University of Nevada

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Fractal Applications to Soil Hydraulic Properties

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Hydrology/Hydrogeology.

by

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ABSTRACT

Three manuscripts are presented describing the application of fractal mathematics to soil hydraulic properties. The results indicate that the pore space of many field soils can best be represented by a self-similar or fractal geometry. The solid phase of the soil, i.e. the soil grains, is less amenable to fractal scaling and only a narrow range of typically encountered field soils are likely to be well described by fractal geometry. In the first manuscript, a simple self-similar model, the Sierpinski carpet is used to represent the pore number and pore size distribution of typical field soils. The soil water retention function is theoretically developed and shown to be equivalent to the power law empirical model for retention developed by Brooks and Corey (1964) and Campbell (1974). These results are extended to data reported on field soils and a relation between the fractal dimension and soil texture is presented. The results indicate an increasing fractal dimension with finer soil texture.

In the second manuscript, fractal scaling arguments are used to develop a theoretical basis for Arya and Paris' (1981) curve fitting coefficient, α. This term is shown to be equivalent to the fractal dimension of the pore trace and is consistent with the ideas of a scale-dependant tortuosity (Wheatcraft and Tyler, 1988). To estimate the fractal dimension, the fractal scaling often seen in particle size distributions (PSD), is utilized as a surrogate measure of the pore space fractal dimension. Ten soils were analyzed for fractal behavior, five reported by Arya and Paris (1981) and five reported by Mualem (1976). Of these ten soils, nine clearly showed fractal scaling in their particle number verses size. Good agreement between measured and predicted water retention was observed in nine out of the ten soils when the fractal dimension of the pore trace was estimated from the particle size distribution.

The third manuscript examines, in detail, the concepts of self-similarity in particle size distributions. It is shown that self-similarity in grain number verses grain size may be limited to a narrow range of typically encountered field soils. Theoretical results show that the fractal dimension for PSDs must range between 0.0 and 3.0, with typical soils ranging from 2 to 3. The analysis suggests that most soils do not behave as fractal *porous media,* rather only the void space (porosity) of the soil displays fractal behavior. This conclusion suggests that fractal will play an important role in the estimation of hydraulic and transport properties of soils.

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CHAPTER 1

INTRODUCTION

In the following chapters, three manuscripts are presented discussing the concepts of fractal mathematics and its applications to the understanding of soil hydraulic properties, specifically the retention of water under capillary tensions.The burgeoning field of fractal mathematics has seen little application to date in the study of soil physical properties and it is hoped that the work presented here will shed new light on the understanding of fluid/solid interactions in the strongly disordered material we call soil. The results of this research suggest in many cases, orderly prediction of hydraulic properties can be made when self-similar transformations such as those embodied in fractal concepts are invoked.

The work to follow represents three separate manuscripts on the study of fractal behavior in soils. The manuscripts represent a continuum of ideas, beginning with a theoretical approach to soil porosity, followed by an application to an existing model of soil porosity and grain structure and finishing with a discussion of the relationship between soil grain structure and soil porosity. The first (Chapter 2) presents a simple theoretical model of fractal pore size and pore number distribution which is theoretically equivalent to the empirical water retention relations developed by Brooks and Corey (1964) and Campbell (1974). This paper will appear in Water Resources Research in the summer of 1990.

The second paper (Chapter 3) represents an application of fractal mathematics to the understanding of the scale dependant tortuosity embodied in the the Arya and Paris (1981) model of water retention. This paper clearly shows that the empirical coefficient used in the Arya and Paris model is the fractal dimension of the pore trace. The paper then suggests a relation between the grain and pore size distribution to estimate this fractal dimension. This paper appeared in Soil Science Society of America Journal in July–August, 1989.

The third and final manuscript (Chapter 4) looks into detail at the implications of the relationship between grain and pore size for field soils. The simple relation suggested in Chapter 3 is shown to be a special case resulting in the development of both fractal scaling in the grain size distribution and the pore size distribution. This special case results in the development of a *fractal porous medium,* which is shown to be limited to a narrow rang of typically encountered field soils. This manuscript will be submitted to Soil Science Society of America Journal.

CHAPTER 2

FRACTAL PROCESSES IN SOIL WATER RETENTION

Scott W. Tyler
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ABSTRACT

Numerous empirical models exist for soil water retention and unsaturated hydraulic conductivity data. It has generally been recognized that the empirical fitting coefficients in these models are somehow related to soil texture. However, the fact that they are empirical means that elaborate laboratory experiments must be performed for each soil to obtain values for the parameters. Moreover, empirical models do not shed insight into the fundamental physical principles that govern the processes of unsaturated flow and drainage. We propose a physical conceptual model for soil texture and pore structure that is based on the concept of fractal geometry. The motivation for a fractal model of soil texture is that some particle size distributions in granular soils have already been shown to display self-similar scaling that is typical of fractal objects. Hence it is reasonable to expect that pore size distributions may also display fractal scaling properties. The paradigm that we use for the soil pore size distribution is the Sierpinski carpet, which is a fractal that contains self similar "holes" (or pores) over a wide range of scales. We evaluate the water retention properties of regular and random Sierpinski carpets and relate these properties directly to the Brooks and Corey (or Campbell) empirical water retention model. We relate the water retention curves directly to the fractal dimension of the Sierpinski carpet and show that the fractal dimension strongly controls the water retention properties of the Sierpinski carpet "soil". Higher fractal dimensions are shown to mimic clay-type soils, with very slow dewatering characteristics and relatively low fractal dimensions are shown to mimic a sandy soil with relatively rapid dewatering characteristics. Our fractal model of soil water retention removes the empirical fitting parameters from the soil water retention models and provides parameters (fractal dimension) which are intrinsic to the nature of the fractal porous structure. The relative permeability functions of Burdine and Mualem are also shown to be fractal directly from the fractal water retention results.

INTRODUCTION

The analysis and estimation of soil water retention properties has received considerable attention for more than 60 years (Gardner et al., 1922). The literature contains many articles describing the retention process of various soils (for example, Mualem, 1976), the relationship between soil texture and retention properties (e.g., Gupta and Larson, 1979), and the functional form of the retention data (Brooks and Corey, 1964; van Genuchten, 1980; etc.).

In recent years, considerable attention has been focused towards the development of functional relationships between water content and capillary pressure for modeling purposes and the development of simple methods to predict the hydraulic conductivity of the soils (van Genuchten and Nielsen, 1985). Many functional forms have been proposed to represent soil retention data, each with varying degrees of success. Lacking in most of these analyses however, is a development from fundamental principles. Most functional forms have been empirical, curve-fitting equations which produce the lowest mean-square error and highest r-squared for actual soil data. Other functional forms (for instance, van Genuchten, 1980) have been developed to be compatible with the conductivity models developed by Burdine (1953) and Mualem (1976). Such models, however, tend to require a greater number of empirical curve fitting parameters and there is less emphasis on their physical significance. A fundamental problem with all empirically-based relationships is that laboratory measurements must be made for each soil to determine the degree of confidence in the empirical relation. If relationships can be obtained that have a physical basis, it may be possible to determine retention parameters via indirect methods such as textural analysis.

The Brooks and Corey (1964) or Campbell (1974) power-law function has been widely used to predict the water retention properties of soils. The two functions are:

$$\left(\frac{\Theta}{\Theta_s}\right) = \left(\frac{\psi}{\psi_a}\right)^{-\lambda} \quad \text{Brooks – Corey} \tag{1a}$$

$$\left(\frac{\psi}{\psi_a}\right) = \left(\frac{\Theta}{\Theta_s}\right)^{-b_c} \quad \text{Campbell} \tag{1b}$$

where Θ_s and ψ_a are the saturated water content and air entry pressure, respectively. These two functions are equivalent as:

$$b_c = \frac{1}{\lambda} \tag{2}$$

The fitting parameter, b_c, has been called the pore-size distribution index although little consideration has been given to its physical significance.

The power-law function (Eq. 1a or 1b) has considerable advantages in its simplicity and integrability in conductivity models. The function has been shown to accurately fit retention properties in the dry regions, however it often fails to represent the slope of many experimental results near saturation.

The value of b_c has been shown by many workers to be related to soil texture. Average values of b_c reported by Clapp and Hornberger (1978) range from 4.05 for sand to 11.5 for clay soils. No theoretical model has been developed to predict b_c based from basic soil properties, however. Our objective in this paper is to develop an approach to understanding the physical significance of the fitting exponent and to simplify the actual soil pore structure for the prediction of unsaturated conductivity.

In the following sections, we employ simple fractal models to show the relationship of b_c to the geometry and distribution of soil pore structure. Using these models, based upon the Sierpinski carpet, we suggest that the geometry of complex porous media may be represented with fractal scaling concepts. In the first section, we introduce fractal concepts and develop retention properties for Sierpinski carpets, a pure fractal form. In the second section we then develop an approximate solution to pore–size/pore–number distribution for the Sierpinski carpet and show the soil retention equation of Brooks and Corey (1964) and Campbell (1974) to be based upon fractal concepts of pore distribution. The new fractal model of retention (instead of a fitting equation) is then used to estimate the hydraulic conductivity of porous media based upon the models of Burdine (1953) and Mualem (1976).

FRACTAL CONCEPTS

Fractals and the concepts of self–similar scaling have been applied to a wide range of natural processes (Feder, 1988). The basic premise upon which fractal concepts are based is the notion of self–similarity. The term self–similarity (or statistical self–similarity) implies that regular (or statistically regular) patterns appear in nature at all scales of observation. For example, a coastline exhibits statistical self-similarity since irregularities (bays, estuaries, wave scallops) can be found at any scale of observation. It has been found that the traditional notion of length in this case is meaningless without specifying the scale at which the length measurement was made. The length of the coastline, $L(\epsilon)$, as a function of the scale of measurement, ϵ, has been found (Mandelbrot, 1983) to obey a power–law relation of the form:

$$L(\epsilon) = F\epsilon^{1-D} \tag{3}$$

where D represents the fractal dimension and F is a constant. It is clear that the dimension may be estimated (3) from the slope of a log–log plot of L versus ϵ. The reader is referred to Feder (1988); Wheatcraft and Tyler (1988); and Jacquin and Adler (1987) for a more complete review of fractal concepts as they apply to geologic systems.

In recent years, a great deal of attention has been placed on fractal scaling in porous media. In detailed studies of solid–pore interfaces, Katz and Thompson (1985); Thompson et al. (1987); Krohn (1988a, b) have shown that these interfaces can often be described with fractal scaling. They found that fractal scaling was valid at microscopic scale for several types of geologic porous media.

Similar power–law relations have been reported for particle–size distributions (Turcotte, 1986; Tyler and Wheatcraft, 1989) where instead of a measurement of length, the measured quantity is the number of particles greater than a specific size, ϵ. In these approaches, the scale dependent measure, $L(\epsilon)$ is the number of particles, while the measurement scale, ϵ, is a characteristic particle diameter.

APPLICATION TO SYNTHETIC MEDIA

To investigate the impacts of fractal scaling upon water retention relations, a two–dimensional representation of a porous medium was developed based upon the fractal Sierpinski carpet. The Sierpinski carpet represents a fractal geometric pattern in which successively smaller "holes" are cut out of the plane to produce a pattern which is self–similar at all scales smaller than the initial size of the carpet. The recursion, if carried to infinity, yields a pattern which is everywhere filled with holes.

The carpet of size a by a is formed by initially subdividing the carpet into b_1^2 subsquares, each of size $\frac{a}{b_1}$ by $\frac{a}{b_1}$. From the original carpet, l_1, subsquares are removed and represent a pore of size $\frac{a}{b_1}$ by $\frac{a}{b_1}$. The remaining $b_1^2 - l_1$ subsquares are then each divided into b_1^2 subregions and l_1 subsquares are removed from each of the original subregions. Such a recursion algorithm, if carried out to M steps where M is very large, results in a carpet everywhere filled with holes (pores) of all sizes, with a predominance of small holes. Figure 1 shows a typical Sierpinski carpet carried to four levels of recursion and begins to show the large number of small holes developed by the algorithm. In this carpet, the dark areas represent pore spaces.

Once the carpet is generated, its fractal scaling is borne out in its porosity. The number of squares of size $(\frac{a}{b})^2$ needed to cover the pores equal to or larger than $(\frac{a}{b})^2$ is given by l. Subscripts are removed from both b and l to indicate that they may take on a range of values. The number needed to cover the solid phase is therefore $b^2 - l$. It can be shown (Mandelbrot, 1983) that there exists a power–law scaling between the scale of measurement (b) and the solid phase of the form:

$$b^D = b^2 - l \tag{4a}$$

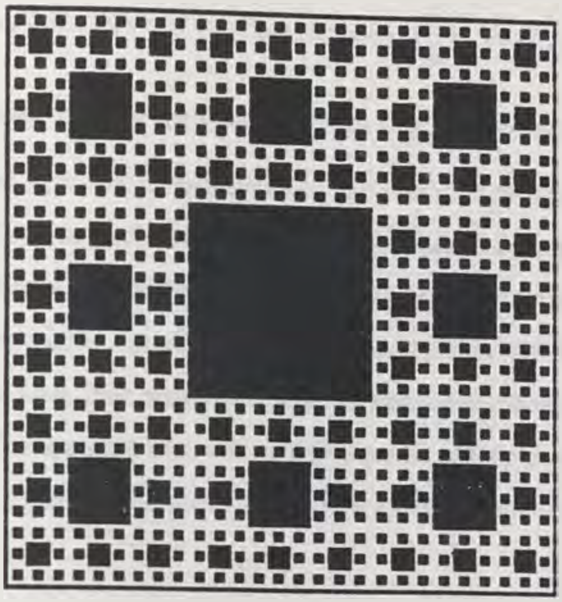


Figure 1. Sierpinski carpet carried to four levels of recursion. The fractal dimension for this carpet is ~1.893.

where D represents the fractal dimension of the carpet. The fractal dimension is easily obtained by rearranging Eq. (4a) as:

$$D = \frac{\log(b^2 - l)}{\log(b)} \qquad (4b)$$

The fractal dimension is determined by the recursion algorithm. As an example, Figure 1 shows a carpet with four levels of recursion. The initial algorithm divided the carpet into (3^2) subregions and one subregion removed. Using Eq. (4b), the fractal dimension is approximately 1.893. At the second iteration, the region is subdivided into (9^2) subregions and 1 subregion is removed from the remaining 8 solid regions. The number of subregions of size $(\frac{a}{9})^2$ needed to cover the open area larger than $(\frac{a}{9})^2$ may be obtained by rearranging Eq. (4a):

$$l = b^2 - b^D \tag{5}$$

In this case, $l = (9^2) - (9)^{1.893}$ or 17 squares of size $(\frac{a}{9})^2$. The fractal scaling described by Eq. (5) may be used to estimate the total cross-sectional area of pores greater than any arbitrary size $(\frac{a}{b})^2$.

Our subsequent hydraulic investigations on the retention properties of Sierpinski carpets are based upon a class of porous media shown in Figure 2. The Sierpinski carpet forms a lattice-like bundle of capillary tubes with tube size and distribution determined by the recursion algorithm. This approach is in contrast to that proposed by Adler (1986) and Adler and Jacquin (1987) in which the stippled phase of Figure 2 was chosen to have finite permeability and the void phase or holes shown in Figure 2 was considered impermeable. Our approach allows direct development of simple analytical solutions to retention and conductivity and eliminates the need for numerical iterations, although it ignores pore interactions.

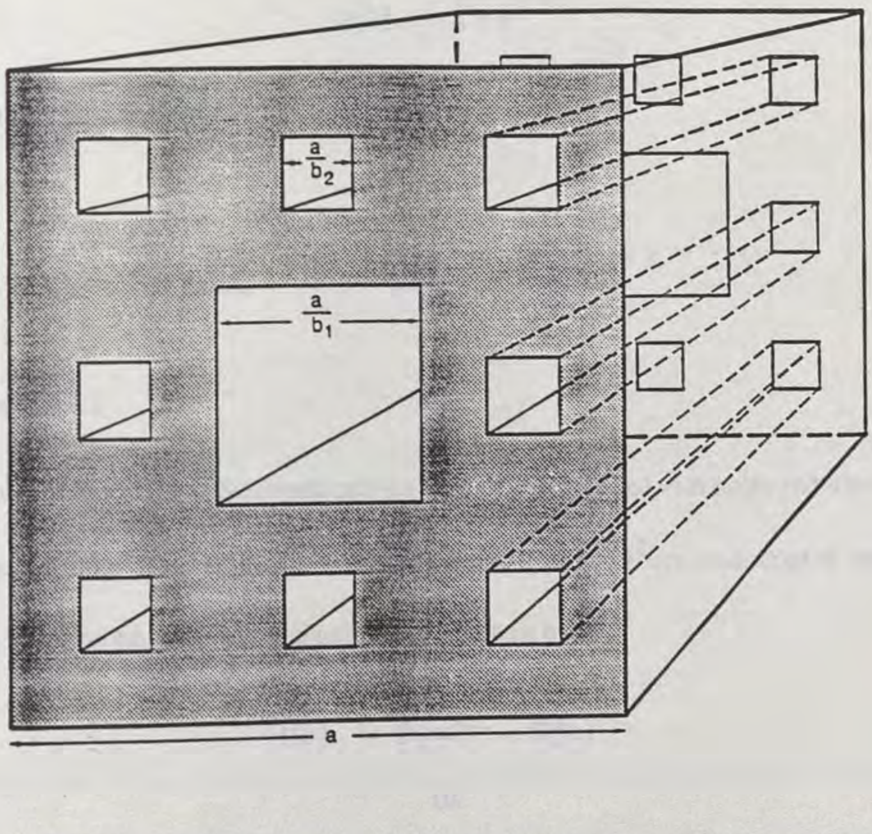


Figure 2. Conceptual view of a Sierpinski carpet used to simulate a porous medium

The areal porosity of the carpet, $\Phi(b)$ as measured by squares of characteristic size $(\frac{a}{b})^2$ is given by:

$$\Phi(b) = \frac{(\frac{a}{b})^2\, l}{a^2} = \frac{l}{b^2} \tag{6}$$

In Eq. (6) only holes (pores) of size greater than $(\frac{a}{b})^2$ are counted towards the porosity. It is important to note that l represents the number of squares of size $(\frac{a}{b})^2$ needed to cover the open area or porosity. Pores which are smaller than $(\frac{a}{b})^2$ are not counted. The size defined by $(\frac{a}{b})^2$ is analogous to the resolving power of a microscope. Pores smaller than the resolving power are not discernible and are therefore not included in the porosity estimation.

Combining Eqs. (4a) and (6) yields a simple form for the porosity:

$$\Phi(b) = 1 - b^{D-2} \tag{7}$$

In the limit as b goes to infinity:

$$\lim_{b \to \infty} \Phi(b) = \lim_{b \to \infty} (1 - b^{D-2}) = 1 \tag{8}$$

where $1 \leq D < 2$.

In contrast to soils, true Sierpinski carpets are everywhere filled with holes and therefore have a porosity equal to unity. If pores greater than a characteristic size $(\frac{a}{b})^2$ are dewatered by applying a tension to the water column, the water content $\Theta(b)$ is given by:

$$\Theta(b) = \Phi(\infty) - \Phi(b)$$

or

$$\Theta(b) = 1 - \Phi(b) = b^{D-2} \tag{9}$$

The water retention characteristics for the Sierpinski carpet can easily be developed given that the water content, as a function of b, may be estimated from Eq. (9). First, we assume that the pores, as a function of the characteristic size, b, will drain at a pressure prescribed by the capillary rise equation:

$$\psi(b) = \frac{2\sigma\cos\beta}{R(b)} \tag{10}$$

where R represents the radius of the pore, σ is the surface tension of the fluid interface and β is the contact angle.

In order to simplify the analysis, we next assume that instead of pores with square cross sections, the pores are cylindrical. The pore radius, $R(b)$, is chosen such that the circular cross section is equivalent to the original square area, i.e.:

$$\left(\frac{a}{b}\right)^2 \equiv \pi\left[R(b)^2\right]$$

or

$$R(b) = \frac{1}{\sqrt{\pi}}\left(\frac{a}{b}\right) \tag{11}$$

Replacing Eq. (11) in Eq. (10) and solving for b yields:

$$b = \frac{\psi(b)a}{2\sigma\sqrt{\pi}\cos\beta} \tag{12}$$

By replacing b in Eq. (9) with the expression in Eq. (12) we yield a simple expression for the water retention relation:

$$\Theta(b) = \left(\frac{a}{2\sigma\sqrt{\pi}\cos\beta}\right)^{D-2} (\psi)^{D-2} \tag{13}$$

Most retention data display a region of complete saturation between atmospheric pressure and some critical tension, often referred to as the air entry pressure, ψ_a. If we assume that the carpet remains fully saturated until ψ_a is exceeded, Eq. (13) may be written in this range as:

$$\Theta(b_1) = 1 = \left(\frac{a}{2\sqrt{\pi}\sigma\cos\beta}\right)^{D-2} (\psi_a)^{D-2} \tag{14}$$

where b_1 represents the largest pore size present in the carpet and ψ_a represents the traditional definition of an air entry pressure. Dividing Eq. (14) into Eq. (13) yields the familiar Brooks and Corey or Campbell power–law expression for the water retention function:

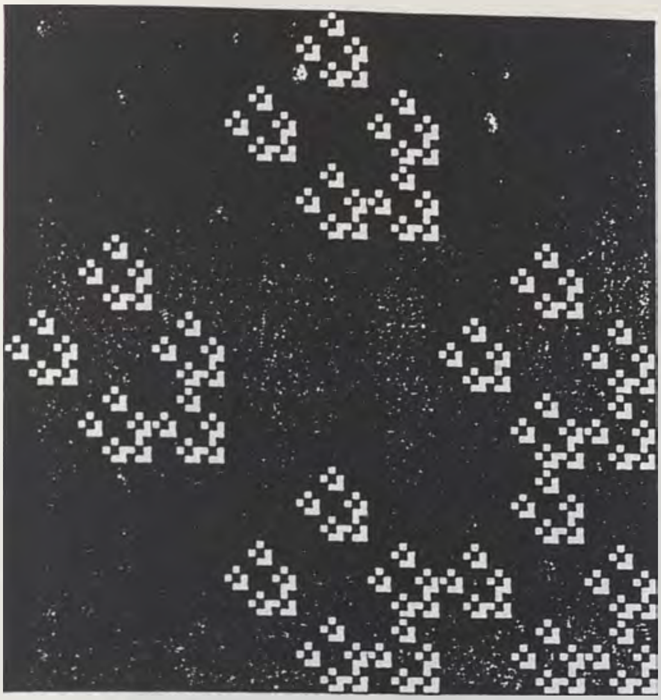
$$\psi = \psi_a\left(\frac{\Theta}{\Theta_s}\right)^{\frac{1}{D-2}} \tag{15}$$

where D is the fractal dimension of the carpet, $\frac{1}{D-2}$ is equivalent to $-b_c$ of the Campbell (1974) model or $-\frac{1}{\lambda}$ of the Brooks and Corey (1964) model, and θ_s is equal to unity.

Water retention data were calculated for four example carpets shown in Figures 3a–d using Eqs. (9) and (15). The carpets are shown in increasing order of fractal dimensions. The first two carpets (3a and 3b) both have equivalent b_1 values of 3 and result in initially large pores. In carpet 3a, l_1 is equal to 4, while carpet 3b has an initial l_1 value of 1. This results in carpet 3a draining a significant portion of its water content under low capillary pressures. Carpet 3b dewaters more slowly since fewer larger pores are present.

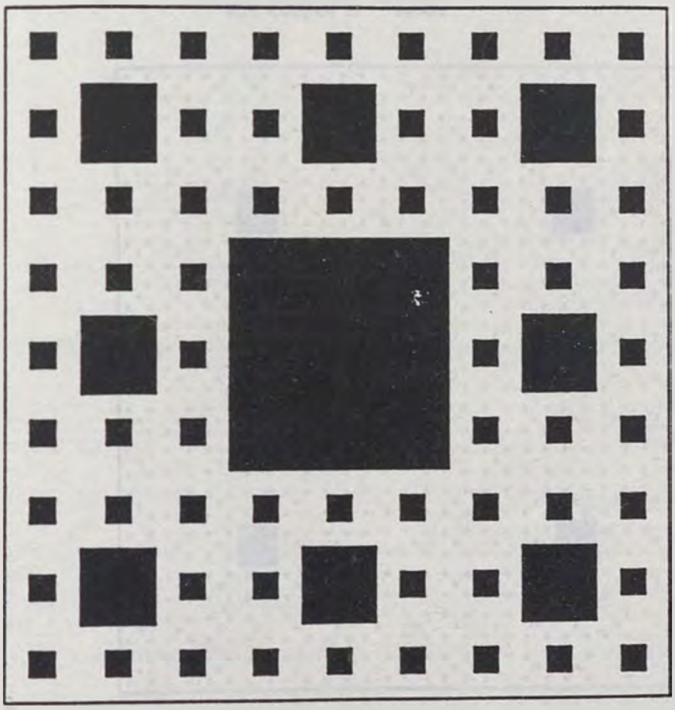
Carpets 3c and 3d both begin with b_1 equal to 15. This results in a smaller initial pore size than the carpets shown in Figures 3a–b. Consequently, the air entry pressure is higher by the factor $\frac{b_1(3c,\ 3d)}{b_1(3a,\ 3b)}$ *or* $\frac{15}{3}$. The rate of drainage as a function of capillary pressure is larger in carpet 3c than 3d since the initial number of pores of size b_1 is larger in carpet 3c than in carpet 3d.

Water retention curves for the Sierpinski carpets shown in Figures 3a–d are shown in Figure 4. Recalling from Eq. (4) that the value of D represents a measure of the ratio between characteristic pore size (b^2) and pore area (l); the fractal dimension may be considered a measure of the soil texture. A value approaching unity suggests that the pore space is dominated by large pores, i.e., $b^2 - l$ approaches b. The cross section of the carpet takes on a very sparse appearance at large scales of measurement. In essence, the pore space of carpet does a poor job of filling the plane. As D approaches two, the pore space grows only slowly with decreasing measurement scale, i.e., large b. For two carpets with equal b_1 values (corresponding to equal air entry pressures), the carpet with the largest l_1 value will dewater faster with respect to capillary pressure since a larger percentage of the water is held in large voids. For carpets with dissimilar values of b_1, initial dewatering will occur first in the carpet with the smaller b_1 value (Eq. 14). The fractal dimension therefore, strongly controls the water retention properties of the Sierpinski carpet. Each Sierpinski carpet shown in Figures 3a–d has a slightly different generating algorithm and therefore a different fractal dimension. The retention curves presented in Figure 4, however, show the importance of the fractal dimension in determining the shape of the retention function. For a fractal



Carpet generator is defined by: 

Figure 3a. Sierpinski carpet for Soil A. The fractal dimension of the carpet is ~1.46.



Carpet generator defined by: 

Figure 3b. Sierpinski carpet for Soil B. The fractal dimension of the carpet is ~1.89.

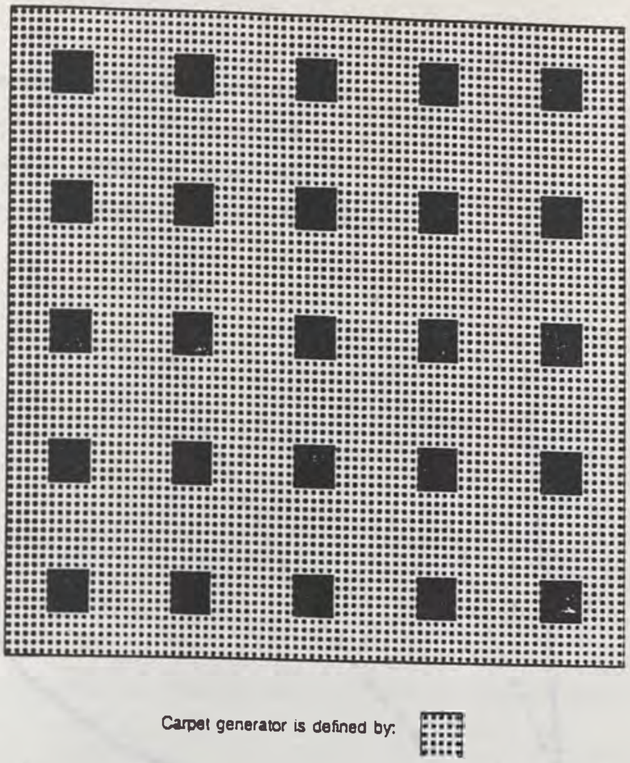


Figure 3c. Sierpinski carpet for Soil C. The fractal dimension of the carpet is ~1.96.

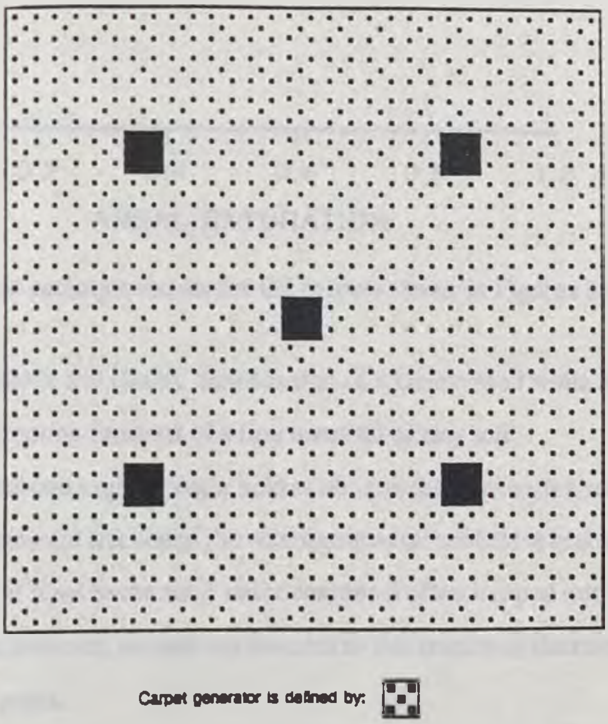


Figure 3d. Sierpinski carpet for Soil D. The fractal dimension of the carpet is ~1.99.

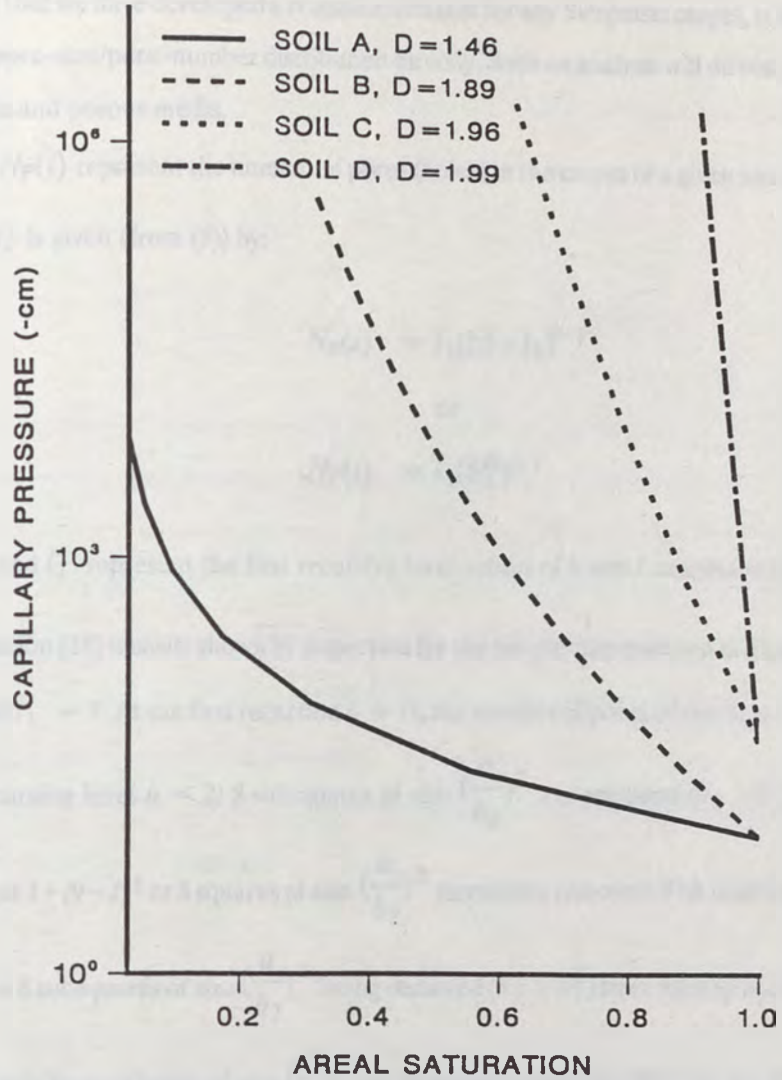


Figure 4. Calculated retention curves for the carpets shown in Figures 3a–d.

dimension near 1.5, the synthetic soil closely matches that of a coarse sand while a fractal dimension approaching 2 produces a retention function of a fine textured or clay soil.

The analysis presented above neglects water held as films on the pore walls already drained. Water held in films will become a dominant fraction of the volumetric water content in real soils at low capillary pressures. The contribution of films to the total water content is often lumped into the residual water content term. In this analysis, however, we limit our concern to that portion of the retention curve where water is filling the capillary pores.

PORE NUMBER DISTRIBUTION

Now that we have developed a retention relation for any Sierpinski carpet, it is of interest to investigate the pore-size/pore-number distribution directly. Such an analysis will directly lead to applications in real soils and porous media.

Let $N_P(i)$ represent the number of pores (holes) in the carpet of a given size $R(i)$. At each recursion, $N_P(i)$ is given (from (5)) by:

$$N_P(i) = l_1(b_1^2 - l_1)^{i-1}$$

or

$$N_P(i) = l_1(b_1^D)^{i-1} \qquad (16)$$

where b_1 and l_1 represent the first recursive level values of b and l, respectively.

Equation (16) is easily shown by inspection for the simple carpet shown in Figure 1. For this carpet, $b_1 = 3$ and $l_1 = 1$. At the first recursion $(i = 1)$, the number of pores of size b, is $1 \bullet (9-1)^0$ or 1. At the second recursion level $(i = 2)$ 8 subsquares of size $(\frac{a}{b_2})^2$ are removed $(b_2 = b_1^2)$. Equation (16) also predicts that $1 \bullet (9-1)^1$ or 8 squares of size $(\frac{a}{b_2})^2$ have been removed. The third level of recursion $(i = 3)$ results in 8 subsquares of size $(\frac{a}{b_3})^2$ being removed $(b_3 = b_1^3)$ from each of the previous subsquares. This results in 64 new holes of size $(\frac{a}{b_3})^2$, in agreement with Eq. (16), i.e., $1 \bullet (9-1)^2$.

The total number of squares (pores), N_P, of size $(\frac{a}{b_i})^2$ or greater is simply:

$$N_P((\frac{a}{b})^2 > (\frac{a}{b_i})^2) = \sum_{j=1}^{i} l_1(b_1^D)^{j-1}$$

or

$$N_P[(\frac{a}{b})^2 > (\frac{a}{b_i})^2] = \frac{l_1}{b_1^D} \sum_{j=1}^{i} b_1^{Dj} \qquad (17)$$

Recalling that the Sierpinski generating algorithm is developed using:

$$b_i = (b_1)^i \tag{18}$$

and using Eq. (11) allows us to rewrite Eq. (17) to show the relation between the number of pores equal to or larger than a characteristic radius, $R(i)$:

$$N_P(R > R(i)) = l_1 \left(\frac{a}{\sqrt{\pi} b_1} \right)^D \sum_{j=1}^{i} R(j)^{-D} \tag{19}$$

It is interesting to compare Eq. (19) and Eq. (9). While the porosity or open area described in Eq. (9) shows power-law scaling based solely upon the smallest pore size, the cumulative number of pores (Eq. 19) does not show simple power scaling behavior due to the summation process. Equation (19) shows that the number of pores greater than a characteristic size, $R(i)$ is a polynomial containing information from all pore-size classes larger than $R(i)$. As such, Eq. (19) is not as useful or simple as Eq. (9). Fortunately, it is obvious that the i^{th} term in the series in Eq. (19) dominates the sum. This fact is easily visualized in Figure 1 where the ratio of the number of smallest pores to the total number of pores goes as 8/9, 64/73, 512/585. Approximately 88 percent of the pores are of the smallest size. Such dominance suggests that an approximation of the form:

$$N_P(R > R(i)) \mathrel{\tilde{\propto}} R(i)^{-D} \tag{20}$$

could be used in place of Eq. (19). Equation (20) will be advantageous since the pore number distribution is solely dependent upon the smallest pore size, and unlike Eq. (19), is not subject to the distribution of pore sizes larger than $R(i)$.

To show the validity of Eq. (20), two examples used are shown in Figures 5a and 5b. Pore number relations for two soils representing a sand (D = 1.465) and a clay (D = 1.994) were generated using Eq. (19) and are represented by crosses. Shown plotted along with the calculated data is Eq. (20) using the true input fractal dimension. From the two plots, it is clear that little error is associated with the use of Eq. (20) and errors are likely to be masked by errors in measurement. It is also interesting to note that Eq. (20) was fit to the calculated data to determine the error in fractal dimension estimation. For the sandy soil (Figure 5a), the estimated dimension was 1.464 while the actual dimension was 1.465. The clay soil dimension was estimated to be 1.992 while the input value was 1.998. Such errors in fractal dimension

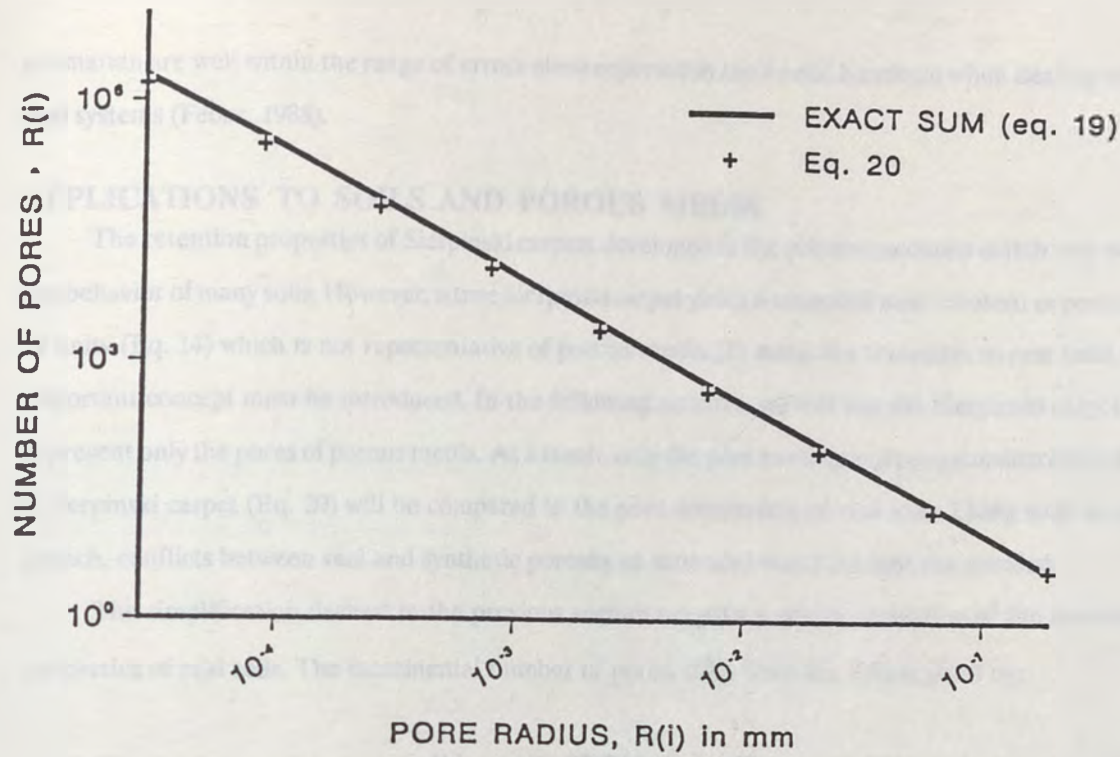


Figure 5a. Comparison of exact pore size/number distribution with approximation developed in (20) for a carpet of fractal dimension of ~1.465.

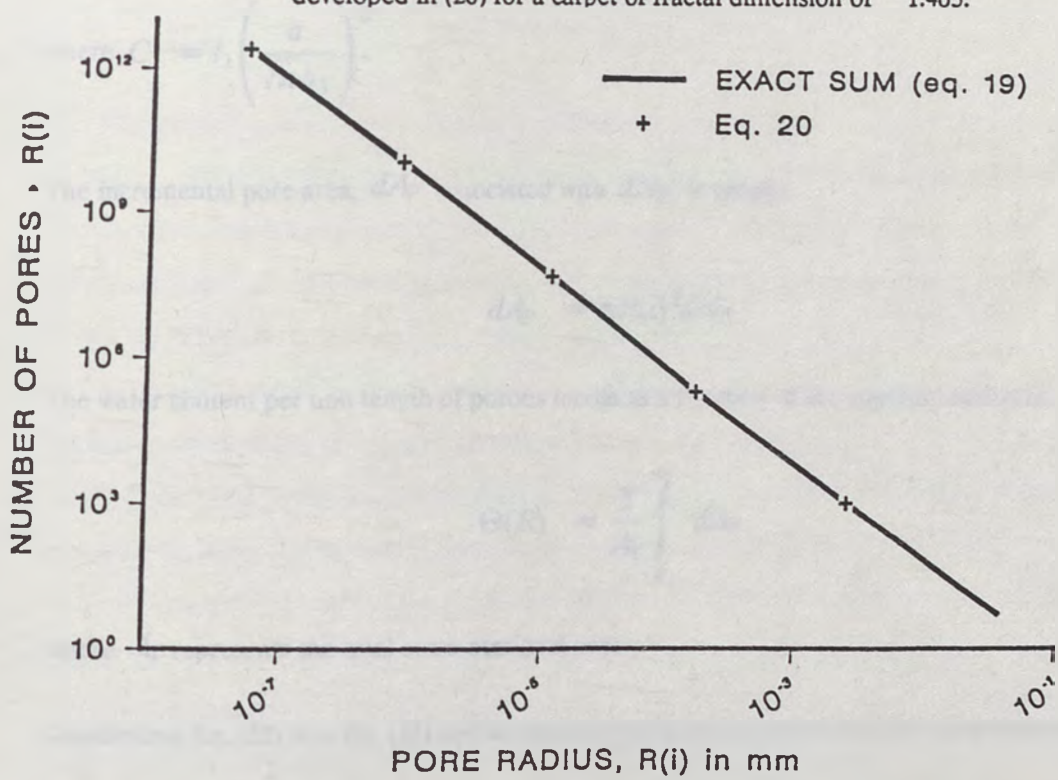


Figure 5b. Comparison of exact pore size/number distribution with approximation developed in (20) for a carpet of fractal dimension of ~1.998.

estimation are well within the range of errors often reported in the fractal literature when dealing with real systems (Feder, 1988).

APPLICATIONS TO SOILS AND POROUS MEDIA

The retention properties of Sierpinski carpets developed in the previous sections match very well the behavior of many soils. However, a true Sierpinski carpet yields a saturated water content or porosity of unity (Eq. 14) which is not representative of porous media. To make the transition to real soils, an important concept must be introduced. In the following sections, we will use the Sierpinski carpet to represent only the pores of porous media. As a result, only the pore number and pore size distribution of a Sierpinski carpet (Eq. 20) will be compared to the pore distribution of real soils. Using such an approach, conflicts between real and synthetic porosity or saturated water content are avoided.

The simplification derived in the previous section suggests a simple derivation of the retention properties of real soils. The incremental number of pores, dN_P from Eq. (20) is given by:

$$dN_P = -CDR(i)^{-D-1}\ dR \tag{21}$$

where $C = l_1 \left(\frac{a}{\sqrt{\pi} b_1} \right)^D$.

The incremental pore area, dA_P associated with dN_P is simply:

$$dA_P = \pi R(i)^2 dN_P \tag{22}$$

The water content per unit length of porous media as a function of the capillary radius is:

$$\Theta(R) = \frac{1}{A_T} \int_R^o dA_P \tag{23}$$

where A_T represents the total cross-sectional area.

Substituting Eq. (22) into Eq. (23) and integrating yields an expression for the water content:

$$\Theta(R) = \frac{1}{A_T} \left(\frac{\pi DC}{2-D} \right) R^{2-D} \tag{24}$$

The saturated water content, Θ_s is given by:

$$\Theta_s = \frac{1}{A_T}\int_{R_a}^{o} dA_P = \frac{1}{A_T}\left(\frac{\pi DC}{2-D}\right)R_a^{2-D} \tag{25}$$

where R_a represents the pore size associated with the air entry pressure. Dividing Eq. (24) and Eq. (25) and relating R to ψ via the equation of capillary rise yields the Brooks and Corey or Campbell power–law expression for the water retention function:

$$\frac{\Theta}{\Theta_s} = \left(\frac{\psi}{\psi_A}\right)^{D-2} \tag{26}$$

where D–2 is analogous to Campbell's $-\frac{1}{b_c}$ and θ_s represents the saturated volumetric water content of porous medium.

These results indicate that the power–law function suggests fractal scaling of pore size and number of pores. Clapp and Hornberger (1978) report values of b to range from 4.05 to 11.4. These correspond to fractal dimensions ranging from 1.75 to 1.91 in good agreement with the Sierpinski carpet range of dimensions. These results closely agree with the data shown in Figure 4 and suggest that finer textured soils will have characteristically higher fractal dimensions.

The simulation of soil water retention with the geometrically regular Sierpinski carpet improves the basic understanding of retention properties by suggesting a simple scaling relation for pore size and pore number. The power–law model of either Brooks and Corey or Campbell has gained wide acceptance for modeling retention data, particularly outside of the nearly saturated region. In this range, the analogy between real soils and Sierpinski carpets indicates that the pore size/number distribution may follow fractal scaling concepts. In soils which do not display a distinct air entry pressure, the pore size/number relations do not appear to follow fractal concepts. In these soils, particular fine textured soils, other forces beside capillary forces may be influencing the retention properties. Such processes as consolidation may also affect experimental results of retention. The use of Eq. (20) to estimate the pore–number relation may also affect the simulated retention curves, particularly at the very wet end of the curve.

It is important to recognize the physical significance of the parameters a, b_1, and l_1 in the fractal dimension calculation. The term a can be directly linked to the concept of a representative elementary volume (REV). Above the scale defined by a^2, the medium's pore-number and pore-size distributions are additive and the synthetic porous medium resembles a tiled floor (often called a spatially periodic porous medium). Below the size defined by a^2, any samples of the pore structure will fail to provide a complete picture of the pore structure. In this manner, the carpet size, a, defines the REV for the pore structure. The characteristic length of the REV of the porous medium is simply defined as the square root of the ratio of a^2 to the areal porosity of the actual soil or porous medium. The characteristic scale may also be considered as the "window of observation" (Cushman, 1987) through which the porous medium is viewed. Similarity at scales above this window (be they fractal or spatially periodic) cannot be discerned. The term b_1 is directly linked to the air entry pressure and is a measure of the characteristic size of the largest pores in the soil. The term l_1 is related to the rate of change of the water content as a function of the capillary pressure. The relationship between these two initial conditions, as defined by the fractal dimension (Eq. 4b) then defines the retention properties over the full range of tension and water content for a fractal porous medium. This behavior suggests that for many soils displaying power law scaling, particularly unstructured soils, retention data collected over a small range in tensions can be used to provide insight into the pore structure over a much wider range of tensions. If fractal scaling of the pore structure can be shown, higher confidence can be placed on such extrapolations.

EXTENSIONS TO HYDRAULIC CONDUCTIVITY

Several theoretical models for prediction of hydraulic conductivity from water retention data of soils have been developed since the pioneering work of Childs and Collis-George (1950). Of these various models, the work of Burdine (1953) and Mualem (1976) have been most widely applied due to their suitability in deriving a closed form conductivity function (van Genuchten and Nielsen, 1985). These two methods differ significantly in their approaches towards pore interaction terms and, as a result, develop different integral expressions for the relative hydraulic conductivity. Closed form solutions of these integral expressions have been presented by Brooks and Corey (1964), Campbell (1974), Mualem (1976) and van Genuchten (1980) using the empirical power-law model. When the power-law model is used, the two techniques produce a power-law scaling for relative conductivity, both as a function of capillary pressure and water content. The two models differ slightly in the magnitude of the exponent. In light of the relationship between the empirical pore distribution parameter (Brooks and Corey's λ or Campbell's b_c) and the fractal dimension developed in this paper in Eq. (26), it is simple to insert this relationship into the relative conductivity models developed by Burdine (1953) and Mualem (1976). The functional relationships, in terms of water content and capillary pressure, are shown in Eqs. (27a, b) for the Burdine formulation and Eqs. (28a, b) for the Mualem formulation.

Burdine Model:

$$K_R(\Theta) = \left(\frac{\Theta}{\Theta_s}\right)^{\frac{2D-6}{D-2}} \tag{27a}$$

$$K_R(\psi) = \left(\frac{\psi}{\psi_a}\right)^{2D-6} \tag{27b}$$

Mualem Model:

$$K_R(\Theta) = \left(\frac{\Theta}{\Theta_s}\right)^{\frac{\frac{5}{2}D-7}{D-2}} \tag{28a}$$

$$K_R(\psi) = \left(\frac{\psi}{\psi_a}\right)^{\frac{5}{2}D-7} \tag{28b}$$

The conductivity models shown in Eqs. (27a, b) and (28a, b) represent the relative conductivity of a soil displaying fractal scaling in its pore size/number distribution. The conductivity is defined by the fractal dimension which, in turn, is controlled by b_1 and l_1. Both b_1 and l_1 control the retention curve behavior in the vicinity of the air entry pressure. These relationships suggest that data collected near saturation can be used to extrapolate soil behavior far beyond these low tension regimes. Since it is often more difficult to measure conductivity properties at very negative tensions, the fractal analysis presented in this paper suggests that behavior in the region may be estimated under the assumption of fractal or self–similar behavior.

CONCLUSIONS

The use of a simple fractal geometry (Sierpinski carpet) clearly shows the fractal scaling properties of soil pore size and distribution underlying the power function equation developed by Brooks and Corey and Campbell to water retention properties. For those soils where Eq. (1a) or (1b) are appropriate, the pore–size distribution is shown to be fractal. The fractal dimensions of typical soils range from 1.71 to 1.95 with the highest dimensions associated with the finest textured soils. The power–law function, while in use for many years, had not previously been physically based and little could be inferred from the empirical fitting coefficient. The application of fractal approaches suggests that a more complete understanding of the pore–size/pore–number distribution may be obtained by applying simple models such as the Sierpinski carpet to complex soils and porous media.

For soils displaying fractal pore distributions, the analyses shown in this paper suggest that soil retention properties may be estimated over a wide range of capillary pressures, based upon very limited data. The relationship of the fractal dimension to the initial recursion parameters b_1 and l_1 suggest that retention data can be estimated from soil behavior near saturation. The characteristic length of the carpet, a, may be shown to be directly related to the representative elementary volume concept for porous media. Both b_1 and l_1 represent soil behavior around the air entry pressure (the air entry pressure and the slope of the retention curve at the air entry pressure, respectively). Although there are experimental difficulties in this region, the fractal model suggested in this paper may reduce the need to conduct retention experiments over the wide range of capillary pressures traditionally conducted. Experiments conducted over a narrow range of tensions can easily be extended under the assumption of fractal scaling of the pore structure. This reduction in time–consuming laboratory efforts may lead to the concentration in efforts towards a better understanding of the impacts of variability of retention and conductive properties of field soils.

ACKNOWLEDGMENTS

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CHAPTER 3

APPLICATION OF FRACTAL MATHEMATICS TO SOIL WATER RETENTION ESTIMATION

Scott W. Tyler
Stephen W. Wheatcraft

ABSTRACT

In this paper, we present an analysis correlating the fitting parameter α in the Arya and Paris (1981) soil water retention model to physical properties of the soil. Fractal mathematics are used to show that α is equal to the fractal dimension of the pore trace and expresses a measure of the tortuosity of the pore trace. The fractal dimension of the particle-size distribution can be easily measured and related to the α parameter of the Arya and Paris model. By suggesting a physical significance of the coefficient, the universality of the model is greatly improved. Soil water retention data, estimated strictly from particle-size distributions, are proven to match measured data quite well. The fractal dimension of pore traces range from 1.011 to 1.485 for all but one soil tested.

INTRODUCTION

Numerous attempts have been made to relate particle-size distribution to soil water retention data (Hall et al., 1977; Clapp and Hornberger, 1978; Gupta and Larson, 1979). The relative ease with which this data may be attained, as well as the similarity in shape of the retention and cumulative distribution curves for soils, suggested such efforts were justified. In recent years, two approaches, Arya and Paris (1981) and Haverkamp and Parlange (1986), have been presented that show significant promise in relating the particle-size distribution data to the retention curve in non-swelling soils. These simplified techniques provide a valuable method for bridging the data availability gap. Traditional soil surveys provide a wealth of spatially variable particle-size distribution data, but are often lacking in retention data. Studies of flow and transport in field soils, however, require input data on the variability of hydraulic properties (retention and conductivity data). Techniques that can bridge this data gap are therefore critical in estimating the impacts of agriculture and industry on soil and ground-water quality.

The Arya and Paris model (reviewed below) has received some criticism (Haverkamp and Parlange, 1982, 1986; Arya and Paris, 1982) for its empiricism. In this paper, we significantly reduce this empiricism. We propose that the technique is physically sound and provides a valuable alternative when it is necessary to estimate soil water retention when such data are lacking. This paper focuses on the technique of Arya and Paris (1981) to apply physical significance and develop estimative techniques for their curve–fitting parameter, α. The physical significance is based upon the concepts of fractal mathematics and scaled similarities.

MODEL DESCRIPTION

Arya and Paris (1981) present a "physicoempirical" approach to water retention data. The underlying assumption of the model is that the soil particle size is related to a corresponding pore diameter. This forms the "physical" basis for the model. The model treats the soil as a bundle of capillary tubes. Each capillary tube corresponds to a user–defined particle–size class. The capillary tube volume is taken to be a function of the particle size, the weight fraction of the particle size and an empirical fitting coefficient, α. The fitting parameter is derived from least squares regression of the predicted water content to the measured water content. This term forms the "empirical" side of the model. As correctly pointed out by Haverkamp and Parlange (1982), this technique is somewhat sensitive to the user– defined particle–size classes. In this paper, we recognize this limitation, however, we show that the model is conceptually based on physical principles.

Upon division of the particle–size distribution data into M size fractions, the solid mass in the i^{th} particle class is equated to the mass of N_i spherical particles of radius R_i. Their volume, Vp_i is given by:

$$V_{p_i} = \frac{4}{3} \; N_i \; \pi \; R_i^3 \tag{1}$$

The volume of the voids, V_{v_i}, is represented by a single capillary tube of radius r_i:

$$V_{v_i} = \pi \; r_i^2 \; h_i \tag{2}$$

where h_i is the capillary tube length. The length (h_i) of a straight capillary tube in a cubic close–packed arrangement, as measured in units of R_i, is simply the product of the number of particles of radius R_i multiplied by their diameter ($2R_i$). Arya and Paris (1981) assume that the particles are not spherical and represent the true capillary pore length of the R_i class as:

$$h_i = 2 \; R_i \; N_i^{\alpha} \tag{3}$$

where α is an empirical constant between one and two. The use of Eq. (3) is justified by Arya and Paris to account for the nonspherical nature of the particles.

By combining Eqs. (1), (2), and (3), the authors arrive at an expression for the capillary tube radius, r_i:

$$r_i = R_i \left\{ \frac{2}{3} e \; N_i^{\,(1-\alpha)} \right\}^{1/2} \tag{4}$$

where e is the sample void ratio. The void ratio for each pore class is assumed constant and equal to the bulk sample void ratio. The tube radius is then related to the capillary pressure, ψ_i, via the equation of rise in a capillary tube. The water content corresponding to the capillary pressure ψ_i, is evaluated by summing the available pore space contained in all particle classes from the smallest class to the i^{th} class.

The technique does not account for hysterisis nor entrapped air. The latter condition may be overcome by assuming that any entrapped air is equally divided among all pore classes (Haverkamp and Parlange, 1986). By the application of the equation of capillary rise (and assuming a zero contact angle), the technique most closely models the main drying curve for a nonswelling soil.

EVALUATION OF THE α TERM

The power law relationship given by Eq. (3) indicates that as R_i decreases (assuming N_i increases proportionally) h_i will grow exponentially. This apparent length-increase behavior has received considerable attention through the subject of fractal mathematics (Mandelbrot, 1983).

The development of fractal mathematics has shown that highly irregular features such as coastlines, rivers and fractures (and capillary tubes) can be categorized and quantified in a fundamentally new and different way (Mandelbrot, 1983). One of the most important features of a fractal object is that its "degree of irregularity" is independent of scale.

In normal Euclidean geometry, we can easily measure the length of a straight line of length L. If our measuring unit is of length ϵ ($\epsilon \leq L$), then:

$$L(\epsilon) = N\epsilon^1 = constant \tag{5}$$

where N is the number of measuring units needed to cover the straight line and the exponent of unity is consistent with the topologic dimension of a line. This analogy may be expanded to two or three dimensions (a plane or cube, respectively) where the exponent reflects exactly the topological dimension (plane = 2.0 and cube = 3.0).

If the line we are measuring is irregular, Eq. (5) remains true except that the length, $L(\epsilon)$, for any ϵ is not constant. It has been found that the following relationship holds true for irregular lines, such as a coastline:

$$F = N\epsilon^{D} = constant \tag{6}$$

By analogy with Eq. (5), F is taken to be a measure of the line length which is independent of ϵ, and D is taken to be the dimension which yields the constant length (F) of the line.

By combining Eqs. (5) and (6), we obtain:

$$L(\epsilon) = F\epsilon^{1-D} \tag{7}$$

which can be thought of as a transformation relationship between the topological dimension of one and the fractal dimension of D for a fractal line. The reader is referred to Mandelbrot (1983) and Feder (1988) for a more complete review of fractal concepts.

Instead of the straight capillary tube approach of Arya and Paris (1981), fractal measures can be used to evaluate the pore length as a function of measuring scale. In this approach, smaller and larger particles line the pore wall. Figure 1a represents a Euclidean (nonfractal) pore wall whose length is independent of the scale at which it is measured. The volume of the pore space is easily measured and constant. Figure 1b, however, represents a more realistic soil pore. Both large and small grains line the pore. If we magnified any portion of the pore, we would see yet more small grains lining the pore wall. The volume of such a pore is therefore a function of the scale at which it is measured. Its length is also a function of scale. The pore length and radius are "measured" in a retention experiment by varying the tension applied to the soil water. An equivalent radius may be estimated from Laplace's equation, while pore length may be estimated from the change in water content. The left hand trace in Fig. 1b is shown measured with a ruler length of ϵ_1, and the total pore length, h_1, is simply $N_1 \epsilon_1$. The right hand side of Fig. 1b shows the same pore as measured with a smaller ruler size, ϵ_2. As we decrease the ruler size from ϵ_1 to ϵ_2, it is clear that the estimated pore trace length, h_2 will be larger than h_1 since we can more closely follow the irregularities of the pore wall. In contrast, as ϵ is made smaller in Fig. 1a, the estimate of the

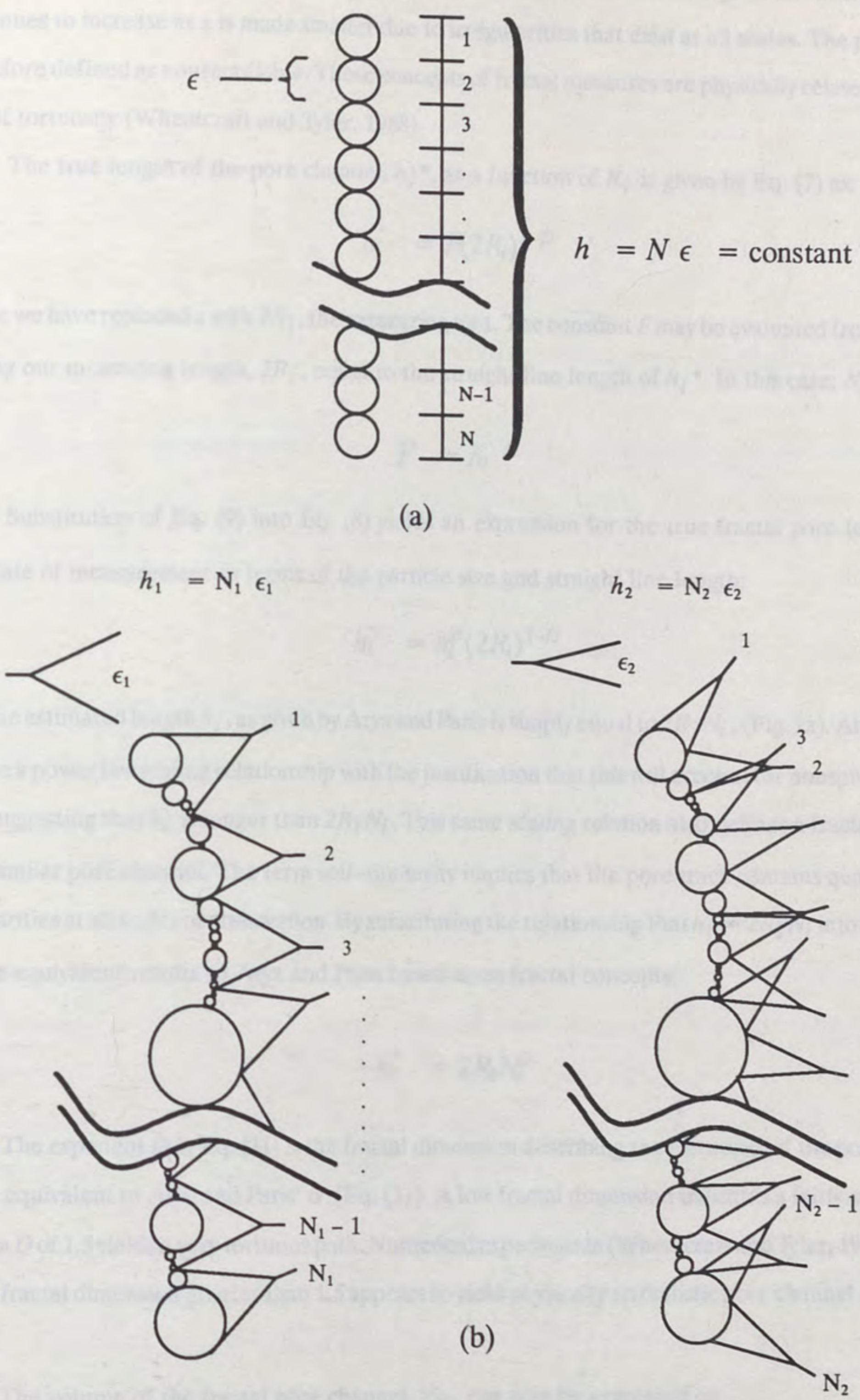


Figure 1.(a) Straight capillary tube comprised of uniform spheres. (b) Fractal capillary tube in which length is a function of measurement scale ($2R_i$).

line length (or pore trace) does not change. For the fractal pore trace in Fig. 1b however, the length continues to increase as ϵ is made smaller due to irregularities that exist at all scales. The pore trace is therefore defined as nonrectifiable. These concepts of fractal measures are physically related to a measure of tortuosity (Wheatcraft and Tyler, 1988).

The true length of the pore channel, $h_i{}^*$, as a function of R_i is given by Eq. (7) as:

$$h_i^* = F(2R_i)^{1-D} \tag{8}$$

where we have replaced ϵ with $2R_i$, the measuring unit. The constant F may be evaluated from Eq. (6) by setting our measuring length, $2R_i$, equal to the straight line length of $h_i{}^*$. In this case; $N_i = 1$ and:

$$F = h_i^{\ D} \tag{9}$$

Substitution of Eq. (9) into Eq. (8) yields an expression for the true fractal pore length at the i^{th} scale of measurement in terms of the particle size and straight line length:

$$h_i^* = h_i^D(2R_i)^{1-D} \tag{10}$$

But the estimated length h_i, as given by Arya and Paris is simply equal to $2R_iN_i$; (Fig. 1a). Arya and Paris invoke a power law scaling relationship with the justification that this will account for nonspherical particles suggesting that h_i is longer than $2R_iN_i$. This same scaling relation also defines a fractal scaling or self–similar pore channel. The term self–similarity implies that the pore trace contains quantifiable irregularities at all scales of observation. By substituting the relationship that $h_i = 2R_iN_i$ into Eq. (10), we obtain equivalent results to Arya and Paris based upon fractal concepts:

$$h_i^* = 2R_iN_i^D \tag{11}$$

The exponent D in Eq. (11) is the fractal dimension describing the tortuosity of the pore channels and is equivalent to Arya and Paris' α (Eq. (3)). A low fractal dimension indicates a fairly straight path while a D of 1.5 yields a very tortuous path. Numerical experiments (Wheatcraft and Tyler, 1988) indicate that a fractal dimension greater than 1.5 appears to yield physically unrealistic pore channel representations.

The volume of the fractal pore channel, V_{v_i} can now be expressed as:

$$V_{v_i} = \pi \; r_i^2 \; N_i^D 2R_i \tag{12}$$

We now replace Eq. (2) with Eq. (12) in the Arya and Paris model which results in a similar equation for the effective pore radius:

$$r_i = R_i \left[\frac{2}{3} e \; N_i^{\;1-D} \right]^{1/2} \tag{13}$$

The difference, however, is that the exponent D is a true measure of scale–dependent tortuosity of the pore trace.

The capillary pressure, ψ_i, in terms of equivalent height of water for the corresponding pore radius, r_i, is obtained using the equation of rise in a capillary tube and is given below:

$$\psi_i = \frac{2\gamma \cos\beta}{\varrho_w g R_i} \left[\frac{2}{3} e \; N_i^{\;1-D} \right]^{-1/2} \tag{14}$$

where γ is the surface tension of water, β is the contact angle, ρ_w is the density of water and g is the acceleration due to gravity.

In order to apply Eq. (14), it is necessary to assume that the bulk sample void ratio is equivalent to the void ratio of each particle class. This assumption, in light of the power law scaling used to estimate the pore length, has interesting ramifications. Most pore–size/grain–size relations (for example, see Haverkamp and Parlange, 1986) assume a linear relationship of the form: $r_i = CR_i$, where C is a constant indicative of the grain packing. Such linear behavior is not preserved in porous media exhibiting fractal pores.

Assuming a constant void ratio for all pore classes, Eqs. (1) and (12) may be used to show the relationship of pore size to grain size. The void ratio is given as:

$$e = \frac{\pi \; r_i^2 \; N_i^D \; 2R_i}{\frac{4}{3} \; \pi N_i R_i^3} \tag{15}$$

Rearranging terms yields:

$$r_i = C\sqrt{N_i^{1-D}} \;\; R_i \tag{16}$$

where: $C = \sqrt{\frac{2}{3}e}$.

The relationship between pore and grain size is now a function of the distribution and number of soil grains. Equation (16) shows several interesting features. For soils exhibiting nonfractal pores (D = 1), Eq. (16) reduces to the traditional linear proportionality of grain size to pore size. The same is also true if the soil exhibits a constant number of grains in each particle–class size. Such a soil would be dominated (on a weight basis), however, by a few large grains.

Most soils, however, show an increasing number of grains with decreasing grain size. For such soils, Eq. (16) indicates a more rapid decrease in effective pore radius with decreasing grain size than a purely linear decrease. Such behavior is consistent with fractal pore traces. Such traces will show increased tortuosity as the scale of observation is decreased. This increased tortuosity effectively lengthens the pore, resulting in a smaller cross–sectional area (pore radius) necessary to maintain the assumption of a constant void ratio.

ESTIMATION OF THE FRACTAL DIMENSION

Arya and Paris (1981), in their study of soils and soil moisture, based their estimates of α (our D) on a mean squared difference between measured and predicted capillary pressures. Water content at each capillary pressure was estimated based upon the particle mass at the i^{th} particle size and the sample void ratio. Based upon this, Arya and Paris report good agreement between measured and predicted water retention data on most of the 15 soils reported. Their fitting coefficient, α, was found to vary between 0.9 and 1.5.

If fractal concepts are to be of use in estimating α, it is necessary that an estimation technique be available for the fractal dimension. Turcotte (1986) has shown that particle sizes of geologic material exhibit fractal behavior of the form:

$$NR_i^D = constant \qquad (17)$$

where N is the total number of particles of radius greater than R_i and D is the fractal dimension of the particle–size distribution. The fractal dimension defines the distribution of particles by size. For $D = 0$, the distribution is composed solely by particles of equal diameter. When the fractal dimension is equal to 3.0, the number of particles greater than a given radius doubles for each corresponding decrease in particle mass by one–half (or particle radius decrease of $(\frac{1}{2})^{1/3}$). A fractal dimension between 0 and 3.0 therefore reflects a greater number of larger grains, while a dimension greater than 3.0 reflects a distribution dominated by smaller particles. Turcotte (1986) presents data on the fractal dimension of 21 particle–size distributions. Of the distributions presented, those representing soils had fractal dimensions approaching 3.0.

It is possible to estimate the fractal dimension of the particle–size distribution by plotting the cumulative number of particles larger than a given sieve size. Equation (17) may be rearranged and plotted on a log–log scale. The fractal dimension D will be equivalent to the negative of the slope. Since mechanical sieving yields a distribution of particle sizes between two successive sieves and it is impractical to count the number of particles directly, it is necessary to choose a "representative" particle radius for a given sieve size. For this analysis, this radius was chosen as the arithmetic mean between two successive sieve sizes. The number of particles assigned to each sieve was calculated by dividing the retained weight by the weight of a particle of mean radius between the two successive sieve sizes. The particle density was assumed to be 2.65 gr/cm^3 for all analyses in this paper.

The choice of an arithmetic mean particle size to estimate the fractal dimension is consistent with the Arya and Paris method of representative radii of successive pore classes. Other averages (harmonic, geometric, etc.) could be used, however, it would also be necessary to estimate R_i (Eq. (14)) using an equivalent averaging process. The use of the mean particle radius will have some affect on the estimated fractal dimensions. We are currently investigating various other techniques to more uniquely define the dimension from particle–size distributions.

In our case, it is necessary to estimate the fractal dimension of the one–dimensional trace of the pore channel. The scale of the measurement unit ($2R_i$) will determine the pore channel length. Mandelbrot et al. (1984) has suggested that the difference between an object's fractal dimension and its traditional topologic dimension (D_T) be denoted as the fractal increment, D_i. It has also been suggested (Mandelbrot et al., 1984) that this fractal increment can be used to estimate the fractal dimension of lower topological–dimensioned objects taken from the original fractal process. For example, the fractal dimension of a profile taken across a fractal surface will have the same fractal increment as the surface. If the fractal dimension of the surface is 2.5 (a highly irregular surface), the fractal increment is then 0.5 and the profile's fractal dimension would be 1.5 (1 + 0.5). This "slit island" technique has been used extensively in the literature to estimate the fractal dimension of surfaces and volumes from transects, cross sections, and contours. In this study, we make the assumption that the fractal increment ($D - D_T$) obtained from the grain–size distribution may be used to estimate the fractal dimension of the pore trace where D is the fractal dimension of the particle–size distribution. Since a particle–size distribution represents a three-dimensional collection of soil grains (D_T = 3), the fractal increment is given by D-3. The fractal dimension of the pore trace (viewed as a one-dimensional transect through a three-dimensional soil matrix) is then $1 + D_i$. The fractal dimension of the pore trace can range from one (a pore trace whose length is independent of measurement scale) to two (a trace which completely fills the plane).

MODELS AND MATERIALS

Arya and Paris (1981) present complete cumulative distribution data and water retention data for five soils (Soils B-F). Both particle-size and water retention data were digitized from their original paper for calculation and comparison. In addition, five soils cataloged in Mualem (1976) were analyzed. The soils ranged in texture from sand (Oakley Sand) to silty clay loam (Arya and Paris' Soil B). Cumulative number of particles (N) were calculated based upon 1 gram samples and an assumed particle density of 2.65 gr/cm^3. Particle-size classes (R_i) were chosen as the mean radius between successive sieve sizes.

The fractal dimensions (D) of the particle-size distributions were calculated from the slope of the log particle size versus log number of particles. A least squares regression was used to estimate D from the log-log plot.

The capillary pressure for the i^{th} particle-size class was calculated using Eq. (14) assuming a zero contact angle and fluid properties at 25°C. The soil water content θ_i, for each ψ_i was calculated as in Arya and Paris (1981) by summing the available pore space from the smallest size class up to the i^{th} class. The void ratios were estimated from bulk density or saturated water content data depending upon the data source. The resulting pairs of capillary pressure and water content (ψ_i, θ_i) were fitted using van Genuchten's retention model (1980):

$$\theta = \theta_r + \frac{\theta_s - \theta_r}{(1 + (a\psi)^n)^m} \tag{18}$$

where a, n, and m, and θ_r were taken to be fitting parameters.

The fitted equation was used to estimate root mean squared (RMS) error between measured and predicted water retention data using the fractal approach to the Arya and Paris model. RMS error was calculated as:

$$RMS = \left[\frac{1}{N_p - 4} \sum (\theta_{measured} - \theta_{predicted})^2\right]^{1/2} \tag{19}$$

where N_p is the number of measured pairs of water content and pressure head, with four degrees of freedom removed due to the fitting parameters used in Eq. (18). For two soils, saturation data were used to improve the fit of the data. In these cases, RMS error was calculated in terms of saturation(s) divided by the arithmetic mean of reported saturated water content and bulk density–derived porosity.

RESULTS AND DISCUSSION

Fractal particle–size relations are shown in Figs. 2a–d, 3a–d, and 4a, b for the 10 soils analyzed. With the exception of the Sable de Riviere sand (Fig. 4b), the soils showed clear fractal behavior according to Eq. (17). The fractal dimension of the distributions ranged from 2.7 to 3.485, while fractal dimensions of the estimated pore traces ranged from 0.7 to 1.485. The fractal dimensions (D) of all but the Sable de Riviere sand were greater than 3.0. It is interesting to note that the power function fit is rather poor for this soil. Table 1 shows the distribution of fractal dimensions for the soils analyzed. The fractal dimensions of the particle– size distributions were generally larger than those reported by Turcotte (1986). It is not clear if these differences are related to the choice of the mean particle radius or the more coarse texture of soil investigated by Turcotte (1986)

Calculated water retention data for each of the 10 soils are shown along with reported retention data in Figs. 5 through 14. Of the 10 soils analyzed, 7 showed good to excellent agreement with the measured retention data. The three soils showing poor agreement were the Sable de Riviere, Oakley Sand, and Arya and Paris Soil F. These are the coarsest textured soils investigated. These soils generally also had the lowest fractal dimensions. Since the pore trace dimension of the Sable de Riviere sand was less than 1.0, implying nonfractal behavior, water retention data was calculated using both $D = 0.7$ and $D = 1.0$ (Fig. 14). It is apparent, however, that a poor agreement exists using either $D = 0.7$ or $D = 1.0$. The reasons for this are unclear, however, the low fractal dimension of the particle–size distribution (2.7) indicates that the soil is predominantly composed of larger particles, with a significantly decreasing number of smaller size particles. This is not surprising since the soil is a sand and therefore it may not be appropriate to apply the scale–dependent tortuosity concepts inherent in the Arya and Paris model.

This soil was also poorly fit by Eq. (17) and therefore the assumption of fractal scaling for this soil is suspect. This soil exhibits a very sharp retention curve implying a very narrow range of pore sizes which may not be distinguishable with a relatively coarse sieve analysis. These results are consistent with Schuh et al. (1988) who reported fitted values of α to be highly variable near saturation for coarse textured soils. Schuh et al. (1988) also suggests that α (in our study D) was not constant over all pore classes and generally increased with decreasing pore size. Although the fractal approach presented in this paper permits

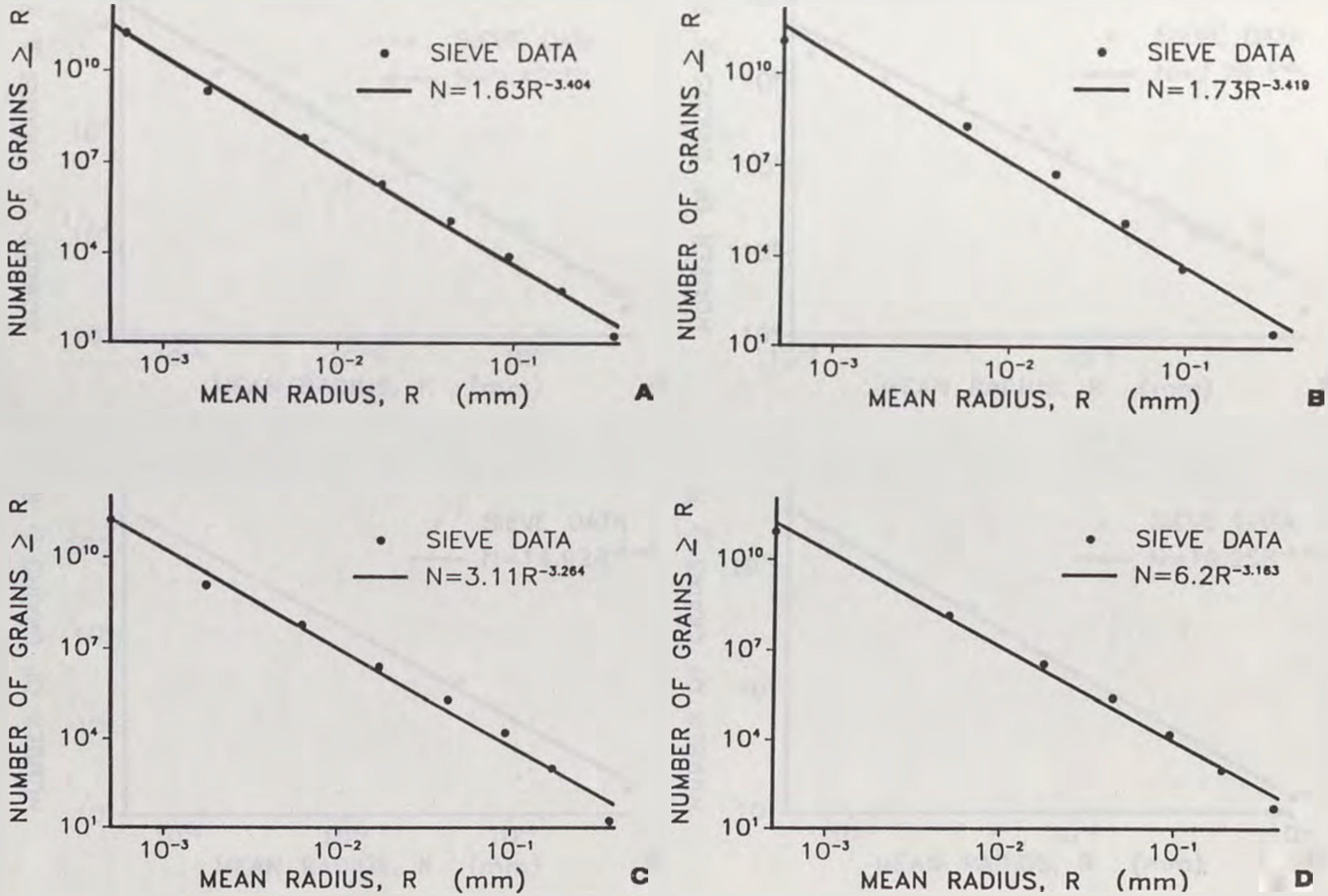


Figure 2a–d. Fractal particle distributions for (a) Ayra and Paris soil B; (b) Arya and Paris soil C; (c) Ayra and Paris soil D; and (d) Arya and Paris soil E.

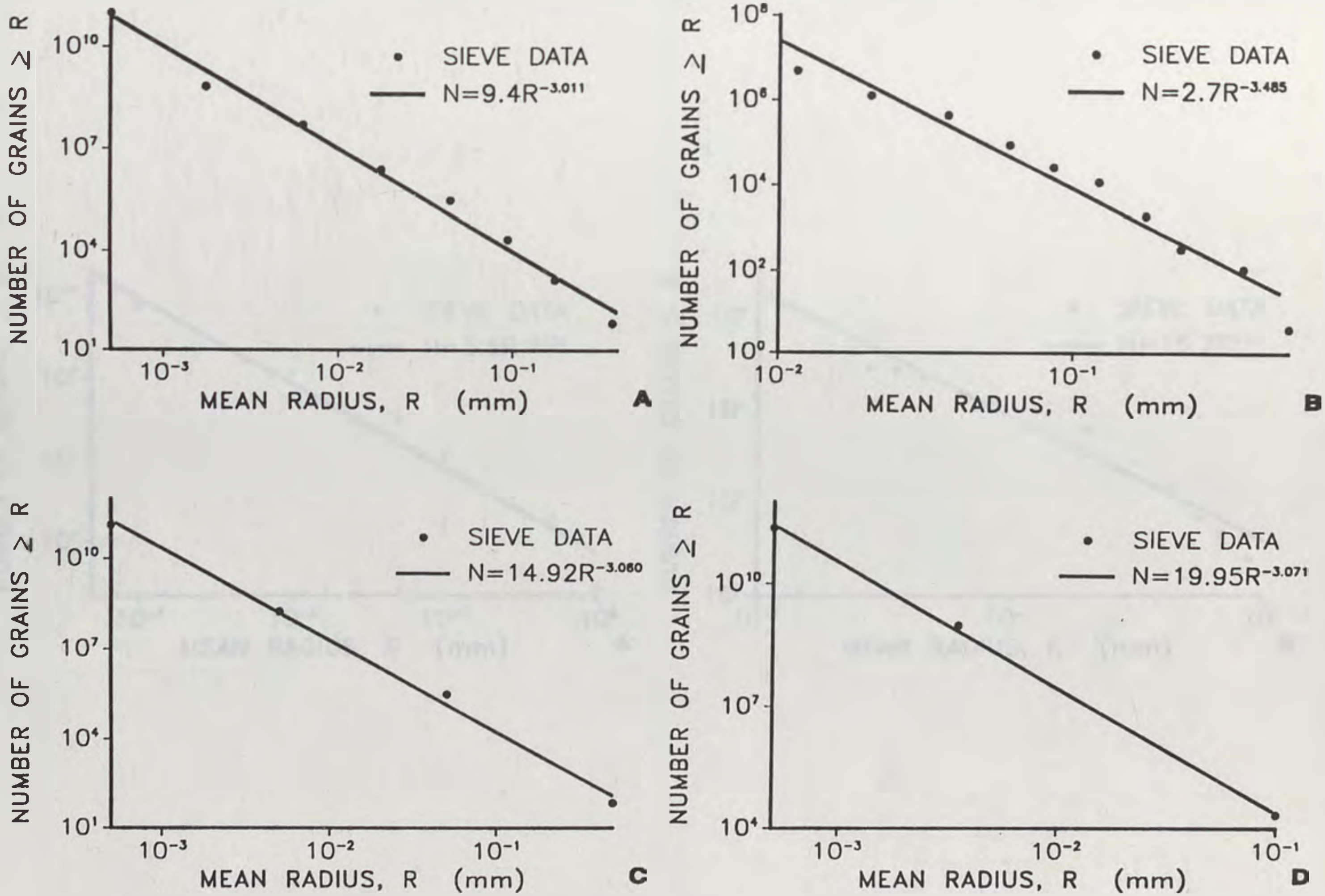
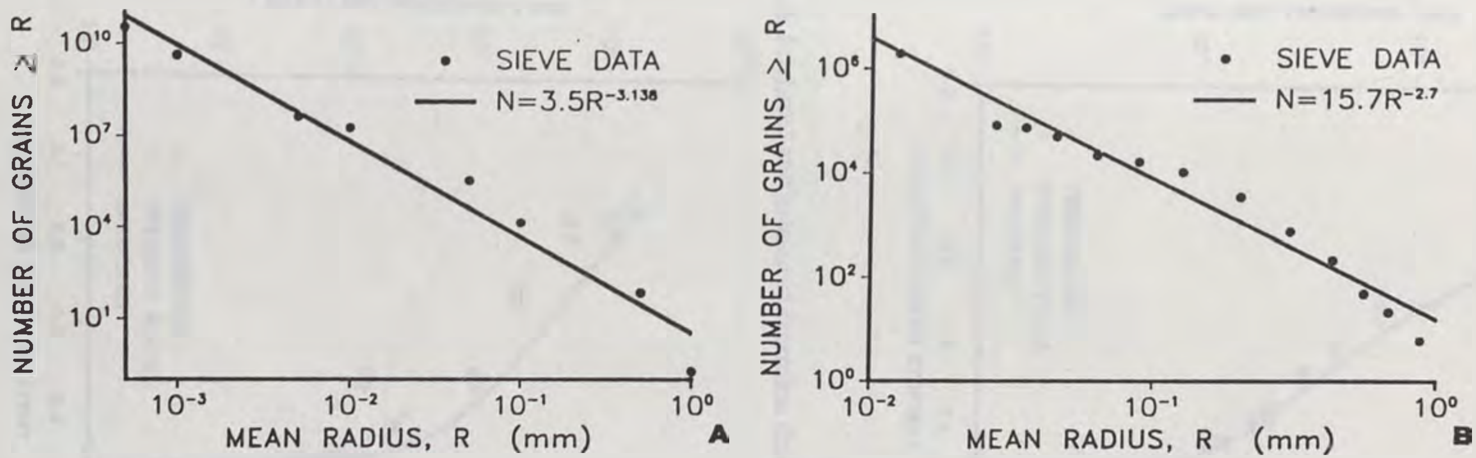


Figure 3a–d. Fractal particle distribution for (a) Arya and Paris soil F; (b) Columbia silt; (c) Gilat sandy loam; and (d) Yolo ligtht clay.



Fgiure 4a,b. Fractal particle distribution for (a) Oakley sand and (b) Sable de Riviere sand.

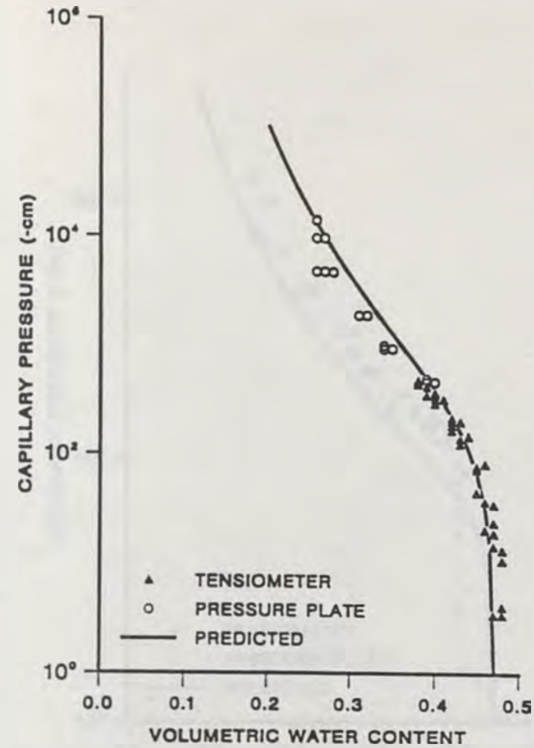


Figure 5. Predicted and measured retention data for Arya and Paris soil B.

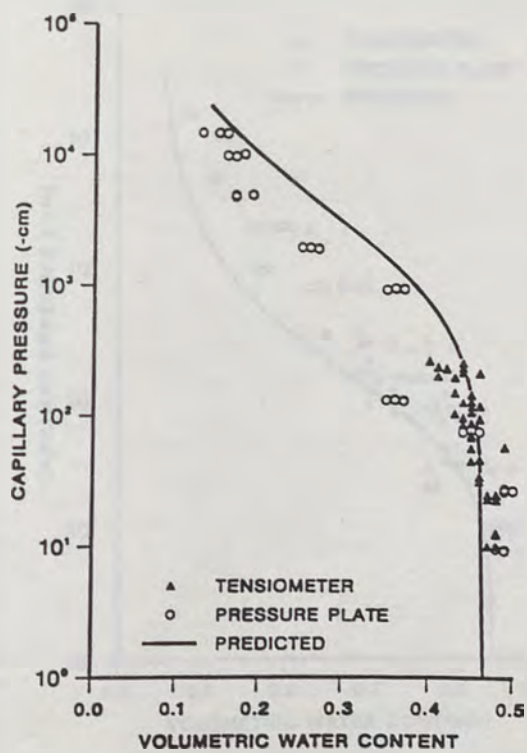


Figure 6. Predicted and measured retention data for Arya and Paris soil C.

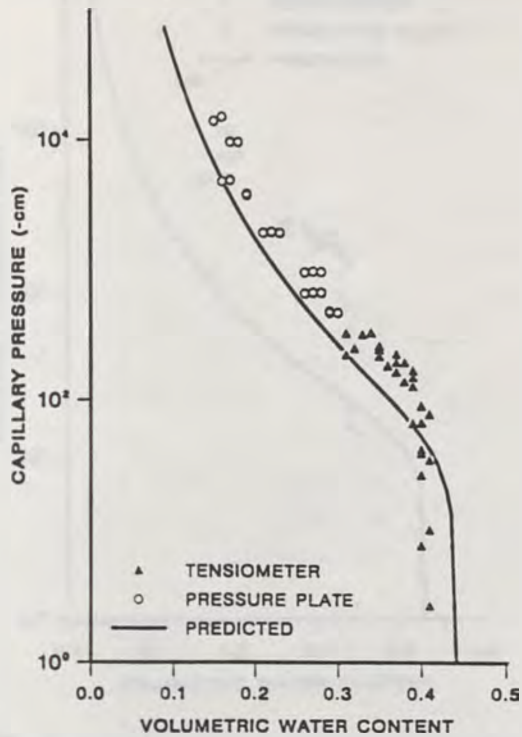


Figure 7. Predicted and measured retention data for Ayra and Paris soil D.

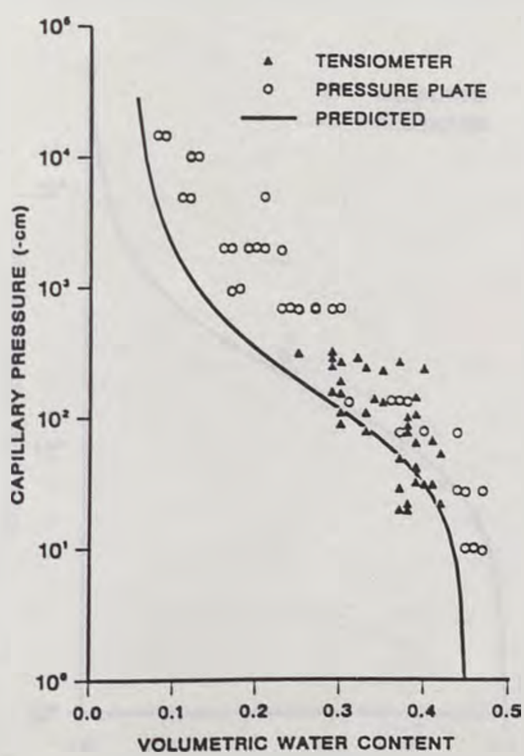


Figure 8. Predicted and measured retention data for Ayra and Paris soil E.

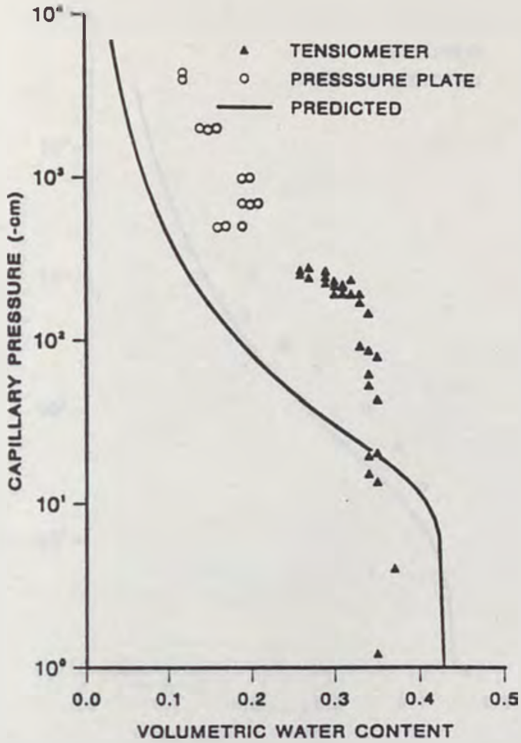


Figure 9. Predicted and measured retention data for Arya and Paris soil F.

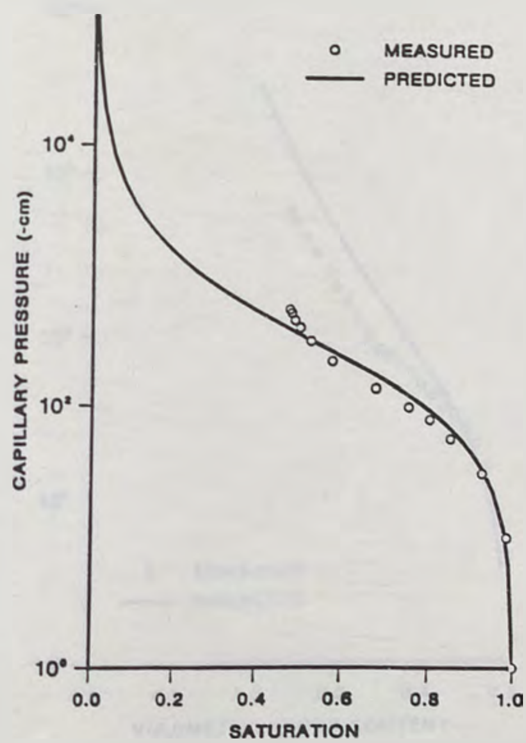


Figure 10. Predicted and measured retention data for Columbia Silt.

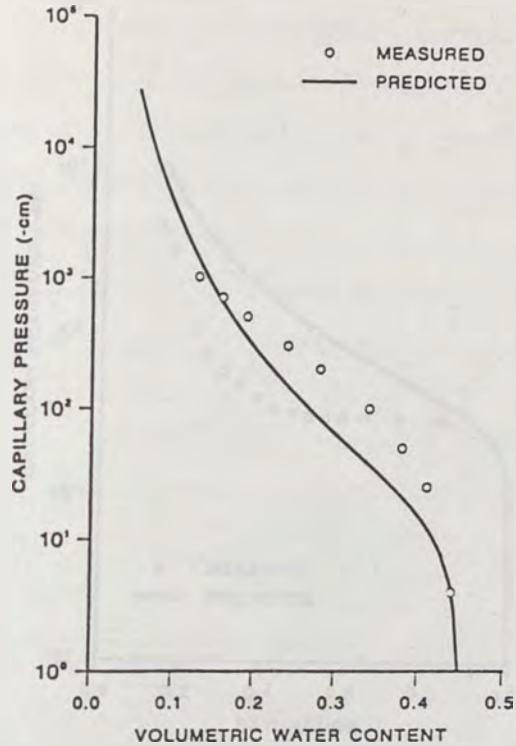


Figure 11. Predicted and measured retention data for Gilat sandy loam.

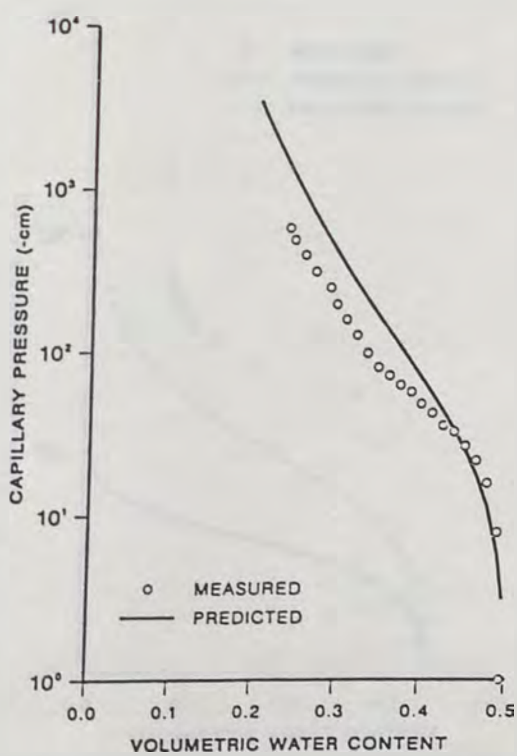


Figure 12. Predicted and measured retention data for Yolo light clay.

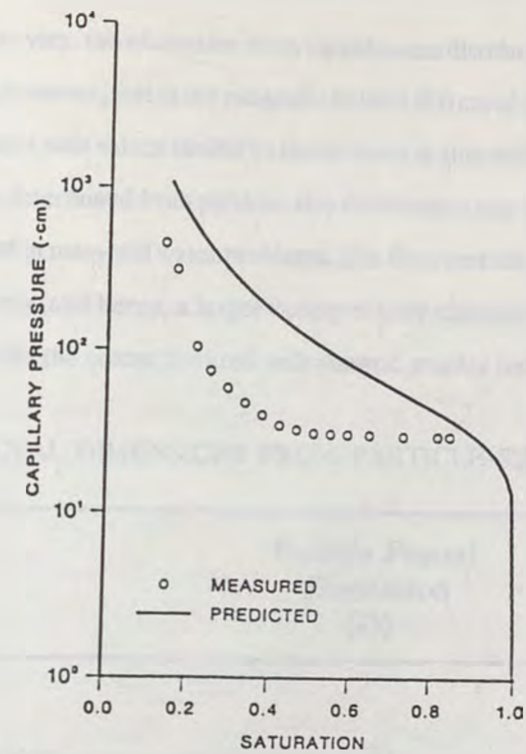


Figure 13. Predicted and measured retention data for Oakley sand.

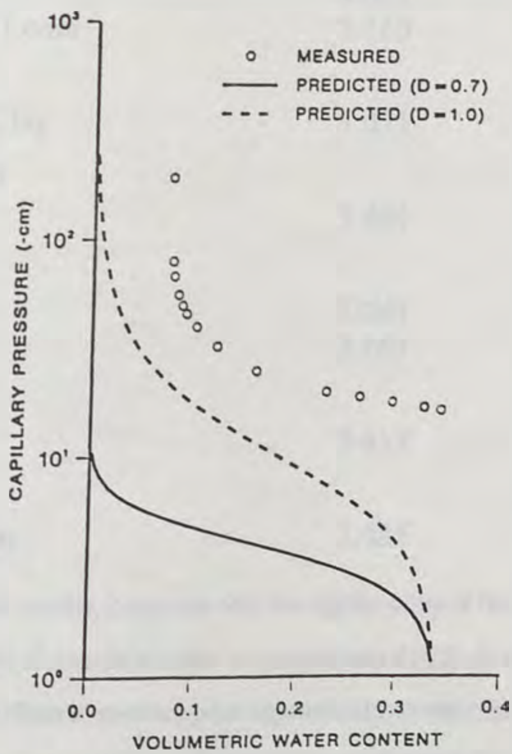


Figure 14. Predicted and measured retention data for Sable de Riviere sand.

the fractal dimension to vary, the estimation from particle-size distribution may not be straightforward. It is important to note, however, that in the range of -10 to -1,000 cm of tension, Schuh et al. (1988) found that α was fairly constant with values similar to those found in this study. Such behavior suggests that a constant value of α as determined from particle-size distribution may be appropriate over the range of tensions typically found in many soil water problems. The finer textured soils which showed a wider distribution in particle sizes (and hence, a larger variety of pore classes) showed characteristically higher fractal dimensions, while the coarse textured soils showed smaller fractal dimensions.

TABLE 1. FRACTAL DIMENSIONS FROM PARTICLE-SIZE DATA.

Soil Type	Particle Fractal Dimension (D)	Pore Fractal Dimension (D - 2)
Sand		
Sable de Riviere	2.70	0.70
Oakley Sand	3.138	1.138
Sandy Loam		
A/P Soil F	3.011	1.011
Gilat Sandy Loam	3.160	1.160
Clay Loam		
Yolo Light Clay	3.071	1.071
Silty Clay Loam		
A/P Soil B	3.404	1.404
Loam		
A/P Soil D	3.264	1.264
A/P Soil E	3.163	1.163
Silty Loam		
A/P Soil C	3.419	1.419
Silt		
Columbia Silt	3.485	1.485

Based upon these results, it appears that the applicability of the fractal model can be determined by inspection of the plot of particle number versus particle size. Soils exhibiting fractal increments close to or less than zero are likely to produce poor approximations while increments in the range of 0.1 to 0.5 should produce accurate retention data. Particular interest is called to the Yolo Light Clay Soil (Figs. 3d and 12). The fractal dimension as well as the water retention data were calculated using only three particle-size classes: sand, silt, and clay. Even with such minimal data, the calculated retention data closely fits the measured data. It is important to note that the Oakley Sand and Columbia Silt are plotted as saturation. Based upon the bulk densities reported for these soils, the retention experiments did not completely saturate the soils. Table 2 shows the RMS error for each of the soils analyzed.

The apparent residual water contents predicted with the fractal model were generally less than the reported water contents at high tensions. This is not unreasonable since the Arya and Paris model assumes complete desorption of all pores of a given class size at the critical pressure. At low tensions, this assumption appears reasonable, however, at high tensions, a significant percentage of water may be held as films and in dead-end or poorly-connected pores. As a result, the model will tend to underpredict the water content in the high tension regions. Fortunately, the range of interest for many flow and transport problems is at low to intermediate tensions where the model appears to be most accurate.

The sensitivity of the analysis to the fractal dimension is shown in Fig. 15. Water retention data were calculated for Arya and Paris' Soil B for three different values of D (1.0, 1.2, and 1.6) and are shown in comparison to the retention data generated with the fractal dimension estimated from the particle-size distribution ($D = 1.404$). Figure 15 shows the extreme sensitivity to the fractal dimension and clearly shows that the fractal dimension obtained independently from the particle-size distribution is appropriate to estimate the retention data.

TABLE 2. CALCULATED RMS ERROR (IN WATER CONTENT).

Soil	RMS (cm^3/cm^3)
Arya and Paris Soil B	0.014
Arya and Paris Soil C	0.0403
Arya and Paris Soil D	0.037
Arya and Paris Soil E	0.077
Arya and Paris Soil F	0.141
Columbia Silt	0.023*
Gilat Sandy Loam	0.064
Yolo Light Clay	0.038
Oakley Sand	0.137*
Sable de Riviere Sand	0.249 (D = 0.7)

* Calculated from saturation data and the average of calculated and reported saturated water content.

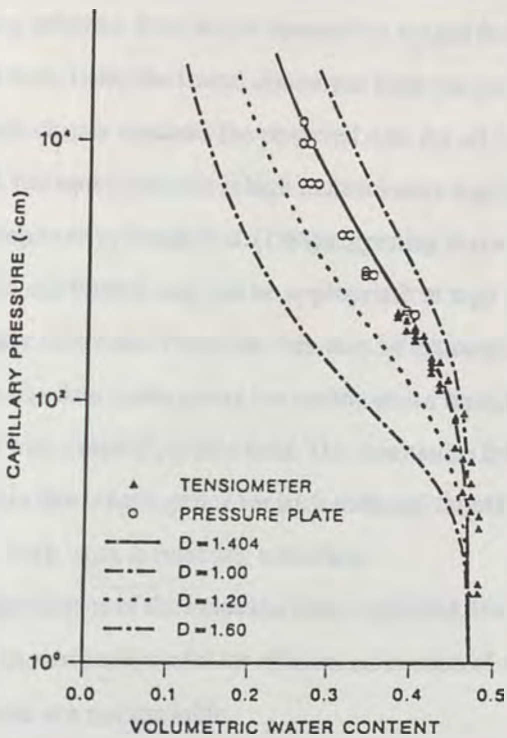


Figure 15. Sensitivity of predicted water content to the fractal dimension.

CONCLUSIONS

The empirical constant (α) used in the Arya and Paris model is shown to be equivalent to the fractal dimension of a tortuous fractal pore. This analysis significantly reduces the empiricism of their water retention model. A relationship is developed to estimate the pore trace fractal dimension and hence, the pore channel length which is strictly scale-dependent based upon the mean particle radius. The concepts of fractal pore spaces also imply a nonlinear relationship between grain size and equivalent pore radius in conflict with traditional analysis.

A simple method is presented to relate the fractal dimension of the particle-size distribution to the dimension of the fractal pore trace. The relationship between particle-size distribution and pore geometry reduces the model input variables to bulk density and cumulative particle-size distribution data.

Ten soils in which retention and particle-size data were available were analyzed to obtain both the fractal dimension and, subsequently, the water retention data using the Arya and Paris model. The soil textures ranged from sand to silty clay loam to silt. The fractal dimension of the particle–size distributions generally increased with decreasing texture. Of the soils investigated, all but one soil showed clear fractal or power law scaling behavior. Pore fractal dimensions ranged from less than 1.0 to 1.4, in agreement with other reported data. Using the fractal dimension from the particle–size distribution, the estimated water retention data closely matched the observed data for all but the three coarsest soils. For most of the soils analyzed, the water contents at high tensions were slightly underestimated. The results are consistent with those reported by Schuh et al. (1988) suggesting that a constant value of α (equivalent to the fractal dimension in our model) may not be appropriate at high soil water tensions.

These results indicate that water retention data may be estimated with reasonable accuracy for soils in which the particle–size data shows power law scaling with a fractal dimension of greater than 3.0. Such soils are those with a wide range of particle sizes. The results also indicate that a nonlinear relationship between pore and grain size is appropriate for such soils and should be incorporated into retention and conductivity models. Such work is currently underway.

With the physical significance of the empirical term explained, the Arya and Paris model is shown to be physically based and should prove useful for efficient estimation of water retention data where field or laboratory measurements are not available.

ACKNOWLEDGMENTS

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CHAPTER 4

FRACTAL SCALING OF SOIL PARTICLE-SIZE DISTRIBUTIONS (PSD) ANALYSIS AND LIMITATIONS

S.W. Tyler
S.W. Wheatcraft

ABSTRACT

In this paper, the concepts of fractal particle size distributions (PSD) are developed and related to actual soil PSD. Using the assumption of fractal scaling of the number of soil grains of a given size, the corresponding relations of mass distribution, a more commonly used relation, are developed. These developments are used to provide limits on the range of the fractal dimension, D and also on the range of soils which can truly exhibit fractal scaling. The results indicate that fractal scaling in soil PSD's comprise a fairly limited range of textures and fractal scaling may not be applicable to many soils. Based upon these results, we revisit our earlier work (Tyler and Wheatcraft, 1989) and discuss the results of apparent fractal scaling of soil PSD studied in the work. The fractal scaling and magnitude of the fractal dimensions found in our previous work are shown to be more representative of the plotting and fitting algorithms rather than the actual fractal nature of the PSD. As a result, the technique of estimation of the pore-trace fractal dimension, a hydraulic property of soils crucial to the estimate of water retention from PSD data alone, is not robust.

INTRODUCTION

PSD and textural analysis form one of the most common describers of field soils. The textural triangle (USDA, 1975) provides a natural language through which scientists, agronomists and the lay public may succinctly describe the major physical properties of soils. To the soil scientist, the PSD has been used routinely to predict physical properties such as water retention (Clapp and Hornberger, 1979), bulk density, permeability and porosity. These approaches generally require the PSD to be quantified by certain parameters such as the mean grain diameter, the uniformity coefficient, the liquid limit, and in recent years, the fractal dimension (Turcotte, 1986; Tyler and Wheatcraft, 1989). In this paper, we will examine in detail, the derivation of fractal scaling for PSDs, its implications, and its limitations. Our primary objective is to relate the fractal dimension of the PSD to textural and other soil descriptors where possible, and to clearly define the range of fractal applicability.

Fractal Scaling of the PSD

Fractals (Mandelbrot, 1983) have become a widely accepted scaling treatment of physical systems. The concepts of fractals are based on the notion of scale-invariant transformations. Scale-invariant transformations map objects or processes onto themselves, independent of scale. As a result, fractal objects appear similar (either in shape or with respect to some intrinsic statistical property) at all scales of observation. In a soil PSD, the concept of fractal scaling suggests that at any scale of observation the solid phase of the soil will appear similar. This diverges somewhat from the more traditional Miller and Miller scaling, in which two different soils display similar pore geometry and are related by a linear scaling coefficient. In a fractal soil, similarity in shape are exhibited in a single soil over a wide range of scales of observation. For example, take a slice through a soil core (assume the soil pores have been treated with epoxy such that the physical structure of the soil is rigid). If the soil grains display fractal scaling in its PSD, the ratio of the area comprising the solid phase would appear constant at any scale of magnification. The term "at any scale of magnification" naturally has limits, and we will discuss these limits further on in the paper. The behavior of a non-fractal soil will be significantly different. For example, if we investigate a soil made up of single grain size, as we increase our magnification, we will either zoom in on a grain or void with the resulting void ratio approaching zero or one, respectively. In the fractal soil, the higher magnification shows no change in void ratio. This is equivalent to smaller grains filling the void spaces between the larger grains. Such a physical description of soil is, on the surface, very consistent with our intuitive understanding of soil as a porous medium. As we shall see later, rigorous fractal or self-similarity in particle size may apply to only a small fraction of real soils. In addition fractal scaling in both particle size and pore size is required to consider the soil to be a fractal porous medium. Fortunately, understanding the hydraulic behavior of soils is most closely tied to the pore space, which has been shown to be fractal in many cases (Katz and Thompson, Tyler and Wheatcraft, 1989, 1990, and Jacquin and Adler, 1987). In Figure 1, we show a typical cross section through a soil displaying a fractal number distribution in its pore size distribution. In this figure, we see pores of a wide range of sizes. In the two-dimension plane, the fractal dimension of the pore projection in Figure 1 was calculated as 1.80.

Fractal scaling arguments and analysis by Tyler and Wheatcraft (1990) and Toledo et al. (1990) suggest that the shaded area of Figure 1, A, (R) given by N^*R^2, where N is the number of tiles of characteristic size R, is given by:

$$A(r > R) = C\left[1 - \left(\frac{R}{\lambda}\right)^{2-D}\right] \tag{1a}$$

where C and λ are constants relating to the shape factors and total range of scale, and D is the fractal dimension. Note that if D is equal to 2.0 (the limiting case for this figure), the pore area is zero and is

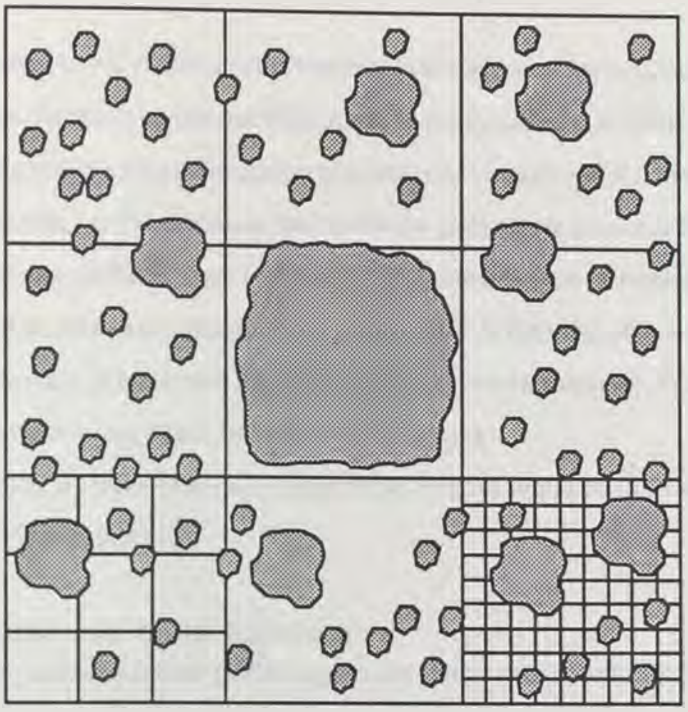

Figure 1. Hypothetical soil pore distribution. Areal fractal dimension is 1.89.

therefore independent of the measuring scale. If D is less than 2.0, the pore area grows as the measuring scale is decreased. The same analogy can easily be carried to a soil PSD. At a given sieve radius R_1, soil grains of size R_1 or greater are retained. Grains greater than R_1 are not detected. Picturing the sieve openings as tiles of size R_1, by R_1, the number of sieve openings covered by soil grains will follow equation (1a). As the sieve size is decreased, A(R) grows (more slowly as R becomes very small) and asymptotically reaches a constant value, related to the porosity. For simplicity, we assume that at $R = 0$, the soil grains are arranged such that they are everywhere one-layer deep.

To carry the analogy to three dimensions, the volume, $V(r > R)$ of cubes of size R needed to fill the soil grains of size R or larger is given by:

$$V(r > R) = C\left(1 - \left(\frac{R}{\lambda}\right)^{3-D}\right) \tag{1b}$$

In the limiting case, D = 3, we see that the volume of the soil grains does not increase with decreasing scale of resolution. In this case, the soil PSD would be composed of soil grains of strictly one size.

It is also possible that the fractal dimension will change as a function of R. For example, soil PSDs are bounded distributions, i.e. the maximum and minimum grain size is generally fixed by the analysis techniques, such a sieving on the < 2mm fraction. If a soil contains little or no clay size fragments for example, the fractal dimension will approach 3 in equation (1b). At the value of 3, no new detail (grains) are discernable as the scale of resolution is further decreased. Below this scale, the distribution of soil grains is not scale–invariant, but rather translationally invariant.

In the next section, we will look in more detail at the range of values that D, the fractal dimension, may take on in real soil distributions.

Analysis of Number and Mass Relations

Turcotte (1986) and Mandelbrot (1983) suggest the fractal relationship for PSD's of the form:

$$N(r > R) \propto R^{-D} \tag{2}$$

where N(r > R) is the number of particles of size R or greater and D is the fractal dimension. For D = 0, the number is independent of the scale of observations, while D > 0 implies additional detail with a finer scale of observation, i.e., a smaller value of R. The relationship is directly analogous to the distribution of pores of the Menger sponge (Mandelbrot, 1983) or Sierpinski carpet (Tyler and Wheatcraft, 1990). In most soils analysis, it is not feasible to count the number of grains of a particular size and in addition, the soil grains span a spectrum of sizes and shapes. Instead, we measure the mass of grains whose size ranges between an upper and lower boundary defined by the sieve diameters or settling times (Gee and Bauder, 1986). To calculate the number of particles, N(r > R), we must make an estimate of some characteristic grain size, $\bar{R}$ whose volume and density is used to calculate $N(r > \bar{R})$ via:

$$N(R_1 < r < R_2) = \frac{M(R_1 < r < R_2)}{\frac{4}{3}\pi \bar{R}^3 \rho_p} \tag{3}$$

where $M(R_1 < r < R_2)$ represent the mass of soil grains between two upper and lower sieve diameters and ρ_s is the grain density. As will be discussed further in this paper, such an analysis may lead to physically unrealistic results.

The cumulative number of soil grains greater than the characteristic size $\bar{R}$ is given by:

$$N(r > \bar{R}) = \sum_{R=R_L}^{\bar{R}} N\ (R_i < r < R_{i+1}) \tag{4}$$

where R_L is the largest grain radius present in the distribution. Immediately, we see from equation (3) that our calculation of $N(r > \bar{R})$ is strongly dependent upon our choice of $\bar{R}$, especially when $\bar{R}$ is small, since it is cubed in equation (3).

This approach is somewhat different than that stated in equations (1a) and (1b). Equations (1a) or (1b) state that the area covered by grains of size R or larger is calculated by tiling, or covering the area with tiles of size R. Equation (2), on the other hand, states strictly that the number of grains larger than R will be proportional to R^{-D}. From equation (1b), the total number of cubes of size R needed to fill the volume of grains to a size R is given by:

$$\begin{aligned} N\ (r > R) &= \frac{V(r > R)}{\frac{4}{3}\pi R^3} \\ N\ (r > R) &= C^{*}\left[R^{-3} - \lambda^{D-3}R^{-D}\right] \end{aligned} \tag{5}$$

where $C^{*} = \dfrac{C}{\frac{4}{3}\pi}$.

Equation (5) represents strict self similarity while equation (2) is an approximation. Using these two approaches will yield different values of D for the same set of data.

To avoid these difficulties, it is more appropriate to investigate the PSD's in terms of mass, a more easily measured quantity. From equation (1b), the mass $M(r > R)$ is simply:

$$M(r > R) = \varrho_p C\left(1 - \left(\frac{R}{\lambda}\right)^{3-D}\right) \tag{6}$$

where ρ_p represents the average grain density. In most cases, the grain density vs. grain size is not measured and is therefor taken to be constant. Such an assumption may also introduce significant error.. The total mass, M_T is:

$$M_T = M(r > 0) = \varrho_p C\left[1 - \left(\frac{0}{\lambda}\right)^{3-D}\right] \tag{7}$$

$$M_T = \varrho_p C$$

Equation (6) may be normalized by equation (7) to yield:

$$\frac{M(r > R)}{M_T} = 1 - \left(\frac{R}{\lambda}\right)^{3-D} \tag{8}$$

The constant λ may be easily evaluated if we choose an upper limit R_L for fractal behavior. For a typical PSD, R_L is often the very coarse sand fraction. At $R = R_L$, $\frac{M(r > R_L)}{M_T}$ is zero and λ must be equivalent to R_L.

Equation (8) is easily rearranged to provide the more typically reported PSD data of "% mass less than" or $M(r < R)$ by noting that:

$$\frac{M(r < R)}{M_T} = 1 - \frac{M(r > R)}{M_T}$$

or (9)

$$\frac{M(r < R)}{M_T} = \left(\frac{R}{R_L}\right)^{3-D}$$

This result is equivalent to Turcotte (1986) in form, however, strict self similarity has been preserved in the analysis. In Turcotte (1986), strict geometric similarity is not preserved in grain number verses characteristic size, however, mass based similarity is preserved.

Equations (2) and (9) provide insight into the limiting values of the fractal dimension of the PSD which have not been clearly reported in the past. Equation (2) suggests that D may range from zero to infinity. For a negative value of D, the cumulative number of grains greater than R decreases as R is decreased; a non-physical situation. At D = 0, the distribution is independent of observation size, R and behaves in a Euclidean or regular geometry. In the case of equation (2), the number of grains remains unchanged as R is decreased, while equation (5) shows strict euclidean behavior. At $D = 0$, equation (5) reduces approximately to:

$$N(r > R) \cdot R^3 = constant \tag{10}$$

which is consistent with traditional geometry.

Equation (9) on the other hand, bounds the fractal dimension at 3. For a value of D greater than 3, the cumulative mass exceeds the total mass with decreasing observation size, another physically impossible situation. At the limiting value of 3, the distribution is also independent of scale. Equation (9) does not, however, provide a lower bound on the value of D. Combining the limiting values obtained via observation of the number and mass based relations (equations (2) and (9), respectively), the range of soil PSD fractal behavior is strictly limited to:

$$0 < D < 3 \tag{11}$$

These results will be used in a later section to discuss the apparent anomalous results of fractal scaling of PSD reported in Tyler and Wheatcraft (1989).

Applications to Real Soil PSD's

In the past, use of equation (2) has led to conclusions regarding scale–invariant behavior of soils and particulates (Turcotte, 1986; Tyler and Wheatcraft, 1989). In Tyler and Wheatcraft (1989), log–log plots of particle numbers verses effective or average particle radius most often showed linear behavior. In addition, for all but one soil tested, the fractal dimensions estimated from least squares regression exceeded 3.0. We suggested at that time that the particle fractal dimension could be directly used to estimate the fractal dimension of the pore tortuosity term of the Arya and Paris (1981) model. As we shall see in this manuscript, the robustness of such an extrapolation from particle size to pore geometry appears limited.

For most soils in the textural triangle, the size of the fragments range from 2 mm to less than 2 microns. For a given soil sample, one can expect to find the mass of solids to be distributed over this range. In general, coarse textured soils are more likely to have fragments of 2 mm or greater comprising a large percentage of the total weight, while fine textured soils may have very little mass concentrated at the 2 mm size. However, most fine textured soils will have some fragments in the sand size range.

With this preface on the nature of soils typically encountered, we must re–look at the scaling implications of equation (9). For example, if we encounter a soil with some fraction of sand size fragments, we begin to fix the upper bound of equation (9). Given a fixed point (the characteristic grain size below which 100 percent of the mass is found), then it is clear from equation (9) that the only variable remaining is the fractal dimension, which we have constrained to $0 < D < 3$. By choosing various values of D, we can easily show the dependance of texture upon D, and also the limited range of soil textural classes which truly follow the fractal scaling arguments. Figure 2a shows equation (9) for soils with largest grain diameters of 2 mm (radius of 1 mm), the top of the sand classification (USDA, 1975), while Figure 2b shows the same fractal dimensions with largest fragments of only 1 mm in diameter. In each figure, a fairly narrow range of fractal dimensions span the commonly encountered textures. Given the curves of Figures 2a–b, it is a simple matter to pick off (or calculate from equation (9)) the percent passing the silt and clay size fractions and translate these distributions to the textural triangle shown in Figures 3a and b. For soils with coarsest fragments (coarse sand) at the 2 mm diameter true–self similar or scale–invariant behavior follows a semi–circle from sand to clay soils. The same is true for finer–textured soils (largest grain diameter = 1 mm), however, the self–similar curve moves inward slightly towards the silts. It is clear that fractal dimensions approaching 3 are associated with fine–textured soils. Well graded soils such as loams, and silty loam, however, are not self similar (Tyler and Wheatcraft, 1989) in their particle–size distributions as described by equation (9). Given these restrictions in soil texture, it becomes apparent that difficulties may arise in using fractals as unique descriptors of particle–size distributions.

As a counter example, the particle–size distribution for sandy clay loam, clay loam and silty clay loam are shown in Figure 4. In each case, fractal dimensions may be estimated for these soils via a least squares regression on the log transformed PSD data. For these three soils, the least squares estimated fractal dimensions are nearly identical (2.82) yet Figure 5 shows the relation of these soils to equation (9) (with $R_L = 1$ mm) on the textural triangle. It is clear from the figure that these soils do not fit the scale invariance requirement, and worse, all suggest (via regression) poor sensitivity relating the estimated fractal dimension and soil texture.

Re–evaluation of the PSD Dimensions

In an earlier work (Tyler and Wheatcraft, 1989), we suggested that PSD's would be valuable to predict the fractal dimension of the pore trace as defined by the Arya and Paris (1981) model of soil water retention. It appeared from plots of $N(r > \bar{R})$ verses $\bar{R}$ that clear fractal behavior described by equation (2) were evident in at least 9 of the ten soils analyzed . In each of these 9 soils, the estimated fractal

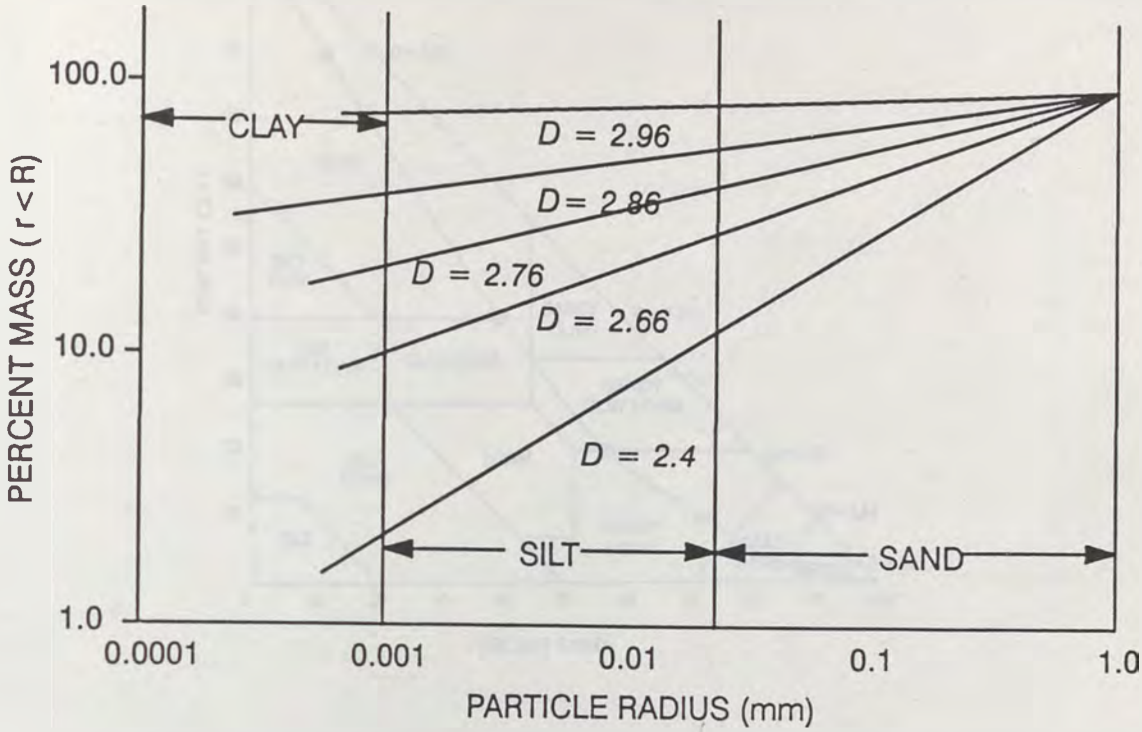

Figure 2a. Mass verses particle radius for R_L = 1.0 mm.

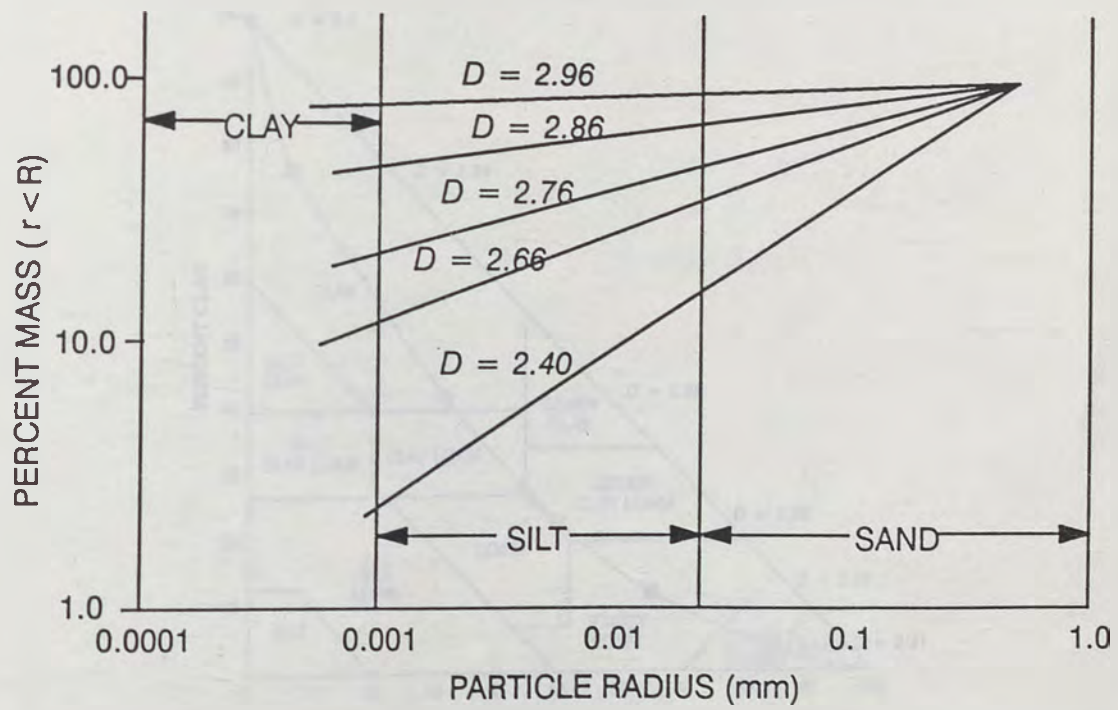

Figure 2b. Mass verses particle radius for R_L = 0.5 mm.

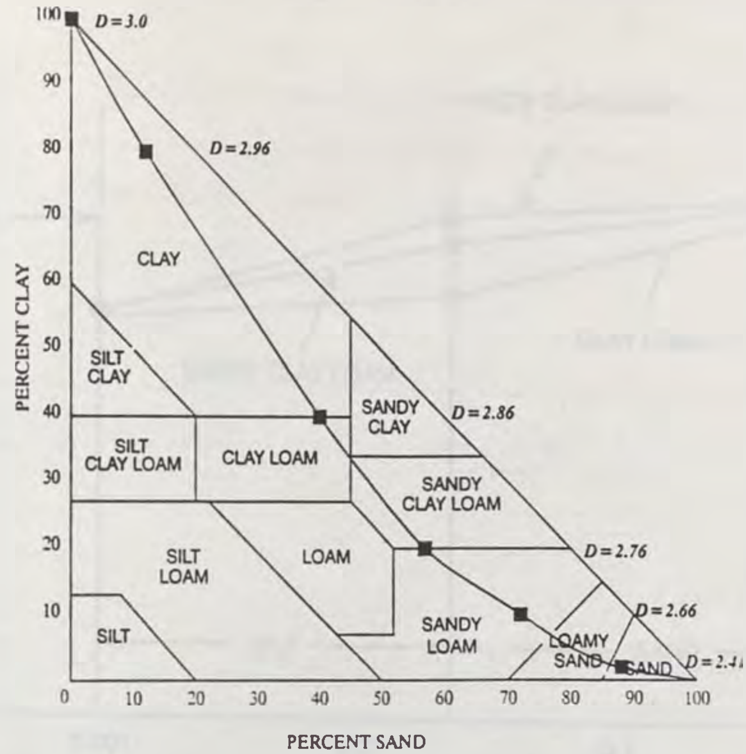

Figure 3a. Soils displaying fractal scaling in PSD; $R_L = 1.0$ mm.

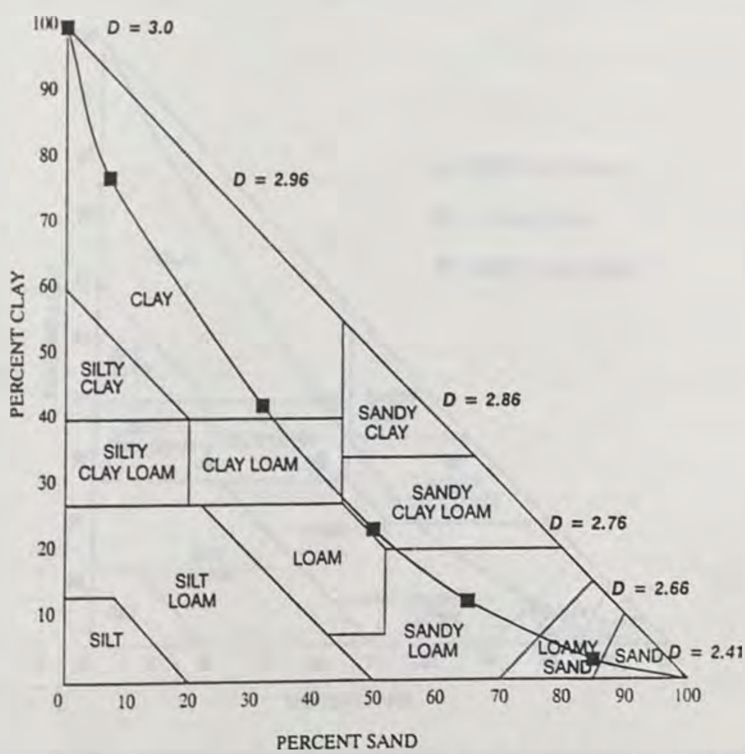

Figure 3b. Soils displaying fractal scaling in PSD, $R_L = 0.5$ mm.

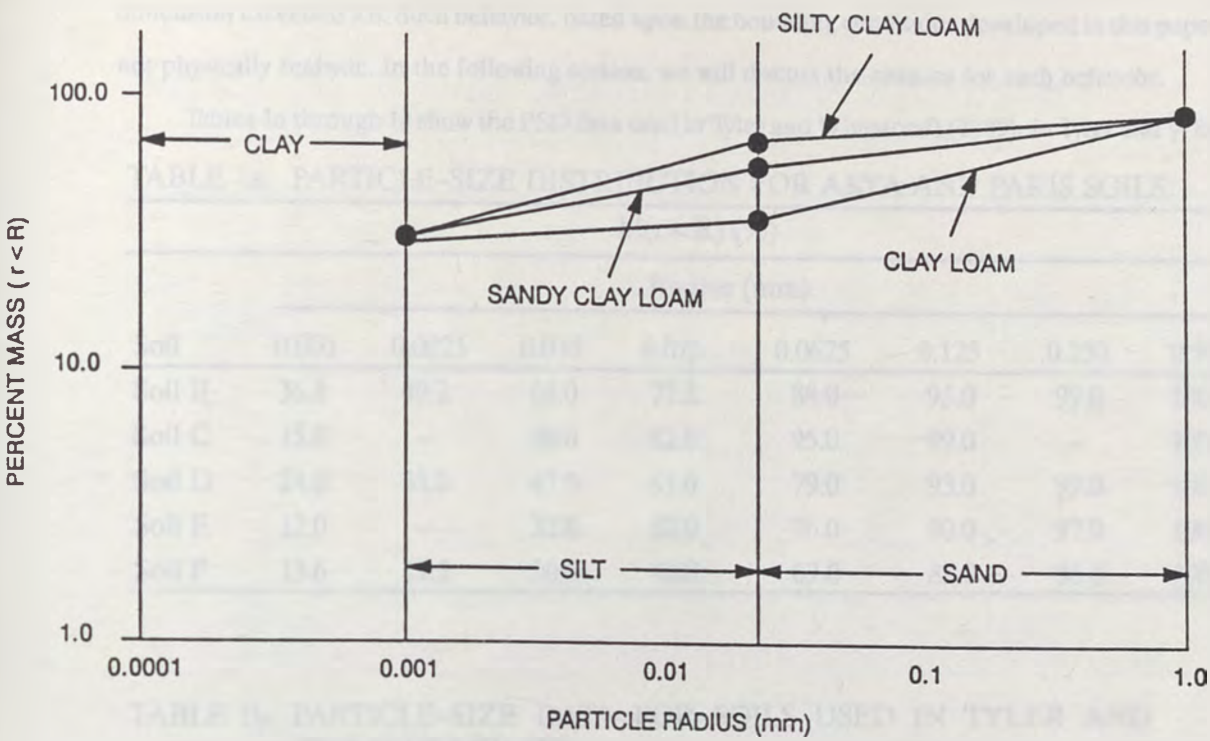

Figure 4 PSDs for Sandy clay loam, clay loam and silty clay loam.

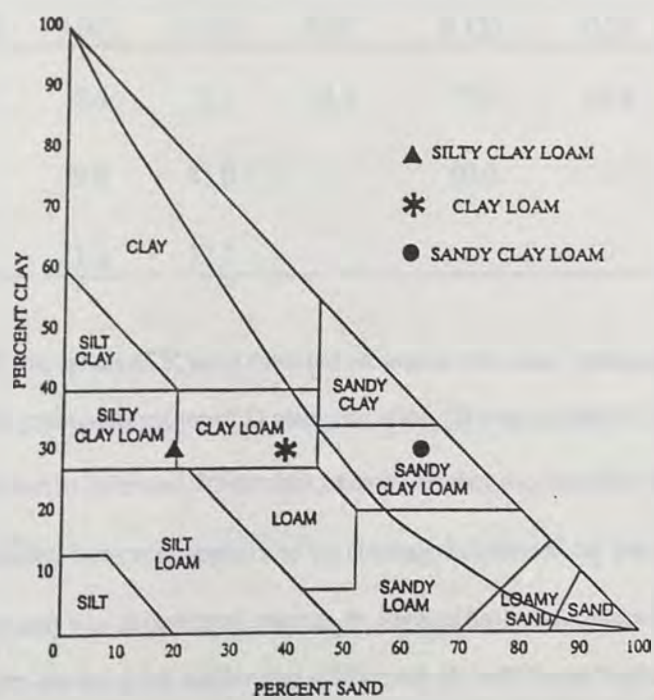

Figure 5. The relationship of sandy clay loam, clay loam and silty clay to soils displaying fractal scaling with $R_L = 1.0$ mm

dimension exceeded 3.0. Such behavior, based upon the bounding conditions developed in this paper, is not physically realistic. In the following section, we will discuss the reasons for such behavior.

Tables 1a through 1c show the PSD data used in Tyler and Wheatcraft (1989). In Tyler and Wheat-

TABLE 1a. PARTICLE–SIZE DISTRIBUTION FOR ARYA AND PARIS SOILS.

	M(r < R) (%)							
	Radius (mm)							
Soil	0.001	0.0025	0.010	0.025	0.0625	0.125	0.250	0.500
Soil B	36.8	49.2	66.0	77.2	88.0	95.0	99.0	100.0
Soil C	15.0	–	50.0	82.0	95.0	99.0	–	100.0
Soil D	24.0	31.0	47.0	61.0	79.0	93.0	99.0	100.0
Soil E	12.0	–	32.0	52.0	76.0	90.0	97.0	100.0
Soil F	13.6	17.2	30.0	42.8	69.0	88.0	96.0	100.0

TABLE 1b. PARTICLE–SIZE DATA FOR SOILS USED IN TYLER AND WHEATCRAFT, 1989.

	M(r < R) (%)							
	Radius (mm)							
Soil	0.0005	0.001	0.025	0.05	0.125	0.25	1	5
Oakley Sand	4.0	8.6	12.1	32.5	77.6	92.9	102.8	103.0
Gilat Sandy Loam	–	19.0	45.0	–	90.0	–	100.0	–
Yolo Light Clay	–	31.2	77.2	–	–	–	101.0	–

craft (1989), some of the values of $\bar{R}$ were rounded off and in one case, incorrectly reported (Yolo light clay). As a result, the estimated values of D using equation (2) may be slightly different than those reported earlier. As shown in Tyler and Wheatcraft, strong linearity was noted in log–log plots of $N(r > \bar{R})$ verses $\bar{R}$. Such linearity, however, appears to be strongly influenced by the range of the values of $N(r > \bar{R})$ which is typically 6 to 10 orders of magnitude. Figures 6a–b show $N(r > \bar{R})$ for the 10 soils investigated and, the slopes are all quite similar and confidently fit with linear regression. These data were calculated using equation (2) and plotted as $N(r > \bar{R})$ verses arithmetic average of two successive sieve

TABLE 1c. PARTICLE–SIZE DATA FOR COLUMBIA SILT AND SABLE de RIVIERE.

	M(r < R) (%)									
	Radius (mm)									
Soil	0.016	0.025	0.05	0.0715	0.1	0.1425	0.21	0.25	0.5	0.585
Columbia Silt	6.0	14.0	34.6	50.0	66.0	80.0	90.0	93.0	99.3	100.0

	M(r < R) (%)												
	Radius (mm)												
Soil	0.025	0.03	0.04	0.05	0.075	0.1	0.15	0.25	0.35	0.5	0.6	0.75	1.0
Sable de Riviere Sand	4.0	4.2	5.0	8.0	9.5	13.5	29.0	54.0	70.5	84.8	90.0	95.5	100.0

sizes. Using the arithmetic average assumes that the PSD is not continuous with equal size particles in between each sieve size. In reality, the PSD is continuous with soil grains of all sizes making up the mass between successive sieve sizes. Therefore, the number method based (equation (2)) introduces errors from the outset.

To eliminate these initial errors, one may use equation (9) directly and plot percent mass passing, $M(r < R)$ verses R where R now correctly represents the upper sieve radius. Figures 7a–b show the 10 soils of Table 1 fitted using equation (9). The data are no longer clearly linear, and show a wide degree of scatter. The estimated fractal dimensions from the slopes of both Figures 6a–b and 7a–b and their corresponding R^2 values are given in Table 2. There is a significant discrepancy between the estimated fractal dimensions when the two methods are compared.

The basis for this discrepancy lies in the range of the ordinate axis and the errors introduced in estimating the number of grains of any characteristic size for the number based approach. Using the number based approach, apparent linearity is almost guaranteed due the range (up to 10 orders of magnitude) of the ordinate when compared to the range of the abscissa (generally 3 orders of magnitude). Slopes and, therefore, fractal dimensions, are dominated in the least squares analysis by the values of $N(r > \bar{R})$ at the smallest value of $\bar{R}$. Therefore, choice of $\bar{R}$ for the clay–size fraction will control the value of the slope of the fitted line. The choice of $\overline{R}$ for the clay fraction will dominate the calculated

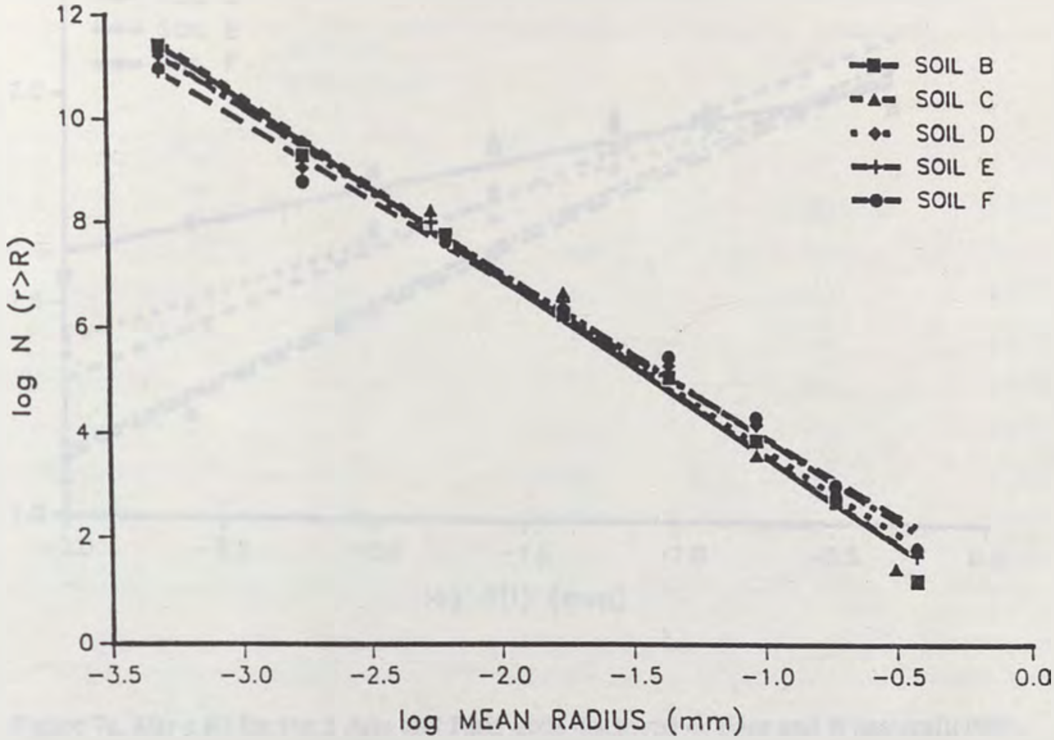

Figure 6a. $N(r > \overline{R})$ for the 5 Ayra and Paris soils discussed in Tyler and Wheatcraft(1989).

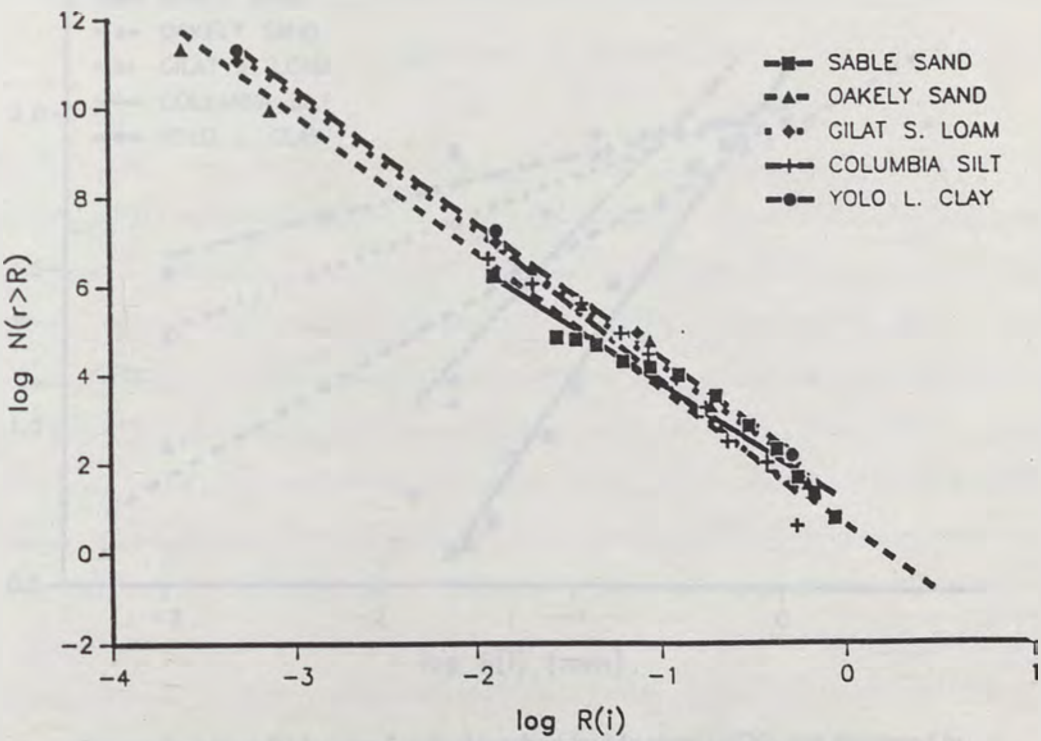

Figure 6b. $N(r > \overline{R})$ for the 5 soils described by Mualem (1976) and discussed in Tyler and Wheatcraft (1989)

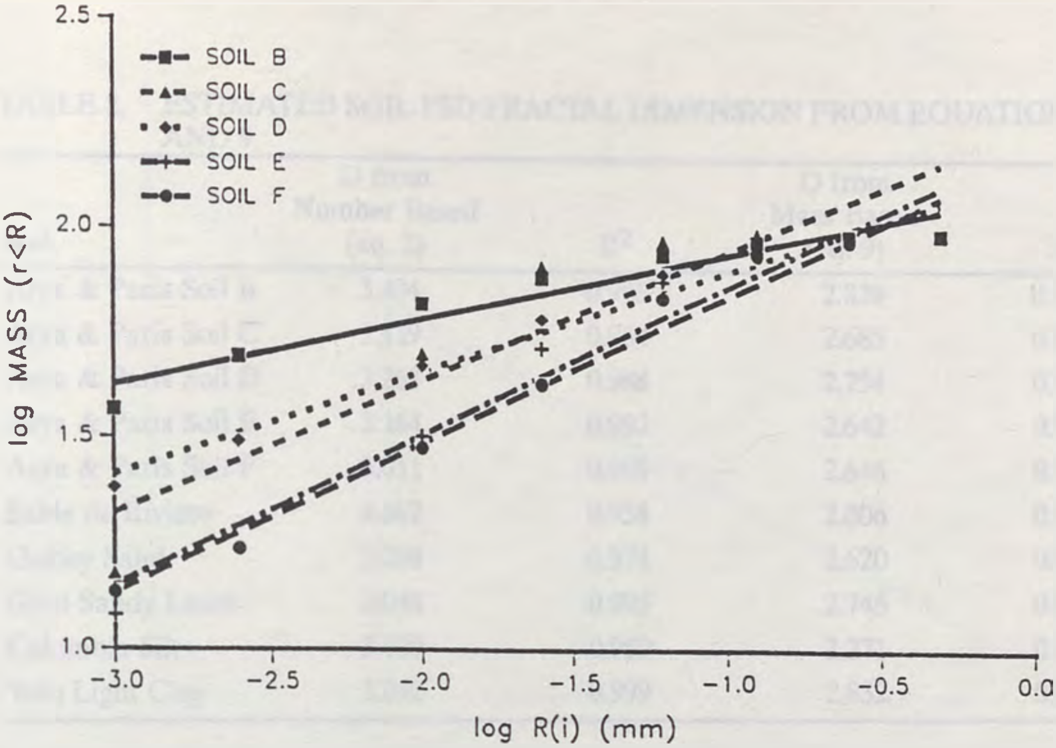

Figure 7a. M(r < R) for the 5 Ayra and Paris' soils discussed in Tyler and Wheatcraft(1989).

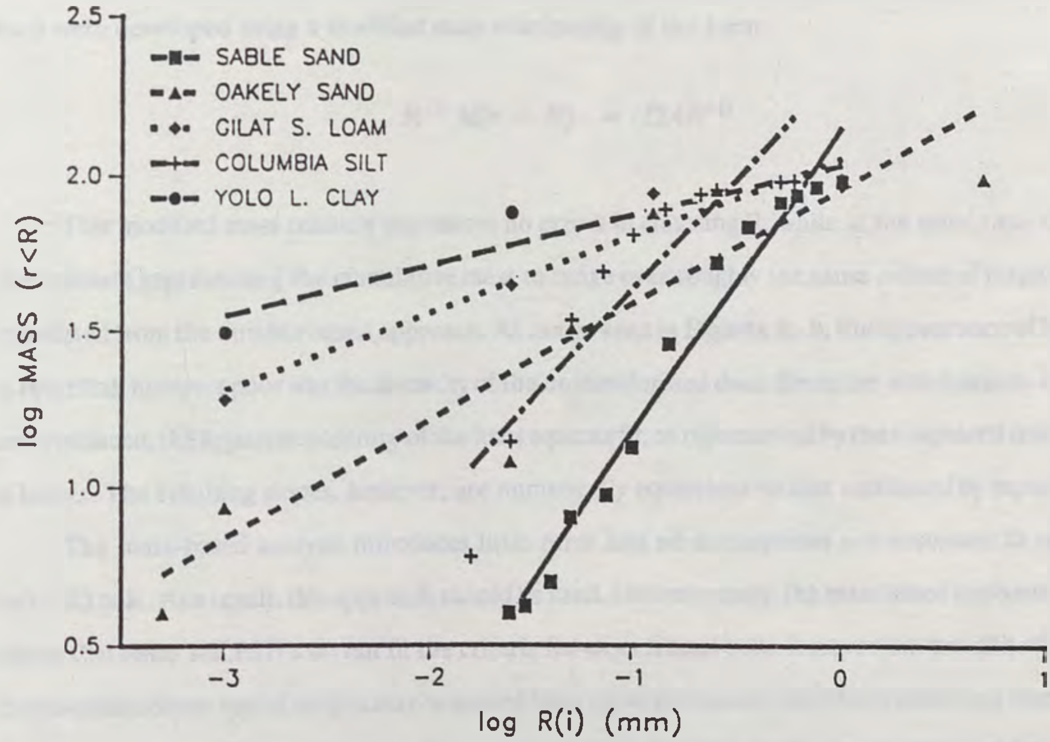

Figure 7b. M(r < R) for the 5 soils described by Mualem (1976) and discussed in Tyler and Wheatcraft (1989).

TABLE 2. ESTIMATED SOIL PSD FRACTAL DIMENSION FROM EQUATIONS 2 AND 9.

Soil	D from Number Based (eq. 2)	R^2	D from Mass Based (eq. 9)	R^2
Arya & Paris Soil B	3.404	0.995	2.839	0.938
Arya & Paris Soil C	3.419	0.984	2.685	0.835
Arya & Paris Soil D	3.263	0.988	2.754	0.970
Arya & Paris Soil E	3.164	0.992	2.642	0.951
Arya & Paris Soil F	3.011	0.989	2.646	0.977
Sable de Riviere	2.682	0.958	2.006	0.972
Oakley Sand	3.098	0.974	2.620	0.875
Gilat Sandy Loam	3.046	0.995	2.745	0.943
Columbia Silt	3.499	0.959	2.271	0.839
Yolo Light Clay	3.042	0.999	2.832	0.887

value of $N(r > \bar{R})$ since its value is cubed (equation (3)) while the magnitude of the actual mass of the clay fraction will have much less of an impact on $N(r > \bar{R})$. To more clearly demonstrate this behavior, Figures 8a–b were developed using a modified mass relationship of the form:

$$R^{-3}\, M(r < R) \;=\; DAR^{-D} \qquad (12)$$

This modified mass relation introduces no errors in choosing R, while at the same time allowing the ordinate representing the cumulative mass to range over roughly the same orders of magnitude as calculated from the number based approach. As can be seen in Figures 8a–b, the appearance of linearity is returned, however poor was the linearity of the untransformed data. Since the non–linearity is apparently reduced, the apparent accuracy of the least squares fit, as represented by the r–squared coefficient, is better. The resulting slopes, however, are numerically equivalent to that estimated by equation (9).

The mass–based analysis introduces little error and no assumptions are necessary to calculate $m(r < R)$ or R. As a result, this approach should be used. Unfortunately, the mass based approach clearly shows that many soil PSD's do not fit the criteria for strict fractal behavior over the breadth of the size distributions. Some useful insight may be gained from those portions of the PSD's exhibiting fractal scaling, particularly in the interpolation of intermediate mass fractions or the interpolation to finer increments between sieve cuts. It appears unlikely, however, that the fractal dimension obtained from the PSD is directly applicable to the estimation of the pore–trace dimension needed in the model of soil

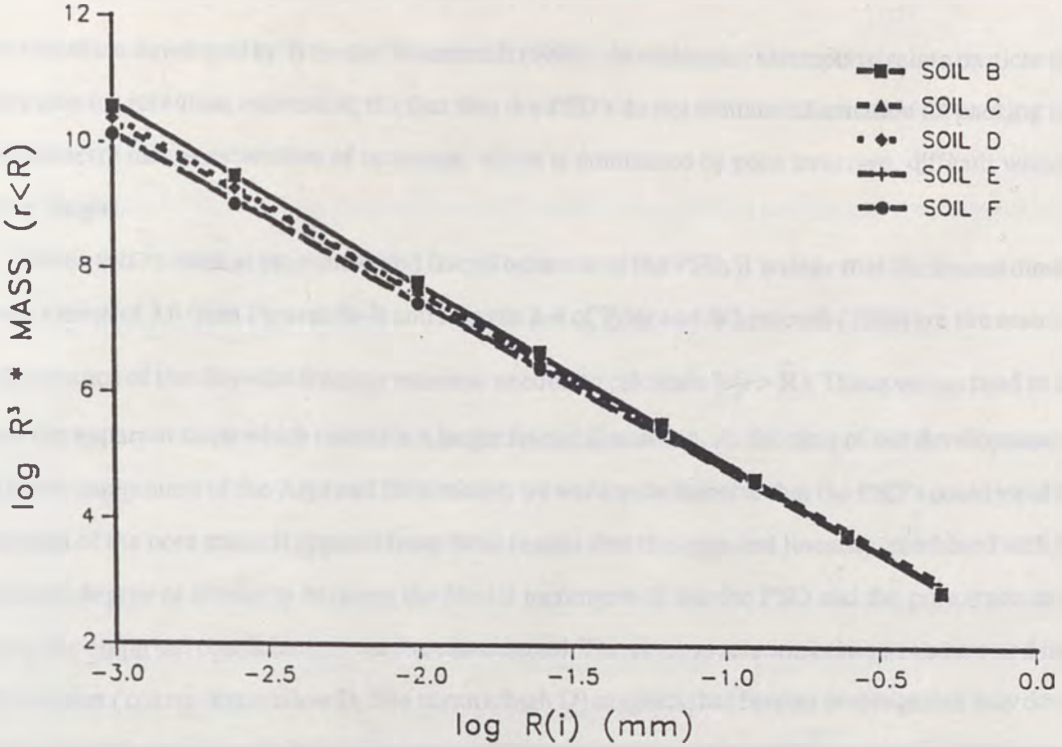

Figure 8a. R^3 M(r < R) verses R for the 5 Ayra and Paris soils.

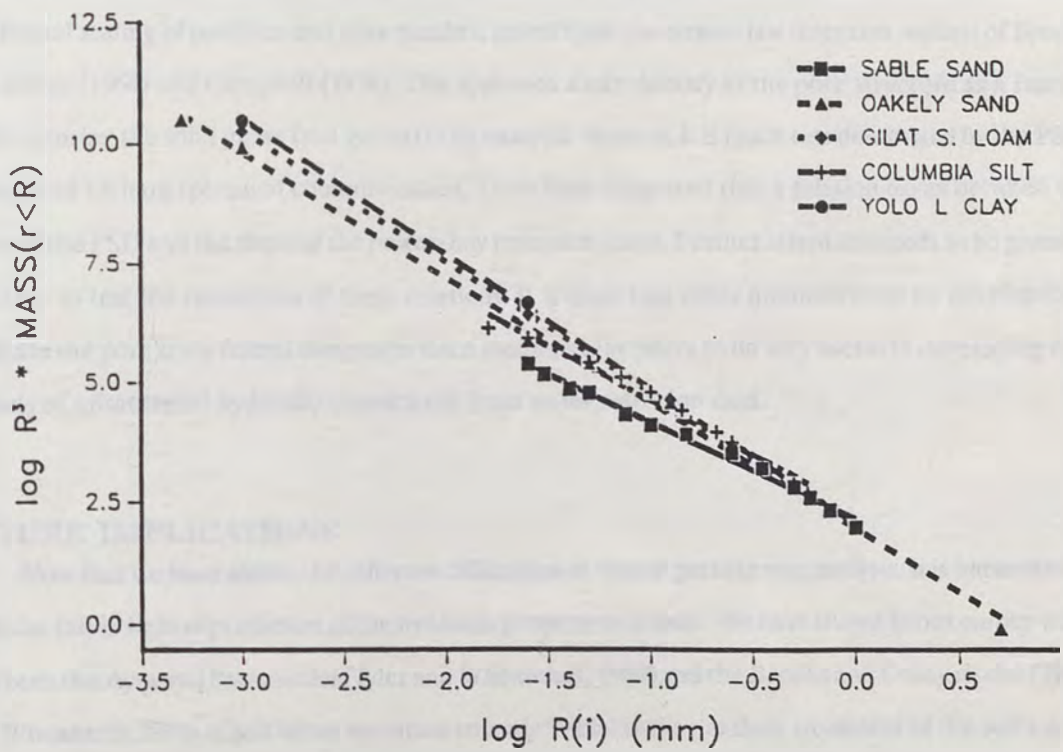

Figure 8b. R^3 M(r < R) verses for the 5 soils described by Mualem (1976).

water retention developed by Tyler and Wheatcraft (1990). As with other attempts to relate particle size to pore size for retention estimation, the fact that the PSD's do not contain information on packing and pore geometry makes estimation of retention, which is dominated by pore structure, difficult without further insight.

Given this re-look at estimation and fractal behavior of the PSD, it is clear that the fractal dimensions in excess of 3.0 from Figures 6a-b and Figures 2-4 of Tyler and Wheatcraft (1989) are the result of the dominance of the clay-size fraction estimate needed to calculate $N(r > \bar{R})$. These values tend to decrease the apparent slope which results in a larger fractal dimension. At the time of our development of the fractal component of the Arya and Paris model, we were quite hopeful that the PSD's could yield the dimension of the pore trace. It appears from these results that the apparent linearity, combined with the significant degree of similarity between the fractal increment of the the PSD and the pore trace as reflecting the empirical coefficient, α, may be coincidental. The obvious relation between texture and fractal dimension (coarse texture/low D, fine texture/high D) suggests that further investigation may develop a relationship between the hydraulic nature of the pore space and the solid phase, but it will be necessary to include further information regarding the nature of the packing of the solid phase. In a recent paper, (Tyler and Wheatcraft, 1990) we have suggested a simple relationship between water retention and fractal scaling of pore size and pore number, based upon the power-law retention models of Brooks and Corey (1964) and Campbell (1974). This approach looks directly at the pore structure as a fractal, while ignoring the solid phase (soil grains) in its analysis. As such, it is much less dominated by the PSD. Chang and Uehara (personal communication, 1989) have suggested that a relation exists between the slope of the PSD and the slope of the power-law retention curve. Further attention needs to be given to this area to test the robustness of these relations. It is clear that other methods must be developed to estimate the pore trace fractal dimension since methods may prove to be very useful in developing new models of unsaturated hydraulic conductivity from water retention data.

FUTURE IMPLICATIONS

Now that we have shown the inherent difficulties in fractal particle size analysis, it is important to consider this in light of prediction of the hydraulic properties of soils. We have shown in our earlier work that both the Ayra and Paris model (Tyler and Wheatcraft, 1989) and the Brooks and Corey model (Tyler and Wheatcraft,1990) of soil water retention embody fractal scaling in their treatment of the soil's pore structure. We have also suggested that the apparent fractal PSD data could be used to estimate the fractal nature of this pore structure. Bridging this gap between fractal particles sizes and fractal pore structure in reality imposes a very strong restriction on the porous medium, i.e. *the porous medium must be fractal*. Such an assumption may be too stringent for most soils. We examine the necessary conditions for such behavior below.

For the case of fractal porous medium, both the solid phase (PSD) and the void phase (pores) must follow a power law scaling of the form:

$$N_s\ (r > R_s) \propto R_s^{-D} \tag{13}$$

$$N_v\ (r > R_v) \propto R_v^{-D} \tag{14}$$

where the subscripts s and v stand for solid and void, respectively. In order that the porous medium display fractal scaling in its intrinsic properties, i.e. porosity and bulk density, etc., the following relationship between the characteristic scales must be operative:

$$R_v = \lambda\ R_s \tag{15}$$

where λ is a linear scaling coefficient. By fixing the relation between the solid and void scales, the bulk medium can be described with either the solid or void phase properties. With this relation invoked, the porous medium will be self similar, i.e. the porous structure will look the same independent of the scale of observation. The implications of fractal scaling for both solid and void space at the same time are a medium whose porosity will grow in increasing scale of observation while concurrently, the bulk density will decrease with increasing scale of observation.. These properties will necessarily scale as a power function of the scale of observation. Invoking this rather special case of fractal behavior, although appealing from the standpoint of estimating hydraulic behavior from grain size behavior, may be too limiting.

At this time, the evidence is quite clear that many natural porous media display fractal scaling in their pore structure, but the jury is not yet in on the evidence of fractal scaling in their interactions with the solid phase i.e. the traditional definitions of a porous medium. Although at first disappointing, many of our studies concentrate on the way in which fluids move through the void spaces of soils and aquifers and the applicability of fractal scaling to this structure is significant. The fractal behavior of water retention, pore trace tortuosity, and differential advection of solutes gives the soil physicist and hydrogeologist a powerful new tool to understand transport in spatially variable soils and aquifers. We must continue to explore techniques to extract the necessary scaling information (fractal dimension, range of fractal scaling, etc.) from easily measured quantities in the field. Such approaches may rely on both PSD data and information on the porous media behavior, such as bulk density. From information on both the solid phase and the bulk behavior, the pore structure behavior may be extractable. It is clear that these as well as other approaches need to be initiated.

CONCLUSIONS

In this review of fractal scaling of soil particle–size distributions, we have developed bounding values for the range of physically plausible fractal dimensions. Dimensions previously reported in our work routinely exceeded these boundaries and are the result of error introduced in using the traditional number–based approach to estimate the fractal dimensions. Based upon these introduced errors, we suggest using a mass based approach to estimate the fractal dimension of the PSD. Using this unbiased approach, we have shown that strict fractal or self similar behavior in soil PSD's are restricted to a narrow spectrum of soils found in nature. As a result, the estimation of the pore–trace fractal dimension from the PSD as suggested in our earlier work (Tyler and Wheatcraft, 1989) does not appear to be a robust estimator. Research to date strongly supports the notion that many soils and porous media display fractal scaling in their pore space, however, such is not a sufficient condition to insure that the solid phase (PSD) is also fractal. Further research is, therefore, needed to develop techniques to extract the pore–trace dimension since it should also be useful in development of models relating retention, unsaturated conductivity and other hydraulic properties critical to the understanding of fluid and contaminant transport in soils and aquifers.

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