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# Hands-off Control for Discrete-time Linear Systems subject to Polytopic Uncertainties<sup>\*</sup>

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## Abstract:

This paper develops approaches to the hands-off control problem subject to performance constraints for discrete-time linear systems. The approaches minimize the  $l_1$ -norm of the control input to acquire the hands-off property, while satisfying the performance constraints that are given in terms of the quadratic cost of states and inputs with respect to the optimal solution to the finite-horizon linear quadratic regulator problem. We consider three kinds of the input and state matrices for the system; 1) known, 2) uncertain but contained in a known discrete set, and 3) uncertain but contained in a known polytopic uncertainty set. For the first two cases, we show that each problem is formulated as an  $l_1$  optimization that is expressed as a second-order cone programming. We also show that the last case leads to a second-order cone programming after relaxations. A numerical example is included to illustrate the validity of the proposed approach.

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*Keywords:* Robust control, Uncertainty, Linear optimal control, Convex programming, Discrete-time systems

## 1. INTRODUCTION

Control effort minimization is a fundamental requirement in practical control systems for saving fuel/electricity consumption and reducing noise and vibration (Chan, 2007; Dunham, 1974). For such problems, sparse control which takes input value mostly zero is effective. Thus, a novel design method called *maximum hands-off control* that produces a control input with the minimum support per unit of time has been proposed (Nagahara et al., 2016a).

The maximum hands-off control problem is initially formulated as an  $L^0$  optimal control for continuous-time systems to bring an arbitrary state to the origin. Although the  $L^0$  minimization problem is difficult to solve due to the non-convexity and the non-smoothness of the problem, it is proved that the set of  $L^0$  optimal solutions is equivalent to that of  $L^1$  optimal solutions under a uniqueness assumption called normality (Nagahara et al., 2016a). This property is important in view of computation;  $L^1$  optimal control problem can be formulated as a convex optimization problem, which is easily solved by numerical methods (Boyd and Vandenberghe, 2004). For discrete-time sys-

tems, the equivalence between  $l_0$  and  $l_1$  sparsity-promoting problems is also investigated in (Nagahara et al., 2016b). Here, we note that even though the equivalence does not hold always, we may still obtain a hands-off control that has a short support per unit of time by minimizing  $l_1$ -norm because it was shown that the  $l_1$ -norm is the convex envelope of the  $l_0$ -norm (Fazel, 2002) and  $l_1$ -minimization is known to lead to sparse solutions.

*Robustness* against uncertainty is a critical requirement in a lot of practical real-world control systems. Thus, robust control has been one of the most appealing approaches in the realm of control theory over 30 years. One of the most well-known approaches in robust control is  $H_\infty$  control, in which the uncertain system is modeled as a set of systems. It finds a control that achieves a control objective for all possible systems in the set; see (Zhou et al., 1996) for example.

In this paper, we consider the hands-off control problem that minimizes the  $l_1$  norm of the control input subject to uncertainties with performance constraint. The considered uncertainty takes the form of polytopic uncertainty, which is modeled by the convex hull of multiple possible systems (Badgwell, 1997; Bemporad and Morari, 1999). On the other hand, the considered performance constraint is obtained by relaxing the optimal cost of the finite-horizon linear quadratic regulator (LQR) problem. For such problem, the robust control design is formulated in

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terms of linear matrix inequalities (LMIs) (Boyd et al., 1994), which represents a convex optimization problem and can be solved numerically by optimization softwares such as YALMIP in MATLAB (Löfberg, 2004; Löfberg, 2012; Löfberg, 2009).

The main contribution of this paper is twofold. (i) the inclusion of a design parameter that allows us to specify the degree of possible cost relaxation under which the input sparsity is sought for, (ii) the consideration of uncertainties in the system model. This is achieved by adopting a relaxed constraint for the terminal state, instead of forcing the terminal state to be zero as in (Nagahara et al., 2016a).

The remainder of this paper is organized as follows: Section 2 provides the notation and an overview of linear quadratic regulator problem, followed by a presentation of the basic problem formulation for the nominal system in Section 3. Section 4 is the main part of this paper, which considers the hands-off control for uncertain systems. Following to some remarks in Section 5, Section 6 presents a numerical example. Finally, Section 7 concludes the paper.

## 2. MATHEMATICAL PRELIMINARIES

### 2.1 Notation

The set of real numbers is denoted by  $\mathbb{R}$ . The set of vectors with length  $n$  is denoted by  $\mathbb{R}^n$ , and the set of matrices of size  $n \times m$  is denoted by  $\mathbb{R}^{n \times m}$ . The vector of ones whose length is  $n$  is denoted by  $\mathbf{1}_n$ . The identity matrix of size  $n$  is denoted by  $I_n$ . The subscript  $n$  is dropped when the size is clear. For matrices  $M$  and  $N$ ,  $M \otimes N$  indicates the Kronecker product.

### 2.2 A review of linear-quadratic regulator problem

This subsection provides a brief overview on the finite-horizon LQR problem (Boyd, 2008), based on which this paper proposes approaches to hands-off control problems.

Consider a discrete-time linear system

$$x[t+1] = Ax[t] + Bu[t], \quad x[0] = x_0, \quad (1)$$

where  $A \in \mathbb{R}^{n_x \times n_x}$  and  $B \in \mathbb{R}^{n_x \times n_u}$  form a controllable pair,  $x[t] \in \mathbb{R}^{n_x}$  represents the system state, and  $u[t] \in \mathbb{R}^{n_u}$  represents the control input.

The objective of the finite-horizon LQR problem is to find a sequence of control inputs that minimizes the following quadratic cost function:

$$J(u) = x^T[T_f]Q_f x[T_f] + \sum_{t=0}^{T_f-1} x^T[t]Qx[t] + u^T[t]Ru[t],$$

$$Q = Q^T > 0, \quad Q_f = Q_f^T > 0, \quad R = R^T > 0, \quad (2)$$

where  $T_f$  is the time horizon. Generally,  $Q$  and  $Q_f$  are required to be positive semidefinite. However, we restrict our attention to positive definite weight matrices for convenience.

For such a problem, it is well-known that the optimal control input is (Chow, 1975)

$$u^*[t] = -F[t]x[t], \quad t = 0, 1, \dots, T_f - 1,$$

$$F[t] = (B^T P[t+1]B + R)^{-1} B^T P[t+1]A, \quad (3)$$

where  $P[t]$  is the solution to

$$P[t] = A^T P[t+1]A + Q$$

$$- A^T P[t+1]B (B^T P[t+1]B + R)^{-1} B^T P[t+1]A, \quad (4)$$

$$P[T_f] = Q_f.$$

Moreover, the corresponding optimal cost is given by

$$J_{\text{LQR}} := J(u^*[t]). \quad (5)$$

### 2.3 Some matrix inequalities

To treat polytopic uncertainties efficiently in Section 4, the following relaxations will be used:

*Lemma 1.* (Kiefer (1959)). Let  $M_i = M_i^T > 0$  and  $\lambda_i \geq 0$  for  $i = 1, 2, \dots, p$  satisfy  $\sum_{i=1}^p \lambda_i = 1$ . Then

$$\left( \sum_{i=1}^p \lambda_i N_i \right)^T \left( \sum_{i=1}^p \lambda_i M_i \right)^{-1} \left( \sum_{i=1}^p \lambda_i N_i \right) \leq \sum_{i=1}^p \lambda_i N_i^T M_i^{-1} N_i$$

The equality holds if and only if  $N_1^T M_1^{-1} = \dots = N_p^T M_p^{-1}$ .

It should be emphasized that  $N_i$  is not required to be symmetric or square. In addition, according to the original reference, the results of Lemma 1 holds under the assumption that  $\lambda_i > 0$ . However, we can trivially include the case with  $\lambda_i = 0$ .

*Corollary 2.* Let  $M_i = M_i^T > 0$  and  $\lambda_i \geq 0$  for  $i = 1, 2, \dots, p$  satisfy  $\sum_{i=1}^p \lambda_i = 1$ . Then

$$\left( \sum_{i=1}^p \lambda_i M_i \right)^{-1} \leq \sum_{i=1}^p \lambda_i M_i^{-1}.$$

The equality holds if and only if  $M_1 = M_2 = \dots = M_p$ .

**Proof.** Let  $N_i = I$  for all  $i$  in Lemma 1.

*Corollary 3.* Let  $L = L^T > 0$  and  $\lambda_i \geq 0$  for  $i = 1, 2, \dots, p$  satisfy  $\sum_{i=1}^p \lambda_i = 1$ . Then

$$\left( \sum_{i=1}^p \lambda_i N_i \right)^T L \left( \sum_{i=1}^p \lambda_i N_i \right) \leq \sum_{i=1}^p \lambda_i N_i^T L N_i.$$

The equality holds if and only if  $N_1 = N_2 = \dots = N_p$ .

**Proof.** Let  $M_i = L^{-1}$  for all  $i$  in Lemma 1.

## 3. HANDS-OFF CONTROL PROBLEM FOR KNOWN SYSTEM

Using the results in Section 2.2, this section proposes the hands-off control problem for the system (1) with a known controllable pair of  $A$  and  $B$ . More specifically, the problem is set up so as to minimize the  $l_1$ -norm of the control input while satisfying the control performance condition that specifies the degree of relaxation compared with the optimal cost of the LQR problem in (5).

*Problem 4.* (Hands-off Control Problem with Performance Constraint cf. Nagahara et al. (2016a)):

For the linear system

$$x[t+1] = Ax[t] + Bu[t],$$

$$x[0] = x_0, \quad t = 0, 1, \dots, T_f - 1, \quad (6)$$

with the controllable pair of  $(A, B)$ , find a sequence of control inputs  $u[t]$  that minimizes the  $l_1$ -norm of the control input

$$\sum_{t=0}^{T_f-1} \sum_{i=1}^{n_u} |u_i[t]|, \quad (7)$$

where  $|\cdot|$  denotes the element-wise absolute value, subject to

$$J(u) := x^T [T_f] Q_f x [T_f] + \sum_{t=0}^{T_f-1} x^T [t] Q x [t] + u^T [t] R u [t] \leq J^*, \quad (8)$$

where

$$J^* := \gamma J_{\text{LQR}}, \quad \gamma \geq 1. \quad (9)$$

In (9),  $J_{\text{LQR}}$  is defined as in (5) and specifies the control performance condition.

*Remark 5.* The parameter  $\gamma$  is used to make a balance between the sparsity of the control input and the deviation from the optimal cost (5). If  $\gamma = 1$ , then there is no freedom to minimize the norm of the control inputs, and the solution to the problem coincides with (3). As  $\gamma$  becomes larger, the  $l_1$ -norm of the hands-off control inputs may become smaller, but the performance degrades more and more compared with (5).

To solve Problem 4, let us first simplify the expressions in (7)-(8) by defining

$$\begin{aligned} \bar{x} &:= \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[T_f] \end{bmatrix}, \quad \bar{u} := \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[T_f - 1] \end{bmatrix}, \\ \hat{A} &:= \begin{bmatrix} I_{n_x} & 0 & \cdots & 0 \\ A & & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{T_f-1} & A^{T_f-2} & \cdots & I_{n_x} \end{bmatrix}, \\ \bar{A} &:= \begin{bmatrix} 0_{n_x \times n_x T_f} \\ \hat{A} \end{bmatrix}, \quad \bar{B} := I_{T_f} \otimes B, \quad A_0 := \begin{bmatrix} A \\ 0_{(n_x-1)T_f \times n_x} \end{bmatrix}, \\ \hat{Q} &:= \text{diag}[I_{T_f-1} \otimes Q, Q_f], \quad \bar{Q} := \text{diag}[Q, \hat{Q}], \quad \bar{R} := I_{T_f} \otimes R, \\ G_1 &:= \bar{A} \bar{B}, \quad G_2 := \begin{bmatrix} I_{n_x} \\ \hat{A} A_0 \end{bmatrix}. \end{aligned} \quad (10)$$

Then, the cost (7) can be rewritten as

$$\sum_{t=0}^{T_f-1} \sum_{i=1}^{n_u} |u_i[t]| = \mathbf{1}_{n_u T_f}^T \bar{u}. \quad (11)$$

Also, the vector of the states is expressed as

$$\bar{x} = G_1 \bar{u} + G_2 x_0, \quad (12)$$

and thus  $J(u)$  in (8) can be expressed as

$$\begin{aligned} J(u) &= \bar{x}^T \bar{Q} \bar{x} + \bar{u}^T \bar{R} \bar{u} \\ &= \begin{bmatrix} \bar{u} \\ x_0 \end{bmatrix}^T \begin{bmatrix} G_1^T \bar{Q} G_1 + \bar{R} & G_1^T \bar{Q} G_2 \\ G_2^T \bar{Q} G_1 & G_2^T \bar{Q} G_2 \end{bmatrix} \begin{bmatrix} \bar{u} \\ x_0 \end{bmatrix}. \end{aligned} \quad (13)$$

Based upon (11)-(13), Problem 4 can be reformulated as a second-order cone programming as follows:

$$\begin{aligned} \min_w & q_0^T w \\ \text{s.t.} & w^T P_1 w + q_1^T w + r_1 \leq 0, \\ & q_2^T w \leq 0, \quad q_3^T w \leq 0, \end{aligned} \quad (14)$$

where

$$\begin{aligned} w &:= \begin{bmatrix} \bar{u} \\ v \end{bmatrix}, \quad q_0 := \begin{bmatrix} 0_{n_u T_f} \\ \mathbf{1}_{n_u T_f} \end{bmatrix}, \\ P_1 &:= \begin{bmatrix} G_1^T \bar{Q} G_1 + \bar{R} & 0 \\ 0 & 0 \end{bmatrix}, \quad q_1 := \begin{bmatrix} 2G_1^T \bar{Q} G_2 x_0 \\ 0 \end{bmatrix}, \\ r_1 &:= x_0^T G_2^T \bar{Q} G_2 x_0 - J^*, \\ q_2 &:= \begin{bmatrix} I_{n_u T_f} \\ -I_{n_u T_f} \end{bmatrix}, \quad q_3 := \begin{bmatrix} -I_{n_u T_f} \\ -I_{n_u T_f} \end{bmatrix}. \end{aligned} \quad (15)$$

This second-order cone programming (14)-(15) can be solved using existing numerical softwares such as YALMIP on MATLAB (Löfberg, 2004; Löfberg, 2012; Löfberg, 2009). Here, the constraint  $w^T P_1 w + q_1^T w + r_1 \leq 0$  corresponds to (8) and guarantees the satisfaction of the performance condition, while the other two constraints,  $q_2^T w \leq 0$  and  $q_3^T w \leq 0$ , determine the bounds on the absolute value of  $u[t]$ .

*Remark 6.* Unlike the original paper (Nagahara et al., 2016a), where a continuous-time setup is considered, it is not necessary to impose the constraint  $\max_i |u_i[t]| \leq 1$  in a discrete-time setup. This is because without such a constraint, a continuous-time setup produces an optimal control input of a Dirac delta function, while a discrete-time setup guarantees the boundedness of  $|u_i[t]|$  as long as (11) is bounded.

#### 4. HANDS-OFF CONTROL PROBLEM FOR UNCERTAIN SYSTEMS

This section considers the hands-off control problem for the system (1) but with uncertainties in the state matrix  $A$  and input matrix  $B$ . As in the previous section, we impose the control performance condition that specifies the degree of relaxation compared with the optimal cost of the LQR problem.

##### 4.1 Discrete Uncertainties

Let us start with the system (1) where the pair  $(A, B)$  is uncertain but contained in a known discrete set, i.e.,

$$(A, B) \in \mathcal{S}_d := \{(A, B) = (A_j, B_j), j = 1, \dots, n\}, \quad (16)$$

where  $n$  is the number of scenarios and  $(A_j, B_j)$  are controllable pairs for all  $j = 1, \dots, n$ .

For such systems, the performance condition is specified using the following upper bound

$$J_d^* := \max_{j=1, \dots, n} \gamma_j J_{\text{LQR}, j}, \quad \gamma_j \geq 1, \quad (17)$$

in place of (9), where  $J_{\text{LQR}, j}$  is the optimal cost (5) corresponding to the scenario  $(A_j, B_j)$  in (16), and  $\gamma_j$  specifies the degree of relaxation for each scenario. In this way, the existence of the control input satisfying the performance condition is guaranteed, and the parameter  $\gamma$  can be used to balance between the sparsity of the input and the deviation from the optimal cost in the worst-case scenario.

The constraint of performance condition (8) needs to be satisfied in any of the  $n$  scenarios, thus the second-order cone programming of this problem replaces the first constraint of (14) by  $n$  constraints, each of which corresponds to one of  $n$  scenarios.

Therefore, the following second-order cone programming provides the solution to the hands-off control problem subject to discrete uncertainties.

$$\begin{aligned} & \min_w q_0^T w \\ & \text{s.t. } w^T P_{1j} w + q_{1j}^T w + r_{1j} \leq 0, \quad j = 1, \dots, n, \\ & \quad q_2^T w \leq 0, \quad q_3^T w \leq 0, \end{aligned} \quad (18)$$

where  $w$ ,  $q_0$ ,  $q_2$  and  $q_3$  are defined as in (15), and

$$\begin{aligned} P_{1j} &= \begin{bmatrix} G_{1,j}^T \bar{Q} G_{1,j} + \bar{R} & 0 \\ 0 & 0 \end{bmatrix}, \quad q_{1j} = \begin{bmatrix} 2G_{1,j}^T \bar{Q} G_{2,j} x_0 \\ 0 \end{bmatrix}, \\ r_{1j} &= x_0^T G_{2,j}^T \bar{Q} G_{2,j} x_0 - J_d^*, \end{aligned} \quad (19)$$

and  $G_{1,j}$  and  $G_{2,j}$  are defined as in (10) for the each scenario  $j$  of the pair  $(A_j, B_j)$ . Thus, we have  $n$  quadratic constraints and  $4n_{u_x} T_f$  linear constraints.

The formulation in (18)-(19) minimizes the  $l_1$ -norm of the control input while guaranteeing the performance condition satisfaction for any of the  $n$  scenarios in (16).

*Remark 7.* It is known that the set of these  $n$  constraints in (18) is equivalent to

$$w^T P_1 w + q_1^T w + r_1 \leq 0, \quad \forall (P_1, q_1, r_1) \in \mathcal{S}_p, \quad (20)$$

where

$$\mathcal{S}_p = \left\{ (P_1, q_1, r_1) \mid (P_1, q_1, r_1) = \left. \begin{aligned} & \sum_{j=1}^n \lambda_j (P_{1j}, q_{1j}, r_{1j}) \lambda_j \geq 0, \quad \sum_{j=1}^n \lambda_j = 1 \end{aligned} \right\}. \quad (21)$$

We will use this equivalence in the following subsection.

#### 4.2 Polytopic Uncertainties

This subsection considers the case where the pair  $(A, B)$  contains polytopic uncertainties, i.e.,

$$[A \ B] \in \Omega_p,$$

$$\Omega_p := \left\{ [A(\lambda) \ B(\lambda)] = \sum_{i=1}^p \lambda_i [A_i \ B_i], \quad \sum_{i=1}^p \lambda_i = 1, \quad \lambda_i \geq 0 \right\}, \quad (22)$$

where  $A_i$  and  $B_i$  are constant matrices satisfying the controllability of  $(A_i, B_i)$  for all  $i$ , and  $\lambda_i$ s are time-invariant uncertainties.

Unlike the case of discrete uncertainties, systems with polytopic uncertainties have infinite number of scenarios that replaces (8). Thus, to treat polytopic uncertainties efficiently, the constraint (8) is relaxed using an upper bound on the cost function (13) that is easy to compute.

For this purpose, first notice that from the definitions in (10), it holds that

$$\begin{aligned} G_1^T \bar{Q} G_1 &= \bar{B}^T \bar{A}^T \bar{Q} \bar{A} \bar{B} = \bar{B}^T \hat{A}^T \hat{Q} \hat{A} \bar{B}, \\ G_2^T \bar{Q} G_1 &= (\hat{A} A_0)^T \hat{Q} \hat{A} \bar{B} = A_0^T \hat{A}^T \hat{Q} \hat{A} \bar{B}, \\ G_2^T \bar{Q} G_2 &= A_0^T \hat{A}^T \hat{Q} \hat{A} A_0 + Q. \end{aligned} \quad (23)$$

Accordingly, the matrix characterizing  $J(u)$  in (13) can be expressed as

$$\begin{aligned} & \begin{bmatrix} G_1^T \bar{Q} G_1 + \bar{R} & G_1^T \bar{Q} G_2 \\ G_2^T \bar{Q} G_1 & G_2^T \bar{Q} G_2 \end{bmatrix} \\ &= \begin{bmatrix} \bar{B} \\ A_0 \end{bmatrix}^T \begin{bmatrix} \hat{A}^T \hat{Q} \hat{A} & \hat{A}^T \hat{Q} \hat{A} \\ \hat{A}^T \hat{Q} \hat{A} & \hat{A}^T \hat{Q} \hat{A} \end{bmatrix} \begin{bmatrix} \bar{B} \\ A_0 \end{bmatrix} + \begin{bmatrix} \bar{R} \\ Q \end{bmatrix}. \end{aligned} \quad (24)$$

Next, define

$$\begin{aligned} \tilde{A} &:= I - \begin{bmatrix} 0 & 0 \\ I_{T_f-1} \otimes A & 0 \end{bmatrix}, \quad \tilde{A}_i := I - \begin{bmatrix} 0 & 0 \\ I_{T_f-1} \otimes A_i & 0 \end{bmatrix}, \\ \hat{A}_i &:= \begin{bmatrix} I & 0 & \dots & 0 \\ A_i & & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_i^{T_f-1} & A_i^{T_f-2} & \dots & I \end{bmatrix}, \quad A_{0,i} := \begin{bmatrix} A_i \\ 0_{(n_x-1)T_f \times n_x} \end{bmatrix}. \end{aligned} \quad (25)$$

Then, it follows that

$$\hat{A} = \tilde{A}^{-1} = \left( \sum_{i=1}^p \lambda_i \tilde{A}_i \right)^{-1}, \quad \hat{A}_i = \tilde{A}_i^{-1}, \quad A_0 = \sum_{i=1}^p \lambda_i A_{0,i}. \quad (26)$$

Moreover, define

$$\bar{B}_i := I_{T_f} \otimes B_i. \quad (27)$$

Then, (22), (25) and (27) yield

$$\begin{bmatrix} \bar{B} & 0 \\ 0 & A_0 \end{bmatrix} = \sum_{i=1}^p \lambda_i \begin{bmatrix} \bar{B}_i & 0 \\ 0 & A_{0,i} \end{bmatrix}. \quad (28)$$

On the other hand, from Corollary 2 and (26), it holds that

$$\begin{aligned} \hat{A}^T \hat{Q} \hat{A} &= \tilde{A}^{-T} \hat{Q} \tilde{A}^{-1} = \left( \sum_{i=1}^p \sum_{j=1}^p \lambda_i \lambda_j \tilde{A}_i^T \hat{Q}^{-1} \tilde{A}_j \right)^{-1} \\ &\leq \sum_{i=1}^p \sum_{j=1}^p \lambda_i \lambda_j \left( \tilde{A}_i^T \hat{Q}^{-1} \tilde{A}_j \right)^{-1} = \sum_{i=1}^p \sum_{j=1}^p \lambda_i \lambda_j \hat{A}_i^T \hat{Q} \hat{A}_j. \end{aligned} \quad (29)$$

From (29), it follows that

$$\begin{bmatrix} \hat{A}^T \hat{Q} \hat{A} & \hat{A}^T \hat{Q} \hat{A} \\ \hat{A}^T \hat{Q} \hat{A} & \hat{A}^T \hat{Q} \hat{A} \end{bmatrix} \leq \sum_{i=1}^p \sum_{j=1}^p \lambda_i \lambda_j \begin{bmatrix} \hat{A}_i^T \hat{Q} \hat{A}_i & \hat{A}_i^T \hat{Q} \hat{A}_j \\ \hat{A}_j^T \hat{Q} \hat{A}_i & \hat{A}_j^T \hat{Q} \hat{A}_j \end{bmatrix}. \quad (30)$$

Due to the fact that  $\mathbf{1}^T \mathbf{1}$  and  $M = M^T$  are both positive semidefinite,  $(\mathbf{1}^T \mathbf{1}) \otimes M$  is positive semidefinite.

Hence, based on Corollary 3 together with (28) and (30), we can deduce

$$\begin{aligned} & \begin{bmatrix} \bar{B} & 0 \\ 0 & A_0 \end{bmatrix}^T \begin{bmatrix} \hat{A}^T \hat{Q} \hat{A} & \hat{A}^T \hat{Q} \hat{A} \\ \hat{A}^T \hat{Q} \hat{A} & \hat{A}^T \hat{Q} \hat{A} \end{bmatrix} \begin{bmatrix} \bar{B} & 0 \\ 0 & A_0 \end{bmatrix} \\ &\leq \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \lambda_i \lambda_j \lambda_k \begin{bmatrix} \bar{B}_k^T \hat{A}_j^T \hat{Q} \hat{A}_i \bar{B}_k & \bar{B}_k^T \hat{A}_j^T \hat{Q} \hat{A}_i A_{0,k} \\ A_{0,k}^T \hat{A}_j^T \hat{Q} \hat{A}_i \bar{B}_k & A_{0,k}^T \hat{A}_j^T \hat{Q} \hat{A}_i A_{0,k} \end{bmatrix}. \end{aligned} \quad (31)$$

Thus, an upper bound on the cost function  $J(\bar{u})$  is obtained as a function of the control input:

$$J(\bar{u}) \leq \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \lambda_i \lambda_j \lambda_k J_{ijk}(\bar{u}), \quad (32)$$

where

$$\begin{aligned} J_{ijk}(\bar{u}) = & \bar{u}^T \left( \bar{B}_k^T \hat{A}_j^T \hat{Q} \hat{A}_i \bar{B}_k + \bar{R} \right) \bar{u} \\ & + 2x_0^T \left( A_{0,k}^T \hat{A}_j^T \hat{Q} \hat{A}_i \bar{B}_k \right) \bar{u} \\ & + x_0^T \left( A_{0,k}^T \hat{A}_j^T \hat{Q} \hat{A}_i A_{0,k} + Q \right) x_0. \end{aligned} \quad (33)$$

So the constraint (8) is relaxed as

$$\sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \lambda_i \lambda_j \lambda_k J_{ijk}(\bar{u}) \leq J_p^*. \quad (34)$$

Here, it is assumed that  $J_p^*$ , which characterizes the trade-off between the sparsity of control input and systems' performance, is given. The selection of the performance parameter  $J_p^*$  is discussed in Section 5.

The corresponding second-order cone programming formulation of (14) replaces  $P_1$ ,  $q_1$  and  $r_1$  by

$$\begin{aligned} P_1 &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \lambda_i \lambda_j \lambda_k P_{1ijk}, \quad q_1 = \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \lambda_i \lambda_j \lambda_k q_{1ijk}, \\ r_1 &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \lambda_i \lambda_j \lambda_k r_{1ijk}, \end{aligned} \quad (35)$$

where

$$\begin{aligned} P_{1ijk} &= \begin{bmatrix} \bar{B}_k^T \hat{A}_j^T \hat{Q} \hat{A}_i \bar{B}_k + \bar{R} & 0 \\ 0 & 0 \end{bmatrix}, \\ q_{1ijk} &= \begin{bmatrix} 2\bar{B}_k^T \hat{A}_j^T \hat{Q} \hat{A}_i A_{0,k} x_0 \\ 0 \end{bmatrix}, \\ r_{1ijk} &= x_0^T \left( A_{0,k}^T \hat{A}_j^T \hat{Q} \hat{A}_i A_{0,k} + Q \right) x_0 - J_p^*. \end{aligned} \quad (36)$$

Using (35) with Remark 7, the second-order cone programming formulation for the hands-off control for polytopic uncertainties is

$$\begin{aligned} & \min_w q_0^T w \\ \text{s.t. } & w^T P_{1ijk} w + q_{1ijk}^T w + r_{1ijk} \leq 0, \quad i, j, k = 1, \dots, p, \\ & q_2^T w \leq 0, \quad q_3^T w \leq 0, \end{aligned} \quad (37)$$

where  $w$ ,  $q_0$ ,  $q_2$  and  $q_3$  are defined as in (15). This optimization problem has  $p^3$  quadratic constraints and  $4n_{u_x} T_f$  linear constraints.

*Remark 8.* It is straightforward to show that the number of quadratic constraints in (37) can be reduced from  $p^3$  to  $p^2(p+1)/2$  by using the symmetry of  $i$  and  $j$  in  $P_{1ijk}$ ,  $q_{1ijk}$  and  $r_{1ijk}$ .

## 5. DISCUSSIONS

This section briefly discusses the computational cost and some concerns regarding the performance condition of the proposed approach.

### 5.1 Computational Cost

As we have seen, the computational cost of (37) increases quadratically with respect to the number of vertices of polytopic uncertainty. However, this computational cost

of (37) can be reduced by further relaxing the constraints. An approach is to find a pair  $(\bar{P}_1, \bar{q}_1, \bar{r}_1)$  such that  $\forall w$ ,

$$\begin{aligned} & w^T \bar{P}_1 w + \bar{q}_1^T w + \bar{r}_1 \leq 0 \\ \Rightarrow & w^T P_{1ijk} w + q_{1ijk}^T w + r_{1ijk} \leq 0, \quad \forall i, j, k = 1, \dots, p. \end{aligned} \quad (38)$$

If such a pair is found, then the number of quadratic constraints is reduced from  $p^3$  to one. To find such  $(\bar{P}_1, \bar{q}_1, \bar{r}_1)$ , an inner Dikin ellipsoid, an inner Löwner John ellipsoid (Henrion et al., 2001), or other inner approximations for the intersection of ellipsoids  $(P_{1ijk}, q_{1ijk}, r_{1ijk})$  (Boyd et al., 1994) can be used.

### 5.2 Performance Condition

As in the case with discrete uncertainties, it is possible to choose the performance condition  $J_p^*$  using the exact upper bound on  $\sum_i \lambda_i J_i(\bar{u})$  by solving a minimax constrained problem. However, Section 4.2 proposes to relax the constraint in Problem 4 by using the upper bound of quadratic cost instead of the quadratic cost itself. Thus, there is no reason to use the exact upper bound on  $\sum_i \lambda_i J_i(\bar{u})$ .

One option is to choose  $J_p^*$  sufficiently large. For example, we may compare the performance with the nominal by using  $J_{LQR}$  corresponding to the nominal system (e.g.,  $A = \sum_{i=1}^p A_i/p$  and  $B = \sum_{i=1}^p B_i/p$ ) and then setting  $J_p^* = \gamma J_{LQR}$  with a relatively large  $\gamma$ . Note that such  $J_p^*$  may lead to infeasible programming, if selected  $\gamma$  is not sufficiently large. In this case, increase the value of  $\gamma$ .

One other method for choosing  $J_p^*$  is setting it in such a way that the existence of a feasible control input is guaranteed. For example, if both  $A$  and  $B$  are subject to polytopic uncertainty as in (22), then we may compute  $\bar{u}$  that minimizes  $J_{ijk}(\bar{u})$  for each  $i, j$ , and  $k$  and let

$$J_p^* = \gamma \max_{i,j,k} J_{ijk}(\bar{u}_{\text{approx}}), \quad \gamma \geq 1 \quad (39)$$

where  $\bar{u}_{\text{approx}} = \arg \max_{i,j,k} J_{ijk}^*(\bar{u})$ . Then it is guaranteed that there exists a control input  $\bar{u} = \bar{u}_{\text{approx}}$  that satisfies

$$\sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \lambda_i \lambda_j \lambda_k J_{ijk}(\bar{u}) \leq J_p^*. \quad (40)$$

Alternatively, we could evaluate how much control effort is needed to improve the performance compared with the worst case of uncertainties without control inputs by setting  $J_p^*$  as follows:

$$J_p^* = \eta J(0), \quad \eta \leq 1, \quad J(0) = \max_A x_0^T G_2^T \bar{Q} G_2 x_0. \quad (41)$$

## 6. NUMERICAL EXAMPLE

In this section, we apply the results obtained in Subsection 4.2. to a discrete-time linear system subject to polytopic uncertainties. The example is taken from (Ding and Ping, 2013), which considers the model of a continuous stirred tank reactor for an exothermic, irreversible reaction. The polytope representing uncertainties of the considered plant is characterized by 4 vertices as follows:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.8227 & -0.00168 \\ 6.1233 & 0.9367 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.9654 & -0.00182 \\ -0.6759 & 0.9433 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0.8895 & -0.00294 \\ 0.9447 & 0.9968 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0.8930 & -0.00062 \\ 2.7738 & 0.8864 \end{bmatrix}, \end{aligned}$$

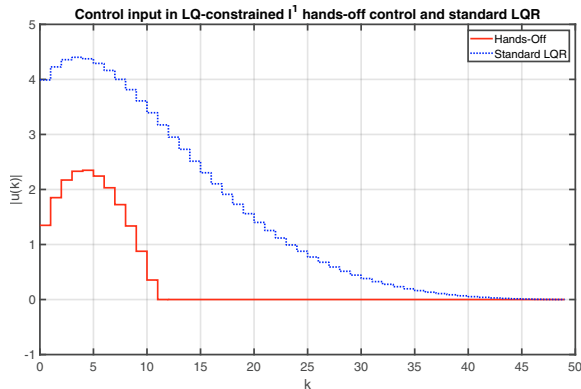


Fig. 1. Control input  $u[t]$ : hands-off control (real line) and standard LQR (dotted line)

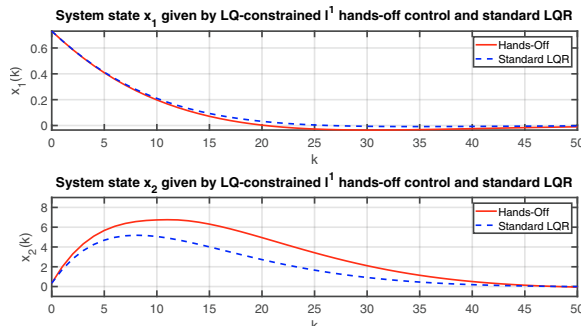


Fig. 2. State  $x[t] = (x_1[t], x_2[t])$ : hands-off control (real line) and standard LQR (dotted line)

$$B_1 = [-0.000092 \ 0.1014]^T, \quad B_2 = [-0.000097 \ 0.1016]^T, \\ B_3 = [-0.000157 \ 0.1045]^T, \quad B_4 = [-0.000034 \ 0.0986]^T,$$

where each pair  $(A_i, B_i)$  is controllable. The initial state of the plant is a 2-dimensional random vector whose elements are selected from a uniform distribution over  $(0, 1)$ . The horizon length  $T_f$  is set to 50. Weighting matrices are selected as  $Q = I$ ,  $Q_f = I$ , and  $R = 1$ . We set the constant  $\gamma$  in (17) to 1.5. For the true system, we set  $\lambda_1 = 0.1$ ,  $\lambda_2 = 0.2$ ,  $\lambda_3 = 0.3$ , and  $\lambda_4 = 0.4$ .

The optimization is set up based on (35) and (37) and solved using YALMIP with solver option of `fmincon` on MATLAB (Löfberg, 2004; Löfberg, 2012; Löfberg, 2009). The optimal control input is illustrated in Fig. 1. In this figure, we also plot the standard finite-horizon LQR control as in (3). Compared with the standard LQR control, the hands-off control is sufficiently sparse.

Moreover, we apply the obtained control input to the considered plant in order to assess the behavior of state trajectories. The state trajectories generated by the standard LQR control and the hands-off control are shown in Fig. 2. According to this figure, the proposed hands-off control, which is much sparser than the LQR control, leads to a comparable performance with the LQR control. One can obtain different sparsity and robustness properties by manipulating  $\gamma$  and weighting matrices.

## 7. CONCLUSIONS

This paper has proposed approaches to constrained hands-off control problem for discrete-time linear systems for three different scenarios. Such a problem has been formulated as minimization of the  $l_1$ -norm of the control

input that satisfies given performance conditions. It has been shown that this optimization problem is simplified to second-order cone programming. Moreover, it has been illustrated through a numerical example that the proposed approach gives a sparse control input while the system performance is fairly close to the standard finite-horizon LQR performance as desired.

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