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Linear Hybrid Dynamical Control of Digital Displacement Units

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ABSTRACT

This paper concerns development of control strategies for energy efficient fluid power digital displacement machines® (DDM). The DDM technology yields an efficient reduction in displacement levels by deactivating independently cylinder pressure chambers by electromagnetically controlled on-off valves. Since the continuous dynamics of each pressure chamber is activated/deactivated discretely at fixed shaft positions, the DDM dynamics belongs to the class of hybrid dynamical systems. However, control development for hybrid system is in general very complex due to the use of Lyapunov stability theory for both continuous and discrete systems. This paper shows stability based on hybrid dynamical theory for a linear continuous plant actuated by a DDM, which dynamics has been approximated by a linear discrete model. It is shown that the control design problem is identically to that of a fully discrete description of the system based on a zero-order-hold approximation of the continuous design. Furthermore, the problem of having a nonlinear plant, multiple DDMs or an angle dependent asynchronous control update rate for a variable speed DDM is addressed, where measures of stability analysis is discussed.

KEYWORDS: Fluid Power, Digital Displacement, Hybrid system, Control model, Event-driven

1 INTRODUCTION

Development of control strategies for digital displacement machines (DDM) is considered an important aspect with respect to ensuring proper operation of the hydraulic system actuated by the digital pump/motor units. However, this is a complicated task since the continuous dynamics of each pressure chamber is activated/deactivated discretely as function of the shaft angle. As a result, the system dynamics belongs to the class of hybrid dynamical system, interconnecting continuous and discrete dynamics. This may explain why most state-of-the-art control strategies neglect the dynamics of the DDM when designing the control system for the actuated plant [1, 2, 3, 4, 5, 6]. A continuous approximation of a DDM is proposed by Pedersen et. al [7], but is only valid for a relatively high number of cylinders, at high displacements and for relatively low frequency excitations. A discrete approximation has been derived by Johansen et. al. [8] and is used for closed loop control in [9, 10, 11]. Similar to the continuous model, the discrete model is only valid for a relatively high number of cylinders, high displacements (linearly). Additionally, it is only applicable for a single machine with fixed shaft speed. To increase the validity and accuracy, hybrid dynamical models have been proposed by Pedersen et. al [12, 13] and Sniegucki et. al [14]. Model based control design for hybrid systems is based on Lyapunov stability theory for both nonlinear continuous and discrete system and is thus challenging even for relatively simple systems [15, 16, 17, 18, 19]. This paper presents a linear representation of the hybrid dynamical formulation presented in [12] and shows stability of a linear continuous actuated plant with use of Lyapunov stability theory for hybrid systems. Since the considered hybrid system consist of linear continuous and discrete state equations, a linear state feedback controller is sufficient to guarantee stability. It is shown that the control tuning problem may be performed by classical DLTI pole placement method, where the continuous plant dynamics is approximated with a zero-order-hold input. Furthermore, the challenge of having a state (angle) dependent sampling rate, nonlinear continuous plant and multiple DDMs is discussed as further work.

2 NONLINEAR MATHEMATICAL MODEL

The hybrid dynamical model is constructed based on a description of the nonlinear dynamics of the DDM. The model equations are only presented briefly, where a more detailed description is found in [10, 20, 12, 13]. The same nonlinear model describing the DDM dynamics may be found in [21]. The nonlinear mathematical model is established based on the illustration of the DDM and a single pressure chamber shown in Fig. 1.

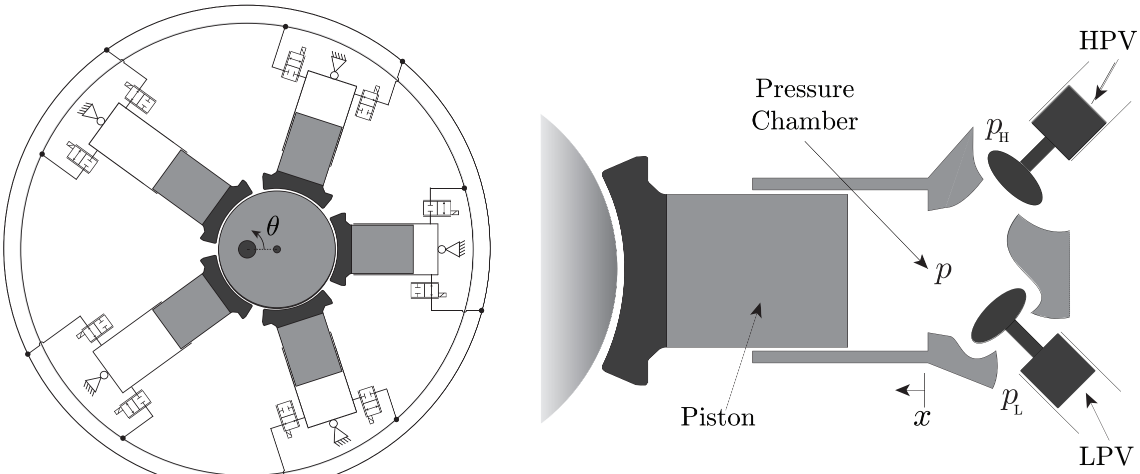

Figure 1: Illustration of the radial piston type digital displacement machine (5 cylinder machine) and definition of variables used for the mathematical model [20, 22].

The piston displacement for the i'th cylinder, x_i, is described as function of the shaft angle by

$$x_i = r_e\,(1-\cos(\theta_i)) \qquad \theta_i = \theta + \frac{2\,\pi}{N_c}\,(i-1) \qquad i \in \{1,\ldots,N_c\} \tag{1}$$

where r_e is the eccentric shaft radius and N_c is the number of cylinders. The displacement volume is thus given as $V_d = 2\,r_e A_p$, where A_p is the piston area. The chamber volume for the i'th cylinder, V_i, and its time derivative are then given by

$$V_i = \frac{V_\mathrm{d}}{2}\,(1-\cos(\theta_i)) + V_0 \qquad \dot{V}_i = \frac{V_\mathrm{d}}{2}\,\sin(\theta_i)\,\dot{\theta} \tag{2}$$

where V_0 is the minimum chamber volume. The continuity equation is used to describe the pressure build-up for the i'th cylinder given by

$$\dot{p}_i = \frac{\beta_\mathrm{e}(p_\mathrm{i})}{V_\mathrm{i}}\,\left(Q_{\mathrm{H},i} - Q_{\mathrm{L},i} - \dot{V}_i\right) \tag{3}$$

where β_e is the pressure dependent effective bulk modulus. Q_L and Q_H are the flows through the low and high pressure valve respectively. The orifice equation is used to describe the flows through the valves and are given to be

$$Q_{\mathrm{L},i} = \frac{x_{\mathrm{L,i}}}{k_\mathrm{f}}\,\sqrt{|p_i - p_\mathrm{L}|}\,\mathrm{sign}\,(p_i - p_\mathrm{L}) \qquad Q_{\mathrm{H},i} = \frac{x_{\mathrm{H,i}}}{k_\mathrm{f}}\,\sqrt{|p_\mathrm{H} - p_i|}\,\mathrm{sign}\,(p_\mathrm{H} - p_i) \tag{4}$$

where $x_\mathrm{L} \in [0,1]$ and $x_\mathrm{H} \in [0,1]$ are normalized valve plunger positions, while k_f is the valve flow coefficient. When considering the fundamental machine dynamics it is deemed sufficient to model the valves as a simple first order system given as

$$\dot{x}_{\mathrm{H,i}} = \frac{1}{\tau_\mathrm{v}}\,(u_{\mathrm{H,i}} - x_{\mathrm{H,i}}) \qquad \dot{x}_{\mathrm{L,i}} = \frac{1}{\tau_\mathrm{v}}\,(u_{\mathrm{L,i}} - x_{\mathrm{L,i}}) \tag{5}$$

where u_H and u_L are the inputs to the high and low pressure valve respectively, while t_v is the valve time constant. The torque contribution from the i'th pressure chamber is derived to be given by

$$\tau_i = \frac{d\,V_i\,(\theta_i)}{d\theta}\,p_i = \frac{V_\mathrm{d}}{2}\,\sin(\theta_i)\,p_i \tag{6}$$

For a full stroke operated DDM, the binary valve inputs may only be altered discretely at fixed shaft positions. Since the nonlinear model comprises several states for each cylinder chamber, using the nonlinear model for control development is considered extremely difficult. Therefore, a significantly simpler model of the DDM is used which neglects the pressure dynamics of the individual chamber and thereby directly describes the output torque or flow.

3 DISCRETE DYNAMICAL DDM MODEL

Considering the pressure build-up in each chamber to be significantly faster than the remaining machine dynamics, the flow and torque throughput may be approximated by [23, 9, 11, 10]

$$\begin{aligned} Q_\mathrm{H} &\approx \frac{dV(\theta)}{d\theta}\,\frac{d\theta}{dt} = \frac{V_\mathrm{d}}{2}\,\sin(\theta)\,\dot{\theta} = \mathcal{D}(\theta)\,\dot{\theta} \\ \tau &= \frac{dV(\theta)}{d\theta}\,p \approx \frac{V_\mathrm{d}}{2}\,\sin(\theta)\,p = \mathcal{D}(\theta)\,p \end{aligned} \tag{7}$$

Since the binary input may only be altered once for every cylinder per revolution, the sampling angle is given by $\theta_\mathrm{s} = 2\,\pi/N_\mathrm{c}$. The displacement fraction between samples may then be evaluated by

$$\mathcal{D}[k] = \frac{\Delta V[k]}{\theta_\mathrm{s}} = \frac{\left(V\,(\theta[k+1]) - V\,(\theta[k])\right)N_\mathrm{c}}{2\,\pi} \tag{8}$$

where the discrete control update angle is given as

$$\theta[k] = \phi_0 + \theta_s\,(k-1) \qquad\qquad i \in \{1,...,N_c\} \tag{9}$$

where ϕ_0 is the local shaft angle where the LPV is closed to initiate an active stroke. In this paper a digital displacement motor is considered, but the presented method also applies for a pump. The change is volume for the motoring stroke is evaluated by

$$\Delta V[k] = \begin{cases} 0 & \theta[k], \theta[k+1] \notin [0\,;\phi_H] \\ V(\theta[k+1]) - V(\theta[k]) & \theta[k], \theta[k+1] \in [0\,;\phi_H] \\ V(\theta[k+1]) - V(0) & \theta[k] < 0 < \theta[k+1] \\ V(\phi_H) - V(\theta[k]) & \theta[k] < \phi_H < \theta[k+1] \end{cases} \tag{10}$$

where ϕ_H is the angle where the motoring stroke is ended by closing the HPV. The discrete state model comprises of memory states of the input decisions (active or inactive) and are given by the matrix-vector representation as

$$\underbrace{\begin{bmatrix} u(k) \\ u(k-1) \\ u(k-2) \\ \vdots \\ u(k-m+1) \end{bmatrix}}_{z(k+1)} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A_u} \underbrace{\begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(k-m+1) \\ u(k-m) \end{bmatrix}}_{z(k)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B_u} u(k) \tag{11}$$

$$\mathcal{D}(k) = \begin{bmatrix} \mathcal{D}[1] & \mathcal{D}[2] & \ldots & \mathcal{D}[m] \end{bmatrix} z(k) + \mathcal{D}[0]\,u(k)$$

The chamber pressure buildup as function of the shaft angle is described, by neglecting the fast transient flow dynamics and gives

$$\frac{dp}{dt}\frac{dt}{d\theta} = \frac{dp}{d\theta} = -\frac{\beta_e}{V(\theta)}\frac{dV(\theta)}{d\theta} = \underbrace{-\frac{\beta_e}{\frac{V_d}{2}\,(1-\cos(\theta)) + V_0}\frac{V_d}{2}\sin(\theta)}_{f_p(\theta)} \tag{12}$$

A discrete representation of the chamber pressure may then be written as [14].

$$p(k+1) = \begin{cases} p_H & if\ \ (\bar{x}_L = 0 \wedge \bar{x}_H = 1) \vee p(k+1) > p_H \\ p(k) + f_p(\theta[k]) & if\ \ \bar{x}_L = 0 \wedge \bar{x}_H = 0 \\ p_L & if\ \ (\bar{x}_L = 1 \wedge \bar{x}_H = 0) \vee p(k+1) < p_L \end{cases} \tag{13}$$

By normalizing the discrete pressure by $p(k) = \bar{p}(k)\,p_H(k)$, the torque throughput of the discrete model is given by $\tau(k) = \mathcal{D}(k)\,\bar{p}(k)\,p_H(k)$. The resulting output maps for the torque and flow thus becomes

$$\tau_m(k) = \underbrace{\begin{bmatrix} \mathcal{D}[1]\,\bar{p}[1] & \mathcal{D}[2]\,\bar{p}[2] & \ldots & \mathcal{D}[m]\,\bar{p}[m] \end{bmatrix} p_H}_{C_\tau} z(k) + \underbrace{\mathcal{D}[0]\,\bar{p}[0]\,p_H}_{D_\tau} u(k)$$

$$Q_m(k) = \underbrace{\begin{bmatrix} \mathcal{D}[1] & \mathcal{D}[2] & \ldots & \mathcal{D}[m] \end{bmatrix} \omega}_{C_q} z(k) + \underbrace{\mathcal{D}[0]\,\omega}_{D_q}\ u(k) \tag{14}$$

For simplicity ω and p_H are considered constant in this paper, which yields the same model as when linearizing at zero displacement. A validation of the discrete model is done

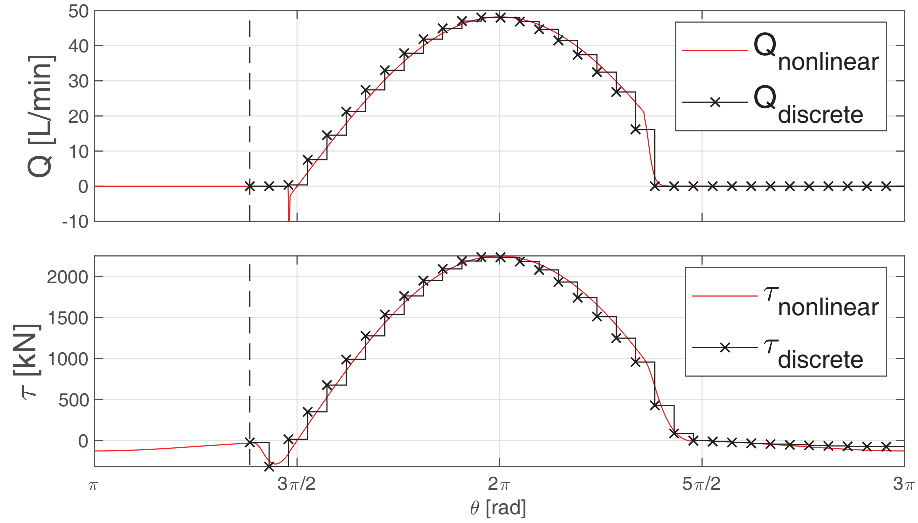

Figure 2: Comparison of discrete and non-linear model response with 42 cylinders. $\omega = 100\,\mathrm{rpm}$, $p_\mathrm{H} = 300\,\mathrm{bar}$, $p_\mathrm{L} = 10\,\mathrm{bar}$ [24].

by an impulse response comparison between the discrete and nonlinear model, where the results are shown in Fig. 2. It is seen that the discrete model is fairly accurate in describing the DDM dynamics as long as the number of cylinders are fairly high. This discrete model representation of the DDM dynamics is combined with the continuous plant dynamics to obtain a hybrid dynamical description of the hydraulic system.

4 LINEAR HYBRID DYNAMICAL MODEL

Control of a continuous plant actuated by a DDM with discretely updated inputs, may be classified as a hybrid dynamical system. A hybrid system comprises of both continuous differential equations and discrete difference equations. A hybrid system is in general formulated as [19]

$$\mathcal{H}: \quad x \in \mathbb{R}^n \quad \begin{cases} \dot{x} & \in F(x), \qquad\qquad x \in C \\ x^+ & \in G(x), \qquad\qquad x \in D \end{cases} \tag{15}$$

$\dot{x}$ denotes the state time derivative and x^+ denotes the state value after a jump. The sets and map used to describe the hybrid dynamical system are:

- The flow set C $\subset \mathbb{R}^n$
- The flow map F: C $\rightarrow \mathbb{R}^n$
- The jump set: D $\subset \mathbb{R}^n$
- The jump map: G: D $\rightarrow \mathbb{R}^n$

As long as the state x belongs to the flow set C, x is described by the differential inclusion given by the flow map F and when x belongs to the jump set D, x is described by the difference inclusion given by the jump map G.

In this paper, the discrete linear time invariant (DLTI) approximation of the DDM dynamics derived in (11) and (14) is used for stability analysis in the sense of a hybrid dynamical systems. However, to enable a description in the hybrid domain where the dynamics is described in serial connection (either flowing or jumping) requires a reformulation given by

$$\underbrace{\begin{bmatrix} z(k) \\ z(k-1) \\ z(k-2) \\ \vdots \\ z(k-m) \end{bmatrix}}_{z^+} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A_z} \underbrace{\begin{bmatrix} z(k-1) \\ z(k-2) \\ \vdots \\ z(k-m) \\ z(k-m-1) \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B_z} u(k) \tag{16}$$

$$\begin{aligned} \tau_\mathrm{m} &= \underbrace{\begin{bmatrix} \mathcal{D}[0]\,\bar{p}[0] & \mathcal{D}[1]\,\bar{p}[1] & \ldots & \mathcal{D}[m]\,\bar{p}[m] \end{bmatrix}}_{C_{z\tau}} p_\mathrm{H}(k)\, z^+ \\ Q_\mathrm{m} &= \underbrace{\begin{bmatrix} \mathcal{D}[0] & \mathcal{D}[1] & \ldots & \mathcal{D}[m] \end{bmatrix}}_{C_{zq}} \omega(k)\, z^+ \end{aligned} \tag{17}$$

In this paper, a linear actuated plant with continuous dynamics given by $\dot{x} = A\,x + B_\mathrm{p}\,u_\mathrm{p}$ is considered, where u_p is either the torque, τ_m, or flow, Q_m, dependent on the controlled variable (speed or pressure). The linear hybrid model of the continuous plant actuated by the DDM is given to be

$$\left.\begin{aligned} \dot{x} &= A\,x + B\,z \\ \dot{z} &= 0 \\ \dot{\chi} &= 1 \end{aligned}\right\} \chi \in [0,T] \qquad \left.\begin{aligned} x^+ &= x \\ z_\mathrm{u}^+ &= A_\mathrm{z}\,z + B_\mathrm{z}\,u \\ \chi^+ &= 0 \end{aligned}\right\} \chi = T \tag{18}$$

where χ is a timer generating the control updates when reaching $T = 2\,\pi/(\omega\,N_\mathrm{c})$. The input matrix B is defined as $B_\mathrm{p}\,C_z$, where $C_\mathrm{z} = C_{z\tau} \wedge C_{z\tau}$ depending on whether the output is torque or flow. Similarly, $C_\mathrm{u} = C_\tau \wedge C_q$ for the DLTI model.

5 STABILITY AND FEEDBACK CONTROL

Control design of hybrid dynamical systems is done by stability analysis featuring a Lyapunov function candidate. The objective is to show Uniformly Globally Pre-asymptoically Stability (UGPaS) for the hybrid system with respect to the set $\mathcal{A}$. The definition of UGPaS for the hybrid dynamical system $\mathcal{H}$ is defines as [19]:

(Sufficient Lyapunov Conditions) Let $\mathcal{H} = (C,F,D,G)$ be a hybrid system and let $\mathcal{A} \subset \mathbb{R}^n$ be closed. If V is a Lyapunov function candidate for $\mathcal{H}$ and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and a continuous positive definite function $\rho : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned} &\bullet\quad \alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) && \forall x \in C \cup D \cup G(D) \\ &\bullet\quad \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_\mathcal{A}) && \forall x \in C, \forall f \in F(x) \\ &\bullet\quad V(g(x)) - V(x) \leq -\rho(|x|_\mathcal{A}) && \forall x \in D, \forall g \in G(x) \end{aligned} \tag{19}$$

Then $\mathcal{A}$ is Uniformly Globally p re-Asymptotically Stable for $\mathcal{H}$.

where $\mathcal{A} = \{0\} \times \{0\} \times [0, T_s]$ for the states $\begin{bmatrix} x & z & \chi \end{bmatrix}$. Since the hybrid system is linear, it may be optimally controlled by a linear controller. Inserting the control law $u = K x_1$, where $x_1 = \begin{bmatrix} x & z \end{bmatrix}^T$ and $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ yields the closed loop system dynamics given by

$$\left.\begin{aligned} \dot{x} &= Ax + Bz \\ \dot{z} &= 0 \\ \dot{\chi} &= 1 \end{aligned}\right\} \chi \in [0, T] \qquad \left.\begin{aligned} x^+ &= x \\ z^+ &= A_z z + B_z K x_1 \\ \chi^+ &= 0 \end{aligned}\right\} \chi = T \tag{20}$$

The flow and jump maps may be written in a compact form as

$$\begin{aligned} f(x_1) &= \begin{bmatrix} A_f x_1 \\ 1 \end{bmatrix} & g(x_1) &= \begin{bmatrix} A_g x_1 \\ 0 \end{bmatrix} \\ A_f &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} & A_g &= \begin{bmatrix} I & 0 \\ B_z K_1 & A_z + B_z K_2 \end{bmatrix} \end{aligned} \tag{21}$$

The stability analysis is then made through the use of the following definitions [19]

$$H := e^{(A_f T)} A_g \qquad\qquad W(x_1) := x_1^T P x_1 \tag{22}$$

where P is a positive definite and symmetric matrix. It follows that $x_1(t+T, j+1) = H x_1(t, j)$, where H describes the closed loop system dynamics. The dissipation of energy may be shown by evaluating the change in $W(x_1)$ due to a jump, which must satisfy (23) for $\varepsilon > 0$

$$W(Hx_1) - W(x_1) = x_1^T (H^T P H - P) x_1 \leq -\varepsilon\, |x_1|^2 \tag{23}$$

Meaning that $H^T P H - P \prec 0$ or equivalently $|\lambda(H)| < 1$ [19]. Considering the Lyapunov function candidate

$$\begin{aligned} V(x) &:= e^{(-\sigma x_2)} W\left(e^{(A_f(T-x_2))} x_1\right) \\ &:= \underbrace{e^{(-\sigma x_2)}}_{\succ 0} x_1^T \underbrace{\left(e^{(A_f(T-x_2))}\right)^T P \left(e^{(A_f(T-x_2))}\right)}_{\succ 0} x_1 \end{aligned} \tag{24}$$

where $\sigma > 0$. It may be established that the Lyapunov candidate function, $V(x)$, fulfills the requirements in (19) for UGPaS. Since the exponential functions and P are always positive definite, it can be verified that

$$\underline{c}\, |x|^2_{\mathcal{A}} \leq V(x) \leq \bar{c}\, |x|^2_{\mathcal{A}} \qquad\qquad \forall x \in C \cup D \tag{25}$$

where $\underline{c} > 0$ and $\bar{c} > 0$, given that $\underline{c} < \bar{c}$ and thus the first requirement in (19) is fulfilled. For the second requirement in (19), it may be established that

$$\begin{aligned} \langle \nabla V(x), f(x) \rangle &= -\underbrace{\sigma\, e^{(-\sigma x_2)}}_{\succ 0} x_1^T \underbrace{\left(e^{(A_f(T-x_2))}\right)^T P \left(e^{(A_f(T-x_2))}\right)}_{\succ 0} x_1 \leq -\beta |x_1|^2 \\ \langle \nabla V(x), f(x) \rangle &= -\beta |x|^2_{\mathcal{A}} \qquad\qquad \forall x \in C \end{aligned} \tag{26}$$

where $\beta > 0$. The third requirement may be established to yield (27), when considering that $x_2 = T$ when jumping.

$$\begin{aligned} V(g(x)) - V(x) &= e^{(-\sigma x_2)} \left(W \left(e^{(A_\mathrm{f}(T-x_2))} A_\mathrm{g} x_1 \right) - W(x_1) \right) \\ &= e^{(-\sigma T)} \left(W(H x_1) - W(x_1) \right) \\ &= \underbrace{e^{(-\sigma T)}}_{\succ 0} \left(-\varepsilon |x_1|^2 \right) \leq -\gamma |x|_{\mathcal{A}}^2 \qquad\qquad \forall x \in D \end{aligned} \tag{27}$$

The hybrid dynamical system is hence shown to be UGPaS for those values of K such that $|\lambda(H)| < 1$.

6 CONTROLLER SYNTHESIS

The objective for control design is hence to determine the controller gain K, such that $|\lambda(H)| < 1$. When evaluating H, it is found that the closed loop dynamics is given to be

$$H = \underbrace{\begin{bmatrix} A_\mathrm{d} & B_\mathrm{d} C_\mathrm{z} \\ 0 & I \end{bmatrix}}_{e^{(A_\mathrm{f} T)}} \underbrace{\begin{bmatrix} I & 0 \\ B_\mathrm{z} K_1 & A_\mathrm{z} + B_\mathrm{z} K_2 \end{bmatrix}}_{A_\mathrm{g}} = \begin{bmatrix} A_\mathrm{d} + B_\mathrm{d} C_\mathrm{z} B_\mathrm{z} K_1 & B_\mathrm{d} C_\mathrm{z} \left(A_\mathrm{z} + B_\mathrm{z} K_2 \right) \\ B_\mathrm{z} K_1 & A_\mathrm{z} + B_\mathrm{z} K_2 \end{bmatrix} \tag{28}$$

where A_d and B_d are the discrete system matrices for the continuous plant. The objective is hence to determine the controller gains $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ to yield the desired pole-locations. It may be found that the control design problem is the same as for a sample-and-hold DLTI description of the control system. The open loop description of the DLTI model yields

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A_\mathrm{d} & B_\mathrm{d} C_\mathrm{u} \\ 0 & A_\mathrm{u} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} B_\mathrm{d} D_\mathrm{u} \\ B_\mathrm{u} \end{bmatrix} u(k) \tag{29}$$

Inserting the control law $u(k) = \begin{bmatrix} K_1 & K_2' \end{bmatrix} \begin{bmatrix} x(k) & z(k) \end{bmatrix}^T$ yields

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} A_\mathrm{d} + B_\mathrm{d} D_\mathrm{u} K_1 & B_\mathrm{d} \left(C_\mathrm{u} + D_\mathrm{u} K_2' \right) \\ B_\mathrm{u} K_1 & A_\mathrm{u} + B_\mathrm{u} K_2' \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} \tag{30}$$

Comparing by coefficient between (28) and (30) reveals that $K_2 = \begin{bmatrix} K_2' & 0 \end{bmatrix}$ in which case the closed loop system dynamics are equivalent to that given in (28). Therefore, one state in the hybrid formulation becomes indifferent, since the current updated input may not be used to generate the current input. For this linear system, a hybrid formulation is hence not needed and the significantly simpler DLTI description may be used for control design. However, if a nonlinear continuous plant, varying speed DDM or multiple DDMs in a transmission is considered, a hybrid dynamical system formulation is likely a necessity. However, the identification of a Lyapunov function may be very difficult to find. An important problem is variable speed operation of the DDM, where the objective is to show that $|\lambda(H)| < 1$ where H is varying as function of the rotational speed being a state.

$$H = e^{A_\mathrm{f} T} A_\mathrm{g} \qquad\qquad T = \frac{2\pi}{\omega N_\mathrm{c}} \tag{31}$$

It has been attempted to transform the system dynamics to the shaft position domain, which results in a nonlinearity in the continuous plant dynamics due to

$$\frac{dx}{dt} = f(x(t), u(t)) \qquad \frac{dx}{dt}\frac{dt}{d\theta} = \frac{dx}{d\theta} = \frac{1}{\omega(\theta)} f(x(\theta), u(\theta)) \qquad \omega \neq 0 \tag{32}$$

The introduction of the nonlinearity greatly increases the complexity of finding a Lyapunov function if it is not linearized. Also the theory of dynamical systems on time-scales has been found insufficient in solving the problem of variable speed operation, since the time-scale (speed) has to be known priori to ensure stability. Therefore, further research in stability proof and controller synthesis for variable speed operated DDM(s) and nonlinear actuated plants is required.

7 CONCLUSION

This paper concerned control development for digital hydraulic pump/motor units, which is highly challenged by the non-smooth dynamical behavior. The paper presented a linear hybrid dynamical description of a linear continuous plant actuated by a digital displacement machine with discretely updated input. A stability proof was conducted for a linear feedback controller through the use of a control Lyapunov function. It was further shown that the control design problem for the hybrid system is identical to that of using a purely discrete model with zero-order-hold input to the discrete approximated continuous plant. It was further discussed how the hybrid formulation may be used for control of a nonlinear plant, variable speed DDM operation and multiple DDMs in a transmission, where the purely discrete formulation is no longer applicable. However, the difficultly of finding a Lyapunov function for a hybrid system is considered very challenging for any nonlinear or time varying plant. Therefore, a great amount of further research is necessary to solve these challenging mathematical problems.

8 ACKNOWLEDGMENT

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References

[1] M. Ehsan, W. Rampen, and S. Salter, "Modeling of digital-displacement pumpmotors and their application as hydraulic drives for nonuniform loads," *Journal of Dynamic Systems, Measurement and Control, vol. 122, no. 1, pp. 210215*, 2000.

[2] M. Heikkila and M. Linjama, "Displacement control of a mobile crane using digital hydraulic power management system," *Mechatronics 23(4), pp. 452 - 461*, 2013.

[3] S. Nordaas, M. K. Ebbesen, and T. O. Andersen, "The potential of a digital hydrulic winch drive system," *Proc. of the Ninth Workshop on Digital Fluid Power, Aalborg, Denmark*, 2017.

[4] S. Nordaas, M. K. Ebbesen, and T. O. Andersen, "Feasibility study of a digital hydraulic winch drive system," *Proc. of the Ninth Workshop on Digital Fluid Power, Aalborg, Denmark*, 2017.

[5] B. S. R. Armstrong and Q. Yuan, "Multi-level control of hydraulic gerotor motor and pumps," *Proceedings of the 2006 American Control Conference, Minneapolis, Minnesota, USA*, June 2006. Proceedings of the 2006 American Control Conference Minneapolis, Minnesota.

[6] X. Song, "Modeling an active vehicle suspension system with application of digital displacement pump motor," *Proceedings of the ASME 2008 International Design Engineering Techical Conference/Computers and Information in Enegineering Conference, Brooklyn - New York, Vol. 5, pp. 749-753*, 2008.

[7] N. H. Pedersen, P. Johansen, and T. O. Andersen, "Feedback control of pulse-density modulated digital displacement transmission using a continuous approximation," *Submitted to IEEE/ASME Transactions on Mechatronics*, 2018.

[8] P. Johansen, D. B. Roemer, T. O. Andersen, and H. C. Pedersen, "Discrete linear time invariant analysis of digital fluid power pump flow control," *Journal of Dynamic Systems, Measurement and Control*, 2017.

[9] N. H. Pedersen, P. Johansen, and T. O. Andersen, "Lqr feedback control development for wind turbines featuring a digital fluid power transmission system," 2016.

[10] N. H. Pedersen, P. Johansen, and T. O. Andersen, "Optimal control of a wind turbine with digital fluid power transmission," *Nonlinear Dynamics*, 2018.

[11] N. H. Pedersen, P. Johansen, and T. O. Andersen, "Event-driven control of a speed varying digital displacement machine," *Proceedings of ASME/BATH FPMC Symposium on Fluid Power and Motion Control, Sarasota, Florida, USA*, 2017.

[12] N. H. Pedersen, P. Johansen, and T. O. Andersen, "Non-linear hybrid control oriented modelling of a digital displacement machine," *Proc. of 9th Workshop on Digital Fluid Power, Aalborg, Denmark*, 2017.

[13] N. H. Pedersen, P. Johansen, and T. O. Andersen, "Four qudrant hybrid control oriented dynamical system model of digital displacement units," *Proc. of the Bath/ASME Symposium on Fluid Power and Motion Control, Bath, UK*, 2018.

[14] S. M, G. M, and K. U, "Optimal control of digital hydraulic drives using mixed-integer quadratic programming," September 2013. 9th IFAC Symposium on Nonlinear Control System, Toulouse.

[15] R. G. Sanfelice and A. R. Teel, "Lyapunov analysis of sample-and-hold hybrid feedbacks," 2005+.

[16] A. R. Teel, R. G. Sanfelice, and R. Goebel, "Hybrid control systems," 2009.

[17] R. G. Sanfelice, "Feedback control of hybrid dynamical systems," *Encyclopedia of Systems and Control*, 2015.

[18] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE control systems magazine*, 2009.

[19] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems - Modeling, Stability and Robustness*. Princeton, 2012.

[20] N. H. Pedersen, P. Johansen, and T. O. Andersen, "Challenges with respect to control of digital displacement hydraulic units," *Modeling, Identification and Control*, 2018.

[21] N. H. Pedersen, P. Johansen, and T. O. Andersen, "Analysis of the non-smooth dynamical effects and suitable operation and control strategies for digital displacement units," *Proc. of the 10th Digital Fluid Power Conference, Tampere, Finland*, 2019.

[22] N. H. Pedersen, *Development of Control Strategies for Digital Displacement Units*. PhD thesis, Aalborg University, 2018.

[23] P. Johansen, D. B. Roemer, H. C. Pedersen, and T. O. Andersen, "Discrete linear time invariant analysis of digital fluid power pump flow control," *Journal of dynamic system measurement and control, ASME, accepted for publication*, 2016.

[24] N. H. Pedersen, *Development of Control Strategies for Digital Displacement Units*. PhD thesis, Deparment of Energy Technology, Aalborg University, 2018.