whole class realized learning trajectory emerged around student understanding of the meaning of the variable in various contexts of expressions, equations, and functions (Figure 25). In the first part of this section, a description of the overall realized learning trajectory that emerged is given. The following sections will provide a detailed explanation of each interpretation of the idea of variable.

**Figure 26.** Realized Learning Trajectory

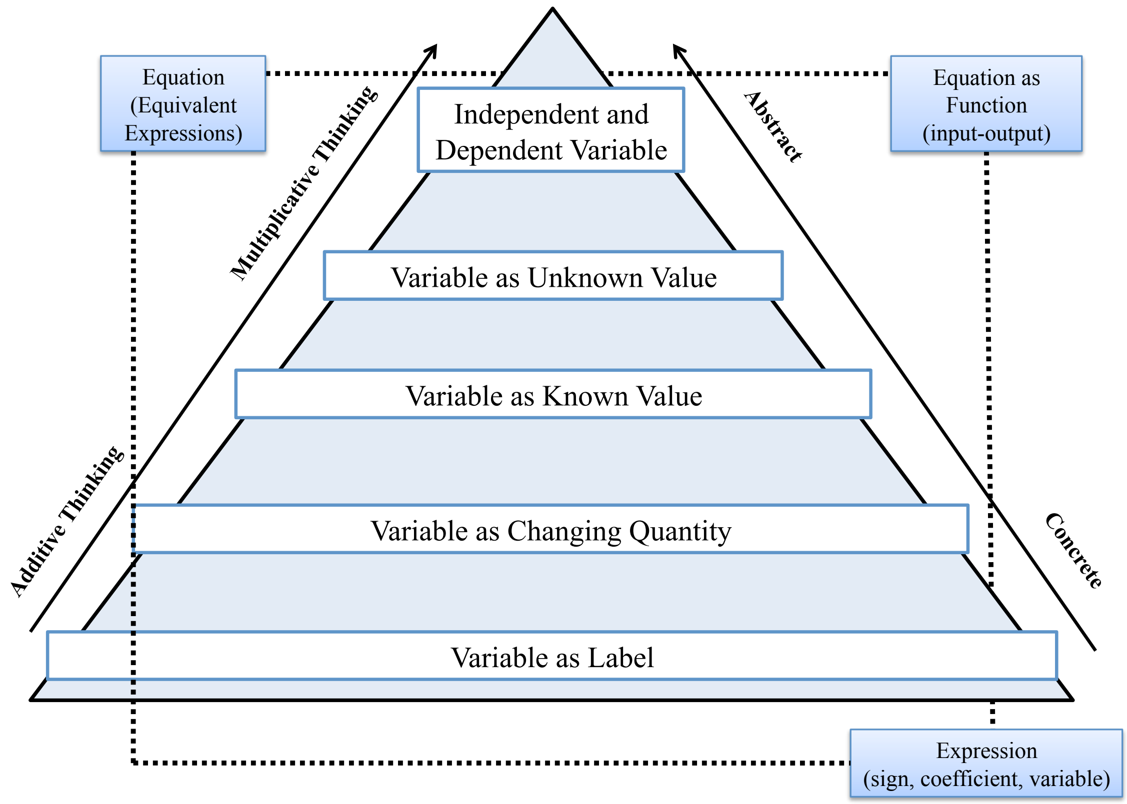
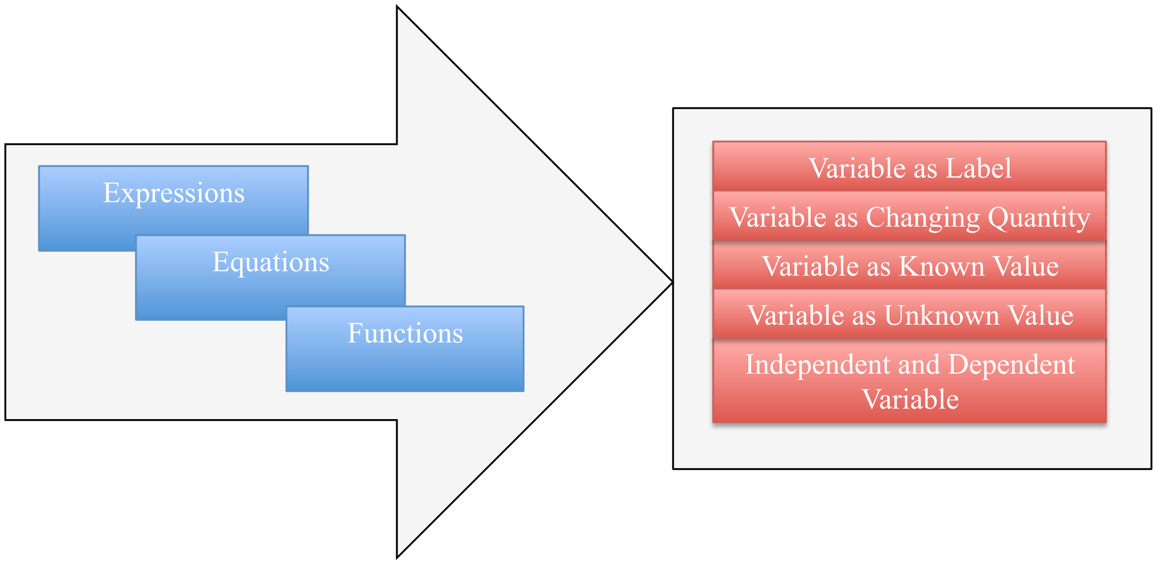



*Figure 26.* This is a depiction of the realized learning trajectory that emerged as a result of the teaching experiment. The class interpretation of variable evolved as they studied expressions, equations, and functions, the sequence of the instructional unit.

The students' interpretation of variable evolved as they explored the meaning of expressions and how the meaning of an expression is linked to the meaning of equation. This initial changing interpretation of a variable became the critical foundation for students to later understand how a variable was interpreted in a function. Figure 27 shows how the changing meaning of variables emerged from the lesson sequence of expressions, equations, and functions how these interpretations of variables were present in misconceptions and errors in the students' written work.

**Figure 27.** The Lesson Sequence, Learning Trajectory, and Sample Misconceptions




**A sample of misconceptions and errors from student work for:**

| | | |
|---|---|---|
| Variable as Label | $h$ = hexagons<br>$bm$ for blue marbles<br>$3a + 4a + 7b = 14f \rightarrow$<br>3 apples + 4 apples + 7 bananas = 14 fruits | The implication is that the students are labeling objects with variables. |
| Variable as Changing Quantity | Modeling "The Cost of 5 balls with an expression that can be used at different soccer ball shops", the student wrote: $5 = d$. | The implication is that this is modeling the number of balls (using a known value) and not the cost of each ball (a changing quantity) and so this student does not understand that the cost is a changing quantity and the expression should be $5d$. |
| Variable as Known Value | Given $4c + 3f$ for $c = 2$ and $f = 5$, student wrote:<br>$4 \bullet c(2) = 8$ and $3 \bullet f(5) = 15$ then $8+15 = 23$ | The implication is that the student does not understand that the variable no longer needs to be written once the value of the variable is known. |
| Variable as Unknown Value | Given the equation $21 = 7u$, student subtracted 7 from both sides to get $u = 14$. | The implication is that the student was attempting to solve for the unknown variable but used the wrong operation and needs to understand that $7u$ means |

| | | |
|---|---|---|
| | | 7• $u$. |
| Independent and Dependent Variable | When asked to write an equation for a given situation that included an independent and dependent variable, the student wrote: $5x + 20$, instead of $y = 5x + 20$. | The implication is that the student does not understand that there is an independent and dependent variable because this is an expression and not an equation with an input and output. |

*Figure 27*. The changing meanings of a variable emerged through the hypothetical learning trajectory of expressions, equations, and functions and were present in the student misconceptions and errors.

Within this sequence of three phases, students' mathematical thinking revealed five meanings of variable: variable as label, variable as changing quantity, variable as known value, variable as unknown value, and independent and dependent variable. As the meanings of variable became more refined, the type of thinking moved from additive to multiplicative and the depth of understanding moved from concrete to abstract (Figure 26).

Although the instructional unit was based on the progression from expressions to equations to functions, the sixth-grade students were not fully able to understand expressions, equations, and functions until the meaning of a variable was made explicit. For example, students learned that variables could be labels, changing quantities, known values, unknown values, and independent or dependent. The following sections describe and provide examples of each interpretation of variable, forms of student reasoning, key mechanisms that shifted student mathematical thinking, and types of thinking.

**Variable as label.** Students first learned variables as a way of labeling certain known quantities or for keeping a record of a specific quantity (Figure 28). By learning a variable as a label, students demonstrated algebraic thinking in that they were able to understand that a quantity could be labeled with a variable.

**Figure 28.** Summary of Variable as Label

| Changing Concept of Variable | Conditions | Focus of Classroom Activity | Forms of Reasoning | Key Mechanisms that Shifted Student Thinking | Type of Thinking |
|---|---|---|---|---|---|
| 1. Variable as Label | Quantity is known<br><br>Using a variable to keep a record of a specific quantity | Finding the total of two groups by writing an expression or an equation.<br>***adults* + *children* = *Total***<br>***a* + *c* = *T***<br><br>Using a sum of like and unlike terms to find the total amount.<br>**8 *red M&Ms* + 4 *blue M&Ms***<br>**8*r* + 4*b*** | Abbreviating the name of a group with the first letter of the word.<br><br>Using a letter to represent one object or one group with the same classification. | Asking students to share different representations of the expression.<br>***adults* + *kids* = *total***<br>***a* + *k* = *t***<br>**2 *adults* + 5 *kids***<br>**2*a* + 5*k***<br><br>Directing students to come up with a more efficient way to represent the same expression.<br>***r* + *r* + *r* + *r* + *r* =**<br>**2*r* + 3*r* =**<br>**5*r***<br><br>Circling sign, coefficient, and variable to combine like terms. | Additive<br>**Finding the Total**<br>***r* + *r* = 2*r***<br><br>Multiplicative<br>**5 • *r***<br>**where *r* = 1**<br><br>Algebraic<br>**Generalizing Arithmetic to Algebra**<br><br>Expressions and Equations<br>**Expression as a sum of like and unlike terms**<br><br>**Equation as computing a total** |

*Figure 28.* Conditions, focus of classroom activity, forms of reasoning, key mechanisms that shifted student thinking, and types of thinking observed as students learned variable as a label.

***Quantity is known.*** Students reasoned that a variable is a label for a known quantity. For example, students began using variables to label a category. Specifically, students were asked to find the total of two groups, boys and girls, by writing an algebraic expression or equation (Figure 29).

**Figure 29.** Student Work Example 1 (Variable as Label)

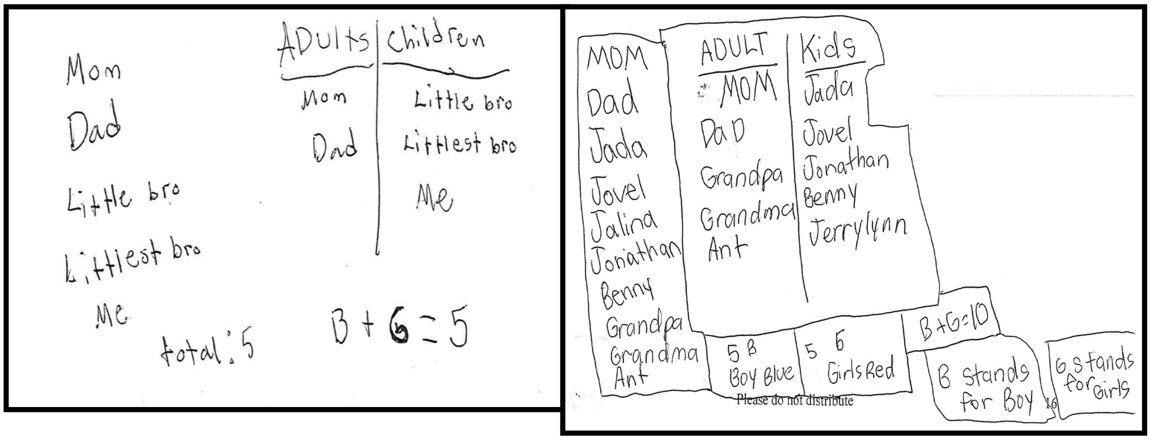



*Figure 29.* Two examples of student work that show that students interpreted variables as labels.

Student Work Example 1 contains two different examples of student thinking where the variable is a label. The example on the left shows that the student labeled 2 boys with *B* and 3 girls with *G*. In the example on the right, the student labeled 5 boys with *B* and 5 girls with *G*. Both students in this example used a variable as a label for boys and a different variable as a label for girls where the variable is a naming specific quantity.

**Figure 30.** Student Work Example 2 (Variable as a Label)

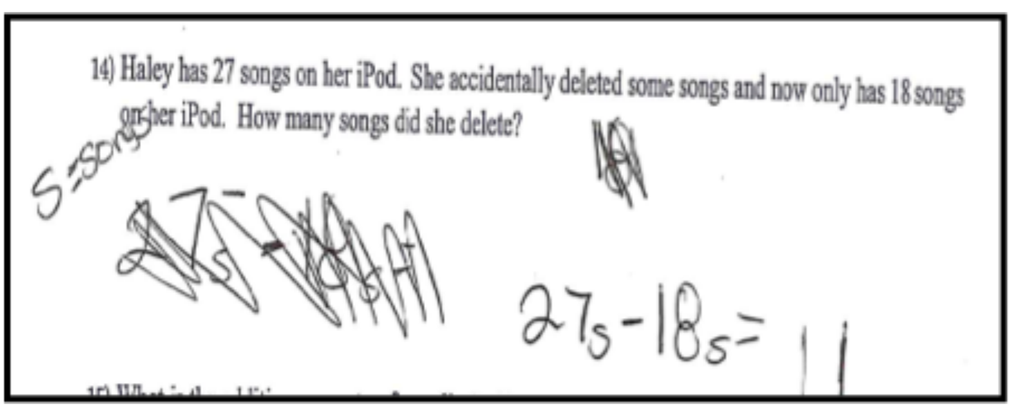



*Figure 30.* An example of student work that shows how a student labeled an item with a variable.

Student Work Example 2 (Figure 30) is another example of how a student labeled songs with the variable *s*. The variable in this example is used to label songs where 27*s* is interpreted as 27 *songs* and 18*s* is interpreted as 18 *songs*. These examples demonstrate how students used variables as labels to replace a specific category and also show that students were thinking additively and algebraically. Additive thinking occurred because students were adding to find the total and the variable replaced the known group or named a specific amount. These examples also show that students were thinking arithmetically and moving towards algebraic thinking by replacing a known quantity with a variable as a label.

***Keeping a record.*** The next example shows the second case in which students used a variable as a label to keep a record of a certain item or object.

**Figure 31.** Student Work Example 3 (Variable as Label)

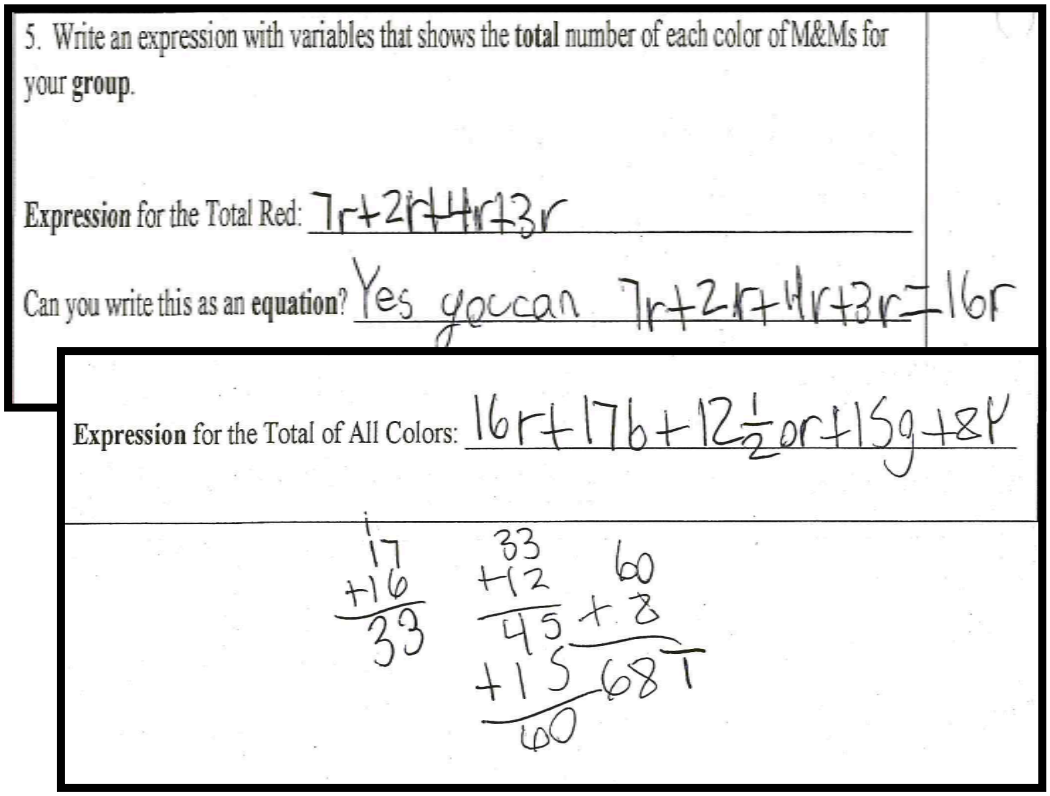



*Figure 31.* In these examples, students used a variable to label one object.

The purpose of the variables in Student Work Example 3 (Figure 31) is to label colors or certain shapes. For instance, 7*r* means "7 red" where the variable *r* labels the color red. In another activity, students recorded that a soccer ball has 20 hexagons, and wrote an *h* to label and replace the word *hexagon*.

Teacher: A is going to write what she has.

A: I have 16 h's plus 4 h's and that equals 20

Teacher: 20 what?

A: 20 h's

Teacher: Which represents? What does h mean?

A: hexagons

Teacher: Why did you write it 20 times? How many hexagons are on the soccer ball?

A: 20

Teacher: If I write a single h, what does that represent?

A: 1 hexagon

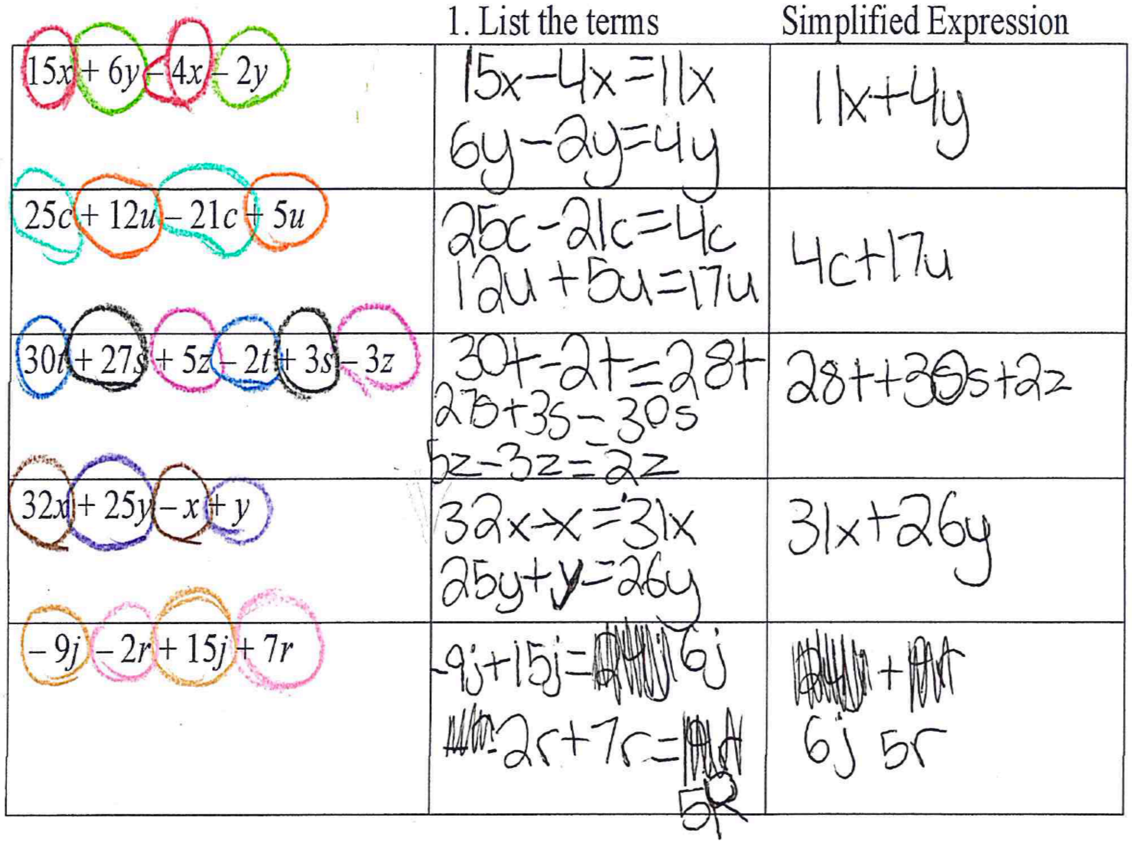
Teacher: So to show 20 hexagons we write?

A: 20h

The *h* is labeling 1 hexagon, so 20*h* means 20 hexagons. In these examples students were thinking both arithmetically by finding a total, and also multiplicatively because they were beginning to grasp the concept of 20*h* as 20 • *h* where *h* is one hexagon.

***Expressions and equations with variable as label.*** Throughout the instructional unit, students learned about expressions and equations. In this stage, variable as label, students understood that an expression is a sum of like and unlike terms and viewed an equation as finding the total amount (Figures 32 and 33).

**Figure 32.** Variable as Label and Expressions



| | 1. List the terms | Simplified Expression |
|---|---|---|
| $15x + 6y - 4x - 2y$ | $15x-4x=11x$ <br> $6y-2y=4y$ | $11x+4y$ |
| $25c + 12u - 21c + 5u$ | $25c-21c=4c$ <br> $12u+5u=17u$ | $4c+17u$ |
| $30t + 27s + 5z - 2t + 3s - 3z$ | $30t-2t=28t$ <br> $27s+3s=30s$ <br> $5z-3z=2z$ | $28t+30s+2z$ |
| $32x + 25y - x + y$ | $32x-x=31x$ <br> $25y+y=26y$ | $31x+26y$ |
| $-9j - 2r + 15j + 7r$ | $-9j+15j=6j$ <br> $-2r+7r=5r$ | $6j$ $5r$ |

*Figure 32*. An example of student work where an expression as a sum of like and unlike terms.

**Figure 33.** Variable as Label and Equations

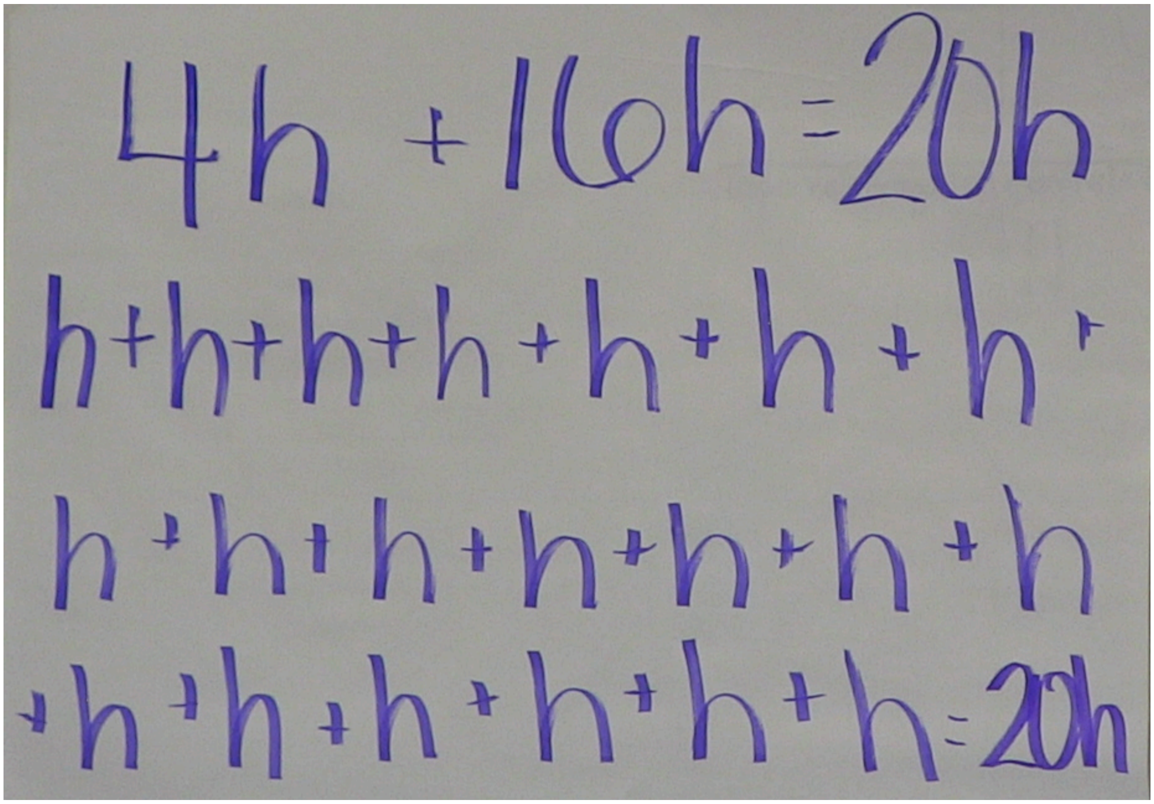



*Figure 33*. An example of student work in which an equation computes the total.

Students viewed the equal sign to mean "compute" and not as "the same on both sides." The variables in both examples are labels of objects, known quantities, or specific categories. During this phase, student thinking evolved from arithmetic to algebraic and students transitioned from additive to multiplicative thinking. However, the idea of variable as a label was present and students needed to view variables as changing quantities. The next section describes student thinking about a variable as a changing quantity.

**Variable as changing quantity.** After interpreting a variable as a label, students understood how to add and subtract like terms, two or more terms that have the same variable raised to the same exponent. However, the idea of a variable as a changing quantity was unclear because, thus far, students had been given a known value and a variable was assigned to that value. A variable as a changing quantity was presented to students within the context of the cost of an item (Figure 34).

**Figure 34.** Summary of Variable as Changing Quantity

| Changing Concept of Variable | Conditions | Focus of Classroom Activity | Forms of Reasoning | Key Mechanisms that Shifted Student Thinking | Type of Thinking |
|---|---|---|---|---|---|
| 2.<br>Variable as Changing Quantity | Variable represents changing values of a specific quantity | Writing an expression that can be used to find the total cost.<br>**4c + s**<br>**where 4c is the cost of 4 packages of 6 cupcakes and s is the cost of a single cupcake** | Symbolizing the price with a variable and acknowledging that this price may change.<br>**c is the cost of a package of 6 cupcakes** | Suggesting to students that the expression is a formula that can be used at any store. | Multiplicative<br>**4 • c**<br>**where c is a changing quantity**<br><br>Algebraic<br>**Using symbols in a Meaningful Way to model a formula**<br><br>Expressions and Equations<br>**Expression as modeling the cost of a known quantity**<br><br>**Equation as computing the total cost** |

*Figure 34.* Conditions, focus of classroom activity, forms of reasoning, key mechanisms that shifted student thinking, and types of thinking observed as students learned variable as a changing quantity.

***Changing values.*** Students began thinking that a variable could be a changing quantity after writing an expression for the cost of a specific quantity of cupcakes (Figure 35).

**Figure 35.** Student Work Example 4 (Variable as Changing Quantity)

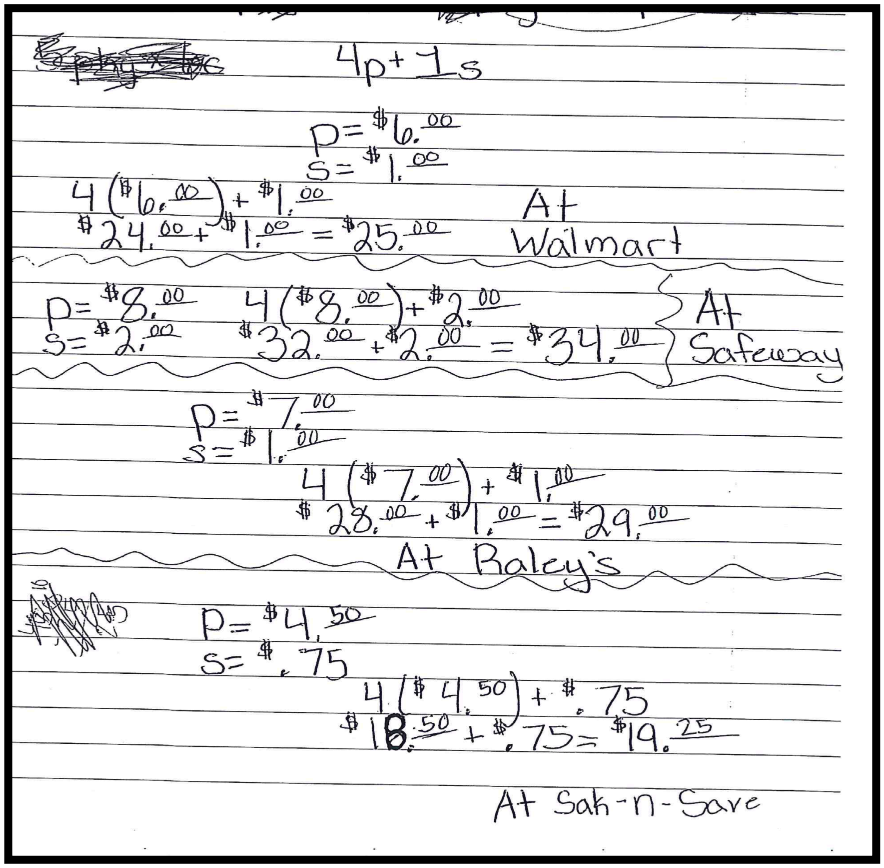



*Figure 35.* Student work that shows how students began to interpret a variable as a changing quantity.

In Student Work Example 4, students tried to make sense of letting the variables be different or changing values. The context of price was important in that students already understood that prices change at different stores.

L: In the cupcake problem, the variable was the cost of the cupcakes.

Teacher: Good. Anyone else want to add to that?

I: A quantity was assigned to the variable.

Teacher: Good. We have a letter or symbol that represents a quantity. What about this quantity though? Remember when we are talking about the cupcakes?

Class: The cost

Teacher: We had different stores. A quantity that can what?

Class: Change

Teacher: Good. A quantity that can change.

J: At every single store the cost changed.

At this point, students were thinking multiplicatively because they knew that $4p$ is $4 \cdot p$. Students' algebraic thinking consisted of using symbols in a meaningful way to write a formula. Also, it is important to note that students were unsure of how to write an algebraic expression, but understood how to write a formula (Figure 36).

Teacher: So, can I use this formula at any store?

C: Yes, as long as you are buying cupcakes. If you are buying cupcakes, then you need the price of the 6 pack of cupcakes and the price of a single cupcake.

Teacher: Okay, anyone have anything to add to that? Or anything different?

M: I disagree because some stores might not have packs of six.

Teacher: Anything different?

G: I say you can because you are going to have to buy the exact same thing. You are going to have to buy 1 cupcake and 4 packages of six but it will just be a different cost.

**Figure 36.** Student Work Example 5 (Variable as Changing Quantity)

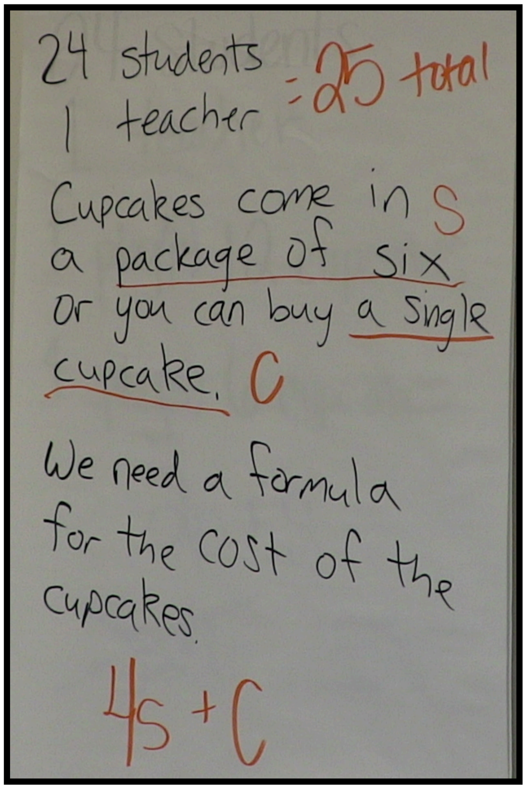
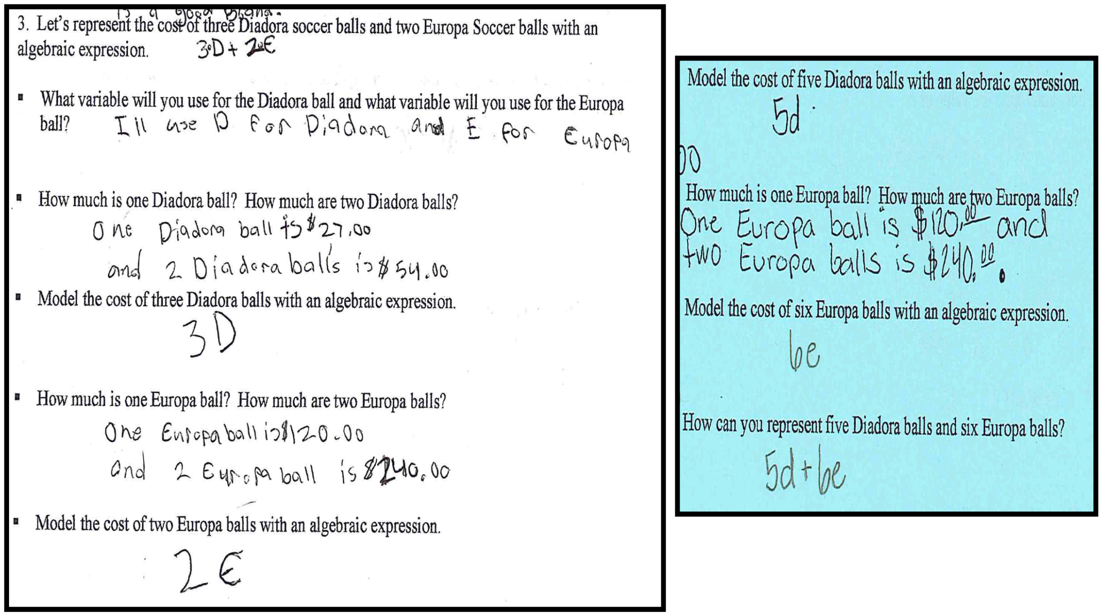



*Figure 36.* Student work that shows how students wrote an algebraic expression and how they interpreted a variable as a changing quantity.

The word *formula* was common to the everyday language used in their classroom and, therefore, students were able to transition to writing a formula with variables to recognizing that a formula can also be called an "expression".

***Expressions and equations with variable as changing quantity.*** As students learned that a variable could be a changing quantity, they transitioned from viewing an expression as combining like and unlike terms, to an expression that models the cost of a known quantity (Figure 37).

**Figure 37.** Expressions and Variable as Changing Quantity



3. Let's represent the cost of three Diadora soccer balls and two Europa Soccer balls with an algebraic expression. 3D + 2E

- What variable will you use for the Diadora ball and what variable will you use for the Europa ball? I'll use D for Diadora and E for Europa
- How much is one Diadora ball? How much are two Diadora balls?
  One Diadora ball is $27.00
  and 2 Diadora balls is $54.00
- Model the cost of three Diadora balls with an algebraic expression.
  3D
- How much is one Europa ball? How much are two Europa balls?
  One Europa ball is $120.00
  and 2 Europa ball is $240.00
- Model the cost of two Europa balls with an algebraic expression.
  2E

Model the cost of five Diadora balls with an algebraic expression.
5d

How much is one Europa ball? How much are two Europa balls?
One Europa ball is $120.00 and two Europa balls is $240.00.

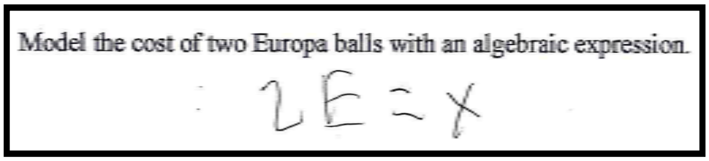
Model the cost of six Europa balls with an algebraic expression.
6e

How can you represent five Diadora balls and six Europa balls?
5d + 6e

*Figure 37.* These examples of student work show that students used an expression to model the cost of a known quantity.

In Figure 37, the coefficients in the expressions are representing the known quantity and the variable is representing the price. For example, in *3D*, the coefficient, *3*, represents three Diadora balls, and the variable, *D*, represents the cost of one Diadora ball. Thus, *3D* represents the total cost of three Diadora balls. Students thought about variables and coefficients in a similar fashion in equations; however, they continued to see an equation as computing the total amount. As students interpreted that a variable could be a changing quantity, they continued to view the equal sign as equivalent to a compute sign (Figure 38).

**Figure 38.** Equations and Variable as Changing Quantity



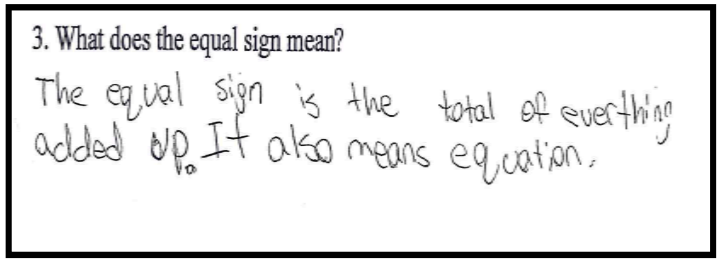




*Figure 38.* These examples of student work show how students used an equation to compute the total cost of a known quantity. In these examples, students interpret an equation as the total of an arithmetic or algebraic problem.

**Variable as known value.** Students' understanding of a variable as a changing quantity and a variable as a known value occurred simultaneously. For example, students viewed variables as changing quantities and were immediately able to substitute known values for the variables (Figure 39). Specifically, students were given the price for one package of 6 cupcakes and for a single cupcake at three different grocery stores and were asked to find the total cost of 24 cupcakes by substituting the known values for the variables into the predetermined expression, $4c + s$, where $c$ is the price of one package of 6 cupcakes and $s$ is the price of a single cupcake.

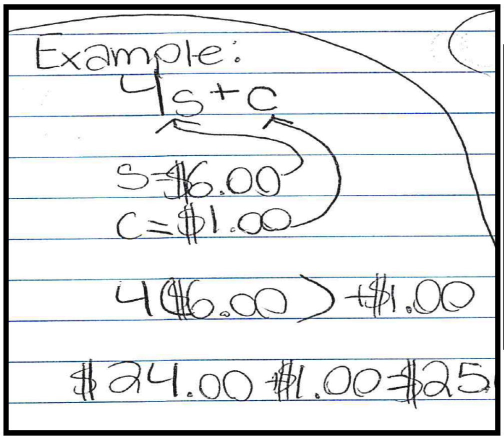
**Figure 39.** Summary of Variable as Known Value

| Changing Concept of Variable | Conditions | Focus of Classroom Activity | Forms of Reasoning | Key Mechanisms that Shifted Student Thinking | Type of Thinking |
|---|---|---|---|---|---|
| 3.<br>Variable as Known Value | Value for variable is given<br><br>Known value can be substituted for the changing quantity | Providing the price of an item at different stores and asking students to find the total cost using the expression.<br>**At Walmart, one package of six cupcakes is \$6 and a single cupcake is \$1.**<br>**At Safeway, one package of six cupcakes is \$8 and a single cupcake is \$2.**<br>**At Raley's, one package of six cupcakes is \$7 and a single cupcake is \$1.** | Substituting a given value for the variable.<br><br>Understanding that the coefficient is multiplied by the value of the variable.<br>**4*c* means 4 • *c*** | Developing the rule of "plug and chug" where once you plug in the value, the variable disappears and terms can be combined.<br>**4*c* + *s* for *c* = 6 and *s* = 1**<br>**4(6) + 1 = 24 + 1 = 25** | Multiplicative<br>**4*c* means 4 • *c***<br><br>Additive<br>**After plug and chug, combine by adding or subtracting**<br><br>Algebraic<br>**Using symbols in a Meaningful Way to evaluate an expression**<br><br>Expressions and Equations<br>**Using an expression to substitute values for variables**<br><br>**An equation is the final computation of the total of the expression** |

*Figure 39.* Conditions, focus of classroom activity, forms of reasoning, key mechanisms that shifted student thinking, and types of thinking observed as students learned variable as a known value.

***Known value.*** In order for a variable to be a known value, the value for the variable must be given and this known value must be substituted for the variable. Figure 39 shows how students thought about substituting the known value for the variable by drawing arrows. Additionally, students were thinking multiplicatively. For example, they understood that 7*u* means 7 • *u* and once a value is substituted for *u*, it must be multiplied by 7.

**Figure 40.** Student Work Example 6 (Variable as Known Value)



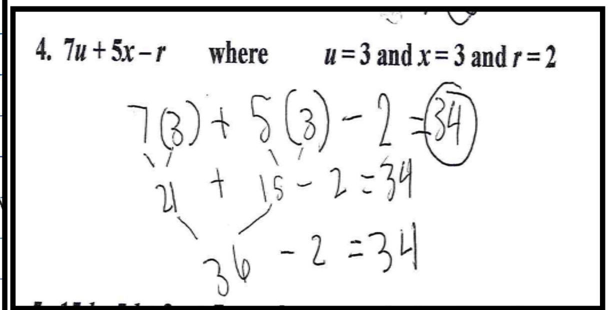




*Figure 40*. This student work shows that students interpreted variables as known values because the value for each variable was given.

The idea of "plug and chug" solidified student thinking about substituting a known value for a variable. Before students learned plug and chug, they did not understand that the variable in an expression was completely replaced by the known value.

Teacher: So some of us are getting it and some aren't. We will keep practicing. When I have 4*s* what does that mean what do I do?

J: Don't you multiply it?

Teacher: Who agrees with J? Do we multiply? 4*s* means 4 times *s*?

*Most raise their hand*

Teacher: Raise your hand if you think its something else. Those of you who didn't raise your hands, are you unsure?

B: Yes

Teacher: Now what do I do for *s*?

S: The *s* stands for 1 isn't it?

Teacher: That is when the variable is by itself. If I now know that it is this…

J: It would be 4 times 2 dollars and 50 cents.

Teacher: 4 times 2 dollars and 50 cents. Do you remember what this is called?

Class: Plug and Chug

Teacher: Plug and chug means plug in what you are given and then chug out your answer.

In this discussion about plug and chug, the students were thinking multiplicatively because they agreed that $4s$ means $4 \cdot s$. Additionally, Student J concluded that the variable is a known value because the price, $4.50, is given in the problem context. Further, written cues, such as drawing arrows or erasing the variable and immediately replacing it with a quantity, in conjunction with the verbalization of plug and chug, assisted students with learning replacement of a variable with a known value (Figure 41).

**Figure 41.** Student Work Example 7 (Variable as Known Value)

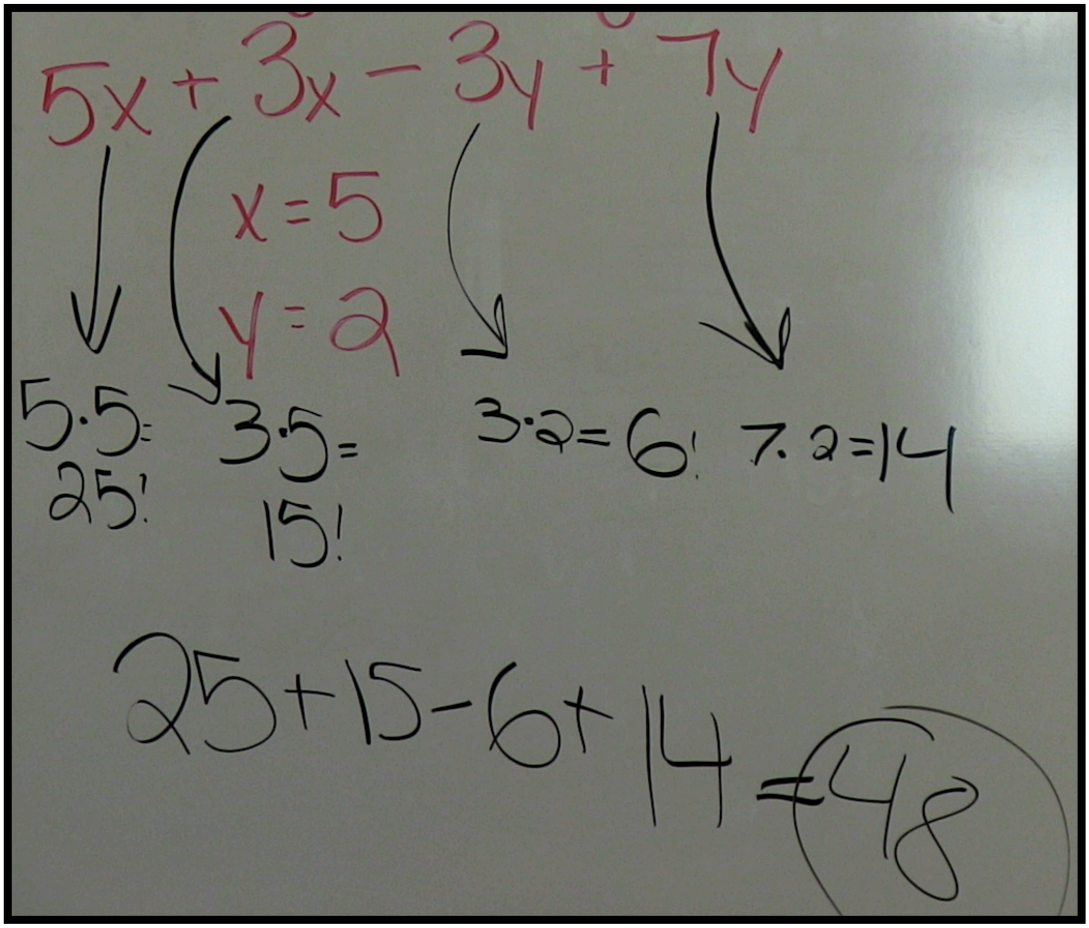



*Figure 41*. This example shows arrows as written cues that allowed students to visualize substitution of a value for a variable.

Students also struggled with the idea that once values were substituted for the variables, the terms could be combined. For example, students understood that unlike terms could not be added or subtracted; however, they struggled with the idea that once a known value was substituted for the variable, the terms became quantities, and like terms, that could be combined. It was critical to create rules for the students to clarify that although the original expression might have unlike terms that cannot be added or subtracted, once values are substituted for the variables, simplification of the expression by combining like terms became possible. Figure 42 shows the rules created by the teacher to support the students' learning of variable as a known value.

**Figure 42.** Rules for Variable as Known Value

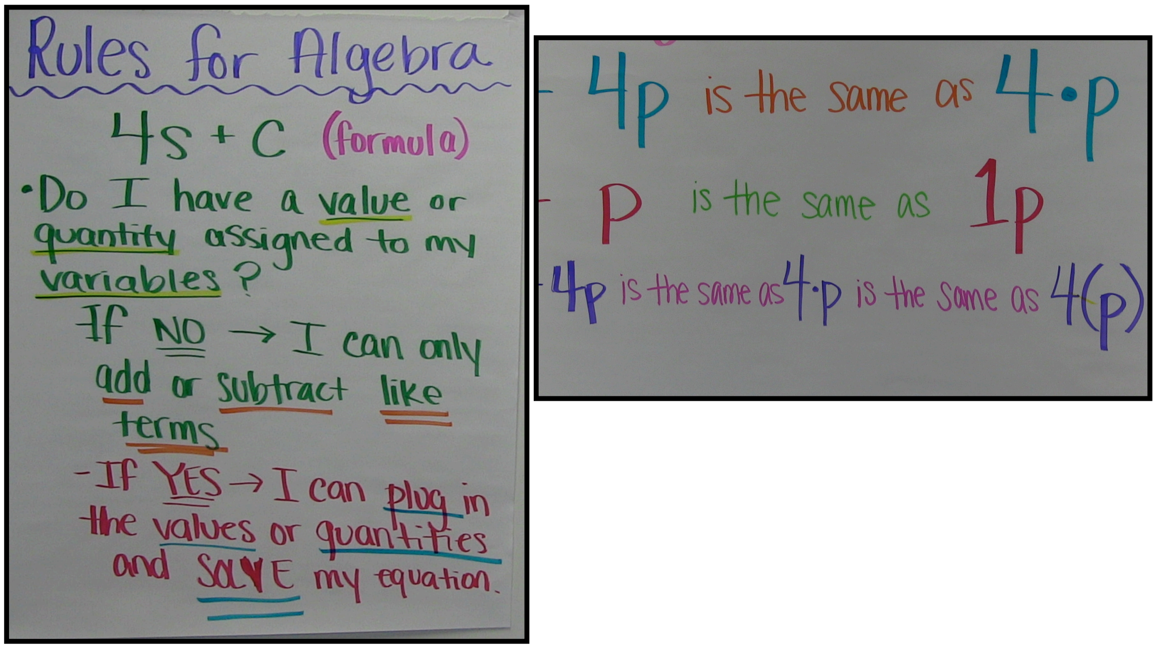
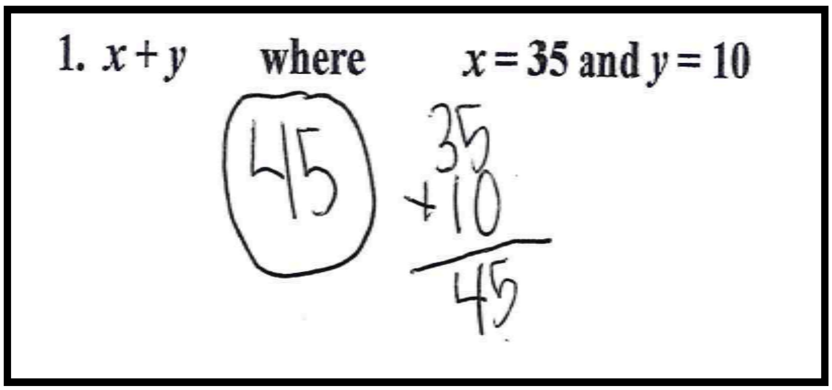



*Figure 42*. The teacher wrote these rules for algebra to assist students with learning the variable as a known value.

***Expressions and equations with the variable as a known value.*** Students continued to think additively, multiplicatively, and algebraically during this stage of the instructional unit. Additive thinking occurred as students substituted the known values into an expression and then were able to combine the like terms. Multiplicative thinking was present as students substituted the known value for the variable in an expression, such as *4p,* where the coefficient must be multiplied by the known value of the variable. At this point, students were naturally thinking algebraically because meaningful substitution of known values for variables took place to simplify an expression. Moreover, students began with concrete mathematical situations and were able to transition the problem into abstract expressions or equations. For instance, students modeled the cost of a 6 pack of cupcakes and a single cupcake with an algebraic expression and then were able to find the total cost using given prices for the packages of cupcakes. Once students learned that a variable could be a known value, they viewed an expression as an instrument for substitution (Figure 43). An equation simply became the final computation after the plug and chug was complete (Figure 44).

**Figure 43.** Expressions and the Variable as Known Value



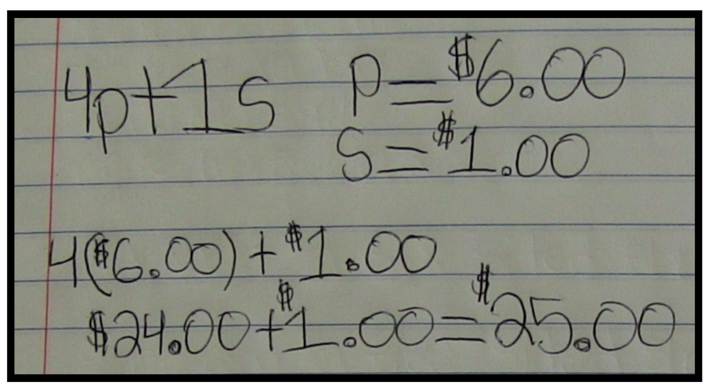




*Figure 43.* An example of student work that shows using an expression to substitute values for variables.

**Figure 44.** Equations and Variable as Known

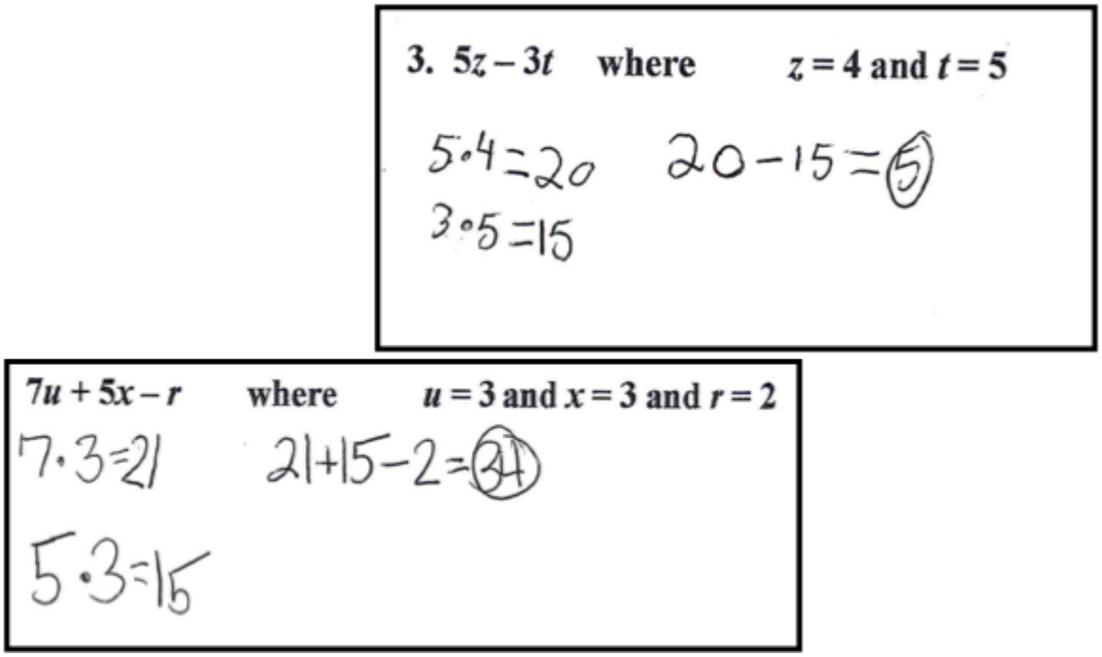



*Figure 44*. These examples demonstrate that students viewed an equation as the final computation of an expression.

**Variable as unknown value.** Students transitioned from viewing the variable as a known value to the variable as an unknown value. In this situation, students had to find the value of the variable by solving for the unknown. Figure 45 provides a summary of the variable as an unknown value. At this point, students were able to make connections between variable as a label, variable as a changing quantity, variable as a known value, and variable as an unknown value, demonstrating relational thinking by understanding the relationship between these ideas of variables.

**Figure 45.** Summary of Variable as Unknown Value

| Changing Concept of Variable | Conditions | Focus of Classroom Activity | Forms of Reasoning | Key Mechanisms that Shifted Student Thinking | Type of Thinking |
|---|---|---|---|---|---|
| 4.<br>Variable as Unknown Value | Must solve for the unknown<br><br>Unknown variable must be isolated | Balancing a scale to solve equations and Algebra Touch App for isolating the variable.<br>$t + 2 = 6$<br><br>Using the area of a rectangle formula to find the width.<br>$A = lw$<br>**if $A$ is 18 and $l$ is 6, find $w$.**<br><br>Learning distributive property where the variable is number of family members and cost is known.<br>**Let $n$ be the number of family members. Every member of the family needs to buy a backpack that costs $90 and a sleeping bag that costs $60.**<br>**1. Total Cost = $n(90 + 60)$**<br>**2. Total Cost = $90n + 60n$** | Using opposite operations to solve an equation.<br><br>The scale must be balanced.<br><br>The equal sign means that both sides have the same value.<br><br>Identifying when to use AP and when to use MP.<br>**AP is Addition Property of Equality and MP is Multiplication Property of Equality.** | "Making moves" to isolate the variable.<br>$t + 2 = 6$<br>$\underline{\ \ -2 = -2}$<br>$t = 4$<br>**Subtracting 2 is making "one move" to isolate $t$.**<br><br>Plug and chug can be used to check the answer to an equation.<br>**$t + 2 = 6$, so $t = 4$**<br>**Check: $4 + 2 = 6$**<br>**$6 = 6$**<br><br>Distributive property is necessary for simplifying an algebraic expression into an equivalent expression.<br>$2(x - 3) = 2x - 6$<br><br>In arithmetic, distributive property is not necessary, but produces equivalent expressions.<br>$2(3 + 4) = 2(7) = 14$<br>**but this is also:**<br>$2(3 + 4) = 2(3) + 2(4) = 6 + 8 = 14$ | Additive<br>**$90n + 60n = 150n$**<br><br>Multiplicative<br>**$150n$ means $150 \cdot n$**<br><br>Algebraic<br>**Study of Structure where the equal sign means "the same as"**<br><br>Expressions and Equations<br>**Equation is the same quantity on both sides**<br><br>**Equation as balanced**<br><br>**Equation is seen as equivalent expressions** |

*Figure 45.* Conditions, focus of classroom activity, forms of reasoning, key mechanisms that shifted student thinking, and types of thinking observed as students learned variable as an unknown value.

To fully understand the variable as an unknown value, students had to realize that the value of the variable was not provided. In other words, it was necessary to solve for or isolate the variable to solve the equation (Figure 46 and Figure 47). The following discussion demonstrates how the teacher presented and the class interpreted when a variable was an unknown value.

Teacher: What is the third definition for variable?

L: An unknown variable

Teacher: Anyone want to add or change that definition? I, what did we talk about when it came to x? We have $6 + x = 9$. What do we have to do?

I: Solve.

Teacher: Solve for the unknown *x*. Who else wants to add to that? The third definition is a letter or symbol that represents an unknown. That example is $6 + x = 9$ and $x = ?$ Do we know what x is.

J: Yes. You subtract 6 from 9 and get 3.

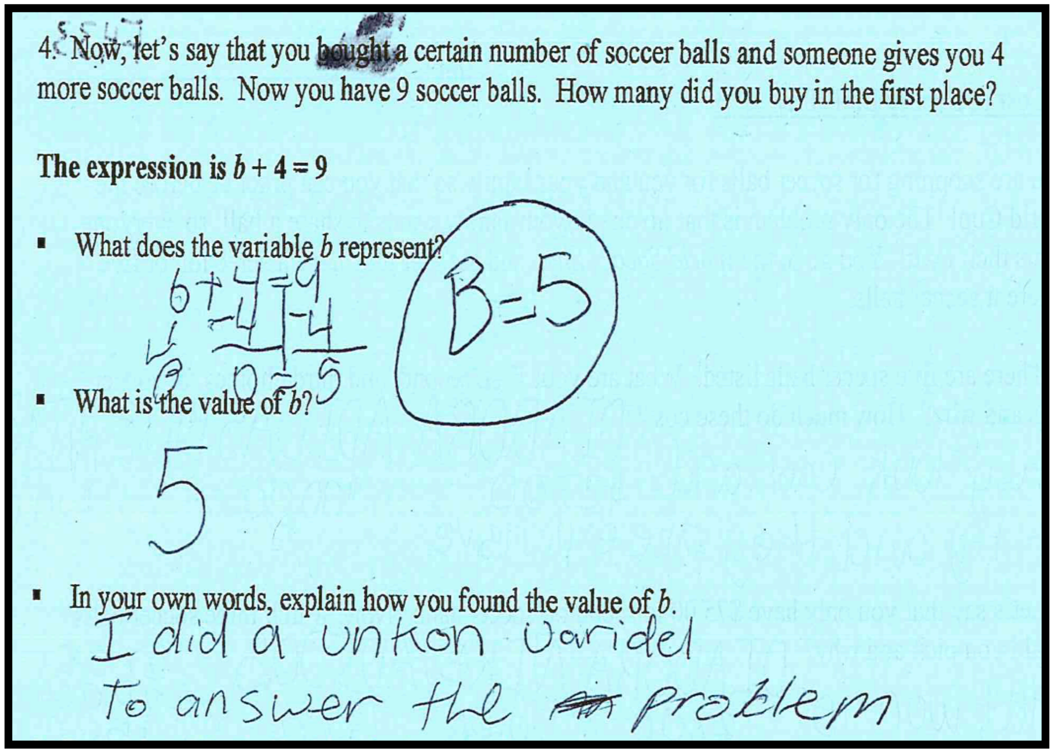
Teacher: When we first see this do we know what *x* is?

Class: No.

Teacher: Just looking at the problem, we don't know what *x* is so we need to solve for *x* or solve for the variable. Unknown variable.

The teacher explained to the students that a variable is unknown if the value for the variable is not given. In other words, students understood that if they had to solve for the variable, then it was an “unknown variable”. If they did not have to solve for the variable, meaning that the value for the variable was provided, then the variable was interpreted as a “known variable”.

**Figure 46.** Student Work Example 8 (Variable as Unknown Value)



4. Now, let's say that you bought a certain number of soccer balls and someone gives you 4 more soccer balls. Now you have 9 soccer balls. How many did you buy in the first place?

**The expression is *b* + 4 = 9**

- What does the variable *b* represent?

$b + 4 = 9$
$\;\;\;-4\;\;\;-4$
$0 = 5$

B=5

- What is the value of *b*?

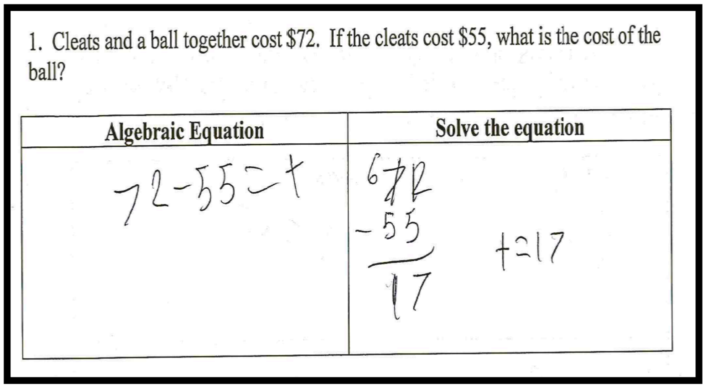
5

- In your own words, explain how you found the value of *b*.

I did a unkon varidel
To answer the ~~an~~ problem

*Figure 46*. This example shows how a student solved for the unknown variable.

**Figure 47.** Student Work Example 9 (Variable as Unknown Value)



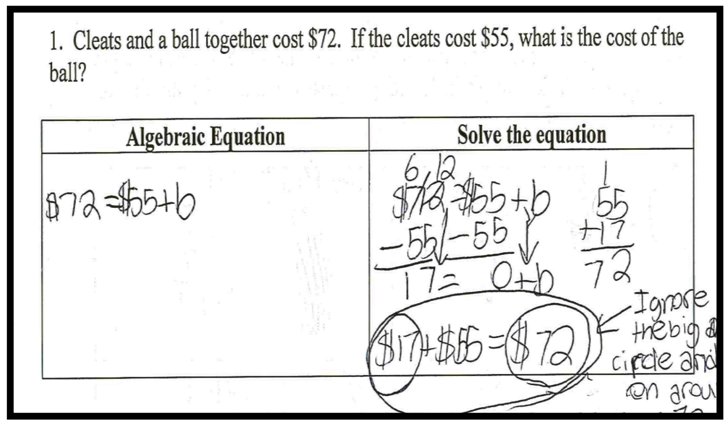




*Figure 47.* Examples of how students solved the same equation by isolating the unknown variable that represents the cost of the ball.

In Figure 47, the two examples of student work show how students modeled the cost of the ball in an algebraic equation. In order to write the algebraic expressions, students had to know that the cost of the ball is represented by a variable that has an unknown value. Further, in both examples, after writing an algebraic equation, the students had to solve for the unknown value of the variable.

***Balancing a scale.*** As students learned that a variable could be an unknown value, they also began to see the equal sign as "the same on both sides" or "balanced". Figure 48 is a photo of an interactive pan scale with different color shapes. This visualization not only made students think about balance, but also forced them to think of opposite operations for solving equations and isolating a variable.

**Figure 48.** Scale and the Variable as Unknown Value

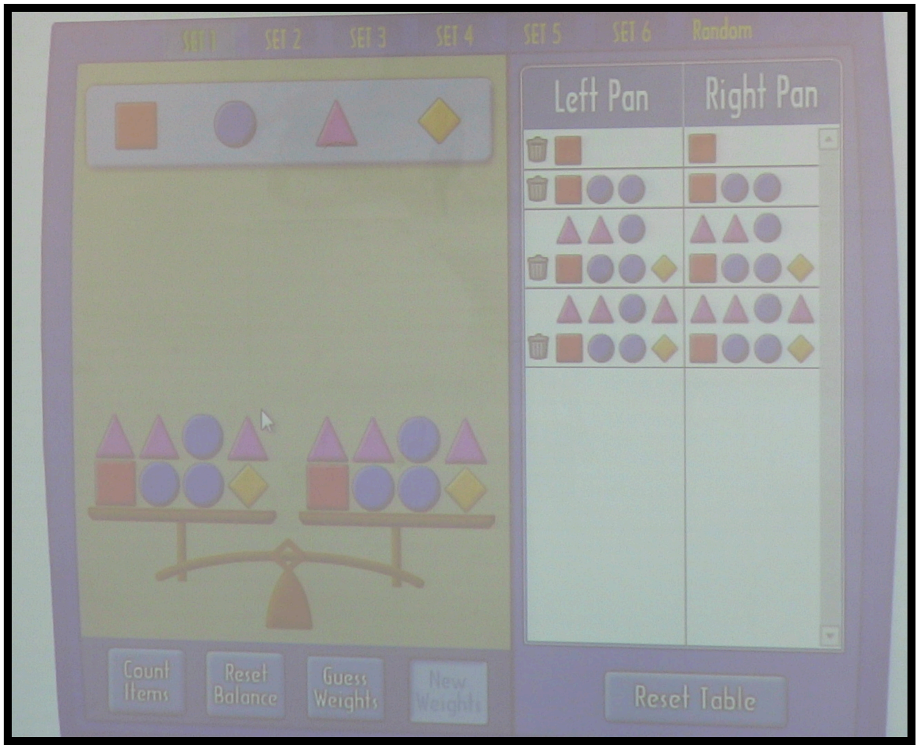



*Figure 48*. A pan scale with different color shapes (retrieved from Illuminations, 2013) used to demonstrate the equality of both sides of an equation.

Moreover, the teacher made the idea of balance and equal explicit to the students by engaging students in the following discussion about balance and equality.

Teacher: What is this?

J: scale

Teacher: What do we use scales for?

G: To measure

Teacher: What is the goal?

O: For it to be equal.

Teacher: What is another word for that?

I: Balanced

Teacher: Balanced. Good. I have four shapes up there. A square, circle, triangle, and diamond. If I have a red square on this side. How do I make it balanced?

Class: Put a red square on the other side.

Teacher: If I add a blue circle or two blue circles, what do I have to do to the other side?

Class: Two blue circles.

Teacher: If I add a yellow diamond and another purple triangle, and another triangle, and a circle.

Class: Yellow diamond, purple triangle, purple triangle, and circle.

J: It's so easy.

Teacher: This shows that when we have a scale you want to make sure it is balanced. That is going to be the same concept when it comes to this third definition of variable. If I am given a problem like $13 = x - 1$, according to this rule, what do I have to do? Raise your hands.

J: Solve for the unknown variable.

Teacher: Solve for the unknown variable or solve for $x$. And I wrote $x$ because on the standardized test most of the time the variable will be $x$. When I do this, I want to think of the scale and remember that both sides are balanced.

Once students visualized the scale, they were able to solve for an unknown variable by balancing an equation (Figure 49). At this point, students were thinking algebraically by studying the structure of an equation and knowing that both sides must have the same value. For example, by viewing the equation in Figure 49 as two sides of a balanced pan scale, students interpreted the equal sign to mean that both sides must be the same or have equal numerical value. More specifically, students realized that the variable, *a*, must have the value of 8 in order for both sides of the scale to have the same value of 13.

**Figure 49.** Student Work Example 10 (Variable as Unknown Value)

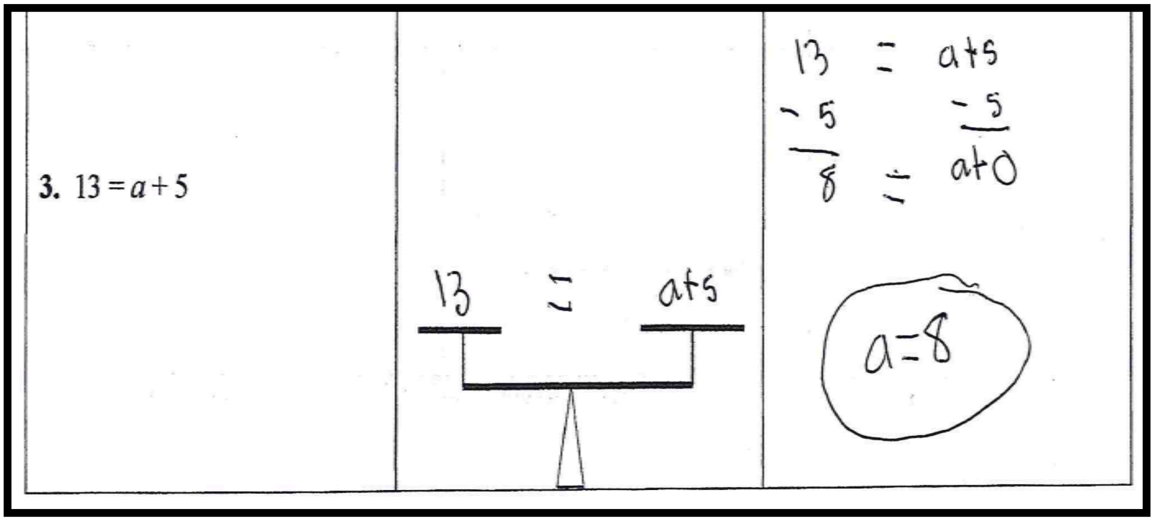



*Figure 49*. An example that shows how a student used visualizing a scale to balance an equation.

***Opposite operations.*** Moreover, students learned that the opposite of addition is subtraction and the opposite of subtraction is addition. Equations are balanced using opposite operations (Figure 50). In this example, a student explained how she subtracted *5* from both sides of the equation since subtraction is the opposite of addition to solve for the unknown variable.

**Figure 50.** Student Work Example 11 (Variable as Unknown Value)

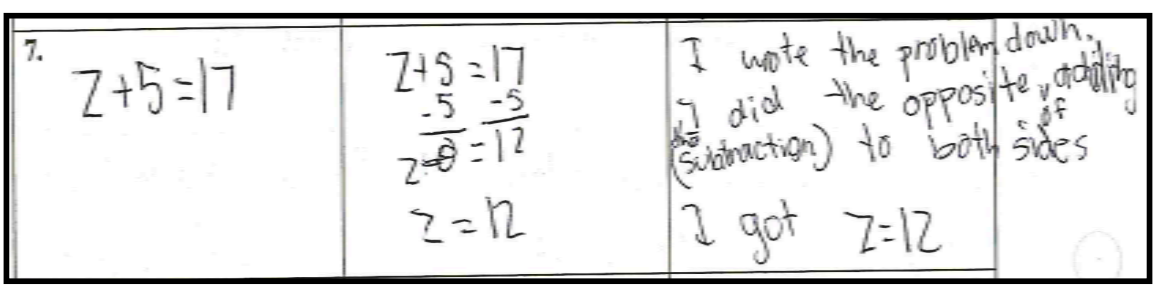



*Figure 50*. An example that shows how a student used opposite operations to solve for the unknown value of the variable.

The whole class discussion below also encompasses how students understood using opposite operations.

| | |
|---|---|
| Teacher: | Let's start with $18 + x = 24$. I have 18 plus $x$ equals 24. I want to do the opposite of whatever symbol this is. Do I have a symbol in front of it? |
| | *(pointing to x)* |
| Class: | No. |
| Teacher: | What does that mean? |
| Class: | Positive. |
| Teacher: | So what is my opposite operation? |
| Class: | Negative |
| Teacher: | Negative or subtraction. So whatever I do to one side I have to… |
| Class: | do to the other |
| Teacher: | Alright, so then that leaves me with 18 minus 18 |
| Class: | Zero |

*Teacher writes out how to subtract 18 from 24 using regrouping.*

| | |
|---|---|
| Teacher: | That leaves me with $x$ equals … |
| Class: | $x$ equals 6 |
| Teacher: | Now I am going to plug it into my original problem and see if I got it right. $18 + 6 = 24$. What is $18 + 6$? |
| Class: | 24 |
| Teacher: | 24 equals 24 |

In this discussion, students used the terms *positive* and *negative* to describe opposite operations. They also knew that negative is interpreted as subtraction and positive is interpreted as addition. Moreover, they realized that whatever operation is applied to one side of an equation, it also must be applied to the other side.

When solving for the unknown, students also identified the Addition Property of Equality or the Multiplication Property of Equality. This is shown in Figure 51 where AP is the Addition Property of Equality and MP is the Multiplication Property of Equality.

**Figure 51.** Student Work Example 12 (Variable as Unknown Value)

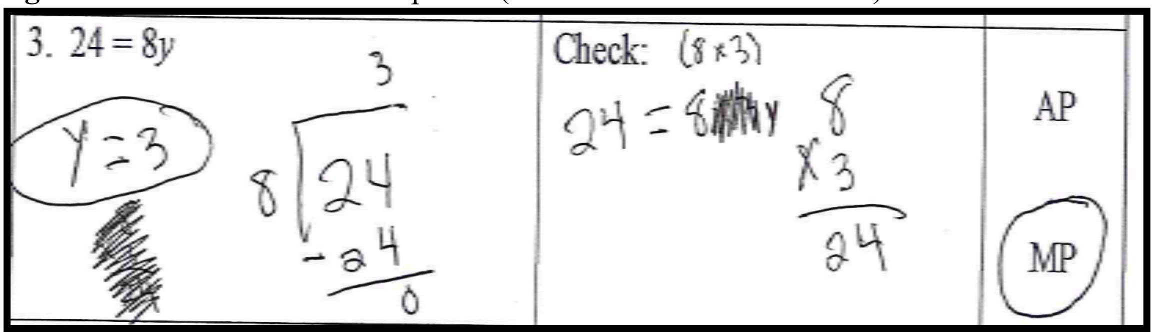



*Figure 51*. This is an example that demonstrates how a student solved an equation by identifying the Addition Property of Equality (AP) or the Multiplication Property of Equality (MP).

The use of reciprocals when solving for the unknown value of the variable was challenging for students. The class discussed the reciprocal of a number and tried to make sense of what it means.

O: One fourth c equals five. I multiplied 1/4 by 20 and got 5.

S: How did you get 20?

O: It takes 1/4 to equal 1.

Teacher: Are you looking for a specific word we learned yesterday?

Class: Reciprocal.

Teacher: What is a reciprocal?

C: When you change the fraction around?

Teacher: And what goes where?

C: The numerator goes on the denominator and the denominator goes on the numerator.

Teacher: Good, so can you use the reciprocal to find your answer in problem number 2?

Class: Yes.

Teacher: Yes, right, so what are you doing with that reciprocal?

O: 4 wholes.

Teacher: The reciprocal is 4 wholes. What did you do to the other side of the equal sign?

O: Multiplied it.

Teacher: By what? Did you multiply 1/4 by 4/1?

O: No

Teacher: Did you just know that 1/4 of 20 was 5?

O: Yes

Teacher: Does anyone have a different way of solving it? He's right.

C: First, I found the reciprocal of 1/4 is 4 and then I did the magic trick and I multiplied them and got 20 and got 20 is equal to c.

The last line shows that Student C understood that using a reciprocal is a “magic trick” because multiplying by the reciprocal eliminates the fraction. However, Student O was still solving for the unknown, but not understanding that he is using a reciprocal; only that 1/4 of 20 is 5.

In the next example of student work (Figure 52), a student solved the equation correctly by using a reciprocal. Moreover, the student wrote two equivalent equations,

$6 \div \frac{1}{3} = w$ and $\frac{1}{3} \bullet w = 6$, which demonstrated her understanding that multiplication and division are opposite operations.

**Figure 52.** Student Work Example 13 (Variable as Unknown Value)

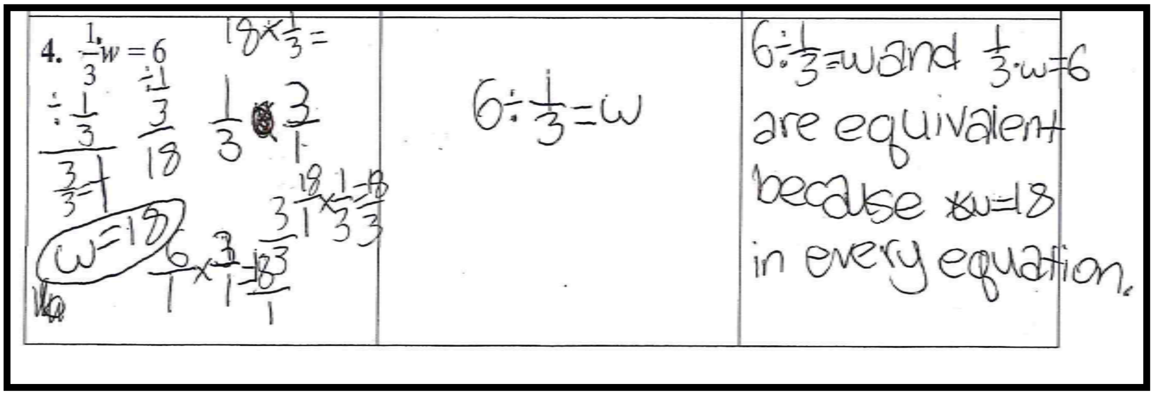



*Figure 52*. This student used the reciprocal, which demonstrated her understanding of opposite operations and equivalent equations.

***Distributive property.*** Students also had to understand the Distributive Property in order to solve equations for the unknown value of the variable. A snapshot of how students made sense of this property is given in the following figures (Figure 53, Figure 54, and Figure 55). In addition, students discussed the Distributive Property. Student E was able to factor out the *n* to explain the Distributive Property to his peers.

C: If we knew what n was then we could multiply it by 150.

Teacher: Okay so what is an equivalent equation to this?

Class: *n* times 150

F: $60n + 90n$

Teacher: Okay, so that is $150n$. What do the 60 and 90 represent?

F: The cost of backpacks and sleeping bags.

Teacher: Alright. Good. How else can we write that thinking of the distributive property? Talk in your groups. I want to see the distributive property.

*Students talk in groups*

E: *n* and then inside the little parenthesis, it could be 90 + 60, so you could multiply the number of people by each of those and then add them up. Want me to show you? Like *n* and then in parenthesis 90 + 60, so *n* would have to multiply by 90 and by 60. The numbers that we get right here, we just have to multiply it.

The student work (Figure 52) shows more examples of how the students were making sense of the Distributive Property.

**Figure 53.** Example of Distributive Property

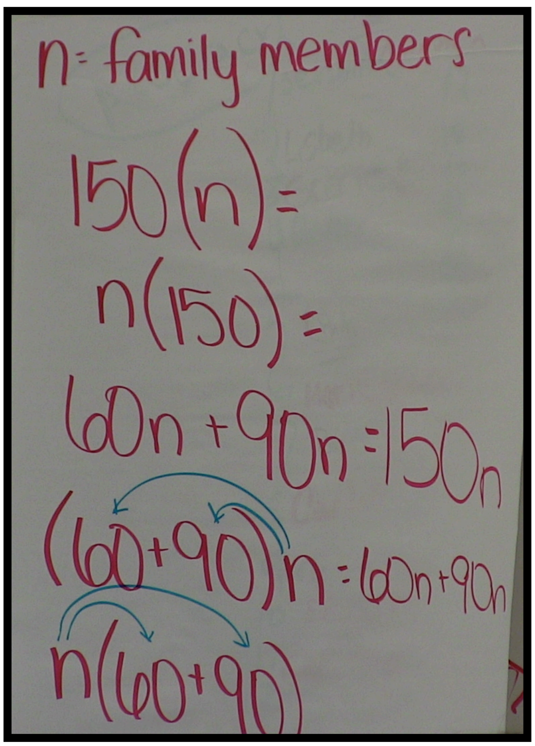



*Figure 53*. This example shows how students drew arrows to make sense of the Distributive Property.

Again, students used arrows to show the "distribute" step. It is also important to note here that students engaged in additive, multiplicative, and algebraic thinking when using the Distributive Property. Additionally, students came to the conclusion that it was not necessary to use the Distributive Property if no variables were present in the given expression (Figure 54).

**Figure 54.** Student Work Example 14 (Variable as Unknown Value)

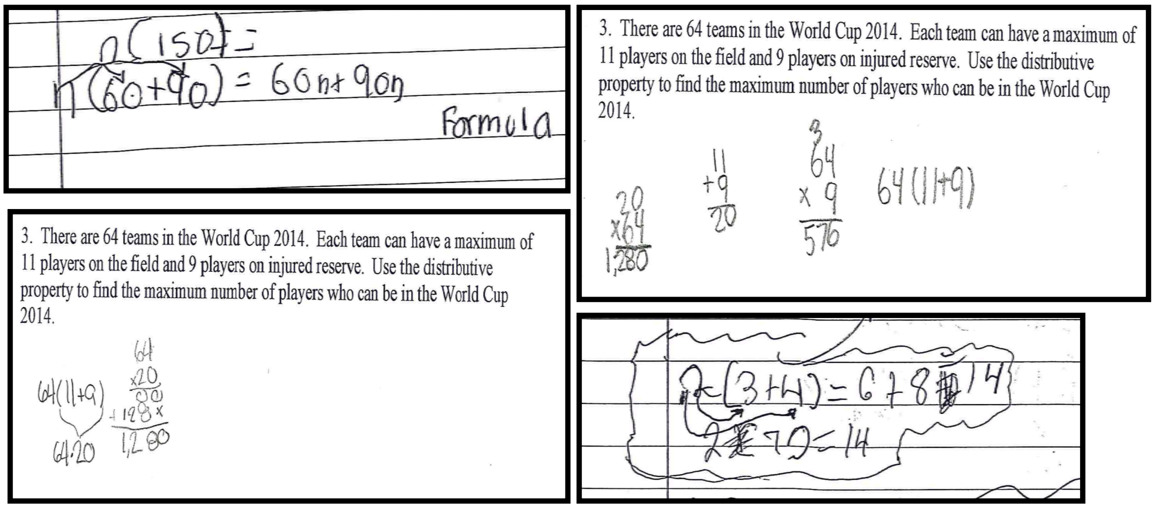



*Figure 54.* These student work examples show students' use of the Distributive Property.

The student work in Figure 55 shows that the student was able to break down an expression by using the Distributive Property and arrive at an equivalent expression.

**Figure 55.** Student Work Example 15 (Variable as Unknown Value)

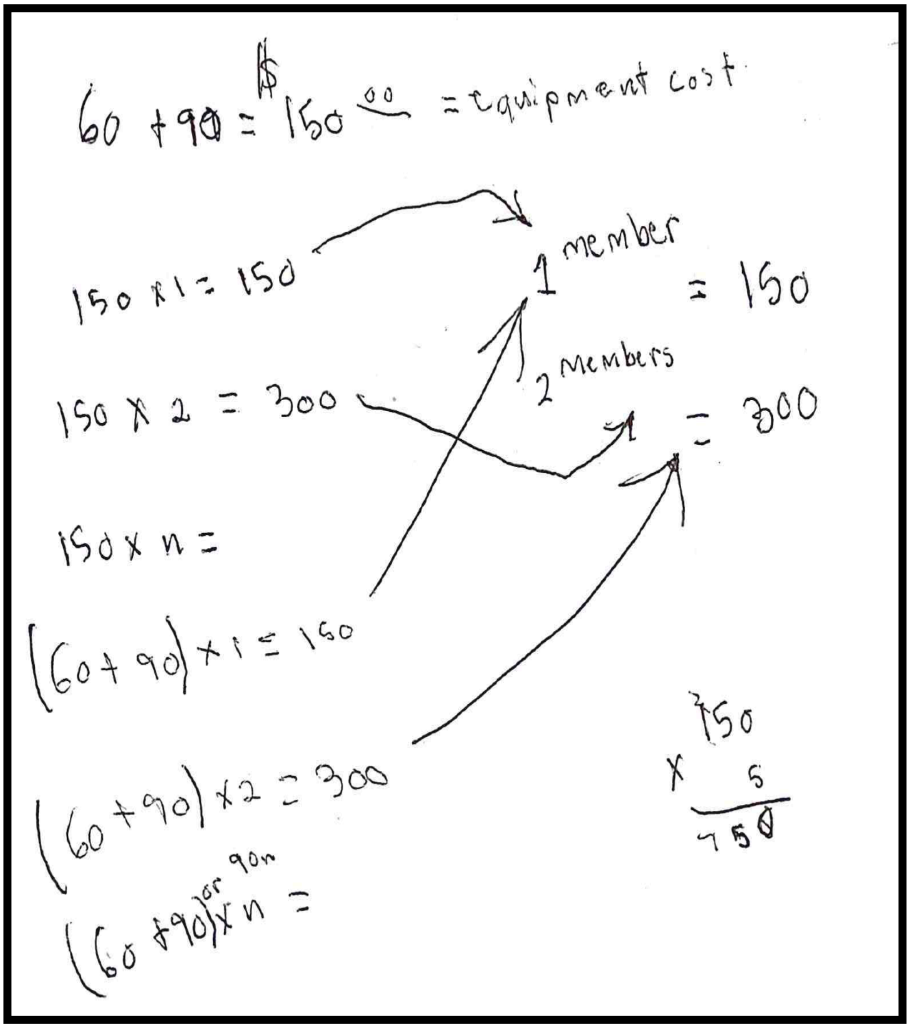
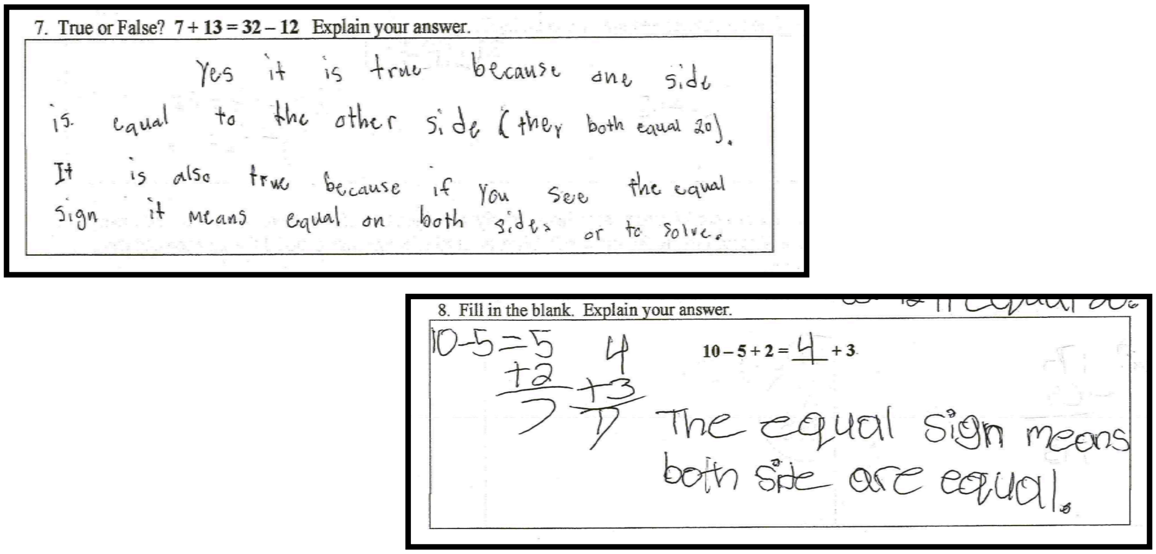



*Figure 55*. This example shows how a student broke down the expression by using the Distributive Property. She also demonstrated her understanding of how the expressions are equivalent by drawing arrows to the same quantities.

***Expressions and equations with the variable as an unknown value.*** As students interpreted the variable as an unknown value, they transitioned from viewing the equal sign as a compute operation to seeing an equal sign as meaning the same on both sides. At this point in the unit, students understood that an equation means that both sides are equal or of the same quantity (Figure 56).

**Figure 56.** Equations as an Equality Relationship


7. True or False? 7 + 13 = 32 – 12 Explain your answer.

Yes it is true because one side is equal to the other side (they both equal 20). It is also true because if you see the equal sign it means equal on both sides or to solve.

8. Fill in the blank. Explain your answer.

10 – 5 = 5 +2 = 7   4 +3 = 7

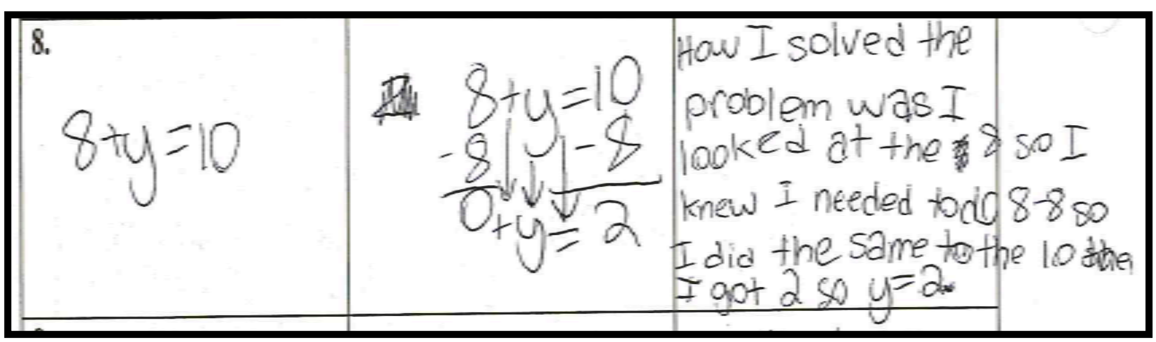
10 – 5 + 2 = 4 + 3

The equal sign means both side are equal.

*Figure 56.* Initially, students understood the meaning of the equal sign to be computing. This example demonstrates that students now understood that the equal sign means that both sides are the same or equal.

Students also saw an equation as both sides are balanced. This helped students see that the same had to be done on both sides to keep the equation balanced (Figure 57).

**Figure 57.** Equations as a Balanced Relationship


8.

8+y=10

8+y=10
-8   -8
0+y=2

How I solved the problem was I looked at the 8 so I knew I needed to do 8-8 so I did the same to the 10 then I got 2 so y=2

*Figure 57.* This example demonstrates how and why a student balanced an equation. At this point, students no longer needed a scale to visualize balance.

Equations were also seen as equivalent expressions. Thus, the expression on the right side of the equal sign must represent the same quantity as the expression on the left of the equal sign (Figure 58).

**Figure 58.** Equation as Equivalent Expressions

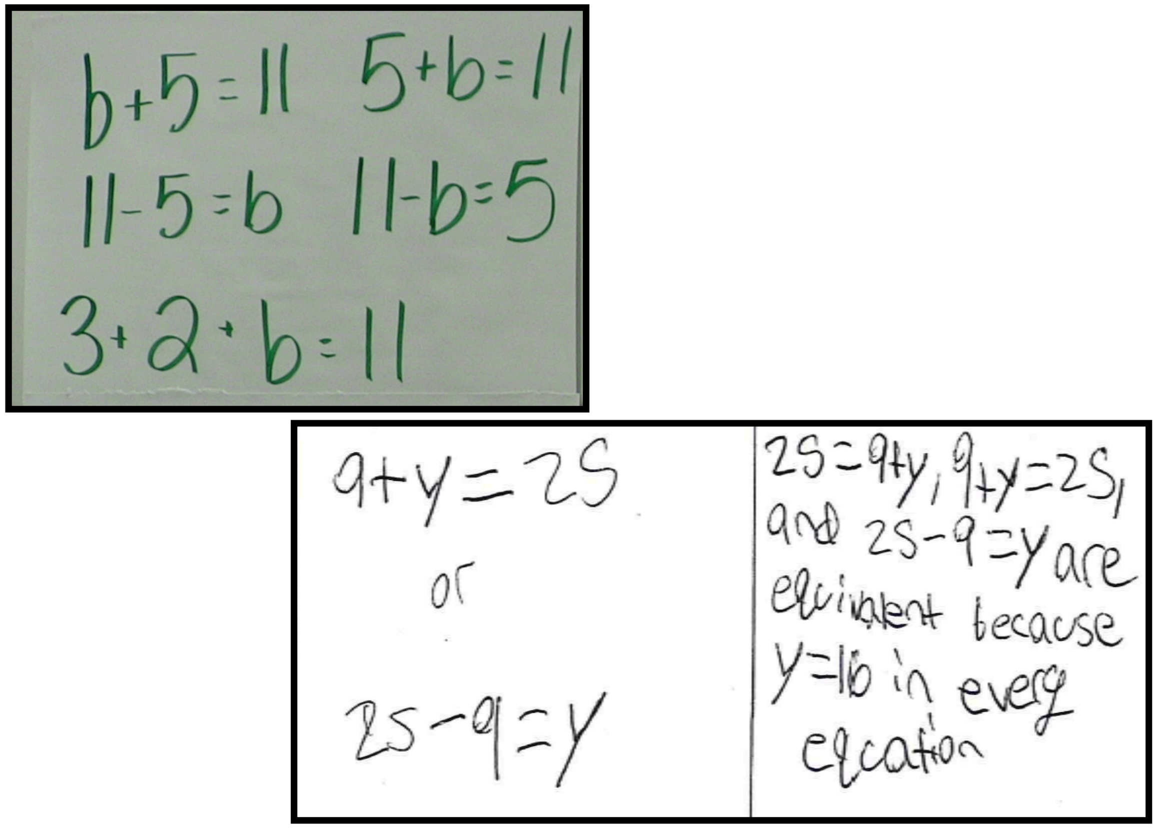



*Figure 58*. These examples show that students understood that an expression on the right of the equal sign represented the same quantity or an equivalent expression to the expression on the left of the equal sign.

**Independent and dependent variable.** Up until this point in the instructional unit, students understood that a variable could be a label or object, a changing quantity, a known value, or an unknown value. The following whole class discussion demonstrates how students had interpreted variables up to this point in the instructional unit.

Teacher: What was the first definition that we ever learned for variable?

J: A letter or symbol that represents a quantity.

C: A variable is a symbol or a letter that you use to represent the thing that you are adding.

L: A letter or symbol that represents an object.

Students were introduced to independent and dependent variables by writing a formula for the perimeter of a square. This activity forced students to relate two variables in one equation. Learning independent and dependent variables promoted relational thinking in that students had to see the relationship between these variables to learn functions. Figure 59 shows the progression of how students came to understand the meaning of independent and dependent variables.

**Figure 59.** Summary of Independent and Dependent Variable

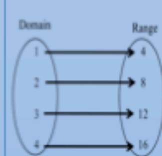
| Changing Concept of Variable | Conditions | Focus of Classroom Activity | Forms of Reasoning | Key Mechanisms that Shifted Student Thinking | Type of Thinking |
|---|---|---|---|---|---|
| 5.<br>Independent and Dependent Variable | A relationship exists between the independent and dependent variable<br><br>If the input is known, the output can be found<br><br>If the output is known, the input can be found | Examining the formula for the perimeter of a square.<br>**$P = 4s$**<br><br>Modeling a situation with a function using an arrow diagram, algebraic function, and graph.<br>**You are saving money in your piggy bank. You already have $20 and you get $5 everyday for doing your chores. How much will you have in your piggy bank after $d$ days?** | Using a verbal phrase for the formula.<br>Writing the formula.<br>**$y = 5d + 20$**<br><br>Using an arrow diagram.<br><br>Graphing the ordered pairs.<br> | Picking any number for $x$ and plugging this into the equation gives a point on the line.<br><br>The arrow diagram gives the points on the line.<br><br>Functions can be represented as equations, arrow diagrams, and graphs. | Algebraic<br>**Studying the patterns through the formula and the Domain and Range**<br><br>Functional<br>**Building and generalizing the relationship between the function as a formula, arrow diagram, and graph.**<br><br>Expressions and Equations<br>**Equation is seen as the same on both sides and balanced**<br><br>**An equation relates two variables** |

*Figure 59.* Conditions, focus of classroom activity, forms of reasoning, key mechanisms that shifted student thinking, and types of thinking observed as students learned variables as independent and dependent.

The following is a whole class discussion that occurred after students had time to work and think in small groups about writing a formula for the perimeter of a square. This discussion continues to demonstrate that students understood equivalent expressions and that students were thinking additively, multiplicatively, and algebraically.

Teacher: What is one of the equations you came up with?

J: $4s = p$

C: *n* times 4 equals *p*

E: $a + a + a + a$ equals *p*

M: Also known as $4a$

U: $4d$ equals *p*

L: $4x$ equals *p*

F: $4y$ equals *p*

Teacher: Are there any others that are not up here?

D: *b* times 4 equals *p*

Teacher: Are there any others without just changing the variables?

I: $2x + 2x = p$

G: $3a + a = p$

Teacher: So, what are all of these equations?

A: Equivalent equations

Teacher: What does equivalent mean?

Class: equal

Teacher: Equal or the same. $4s = p$ is called a function. It has input and output values. What that means is that I can input a number here and then get an output here. If I input, I can input any number and get an output. The plug and chug.

***Arrow diagram.*** Students had to learn a variable as a changing quantity, a variable as a known value, and a variable as an unknown value to understand independent and dependent variables. For example, they had to be able to plug and chug the known input value to find the output value in a function. Moreover, students had to know that a variable could be a changing quantity and that a relationship exists between the independent and dependent variable (Figure 60). Students had also become comfortable using the language of the mathematics, such as input and domain and output and range.

**Figure 60.** Student Work Example 16 (Independent and Dependent Variable)

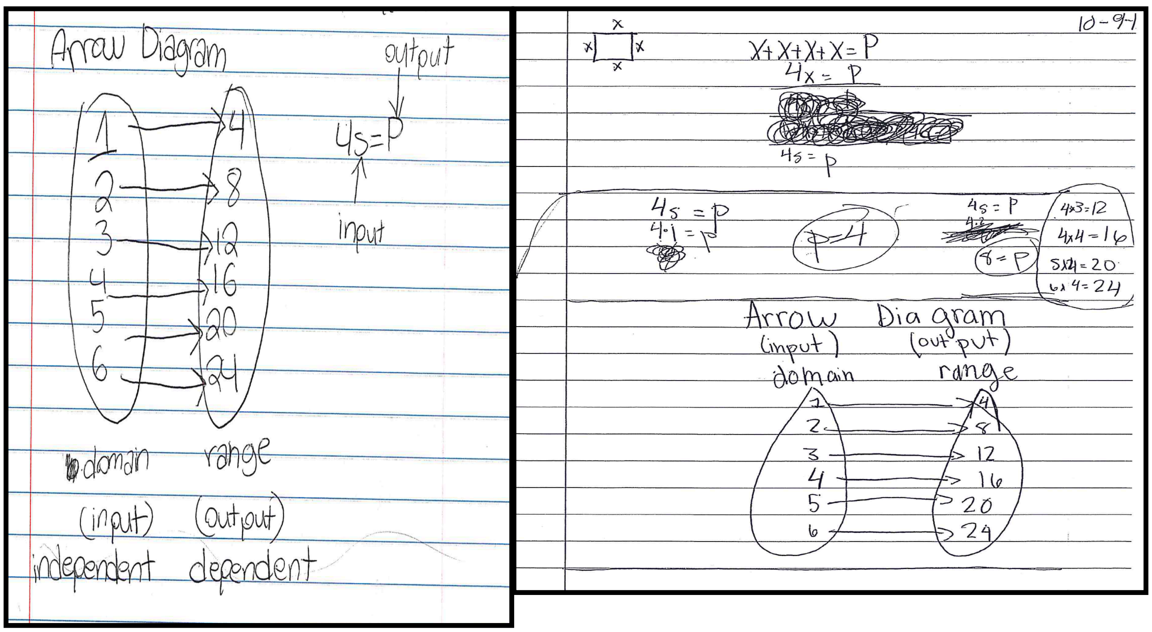
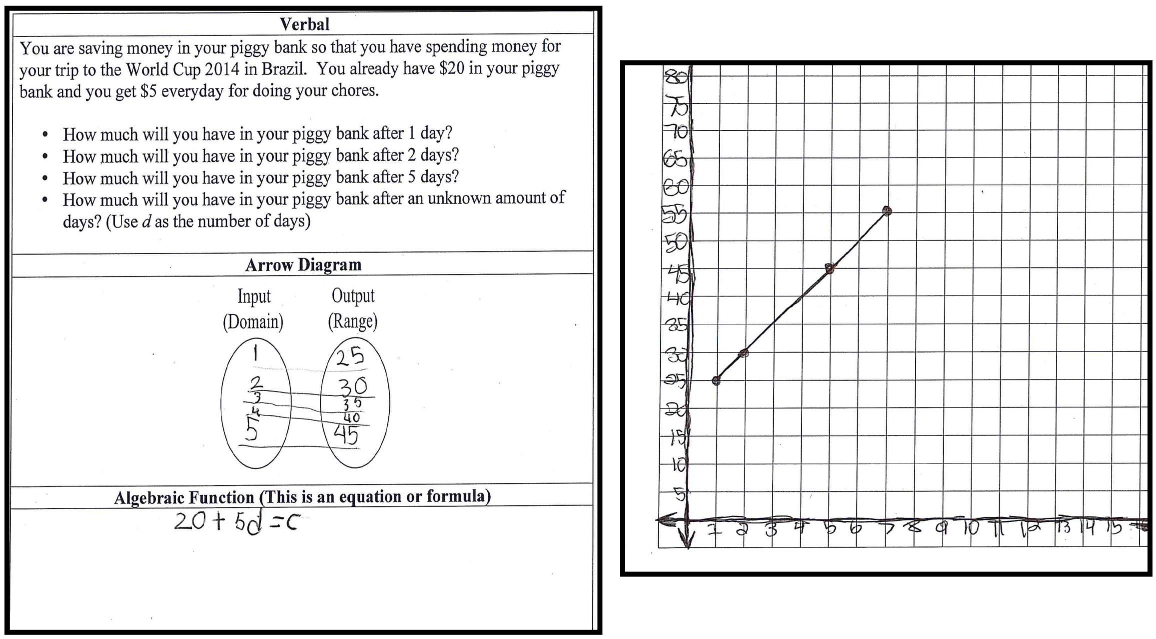



*Figure 60*. Examples of student work that show their understanding of functions. The arrow diagram contains the inputs and outputs for the perimeter of a square example.

***Verbal function to algebraic function.*** Students continued to learn that there are different ways to represent the same function. A function could be represented as an arrow diagram, with the inputs and outputs, a verbal function, and an algebraic function. Also, students were able to graph the ordered pairs formed by the values of the independent and dependent variables (Figure 61).

**Figure 61.** Student Work Example 17 (Independent and Dependent Variable)


**Verbal**

You are saving money in your piggy bank so that you have spending money for your trip to the World Cup 2014 in Brazil. You already have $20 in your piggy bank and you get $5 everyday for doing your chores.

- How much will you have in your piggy bank after 1 day?
- How much will you have in your piggy bank after 2 days?
- How much will you have in your piggy bank after 5 days?
- How much will you have in your piggy bank after an unknown amount of days? (Use *d* as the number of days)

**Arrow Diagram**

Input (Domain) | Output (Range)

1, 2, 3, 4, 5 | 25, 30, 35, 40, 45

**Algebraic Function (This is an equation or formula)**

20 + 5d = c

*Figure 61.* An example of student work where a function is represented verbally, in an arrow diagram, algebraically, and graphically.

***Expressions and equations with independent and dependent variables.*** Students continued to think of an equation as the same on both sides and balanced. However, students now began to see an equation as relating two variables, independent and dependent (Figure 62).

**Figure 62.** Student Work Example 18 (Independent and Dependent Variable)

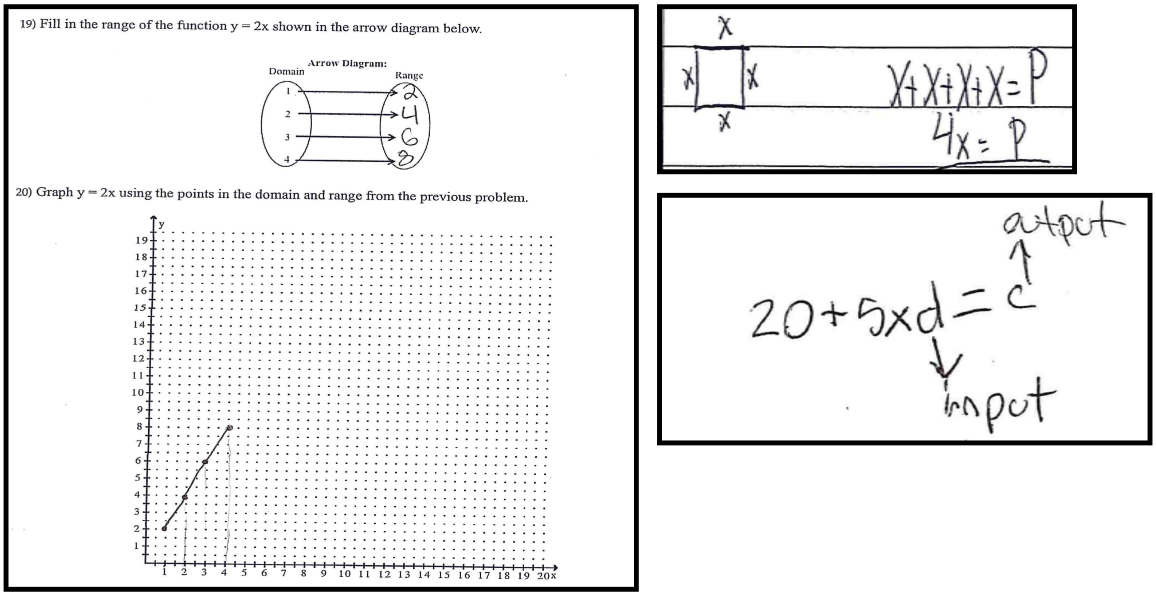



*Figure 62.* Examples of student work that demonstrate how students related a function, arrow diagram, graph, and equation.

It was essential that students knew how to work with the different types of variables in order to understand functions. Also, students used additive, multiplicative, and algebraic thinking to generalize the relationship between a function, arrow diagram, and graph. Students also had to understand that the two sides of an equation had to have the same value and this helped them to realize that a relationship exists between independent and dependent variables.

## Pretest and Posttest Results

Among sixth grade students participating in the algebra teaching experiment ($N$ = 22), there was a statistically significant difference in the mean score of the pretest ($M$ = 6.09, $SD$ = 2.35) and the mean score on the posttest ($M$ = 14.27, $SD$ = 3.56), $t(21) = -12.72$, $p < .001$, $r$ =.94. Therefore, the null hypothesis that there was no difference in the scores on the pretest and the posttest is rejected. Further, the effect size value ($r$ = .94) suggests a high practical significance. Figure 63 displays the individual student scores on the pretest and the posttest.

**Figure 63.** Student Pre and Post Test Scores

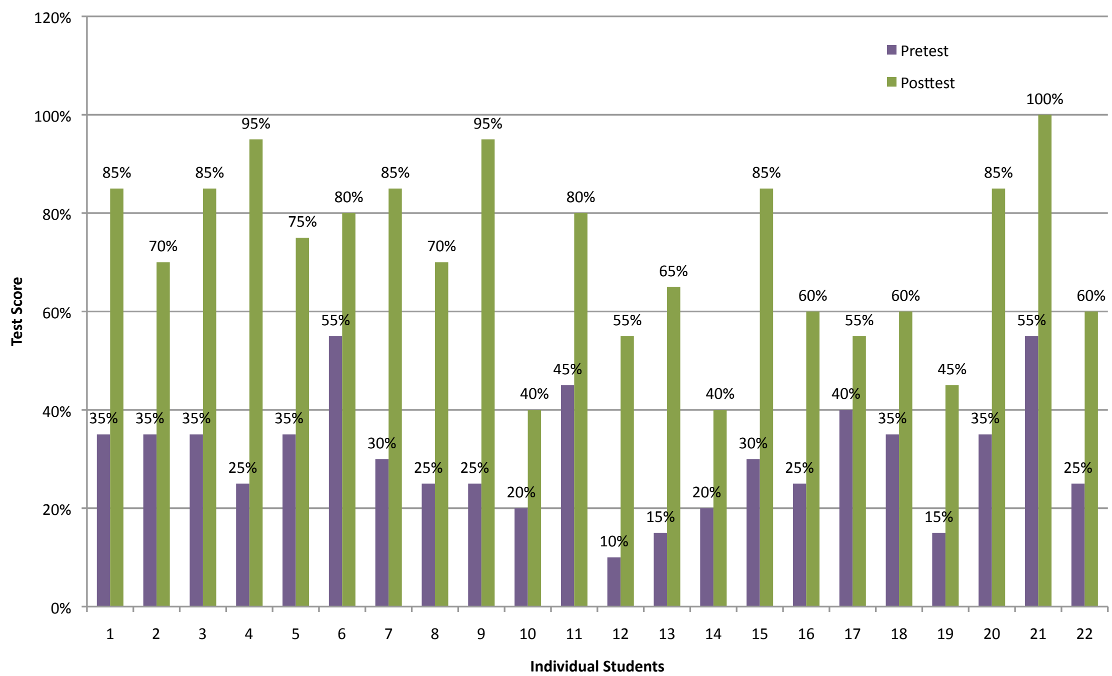



*Figure 63*. A graph displaying the results of the pretest and posttest scores for each student that participated in the teaching experiment.

## Making Connections

### Arithmetic to Algebra

Many tasks in the instructional unit forced students initially to think arithmetically to evolve to thinking algebraically. For instance, when learning about equivalent equations, it was easier for students to relate to an arithmetic situation. Once students realized this relationship between arithmetic and algebra, they were able to transition to algebra by substituting a variable for a known value (Figure 64).

**Figure 64.** Student Work Example 19 (Arithmetic to Algebra)

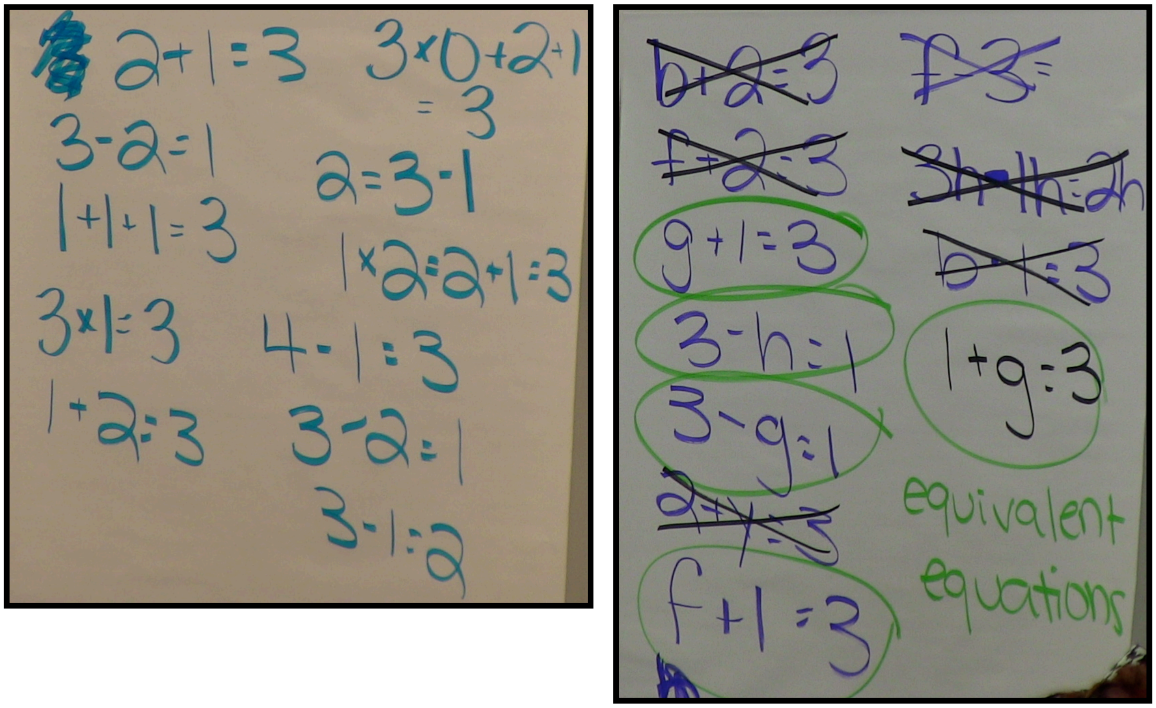



*Figure 64.* An example generated by the whole class of the transition from thinking arithmetically to thinking algebraically.

**Knowing the Value of the Variable versus Finding the Value of the Variable**

As students progressed through the learning trajectory, they discovered that a relationship existed between knowing the value of the variable and finding the value of the variable. This relationship involved the variable as a known value and the variable as an unknown value (Figure 65). Once students understood this relationship, they were able to check their answers by substituting a known value for the variable in the original equation.

**Figure 65.** Relationship between Variable as Known Value and Variable as Unknown Value

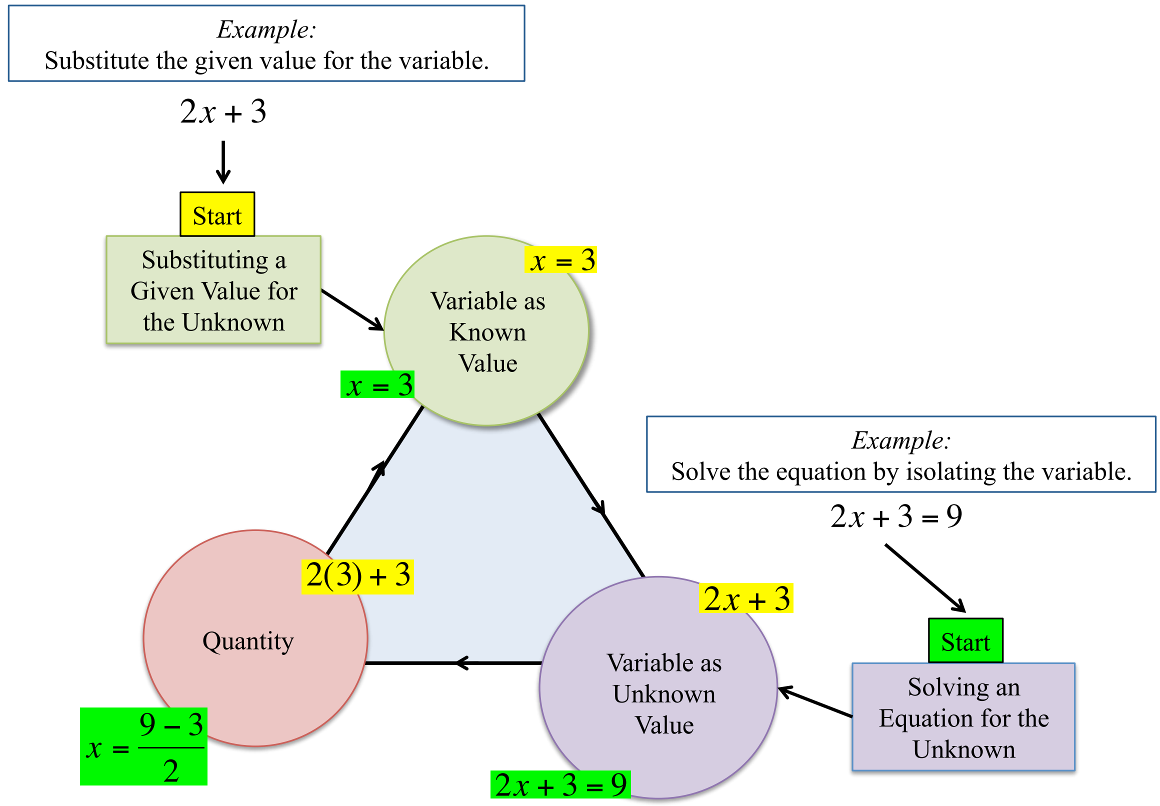



*Figure 65*. This figure demonstrates the relationship between finding the value of the variable and knowing the value of the variable.

## Supporting and Organizing Student Learning

### Modifications Made to All Lessons in the Instructional Unit

The tasks in the instructional unit were modified and reorganized based on student learning. The following figure (Figure 66) shows a visual representation of how a lesson was transformed throughout and after the unit. Version 1 is an outline of the lesson. Version 2 shows the same lesson with examples. Version 3 is similar to Version 2, except the big ideas, small group discussion, and whole group discussion are made explicit for the teacher. The final version of this lesson was completed after the final lesson in the instructional unit was taught. This lesson, and every lesson in the final version of the unit (see Appendix B), contains possible misconceptions and remedies, whole class discussions, questions to guide student thinking, small group discussions, the process to shift student thinking, and the big ideas intended to evolve from the whole class discussion.

**Figure 66.** Lesson Plan Transformation

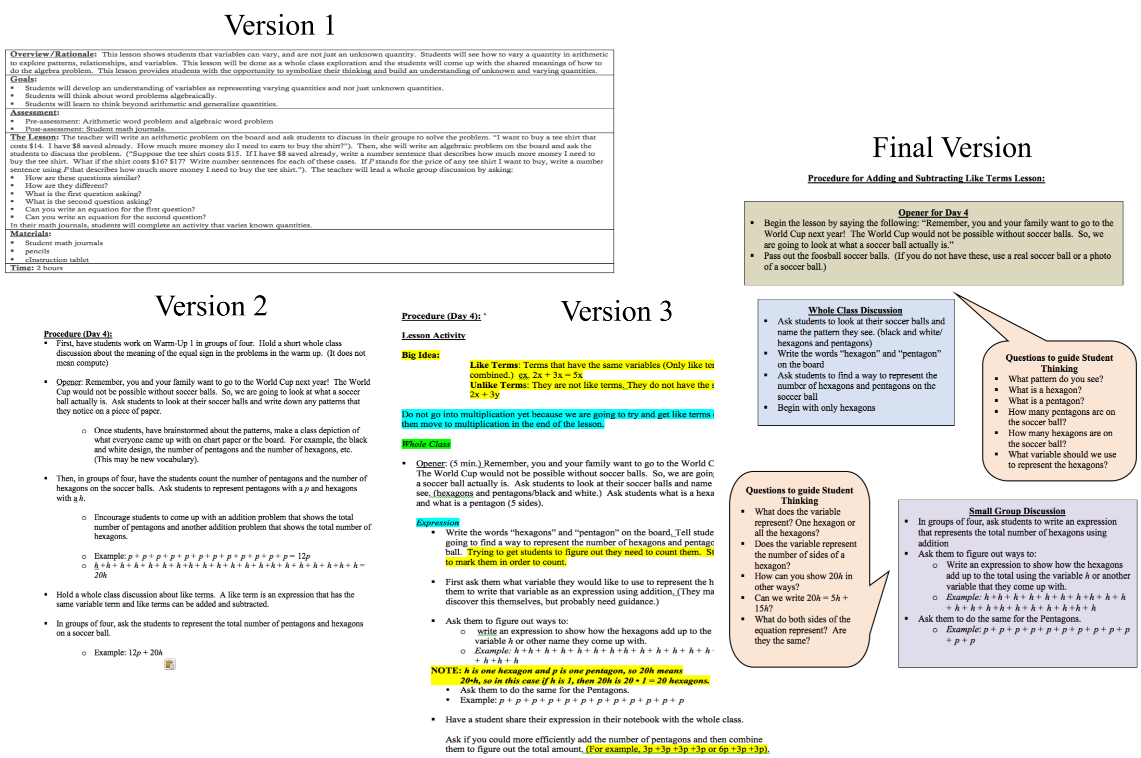



*Figure 66.* A visual representation of how the lesson plans transformed throughout the instructional unit.

**Documentation of Modifications to Individual Lessons in the Instructional Unit**

Although every lesson in general was modified to support student learning through discussions, daily decisions were also made to modify the mathematical tasks in the instructional unit. These decisions were logged in a table that documented a given teaching episode in terms of mathematical meaning, errors and misconceptions, activity that led to misconceptions, context of small group discussions, context of whole group discussions, the role of the teacher and teacher conceptions, and reasons for modifications in the lessons. Figure 67 shows an example of lesson logging for teaching episodes on Day 5 and Day 6.

**Figure 67.** Documentation of a Lesson Log Example

| **Date/ Activity** | **Mathematical Meaning** | **Errors/ Misconceptions** | **Activity that led to misconception** | **Context of small groups** | **Context of whole group** | **Role of teacher/ teacher conceptions** | **What did we change and why?** |
|---|---|---|---|---|---|---|---|
| **Day 5 9/13/13 Adding and Subtracting Like Terms** Students will develop an understanding of variables in mathematics and will learn that like terms can be added and subtracted. Students will also learn to model patterns with algebra. | Simplifying<br><br>Like Terms<br><br>Unlike Terms<br><br>Expression<br><br>Equation | Combining like terms: $3a + 4a + 7b = 14f$<br>3 apples + 4 apples + 7 bananas = 14 fruits<br><br>Subtraction of like terms: All like terms are added.<br><br>Like Terms: Writing the variable in front of the coefficient.<br>ex: R3 +B2 +G7 Should be 3R + 2B + 7G<br><br>Like term is correct and unlike term is wrong.<br><br>Like terms have the same coefficient and same variable ex: $20h$ and $20h$ are like terms, but $5h$ and $15h$ are not like terms. | More Practice (Day 5) worksheet.<br><br>Identifying the like terms and the unlike terms and then writing an algebraic expression for each situation. | Discussion about adding and subtracting like variables.<br><br>Small groups were able to write the expression, but not simplify the expression. | Discussion about equality and why both sides of an equation have to be the same.<br><br>Simplifying the expression to an equivalent expression.<br><br>Used an arrow (→) to show simplification instead of an equal sign (=) | The lesson went long (2 hours).<br><br>Too much information for them.<br><br>Simplifying is a new idea, so that is why she used the arrow to show simplifying first.<br><br>Teacher thinks it is better to start with simplifying abstract problems and then go to context problems.<br><br>Need more review of equal sign.<br><br>The teacher led the whole class discussion and helped students come up with the equal sign means the same on both sides. | Simplifying was not originally part of this lesson. Introduced the idea of simplifying. *(I think the arrow is going to confuse them/ need to use equal sign)*<br><br>Did not get to the cost of a soccer ball.<br><br>Spent more time on simplifying and equality than expected. |

*Figure 67.* An example of how the teaching episodes on Days 5 and 6 were documented. These logs are a snapshot of each lesson and demonstrate how the tasks were modified to support learning.

**Organizing Student Learning**

In addition to logging the events of each teaching episode, the researcher also used a checklist to show student learning over time. The researcher and teacher examined each student's written work to record student learning of the algebra topics. Figure 68 is the checklist that shows the learning over two days of the teaching experiment. Throughout the teaching experiment, topics were added each day to this checklist and written work was analyzed for correctness in order for the teacher and the researcher to understand not only which students did not understand a particular topic, but also to show the learning of the whole class.

**Figure 68.** Checklist for Organization of Student Learning

| Day 2 | | | Day 3(warm-up/individual) | | Day 3(group) | | |
|---|---|---|---|---|---|---|---|
| Variable | Expressions | Equations | Expression vs. Equation | Missing Numbers | Variable | Expression | Equation |
| [redacted] | | | | | | | |
| x | x | x | x | x | x | x | x |
| | | | x | x | x | x | x |
| x | x | x | x | x | x | x | x |
| x | x | x | a | a | a | a | x |
| [redacted] | | | | | | | |
| x | | | x | | x | x | x |
| x | x | x | x | x | x | x | x |
| [redacted] | | | | | | | |
| x | x | x | x | x | x | | x |
| x | | | x | x | x | | x |
| | | | x | x | x | | x |
| | | | x | x | x | | x |
| [redacted] | | | | | | | |
| x | x | x | x | x | x | x | x |
| x | x | x | x | x | x | x | x |
| | | | x | x | x | x | x |
| | | | x | | x | x | x |
| [redacted] | | | | | | | |
| | | | x | x | x | x | x |
| | | | x | x | x | x | x |
| x | x | x | x | x | x | x | x |
| x | x | x | x | x | x | x | x |
| [redacted] | | | | | | | |
| a | a | a | x | x | x | x | x |
| a | a | a | x | x | x | x | x |
| x | x | x | x | x | x | x | x |
| x | | | x | x | x | x | x |
| [redacted] | | | | | | | |

*Figure 68*. This checklist shows student learning over Day 2 and Day 3 of the teaching experiment. An "x" means that the student completed the task relating to the topic correctly, an "a" means that the student was absent, and a "blank" means that the student did not yet understand the topic.

## Summary

This chapter began by providing a context of the setting in which the teaching experiment took place. The first portion of this chapter answered the first research question by detailing the whole class realized learning trajectory that emerged through the design research study. The second portion of this chapter provided an analysis of the means of supporting and organizing student learning of algebra, including explanations of the role of the tasks in supporting learning and what design decisions were made to modify the tasks in the instructional unit.

# CHAPTER V: CONCLUSIONS AND DISCUSSION

This study contributes to the research on understanding how to support the learning of beginning students in algebra by providing implications for student learning, teaching, and curriculum development in mathematics. The product of the analysis was a historical explanation that detailed the pattern that emerged from the teaching experiment (Cobb et al., 2003). The practical and theoretical relevance of the results are discussed in the following sections.

## Practical Relevance

In this teaching experiment, a learning trajectory evolved based on the students' changing interpretations of a variable. This learning trajectory has practical implications for student learning, teaching, and curriculum development. The learning trajectory, including the conditions, the focus of the classroom activity, the forms of reasoning, the key mechanisms that shifted student thinking, and the type of thinking are presented in Figure 69.

**Figure 69.** Learning Trajectory for Beginning Algebra

| **1. Variable as Label** | |
|---|---|
| **Conditions** | ▪ Quantity is known<br>▪ Using a variable to keep a record of a specific quantity |
| **Focus of Classroom Activity** | ▪ Finding the total of two groups by writing an expression or an equation<br>***adults + children = Total***<br>***a + c = T***<br>▪ Using a sum of like and unlike terms to find the total amount<br>**8 *red M&Ms* + 4 *blue M&Ms***<br>**8*r* + 4*b*** |
| **Forms of Reasoning** | ▪ Abbreviating the name of a group with the first letter of the word<br>▪ Using a letter to represent one object or one group with the same classification |
| **Key** | ▪ Asking students to share different representations of the |

| | |
|---|---|
| **Key Mechanisms that Shifted Student Thinking** | ▪ Asking students to share different representations of the expression<br>***adults + kids = total***<br>***a + k = t***<br>**2 *adults* + 5 *kids***<br>**2*a* + 5*k***<br>▪ Directing students to come up with a more efficient way to represent the same expression<br>***r + r + r + r + r =***<br>**2*r* + 3*r* =**<br>**5*r***<br>▪ Circling sign, coefficient, and variable to combine like terms |
| **Type of Thinking** | ▪ Additive<br>**Finding the Total**<br>***r + r* = 2*r***<br>▪ Multiplicative<br>**5 • *r***<br>**where *r* = 1**<br>▪ Algebraic<br>**Generalizing Arithmetic to Algebra**<br>▪ Expressions and Equations<br>**Expression as a sum of like and unlike terms**<br>**Equation as computing a total** |
| **2. Variable as Changing Quantity** | |
| **Conditions** | ▪ Variable represents changing values of a specific quantity |
| **Focus of Classroom Activity** | ▪ Writing an expression that can be used to find the total cost<br>**4c + s**<br>**where 4c is the cost of 4 packages of 6 cupcakes and s is the cost of a single cupcake** |
| **Forms of Reasoning** | ▪ Symbolizing the price with a variable and acknowledging that this price may change<br>**c is the cost of a package of 6 cupcakes** |
| **Key Mechanisms that Shifted Student Thinking** | ▪ Suggesting to students that the expression is a formula that can be used at any store |
| **Type of Thinking** | ▪ Multiplicative<br>**4 • c**<br>**where c is a changing quantity**<br>▪ Algebraic<br>**Using symbols in a Meaningful Way to model a formula**<br>▪ Expressions and Equations<br>**Expression as modeling the cost of a known quantity**<br>**Equation as computing the total cost** |

| | |
|---|---|
| **Conditions** | ▪ Value for variable is given<br>▪ Known value can be substituted for the changing quantity |
| **Focus of Classroom Activity** | ▪ Providing the price of an item at different stores and asking students to find the total cost using the expression.<br>**At Walmart, one package of six cupcakes is $6 and a single cupcake is $1.**<br>**At Safeway, one package of six cupcakes is $8 and a single cupcake is $2.**<br>**At Raley's, one package of six cupcakes is $7 and a single cupcake is $1.** |
| **Forms of Reasoning** | ▪ Substituting a given value for the variable<br>▪ Understanding that the coefficient is multiplied by the value of the variable<br>**4*c* means 4 • *c*** |
| **Key Mechanisms that Shifted Student Thinking** | ▪ Developing the rule of "plug and chug" where once you plug in the value, the variable disappears and terms can be combined<br>**4*c* + *s* for**<br>***c* = 6 and *s* = 1**<br>**4(6) + 1 = 24 + 1 = 25** |
| **Type of Thinking** | ▪ Multiplicative<br>**4*c* means 4 • *c***<br>▪ Additive<br>**After plug and chug, combine by adding or subtracting**<br>▪ Algebraic<br>**Using symbols in a Meaningful Way to evaluate an expression**<br>▪ Expressions and Equations<br>**Using an expression to substitute values for variables**<br>**An equation is the final computation of the total of the expression** |
| **4. Variable as Unknown Value** | |
| **Conditions** | ▪ Must solve for the unknown<br>▪ Unknown variable must be isolated |
| **Focus of Classroom Activity** | ▪ Balancing a scale to solve equations and Algebra Touch App for isolating the variable.<br>***t* + 2 = 6**<br>▪ Using the area of a rectangle formula to find the width.<br>***A* = *lw***<br>**if *A* is 18 and *l* is 6, find *w*.**<br>▪ Learning distributive property where the variable is number of family members and cost is known.<br>**Let *n* be the number of family members. Every member of the family needs to buy a backpack that costs $90 and a sleeping bag that costs $60.**<br>**1. Total Cost = *n*(90 + 60)**<br>**2. Total Cost = 90*n* + 60*n*** |
| **Forms of** | ▪ Using opposite operations to solve an equation. |

| | |
|---|---|
| **Forms of Reasoning** | ▪ Using opposite operations to solve an equation.<br>▪ The scale must be balanced.<br>▪ The equal sign means that both sides have the same value.<br>▪ Identifying when to use AP and when to use MP.<br>**AP is Addition Property of Equality and MP is Multiplication Property of Equality.** |
| **Key Mechanisms that Shifted Student Thinking** | ▪ "Making moves" to isolate the variable<br>$\boldsymbol{t + 2 = 6}$<br>$\underline{\mathbf{-2 = -2}}$<br>$\boldsymbol{t = 4}$<br>**Subtracting 2 is making "one move" to isolate *t*.**<br>▪ Plug and chug can be used to check the answer to an equation<br>**$t + 2 = 6$, so**<br>**t = 4**<br>**Check: 4 + 2 = 6**<br>**6 = 6**<br>▪ Distributive property is necessary for simplifying an algebraic expression into an equivalent expression<br>$\mathbf{2(x - 3) = 2x - 6}$<br>▪ In arithmetic, distributive property is not necessary, but produces equivalent expressions<br>**2(3 + 4) = 2(7) = 14 but this is also:**<br>**2(3 + 4) = 2(3) + 2(4) = 6 + 8 = 14** |
| **Type of Thinking** | ▪ Additive<br>$\mathbf{90n + 60n = 150n}$<br>▪ Multiplicative<br>**150*n* means 150 • *n***<br>▪ Algebraic<br>**Study of Structure where the equal sign means "the same as"**<br>▪ Expressions and Equations<br>**Equation is the same quantity on both sides**<br>**Equation as balanced**<br>**Equation is seen as equivalent expressions** |
| **5. Independent and Dependent Variable** | |
| **Conditions** | ▪ A relationship exists between the independent and dependent variable<br>▪ If the input is known, the output can be found<br>▪ If the output is known, the input can be found |
| **Focus of Classroom Activity** | ▪ Examining the formula for the perimeter of a square.<br>$\boldsymbol{P = 4s}$<br>▪ Modeling a situation with a function using an arrow diagram, algebraic function, and graph.<br>**You are saving money in your piggy bank. You already have \$20 and you get \$5 everyday for doing your chores. How much will you** |

| | |
|---|---|
| **Forms of Reasoning** | ▪ Using a verbal phrase for the formula<br>▪ Writing the formula<br>$\boldsymbol{y = 5d + 20}$<br>▪ Using an arrow diagram<br>▪ Graphing the ordered pairs |
| **Key Mechanisms that Shifted Student Thinking** | ▪ Picking any number for $x$ and plugging this into the equation gives a point on the line.<br>▪ The arrow diagram gives the points on the line.<br>▪ Functions can be represented as equations, arrow diagrams, and graphs. |
| **Type of Thinking** | ▪ Algebraic<br>**Studying the patterns through the formula and the Domain and Range**<br>▪ Functional<br>**Building and generalizing the relationship between the function as a formula, arrow diagram, and graph.**<br>▪ Expressions and Equations<br>**Equation is seen as the same on both sides and balanced**<br>**An equation relates two variables** |

*Figure 69.* A learning trajectory that details the changing concept of variable and how students learned the CCSS for Expressions and Equations (NGA/CCSSO, 2010).

**Implications for Student Learning**

The progression of learning a variable as a label, a variable as a changing quantity, a variable as a known value, a variable as an unknown value, and independent and dependent variables was intertwined with learning expressions, equations, and functions. As students learned to work with variables as labels, they no longer thought about the numbers that the variable might represent and, thus, achieved manipulation of opaque formalisms (Kaput, 1995). Moreover, a transition from additive to multiplicative thinking existed as the interpretation of a variable changed from a variable as a label to a variable as a changing quantity.

Initially, students viewed the equal sign as an action symbol, meaning that they should compute something on the left of the equal sign and place the answer on the right of the equal sign (Carpenter, Franke, & Levi, 2003). The learning trajectory documents that student understanding of the equal sign as a relationship between two quantities (Blanton, 2008) occurred as they interpreted a variable as an unknown value. This finding concurs with Van de Walle et al. (2011) in that students cannot understand how to solve an algebra equation without knowing the meaning of the equal sign and variables.

Additionally, this study supports previous research (Drijvers et al., 2011; Markovits et al., 1986) indicating that students learn functions by understanding the representations and components of a function and how they are related. The learning of functions took place after a conceptual understanding of a variable as a label, a variable as a changing quantity, a variable as a known value, and a variable as an unknown value was present because understanding independent and dependent variables and, thus, functions involves these first four ideas of variables. Therefore, the results of this study contribute to the theory that student learning of algebra should be introduced through the use of variables not only as changing quantities and unknowns (Fey & Good, 1985; Usiskin, 1988), but also as labels, known values, and independent and dependent.

The learning trajectory for beginning algebra (Figure 69) contains the type of thinking that occurred during each conception of a variable. Similar to Kieran's (1989) findings, students understood the variable as an unknown value and solving an equation by using the systemic structure, the equivalence of the left-hand and right-hand sides of an algebraic equation. Using the systemic structure involved relational thinking in that students knew the properties and ways of thinking about operations (Carpenter, Levi, Franke, & Zeringue, 2005) and, also, that the equal sign represents equivalence between two expressions (Carpenter et al., 2003; Molina et al., 2005). In this learning trajectory, visual salience of algebra, the immediate connection between both sides of an equation

(Kirshner & Awtry, 2004), is present during the interpretation of a variable as a label and combining like terms, where students were thinking additively. This visual structure was also evident as students learned the distributive property where the left-hand side of the equation follows visually from the right-hand side of the equation.

**Implications for Teaching**

During this study, the teacher implemented, and the curriculum was structured, in such a way that encouraged discussions and established a classroom community of mathematical thinking and learning in which students felt comfortable sharing mathematical ideas (Cobb, Yackel, & Wood, 1992). Additionally, the learning trajectory contains teaching intentions for algebra in the key mechanisms that shifted student thinking. These key mechanisms are important for teaching in that each mechanism has to develop before the next phase in the trajectory can occur. The key mechanisms represent essential, small elements of comprehension that, when realized, shift student thinking from an initial understanding to making sense of the algebra task. By using the learning trajectory for beginning algebra and focusing on the key mechanisms that shifted student thinking, teachers can anticipate these instances of learning and plan and modify lessons accordingly.

Although objectives for teachers of what students are expected to learn are provided in the CCSS for Mathematics (NGA/CCSSO, 2010), teachers need to know the big mathematical themes and be able to present these as interconnected topics (Ma, 2010). In this teaching experiment, the big mathematical themes were expressions, equations, and functions, similar to the headings in CCSS for Expressions and Equations (NGA/CCSSO, 2010). However, the findings of this study suggest that this sequence is embedded within the progression of the changing concept of a variable. Blanton (2008) and Blanton and Kaput (2003) recommended teaching beginning algebra by making known quantities unknown and varying known quantities. The findings of this study propose teaching expressions and equations by beginning with the concept of a variable as a label and continuing through a variable as a changing quantity, a variable as a known value, and a variable as an unknown value, concluding with functions in conjunction with independent and dependent variables.

**Implications for Curriculum Development**

The instructional unit (Appendix B) used in this study shaped the learning that took place (Elsaleh, 2010) and contains student misconceptions that are intended to inform teachers in anticipation of teaching each lesson. The focus of the classroom activities and example tasks are provided within each concept of a variable in the learning trajectory for beginning algebra (Figure 69). These activities can be modified in future teaching episodes based on student learning. For example, the tasks given in the learning trajectory can be used in algebra curriculum development as a guideline that generates additional problems to promote the key mechanisms that shift student thinking. These classroom activities and tasks, along with the forms of reasoning, convey a progressively more sophisticated understanding of expressions, equations, and functions transpired through the embedded conceptions of a variable.

The instructional unit leads to the development of big mathematical ideas by linking content and focusing on important mathematics (NCTM, 2000). Since algebraic thinking is developed through good questions that press students to articulate their mathematical understanding (Blanton, 2008), the curriculum developed in this teaching experiment was modified to make the big ideas, questions to guide student thinking, and the process to shift student thinking explicit. The curriculum makes the big ideas explicit to give advanced notice of the main objectives of each lesson and to guide the teacher toward these goals. Further, the instructional unit shaped the learning that took place (Elsaleh, 2010) in that it was based on small group and whole class discussions. Each lesson contained a context that gave students a means for discussion and a purpose for doing the mathematics. Thus, by following this curriculum, the students learned algebra by engaging with the content in the instructional unit and by social interaction with their teacher and peers.

## Theoretical Relevance

This teaching experiment employed design research, which aims to develop *empirically grounded theories* through the study of the learning process as well as how to support that process (diSessa & Cobb, 2004; Gravemeijer, 1994). In this study, the learning theory that emerged was the variable schema for learning beginning algebra. The implications of this learning trajectory are discussed in the following section. In addition to this theory, an alternate perspective on Kaput's (1999) five forms of algebraic thinking is introduced.

### Implications of the Variable Schema

The sixth grade CCSS for Expressions and Equations (NGA/CCSSO, 2010) are organized through the progression of expressions, equations (and inequalities), and functions. Initially, the instructional unit developed for this teaching experiment was also based on expressions, equations, and functions. However, as the unit progressed, it became clear that students' interpretations of variables governed their collective learning of the algebra tasks presented in the unit. This schema is as follows: variable as label, variable as changing quantity, variable as known value, variable as unknown value, and independent and dependent variable. Figure 70 is a representation of how the CCSS for Expressions and Equations (NGA/CSSO, 2010) evolved during the teaching experiment.

**Figure 70.** Evolution of the CCSS for Expressions and Equations

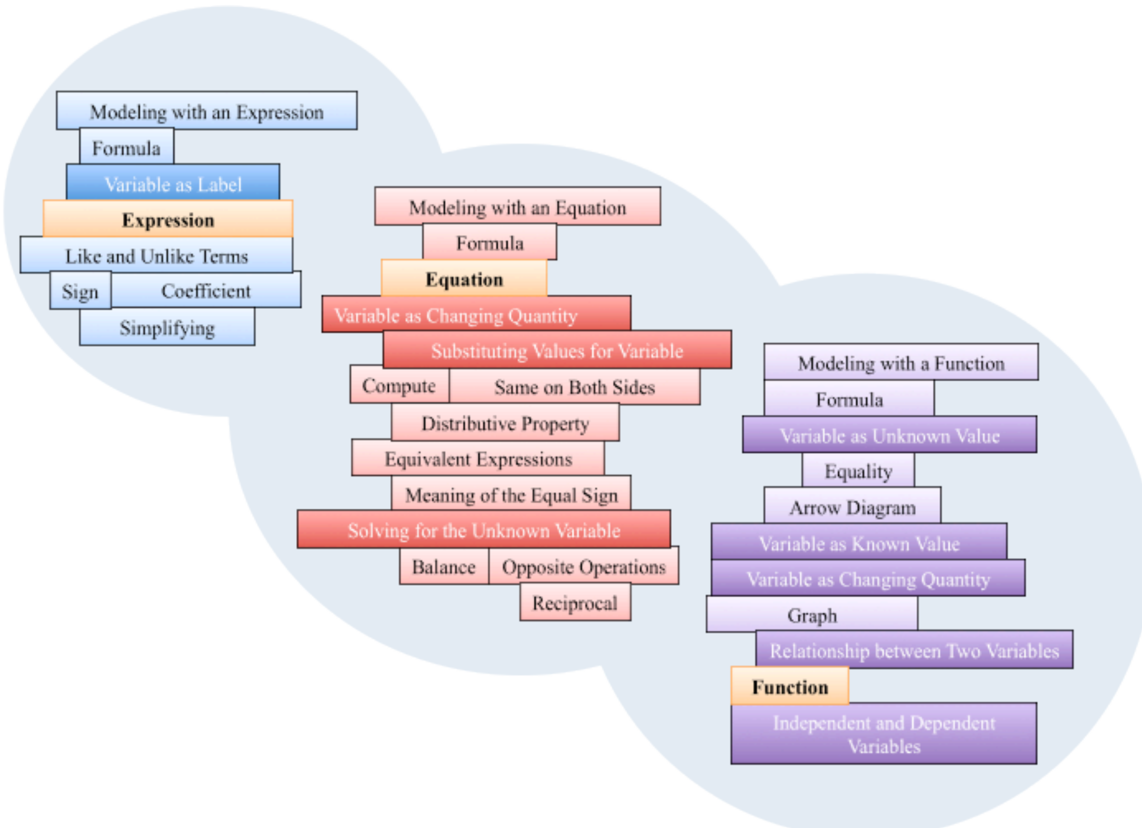



*Figure 70*. The evolution of learning expressions, equations, and functions as outlined in the CCSS (NGA/CSSO, 2010) with the inclusion of the variable schema for learning beginning algebra.

The variable schema for learning beginning algebra should be explored further to better understand how students interpret different meanings of variables and how these interpretations contribute to their learning of expressions, equations, and functions. Moreover, it might be beneficial for in-service and pre-service teacher trainings to include these ideas of variables and how this schema is related to the expression, equation, and function progression of the CCSS (NGA/CSSO, 2010).

**New Perspective of the Framework for Algebraic Thinking**

The five forms of algebraic thinking (Kaput, 1999) were present during the teaching experiment. These forms of algebraic thinking did not transpire in two levels (as in Figure 4); alternatively, the learning of expressions and equations emerged from the process of mathematical modeling through the meaningful use of symbols. Figure 71 introduces this new perspective for a framework for algebraic thinking.

**Figure 71.** New Framework for Algebraic Thinking

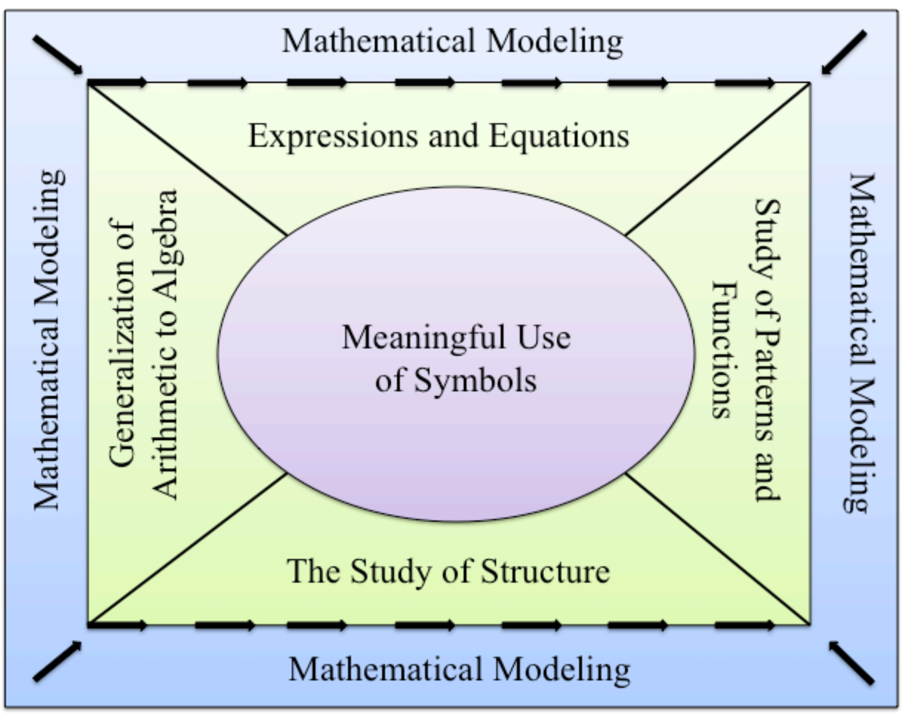



*Figure 71*. A new, alternate, framework for algebraic thinking that is built around the meaningful use of symbols and stems from the process of mathematical modeling.

The basis of this new framework for algebraic thinking is the meaningful use of symbols, manifesting from the process of mathematical modeling. Students modeled real situations using mathematics by placing an algebra problem in context and this helped them make sense of the mathematics and supported conceptual understanding of these abstract representations (Earnest & Baiti, 2008). In this framework, generalization of arithmetic to algebra, expressions and equations, the study of structure, and the study of patterns and functions emerge from the process of mathematical modeling and formalize over time (depicted by the arrows in Figure 71), beginning with the generalization of arithmetic to algebra and concluding with the study of patterns and functions. Algebraic thinking is not linear, occurring in separate levels; rather generalization of arithmetic to algebra, expressions and equations, the study of structure, and the study of patterns and functions result from the process of mathematical modeling through the meaningful use of symbols.

The new framework for algebraic thinking (Figure 71) should be explored with different populations of students to determine how and if these forms of algebraic thinking are present. Furthermore, research should be conducted on algebra curricula that incorporate these forms of algebraic thinking, and the training of in-service and pre-service teachers should be studied to determine if and how these trainings incorporate the content knowledge necessary to teach algebra. Finally, future teaching experiments should use the revised learning trajectory as a hypothetical learning trajectory.

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# Appendix A

## Expressions and Equations
## Version 1

| Expressions and Equations |
|---|
| **Summary:**<br>In this unit, students will learn about algebraic expressions. Students will learn to view an algebraic expression as an object and not simply a computation. An expression is a mathematical statement or sentence that expresses calculations with numbers and variables. A variable is a letter that represents a number and can be a changing quantity. Students will learn to write, evaluate, and simplify expressions. They will also learn how to represent real world situations with an algebraic expression and begin to understand that expressions written in different forms can be equivalent expressions. Students will learn how to solve algebraic equations by using the addition law and multiplication law. Students will learn to view the equal sign as an equivalence operation and not as an operation to compute. Students will learn the process of solving for a single variable in an equation by using relational thinking. |
| **Connections to Prerequisite Knowledge:**<br>▪ Numerical expressions<br>▪ Using letters to represent an unknown in a word problem<br>▪ Using whole number exponents to express powers of 10<br>▪ Calculating numerical expressions using order of operations<br>▪ Generating a relationship between an input number, the rule, and an output number |
| **Connections to Subsequent Learning:**<br>▪ Expressions that include numbers and variables<br>▪ Variables as changing and unknown quantities<br>▪ Using whole number exponents in numerical expressions<br>▪ Using order of operations to simplify algebraic expressions<br>▪ Use their understanding of solving equations and equivalence to solve problems using formulas and graphs |
| **Common Core State Standards for Mathematics:**<br>**Apply previous understandings of arithmetic to algebraic expressions**<br>▪ **6.EE.A1**: Write and evaluate numerical expressions involving whole-number exponents.<br>▪ **6.EE.A2**: Write, read and evaluate expressions in which letters stand for numbers.<br>a. Write expressions that record operations with numbers and with letters standing or numbers. *For example, express the calculation "Subtract y from 5" as 5 – y*.<br>b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression 2(8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms.*<br>c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas* $V = s^3$ *and* $A = 6s^2$ *to find the volume and surface area of a cube with sides of length* $s = \frac{1}{2}$.<br>▪ **6.EE.A3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.* |

- **6.EE.A.4**: Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.*

**Reason about and solve one-variable equations**

- **6.EE.B.5**: Understand solving an equation as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use and explain substitution in order to determine whether a given number in a specified set makes an equation or inequality true.
- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.
- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x$ + p = q and p$x$ = q for cases in which p, q and $x$ are all nonnegative
- rational numbers.

**Represent and analyze quantitative relationships between dependent and independent variables**

- **6.EE.C.9**: Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65t to represent the relationship between distance and time.*

## Common Core State Standards for Mathematics:

**Mathematical practice**

- MP1. Make sense of problems and persevere in solving them.
- MP2. Reason abstractly and quantitatively.
- MP3. Construct viable arguments and critique the reasoning of others.
- MP4. Model with mathematics.
- MP5. Use appropriate tools strategically.
- MP6. Attend to precision.
- MP7. Look for and make use of structure.
- MP8. Look for and express regularity in repeated reasoning.

Lesson 1: Expressions versus Equations

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| **Overview/Rationale:** Comparison of what the equal sign means in arithmetic and what the equal sign means in algebra. In arithmetic, the equal sign means "compute" and in algebra the equal sign is a relation of equality between the two sides. Hold a class discussion to come up with examples of what the equal sign is and its different meanings. Does an expression have an equal sign? Students understanding of the equal sign as an equal relationship is essential for developing an understanding of how to solve algebraic equations. |
| **Goals:**<br>▪ Students will develop an understanding of the equal sign as an equality relationship, rather than a computation.<br>▪ Students will learn that an algebraic expression does not contain an equal sign and an algebraic equation contains an equal sign. |
| **Assessment:**<br>▪ Pre-assessment: A whole class discussion where the teacher asks students to describe the difference between an expression and an equation.<br>▪ Post-assessment: Student math journals. |
| **The Lesson:** The teacher will begin the lesson by writing an expression and an equation on the board (no variables). A whole class discussion will take place where the teacher asks the following questions:<br>How is this expression different from this equation?<br>What does the equal sign mean in the equation?<br>The teacher will write another expression, 2 + 5, and equation, 2 + 5 = __ + 3, on the board. Students will be asked to complete the two problems in their math journals and discuss their answers with their group. The students will have five to ten minutes to discuss in their groups and one person from each group will share their findings, writing on the eInstruction tablet. |
| **Materials:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet |
| **Time:** 1 hour |

# Lesson 2: Variables and Like Terms

**Overview/Rationale:** Introduce through a whole class discussion the different meanings of the word “variable”. Make a class concept map that brainstorms different ideas about variables. Include real-life examples, graphing examples, and variables as unknown examples. Focus on integrating arithmetic in the discussion ($2 + 3 = x$ and thus, $x = 5$). The whole class will come up with the meaning of “like terms” and why like terms are important. What do like terms look like? What can you do with like terms? Variables are used throughout mathematics, as well as, other subjects to describe unknown numbers and changing quantities. It is essential that students understand variables and like terms to succeed in algebra and future math courses.

**Goals:**

- Students will develop an understanding of variables in mathematics.
- Students will learn that like terms can be added and subtracted.

**Assessment:**

- Pre-assessment: A whole class concept map.
- Post-assessment: Student math journals.

**The Lesson:** The teacher will begin the lesson by making a concept map using the word “variable” on the board. The students will come up with their own ideas and definitions of variable. The teacher will lead a discussion about how variables can stand for a mystery number. She will focus on integrating arithmetic and algebra (ex. 3 + 6 = x and thus, x = 9). In their math journals, students will write their own definition of variable and three mathematical examples. Once the students have a solid understanding of variables, the teacher will write a sentence on the board, “Three apples + Nine apples = Twelve apples.” The teacher will ask the students to discuss in their groups why this is a true statement and how to write it using variables. One student from each group will write their variable equation on the board and the equations will be compared. (3a + 9a = 12a). This will lead into a discussion about like terms. Students will be given problems in their math journals for adding and subtracting like terms.

**Materials:**

- Student math journals
- pencils
- eInstruction tablet

**Time:** 2 hours

Lesson 3: Making Known Quantities Unknown

**Overview/Rationale:** In this lesson, students will generalize from arithmetic to algebra. They will begin by working on an arithmetic word problem. Then, students will be presented with an algebraic problem that is similar to the arithmetic problem, except with unknown quantities. Students will work in small groups and discuss how to go about solving the problem. Students must be able to represent unknown quantities using symbols, rather than working out simple computation problems.

**Goals:**
- Students will develop an understanding of variables as representing unknown quantities.
- Students will think about word problems algebraically.

**Assessment:**
- Pre-assessment: Arithmetic word problem and algebraic word problem
- Post-assessment: Student math journals.

**The Lesson:** The teacher will write an arithmetic problem on the board and ask students to discuss in their groups to solve the problem. ("If Ann has $5 and Joe has $8 more than Ann, how much money does Joe have?"). Then, she will write an algebraic problem on the board and ask the students to discuss the problem. ("If Ann has some money and Joe has $8 more than Ann, how much money does Joe have?"). The teacher will lead a whole group discussion by asking:
- How are these questions similar?
- How are they different?
- What is the first question asking?
- What is the second question asking?
- Can you write an equation for the first question?
- Can you write an equation for the second question?

In their math journals, students will complete an activity that makes known quantities unknown.

**Materials:**
- Student math journals
- pencils
- eInstruction tablet

**Time:** 2 hours

# Lesson 4: Varying Known Quantities

**Overview/Rationale:** This lesson shows students that variables can vary, and are not just an unknown quantity. Students will see how to vary a quantity in arithmetic to explore patterns, relationships, and variables. This lesson will be done as a whole class exploration and the students will come up with the shared meanings of how to do the algebra problem. This lesson provides students with the opportunity to symbolize their thinking and build an understanding of unknown and varying quantities.

**Goals:**
- Students will develop an understanding of variables as representing varying quantities and not just unknown quantities.
- Students will think about word problems algebraically.
- Students will learn to think beyond arithmetic and generalize quantities.

**Assessment:**
- Pre-assessment: Arithmetic word problem and algebraic word problem
- Post-assessment: Student math journals.

**The Lesson:** The teacher will write an arithmetic problem on the board and ask students to discuss in their groups to solve the problem. "I want to buy a tee shirt that costs $14. I have $8 saved already. How much more money do I need to earn to buy the shirt?"). Then, she will write an algebraic problem on the board and ask the students to discuss the problem. ("Suppose the tee shirt costs $15. If I have $8 saved already, write a number sentence that describes how much more money I need to buy the tee shirt. What if the shirt costs $16? $17? Write number sentences for each of these cases. If *P* stands for the price of any tee shirt I want to buy, write a number sentence using *P* that describes how much more money I need to buy the tee shirt."). The teacher will lead a whole group discussion by asking:
- How are these questions similar?
- How are they different?
- What is the first question asking?
- What is the second question asking?
- Can you write an equation for the first question?
- Can you write an equation for the second question?

In their math journals, students will complete an activity that varies known quantities.

**Materials:**
- Student math journals
- pencils
- eInstruction tablet

**Time:** 2 hours

Lesson 5: Algebraic Role of the Equal Sign

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| **Overview/Rationale:** This lesson will help students see the role of the equal sign as a relationship between two quantities. Students may already have the misconception that that the equal sign is for performing computations. In this lesson, students will find a variety of pairs of numbers that add to 15. Students will have small group discussion about how many pairs exist and what they are. Then, the whole class will write an algebraic equation showing this sum. It is important for students to develop an algebraic view of equality for future math courses. |
| **Goals:**<br>▪ Students will learn that the equal sign is a relation between two quantities.<br>▪ Students will begin expand their knowledge of variables by writing algebraic equations. |
| **Assessment:**<br>▪ Pre-assessment: Finding pairs of numbers that add to 15 and generalizing to variables.<br>▪ Post-assessment: Student math journals. |
| **The Lesson:** The teacher will ask each group of students to come up with a variety of pairs of numbers that add to 21. (Each group will be given a different sum). Students will be asked to discuss why each pair of numbers adds to 21. For example, 1 + 20 = 2 + 19 = 3 + 18, etc. Once students have a variety of correct pairs of numbers that add to 21, the teacher will ask them to write the same equation using variables. For example, a student may say that x = 1 and y = 20, so x + y = 21. Students will work on problems that develop an algebraic role of the equal sign in their math journals. (Ex. Finding missing numbers.) |
| **Materials:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet |
| **Time:** 1 hour |

Lesson 6: Learning the Equal Sign with Meaning

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| **Overview/Rationale:** In this lesson, the students will be introduced to solving equations. To help students engage in relational thinking, true/false and open number sentence examples will be used. These examples show that the equal sign is a relationship between two quantities and not only an operand. Students will be asked to write their own open sentences and will describe them to other students in small groups. |
| **Goals:**<br>▪ Students will learn that the equal sign is a relation between two quantities.<br>▪ Students will learn that the equal sign is a relationship between two quantities and not only an operand.<br>▪ Students will write their own open sentences. |
| **Assessment:**<br>▪ Pre-assessment: True/false and open number sentences.<br>▪ Post-assessment: Student math journals. |
| **The Lesson:** The teacher will begin the lesson by writing true/false and open number sentences on the board. For example, true/false (5 + 0 = 5, 6 = 3 + 3, 24 – 6 = 19, 9 + 14 = 31 – 8) and open number sentences (32 + ___ = 41 + 5, 6 + 6 + 7 = ___ - 5). Students will discuss in their groups how to solve these true/false sentences and open number sentences. Once students have a good understanding of these number sentences, each group will write their own true/false number sentences and open number sentences and the other groups will solve them using whole class discussions and the eInstruction tablet. Students will play the unknown card game. In their math journals, students write three true/false number sentences and solve them and three open number sentences and solve them, explaining their reasoning. |
| **Materials:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet |
| **Time:** 2 hours |

Lesson 7: Learning Variables with Meaning

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| **<u>Overview/Rationale</u>:** In this lesson, students will learn the mathematician's rule where the same symbol or letter in an equation represents the same number every place it occurs. For example, in the number sentence<br>$\_\_\_ + \_\_\_ + 2 = \_\_\_ + 5 + 1$<br>The ___ must be the number 4 every time. Students will expand their learning of the variable as a placeholder by modeling the following example with algebra: Jane's bookcase contains 17 books. She takes some books to school and the bookcase is left with 12 books. How many books did Jane take? This lesson places the variables in context, and thus, gives the variables meaning. |
| **<u>Goals</u>:**<br>▪ Students will learn the mathematician's rule where the same symbol or letter in an equation represents the same number every place it occurs.<br>▪ Students will learn that variables can represent a real life situation and thus, variables have meaning.<br>▪ Students will expand their knowledge about open number sentences and equality. |
| **<u>Assessment</u>:**<br>▪ Pre-assessment: Open number sentence with mathematician's rule.<br>▪ Post-assessment: Student math journals. |
| **<u>The Lesson</u>:** The teacher will begin the lesson by writing an open number sentence on the board with more than one blank. The answer for each blank will be the same number. Students will solve for the blank by discussing what they are doing with their group. Once each group has come up with an answer, one student from each group will write the answer on the board and explain their group's reasoning. The teacher will write another open number sentence on the board and again ask students to solve for the blank in their groups. Then, the teacher will write an algebraic equation on the board, using a variable, and ask students if this is the same as the blanks. In a whole class discussion, the students and teacher will explore combining like terms and solving for the variable. Students will work on modeling situations with open number sentences and solving for the blank/variable in their journals. |
| **<u>Materials</u>:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet |
| **<u>Time</u>:** 2 hours |

Lesson 8: Learning Variables as Changing Quantities

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| **Overview/Rationale:** This lesson will introduce the students to the graph of a line and will give them the equation of the line. Students will discover that a point on the line makes the quantities on each side of the equation equal. For example, the graph and equation of the line will be given for $y = 3x + 1$. By finding different ordered pairs that work in the equation, the students will see that variables can be changing quantities. |
| **Goals:**<br>▪ Students will learn that equations can have more than one variable.<br>▪ Students will learn that a point (x, y) on a line makes the equation y = mx + b true and a point not on the line makes the equation y = mx + b false.<br>▪ Students will learn that variables can be changing quantities. |
| **Assessment:**<br>▪ Pre-assessment: Picking points on a line and substituting the points on the line into the equation of the line.<br>▪ Post-assessment: Student math journals. |
| **The Lesson:** The teacher will draw a line on a graph and introduce students to the equation of the line. Then, the teacher will ask each group to pick a point on the line and substitute the point (x,y) into the equation and see if it makes a true statement. One student from each group will present their answer and explain why it makes the equation true. A whole class discussion will be held that focuses on the equation of the line and variables as changing quantities. Students will work together on line, equation, and point problems in their math journals. |
| **Materials:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet |
| **Time:** 1 hour |

Lesson 9: Balance and the Equal Sign

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| **Overview/Rationale:** This lesson will help students conceptualize the equal sign as a balance scale, where the two expressions on either side of the equal sign have the same value. This visual conceptualization will attach meaning to the equal sign. Students will transition from thinking the equal sign indicates "the answer is" to the equal sign indicates "the same as." The AlgebraTouch app will be used to during this lesson. Once students understand solving equations on the app, they will work with Hands-On Algebra. This lesson will expand on the last lesson and students will use the manipulatives similar to Hands-On Algebra. In order for a scale to remain balanced, both sides must be equal. |
| **Goals:**<br>▪ Students will expand their knowledge of the equal sign and equality.<br>▪ Students will learn that equations must remain balanced.<br>▪ Students will begin to learn the process of solving an equation for the variable. |
| **Assessment:**<br>▪ Pre-assessment: A whole class discussion about balance and visualizing equations on the AlgebraTouch App.<br>▪ Post-assessment: Student math journals. |
| **The Lesson:** The teacher will begin the lesson by having students come up with their own definition of balance. The discussion will expand to balancing equations. Students will work in their groups on solving simply algebraic equations on the AlgebraTouch App. Students will be asked to write their own equations and solve for the variable. Each group will write an equation and the other groups will solve the equation on the board with explanations. The teacher will show students how to use Hands-On Algebra and students will work on problems in their groups. In math journals, students will solve algebraic equations. |
| **Materials:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet<br>▪ AlgebraTouch App<br>▪ iPads<br>▪ Hands-On Algebra |
| **Time:** 4 hours |

Lesson 10: Decontextualize a Problem

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| **Overview/Rationale:** The whole class will have a discussion about the following problem and how to solve the problem: The perimenter of a garden is 20 ft and the length of the garden is 8 ft more than the width. Find the dimensions of the garden. To decontextualize this problem, students will learn that the perimeter of a rectangle is given by $A = 2l + 2w$ and will express the length in terms of the width to model this problem. Once the whole group comes up with the symbolic representation, students will describe each part of the equation and consider the units. This is contextualizing the algebraic equation. |
| **Goals:**<br>▪ Students will connect their previous knowledge of solving equations with writing equations.<br>▪ Students will represent real life situations with algebra. |
| **Assessment:**<br>▪ Pre-assessment: A whole class discussion about solving the area problem<br>▪ Post-assessment: Student math journals. |
| **The Lesson:** The teacher will begin by presenting the area problem on the board and students will discuss how they would go about solving the problem. A whole class discussion will be held where the teacher asks how to represent length and how to represent width and how to represent perimeter. Once the whole group comes up with the symbolic representation, the students will have small group discussions about how to solve the problem and the units in the problem. The teacher will ask students for their ideas about what numbers to substitute for each variable and how to go about finding the perimeter. The whole class will solve the equation together. In their math journals, students will be given a similar area problem and asked to write the steps about how they would solve for the dimensions. |
| **Materials:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet |
| **Time:** 2 hours |

Lesson 11: Systems Structure of Algebraic Equations (Equivalence of LHS and RHS)

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| **Overview/Rationale:** In arithmetic, students generalize that 5 + 3 = 8 can be expressed as 8 – 3 = 5. This lesson will deepen students understanding of the meaning of the equal sign by looking at the systemic structure of algebraic equations. For instance, generalizing that $x + 3 = 8$ can be expressed as $3 = 8 - x$. During this lesson, the addition property of equality and the multiplication property of equality will be reinforced (i.e. if $a = b$, then $a + c = b + c$ and $ac = bc$). |
| **Goals:**<br>▪ Students will expand their knowledge about equivalence and the equal sign.<br>▪ Students will learn how to write equivalent algebraic equations.<br>▪ Students will continue to learn about the addition property of equality and the multiplication property of equality. |
| **Assessment:**<br>▪ Pre-assessment: Writing an equivalent numerical equation.<br>▪ Post-assessment: Student math journals. |
| **The Lesson:** The teacher will write an arithmetic equation on the board and ask students to write an arithmetic equation that is equivalent to the one on the board. Once students have discussed their answers in their group, explaining why their equation is equivalent to the equation on the board, the teacher will write an algebraic equation on the board and ask students to write an equivalent equation. Students will share their equations with each other and decide if they are actually equivalent equations. Students will solve the equations for the variable and discuss why they get the same answer or not for the variables in the equations. Students will complete an activity in their math journals in which they write equivalent equations. |
| **Materials:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet |
| **Time:** 3 hours |

Lesson 12: Patterns

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| **Overview/Rationale:** Students will begin to notice and describe patterns after seeing pictorial representations on the board. Then, they will create their own growing pattern using pattern blocks. They will keep track of each figure, the number of blocks added, and the total number of blocks. Using this information, students will predict what the 10th figure will look like and how many blocks will be needed to build it. This lesson will lead into writing a rule, or function, for the number of blocks in each figure. |
| **Goals:**<br>▪ Students will learn about patterns.<br>▪ Students will learn that a pattern can be represented by algebra.<br>▪ Students will learn that a function can be a rule for a pattern. |
| **Assessment:**<br>▪ Pre-assessment: Noticing patterns.<br>▪ Post-assessment: Student math journals. |
| **The Lesson:** The teacher will display pictorial representations of patterns on the board and ask students to discuss what they see. A whole class discussion will be held about what a pattern is and how to represent a pattern using algebra. Students will work on representing patterns using numbers and eventually algebra. For example, a pattern may be, x, x^2, x^3, x^4, etc. Each group of students will make their own pictorial pattern that can be represented using numbers and variables. The groups will exchange pictures and write the rule for each pattern. In their math journals, students will complete a pattern activity. |
| **Materials:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet |
| **Time:** 2 hours |

Lesson 13: Functions/ Dependent and Independent Variables

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| **Overview/Rationale:** Students will learn that functions have three components: the domain, the range, and the rule. Students will learn that a function can be represented in different ways such as verbally, in an arrow diagram, algebraically, and graphically. |
| **Goals:**<br>▪ Students will learn about functions.<br>▪ Students will learn that a function can be represented verbally, in an arrow diagram, algebraically, and graphically.<br>▪ Students will learn about dependent and independent variables.<br>▪ Students will learn domain and range. |
| **Assessment:**<br>▪ Pre-assessment: Noticing patterns.<br>▪ Post-assessment: Student math journals. |
| **The Lesson:** The teacher will write a verbal function on the board (The perimeter of a square is the sum of the sides.) Students will come up with an algebraic representation of this statement. Then, each group will share what they came up with (P = s + s + s + s or P = 4s). Then, students will be asked to find P for different values for the sides of the square. Students will be introduced to the arrow diagram and will be asked to plot the points and graph the line. Each group will be given a similar problem and asked to represent using the four forms for a function. Each group will present their functions on chart paper. Students will do a similar activity in their math journals. |
| **Materials:**<br>▪ Student math journals<br>▪ pencils<br>▪ eInstruction tablet<br>▪ Chart Paper |
| **Time:** 3 hours |

Appendix B

# ALGEBRA SIXTH GRADE UNIT

Teacher's Manual

| Common Core State Standards of Mathematics Grade 6<br>Expressions and Equations |
|---|
| • Apply and extend previous understandings of arithmetic to algebraic expressions.<br>• Reason about and solve one-variable equations and inequalities.<br>• Represent and analyze quantitative relationships between dependent and independent variables. |
| • Use properties of operations to generate equivalent expressions.<br>• Solve real-life and mathematical problems using numerical and algebraic expressions and equations. |

Diana L. Moss

## Unit Objectives

### Mathematical Content

The purpose of this instructional unit is for students to conceptually understand algebraic expressions and equations. Students will learn to view an algebraic expression as an object and not simply a computation. An expression is a mathematical statement or sentence that expresses calculations with numbers and variables. A variable is a letter that represents a number and can be a changing quantity. Students will learn to write, evaluate, and simplify expressions. They will also learn how to represent real world situations with algebraic expressions and begin to understand that expressions written in different forms can be equivalent expressions. Students will learn how to solve algebraic equations by using the addition law and multiplications law. Students will learn to view the equal sign as an equivalence relation and not as an operation to compute. Students will also learn the process of solving for a single variable in an equation by using relational thinking.

For students, learning algebra is learning a new language. Therefore, this instructional unit encourages students to use algebraic thinking to solve problems related to a common theme. The lessons are organized to naturally bridge students' previous understandings of arithmetic to new understandings of algebra. The teacher can adapt the lessons to meet the needs of his or her students. The student worksheets and teacher suggestions are intended as tools to facilitate the development of algebraic thinking and the mathematical concept of algebraic expressions and equations. This instructional unit is aligned with the Common Core State Standards for Mathematics.

Common Core State Standards for Mathematics

**Apply previous understandings of arithmetic to algebraic expressions**

- **6.EE.A.1**: Write and evaluate numerical expressions involving whole-number exponents.
- **6.EE.A.2**: Write, read and evaluate expressions in which letters stand for numbers.
    a. Write expressions that record operations with numbers and with letters standing or numbers. *For example, express the calculation "Subtract y from 5" as $5 - y$.*
    b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression $2(8 + 7)$ as a product of two factors; view $(8 + 7)$ as both a single entity and a sum of two terms.*
    c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas $V = s^3$ and $A = 6s^2$ to find the volume and surface area of a cube with sides of length $s = ½$.*
- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.*
- **6.EE.A.4**: Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the*

*expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.*

**Reason about and solve one-variable equations**

- **6.EE.B.5**: Understand solving an equation as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use and explain substitution in order to determine whether a given number in a specified set makes an equation or inequality true.
- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.
- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and $x$ are all nonnegative rational numbers.

**Represent and analyze quantitative relationships between dependent and independent variables**

**6.EE.C.9**: Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation d = 65t to represent the relationship between distance and time.*

**Common Core State Standards for Mathematics:**

**Mathematical practice**

- MP1. Make sense of problems and persevere in solving them.
- MP2. Reason abstractly and quantitatively.
- MP3. Construct viable arguments and critique the reasoning of others.
- MP4. Model with mathematics.
- MP5. Use appropriate tools strategically.
- MP6. Attend to precision.
- MP7. Look for and make use of structure.
- MP8. Look for and express regularity in repeated reasoning.

Algebra Vocabulary

**Addition Property of Equality:** The property that states that if you add or subtract the same number from both sides of an equation, the new equation will have the same solution.

**Area:** The number of square units needed to cover a given surface.

**Base:** When a number is raised to a power, the number that is used as a factor is the base.

**Coefficient:** The number that is multiplied by a variable in an algebraic expression.

**Coordinate Plane:** A plane formed by the intersection of a horizontal umber line called the *x*-axis and a vertical number line called the *y*-axis.

**Difference:** The result when two or more quantities are subtracted.

**Distributive Property:** The property that states if you multiply a sum by a number, you will get the same result if you multiply each addend by that number and then add the products.

**Domain:** The set of all possible input values of a function.

**Equation:** A mathematical sentence that shows that two expressions are equivalent.

**Equivalent:** Having the same value.

**Equivalent Expression:** Equivalent expressions have the same value for all values of the variables.

**Evaluate:** To find the value of a numerical or algebraic expression.

**Exponent**: The number that indicates how many times the base is used as a factor.

**Expression:** A mathematical phrase that contains operations, numbers, and/or variables.

**Factor:** A number that is multiplied by another number to get a product.

**Formula:** A rule showing relationships among quantities.

**Function:** An input-output relationship that has exactly one output for each input.

**Function Table:** A table of ordered pairs that represent solutions of a function.

**Graph:** A set of points and the line segments or arcs that connect the points.

**Graph of an Equation:** A graph of the set of ordered pairs that are solutions of the equation.

**Hexagon:** A six-sided polygon.

**Like Terms:** Two or more terms that have the same variable raised to the same power.

**Line:** A straight path that extends without end in opposite directions.

**Multiplication Property of Equality:** The property that states that if you multiply/divide both sides of an equation by the same number, the new equation will have the same solution.

**Ordered Pair:** A pair of numbers that can be used to locate a point on a coordinate plane.

**Pentagon:** A five-sided polygon.

**Perimeter:** The distance around a polygon.

**Product:** The result when two or more quantities are multiplied.

**Quotient:** The result when two or more quantities are divided.

**Range:** The set of all possible output values of a function.

**Simplify:** To work the problem until nothing else can be done.

**Substitute:** To replace a variable with a number or another expression in an algebraic expression.

**Sum:** The result when two or more quantities are added.

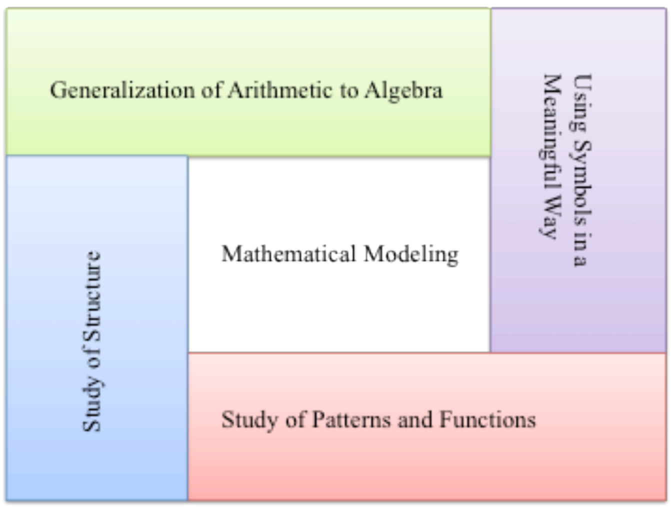
**Term:** The parts of an expression that are added or subtracted.

**Variable:** A symbol used to represent a quantity that can change.

Source: Holt Pre-Algebra

Unit Design Features

This instructional unit is based on Kaput's (1999) five interrelated forms of algebraic thinking. The instructional unit contains the following forms of algebraic thinking: (a) Generalizing Arithmetic to Algebra, (b) Using Symbols in a Meaningful Way, (c) Study of Structure, (d) Study of patterns and functions, and (e) Mathematical Modeling and combining the first four forms.




The lessons in this unit are related through the common theme of *Going to the World Cup*. This theme sets up a context of going to a soccer game where students can use their informal knowledge to explore algebraic expressions and equations. This context lends a purpose for algebraic thinking. The purpose is to solve the *Going to the World Cup* Challenge Problem, which must be done by using algebraic expressions and equations.

*Going to the World Cup* Challenge Problem

| |
|---|
| You and your family want to go to the 2014 World Cup. There are 64 matches at the FIFA World Cup Finals, and, unfortunately, you will not be able to attend every match. Tickets for each match for adults are \$140 and for children are \$70. How much money will it cost for the tickets for your family to go to one World Cup Match? Two World Cup Matches? Three World Cup Matches? Write an algebraic expression for the Total Cost for tickets for a family to attend matches at the World Cup. |

All the data needed for solving this problem is not embedded in the given information. This problem is intended to promote thinking about what is needed to solve the problem. Students will need to think about their own experiences to figure out the necessary information to solve this problem. For example, students will need to figure out how much money they have total, how to raise this amount of money, how many children will be attending, how many adults will be attending, etc. Thus, this context of going to a soccer game was specifically chosen to generate algebraic thinking.

Additionally, the lessons in this unit encourage learning algebra through whole class mathematical discussions (Lamberg, 2012) and providing students with the necessary tools and knowledge so that they are able to solve the challenge problem at the conclusion of the unit. Students do not immediately solve the problems, but engage in algebraic thinking and brainstorm ways to solve the problems. The student worksheets have problems that increase in difficulty level. The worksheets are tools for the teacher to use and serve as models of various types of algebra problems. The teacher may choose appropriate problems for the students and use them to guide the lessons. The teacher may also choose to allow students to model their own problems based on the problems on the worksheets.

Algebraic Thinking Unit Timeline

**Day 1: Pretest**
All students work on the pretest individually.

**Day 2: Introduction Lesson**
Introduce the *Going to the World Cup* theme with a discussion about what information is necessary to solve the problem. Generate a list of ideas of what information is needed to solve the problem. Focus the discussion on the unknown values and algebraic representations of these unknown values.

***Note:*** *The following lessons are related to the context of Going to a Soccer Game. These lessons are intended to engage students in algebraic thinking as presented in the Framework for the Five Forms of Algebraic Thinking. These lessons lead up to the final lesson of solving the Going to a Soccer Game problem.*

**Day 3: Variable, Expression, and Equation Lesson**
Students will create their own definitions to variable, expression, and equation. A Cognitive Content Dictionary Chart (CCD) will be used to come up with a whole class definition of variable, expression, and equation. This lesson is to get students thinking about algebra and using the academic language.

**Day 4: Adding and Subtracting Like Terms Lesson**
Students will develop an understanding of variables in mathematics and will learn that like terms can be added and subtracted. Students will also learn to model patterns with algebra.

**Day 5: Continuation of Adding and Subtracting Like Terms Lesson**
Students will develop an understanding of variables in mathematics and will learn that like terms can be added and subtracted.

**Day 6: Combining Like and Unlike Terms**
This lesson will extend the students' previous understanding of combining like and unlike terms.

**Day 7: Simplifying and Equality Lesson**
This lesson will continue to build students' understanding of combining like and unlike terms. Students will learn how to simplify an expression and transform it into an equation.

**Day 8: More Adding and Subtracting Like Terms Lesson**
This lesson will extend the students' previous understanding of the equal sign as an equal relationship between two sides of an equation. Students will learn how to isolate a variable on one side of an equation using the addition property of equality.

**Day 9: Variable as a Changing Quantity Lesson**
This lesson will extend the students' previous understanding of the variable from representing an object to representing a changing quantity. Students will represent the cost of cupcakes with a variable expression and then solve.

**Day 10: Variable as a Changing Quantity and The Addition Property of Equality Lesson**
This lesson will extend the students' previous understanding of the variable from representing an object to representing a changing quantity. Students will represent the cost of soccer balls with a variable expression and then solve.

**Day 11: Variable as the Unknown and The Addition Property of Equality Lesson**
This lesson will extend the students' previous understanding of the variable from representing a changing quantity to representing an unknown value. Students will represent the cost of soccer balls with a variable expression and then solve.

**Day 12: Variable as the Unknown and The Addition Property of Equality Lesson**
This lesson will extend the students' previous understanding of the variable from representing a changing quantity to representing an unknown value. Students will solve equations using the addition property of equality and will practice with Algebra Touch.

**Day 13: Multiplication Property of Equality Lesson**
This lesson will extend the students' previous understanding of the equal sign as an equal relationship between two sides of an equation. Students will learn how to isolate a variable on one side of an equation using the multiplication property of equality.

**Day 14: Review of Addition Property of Equality and Multiplication Property of Equality Lesson**
This lesson will extend the students' previous understanding of the equal sign as an equal relationship between two sides of an equation. Students will review how to isolate a variable on one side of an equation using the addition and multiplication property of equality.

**Day 15: Structure of Algebraic Equations Lesson**
**(Equivalence of Left hand side and Right hand side)**
This lesson will deepen students understanding of the meaning of the equal sign by looking at the systemic structure of algebraic equations.

**Day 16: Equivalent Equations and Distributive Property Lesson**
This lesson will extend the students' previous understanding of like terms as quantities that can be added and subtracted. Students will learn more about equivalent equations. Students will learn the distributive property.

**Day 17: Solving Algebraic Equations using the Addition Property of Equality and the Multiplication Property of Equality Lesson**
In this lesson, students will learn how to isolate the variable in an algebraic equation using the addition property of equality and the multiplication property of equality.

**Day 18: Functions Lesson**
Students will learn that functions have three components: the domain, the range, and the rule. Students will learn that a function can be represented in different ways such as verbally, in an arrow diagram, algebraically, and graphically. Students will continue to practice substituting values for variables.

**Day 19: Review and Solving the Challenge Problem Lesson**
Students will finish the functions activity from Day 18. Students will practice the vocabulary from this unit and continue to clear up misconceptions in the unit. Students will do the challenge problem from Day 1.

**Day 20: Posttest**
All students work on the posttest individually.

**Day 1: Pretest**

**Lesson Objective:**
All students work on the pretest individually.

**Materials Needed:**
- Pretest
- Pencil
- Scratch paper

Algebra Instructional Unit

Name:________________________________________

**Pretest**

Please show your work and circle your answer.

1) What is an algebraic expression? Give an example.

2) What is an algebraic equation? Give an example.

3) Is 9 a solution of $p - 7 = 2$ ?

4) Evaluate the expression for the given replacement values.
$2x + 5y$ for $x = 4$ and $y = 7$

5) Simplify the expression by combining any like terms.
$8b - 2b$

6) Simplify the expression by combining any like terms.
$5x + 2 - 3x + 8$

7) If David has $15 and Joanna has $6 less than David, how much money does Joanna have?

8) If David has some money and Joanna has $6 less than David, how much money does Joanna have?

9) In algebra, what is a variable?

10) How many pairs of numbers can you find that add to 10?
Express the number 10 as a sum of two numbers using variables.

11) True or False?
$9 + 15 = 40 - 16$

12) Fill in the blank.
$3 + 2 + 5 = \_\_\_\_ - 2$

13) Fill in the blanks.
$\_\_\_ + 3 + \_\_\_ = 8 + 1 + \_\_\_$

14) Haley has 27 songs on her iPod. She accidentally deleted some songs and now only has 18 songs on her iPod. How many songs did she delete?

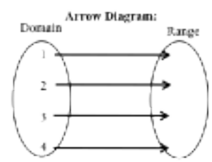
15) What is the addition property of equality? Give an example.

16) What is the mulitplication property of equality? Give an example.

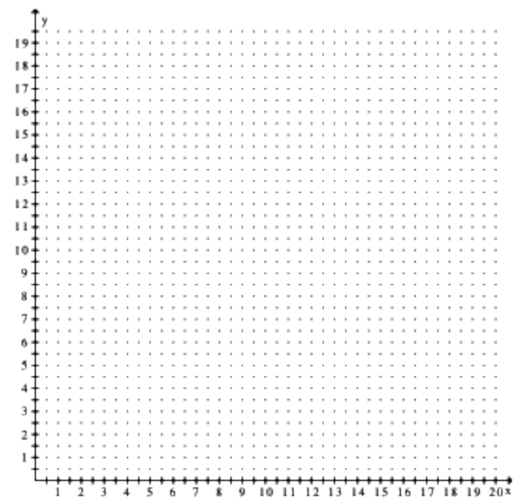
17) Solve the equation.
$2x + 10 = 14$

18) Fill in the blank.
If 2 + 3 = 5, then 5 - ____ = 2

19) Fill in the range of the function y = 2x shown in the arrow diagram below.




20) Graph y = 2x using the points in the domain and range from the previous problem.



## Day 2: Introduction Lesson

**Lesson Objective:**
Introduce the *Going to the World Cup* theme with a discussion about what information is necessary to solve the problem. Generate a list of ideas of what information is needed to solve the problem. Focus the discussion on the unknown values and algebraic representations of these unknown values.

**CCSS:**
- **6.EE.A.2**: Write, read and evaluate expressions in which letters stand for numbers.
- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

**Materials Needed:**
- World Cup Video
- Day 2: Introduction Lesson Student Sheet
- Math Notebook
- Pencil

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Variable | An entire equation with letters like a + b = c | Make explicit that a variable is one letter |
| | A word for a number | It is only one letter |
| Expression | Only has a plus sign | Show examples with subtraction |
| | Answer to an equation | Show the difference between equation and expression |
| | Is a question | Use the word statement or sentence |
| Equation | Getting equation confused with expression | Show more examples |

The lessons in this unit are related through the common theme of *Going to the World Cup*. This theme sets up a context of going to a soccer game where students can use their informal knowledge to explore algebraic expressions and equations. This context lends a purpose for algebraic thinking. The purpose is to solve the *Going to the World Cup* Challenge Problem, which must be done by using algebraic expressions and equations.

***Going to the World Cup* Challenge Problem**

You and your family want to go to the 2014 World Cup. There are 64 matches at the FIFA World Cup Finals, and, unfortunately, you will not be able to attend every match. Tickets for each match for adults are $140 and for children are $70. How much money will it cost for the tickets for your family to go to one World Cup Match? Two World Cup Matches? Three World Cup Matches? Write an algebraic expression for the Total Cost for tickets for a family to attend matches at the World Cup.

**Procedure for Introduction Lesson:**

**Whole Class Discussion**

- Opener: Begin the lesson, by asking the class questions about soccer.
    - What is your favorite pro soccer team?
    - Have you ever been to a pro soccer game?
    - Show the video about next year's World Cup. https://www.youtube.com/watch?v=xuY6l4HhDzE

Get your students interested and curious about the World Cup and Soccer!

**Questions to guide Student Thinking**

- Does everyone's family have the same number of people?
- Can you think of a formula that we can use every time to figure out how many people are going in each family?

**Small Group Discussion**

- On paper, have each student make a list of their family members and total the amount of people that will be going to Brazil.
    - Have the students share in groups of four who would be going to the World Cup.
- Then, ask each student to separate their lists into number of adults and number of children. Ask students if they can model this with variables. For example, $a$ is the number of adults and $c$ is the number of children.
- Ask students to write an expression that shows the total amount of family members going to the World Cup. Ask students to use numbers and then use variables. For example, if a family has 2 adults and 3 children then this can be written as $2 + 3$ (arithmetic) or as $a + c$ (algebra).

**Whole Class Discussion**

- Let students share their different representations of the total amount of people going to the World Cup.

(This discussion may lead into a discussion on expressions versus equations and the equal sign. However, the purpose of this lesson is to introduce the thematic unit and for students to learn about variables and representing unknown quantities.)

**Process to Shift Student Thinking**

- Create a three column table on the board
- Label the columns: “adults,” “children,” and “total”
- Ask students to report the number of adults and children
- Record this on the column table
- Write the words:

“adults + kids” = “total number of people”

**Questions to guide Student Thinking**

- Can you think of a formula that we can use every time to figure out how many people are going in each family?
- Can we write this in a mathematical equation?
- Can we use a letter or a variable to represent the adults?
- Can we use a letter or a variable to represent the children?
- What does the letter $c$ represent? What does the letter $a$ represent? Does it matter what letter you use?
- What does $a + c$ represent?
- How is that different from $a + c = t$?

***Big Ideas that Evolve from Whole Class Discussion***

- Variable
- Expression
- Equation
- Equal Sign
- Formula

**Student Homework: Bring a soccer ball to class tomorrow.**

**Name: ___________________________**

**Day 2: Introduction Lesson Student Sheet**

"A FIFA World Cup™ is a very special experience. Every four years, fans from all over the world flock to the host nation to witness the 64 matches at the FIFA World Cup finals live and in person. They come for many reasons, some to support a favourite team every step of the way, others simply to share in the passion and excitement at the stadiums. For every enthusiastic supporter, being at the FIFA World Cup finals in person is an unforgettable and thrilling experience" (FIFA World Cup™ ).

You and your family want to go to the 2014 World Cup. There are 64 matches at the FIFA World Cup Finals, and, unfortunately, you will not be able to attend every match. Tickets for each game for adults are $140 and for children are $70. How much money will it cost for the tickets for your family to go to one World Cup Game? Two World Cup Games? Three World Cup Games?

INSTRUCTIONS:

*Do not solve the problem above yet, but try to do the following steps.*

1. Make a list of how many total family members will want to go to the World Cup.

2. Separate the list into adults and children.

3. Model with variables the number of adults and the number of children.

4. Can you write an equation for the total amount of adults and children going to the game?

## Day 3: Variable, Expression, Equation Lesson

**Lesson Objective:**
Students will create their own definitions to variable, expression, and equation. A Cognitive Content Dictionary Chart (CCD) will be used to come up with a whole class definition of variable, expression, and equation. This lesson is to get students thinking about algebra and using the academic language.

**CCSS:**
- Addresses the CCSS that involve variable, expression, and equation.

**Materials Needed:**
- Index cards
- Cognitive Content Dictionary Chart on the board or on chart paper
- Day 3: Warm-Up
- Chart paper
- Pencil
- Pens
- Markers

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Equal Sign | The equal sign means only equation | Make explicit that the equal sign means the same as or balanced. |
| | The equal sign means the total | Show the difference between "compute" and "equality" |
| Variable | It is a letter that represents a number, but does "NOT" change. | They will see that a variable can change when we do graphing. Right now, they think it is just and unknown value. |
| Equation | An expression with an equal sign. x + 2 = | It has to equal something, so more discussion about balance and equality. |
| Expressions | Deals with feelings | Not a happy face on board |
| | Only operation is addition | Show examples with other operations |

**Procedure for Variable, Expression, Equation Lesson:**

**Warm Up for Day 3**

- Ask students to work on Warm Up for Day 3 in groups of four. Allow about 15 minutes or until most students are done with the warm up.
- As students are working, walk around and look at student solutions and pick problems to discuss during whole class discussion.

Students might not be used to having Whole Class Discussions:

- *Let them use a microphone*
- *Ask the class if they agree, disagree, or are unsure*
- *Make sure your classroom is a place where students are comfortable*

**Whole Class Discussion**

- Quickly review the expression or equation part of the warm up and ask students to justify their reasoning
- Write a problem from the warm up on the board
- Ask students how they solved it and have them share their strategies
- Discuss the meaning of the equal sign.

**Questions to guide Student Thinking**

- What does the equal sign mean?
- Show me why the missing number is…
- What do you notice about both sides of the equation?
- What quantity is on each side of the equation?

***Big Idea that Evolves from Whole Class Discussion***

- The Equal Sign means "same as" and it indicates an equivalent relationship between both sides of the equation.

**Small Group Discussion**

- In groups of four, ask students to come up with a word definition and an example in mathematics for variable, expression, and equation
- The groups will discuss and write their answers on 3 separate index cards

**Process to Shift Student Thinking**

- Variable, expression, and equation were introduced in the prior lesson
- Students have an idea for each of these
- Small group discussion helps them to formulate their own definitions and examples
- Whole class discussion compares the small group's examples
- Look for similarities and differences
- Come up with a whole class definition with help from the students
- Now, present the formal definitions and compare and contrast the whole class definition

**Whole Class Discussion**

- Ask a volunteer from each group to tape each card to the CCD Chart
- Once all the cards are on the CCD chart, the teacher will lead a discussion to formulate a class definition of variable, expression, and equation

**Questions to guide Student Thinking**

- What similarities do you see in the definitions/examples?
- What differences do you see in the definitions/examples?
- Which definitions/examples are correct? Why?
- Why is a variable only one letter?
- What is the difference between an equation and an expression?
- Can variables change values?
- What does the equal sign mean?

***Big Ideas that Evolve from Whole Class Discussion***

- Variable
- Expression
- Equation
- Equations have an equal sign
- The equal sign means both sides are the same

**Name: ___________________________**

**Day 3: Warm-Up**

1. Circle whether the following is a mathematical "expression" or an "equation".

| | | |
|---|---|---|
| a. 25 + y | expression | equation |
| b. 45 – 6 = 30 + 9 | expression | equation |
| c. 15 – 3 + x | expression | equation |
| d. 20 = 6 + 18 – 4 | expression | equation |

| 2. Find the missing number. | Answer: |
|---|---|
| 7 + 10 = ___ + 3 | |
| ___ + 35 = 46 | |
| 40 – 18 = ___ + 7 | |
| 32 + ___ = 1 + 9 | |

3. What does the equal sign mean?

## Day 4: Adding and Subtracting Like Terms Lesson

**Lesson Objective:** Students will develop an understanding of variables in mathematics and will learn that like terms can be added and subtracted. Students will also learn to model patterns with algebra.

**CCSS:**
- **6.EE.A.2**: Write, read and evaluate expressions in which letters stand for numbers.
- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions.
- **6.EE.A.4**: Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them).
- **6.EE.B.5**: Understand solving an equation as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use and explain substitution in order to determine whether a given number in a specified set makes an equation or inequality true.

**Materials Needed:**
- Soccer Balls (foosballs)
- Chart Paper
- Paper
- Overhead Projector
- Math Notebooks
- Pens and Pencils

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Like Terms | $10h + 12p = 32$ | Teach like terms again using fruit, 5 apples plus 3 bananas |
| | $h + p = hp$ | Again, more on like terms tomorrow. Show how this is multiplication by substituting in values for h and p. |
| | h = hexagons instead of h = 1 hexagon | Some understood that h is representing one hexagon. Clarify tomorrow |
| | $10h \cdot 2h = 20h$ | Teach multiplication of $h \cdot h = h^2$ |

**Procedure for Adding and Subtracting Like Terms Lesson:**

**Opener for Day 4**

- Begin the lesson by saying the following: "Remember, you and your family want to go to the World Cup next year! The World Cup would not be possible without soccer balls. So, we are going to look at what a soccer ball actually is."
- Pass out the foosball soccer balls. (If you do not have these, use a real soccer ball or a photo of a soccer ball.)

**Whole Class Discussion**

- Ask students to look at their soccer balls and name the pattern they see. (black and white/ hexagons and pentagons)
- Write the words "hexagon" and "pentagon" on the board
- Ask students to find a way to represent the number of hexagons and pentagons on the soccer ball
- Begin with only hexagons

**Questions to guide Student Thinking**

- What pattern do you see?
- What is a hexagon?
- What is a pentagon?
- How many pentagons are on the soccer ball?
- How many hexagons are on the soccer ball?
- What variable should we use to represent the hexagons?

**Questions to guide Student Thinking**

- What does the variable represent? One hexagon or all the hexagons?
- Does the variable represent the number of sides of a hexagon?
- How can you show $20h$ in other ways?
- Can we write $20h = 5h + 15h$?
- What do both sides of the equation represent? Are they the same?

**Small Group Discussion**

- In groups of four, ask students to write an expression that represents the total number of hexagons using addition
- Ask them to figure out ways to:
  - Write an expression to show how the hexagons add up to the total using the variable *h* or another variable that they come up with.
  - *Example:* $h + h + h + h + h + h + h + h + h + h + h + h + h + h + h + h + h + h + h + h$
- Ask them to do the same for the Pentagons.
  - *Example*: $p + p + p + p + p + p + p + p + p + p + p + p$

**Whole Class Discussion**

- Ask for volunteers to share what they wrote in their math notebooks with the class using the overhead projector.
- Ask the rest of the class if they have something different of the same.
- Then, prompt them to come up with a more efficient way of adding up the number of hexagons or pentagons to find the total.
    - *Example*: 3p +3p +3p +3p or 6p +3p +3p

Students might not be used to showing their work to the class:

- *Let them use a microphone*
- *It is faster to have students show their notebooks on the overhead than to rewrite their answer on the board*
- *Ask the class if they agree, disagree, or are unsure*

**Process to Shift Student Thinking**

- Students are modeling an algebraic expression using variables
- Today, a variable is a label where $h$ is one hexagon or $p$ is one pentagon
- When students come up with a more efficient way to represent the total number of hexagons, they are combining like terms
- Get multiple representations of the same expression to show that they are the same thing.
- For example, $h + h + 18h$ is the same as $2h + 18h$. Both represent 20 hexagons where $h$ is one hexagon.
- Move this into a discussion about multiplication where we look at $20h$ as $20 \cdot h$.

***Big Ideas that Evolve from Whole Class Discussion***

- Variable as a label
- Expression
- Like terms have the same variables and can be added or subtracted.
- Unlike terms have different variables and can not be added or subtracted.
- Equivalent Expressions and Equality

## Day 5: Continuation of Adding and Subtracting Like Terms Lesson

**Lesson Objective:** Students will develop an understanding of variables in mathematics and will learn that like terms can be added and subtracted.

**CCSS:**

- **6.EE.A.2**: Write, read and evaluate expressions in which letters stand for numbers.
- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions.
- **6.EE.A.4**: Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them).
- **6.EE.B.5**: Understand solving an equation as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use and explain substitution in order to determine whether a given number in a specified set makes an equation or inequality true.

**Materials Needed:**

- Soccer Balls (foosballs)
- Chart Paper
- Paper
- Overhead Projector
- Math Notebooks
- Pens and Pencils
- Day 5: Practice Adding and Subtracting Like Terms

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Like Terms | Writing variable in front of coefficient.<br>ex: R3 + B2 + G7<br>Should be 3R + 2B + 7G | Define coefficients and show more like term examples. |
| | Like term is correct and unlike term is wrong | More examples of like and unlike terms |
| | Like terms have same coefficient and same variable 20h and 20h | Show that different coefficients and same variable are also like terms |
| Variables | Blue marbles as BM<br>Green marbles as GM<br>Red marbles as RM<br>(This could mean multiplication of two variables) | Redefine variable as ONE letter that replaces a quantity. This could become very confusing in later math. |
| Combining Like Terms | 3a + 4a + 7b = 14f<br>3 apples + 4 apples + 7 bananas = 14 fruits | More practice with like terms and variables. |
| Subtraction of Like Terms | All like terms are added. | Show students to be mindful of the operation in the problem.  Do a variety of problems with both addition and subtraction |

**Procedure for Continuation of Adding and Subtracting Like Terms Lesson:**

**Whole Class Discussion**

- Begin the lesson by having a discussion about how the students found the total amount of hexagons.
- Ask for volunteers to come to the overhead or board to show the class.
- Then, ask students to do the same with pentagons
    - Example: $p + p + p + p + p + p + p + p + p + p + p + p = 12p$

**Questions to guide Student Thinking**

- What do we need to do in order to find the total amount of hexagons?
- How can we represent the amount that we are adding and the total amount as an equation?
- What does the total represent?
- Why is it important to indicate the total amount with the same variable?
- Why are we using $h$ for hexagon and p for pentagon?

NOTE:

- *Give the total amount of hexagons (20) and the total amount of pentagons (12) to your students so they don't have to count again*
- *When a student is at the board, make sure the rest of the class is thinking whether they agree, disagree, or are unsure*

**Small Group Discussion, then Share with the Whole Group**

- In groups of four, students will explore like and unlike terms. Here is a list of questions for students to answer:
    - Why did you write $12p$? What does that mean?
    - Could you write the answer $12h$ instead? Why is the variable important?
    - Can you add $p + h$ and get $12p$? Why or why not?

**Share with whole class.**

- 
    - What do you notice about adding variables in a number sentense?
    - Can you add $p + h$ together? Would it make sense?

**Share with whole class.**

**Whole Class Discussion**

- Ask students to think back to the lesson where they were adding adults + children = total number of people.
- Clarify the $T$ represented the total number of people and that is different than the total number of adults + the total number of children. (There is no such thing as adultkids).
- Ask the whole class:
  - Could you write an equation that states $a + c = a + c$? Why or why not?
  - Why is it that when you add like variables your answer has to represent the same variable?
  - Why can't you add unlike variables?

**Small Group Discussion**

- Ask students to represent the total number of pentagons and hexagons in a soccer ball using an algebraic expression.

**Questions to guide Student Thinking**

- How can you write an expression to represent the total number of pentagons and hexagons on a soccer ball? ($12p + 20h$)
- Can these two terms be added together?

**Whole Class Discussion**

- Ask for volunteers to share their expressions.
- As a whole group, do the first few practice problems, then let students work on these together in their small groups.

**Process to Shift Student Thinking**

- Students are now looking at both expressions and equations by combining like terms.
- Make the connection that combining the like terms is "simplifying" the expression.
- Use fruit examples such as two apples plus four apples are six apples, but 10 apples plus 5 bananas is not 15 applebananas. Show this using $a$ for apples and $b$ for bananas.
- Coefficients go in front of the variable.
- Connect to arithmetic that $2 + 5 = 7$, so $2a + 5a = 7a$.

***<u>Big Ideas that Evolve from Whole Class Discussion</u>***

- Like terms have the same variable and can be added and subtracted.
- Unlike terms have different variables and can not be combined
- 7*a* means 7 • *a*
- A variable is one letter that represents a quantity. (Still looking at variable as a label).
- Combining like terms is simplifying an expression.
- Both sides of the equal sign have the same quantity.
- An expression does not have an equal sign.
- An equation has an equal sign.

**Name: ___________________________**

**Day 5: Practice Adding and Subtracting Like Terms**
**(This can be done in small groups or whole class with discussion.)**

**Directions:** Identify the like terms and the unlike terms. Then write an algebraic expression for each situation.

**1. Combine 3 homework assignments with 2 homework assignments plus 1 student.**

Variables:

Like terms:

Unlike terms:

Expression:

**2. Combine 10 fingers plus 6 toes and 4 toes.**

Variables:

Like terms:

Unlike terms:

Expression:

**3. We have a bowl of three apples and a bowl of four apples and 7 bananas. Combine these in one bowl.**

Variables:

Like terms:

Unlike terms:

Expression:

**4. I have three red marbles, 2 blue marbles, and 7 green marbles in a jar.**

Variables:

Like terms:

Unlike terms:

Expression:

**5. Ms. Bedford has 15 dogs and 2 cats, but gives 3 dogs and 1 cat away to Ms. Moss.**

Variables:

Like terms:

Unlike terms:

Expression:

**6. For school, Mr. Moss has 3 math books, 2 English books, and 1 Spanish book. He lets his friend borrow 2 math books.**

Variables:

Like terms:

Unlike terms:

Expression:

**7. You are making a hot air balloon with tissue paper. You have 8 pieces of rectangular tissue paper and 10 pieces of square tissue paper. Your friend takes 2 pieces of rectangular tissue paper and 3 pieces of rectangular tissue paper.**

Variables:

Like terms:

Unlike terms:

Expression:

## Day 6: Combining Like and Unlike Terms

**Lesson Objective:** This lesson will extend the students' previous understanding of combining like and unlike terms.

**CCSS:**

- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

**Materials Needed:**

- Day 6: Assessment for week 1
- Chart Paper
- Paper
- Sharpie Markers
- Black pens
- Math Notebooks
- Overhead projector

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Combining Like Terms | 5x + 6y + 3x – 2y = 12xy (Added and subtracted coefficients and then put xy) | Point out like and unlike terms |
| Simplify | Simplify means to write and equivalent expression. Ex: Can you simplify the expression? 10x – 2x = 5x + 5x – 2x | Go over the vocabulary and what it means to simplify again. |
| Coefficient | Variable without a number in front, even though they understood what a variable is | Ask what the coefficient of *x* is. They might come up with 1. |
| | Same number like 15x and 15y | Explain that coefficient is just the number in front of the variable. |
| Like Terms | Like terms have to have the same coefficient and same variable | Showed that 2x + 5x = 7x so 2x and 7x are like terms even though the coefficient is different. |

**Procedure for Combining Like and Unlike Terms:**

**Day 6: Assessment for Week 1**

- Ask students to do the assessment for week 1 independently. The purpose of this assessment is to evaluate student understanding of the topics covered in the first week of the unit.
- (Time: about 15 minutes)
- Collect assessment before whole class discussion.

Grading:

- *Make a spreadsheet with a checklist that shows the names of each student.*
- *Evaluate each student by topic.*
- *Example: Jane understands expressions vs. equations, so Jane gets a check mark next to her name in the "expressions vs. equations" column*
- *Continue this checklist process throughout the unit.*

**Whole Class Discussion**

- Begin the lesson by reviewing topics from week 1.
- Facilitate the whole class discussion, but allow students to ask questions and lead the discussion.

**Questions to guide Student Thinking**

- In your own words, do you remember the definition of variable?
- What is an example of two like terms? Why?
- What is an example of unlike terms? Why?
- Can unlike terms have different coefficients? Explain.

***Big Ideas that Evolve from Whole Class Discussion***

- What is a coefficient? A coefficient is the number in front of the variable.
- (Need to clarify sign, variable, coefficient.)

**Whole Class Discussion Continued**

- Write an expression on the board from the previous lesson on the number of hexagons or pentagons. (Example: $3h + 17h$)
- Prompt students again to say what the number in front of the variable means
- Ask students how this is different than writing $h3$.
- Then, explain that the number in front of a variable is called a coefficient.
- Write this on the board and ask students to write it in their math notebooks with some examples of algebraic expressions.
- Ask for volunteers to come to the board and show what they wrote in their notebooks using the overhead projector

**Questions to guide Student Thinking**

- What does the number in front of the variable mean?
- What is the sign of the number?
- Do you agree, disagree, or are unsure about this example?

**Process to Shift Student Thinking**

- This lesson is looking at like and unlike terms, but also at equivalent expressions.
- Example: $3x + 2x$ –The coefficients are 3 and 2
- Ask students, when you add these together, what equation would you get? $3x + 2x = 5x$
- Then move to unlike terms.
- Example: $3x + 2y$ –The coefficients are 3 and 2
- Ask students, if you can add these terms together.
- **Pull examples from the students
- Ask for volunteers to explain to the class why or why not.

***Big Ideas that Evolve from Whole Class Discussion***

- Coefficients can be different in like terms
- Coefficients vs. Variables
- The variable is what matters when looking for like terms, not the coefficient
- Simplifying and realizing that this is an equivalent expression

*Allow students to come up with their own examples and definitions. Then, bridge this to formal definitions!*

**Name: ________________________**

**Day 6: Assessment for Week 1**

1. In your own words, what is a variable? Can you give an example?

| In Words | Example |
|---|---|
| | |

2. Is this an expression or an equation? Why?

| $3x + 5 - 6y$ |
|---|
| |

3. Is this an expression or an equation? Why?

| $50 = 2x$ |
|---|
| |

4. Can you write the following using an expression?

| "Five soccer balls and seven basketballs" |
|---|
| |

5. Can you simplify the expressions?

| $10x - 2x$ | $5x + 6y + 3x - 2y$ |
|---|---|
| | |

## Day 7: Simplifying and Equality Lesson

**Lesson Objective:** This lesson will continue to build students' understanding of combining like and unlike terms. Students will learn how to simplify an expression and transform it into an equation.

**CCSS:**
- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

**Materials Needed:**
- Day 7: M&Ms and Algebra Worksheet
- Day 7: Practice Problems
- Day 7: Word Problems
- Small bag of M&Ms for each student
- Chart Paper
- Paper
- Sharpie Markers
- Black pens
- Smart Board
- Overhead Projector

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Combining Like Terms | Instead of writing 4b + 4b = 8b, they wrote b + b + b + ….. eight times | Come up with simpler ways to combine like terms. |
| | 35x + 28y – 6x + 2y They added 35x + 6x instead of subtracted | Circle like terms with different colors and group the sign in front. |
| | 41c +20m – 16m They did not rewrite 41c in their simplified answer | Show students that the simplified form has to be equal to the original |
| | 7y + 5y = 12 instead of 12y | Remind the students that there is a variable and both sides must be the same |
| | 35x + 28y – 6x + 2y = 29x 30y (Dropped the addition sign) | Students combined like terms using column addition and then got two separate answers instead of realizing this is one answer. This is the idea of equality. Circling like terms in a different color will help. |
| | 41c + 20m – 16m = 4m – 41c Reversed the order and then had the wrong signs | Circling like terms with a different color and include the sign. |
| | Or for orange instead of just O | A variable can be only one letter. |
| | 2b + 1b = 41 because his b looked like a 6 (Efrain) | Go over how important it is to write clearly and precisely in math. |
| | Expression for Total Red and Yellow 9 + 13 =21ry | Equality on both sides. |
| Solving Equations | 16 – 7 = n + 1 Some wrote the answer is 9 since 16 – 7 is 9 | Again, need more practice with equality and showing both sides are the same |

**Procedure for Simplifying and Equality Lesson:**

**Whole Class Discussion**

- Show the Map of Brazil on the Smart Board or Overhead.
- Tell students that there are 12 cities on the map. These cities all have to have stadiums for the 2014 World Cup in Brazil.
- Pose this question to the class on the board:
    - Half of the host cities will have their games in brand new stadiums that will be built specifically for the World Cup. The stadium for Brasília, the capital of Brazil, was demolished and will be rebuilt. How many of the host cities do not have to build new stadiums?
- (In other words, there are twelve stadiums and six stadiums will be brand new)

***Big Ideas that Evolve from Whole Class Discussion***

- Combining like terms and representing a situation with an algebraic equation.
- An equation represents the same amount on both sides.


Source: http://www.worldcupbrazil.net/world-cup-2014/stadiums/

**Small Group Discussion**

- Ask students to discuss and write an equation that represents what is going on in this situation.
- (They may write on Chart Paper or in Math Notebooks)

**Questions to guide Student Thinking**

- What variables and what expression can be used to represent the situation?
- How can you show this with an algebraic equation or expression?
- How many total cities are there?
- What is half of that number?
- So if half have to build new stadiums, how many do not have to build stadiums?
- Can you show this using an algebraic equation?

**Process to Shift Student Thinking**

- Some students may still be learning what an equation actually is.
- Try using the word "formula" and say "with variables" to help them sort through this problem.

***Big Idea of Modeling***

12 host cities. Half of the 12 host cities is 6 host cities. The stadium for Brasilia is included in the 6 host cities. If half of the stadiums have to build new stadiums, then the other half do not need new stadiums.

**Whole Class Discussion**

- Once students have had time to discuss the problem, ask each group what they came up with and write their expression on the board.
- Hold a whole class discussion comparing the different methods and solutions that each group came up with.
- For example, let $c$ be the number of cities that do not have to build new stadiums, then $c + 5 + 1 = 12$ or $c + 6 = 12$. Remember, there are five cities that need brand new stadiums and one stadium that must be demolished and rebuilt.

***Big Idea: Equivalent Equations***

**(Possible Answers)**

- $6 + c = 12$ (6 + *number of cities that do not need new stadiums = 12)*
- $6n + 6o$ (6 *new* + 6 *old*)
- $12 - c = 6$
- $12 - c = 5 + 1$
- $c + 5 + 1 = 12$

**Questions to guide Student Thinking**

- Are these equations equivalent?

**Small Group Discussion, then Share with the Whole Group**

- Then ask students to solve for the variable in their notebook.
- (Most will be able to mentally do this), so tell them to write *why* the answer is 6 or whatever answer they came up with. Ask students to show their work and compare their answers.

**Share with whole class.**

**Process to Shift Student Thinking**

- Representing a problem in context with a variable
  - The number of cities that do not have to build new stadiums with the variable $c$.
- Writing an expression
  - The number of cities that do not have to rebuild stadiums is $12 - c$
- Writing an equation
  - $12 - c = 6$
- Solving for the variable to show equality on both sides of the equal sign.
  - Leads to Addition Property of Equality → Next lesson

**Small Group Discussion**

- Each student should have a small bag of M&Ms.
- Ask the students to sort their M&Ms by color
- Pass out the M&Ms and Algebra Work Sheet and allow students to work on it. (Go over the directions. Make certain that students understand that problems 4 and 5 are done in groups.)
- The M&M activity should take about 30 minutes.
- Collect the Worksheets
- If there is more time, then ask students to work on the Day 7: Practice Problems and Word Problems.

**Process to Shift Student Thinking**

- One letter represents each color of M&M
  - Use $b$ for blue and not $bl$
- Think equality when simplifying an expression
  - $2a + 7a = 7a$ and not 7
- Must keep the sign with the term when adding and subtracting
  - Sign, coefficient, variable
- Begin asking students if they need to write 1 in front of a variable
  - $1y = y$

**Name: ________________________**

**Day 7: M&Ms and Algebra Worksheet**

Directions: Open your bag of M&Ms and sort the colors.

| | |
|---|---|
| 1. Write the number of each color below. | |
| Red: | Green: |
| Blue: | Orange: |
| Brown: | Yellow: |

| | |
|---|---|
| 2. What **variable** should you use for each color? | |
| Red: | Green: |
| Blue: | Orange: |
| Brown: | Yellow: |

3. Write an **expression** below for the number of red M&Ms and the number of yellow M&Ms in your bag.

4. Write an expression with variables that shows the **total** number of red M&Ms and yellow M&Ms for your **group**.

**Expression** for the Total Red: ______________________________

Can you write this as an **equation**? ______________________________

**Expression** for the Total Yellow: ______________________________

Can you write this as an **equation**? ______________________________

**Expression** for the Total Red and Yellow: ______________________________

5. Write an expression with variables that shows the **total** number of each color of M&Ms for your **group**.

**Expression** for the Total Red: ______________________________________________

Can you write this as an **equation**? ______________________________________________

**Expression** for the Total Blue: ______________________________________________

Can you write this as an **equation**? ______________________________________________

**Expression** for the Total Brown: ______________________________________________

Can you write this as an **equation**? ______________________________________________

**Expression** for the Total Orange: ______________________________________________

Can you write this as an **equation**? ______________________________________________

**Expression** for the Total Green: ______________________________________________

Can you write this as an **equation**? ______________________________________________

**Expression** for the Total Yellow: ______________________________________________

Can you write this as an **equation**? ______________________________________________

**Expression** for the Total of All Colors: ________________________________________

**Name: ________________________**

**Day 7: Practice Problems**

1. What is a variable?

2. What does the equal sign mean?

3. Add or Subtract

| Add or Subtract | Answer |
|---|---|
| $2x + 19x$ | |
| $a + 27a$ | |
| $46b – 21b$ | |
| $35x + 28y – 6x + 2y$ | |
| $41c + 20m – 16m$ | |

4. Solve for the variable.

| Solve for the variable. | Answer |
|---|---|
| $2 + x = 25$ | |
| $16 – 7 = n + 1$ | |

5. If a hat and coat together cost $185 and the hat costs *n* dollars, write an expression for the cost of the coat.

**Name:**_________________________

**Day 7: Word Problems**

1. Model with an algebra equation. Then solve.

| I have 15 soccer balls and give away some to my friend. I am left with 11 soccer balls. How many did I give away? | |
|---|---|
| **Equation** | **Answer** |
| | |

2. Model with an algebra equation. Then solve.

| Jake has some jerseys and Leah has 4 jerseys. Together they have 16 jerseys. How many jerseys does Jake have? | |
|---|---|
| **Equation** | **Answer** |
| | |

3. Model with an algebra equation. Then solve.

| Three less than a number is 7. What is the number? | |
|---|---|
| **Equation** | **Answer** |

4. Model with an algebra equation. Then solve.

| A customer pays \$60 for soccer cleats after a discount of \$15. What is the original price of the cleats? | |
|---|---|
| **Equation** | **Answer** |

5. Model with an algebra equation. Then solve.

| Josh earned \$72 less than his sister who earned \$93 more than her mom. If they earned a total of \$504, how much did Josh earn? | |
|---|---|
| **Equation** | **Answer** |

## Day 8: More Adding and Subtracting Like Terms Lesson

**Lesson Objective:** This lesson will extend the students' previous understanding of the equal sign as an equal relationship between two sides of an equation. Students will learn how to isolate a variable on one side of an equation using the addition property of equality.

**CCSS:**

- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and $x$ are all nonnegative rational numbers.

**Materials Needed:**

- Day 8: Circling Like Terms Practice
- Chart Paper
- Overhead projector
- Paper
- Color Pens

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Variable | Variable is always an object and not a quantity | Show that a variable can be a changing quantity by using dollars |
| Combining Like Terms | Using the sign after the term and not before the term | Circling the sign, coefficient, variable |
| | Adding and subtracting with negative numbers | Need to teach how to do this even though it is not supposed to be taught until 7th grade according to the CCSS |
| Negative Numbers | Subtraction is different than negative numbers | Have to teach negative numbers |
| Terms | Listing variables not terms. ex: List the terms in 2x + 5y, x and y | Need to go over that a term is the sign, variable, and coefficient |
| Multiplication of 2x | A coefficient in front of a variable means to multiply. | Showed students this using the cost of cupcakes. |

**Procedure for More Adding and Subtracting Like Terms Lesson:**

**Warm-Up**
- Write this example on the board: $2a + 5x + 7y - 2x + 4y - a$
- Ask students to identify the like terms.
- As students say the like terms, circle like terms in the same color (including the sign)

*Make explicit that the sign in front of a term goes with the term!*
$5x - 2y$ has two terms $5x$ and $-2y$

**Whole Class Discussion**
- Discuss the following as a whole group:
  - Coefficients (Note: the coefficient in front of $a$ is -1)
  - See if students know that $2a$ means $2 \cdot a$. See if they can make sense of this.
  - Count the number of terms in the expression.
  - Identify like terms.

**Questions to guide Student Thinking**
- What are the coefficients?
- What does $2a$ mean?
- How many terms are there in the expression?
- What are the like terms?

**Small Group Discussion**
- Hand out Day 8: Circling Like Terms Practice
- Ask students to work on this in their groups.

**Whole Class Discussion**
- Ask for volunteers to share and explain their work with the class. (Show on overhead or write on the board/chart paper)

**Questions to guide Student Thinking**
- Do you agree with this answer?
- Do you have something different?
- If so, can you show what you have?

***Big Ideas that Evolve from Whole Class Discussion***
- Sign, Variable, Coefficient is one term.
- Combine like terms → simplifying the expression.

**Process to Shift Student Thinking**
- Have students repeat: "Sign, Variable, Coefficient"

**Name: ________________________**

**Day 8: Circling Like Terms Practice**

Directions: Circle the like terms with a different color pen. List the terms and then simplify by combining like terms.

1.

| | List the terms | Simplified Expression |
|---|---|---|
| $15x + 6y - 4x - 2y$ | | |
| $25c + 12u - 21c + 5u$ | | |
| $30t + 27s + 5z - 2t + 3s - 3z$ | | |
| $32x + 25y - x + y$ | | |
| $-9j - 2r + 15j + 7r$ | | |

2. Model with an algebra equation. Then solve.

| I have 18 soccer balls and give away some to my friend. I am left with 12 soccer balls. How many did I give away? | |
|---|---|
| **Equation** | **Answer** |
| | |

3. Model with an algebra equation. Then solve.

| Jake has some jerseys and Leah has 21 jerseys. Together they have 37 jerseys. How many jerseys does Jake have? | |
|---|---|
| **Equation** | **Answer** |
| | |

## Day 9: Variable as a Changing Quantity Lesson

**Lesson Objective:** This lesson will extend the students' previous understanding of the variable from representing an object to representing a changing quantity. Students will represent the cost of cupcakes with a variable expression and then solve.

**CCSS:**

- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and $x$ are all nonnegative rational numbers.

**Materials Needed:**

- Chart Paper
- Overhead Projector
- Paper
- Color Pens
- Math Notebooks

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| x versus 1x | Students are still writing a 1 in front of a single variable. This is okay, but shows that they still don't understand it means 1 times x | Reiterated this and wrote it on the rule wall. |
| Adding and Subtracting like terms versus plug and chug | Students did not realize that they could combine once their terms changed to the same unit (money) | Wrote on the rule wall and explained it again. |

**Procedure for Variable as a Changing Quantity Lesson:**

**Whole Class Discussion**

- Tell the students that we are going to have a party and need to by cupcakes for the class.

"We have 24 students in the class plus the teacher, which means that we need 25 cupcakes (adjust for your class). Cupcakes come in packages of six or you can buy a single cupcake."

**Small Group Discussion**

- Ask students to come up with a formula or algebraic expression that can be used to model the *cost* of the cupcakes.
  - Example:
    $4c + s$ where $4c$ is the cost of 4 packages of 6 cupcakes and $s$ is the cost of a single cupcake.

**Questions to guide Student Thinking**

- What does the variable represent?
- So, how do you solve the problem?
- What information do you need?

**Whole Class Discussion**

- Ask for volunteers to show their formula on the board

*Clarify in each formula what the variables mean. It is important to show that this is the* ***price****.*

**Whole Class Discussion**

- Decide on which formula is the best and why. (Needs to be based on the cost of 4 packages of 6 cupcakes and the cost of a single cupcake.)
- Write each of these (use stores in your neighborhood) on separate pieces of chart paper:
  - At Walmart, one package of six cupcakes is \$6 and a single cupcake is \$1.
  - At Safeway, one package of six cupcakes is \$8 and a single cupcake is \$2.
  - At Raley's, one package of six cupcakes is \$7 and a single cupcake is \$1.

**Small Group Discussion, then Share with the Whole Group**

- Ask the students to discuss and write in their math journals how they could use the formula to solve these problems.

**Share with whole class.**

  - What is changing each time you go to a different store? (the price or the cost of the cupcakes)
  - What do your variables mean in this formula? (the price of the cupcakes)
  - Does the price change?

- Ask students to discuss in their groups what a variable is in this scenario.
  (It is a changing quantity – price).

**Share with whole class.**

- Ask students to discuss what a variable means in the M&M activity.
  (A label or object – red M&M).

**Share with whole class.**

**Process to Shift Student Thinking**

- Before this lesson, students view a variable as a label or object.
- By letting a variable represent a price and showing examples of prices at different stores, students shift their thinking to variable as a changing quantity.
- The algebraic expression is a formula that can be used at any store that has packs of 4 cupcakes and single cupcakes.
- Variables can be different letters in the formula as long as they are defined.
- Example: $4c + s$ is the same as $4x + y$ because the variables stand for the same changing quantity.

*Let students' examples and questions lead the discussion. Students will begin to ask each other what makes sense and what doesn't.*
**Remember to use the mic!**

***Big Ideas that Evolve from Whole Class Discussion***

- The formula is the same no matter what the price is for each package of cupcakes.
- A variable can be a changing quantity such as the price of cupcakes.
- A variable can be a label or object such as a red M&M.

**Day 10: Variable as a Changing Quantity and The Addition Property of Equality Lesson**

**Lesson Objective:** This lesson will extend the students' previous understanding of the variable from representing an object to representing a changing quantity. Students will represent the cost of soccer balls with a variable expression and then solve.

**CCSS:**

- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and $x$ are all nonnegative rational numbers.

**Materials Needed:**
- Day 10: Substitution of Values for Variables Warm-Up
- Day 10: Cost of Soccer Ball
- **Day 10: Cost of Soccer ** (Differentiated version that contains Formulas)
- Chart Paper
- Paper
- Color Pens
- Overhead projector

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Plug and Chug | Writing the problem out separately.<br>Ex: 2x + 5y where x =2 and y = 3<br>2•2 = 4<br>5•3 = 15<br>4 + 15 = 19<br>Instead of writing it in one line. Ex: 2(2) + 5(3) = 4 + 15 =19 | Model how to write it again. If students do this in many steps then they may forget a step like adding at the end. In #4 on Warm-Up Day 10 many forgot to subtract the r because they were not writing out the steps clearly. |
| | Writing a x for times instead of parenthesis or a • | More practice with multiplying and changing the symbol to • or () |
| Modeling with Expressions | The word "and" means to add Ex. three diadora and 2 europa 3d 2e (nothing in between) | Make another poster with word problem keywords such as and means to add and less than means to subtract, etc. |
| | The variable is still representing one object and not the unknown. In #4 on the soccer ball ws, 5b + 4b = 9b | Transitioning from the conception of the variable as one object to the variable as an unknown value. (See Junior's ws) |
| | The Cost of 5 diadora balls with an expression is 5 = d | This is modeling the number of balls and not the **cost** of the ball. Ask her how she would figure out the cost of 5 balls if each ball is $27. |
| | Modeling #5 on soccer ws as 24 – 4c = 20<br>Student knew that the answer is 4 cupcakes and tried to show why. | Do another example where once you have the value, you can substitute it in for the variable. |
| Solving one step equation | b + 4 = 9 and the answer is 5. When asked how you found the value they put "I added". So, they added 5 + 4, but really they are subtracting 4 from 9 to find b | More practice with solving one step equations and plugging in the value for the variable. |

**Procedure for Variable as a Changing Quantity and The Addition Property of Equality Lesson:**

**Warm-Up Discussion**
- Review the cupcake problem from yesterday.
- Recall that we came up with the formula $4p + c$ where $p$ was the price of a package of 6 cupcakes and $c$ was the price of a single cupcake.

**Questions to guide Student Thinking**
- How did we find the total cost of 25 cupcakes?
- Can we use this same formula if we go to a different store?
- What happens if we need 30 cupcakes? (There are two ways to do this with our same information: $5p$ OR $4p + 6c$)
- Is this the same formula or a different formula than before?
- Can we use this same formula if we need 30 cupcakes at another store?

***Big Ideas that Evolve from Whole Class Discussion***
- A variable can be one object or a label; or it can be a changing quantity.

**Small Group Discussion**
- Hand out Day 10: Substitution of Values for Variables Warm-Up
- Ask students to work on this in their groups.
- Remind students that $2u$ means 2 times $u$ or $2 \cdot u$

***Big Ideas that Evolve from Discussion***
- When you substitute a given value for a variable, you no longer need the variable and can add or subtract
- $2u$ means 2 times $u$

**Whole Class Discussion**
- Ask students to explain how they did some of the problems on the warm-up by showing their work on the overhead projector.

**Questions to guide Student Thinking**
- Is the variable a known quantity?
- How do you know to multiply?
- Why can you combine terms once you substitute in the value?

*So far, there is not context for this substitution. The cost of a soccer ball activity gives a context to these types of problems.*

**Small Group Discussion**
- Hand out the Day 10: Cost of Soccer Ball Worksheet (The ** worksheets are for students that are still having trouble modeling a situation with algebra. The formulas are given.)
- Ask students to work on this in their groups (groups may be different since worksheets are differentiated).
- Note: the last two problems on the worksheet introduce the idea of the variable as an unknown value.

**Process to Shift Student Thinking**
- The idea of plug and chug (or substitution of a value for variable).
- A variable can be a label or an object.
- A variable can be a changing quantity.
- A variable can be an unknown value.

***Big Ideas that Evolve from Discussion***
- Students begin to see substitution of given values for variables in which the quantity for the variable can change (prices).

**Name:** ______________________________

**Day 10: Substitution of Values for Variables Warm-Up**

Directions: Substitute the given quantity for the variables and then solve. Please show your work and circle your answer.

1. $x + y$ where $x = 35$ and $y = 10$

2. $2f$ where $f = 7$

3. $5z - 3t$ where $z = 4$ and $t = 5$

4. $7u + 5x - r$ where $u = 3$ and $x = 3$ and $r = 2$

5. $15d - 5d + 3w + 7w$ where $d = 2$ and $w = 2$

**Name: ________________________**

**Day 10: Cost of Soccer Ball**

You are shopping for soccer balls for you and your family so that you can practice before the World Cup! The only problem is that no one in your family wants to share a ball, so everyone wants their own! You go to the *World Soccer Shop* and see that they have a selection of five different soccer balls.

1. There are five soccer balls listed. What are your first, second, and third choices for soccer balls and why? How much do these cost?

2. Let's say that you only have $75.00 to spend on soccer balls. Now, which three soccer balls should you pick and why?

3. Let's represent the cost of three Diadora soccer balls and two Europa Soccer balls with an algebraic expression.

- What variable will you use for the Diadora ball and what variable will you use for the Europa ball?
- How much is one Diadora ball? How much are two Diadora balls?
- Model the cost of three Diadora balls with an algebraic expression.
- How much is one Europa ball? How much are two Europa balls?
- Model the cost of two Europa balls with an algebraic expression.
- How can you represent three Diadora balls and two Europa balls?

4. Now, let's say that you bought a certain number of soccer balls and someone gives you 4 more soccer balls. Now you have 9 soccer balls. How many did you buy in the first place?

- How can you model this situation with an algebraic equation? (hint: represent the amount of soccer balls that you bought in the first place with *b*)

- Solve your equation. In your own words, how did you solve for the variable?

5. Remember, Ms. Bedford is bringing cupcakes on Monday. She is going to bring exactly 25 cupcakes and everyone is going to eat only one cupcake. How many students are absent if she still has 4 cupcakes left after Monday?

- How can you model this situation with an algebraic equation? (hint: represent the number of student that are absent with *c*)

- Solve your equation. In your own words, how did you solve for the variable?

| | |
|---|---|
|  Diadora | $27.00 |
|  Adidas blue | $30.00 |
|  Adidas pink | $15.00 |
|  Nike | $20.00 |
|  Europa | $120.00 |

Pricing and Images based on World Soccer Shop Website (2013).

**Name:__________________________**

****Day 10: Cost of Soccer ****

You are shopping for soccer balls for you and your family so that you can practice before the World Cup! The only problem is that no one in your family wants to share a ball, so everyone wants their own! You go to the *World Soccer Shop* and see that they have a selection of five different soccer balls.

1. There are five soccer balls listed. What are your first, second, and third choices for soccer balls and why? How much do these cost?

2. Let's say that you only have $75.00 to spend on soccer balls. Now, which three soccer balls should you pick and why?

3. Let's represent the cost of three Diadora soccer balls and two Europa Soccer balls with an algebraic expression.

The algebraic expression is $3d + 2e$.

- What variable represents the Diadora ball and what variable represents the Europa ball?
- How much is one Diadora ball? How much are two Diadora balls?
- Model the cost of five Diadora balls with an algebraic expression.
- How much is one Europa ball? How much are two Europa balls?
- Model the cost of six Europa balls with an algebraic expression.
- How can you represent five Diadora balls and six Europa balls?

4. Now, let's say that you bought a certain number of soccer balls and someone gives you 4 more soccer balls. Now you have 9 soccer balls. How many did you buy in the first place?

**The expression is $b + 4 = 9$**

- What does the variable *b* represent?

- What is the value of *b*?

- In your own words, explain how you found the value of *b*.

5. Remember, Ms. Bedford is bringing cupcakes on Monday. She is going to bring exactly 25 cupcakes and everyone is going to eat only one cupcake. How many students are absent if she still has 4 cupcakes left after Monday?

**The expression is $25 - c = 4$**

- What does the variable *c* represent?

- What is the value of *c*?

- In your own words, explain how you found the value of *c*.

| | |
|---|---|
|  Diadora | $27.00 |
|  Adidas blue | $30.00 |
|  Adidas pink | $15.00 |
|  Nike | $20.00 |
|  Europa | $120.00 |

Pricing and Images based on World Soccer Shop Website (2013).

## Day 11: Variable as the Unknown and The Addition Property of Equality Lesson

**Lesson Objective:** This lesson will extend the students' previous understanding of the variable from representing a changing quantity to representing an unknown value. Students will represent the cost of soccer balls with a variable expression and then solve.

**CCSS:**

- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and $x$ are all nonnegative rational numbers.

**Materials Needed:**

- Day 10: Cost of Soccer Ball
- Day 11: Assessment for Week 2
- Day 11: Warm Up
- Day 11: Balancing Equations
- Day 11: Solving Equations
- Chart Paper
- Paper
- Color Pens

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Combining Like Terms | 9 – 2 = 7y<br>(Not writing the variable in the 9y – 2y) | Practice writing out simplifying expressions |
| | The problem is 9y – 2y<br>The student rewrote as 9y + 2y = 11y | Pay attention to the original problem. |
| | 9y – 2y = 6y | arithmetic error needs to redo |
| | Combined like terms, but did not write the expression | Practice writing the expression completely |
| | 6z + 10t – 2z – 5t<br>Student wrote 8z + 5t<br>Added and did not subtract | Need to circle the like terms, sign, variable, coefficient |
| Substituting Quantities for variables | 3a + 7b where a = 4 and b = 2<br>Student wrote 12a + 14b (Did not replace variable/ or rewrote variable) | Practice substitution |
| | 3a + 7b where a = 4 and b = 2<br>Student wrote 34 + 72 = 106 | Go over that 3a means 3 times a, so if a is 4, then 3a is 3 times 4 |
| | 3a + 7b where a = 4 and b = 2<br>12 + 14 as the answer | Need to simplify more. Ask student if they should just leave it like that or keep going. |
| | 3a + 7b where a = 4 and b = 2<br>Student wrote 7 + 9 = 16<br>Student added instead of multiplied | 3a means 3 times a and not 3 plus a |
| Equality | 7 + 13 = 32 -12 is false because 7 + 13 equals 20 and not 32 | Viewing each side of the equal sign as one quantity and not seeing the equal sign as "compute" |
| | 10 – 5 + 2 = ___ + 3<br>___ = 7 because 10 – 5 = 5 + 2 = 7 | Viewing each side of the equal sign as one quantity and not seeing the equal sign as "compute" |
| Definition of Equation | An expression without an equal sign | Confusing equation with expression |
| Balancing Equations | 13 = h – 4<br>Student subtracted 4 from 13 and got the answer was 9 | Show student again that they have to do the opposite operation and check answer |
| | Transforming 13 = h – 4 to 13 = 4 - h | Show them with number shat h – 4 is different than 4 – 8. |

**Procedure for Variable as the Unknown and The Addition Property of Equality Lesson:**

**Day 11: Assessment for Week 2**

- Ask students to do the assessment for week 1 independently. The purpose of this assessment is to evaluate student understanding of the topics covered in the first and second weeks of the unit.
- (Time: about 15-25 minutes)
- Collect assessment before whole class discussion.
- Once students finish the assessment, ask them to finish Day 10: Cost of Soccer Ball and do the Day 11: Warm-Up.

**Whole Class Discussion**

- Conclude the soccer ball lesson by asking students for how they solved the last two questions on the soccer worksheet. This will lead into the idea of the variable as an unknown.

**Questions to guide Student Thinking**

- What is your algebraic equation?
- What does the variable stand for?
- What is your answer?
- How would you solve for this variable?

***Big Ideas that Evolve from Whole Class Discussion***

- Three ideas for variable:
  - One object (red M&M)
  - Changing quantity (price)
  - Unknown Value ($2 + x = 5$)

**Whole Class Discussion**

- Use this NCTM Illuminations Link: http://illuminations.nctm.org/ActivityDetail.aspx?ID=33
  - Put something on one side and ask what needs to go on the other side to balance the scale.
- Explain to students that this is the Addition Property of Equality and see if they can come up with the definition in their notebooks.
- Use the questions to guide a discussion.
- Make explicit that Red is Subtraction and Blue is Addition.

**Questions to guide Student Thinking**

- What does a scale like this do?
- What does balanced mean?
- If I put 20 on one side and 15 soccer balls on the other, is the scale balanced?
- What is another word for both sides being the same? (Equal or Equality)
- If I put 2 + 8 on one side and 10 on the other side, is this balanced?
- What if I had $13 = x + 1$. What would $x$ have to be for this to be balanced? (12)
- What did you do to get the answer? (Subtracted 1).

Example:
$$\begin{array}{rcl} 13 & = & x + 1 \\ \underline{-1} & & \underline{-1} \\ 12 & = & x \end{array}$$

*After seeing the Illuminations Link example, students may be able to come up with the definition for Addition Property of Equality. (If you add or subtract the same number from both sides of an equation, the new equation will have the same solution.)*

***Big Ideas that Evolve from Whole Class Discussion***

- The equal sign means that both sides are the same.

**Process to Shift Student Thinking**

- There are three different definitions of variable.
- Balancing a scale is solving an equation.
- Use opposite operations to solve an equation (add and subtract).
- Both sides of an equation need to be the same or equal.
- Whatever you do to one side, you have to do to the other to keep it balanced.
- Focus on writing out the problems neatly and vertically (as in example above).

**Small Group Discussion**

- Ask students to do another example: $y - 5 = 15$.
- Ask them to use different colors to represent addition and subtraction.
- Ask students to work on Day 11 worksheets.

Algebra Sixth Grade Unit by Ms. Moss

**Name: ___________________________**

**Day 11: Assessment for Week 2**

1. In your own words, what is an algebraic equation? Can you give an example?

| In Words | Example |
| --- | --- |
| | |

2. Simplify the expression by combining like terms. Show your work.

| $\mathbf{9y - 2y}$ |
| --- |
| |

3. Simplify the expression by combining like terms. Show your work

| $\mathbf{6z + 10t - 2z - 5t}$ |
| --- |
| |

4. There were 17 cans of soda in my family's refrigerator and now there are only 6. How many cans of soda did my family drink? Write an algebraic equation to model the situation and then solve.

| Algebraic Equation | Answer |
| --- | --- |
| | |

5. Substitute the given quantities in the expression and then solve.

**$3a + 7b$ where $a = 4$ and $b = 2$**

6. Is **20** a solution of $m - 4 = 16$? Why or why not?

7. True or False? **$7 + 13 = 32 - 12$** Explain your answer.

8. Fill in the blank. Explain your answer.

**$10 - 5 + 2 =$ ____ $+ 3$**

**Name: _________________________**

**Day 11: Warm Up**

1. Write an expression for the following sentence: Find the sum of 51 and 16.

2. Express the number 67 as a sum of two numbers. (Find three pairs of numbers.)

3. Express the number 67 as a sum of two **unknown** numbers. Hint: use variables.

Use the Addition Property of Equality to solve for the unknown value.

| | |
|---|---|
| 4. $2 + x = 25$ | 5. $16 - 7 = n + 1$ |
| 6. $97 + 3 = 8 + y$ | 7. $x - 21 = 30$ |

**Name: ________________________**

**Day 11: Balancing Equations**

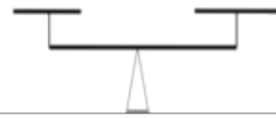
Fill in the blanks: A scale has a 8-ounce weight on each side of it. Is it balanced? _________

A 4-ounce weight is added to the pan on the right side. Is it still balanced? ________________

Name two things you can do to balance the scale. ____________________________________

Directions: Draw a model that shows each equation on a scale. Then state how you would solve the equation without using the scale. (The first problem is done for you.)

| Problem | Model | How to Solve |
|---|---|---|
| **1.** $t + 2 = 6$ |   | $t + 2 = 6$<br>$-2 \quad -2$<br>$t + 0 = 3$<br>$t = 3$ |
| **2.** $4 + x = 10$ |  | |
| **3.** $13 = a + 5$ |  | |

| | | |
|---|---|---|
| **4.** $17 = r + 1$ |  | |
| **5.** $w - 2 = 16$ |  | |
| **6.** $15 + 5 = x + 2$<br>(hint: add like terms first) |  | |
| **7.** $y = 28 - 3$ |  | |
| **8.** $13 = h - 4$ |  | |

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**Name:** ________________________

**Day 11: Solving Equations**

Use the Addition Property of Equality to solve for the unknown value. Use red and blue pens to show addition and subtraction. Check your answer (Replace the variable with your answer and see if both sides are equal).

| | |
|---|---|
| 1. $x + 2 = 25$ | Check: |
| 2. $c - 21 = 30$ | Check: |
| 3. $100 = 8 + y$ | Check: |
| 4. $16 - 7 = n + 1$ | Check: |
| 5. $t - 4 = 20 + 1$ | Check: |

**Day 11: Solving Equations Continued**

Instructions: Model using algebra and then solve the following problems.

1. Cleats and a ball together cost $75. If the cleats cost $56, what is the cost of the ball?

2. A tour book of Brazil weighs 3 pounds. The total weight of a math book and the tour book is 9 pounds. How much does the math book weigh?

3. Cristiano Ronaldo, a Portuguese footballer, earns $12 million. Lionel Messi, an Argentine footballer, and Cristiano Ronaldo earn a total salary of $28 million. How much does Lionel Messi earn?

## Day 12: Variable as the Unknown and The Addition Property of Equality Lesson

**Lesson Objective:** This lesson will extend the students' previous understanding of the variable from representing a changing quantity to representing an unknown value. Students will solve equations using the addition property of equality and will practice with Algebra Touch.

**CCSS:**

- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and $x$ are all nonnegative rational numbers.

**Materials Needed:**

- Day 12: Warm-Up
- Algebra Touch and iPads
- Day 12: Algebra Touch Answer Sheet
- **Day 12: Algebra Touch Answer Sheet** (With examples)
- **Day 12: Solving Equations Worksheet** (With examples and formulas)
- Day 12: Solving Equations Worksheet
- Chart Paper
- Paper
- Color Pens
- 3 Different colors of index cards for each student

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Equation, but do not solve | But we know the answer, why do we need the equation. | We are looking at the variable as an unknown like we did yesterday, so we want to know the equation only. |
| | $b + 5 == 11sb$ | We don't need sb because we know that b is soccer balls. |
| Fill in the blank problems | Equal sign means compute<br>16 – 5 + 3 = ___ + 2<br>Students say this is 14 | Talked about it in whole class discussion and came to conclusion that both sides have to be the same. |
| Combining Like Terms | Adding everything and not doing any subtraction | Sign, variable, coefficient |
| | Dropping the variable<br>6z + 14z = 20 | More practice with this |
| | For the answer writing 15 + 22 and 22 + 15 as if they are different solutions, also dropping variables | Clarify that the order doesn't matter for addition. |
| | Still not writing an expression after combining like terms/ skipping the last step | Review equivalent expressions and show equality. |
| Solving Equations Using the Addition Property of Equality | They don't know how to use negative numbers | Need to teach this |
| | Mistaking a z for a 2 | Cross the z |
| | x + 16 = 1 and student subtracted 1 from both sides and got x + 15 = 0 and then decided that x = 15 | Idea of balancing the equation and checking the answer<br><br>Teach negative numbers |
| | Solving the problem mentally and not writing out the steps | Making more mistakes, need to check answer |
| | Always adding the number to another side even if they are supposed to subtract | Check the answer |
| | 17 + y = 3<br>Subtracted 17 from 3 and then subtracted 3 from 17 | Both sides have to be balanced or equal. |
| | Writing too many zeros when solving equations | Show that they don't need to write the zeros. |

**Procedure for Variable as the Unknown and The Addition Property of Equality Lesson:**

**Warm-Up**

- Present this Context Problem for Solving Equations on the board:

"Mr. Moss has a certain number of soccer balls. Ms. Coy gives Mr. Moss five more soccer balls. Now, Mr. Moss has 11 soccer balls. How many soccer balls did Mr. Moss have in the first place?"

Write an equation that represents the number of soccer balls Mr. Moss had in the first place. **Do not solve.** (Hint: Let the variable $b$ be the number of soccer balls Mr. Moss had in the first place.)

*Give a context problem that provides students with a purpose for writing and solving an equation.*

**Whole Class Discussion**

- Ask for volunteers to SAY (not write because of time) an equation that represents the number of soccer balls Mr. Moss had in the first place.
  - **Examples: $b + 5 = 11$, $11 = 5 + b$, $11 - 5 = b$, $11 - b = 5$,**
- Remind students that when we have to add or subtract to solve for the variable, this is the Addition Property of Equality.

**Questions to guide Student Thinking**

- Are these equations equivalent?
- Why or why not?
- How do you know?
- Can you explain this to your classmates?

***Big Ideas that Evolve from Whole Class Discussion***

- Equivalent Equations have the same solution.

**Whole Class Discussion**

- To clear up the misconceptions from yesterday, redo problems similar to Assessment for Week 2 Using Day 12: Warm-Up.
- Write a problem on the board or chart paper.
- Ask students to solve it in their notebooks individually and AT THE SAME TIME
- Pick volunteers or choose straws for students to come to the board and show what they did.

**Questions to guide Student Thinking**

- How do you know this is correct?
- Can you check this?

NEW IDEA: While students come to the board, give students the different colored index cards. Each student must decide if they agree (green), disagree (red), or unsure (yellow) about what the student on the board is doing. They must choose the index card that represents agree, disagree, or unsure and think about why. As the student on the board is explaining his or her answer, ask the rest of the students to look at what they did and change their thinking if they need to.

**Whole Class Discussion**

- Based on index cards, pick a student that said "disagree" or "unsure" and ask them to come to the board and show what they did to solve the same problem. Try to weed out the misconceptions.
- Continue this process for the Day 12: Warm Up problems.
- On the overhead projector, model on the iPad how to open Algebra Touch
    - go to practice
    - go to "Terms (Like and Unlike)" on the pull down menu.
    - Demonstrate how the app works.
- Then on the drop down menu
    - go to "Isolate" and demonstrate how the app works.

**Glitch** If a student gets a multiplication or division problem, then tell them to click on "Random" until they can use the Addition Property of Equality.

- Hand out the iPads and Day 12: Algebra Touch Worksheet

**If you do not have access to iPads, then you can do modify this lesson without iPads.**

**Small Group Discussion**

- Ask students to go to practice and then "Terms (Like and Unlike)" on the drop down menu. The Worksheet explains that students are to write down the problem that it gives, solve it using the steps, and then check that it is correct using Algebra Touch.

- Then ask students to go to "Isolate" on the drop down menu. The Worksheet explains that students are to write down the problem that it gives, solve it using the steps, and then check that it is correct using Algebra Touch.

- Ask them to show their work and check their answers on the Algebra Touch answer sheet.

When using Algebra Touch for solving One-Step Equations: *Tell students to make ONE move to get the variable by itself. Once they see it is only one move, they focus on the opposite operation.*

**Whole Class Discussion**

- Once students have had an adequate amount of time to do at least 10 problems and fill in the WS, (5 combining like terms and 5 isolate problems), pass out the Day 12: Solving Equations WS and ask them to work on this individually. (Two versions of worksheets)
- Ask for volunteers to do some problems on the board and check to see if different groups are getting the same answers.
- The students not at the board can again use their index cards for agree, disagree, and unsure.

**Process to Shift Student Thinking**

- When moving the numbers and variables around on the iPads, the equation still stays equal.
- "Make one move to solve for the variable."
- Checking answer shows that both sides are still equal.
- Checking answer is the same as plug and chug.
- Red, yellow, and green index cards helps students stay engaged when a classmate is at the board.

**Name:** ___________________________

**Day 12: Warm-Up**

| |
|---|
| 1. In your own words, what is an example of a variable as the unknown? Can you give an example? |

| |
|---|
| 2. Simplify the expression by combining like terms. Show your work. <br><br> $\mathbf{15}\boldsymbol{y} - \mathbf{4}\boldsymbol{y}$ |

| |
|---|
| 3. Simplify the expression by combining like terms. Show your work <br><br> $\mathbf{12}\boldsymbol{z} + \mathbf{7}\boldsymbol{t} - \mathbf{3}\boldsymbol{z} - \mathbf{5}\boldsymbol{t}$ |

| |
|---|
| 4. Substitute the given quantities in the expression and then solve. <br><br> $6a + 3b$ **where** $a = 2$ **and** $b = 5$ |

| |
|---|
| 5. Is **30** a solution of $m - 6 = 24$? Why or why not? |

| |
|---|
| 6. True or False? $5 + 13 = 22 - 4$ Explain your answer. |

| |
|---|
| 8. Fill in the blank. Explain your answer. <br><br> **16 – 5 + 3 = ____ + 2** |

| |
|---|
| 9. Solve for the variable. <br><br> $x + 13 = 17$ |

| |
|---|
| 10. Solve for the variable. <br><br> $32 = r - 5$ |

**Name: ________________________**

**Day 12: Algebra Touch Answer Sheet**

**Combining Like Terms – "Terms (Like and Unlike)"**

**Remember: SIGN, VARIABLE, COEFFICIENT**

| Problem Given on Algebra Touch | Simplify the problem | Explain in WORDS how you simplified it |
|---|---|---|
| **1.** **Example:** $6x + 5y + 3x - 2y$ | $6x + 3x = 9x$ <br> $5y - 2y = 3y$ <br> $6x + 5y + 3x - 2y =$ $9x + 3y$ | I figured out that $6x$ and $3x$ are like terms and $5y$ and $-2y$ are like terms. <br> Then, I combined like terms to get $9x + 3y$. |
| **2.** | | |
| **3.** | | |
| **4.** | | |
| **5.** | | |

**Day 12: Algebra Touch Answer Sheet**

**Solving Equations – "Isolate"**

| **Problem Given on Algebra Touch** | **Solve the problem** | **Explain in WORDS how you solved it** |
|---|---|---|
| **6.**<br>**Example:**<br>$a + 5 = 30$ | $a + 5 = 30$<br>$\underline{\quad - 5 = - 5}$<br>$a + 0 = 25$<br>$a = 25$ | I wrote the problem down.<br><br>I know that I need to solve for the unknown variable, *a*.<br><br>The opposite of adding is subtracting, so I subtracted 5 from both sides.<br><br>My answer is $a = 25$. |
| **7.** | | |
| **8.** | | |
| **9.** | | |
| **10.** | | |

**Name: ______________________**

**Day 12: Solving Equations**

Use the Addition Property of Equality to solve for the unknown value. Check your answer (Replace the variable with your answer and see if both sides are equal).

| | |
|---|---|
| 1. $x + 7 = 24$ | Check: |
| 2. $c - 20 = 52$ | Check: |
| 3. $95 = 8 + y$ | Check: |
| 4. $15 - 3 = n + 1$ | Check: |
| 5. $t - 2 = 24 + 3$ | Check: |

**Day 12: Solving Equations Continued**

Instructions: Write an equation that models the situation using variables and then solve the following problems.

1. Cleats and a ball together cost $72. If the cleats cost $55, what is the cost of the ball?

| Algebraic Equation | Solve the equation |
|---|---|
| | |

2. A tour book of Brazil weighs 4 pounds. The total weight of a math book and the tour book is 10 pounds. How much does the math book weigh?

| Algebraic Equation | Solve the equation |
|---|---|
| | |

3. Cristiano Ronaldo, a Portuguese footballer, earns $13 million. Lionel Messi, an Argentine footballer, and Cristiano Ronaldo earn a total salary of $28 million. How much does Lionel Messi earn?

| Algebraic Equation | Solve the equation |
|---|---|
| | |

**Name:** ________________________

****Day 12: Solving Equations ****

Use the Addition Property of Equality to solve for the unknown value. Circle your answer. Check your answer (Replace the variable with your answer and see if both sides are equal). The first problem is done for you.

| | |
|---|---|
| 1. $x + 7 = 24$<br>$\underline{\quad -7 = -7}$<br>$x + 0 = 17$<br><br>$x = 17$ | Check:<br>$x + 7 = 24$<br>$17 + 7 = 24$<br>$24 = 24$<br><br>$24 = 24$ |
| 2. $c - 20 = 52$ | Check: |
| 3. $95 = 8 + y$ | Check: |
| 4. $15 - 3 = n + 1$ | Check: |
| 5. $t - 2 = 24 + 3$ | Check: |

****Day 12: Solving Equations Continued****

Instructions: Write an equation that models the situation using variables and then solve the following problems. The algebraic equation is given. Solve the equation.

1. Cleats and a ball together cost $72. If the cleats cost $55, what is the cost of the ball?

| Algebraic Equation | Solve the equation |
|---|---|
| $c$ = cost of the ball<br><br>$c + 55 = 72$ | |

2. A tour book of Brazil weighs 4 pounds. The total weight of a math book and the tour book is 10 pounds. How much does the math book weigh?

| Algebraic Equation | Solve the equation |
|---|---|
| $m$ = the weight of a math book<br><br>$10 = m + 4$ | |

3. Cristiano Ronaldo, a Portuguese footballer, earns $13 million. Lionel Messi, an Argentine footballer, and Cristiano Ronaldo earn a total salary of $28 million. How much does Lionel Messi earn?

| Algebraic Equation | Solve the equation |
|---|---|
| $s$ = Lionel Messi's salary<br><br>$13 + s = 28$ | |

**Day 13: Multiplication Property of Equality Lesson**

**Lesson Objective:** This lesson will extend the students' previous understanding of the equal sign as an equal relationship between two sides of an equation. Students will learn how to isolate a variable on one side of an equation using the multiplication property of equality.

**CCSS:**
- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and $x$ are all nonnegative rational numbers.

**Materials Needed:**
- Day 13: Warm-Up
- Day 13: Reading Handout
- Day 13: Solving Equations
- **Day 13: Solving Equations** (with examples)
- Chart Paper
- Paper
- Smart Board
- Overhead projector

**Before you teach, Possible Misconceptions and Remedies:**

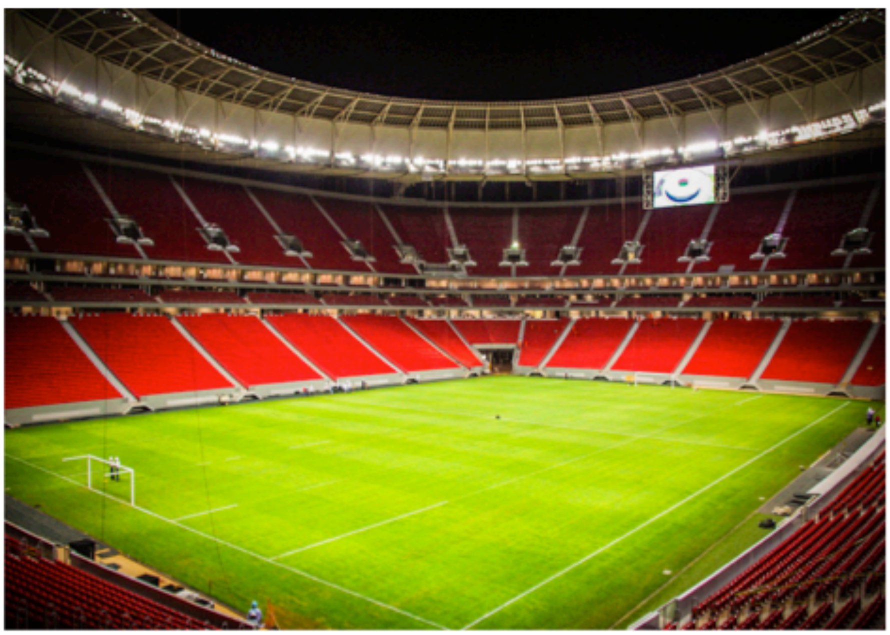
| Idea | Misconception | Remedy |
|---|---|---|
| Checking Solution | c – 20 = 52 and student got that c is 32<br><br>Check: 32 – 20 = 12<br>12 = 12<br><br>Did not realize that it is supposed to be equal to 52 | Rewrite the original problem first, then check |
| Substituting | 4x + 3y where x = 1 and y = 5<br>41 + 35 | 4x means 4 times x |
| | 4x + 3y where x = 1 and y = 5<br>Student wrote 5 + 8 = 13 | 4x does not mean 4 + x |
| | Writing variables even after substitution 4x + 3y where x = 1 and y = 5<br>Answer: 5x + 8y (and adding) | Make explicit that once you have a quantity to substitute you no longer need the variable. |
| Fill in the blank | 22 – 4 = 7 + 11<br>False 22 – 4 = 18 and not 7 + 11 | Viewing each side of the equal sign as one quantity. Seeing the equal sign as equality or balanced and not compute |
| | 23 – 2 + 3 = ___ - 4<br>Student did:<br>23 – (2+3) = ___ - 4 | Go over order of operations again.<br>(in PEMDAS Addition is first) |
| Multiplication Property | 6y = 24<br>Students wrote 64 = 24 | 6y means 6 times y and 64 is not the same as 24 |
| | 6y = 24<br>Student wrote 24 – 6 = 18, y = 18 | Using Addition Property instead of Multiplication Property |
| Addition Property | 25 + 5 = x – 4<br>Student wrote 25 + 5 = 30 – 4 = 26 so x = 26 | Equal sign and balance |
| Combining Like Terms | 25v + 4c – 15v –c<br>Student wrote 10v = 3c | Wrong sign in between, should be a plus sign |
| | 25v + 4c – 15v – c = 40v + 3c<br>Student ignored the – sign | Sign, variable, coefficient |

**Procedure for Multiplication Property of Equality Lesson:**

**Warm-Up and Lesson Opener**

- Ask students to individually do Day 13: Warm-Up.
- Once the students are finished, hand out Day 13: Reading and ask them to read it silently.
- Opener: Show the following video: http://www.youtube.com/watch?v=B9ocFdciC0w
  - Explain to students that this video shows the Stadium Nacional in Brasília.
- Popcorn read or real aloud Day 13: Reading
- Show this picture on the board (www.estadionacionaldebrasilia.com)
- Pose this question to class (Display on board)

**"The lawn in Estadion Nacional has a total area of 12,000 square meters. The length of the grassy field is 120 meters. Find the width."**



**Small Group Discussion**

- Allow students to discuss in their groups how to solve this problem.

**Questions to guide Student Thinking**

- What is the shape of the field?
- How do you find the area of a rectangle?
- Do you know a formula?
- (Area = length • width)

**Whole Class Discussion**

- Ask a representative from each group go up to the board and show how to model this equation and how to solve for the width.
- Remind students about the Addition Property of Equality and then tell them that this is an example of the Multiplication Property of Equality.
- Ask for volunteers to say what they think the Multiplication Property is?

***Big Ideas that Evolve from Whole Class Discussion***

- The area of a rectangle is equal to the length times the width.
- The multiplication property of equality states that you can multiply or divide both sides of an equation by the same number to "isolate" the variable.

*Remind students that any number divided by itself is equal to "1"*

**Small Group Discussion**

- Have each group solve the following problem:
  $12{,}000 = 120w$
- Ask students to work in their groups on Day 13: Solving Equations. (There are two versions to this WS)

**Questions to guide Student Thinking**

- How do you get $w$ by itself?
- What does $120w$ mean?
- How do you know if you have the correct answer?
- Show me.

***Big Ideas that Evolve from Discussions***

- The product is the answer to a multiplication problem.
- Identify when to use the Multiplication Property of Equality (MP) and when to use the Addition Property of Equality (AP).

**Process to Shift Student Thinking**

- The area formula for rectangles can be used for any size rectangle.
- Two properties for isolating the variable (AP and MP)
- Focus on isolating the variable on one side of the equation by making ONE move
- Equal means the same on both sides.
- Realizing that we are substituting in known values for length and width to check answer
- No longer need to write the variable once a value is substituted.
- $1w$ is just $w$ and 120 divided by 120 is 1.

**Name:** ____________________________

**Day 13: Warm-Up**

1. Write an example of an algebraic expression and an example of an algebraic equation.

2. Simplify the expression by combining like terms. Show your work.

$$\mathbf{25}v + \mathbf{4}c - \mathbf{15}v - c$$

3. Substitute the given quantities in the expression and then solve.

$$4x + 3y \quad \text{where } x = 1 \text{ and } y = 5$$

4. Is **27** a solution of $m - 6 = 21$? Why or why not?

5. True or False? $22 - 4 = 7 + 11$ Explain your answer.

6. Fill in the blank. Explain your answer.

$$\mathbf{23 - 2 + 3 = \_\_\_\_ - 4}$$

7. Solve for the variable.

$$\boldsymbol{x + 12 = 20}$$

8. Solve for the variable.

$$\boldsymbol{34 = r - 6}$$

9. Solve for the variable.

$$\boldsymbol{25 + 5 = x - 4}$$

10. Solve for the variable.

$$\boldsymbol{6y = 24}$$

**Day 13: Reading Hand-Out**

"Few Brazilian cities can match the capital Brasilia when it comes to architecture, and the imposing Estadio Nacional Mane Garrincha is a reflection of that, an arena with seating for 68,009 spectators, making it the second largest of the stadiums hosting matches at the 2014 FIFA World Cup Brazil™ .

The city's Estadio Nacional has been all but demolished to make way for the stadium, which boasts a new facade, metal roof and stands, as well as a lowered pitch enabling unobstructed views from every seat.

Founded on carbon neutrality, recycling and complete access via public transport, this environmentally friendly construction project consolidates Brasilia's status as a world leader in sustainable urban planning, creating a valuable legacy for other sectors of the local economy.

The Estadio Nacional will host the Opening Match at the FIFA Confederations Cup Brazil 2013 and seven games at the 2014 FIFA World Cup Brazil, one of them a quarter-final tie.

The stadium will be Brasilia's third, along with the Serejao, the home of Brasiliense, and the Bezerrao, which was recently refurbished and reopened in 2008. Following the world finals the arena will be used to host concerts and major cultural events."
(http://www.fifa.com/worldcup/destination/stadiums/stadium=5002284/index.html)

**Name:________________________**

**Day 13: Solving Equations**

Directions: Solve for the unknown value. Circle AP for Addition Property of Equality or MP for Multiplication Property of Equality. Circle your answer. Check your answer (Replace the variable with your answer and see if both sides are equal).

| **Solve:** | | **Circle:** |
|---|---|---|
| 1. $3x = 12$ | Check: | AP<br><br>MP |
| 2. $t + 6 = 18$ | Check: | AP<br><br>MP |
| 3. $24 = 8y$ | Check: | AP<br><br>MP |
| 4. $n + 4 = 21$ | Check: | AP<br><br>MP |
| 5. $6w = 18$ | Check: | AP<br><br>MP |

**Name:**________________________

****Day 13: Solving Equations****

Directions: Solve for the unknown value. Circle AP for Addition Property of Equality or MP for Multiplication Property of Equality. Circle your answer. Check your answer (Replace the variable with your answer and see if both sides are equal). The first problem is done for you.

| Solve: | | Circle: |
|---|---|---|
| 1. $\frac{3x}{3} = \frac{12}{3}$<br>$1x = 4 \rightarrow$ $x = 4$ | Check:<br>$3x = 12$<br>$3(4) = 12$<br>$12 = 12$ | AP<br>MP |
| 2. $t + 6 = 18$ | Check: | AP<br>MP |
| 3. $24 = 8y$ | Check: | AP<br>MP |
| 4. $n + 4 = 21$ | Check: | AP<br>MP |
| 5. $6w = 18$ | Check: | AP<br>MP |

## Day 14: Review of Addition Property of Equality and Multiplication Property of Equality Lesson

**Lesson Objective:** This lesson will extend the students' previous understanding of the equal sign as an equal relationship between two sides of an equation. Students will review how to isolate a variable on one side of an equation using the addition and multiplication property of equality.

**CCSS:**
- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and $x$ are all nonnegative rational numbers.

**Materials Needed:**
- Day 14: Modeling Problems
- Day 14: Multiplication Property of Equality Warm-Up
- Algebra Touch and iPads
- Day 14: Algebra Touch Answer Sheet
- Chart Paper
- Paper
- Overhead Projector
- Smart Board

**<u>Before you teach, Possible Misconceptions and Remedies:</u>**

| Idea | Misconception | Remedy |
|---|---|---|
| Solving Equations | $x + 4 = 4x$ | Plug in a number for x and show that these are not the same. |
| | 4x = 8<br>and student solved to get x = 12 (added 4 to both sides) | More practice with multiplication property of equality |
| | Students are just doing the math in their heads and not writing out their work | Make the numbers more difficult so they have to show work. |
| | 4x + 3f for c=2 and f=5, student wrote 4•c(2) = 8 and 3•f(5) = 15 then 8+15 = 23<br>Student got the correct answer, but is still leaving the variable once she plugs in a number. | Show her that she no longer needs to write the variable. This could get confusing with functions. |
| | (1/2)x = 10 so x = 5<br>(Focus group, 32:59 in video)<br>They read this as 2x instead of (1/2)x | Plugged in 5 to see that this answer was incorrect. |
| | 4c for c = 2 is 8c | Show her again that once you have a value for the variable you don't need to write the variable. |
| | 10 divided by ½ is 5 | Invert and multiply. |
| Algebra Touch | 8+-2<br>Confusing notation | The app should just do 8-2 |
| | Confusing how you don't have to do the same to both sides of the equation. | The app should make them multiply both sides by the denominator of the fraction… |
| Combining Like Terms | You can't multiply like terms. | Clarify this by showing you can multiply x and y |
| Product | Did not know what the word product means. | Once I told them, they were able to do the first two problems on the warm-up |

**Procedure for Review of Addition Property of Equality and Multiplication Property of Equality Lesson:**

**Whole Class Discussion**

- Pose this question to the class. (Write on board).

**"How much will it cost to buy 2 black soccer balls and 4 white soccer balls if a black soccer ball costs $15 and a white soccer ball costs $10?"**

- As a whole class, come up with the **expression** that represents this cost: 2b + 4w
- **Write this problem on the whiteboard (because we want to be able to erase).**

**Process to Shift Student Thinking**

- At this point, some students may still be writing the variable, even though it is replaced by a value.
- Physically erase the variable and replace with the value.
- This shows that the variable is no longer there.
- Also, provides students with a visual representation of substitution.

**Questions to guide Student Thinking**

- What does *b* stand for?

(The cost of a black soccer ball, $15)

- What does *w* stand for?

(The cost of a white soccer ball, $10)

- If I know that *b* = 15, can I substitute 15 for b in the expression? (point to the expression 2b + 4w)----YES
- If I know that *w* = 10, can I substitute 10 for w in the expression? (point to the expression 2b + 4w)-----YES

**Whole Class Discussion**

- Say this: "Now that I know the values of the variables, I am going to SUBSTITUTE the values in."
- Rewrite the expression 2b + 4w
- Make sure the class realizes that 2b means 2 • b.
- Then, erase the b and the w and write 2•15 + 4•10
- Solve this: 30 + 40 = 70
- The answer is $70 total.

***Big Ideas that Evolve from Whole Class Discussion***

- Clarify that the values REPLACE the variables.

**Small Group Discussion**

- Ask students to work together on Day 14: Multiplication Property of Equality Warm-Up

NOTE: (#3 on the WS checks to see if they understand replacing variables with values)

**Whole Class Discussion**

- Invite a volunteer to come to the board and do problem #3 (this problem checks to see if they understand replacing variables with values)
- Invite a volunteer to come to the board and do problem #7 (this problem introduces multiplying both sides by the same number)
- This should lead into a review about reciprocals and when you multiply reciprocals, you get 1
  - Ex: The reciprocal of 2/3 is 3/2, so 2/3 • 3/2 = 1
  - Also show the reciprocal of ½ is 2.
- Put an equation similar to #7 on the warm up on the board and ask students how they would solve for *x*. Most will say that they would divide both sides by the fraction. This is an opportunity to reinforce that the fraction divided by the fraction is equal to 1.
- Then, ask students if there is another way to isolate the variable.
- If no one comes up with anything, then ask students if they recall learning about reciprocals.
- Use the Algebra Touch App to show students about this property. First, demonstrate using Algebra Touch and then let students work on some problems.
- Students may practice with Day 14: Modeling Problems and Day 14: Algebra Touch Answer Sheet.

**Process to Shift Student Thinking**

- The reciprocal is a flipped fraction.
- What you do to one side of an equation you have to do to the other side to keep it balanced.
- Algebra Touch helps students visualize solving equations by making moves.
- Tell students to only make one move since they are solving one-step equations.
- Important that students realize that they do not need a symbol in between 30*z* to show the product.

***Big Ideas that Evolve from Whole Class Discussion***

- Reciprocals
- Isolate the variable
- Dividing is the same as multiplying by the reciprocal
- Fractions are division

**Name: ________________________**

**Day 14: Multiplication Property of Equality Warm-Up**

1. Write an expression for the following sentence: Find the product of 30 and $z$.

2. Express the number 32 as a product of two numbers. (Find three pairs of numbers.)

3. Find the value of $\mathbf{4c + 3f}$ if we know that $\mathbf{c = 2}$ and $\mathbf{f = 5}$.

Use the Multiplication Property of Equality to solve for the unknown value.

| | |
|---|---|
| 4. $2x = 18$ | 5. $49 = 7n$ |
| 6. $12 = 4b$ | 7. $\frac{1}{2}x = 10$ |

**Name: ________________________**

**Day 14: Modeling Problems**

Instructions: Model using algebra and then solve the following problems.

1. Ryan has $32 for concessions. Anna has three times the amount money that Ryan has. How much money does Anna have? (Let $a$ be the amount of money that Anna has.)

2. The size of Brazil is roughly half of South America. If South America has an area of about 7 million square miles, then what is the area of Brazil? (Let $s$ be the area of Brazil.)

3. If there are 60 soccer matches at the World Cup in 2014 and each of the twelve stadiums will host the same number of matches, then how many matches will be held at each stadium? (Let $m$ be the number of matches held at each stadium.)

**Name: ________________________**

**Day 14: Algebra Touch Answer Sheet**

Directions: Write down the problem given. Solve the problem and show your work. Check your answer using Algebra Touch.

| **Problem Given on Algebra Touch** | **Show your work!** |
|---|---|
| **1.** **Example:** $4x = 36$ | $\frac{4x}{4} = \frac{36}{4}$ <br> $1x = 9 \rightarrow$ $x = 9$ |
| **2.** | |
| **3.** | |
| **4.** | |
| **5.** | |

| **Problem Given on Algebra Touch** | **Show your work!** |
|---|---|
| **6.** | |
| **7.** | |
| **8.** | |
| **9.** | |
| **10.** | |

**Day 15: Structure of Algebraic Equations Lesson**
**(Equivalence of Left hand side and Right hand side)**

**Lesson Objective:** This lesson will deepen students understanding of the meaning of the equal sign by looking at the systemic structure of algebraic equations.

**Materials Needed:**
- Day 15: Warm-Up
- Day 15: Equivalent Equations
- Day 15: More Practice
- Day 15:Reading Hand-Out
- Paper
- Board
- Overhead projector

**CCSS:**
- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

- **6.EE.A.4**: Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Multiplication Property | 21 = 7u<br>7u x 3u = 21u<br>Not substituting the value of 3, treating it as like terms | Show with multiplication that u • u is $u^2$ (use numbers) |
| | 21 = 7u<br>Subtracted 7 from both sides | *7u* means 7 • *u* |
| | (1/4)c = 4c and then solved | Knows that she needs reciprocal, but replaced fraction with reciprocal |

**Procedure for Structure of Algebraic Equations Lesson:**

**Warm-Up**

- Ask students to do the Warm-Up individually. (Based on their work, ask for volunteers to show what they did on problem #2 and problem #5)

**Questions to guide Student Thinking**

- Do you agree, disagree, or are you unsure about what this student did?
- Can you check your answer?
- How?
- Did anyone do this problem in a different way?

**Whole Class Discussion**

- Show the following official emblem on the World Cup Brazil 2014 on the board (on Smart Board)


source: http://www.fifa.com/worldcup/officialemblem/index.html

**Whole Class Discussion Continued**

- Ask the students what they think it means and what they think about it.
- Ask for volunteers to read Day 15: Reading Hand-Out

*This emblem and reading gives meaning to the mathematics. We are setting the problem in a context.*

**Small Group Discussion, then Share with the Whole Group**

- In small groups and in their notebooks, ask the students to write an arithmetic equation that relates the number of green hands, the number of yellow hands, and the total number of hands on the emblem. Ask students to work in their groups to write as many arithmetic equations as possible, such as:
  - $2 + 1 = 3$
  - $3 = 1 + 2$
  - $3 - 1 = 2$
  - $3 - 2 = 1$

**Share with whole class.**

This is generalizing arithmetic.

- Now, pose the following question to the class (On the Smart Board).
  - "Let's say that we didn't know how many green hands were on the emblem, but we did know that there was one yellow hand and the total number of hands is three. Can you write an algebraic equation that shows this scenario?"
- Ask students to come up with algebraic equations in their groups and discuss their answers. Some examples:
  - $x + 1 = 3$
  - $3 = 1 + x$
  - $3 - 1 = x$
  - $3 - x = 1$

This is generalizing algebra.

**Share with whole class.**

Ask the class the following:

- Why is your equation is equivalent to the equation on the board. What does the variable stand for?

**Whole Class Discussion**

- Show the picture of the official emblem again and ask students to count the number of fingers on it. Then, ask how many green fingers and how many yellow fingers. Again, ask students to write arithmetic equations.
- Now, have students come up with their own algebra problem that relates the total number of letters and the number of numbers.
  - There are 20 letters and 4 numbers but let's say we don't know the total, *t*.
  - $20 + 4 = t$
  - $t - 4 = 20$
  - $t - 20 = 4$
- Next ask for volunteers to present their algebra problem to the class and ask them to solve for the variable.

**Questions to guide Student Thinking**

- Does someone have an equivalent equation?
- How do you know?
- What is an equivalent equation?
- Do you agree or disagree with the answer for the variable? Why?

**Small Group Discussion**

- Pass out the Day 15: Equivalent Equations WS and allow students to work on this in small groups.
- If they finish this, then give them Day 15: More Practice

**Process to Shift Student Thinking**

- Essential that students learn that fractions are division and multiplying both sides by a reciprocal is the same as dividing
- Opposite operations
- Arithmetic Equation vs. Algebraic Equation
- Equivalent Equations by moving from arithmetic to algebra

***Big Ideas that Evolve from Whole Class Discussion***

- Studying equivalent equations in arithmetic and moving this to equivalent equations in algebra.

**Name:** ___________________________

**Day 15: Warm Up**

| Directions: Solve the following equations. Show your work and check your answer. |
|---|
| **1.** $b - 6 = 15$ |
| **2.** $\frac{1}{2}c = 5$ |
| **3.** $21 = 7u$ |
| **4.** $70 = 3 + f$ |
| **5.** $3x + 2x = 20$ |

**Name:** ________________________

**Day 15: Equivalent Equations**

**Directions: Solve the equation. Then, write an equivalent equation for the given equation. Why are they equivalent?**

| Solve the Equation | Write an Equivalent Equation | Why are they equivalent? |
|---|---|---|
| **1.** $x - 2 = 4$<br>$+2 = +2$<br>$x + 0 = 6$<br>$x = 6$ | $x - 4 = 2$<br>or<br>$4 + 2 = x$ | $x - 2 = 4$, $x - 4 = 2$, and $4 + 2 = x$ are equivalent because $x = 6$ in every equation. |
| **2.** $25 = 9 + y$ | | |
| **3.** $2x = 8$ | | |

| Solve the Equation | Write an Equivalent Equation | Why are they equivalent? |
|---|---|---|
| **4.** $14 = 2r$ | | |
| **5.** $z + 4 = 10$ | | |
| **6.** $15 = 3x$ | | |

**Name:** ___________________________

**Day 15: More Practice**

**Equivalent** equations mean that the value for the variable is **the same** in each equation.

1. **Write an equation that is equivalent to $x + 15 = 22$.**

**How do you know that the equation that you wrote is equivalent?

2. **Write an equation that is equivalent to $20x = 40$.**

**How do you know that the equation that you wrote is equivalent?

**3. Write an algebra equation for the following problem. Then, try to solve the equation.**

**"If you add 7 to two times a number, you will get 35. What is the number?"**

**Can you check your answer? Show your work.

**4. Write an algebra equation for the following problem. Then, try to solve the equation.**

**"Half of a number is equal to 12. What is the number?"**

**Can you check your answer? Show your work.

**Day 15: Reading Hand-Out**

"The inspiration for this design comes from the iconic photograph of three victorious hands together raising the world's most famous trophy. As well as depicting the uplifting humanitarian notion of hands interlinking, the portrayal of the hands is symbolic of the yellow and green hands of Brazil warmly welcoming the world to Brazilian shores. Victory and union are the two key emotive elements which are vividly depicted through the hands featured in the design. Whilst forming a clear link to the colours of the Brazilian national flag, the green and yellow colours also allude to two of the strongest features of life in Brazil – the golden beaches and beautiful sun reflected in the yellow tones, with green representing the strong tropical interior that Brazil is so famous for. The combination of the strong image, the contemporary typography and striking colours are extremely effective in depicting a modern and diverse host nation."

## Day 16: Equivalent Equations and Distributive Property Lesson

**Lesson Objective:** This lesson will extend the students' previous understanding of like terms as quantities that can be added and subtracted. Students will learn more about equivalent equations. Students will learn the distributive property.

**Materials Needed:**
- Day 16: Warm-Up
- Day 16: Practice
- Day 16: Distributive Property
- Smart Board
- Paper
- Board
- Overhead projector

**CCSS:**
- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression* $3(2 + x)$ *to produce the equivalent expression* $6 + 3x$*; apply the distributive property to the expression* $24x + 18y$ *to produce the equivalent expression* $6(4x + 3y)$*; apply properties of operations to* $y + y + y$ *to produce the equivalent expression* $3y$.
- **6.EE.A.4**: Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions* $y + y + y$ *and* $3y$ *are equivalent because they name the same number regardless of which number* $y$ *stands for.*

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Equivalent Equation | 2x is x + 2 | Plug in a number to show this is untrue |
| | 2 divided by 8 instead of 8 divided by 2 | Show these are not the same value |
| | Equations are equivalent just because they have the same value for the variable | Use opposite operations in the same equation |
| Solving equations with Reciprocal | (1/2)w = 6, inverted the 6 | Check answer |
| Solving Equation | 2x=8 and writing 24 = 8 instead of 2 times 4 | Need to show that with a variable don't need parenthesis but with a number need to show multiplication |
| Words to Equation | Says to add and instead subtracted 7-35 | Vocabulary ELL issue |
| Distributive Property | Ignoring the sign and adding even though it says subtract (9 – 3)5 = 15 + 45 | Show the sign |
| | Forgetting to distribute to the second term 4(x – 2) = 4x – 2 | Draw the arrow and multiply, show another example |
| | Adding in distributive property instead of multiplying 2(y + 9) = 3y + 12 | Distributive property is multiplication first then addition or subtraction |
| | Only distributing to the numbers, but not to the variables 3(y + 9) = 27 + y | Draw the arrows |

**Procedure for Equivalent Equations and Distributive Property Lesson:**

**Whole Class Discussion**

- Show this video about reciprocals: http://www.youtube.com/watch?v=4lkq3DgvmJo (stop at 3 min.)
- Review equivalent equations by having a whole class discussion and asking students to explain what an equivalent equation is and why. (Equivalent means the same, the variable in an equivalent equation has the same value.)

**Warm-Up**

- Hand out Day 16: Warm Up and ask students to work on it independently.
- (If they finish early, hand out Day 16: Practice and ask students to try it)

**Whole Class Discussion**

- Once students finish, review that the opposite operation of addition is subtraction, and the opposite of multiplication is division. Ask students:
  - What is a reciprocal? How would I solve this problem: $\frac{1}{2}x = 8$
  - Remind students that $\frac{1}{2} \cdot 2$ is 1 because this is the same as 2 divided by 2
  - Also, ½ divided by ½ is the same as ½ times 2 (the reciprocal)

*Add anything here that helps students understand inverse operations! Fractions are essential for understanding algebra…*

***Big Ideas that Evolve from Whole Class Discussion***

- The reciprocal of a fraction is its multiplicative inverse, so when you multiply reciprocals you get 1.

**Whole Class Discussion**

- Display this problem (same as Day 16: Practice) on the board and show the pictures. (on Smart Board)
- "Remember, you and your family are going to the World Cup this summer. The hotels are very expensive, so you decide that you will camp near the stadiums and spend your money on tickets instead. Every member of the family needs to buy a backpack that costs $90 and a sleeping bag that costs $60. What is the total cost of this camping equipment?"

source: billericky.hubpages.com

**Small Group Discussion**

- Tell students that we don't know how many members of their family will be going to the World Cup (this is the unknown variable).
- Ask each student to write an algebraic expression for the total cost of a backpack and a sleeping bag.

**Process to Shift Student Thinking**

- Two methods: Let $n$ be the number of family members going.
- The first method is finding the total cost of one backpack and one sleeping bag and then multiplying by $n$. The second method is finding the cost of $n$ backpacks and $n$ sleeping bags and then adding the costs (like terms).
  1. Total Cost = $n(90 + 60)$
  2. Total Cost = $90n + 60n$

  **Students may need to begin this by using an exact number of family members instead of a variable**

***Big Ideas that Evolve from Whole Class Discussion***

- Distributive Property: The property that states if you multiply a sum by a number, you will get the same result if you multiply each addend by that number and then add the products. Example: $2(3 + 4) = 6 + 8 = 14$ OR $2(3 + 4) = 2(7) = 14$

NOTE: *The variable is the number of family members, not the number of sleeping bags or backpacks.*

**Whole Class Discussion**

- Ask students to bring their worksheets to the overhead to explain their reasoning of how they came up with the total cost.
- Then display the following on the board and ask if these are equivalent expressions.
  - $n(90 + 60)$ and $90n + 60n$
- This will lead into a discussion about the distributive property.
- Challenge Problems: Put the following examples on the board and ask students to work in their groups to find the missing number
  - 2(___ + 5) = 30
  - ___ • (6 – 3) = 9

**Questions to guide Student Thinking**

- Do you agree with this student?
- Why or why not?
- What number goes in the blank?
- Could you replace the blank with a variable?
- Does this make the problem easier?
- What are equivalent expressions?
- What does the equal sign mean?

***Big Ideas that Evolve from Whole Class Discussion***

- Distributive property is necessary when there are variables in the expression to multiply; otherwise, use order of operations
- $150n = (60 + 90)n$ which is the same as $60n + 90n$
- Equivalent Expressions
- The equal sign means that both sides of the equation have the same value

**Name:** ___________________________

**Day 16: Warm-Up**

Directions: Solve the equation. Then, write an equivalent equation for the given equation. Why are they equivalent?

| Solve the Equation | Write an Equivalent Equation | Why are they equivalent? |
|---|---|---|
| **1.** $x - 2 = 4$<br>$\underline{\quad + 2 = +2}$<br>$x + 0 = 6$<br>$x = 6$ | $x - 4 = 2$<br>or<br>$4 + 2 = x$ | $x - 2 = 4$, $x - 4 = 2$, and $4 + 2 = x$ are equivalent because $x = 6$ in every equation. |
| **2.** $25 = 9 + y$ | | |
| **3.** $2x = 8$ | | |
| **4.** $\frac{1}{3}w = 6$ | | |

**5.** Write an algebra equation for the following problem. Then, try to solve the equation.

**"If you add 7 to a number, you will get 35. What is the number?"**
(Hint: You don't know the number, so that is your variable.)

**Can you check your answer? Show your work.

**6.** Write an algebra equation for the following problem. Then, try to solve the equation.

**"Half of a number is equal to 12. What is the number?"**
(Hint: You don't know the number, so that is your variable.)

**Can you check your answer? Show your work.

**Name: __________________________**

**Day 16: Practice**

Instructions: Solve the following problem. Make sure to write an algebra problem that shows your thinking.

Remember, you and your family are going to the World Cup this summer. The hotels are very expensive, so you decide that you will camp near the stadiums and spend your money on tickets instead. Every member of the family needs to buy a backpack that costs $90 and a sleeping bag that costs $60. We don't know how many members of your family will be going to the World Cup.

What is the total cost of this camping equipment?

(Hint: Figure this out for 1 family member, 2 family members, and *n* family members.)

**Name:**__________________________

**Day 16: Distributive Property**

1. Use the Distributive Property to evaluate the expressions.

   a. $2(5 + 3)$

   b. $(9 - 3)5$

2. Use the Distributive Property to write an equivalent variable expression.

   a. $4(x - 2)$

   b. $3(y + 9)$

3. There are 64 teams in the World Cup 2014. Each team can have a maximum of 11 players on the field and 9 players on injured reserve. Use the distributive property to find the maximum number of players who can be in the World Cup 2014.

**Day 17: Solving Algebraic Equations using the Addition Property of Equality and the Multiplication Property of Equality Lesson**

**Lesson Objective:** In this lesson, students will learn how to isolate the variable in an algebraic equation using the addition property of equality and the multiplication property of equality.

**CCSS:**
- **6.EE.A.3**: Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3(2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6(4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*

- **6.EE.B.6**: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

- **6.EE.B.7**: Solve real-world mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and $x$ are all nonnegative rational numbers.

**Materials Needed:**
- Day 17: Warm-Up
- Day 17: Practice
- Day 17: Solving Equations
- Paper
- Board/ Smart Board

**Before you teach, Possible Misconceptions and Remedies:**

| Idea | Misconception | Remedy |
|---|---|---|
| Solving Equations | Subtracting and using Addition Property when it is supposed to be multiplication property | Show that 8y means 8 times y. Opposite operations |
| | Guess and check and not writing out work | More difficult problems |
| | Different ways of solving equations | Not really a misconception, but an equivalent equation. |
| Distributive Property | Making it into an equation 3(x + 5) and student writing x = 15 | Do more practice and context problem with distributive property |
| | 7(h + r + 2) Student wrote 7h + 2r, but drew the arrows, doesn't realize that there can be more than two terms in the parenthesis | More examples |

**Procedure for Solving Algebraic Equations using the Addition Property of Equality and the Multiplication Property of Equality Lesson:**

**Warm-Up**

- First, ask students to work individually on the Warm Up.
- They must label whether they are using the Addition Property (AP) or the Multiplication Property (MP) and practice Distributive Property (make sure that they draw arrows to show multiplication)

**Whole Class Discussion**

- Ask for volunteers to show their work to the class on the board.

*Make sure to ask the other students if they agree or disagree and why. Try to get them to talk more to each other and ask each other questions.*

**Whole Class Discussion**

- Show this website on the board: http://store.fifa.com/29649.html
- Display this picture on the Smart Board with the following questions
- Pose the following question: How many USA Home Soccer Jerseys can the class buy for $1,295?
  - From this website, note that one jersey is $85.00
  - A one time shipping fee is $20.00 for the total purchase

The purpose of small group discussion is for students to brainstorm about how to figure out this problem and how to model it using

**Small Group Discussion**

- In groups, have students work on modeling this problem.
- First have them figure out how much one shirt will cost, then how much two shirts will cost, etc.
- Then, remind them that they have to add the $20 shipping fee.

**Questions to guide Student Thinking**

- Do you see a pattern that models this situation?
- How much does one shirt cost?
- How much do two shirts cost?
- What about the one time shipping fee of $20?

**Whole Class Discussion**

- Ask for the groups to volunteer and present on the board how they are thinking about this problem.
- Display the problem on the board again (on Smart Board): How many USA Home Soccer Jerseys can the class buy for $1, 295?
  - From this website, note that one jersey is $85.00
  - A one time shipping fee is $20.00 for the total purchase
- Thus far, the students have brainstormed about how to solve this problem. Formalize the equation, by holding a whole class discussion and coming up with the equation as a group
  - $85j + 20 = 1295$ (Make sure students know that *j* stands for the number of jerseys)
- Now, tell the students that to solve for *j*, they must isolate it on one side of the equation. Ask the class what 85*j* means. (it means 85 times *j*).
- Give students time to think and discuss how to solve for *j* in their groups. Ask for volunteers to come to the board and explain.
- Then, prompt students to explore the addition property of equality by subtracting 2 from both sides of the equation and the multiplication property of equality by dividing both sides of the equation by 85.

**Let students think about this process. The answer is $j = 15$. So, for $1, 295, the class can by 15 jerseys**

**Questions to guide Student Thinking**

- Do you agree or disagree with this student's answer? Why?
- Compare your answer with what this student is doing on the board.

**Small Group Discussion, then Share with the Whole Group**

- In small groups, ask students to work on Day 17 practice in their groups
  - This is what they should come up with on the worksheet:
    - Since it costs an extra $10 per jersey, the new equation is $95j + 20 = 970$

**Share with whole class.**

**Process to Shift Student Thinking**

- Students know how to use the Addition Property and Multiplication Property of Equality, now they are learning to use them together to solve an equation
- As other groups share their answers, students notice that their equation and the other groups' equations are equivalent equations because they have the same answer
- Distributive property does not require solving an equation
- Noticing that there can be more than one term on each side of the equal sign, but you only add or subtract like terms

**Small Group Discussion**

- Ask students to work together on Day 17: Solving Equations WS

***Big Ideas that Evolve from Whole Class Discussion***

- Solving equations by isolating the variable
- The variable is the unknown value
- Finding a pattern so that they can write a formula to easily solve the problem
- To solve, first use Addition Property of Equality and then use Multiplication Property of Equality
- Remind students about opposite operations for balancing equations

**Name:** ________________________

**Day 17: Warm-Up**

Directions: Solve for the unknown value. Circle AP for Addition Property of Equality or MP for Multiplication Property of Equality. Circle your answer. Check your answer.

| **Solve:** | | **Circle:** |
|---|---|---|
| 1. $5x = 30$ | Check: | AP<br>MP |
| 2. $t + 9 = 21$ | Check: | AP<br>MP |
| 3. $32 = 8y$ | Check: | AP<br>MP |

Directions: Use the Distributive Property to Simplify. Circle your solution.

| | |
|---|---|
| 4. $3(x + 5)$ | 5. $6(y - 7)$ |
| 6. $(2 + y)4$ | 7. $4(x - 1)$ |
| 8. $7(h + r + 2)$ | |

**Name: ________________________**

**Day 17: Practice**

We already found that you can buy 15 jerseys plus one time shipping for $1,295. Now, suppose you want a front number AND a back number, but this time you only have $970 to spend. A front number costs an extra $2 and a back number costs an extra $8.

How many numbered jerseys can you buy with $970 if the shipping is still $20 and one jersey costs $85?

1. Think about the problem and do not solve yet. How much does it cost for one jersey with a number on the front and a number on the back? Explain.

2. Now, how much does it cost for *j* jerseys with a number on the front and number on the back?

3. How much does it cost for *j* jerseys with a number on the front and a number on the back plus the shipping fee?

4. Now, try to write an equation and solve. Remember, you have a total of $970.

**Name: ________________________**

**Day 17: Solving Equations**

| Solve the following equations. Check your solutions. | |
|---|---|
| **1.** $b - 6 = 15$ | **Check:** |
| **2.** $26 = 2y$ | **Check:** |
| **3.** $3x + 5 = 20$ | **Check:** |
| **4.** $7 = 3 + 2f$ | **Check:** |
| **5.** $10x + 2 = 32$ | **Check:** |

## Day 18: Functions Lesson

**Lesson Objective:** Students will learn that functions have three components: the domain, the range, and the rule. Students will learn that a function can be represented in different ways such as verbally, in an arrow diagram, algebraically, and graphically. Students will continue to practice substituting values for variables.

**CCSS:**

- **6.EE.B.5**: Understand solving an equation as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use and explain substitution in order to determine whether a given number in a specified set makes an equation or inequality true.

- **6.EE.C.9**: Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

**Materials Needed:**

- Day 18: Warm-Up
- Day 18: Plotting Points Review
- Day 18: Functions
- Paper and graph paper
- Board/ Smart Board

**Before you teach, Possible Misconceptions and Remedies:**

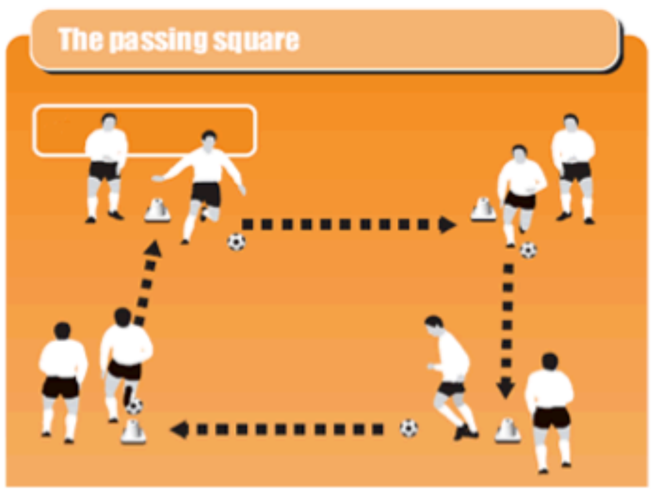
| Idea | Misconception | Remedy |
|---|---|---|
| Distributive Property | 27 – 3x is the same as 3x – 27 | Plug in numbers to show this is not true. |
| | 3(x – 9) = 9x + 27 | Use arrows to simplify. |
| Algebraic Function | Still writing just an expression 5x + 20 | Needs an input and output |
| | Plug and chug, not realizing you plug in input to get output | Practice |

**Procedure for Functions Lesson:**

**Warm-Up**
- Ask students to do the Day 18: Warm-Up independently
- Collect Warm-Up
- As students finish the warm up, hand out Day 18: Plotting Points Review (**as groups finish, they may work on this together**).

*Plotting points and graphing a line is a prerequisite skill for the Functions activity. Do your students remember how to do this?*

**Whole Class Discussion**
- Ask for volunteers to share (using overhead projector) their their work on the Day 18: Plotting Points Review
- As a whole group, do the first few practice problems, then let students work on these together in their small groups.
- Ask students if they know about the soccer square drill.
- Show the following image (on Smart Board)
- Show the following video (Stop video at 40 seconds): https://www.youtube.com/watch?v=RlTOE-eDTrI




Source: http://www.bettersoccercoaching.com/Article-1030-57-Soccer-Drills--Skills-Passing-drills-Passing-square-soccer-drill

**Questions to guide Student Thinking**

- How far away should the cones be placed? (answers vary)
- Should they all be the same distance? (yes, because it is a square)
- What is the perimeter of a square? (**If students do not know this, then ask them to draw a picture and explain the perimeter of a square in their notebooks**)

**Small Group Discussion**

- Ask students to come up with a **formula** for the perimeter of a square without knowing the distance between each cone.

**Process to Shift Student Thinking**

- Try to write a verbal phrase: The perimeter of a square is the sum of the sides.
- Now, write this as an algebraic equation using variables
  - $P = 4s$ or $P = s + s + s + s$
- Then ask students to find the perimeter of the square for different values of $s$

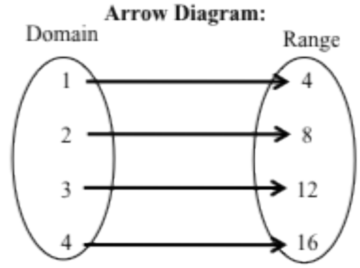
**Questions to guide Student Thinking**

- Where do you think we put the input values? $s$ or $P$? Why?
- Ask for what the value of $s$ might be. (Any positive number). Then find $P$.
- Why does $s$ have to be a positive number?

**Whole Class Discussion**

- Explain to students that $P = 4s$ is a **function** with input and output values.
- Now say, "Let's organize this information in an **arrow diagram**"
- Show the following arrow diagram on the board or a similar arrow diagram using other inputs.
- Explain that input and output can also be the "**domain**" and "**range**"

**Arrow Diagram:**




**<u>Small Group Discussion, then Share with the Whole Group</u>**

- Relate this back to the soccer square.
- Ask students what they think the perimeter (in feet) of the soccer square should be.
- Have some groups come to the board and explain their reasoning to the class.

**Share with whole class.**

- Provide students with graph paper and ask them to plot points from the Arrow diagram (above or the one the whole class came up with).

**Share with whole class.**

**Questions to guide Student Thinking**

- Is the perimeter the input or output? (output)
- If you think the perimeter should be ____ feet, then each side should be what?
- What is $x$ on the graph? (domain or input) (Students may use $P$ and $s$)
- What is $y$ on the graph? (range or output)
- What equation are you graphing? ($P = 4s$)
- Why do some graphs have different points, but the same line?
- Explain your reasoning.

EXAMPLE:
*For instance, if a group thinks that the perimeter should be 40 feet, then each side should be 10 feet. So 10 goes in the domain oval and 40 goes in the range oval.*

**Whole Class Discussion**

- Conclude the discussion by giving the definition of a function:
  - An input-output relationship that has exactly one output for each input.
- Ask students to discuss what this means and then ask for volunteers to explain it, using their graph.
- Summarize the lesson by saying that a function can be represented in the following ways.
- Show the following images on the overhead or Smart Board.

***Big Ideas that Evolve from Whole Class Discussion***

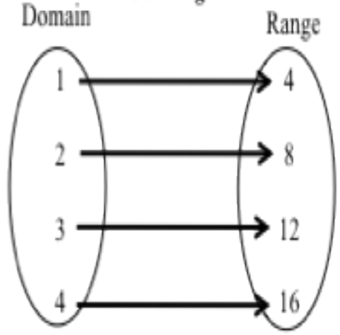
- Function
- Arrow Diagram
- Input
- Output
- Domain
- Range
- For each input, there is only one output.

**Process to Shift Student Thinking**

- You can pick any number for x and plug into the equation to get a point on the line
- Perimeter of a square can be represented as P = 4s, P= s+s+s+s, and P = 2s + 2s, P=s+3s and these are equivalent equations because they represent the same thing
- Input and Output, arrow diagram for organizing information
- The arrow diagram gives you the points on the line
- Functions can be represented as equations, arrow diagrams, and graphs

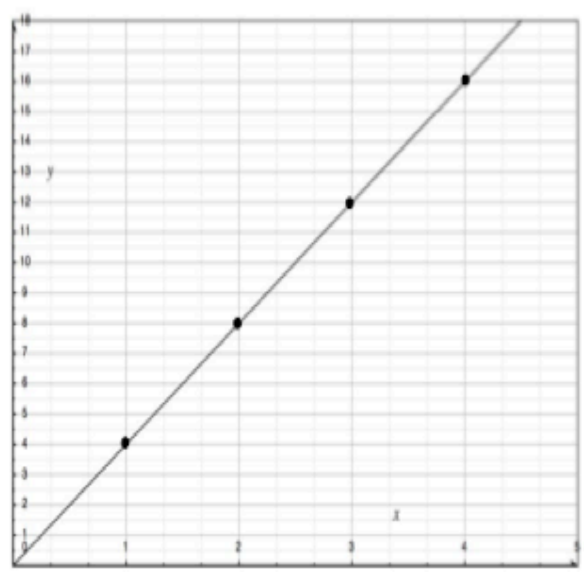
**Verbal:**
The perimeter of a square is the sum of the sides.

**Arrow Diagram:**




**Algebraic:**
$P = s + s + s + s$ or $P = 4s$
Where $P$ is the perimeter of the square and $s$ is the side length.

**Graphical:**



**Name: __________________________**

**Day 18: Warm Up**

1. Draw a line from the equation or expression in Column A to its equivalent equation or expression in Column B.

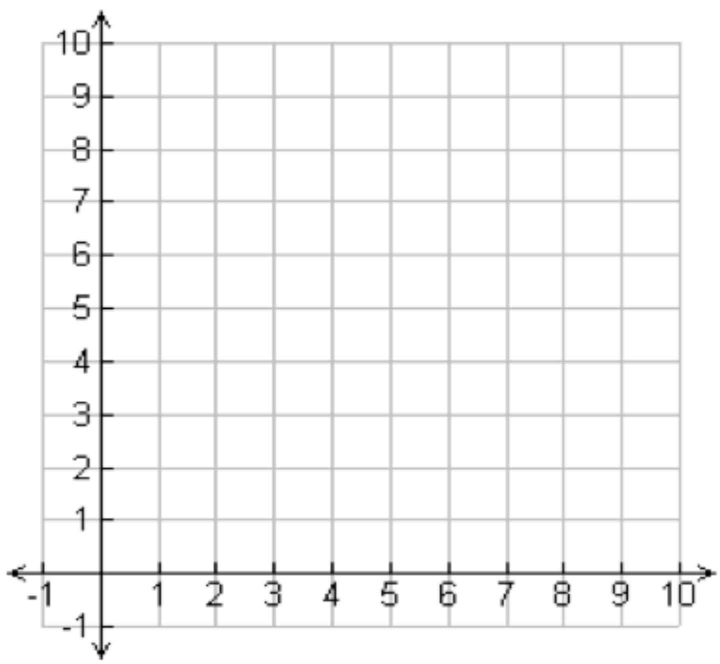
| A | B |
|---|---|
| $x - 27 = 45$ | $72 = 45 + x$ |
| $3x + 6 = 27$ | $3x - 27$ |
| $9(x + 3)$ | $x - 45 = 27$ |
| $3(x - 9)$ | $9x + 27$ |
| $72 - x = 45$ | $x = \frac{27 - 6}{3}$ |

**Name:__________________________**

**Day 18: Plotting Points Review**

1. Plot the following points on the graph and then draw a line through the points.

| *x* | *y* | (*x*,*y*) |
|---|---|---|
| 0 | 3 | (0 , 3) |
| 2 | 5 | (2 , 5) |
| 4 | 7 | (4 , 7) |
| 6 | 9 | (6 , 9) |




2. This is the line $y = x + 3$. Can you explain why? (Hint: plug and chug)

**Name: ________________________**

**Day 18: Functions**

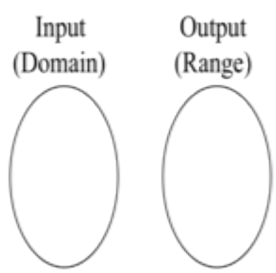
Instructions: Remember, a function can by represented verbally, in an arrow diagram, algebraically, and graphically. Read the verbal function, draw the arrow diagram, write an algebraic function, and graph the function given below.

**Verbal**

You are saving money in your piggy bank so that you have spending money for your trip to the World Cup 2014 in Brazil. You already have $20 in your piggy bank and you get $5 everyday for doing your chores.

- How much will you have in your piggy bank after 1 day?
- How much will you have in your piggy bank after 2 days?
- How much will you have in your piggy bank after 5 days?
- How much will you have in your piggy bank after an unknown amount of days? (Use *d* as the number of days)

**Arrow Diagram**




**Algebraic Function (This is an equation or formula)**

**Graph**

(Use graphing paper)

## Day 19: Review and Solving the Challenge Problem Lesson

**Lesson Objective:** Students will finish the functions activity from Day 18. Students will practice the vocabulary from this unit and continue to clear up misconceptions in the unit. Students will do the challenge problem from Day 1.

**CCSS:**
- Expressions and Equations Grade 6 CCSS

**Materials Needed:**
- Functions Day 18
- Paper and graph paper
- Vocabulary and practice problems for review
- Board/ Smart Board
- Overhead Projector

**Before you teach, Possible Misconceptions and Remedies:**

| | Misconception | Remedy |
|---|---|---|
| Graphing | Starting at 1 instead of the origin | Show the students where the origin is |
| Plug and Chug | Leaving variable in the expression after substituting a value for the variable | Class discussion about what it means to replace the variable with a number |
| Challenge Problem | Some students wrote an expression for the cost for their families and not for the cost of a general family without knowing adults and children | Rewrite the challenge problem to make the goal more clear to the students |
| Math Symbols | The different between x as a variable and x as times. | Use a dot for multiplication |

**Procedure for Review and Solving the Challenge Problem Lesson:**

**Warm-Up**
- Allow students to finish graphing functions from the day before. (Day 18: Functions)

**Whole Class Discussion**
- Ask for volunteers to come to the overhead projector and show their work.
- Focus on the algebraic function, arrow diagram, and the graph.
- Write a review problem on the board
- Ask students to solve in their groups and write their answer in their notebooks. (Each problem should only take about 2 minutes – Use timer on Smart Board)
- When time is up, ask for volunteers to say their answers.
- Tell the other students that they need to ask their questions to the volunteer student (at the overhead or board)
- Also, ask students about vocabulary relevant to the problem
  - Ex: coefficient, expression, variable, like terms, equation, function, replacement values, solution, addition property of equality, multiplication property of equality, arrow diagram, domain, range, input, output
- In the last 15 minutes of class, hand out the *Going to the World Cup* Challenge Problem WS
- Ask students to solve this independently.

**Questions to guide Student Thinking**
- Do you agree or disagree with this student's work?
- Did you do the problem in a different way?
- Can you come to the board and show it to us? Explain.

*Try to get students to discuss (using academic language) anything that they don't understand with each other.*

*Going to the World Cup* Challenge Problem

You and your family want to go to the 2014 World Cup. There are 64 matches at the FIFA World Cup Finals, and, unfortunately, you will not be able to attend every match. Tickets for each match for adults are $140 and for children are $70. How much money will it cost for the tickets for your family to go to one World Cup Match? Two World Cup Matches? Three World Cup Matches? Write an algebraic expression for the Total Cost for tickets for adults and children to attend matches at the World Cup.

**Name: ___________________________**

***Going to the World Cup* Challenge Problem**

You and your family want to go to the 2014 World Cup. There are 64 matches at the FIFA World Cup Finals, and, unfortunately, you will not be able to attend every match. Tickets for each match for adults are \$140 and for children are \$70. How much money will it cost for the tickets for your family to go to one World Cup Match? Two World Cup Matches? Three World Cup Matches? **Write an algebraic expression for the Total Cost for tickets for adults and children to attend matches at the World Cup.**

1. How much money will it cost for the tickets for your family to go to one World Cup Match?

2. How much money will it cost for the tickets for your family to go to two World Cup Matches?

3. How much money will it cost for the tickets for your family to go to three World Cup Matches?

4. Write an algebraic expression for the **Total Cost** for tickets for adults and children to attend matches at the World Cup.

## Day 20: Posttest

**Lesson Objective:**
All students work on the posttest individually.

**Materials Needed:**
- Posttest
- Pencil
- Scratch paper

Algebra Instructional Unit

Name:__________________________________________

**Posttest**

Please show your work and circle your answer.

1) What is an algebraic expression? Give an example.

2) What is an algebraic equation? Give an example.

3) Is 9 a solution of $p - 7 = 2$ ?

4) Evaluate the expression for the given replacement values.
$2x + 5y$ for $x = 4$ and $y = 7$

5) Simplify the expression by combining any like terms.
$8b - 2b$

6) Simplify the expression by combining any like terms.
$5x + 2 - 3x + 8$

7) If David has $15 and Joanna has $6 less than David, how much money does Joanna have?

8) If David has some money and Joanna has $6 less than David, how much money does Joanna have?

9) In algebra, what is a variable?

10) How many pairs of numbers can you find that add to 10?
Express the number 10 as a sum of two numbers using variables.

11) True or False?
$9 + 15 = 40 - 16$

12) Fill in the blank.
$3 + 2 + 5 = \_\_\_\_ - 2$

13) Fill in the blanks.
$\_\_\_ + 3 + \_\_\_ = 8 + 1 + \_\_\_$

14) Haley has 27 songs on her iPod. She accidentally deleted some songs and now only has 18 songs on her iPod. How many songs did she delete?

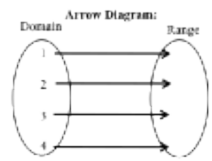
15) What is the addition property of equality? Give an example.

16) What is the mulitplication property of equality? Give an example.

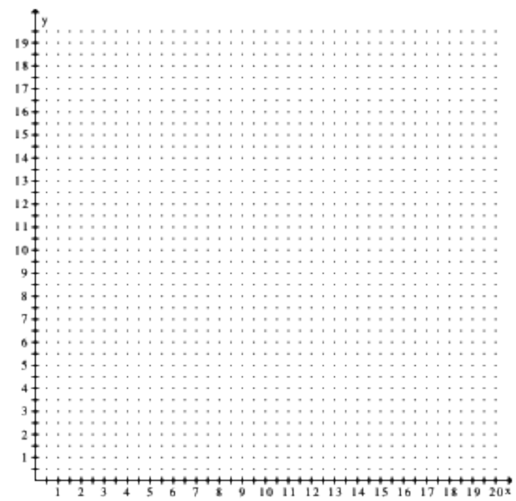
17) Solve the equation.
$2x + 10 = 14$

18) Fill in the blank.
If 2 + 3 = 5, then 5 - ____ = 2

19) Fill in the range of the function y = 2x shown in the arrow diagram below.




20) Graph y = 2x using the points in the domain and range from the previous problem.



Appendix C

**Data and Collection To-Do List**

1. **Documentation**
   - Photograph chart paper, date, number, and organize in file
   - Create log, using a table, of actual lesson and how it was adjusted and what happened during the lesson.

2. **Teacher Log**
   - Create log, using a table, of teacher recommendations.

3. **Student Learning Checklist**
   - Analyze student written work and create checklist
   - Objective of lesson on the top of the checklist and check off student if written work shows grasping of this topic

4. **Class Learning Log**
   - What students understood, what students had difficulty with, and the next steps

5. **Lesson Log**
   - What actually happened in the lesson, how did the lesson change, and why
   - Next steps

6. **Fieldnotes and Transcripts**
   - Type transcript of video and make observational notes in italics (Analysis of this can occur at a deeper level after the study)

7. **Key Mechanisms for Shifts in Student Thinking Log**
   - Document shifts in student thinking about the lesson objectives

8. **Misconceptions**
   - Keep track of misconceptions and errors from student work and ways to remedy these misconceptions.

9. **Revise the lessons and begin revising whole curriculum**
   - Look at tomorrow’s lesson plan and identify what key ideas and potential shifts, modify the lesson based on what happened in previous lesson
   - Give the lesson to the teacher

Appendix D

**Teacher**
**Lesson Reflection Protocol**

**Debriefing Interview after each Teaching Episode and after the Instructional Unit**

1. What are some ways that you have noticed children learning Algebra?
2. In general, what do you do to teach Algebra? What curriculum do you use? What are some strengths and weaknesses?
3. What do you look for as evidence as student learning?
4. What are some design features should be included in the algebra curriculum?
5. How do you think the lesson/unit went?
6. What went well in the lesson/unit? Explain why you think it went well.
7. If you taught this lesson/unit again, what would you do differently?
8. Based on your experience as a teacher, were the students learning? How do you know?
9. What do you plan to teach tomorrow and how will the lesson be adjusted based on today's lesson?
10. Describe anything else that you would like to add about this lesson/unit.