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A Tight Linear Program for Feasibility Check and Solutions to Natural Gas Flow Equations

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***Abstract*—This letter proposed a novel convex optimization model, whose optimal solution is proved to be the precise solution of the natural gas flow equations. Furthermore, a linear program is employed to tightly linearize the nonlinear parts and a corollary is given to check the feasibility. Numerical results show the effectiveness of the proposed method.**

***Index Terms*—Integrated power and natural gas systems, gas flow equations, feasibility testing, constraint satisfaction problem**

I. Introduction

Integrated energy systems, breaking down the barriers among historically independent sectors, can make significant improvements to achieve higher energy efficiency and lower energy costs. For traditional power systems, the integrated energy systems can provide more flexibility enabled by the complementary nature of heterogeneous resources. Since the natural gas industry has prospered in the past twenty-five years due to the incentive support from the Chinese government, a tremendous increase in the development of the integrated power and natural gas systems (IPGSs) has been acknowledged. As a result, the optimal dispatch of IPGSs [1] and the expansion planning for IPGSs [2], [3] are becoming highly interested.

One task for IPGSs is to investigate the problem of *feasibility test* for the stationary [4]-[6]. More specifically, can the system be operated in a way that satisfies the given boundary conditions? This feasibility problem of energy flows can be termed as a constraint satisfaction problem (CSP), which is a homogeneous collection of finite constraints over variables. To perform the feasibility check, generally, an auxiliary optimization model with any objective function should be set up, e.g., min 0, subjected to a series of constraints. If the auxiliary optimization model is feasible, the CSP is feasible; otherwise, it is infeasible.

However, the main challenge to analyze the IPGSs is the disjunctive nature from the gas flow equations. To address this issue, binary variables are introduced to capture the gas flow direction, leading the corresponding auxiliary optimization model to be a mixed integer nonlinear nonconvex programming (MINNP), which is still difficult to solve. Therefore, a piecewise linearization method was introduced in [7] for the nonlinear gas equations, which generated a mixed integer linear programming (MILP). Moreover, a convex relaxation to the gas equations was adopted in [8], resulting in a mixed integer second order cone programming (MISOCP).

Unfortunately, mixed integer programming is essentially *NP-Complete*, which is very time consuming for large-scale problems. This letter only focuses on the analysis of gas flow equations. To further alleviate the computational burden, a convex optimization model is proposed, whose Karush–Kuhn–Tucker (KKT) condition has the same mathematical structure. Furthermore, a tight linear program and a corollary are proposed to check the feasibility and obtain the gas flow solution.

II. Physical Modeling for Natural Gas Networks

Originally, the gas flow equations are described as a set of partial differential equations. In steady state, the gas dynamics along a pipe are ignored and the mass flux is assumed constant, which gives

$$f_{mn}\left|f_{mn}\right| = C_{mn}\left(\pi_m - \pi_n\right), \quad \forall (m,n)\in\mathcal{L}, \forall m,n\in\mathcal{N} \tag{1a}$$

$$\sum_{mn\in GF} f_{mn} - \sum_{mn\in GT} f_{nm} = G_m - D_m, \quad \forall m\in\mathcal{N} \tag{1b}$$

$$\pi_m^{\min} \le \pi_m \le \pi_m^{\max}, \quad \forall m\in\mathcal{N} \tag{1c}$$

$$-f_{mn}^{\max} \le f_{mn} \le f_{mn}^{\max}, \quad \forall (m,n)\in\mathcal{L} \tag{1d}$$

$$G_m^{\min} \le G_m \le G_m^{\max}, \quad \forall m\in\mathcal{N} \tag{1d}$$

where $\mathcal{N}$ is a set of natural gas nodes and $\mathcal{L}$ is a set of natural gas pipelines; m and n are the origin and end nodes for natural gas pipeline mn; C_{mn} is a Weymouth constant describing the loss coefficient of natural gas pipeline mn; f_{mn} is the gas flow in pipeline mn; π_m and π_n are the squared node pressures in a natural gas network; GF is the set of natural gas pipelines with the node m being the origin node and GT is the set of natural gas pipelines with the node m being the end node; G_m and D_m is the gas generation and gas withdrawal at node m, respectively; $\pi_m^{\min}$ and $\pi_m^{\max}$ are the lower and upper bound of squared node pressure on node m, respectively; $f_{mn}^{\max}$ is the maximum allowable gas flow on the pipeline mn.

Moreover, the constraint (1a) describes the squared pressure drop between two nodes is related to the gas flow on the pipeline. Especially, the direction of the gas flow determines the direction of the squared pressure drop. The constraint (1b) refers to the natural gas balance at each node. Constraints (1c)-(1d) are the bound limits for gas flows and squared node pressures.

III. Linear Program for Natural Gas Equations

Solving the CSP in (1) requires setting up an auxiliary optimization model. In this letter, a new auxiliary optimization model to check the feasibility is first established, such that

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$$\min \quad z = \sum_{(m,n)\in\mathcal{L}} \frac{|f_{mn}| f_{mn}^2}{3C_{mn}} \tag{2a}$$

$$s.t. \quad \sum_{mn\in GF} f_{mn} - \sum_{mn\in GT} f_{nm} = G_m - D_m, \quad \forall m\in\mathcal{N} \tag{2b}$$

$$G_m^{\min} \le G_m \le G_m^{\max}, \quad \forall m\in\mathcal{N} \tag{2c}$$

It can be observed that (2b) are linear constraints and the objective function in (2a) is convex since the second derivative is positive semidefinite (i.e., $2|f_{mn}|/C_{mn} \ge 0$). As a result, the above model in (2) is a convex optimization model that is easier to solve to achieve its global optimum in a polynomial time. The Lagrange function of the optimization model (2) is

$$L(f_{mn}) = \sum_{(m,n)\in\mathcal{L}} \frac{|f_{mn}| f_{mn}^2}{3C_{mn}} - \lambda_m\left(\sum_{mn\in GF} f_{mn} - \sum_{mn\in GT} f_{nm} - G_m + D_m\right) \tag{3}$$

The optimal solution and corresponding multipliers (f_{mn}^*, λ_m^*) can be obtained by the KKT condition, such that

$$\frac{\partial L(f_{mn}^*)}{\partial f_{mn}} = |f_{mn}^*| f_{mn}^* / C_{mn} - \lambda_m^* + \lambda_n^* = 0, \quad \forall(m,n)\in\mathcal{L} \tag{4}$$

which gives

$$|f_{mn}^*| f_{mn}^* = C_{mn}(\lambda_m^* - \lambda_n^*), \quad \forall(m,n)\in\mathcal{L} \tag{5}$$

It can be observed that (5) and (1a) have the same mathematical structure, but the multipliers and gas flows may not satisfy the bound limits in (1c) and (1d). To check the feasibility, the following corollary is presented.

<u>Corollary</u>: If the model in (2) is feasible and the corresponding multiplies from (2) satisfy (6), the gas flow equations in (1) are feasible; otherwise, (1) are infeasible.

$$\begin{cases} \max\limits_{\forall m\in\mathcal{N}}\left(\lambda_m^* - \pi_m^{\max}\right) \le \min\limits_{\forall m\in\mathcal{N}}\left(\lambda_m^* - \pi_m^{\min}\right) \\ -\left(f_{mn}^{\max}\right)^2 / C_{mn} \le \lambda_m^* - \lambda_n^* \le \left(f_{mn}^{\max}\right)^2 / C_{mn}, \ \forall(m,n)\in\mathcal{L} \end{cases} \tag{6}$$

Proof: It can be found in (5) that the equation is only related to the difference between two multipliers, and increasing the same value for all the multipliers will not break the equations. Thus, let $\pi_m^* = \lambda_m^* - \delta$, $\forall m\in\mathcal{N}$, where δ is an arbitrary real number, be the squared gas pressure on nodes m and n. (f_{mn}^*, π_m^*) strictly satisfies (1a) and (1b).

To meet the constraints in (1c), it requires that δ should satisfy $\pi_m^{\min} \le \lambda_m^* - \delta \le \pi_m^{\max}$ for $\forall m\in\mathcal{N}$. That means, δ should satisfy $\lambda_m^* - \pi_m^{\max} \le \delta \le \lambda_m^* - \pi_m^{\min}$ for $\forall m\in\mathcal{N}$, which gives

$$\max_{\forall m\in\mathcal{N}}\left(\lambda_m^* - \pi_m^{\max}\right) \le \min_{\forall m\in\mathcal{N}}\left(\lambda_m^* - \pi_m^{\min}\right) \tag{7a}$$

Considering (1d) and substituting (5) into (1d), the limits on gas flows (1d) are shifted to the limits on π, which yields

$$-\left(f_{mn}^{\max}\right)^2 / C_{mn} \le \lambda_m^* - \lambda_n^* \le \left(f_{mn}^{\max}\right)^2 / C_{mn}, \quad \forall(m,n)\in\mathcal{L} \tag{7b}$$

Thus, if the model in (2) is feasible, the constraints in (1a) and (1b) will hold; and meanwhile if the corresponding multiplies from (2) satisfy (7), constraints (1c) and (1d) will hold.

(**Q.E.D.**)

However, the high-order nonlinear optimization still faces difficulties in practice, such as the choice of the starting point, the non-differentiability of the objective function, and the convergence problem. In contrast, linear programs (LPs) are simpler and more robust than nonlinear solvers. Referring to (2a), it has a special structure that it is a separable function with respect to the gas flow variables. Such a function can be tightly linearized by polyhedra.

Since the squared nodal gas pressure is limited, the maximum range of the gas flow can be obtained by (1a) and (1c), where $f_{mn}^{\max} = \sqrt{C_{mn}\left(\pi_m^{\max} - \pi_n^{\min}\right)}$. Let each nonlinear function in (2a) be $z_{mn} = |f_{mn}|(f_{mn})^2/3C_{mn}$ which can be linearized by N_{mn} piecewise linear functions (PWLFs). Let $\left(f_{mn}^{(0)}, f_{mn}^{(2)}, ..., f_{mn}^{(N_{mn})}\right)$ denote the 1+N_{mn} breakpoints of f_{mn} to its range and let $g_{mn}^{(i)} = \left|f_{mn}^{(i)}\right|\left(f_{mn}^{(i)}\right)^2 / 3C_{mn}$ denote the corresponding nonlinear function value of z_{mn} at the i-th breakpoint. According to the λ-formulation [9], the LP model by the tight polyhedra can be formulated as

$$\min \quad \bar{z} = \sum_{(m,n)\in\mathcal{L}}\left(\sum_{i=0}^{N_{mn}} g_{mn}^{(i)}\eta_i\right) \tag{8a}$$

$$s.t. \quad \sum_{mn\in GF} f_{mn} - \sum_{mn\in GT} f_{nm} = G_m - D_m, \quad \forall m\in\mathcal{N} \tag{8b}$$

$$f_{mn} = \sum_{i=0}^{N_{mn}} f_{mn}^{(i)}\eta_i, \quad \forall(m,n)\in\mathcal{L} \tag{8c}$$

$$\sum_{i=0}^{N_{mn}} \eta_i = 1, \quad \eta_i \ge 0, \quad \forall(m,n)\in\mathcal{L} \tag{8d}$$

By solving the above tight LP model in (8), the model in (2) can be approximated. Thus, the optimal solution (f_{mn}^*, λ_m^*) can be obtained, and the feasibility check can be performed by means of the corollary.

***Discussions*:**

(i) Let the approximation error of the piecewise linear function on z_{mn} be ε_{mn}. If the optimal solutions of (2) and (8) are f_{mn}^* and g_{mn}^*, respectively. We have

$$\left|\frac{|f_{mn}^*|(f_{mn}^*)^2}{3C_{mn}} - \frac{|g_{mn}^*|(g_{mn}^*)^2}{3C_{mn}}\right| \le \varepsilon \tag{9}$$

where the approximation error is considered small enough, such that the gas flows f_{mn}^* and g_{mn}^* have the same sign (i.e., the approximation does not change the flow direction). The error on the KKT condition (5) can be expressed as (10). It can be found that the error of KKT condition is bounded, which means that decreasing ε_{mn} by increasing the number of segments will reduce the error on the KKT condition.

$$\left|\frac{|f_{mn}^*| f_{mn}^*}{C_{mn}} - \frac{|g_{mn}^*| g_{mn}^*}{C_{mn}}\right| \le \frac{3\left(|f_{mn}^*| + |g_{mn}^*|\right)\varepsilon}{\left(g_{mn}^*\right)^2 + \left(f_{mn}^*\right)^2 + \left|g_{mn}^* f_{mn}^*\right|} \tag{10}$$

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Moreover, the error of ε_{mn} will not affect the inequalities in (7a) because the same error on λ_m^* lies in both sides of the inequalities. Meanwhile, the error of ε_{mn} will only lead to a bounded error on (7b) according to (10).

(ii) It should be noted that both [7] and the proposed LP model need piecewise linearization. The piecewise linearization is implemented for nonconvex parts of the model in [7], which requires integers. By contrast, it is utilized in the proposed model for convex parts, which thus does not require integers. Moreover, compared (8) with (2), it can be found that the piecewise linearization is only used in the objective function. It is discussed in [10] that the convex optimization (2) can be tightly approximated by the LP with a prescribed accuracy if the number of segments is sufficient.

(iii) It can be observed from (2) that the objective function $|f_{mn}| f_{mn}^2 / 3C_{mn}$ is equivalent to $|\lambda_m - \lambda_n|^{3/2} C_{mn}^{1/2} / 3$, which expects to minimize the difference of multipliers at each pipeline. Hence, it is highly possible that the multipliers can satisfy (7) and this optimal solution is likely feasible.

IV. Numerical Example

The proposed LP model is verified on several real-life natural gas networks in European countries [11], and is compared with the traditional MILP [7] and MISOCP [8]. The three methods are performed on a laptop with an Intel® Core™ i5 Duo Processor T440 (2.30 GHz) and 4 GB RAM, using the commercial solver GUROBI 7.5. In addition, we choose different numbers of segments (# of Seg.) to investigate the impact of the " # of Seg." on the computational performance.

Firstly, under the given boundary conditions, all the test systems are feasible by the three methods (see "F." in Table I). Moreover, 2000 nominations for the 582-node system are selected with different boundary conditions (e.g., gas injection and withdrawal, physical limits, etc.). These boundary conditions are given from the practical gas system operation in winter and summer. The results show that the none nomination is infeasible, which suggests that the corollary is mild in practice. Notably, the infeasibility can be enhanced by the compressors and the configuration of compressor stations.

Secondly, Table I presents the computation time on eight test systems with 10, 50, and 100 segments. For small-scale test systems with a very few number of segments, all the three methods can be realized within acceptable computation time. In contrast, for large-scale test systems or the systems with many segments, the MILP or MISOCP models will require more time to compute. For instance, the computation time of the MILP model on the 584-node system with 50 segments is more than 2 hours and the algorithms of the MILP and MISOCP will fail to solve the 4197-node system within an acceptable period of 5 hours. In contrast, the proposed LP model can solve the gas equations on all the test systems within 5 seconds, which is more than 1000 times faster than the MILP and MISOCP methods.

Finally, the impact of the "# of Seg." on the solution precision is explored. Here, the 14-node system is considered and the results are shown in Table II. Observations in Table II indicate that the gap of the MISOCP is very small which gives a precise optimal solution. Therefore, the MISOCP is chosen as the benchmark. For the LP and MIMP methods, the piecewise linearization may bring errors. Here, Error_V and Error_C are denoted as the maximum error of decision variables and constraints, respectively. With the increase of the number of segments, the errors of both LP and MILP methods become smaller and MILP needs relatively fewer segments than LP. For the precision of 10^{-3}, the LP needs 50 segments while the MILP needs 20 segments. This is because the MILP is adopted to solve a quadratic function, while the LP is used to linearize a cubic function. The linearization for a high-order nonlinear function generally needs more segments. However, the computation time of the MILP with 20 segments is more than 4.76 seconds, whereas the LP with 20 segments only takes 0.15 seconds. This implies that the LP is always fast even though it needs more segments, especially for large-scale systems.

Table I. Computational performance on several systems.

# of Nodes	F.	Time (s)						
		10 Seg.		50 Seg.		100 Seg.		MISOCP
		LP	MILP	LP	MILP	LP	MILP	
11	√	0.09	3.41	0.09	8.32	0.09	19.98	18.54
14	√	0.15	4.76	0.15	12.34	0.18	28.68	45.66
20	√	0.16	7.32	0.17	33.56	0.21	53.99	83.71
24	√	0.18	7.89	0.20	37.21	0.22	74.96	98.54
40	√	0.22	12.56	0.25	78.44	0.29	321.4	260.21
134	√	0.37	43.33	0.39	549.3	0.46	2653	1489.32
582	√	0.63	987.3	0.95	4596	1.48	16879	9825
4197	√	1.33	6487	3.42	>5h	7.72	>5h	>5h

Table II. Impact of the "# of Seg." on the solution precision.

# of Seg.	LP		MILP		MISOCP
	Error_V	Error_C	Error_V	Error_C	gap
10	3.0576	0.3137	0.4463	0.0124	3.45e-4
20	0.4571	0.0628	0.0220	0.0041	
50	0.0819	0.0108	0.0054	0.0036	
100	0.0405	0.0022	0.0039	0.0008	

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