Essays on Risk and Fair Pricing

Hvolby, Thomas

Publication date:
2018

Document Version
Publisher's PDF, also known as Version of record

Link to publication from Aalborg University

Citation for published version (APA):
Hvolby, T. (2018). *Essays on Risk and Fair Pricing*. Aalborg Universitetsforlag. Ph.d.-serien for Det Ingeniør- og Naturvidenskabelige Fakultet, Aalborg Universitet

ESSAYS ON RISK AND FAIR PRICING

BY
THOMAS HVOLBY

DISSERTATION SUBMITTED 2018

AALBORG UNIVERSITY
DENMARK

Essays on Risk and Fair Pricing

Ph.D. Dissertation

Thomas Hvolby

Dissertation submitted January , 2018

Dissertation submitted:	January, 2018
PhD supervisor:	Associate Professor Esben Høg Aalborg University
Assistant PhD supervisor:	Professor Bent Jesper Christensen Aarhus University
PhD committee:	Associate Professor Ege Rubak (chairman) Aalborg University Professor Rolf Poulsen University of Copenhagen Professor Wolfgang Runggaldier University of Padova
PhD Series:	Faculty of Engineering and Science, Aalborg University
Department:	Department of Mathematical Sciences

ISSN (online): 2446-1636
ISBN (online): 978-87-7210-138-5

Published by:
Aalborg University Press
Langagervej 2
DK – 9220 Aalborg Ø
Phone: +45 99407140
aauf@forlag.aau.dk
forlag.aau.dk

Printed in Denmark by Rosendahls, 2018

Abstract

This thesis is concerned with the valuation of contracts on financial markets, specifically the estimation of adjustments to the risk-free price of a derivative due to the inclusion of different types of risks. We consider volumetric risk on power markets and credit risk for general financial contracts. We have an applied focus such that the link from theory to application is thoroughly described, and in Papers A and C we present results from analyses of market data.

In Paper A, within the area of volumetric risk, we examine the correlation structure between wind power production and electricity prices as observed on the Danish power market. We consider different representations of constant and time-varying copula models to describe this non-linear correlation structure, and provide results from the Kolmogorov and the Cramer tests to assess the goodness-of-fit of each model to perform model selection; in our numerical studies a time-varying Gaussian copula provides the best fit to data. We apply the chosen copula specification to simulate contract payoffs for agreements that include paying a fixed price per MWh while receiving a fluctuating wind power production; here we quantify the price adjustment of this agreement associated with joint price and volumetric risk, and find that an adjustment of 7% of the price of a standard forward contract provides a fair price of the derivative.

In Paper B, we examine a change-of-measure approach to the estimation of the credit value adjustments (CVA) for put and call options that are sensitive to wrong way risk (WWR). This approach yields CVAs that depend on a stochastic drift adjustment (the drift that controls the measure change) and given a deterministic approximation of the drift adjustment, the CVA with WWR is of closed form. We compare this method to the formulas specified in the Basel III framework and conclude that Basel III provides a naive estimation of the CVA and is not able to capture right way risk while WWR is not captured desirably.

In Paper C, we examine the estimation of bilateral CVA (BCVA) using a reduced form approach to credit risk modeling. We provide a thorough discussion of the model calibration to quotes on credit default swaps (CDSs)

using the technique of transforming quotes to market implied survival probabilities of the reference credit and calibrating the model such that the survival probabilities suggested by the model fits those implied by the market. We present an analysis of the market quotes and the model calibration for six names and use the resulting parameters to estimate the market price of risk (MPR) associated with the survival probabilities. Further, we present a framework for numerical estimation of the BCVA on CDS contracts traded between two entities with a third entity as the reference credit.

Resumé

Denne afhandling omhandler værdifastsættelse af kontrakter på finansielle markeder, specifikt estimation af prisjusteringer af kontrakters risiko-frie pris grundet forskellige risici. Vi betragter volumenrisiko på energimarkeder og kreditrisiko for generelle finansielle instrumenter. Vi har et anvendelsesorienteret fokus og beskriver sammenhængen mellem teori og anvendelse i detaljer. I artikel A and C præsenterer vi resultater fra analyser på markedsdata.

I artikel A, omhandlende volumenrisiko, undersøger vi korrelationsstrukturen mellem produktionen af vindenergi og elpriser på det danske elmarked. Vi betragter forskellige repræsentationer af konstante og tidsvariate copulamodeller til at beskrive den ikke-lineære korrelationsstruktur og viser resultater fra Kolmogorovs og Cramers tests for at vurdere hvor godt hver model passer på data. Ud fra disse tests vælger vi den model som giver det bedste fit, hvilket er en tidsvariat gaussisk copulamodel. Vi anvender den valgte copulaspecifikation til at simulere udbytte af kontrakter, hvor en fast pris pr. MWh betales mens en fluktuerende vindproduktion modtages. Her kvantificeres prisjusteringen til denne type kontrakt pga. risikoen forbundet med kontraktens pris- og volumenrisiko. Vi estimerer en justering på 7% af prisen på en standard forwardkontrakt giver kontrakten en fair pris.

I artikel B, betragter vi en målskiftetilgang til estimation af *credit value adjustments* (CVA) og anvender tilgangen til put- og calloptioner under tilstedeværelse af *wrong way risk* (WWR). I denne metode er CVA afhængig af en stokastisk justering af prisprocessens drift og givet en deterministisk approksimation af denne justering, har optionernes CVA et lukket-form udtryk. Vi sammenligner det estimerede CVA med formlerne i Basel III aftalen og konkluderer at Basel III giver en naiv estimation af CVA, som ikke er i stand til at tage højde for *right way risk* og ikke ønskeligt tager højde for WWR.

I artikel C, betragter vi bilateral CVA (BCVA) i en *reduced form* tilgang til kreditrisiko. Vi giver en grundig diskussion af kalibrering af modeller til markedsdata på *credit default swaps* (CDSer), hvortil vi bruger en teknik som indvolverer transformation af markedsdata til overlevelsessandsynligheder for det underliggende firma. Modellens parametre bestemmes så modellens overlevelsessandsynligheder passer bedst muligt med disse regnet ud fra markedsdata. Vi analyserer markedsdata for seks firmaer, og viser resultater fra modelkalibreringen. Vi anvender de resulterende modeller til at estimere *market price of risk* (MPR) forbundet med overlevelsessandsynlighederne. Yderligere præsenteres en ramme for numerisk estimation af BCVA på CDSer, som er handlet mellem to firmaer.

Contents

Abstract **iii**

Resumé **v**

I Introduction 1

1 Fair pricing and equivalent martingale measures 3
2 Credit markets and credit risk 3
2.1 Modeling default intensities with a Cox-process 4
2.2 Choosing recovery 6
3 Credit Value Adjustments 9
3.1 Valuing a zero-coupon bond with UCVA 9
3.2 Valuing a general derivative with UCVA and risk-free closeout 12
3.3 Expected positive exposure and wrong way risk 15
3.4 Valuing a general derivative with BCVA and risk-free closeout 15
3.5 Valuing a general derivative with BCVA and replacement closeout 18
4 Scientific contribution of this thesis 21
References 21

II Papers 23

A Joint price and volumetric risk in wind power trading: A copula approach **25**
1 Introduction 27
2 Modeling dependence with copula models 30
2.1 Marginal models 31
2.2 Estimation procedure for the joint model 32
2.3 Time-varying copula models 33

2.4 Quantile dependence 34
2.5 Selection of copula models 35
3 Empirical results 37
3.1 Marginal specifications for spot electricity prices and wind power production 38
3.2 Symmetric vs. asymmetric dependence 40
3.3 Constant copula models 40
3.4 Time-varying copula models 44
4 A simulation study 47
5 Application to pricing and risk management 50
5.1 Example 1 55
5.2 Example 2 57
6 Conclusion 61
A Properties of selected copula models 62
A.1 Elliptical copulas 62
A.2 Archimedian copulas 62
B Additional figures 64
References 65

B Wrong-Way Risk adjusted exposure: Analytical Approximations for Options in Default Intensity Models 69
1 Introduction 71
2 Call and put risk-neutral dynamics 72
3 Expected Positive Exposures under No WWR 74
4 Expected Positive Exposures under WWR 74
5 Proxys of θ_t^s 77
5.1 $\mathbb{Q}$-expectation 77
5.2 Approximation of $\mathbb{Q}^{\mathcal{C}_T}$-expectation 78
6 Potential Future Exposures (PFE) 80
7 Numerical experiments 80
References 83

C Calibration of CIR processes to CVA data and applications to estimation of Market Price of Risk 89
1 Introduction 91
2 Modeling default intensities with a Cox-process 91
3 Valuing a general derivative with UCVA and risk-free closeout 92
4 Valuing a general derivative with BCVA and risk-free closeout 94
5 Valuing a general derivative with BCVA and replacement closeout 94
6 CDS contracts 96
6.1 Formulation of the CDS and its cash flows 96
6.2 Valuing a risk-free CDS 99

7 Modeling risk-free interest rates and default intensities 103
 7.1 Model calibration through implied survival probabilities and implied hazard rates 104
 7.2 The CIR++ model 107
8 Calibration to market data 109
 8.1 Bloomberg CDS data 110
 8.2 Assumptions and formulas for calibration to Bloomberg data 111
 8.3 i^{th} tenor 115
 8.4 Specification of the implied survival probabilities 120
 8.5 Objective function and calibration of parameters 122
 8.6 Calibration routine 123
 8.7 Calibration results 124
 8.8 Challenges in parameter calibration 125
9 Market price of risk 126
 9.1 $\mathbb{Q}$ and $\mathbb{P}$-parameter relations 127
 9.2 Results – monthly price of risk 128
10 Triparty CDSs with BCVA and risk-free closeout 132
 10.1 Simulation of default-intensities and defaults 133
 10.2 Simulation of CDS cash flows 133
11 Premiminary conclusion and further work 135
References 135

Part I

Introduction

1 Fair pricing and equivalent martingale measures

We consider fair pricing as derivatives pricing under the assumption of no-arbitrage. Due to the first fundamental theorem of asset pricing, the existance of an equivalent martingale measure imply that the no-arbitrage condition holds and vice versa, [1].

We consider two markets; the power market in Paper A and credit markets in Papers B and C. In both markets we assume the existance of a unique equivalent martingale measure, denoted the $\mathbb{Q}$-measure.

Under the assumption of the existance and uniqueness of the $\mathbb{Q}$-measure, the discounted value of a financial contract is a martingale process. The current value of any contract can then be calculated by the $\mathbb{Q}$-expectation of the contract's discounted payoff, [1].

In Paper A, we consider fair pricing of contracts that are sensitive to both energy prices and the production of wind power. We assume that the physical measure equals the $\mathbb{Q}$-measure, and therefore we directly observe the $\mathbb{Q}$-dynamics of electricity prices on the power market. We use this dynamic to obtain fair prices of contracts dependent on the electricity prices.

In Paper B we assume price dynamics directly under the $\mathbb{Q}$-measure and do not estimate the dynamics from market data. In Paper C, we assume that the $\mathbb{Q}$-measure dynamics are observed in the credit default swap (CDS) markets. In this case, we extract information from CDS quotes and use these to obtain the necessary $\mathbb{Q}$-dynamics.

2 Credit markets and credit risk

In order to apply fair pricing techniques in Papers B and C, development of pricing formulas on credit markets are necessary. Therefore we devote this section to introducing the estimation of credit risk, i.e. we focus on the risk associated with credit events occurring.

Traditionally two frameworks for estimation of credit risk are applied [2, 3];

- The structural approach where the default of a company is based on the relationship between debt and value of a company
- The reduced form approach: A purely statistical framework, where the default of a company is modeled as the first jump of a stochastic jump process

We only consider models of the reduced form type. The default intensity – equivalent to the intensity of the stochastic jump process – is considered to be stochastic, resulting in a Cox-process set-up suggested by [4].

2.1 Modeling default intensities with a Cox-process

Modeling default times using a Cox-process as the intensity was introduced in [4]. The following is a recap of the results from [4] and [2, pp. 109-117] that we will apply in the CVA pricing frameworks.

Initially, we note that all models considered are of the class of *reduced form models*, meaning that default of the counterparty is not based on corporate finance techniques (valuations of stock, debt, etc.) as in *structural models*, but is modeled completely statistically through a jump process. This process is allowed to have stochastic intensity $\lambda^i(X_t^i)$ instantaneously at time t, where X_t^i is the state variable(s) driving the default intensity of firm i observed at time t. Note that since our purpose is to use the models for derivatives pricing, $\lambda^i(X_t^i)$ is modeled under the pricing measure $\mathbb{Q}$, which is assumed to exist. The process $\lambda^i(X_t^i)$ is henceforth denoted λ_t^i for a shorter notation.

In the Cox-process set-up, we let N_t^i be a counting process with stochastic intensity λ_t^i. Conditioning on no jump occurring prior to or at time t, the counting process $N_{s|t}^i$ for $s \geq t$ is the number of jumps occurring between time t and s. Since we are modeling the default of a company, we are only interested in the first jump of the counting process, since all contracts with the firm are terminated, and insolvency procedures begin. At the default, the creditors of the company will retrieve a fraction on the current amount owed to them by the defaulted company.

We now let τ^i be the first jump time of the $N_{s|t}^i$ process for all $s \geq t$ given that no jump has yet occurred at time t. τ^i is then defined as

$$\tau^i = \inf\left\{ s > t : \int_t^s \lambda_u^i \mathrm{d}u \geq E^i \right\},$$

where $E^i \sim \mathrm{Exp}(1)$ is assumed to be independent of the factors driving λ_s^i. Note that τ^i depends on t, but for simplicity this is suppressed in the notation. We define $\mathcal{G}_s^i = \sigma\{X_u^i : u \leq s\}$, $\mathcal{H}_s^i = \sigma\{\mathbb{1}_{\{\tau^i \leq u\}} : u \leq s\}$, and $\mathcal{F}_s^i = \mathcal{G}_s^i \vee \mathcal{H}_s^i$; here $\mathcal{A} \vee \mathcal{B}$ for any two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ represents the minimal σ-algebra containing both $\mathcal{A}$ and $\mathcal{B}$. Thereby E^i is independent of $\mathcal{G}^i$, λ_s^i is a $\mathcal{G}_s^i$ measurable process, and τ^i is $\mathcal{H}_s^i$ measurable. Now a few properties of the

Cox-process set-up is presented. For an arbitrary $T > t$ it holds that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau^i > T\}}\,\Big|\,\mathcal{G}_T^i\Big] &= \mathbb{Q}\Big(\tau^i > T\,\Big|\,\mathcal{G}_T^i\Big) \\
&= \mathbb{Q}\Big(\inf\Big\{s > t : \int_t^s \lambda_u^i \mathrm{d}u \geq E^i\Big\} > T\,\Big|\,\mathcal{G}_T^i\Big) \\
&\overset{1}{=} \mathbb{Q}\Big(\sup\Big\{\int_t^s \lambda_u^i \mathrm{d}u : s \in (t,T]\Big\} < E^i\,\Big|\,\mathcal{G}_T^i\Big) \\
&\overset{2}{=} \mathbb{Q}\Big(\int_t^T \lambda_u^i \mathrm{d}u < E^i\,\Big|\,\mathcal{G}_T^i\Big) = 1 - F_{\mathrm{Exp}(1)}\Big(\int_t^T \lambda_u^i \mathrm{d}u\Big) \\
&= \exp\Big\{-\int_t^T \lambda_u^i \mathrm{d}u\Big\}. \qquad (1)
\end{aligned}$$

Moreover

$$\mathbb{E}^{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau^i \leq T\}}\,\Big|\,\mathcal{G}_T^i\Big] = \mathbb{Q}\Big(\tau^i \leq T\,\Big|\,\mathcal{G}_T^i\Big) = 1 - \exp\Big\{-\int_t^T \lambda_u^i \mathrm{d}u\Big\} = F_{\tau^i}\Big(T\,\Big|\,\mathcal{G}_T^i\Big).$$

From the CDF of $\tau^i|\mathcal{G}_T^i$ in the above equation, it is seen that $\tau^i|\mathcal{G}_T^i$ represents the first jump of a Poisson process with deterministic intensity $\int_t^T \lambda_u^i \mathrm{d}u$.

From the CDF, the PDF of $\tau^i|\mathcal{G}_T^i$, which is important in CVA pricing, is easily obtained:

$$\begin{aligned}
F_{\tau^i}\Big(s\,\Big|\,\mathcal{G}_T^i\Big) &= 1 - \exp\Big\{-\int_t^s \lambda_u^i \mathrm{d}u\Big\} \quad \text{for all } s \leq T \\
f_{\tau^i}\Big(s\,\Big|\,\mathcal{G}_T^i\Big) &= \lambda_s^i \exp\Big\{-\int_t^s \lambda_u^i \mathrm{d}u\Big\} \quad \text{for all } s \leq T. \qquad (2)
\end{aligned}$$

The last (but important) result we wish to present is the expected value of the product of the indicator that default occurs before some T and a function that itself depends on the time of default. Note that tau^i depends on t; however, this is implicit in the notation. Specifically let $g(s) \geq 0$ be a $\mathcal{G}_s^i$-measurable function for all $s \geq t$. We use that τ^i is defined on the interval (t, ∞) and that the conditional probability density function of τ^i takes the form in (2), and

[1] The event

$$\inf\Big\{s > t : \int_t^s \lambda_u^i \mathrm{d}u \geq E^i\Big\} > T \qquad (*)$$

implies that

$$\int_t^s \lambda_u^i \mathrm{d}u < E_1^i, \qquad \forall s \in (t,T]. \qquad (**)$$

Thereby the probability of the event in $(*)$ can be written as the probability that $(**)$ holds for all $s \in (t,T]$ or more specifically that $(**)$ holds for the supremum value of the integral with $s \in (t,T]$.

[2] Since $\lambda_u^i > 0$ for all $s \in (t,T]$ the integral $\int_t^s \lambda_u^i \mathrm{d}u$ is monotonically increasing in s.

we obtain the following result;

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^i \le T\}} g(\tau^i) \,\middle|\, \mathcal{F}_t^i\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^i \le T\}} g(\tau^i) \,\middle|\, \mathcal{G}_T^i \vee \mathcal{F}_t^i\right] \,\middle|\, \mathcal{F}_t^i\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^i \le T\}} g(\tau^i) \,\middle|\, \mathcal{G}_T^i\right] \,\middle|\, \mathcal{F}_t^i\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T f_{\tau^i}\left(s \,\middle|\, \mathcal{G}_T^i\right) g(s) \mathrm{d}s \,\middle|\, \mathcal{F}_t^i\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \lambda_s^i e^{-\int_t^s \lambda_u^i \mathrm{d}u} g(s) \mathrm{d}s \,\middle|\, \mathcal{F}_t^i\right] \\
&= \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\lambda_s^i e^{-\int_t^s \lambda_u^i \mathrm{d}u} g(s) \,\middle|\, \mathcal{F}_t^i\right] \mathrm{d}s, \qquad (3)
\end{aligned}$$

where the last equality stems from the Fubini-Tonelli theorem due to non-negativity of the integrand[3] and that the integrand is $\mathcal{G}_s^i$-measurable.

2.2 Choosing recovery

Pre-crisis

Three forms of recoveries were widely used in a Unilateral Credit Value Adjustment (UCVA) before the '07-'08 crisis, see [2, pp. 117-122]. A short recap of these is as follows.

- *Recovery of face value* assumes that recovery is a constant fraction $R \in [0, 1]$ of the notional on the derivative. This means that a reference derivative of \$1 notional will return the recovery rate R upon default time τ, if this time is prior to the time of maturity of the contract. This recovery is therefore completely independent of the remaining lifespan and the present value of the contract at the time of default, which seems quite counterintuitive. However, [2] presents this practice as the most widely used in rating-agency studies.

- *Recovery of market value* assumes a constant fraction (the recovery rate) of the market value instantaneously before the default time to be recovered.

- *Recovery of treasury* assumes that for a corporate bond, a fraction of the present value at the time of default of a risk-free bond with the same time to maturity as the derivative is recovered.

[3]We require non-negativity of the default intensity and if the g-function is not defined to be either non-positive or non-negative, we simply split the function into two parts, each defined to be non-negative, and obtain the same result. Specifically we let $g(s) = g^+(s) - g^-(s)$ with $g^+(s) = \max\{g(s), 0\}$ and $g^-(s) = \max\{-g(s), 0\}$.

Post-crisis

The jargon has changed after the financial crisis with the introduction of the Bilateral Credit Value Adjustment (BCVA). Recent literature discusses the closeout amount upon a default, which corresponds to the value of which a fraction (the recovery rate) is retrieved by the surviving party if their counterparty defaults during the lifespan of a derivatives contract. The closeout amount multiplied by the recovery rate equals the total amount recovered by the survivor and is analogue to the recovery in Sec. 2.2. The discussion now lies in the calculation of this closeout amount. The focus in the literature is between two frameworks for calculating the closeout amount: *Risk-free closeout* and *replacement closeout*. The latter is also known as *substitution closeout* or *risky closeout*.

After the introduction of BCVA, the most widely used recovery assumption was that of risk-free closeout, according to [5]. This closeout resembles the recovery of the treasury introduced in Sec 2.2, as the value of the contract at default is determined in a completely risk-free environment (meaning both the defaulted and surviving counterparties are seen as risk-free in the closeout valuation). If the value of the contract is positive from the non-defaulted counterparty's point of view, a fraction of this value is recovered, and if the contract has negative value, the surviving counterparty will give the absolute contract value to the defaulted counterparty's creditors.

In [5] the replacement closeout is introduced. Replacement closeout essentially assumes that when valuing the closeout after a default event the defaulted counterparty is seen as risk-free, whereas the surviving party is seen as defaultable. The closeout convention is also described as valuing the contract from the point of view of a risk-free party, taking the position of the defaulted counterparty in the contract. The argument in replacement closeout is that since the new (risk-free) counterparty would not neglect the risk of the surviving party defaulting before the contract matures, this should not be neglected when valuing the contract at closeout.

There is not complete agreement in the literature when it comes to the choice of closeout, and e.g. [6, pp. 123–124, 278–280], [5, 7], and [8, Ch. 14] debates this issue. In [6] it is generally recommended to use the replacement closeout, due to negative jumps in the portfolio value if this is positive immediately before counterparty default, making the losses for the surviving party greater than one could expect from the mark-to-market up to the default, as well as positive jumps if the portfolio value is negative, resulting in lower recovery for the defaulting counterparty's creditors than the mark-to-market before the default.

In [5, 7, 8], arguments for and against both closeout types are presented and it is pointed out that choosing the wrong closeout can lead to unexpected valuations of contracts at default events. The key argument against the risk-free closeout is similar to those presented in [6], while it is also discussed that for a corporate bond, the valuation will depend on both the borrower and the lender's creditworthiness if risk-free closeout is considered. This is not the case with replacement closeout, in which case the valuation only depends on the creditworthiness of the borrower. The replacement closeout, however, has a disadvantage if the defaulted entity has a high systemic impact, in which case the default may cause a positive jump in the credit spread of the surviving party. This results in a negative jump in the price from the surviving party's point of view, thereby either adding to this party's debt to the defaulter or lowering the recovery. The worst case scenario is that the replacement closeout can contribute to a debt/default spiral followed by the default of a firm with high systemic impact.

Modeling recovery rates

After choosing the appropriate closeout convention, another issue regarding the expected recovery upon default occurs. The defaulted entity's recovery rate is necessary to determine the amount actually retrieved by a surviving creditor. [6, pp. 209-211] argues that the recovery rates are strongly dependent on time, sector and seniority of debt, however the amount of data on recovery rates is sparse. Many studies of CVA consider the recovery rate to be a fixed fraction, which according to the data presented in [6] is a rather unrealistic simplification.

[9] is an early approach to modeling default rates in a portfolio of loans, and a one-factor model to describe default rates is presented. It is claimed that empirical studies have shown a negative correlation between default and recovery rates, meaning that as the probability of default rises the recovery rate tends to lower. An explanation for this is provided by [10] that argues that e.g. decreased consumption or investment can result in both a higher probability of default and a decreased value of posed collateral. This can result in a lower recovery rate, or equivalently a higher *loss-given-default* (LGD), which is defined as one minus.

More recently [11] present a model for the LGD, also for loan portfolios. This model is designed to take into account the dependence of LGD on the number of defaults in a year. The essence of the model is that given the number of defaults the average LGD is normally distributed, and given the default rate the number of defaults is binomially distributed with probability parameter corresponding to the default rate. The normality of the average LGD is assumed in order to allow this to break the lower bound of zero and the upper bound of one that one would usually assume for the average LGD, due to negative observations in the data considered. If one, however, is considering data where these bounds are not exceeded, a distribution that only allows values from zero to one is more intuitive.

A model for stochastic LGDs in a structural model framework is presented in [12], however, the model may also be applied in a reduced form framework. Here it is assumed that the LGDs are correlated and follow beta distributions, which ensures that these are limited between zero and one.

An examination of recovery rates with a focus on structural models of credit risk is provided in [13–15]. It is shown in [14] that when considering a Merton type structural model of default the assumption of constant recovery is not satisfied. This is shown by simulating from the Merton type model (in which the recovery can be derived from other quantities in the model) and calibrating the model to the simulated data. The models discussed in these articles are however not applicable to a reduced form model framework since they depend on the relationship between a company's debt, value and the recovery value if this company defaults.

3 Credit Value Adjustments

3.1 Valuing a zero-coupon bond with UCVA

Assume that our point of view is through an institution that wants to calculate the value of a derivative. Assume further that the institution's counterparty in the trade can default, but the institution itself cannot.[4] Henceforth the institution will be denoted b and the counterparty c. The value of a zero-coupon bond (ZCB) is the risk-neutral expectation of discounted future payoffs, which for a zero coupon bond is \$1 given no default and is explained by the function $R^c(\tau^c)$ – called the recovery given default of the counterparty – at the time of counterparty default, τ^c, given this is prior to maturity T. We allow for the recovery function to be defined at any time s, such that $R^c(s)$ gives the recovery value if the counterparty default occurs at time s. Note the recovery $R^c(s)$ is here the entire closeout value, and is distinct from the recovery rate at default, which is merely a fraction of the entire recovery. We now define $\mathcal{G}_s^r = \sigma\{X_u^r : t \leq u \leq s\}$ as the sigma-algebra generated by the factors driving the risk-free rate, and introduce the sigma-algebra containing all information about counterparty default, default intensity and the risk-free rate as

$$\mathcal{F}_s^{c,r} = \mathcal{F}_s^c \vee \mathcal{G}_s^r = \mathcal{G}_s^c \vee \mathcal{H}_s^c \vee \mathcal{G}_s^r.$$

By using the mean value of the indicator in Eq. (1) and the PDF of the default time τ^c from Eq. (2), as well as the law of iterated expectations with

$$\mathcal{F}_t^{c,r} \subseteq \mathcal{G}_T^c \vee \mathcal{H}_t^c \vee \mathcal{G}_t^r \subseteq \mathcal{G}_T^c \vee \mathcal{H}_t^c \vee \mathcal{G}_T^r$$

[4]The more realistic assumption is that the probability of default for the institution is negligible compared to that of the counterparty. This was usual before the financial crisis especially for trades between a firm and a large bank, since large banks were typically considered as default free or too-big-to-fail.

for any $T \geq t$, it is obtained that[5]

$$
\begin{aligned}
&P^{\mathrm{UCVA}}_{b,\{c\}}(t,T) \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s} \left(1 \cdot \mathbb{1}_{\{\tau^c > T\}}\right) + e^{-\int_t^{\tau^c} r_s \mathrm{d}s} \left(R^c(\tau^c) \cdot \mathbb{1}_{\{\tau^c \leq T\}} \right) \,\middle|\, \mathcal{F}_t^{c,r} \right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^c > T\}} \,\middle|\, \mathcal{G}_T^c \vee \mathcal{H}_t^c \vee \mathcal{G}_T^r \right] \right. \\
&\qquad\qquad \left. + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^c} r_s \mathrm{d}s} R^c(\tau^c) \mathbb{1}_{\{\tau^c \leq T\}} \,\middle|\, \mathcal{G}_T^c \vee \mathcal{H}_t^c \vee \mathcal{G}_T^r \right] \,\middle|\, \mathcal{F}_t^{c,r} \right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s + \lambda_s^c) \mathrm{d}s} + \int_t^T e^{-\int_t^s r_u \mathrm{d}u} R^c(s) \lambda_s^c e^{-\int_t^s \lambda_u^c \mathrm{d}u} \mathrm{d}s \,\middle|\, \mathcal{F}_t^{c,r} \right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s + \lambda_s^c) \mathrm{d}s} \,\middle|\, \mathcal{F}_t^{c,r} \right] + \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\lambda_s^c e^{-\int_t^s (r_u + \lambda_u^c) \mathrm{d}u} R^c(s) \,\middle|\, \mathcal{F}_t^{c,r} \right] \mathrm{d}s. \quad (4)
\end{aligned}
$$

The subscript $(b, \{c\})$ indicates that we are considering the price from the bank's point of view and that only the counterparty c is default-risky. Henceforth all default-risky entities will be specified in brackets.

A remark on the pricing formula (4) is appropriate. If the recovery upon default is zero, i.e. the bond is worthless instantly upon counterparty default, the pricing equation of the defaultable bond resembles that of a risk-free ZCB with the discount rate being the risk-free rate plus the intensity process. This gives a lower price of the defaultable bond than that of the risk-free bond due to "harder" discounting. In the case of zero recovery the credit spread on a bond issued by the company would directly correspond to the default intensity λ_s^c. In the following, the situation with non-zero recovery is considered. Note that the replacement closeout does not make sense in this set-up, since the unilateral CVA scheme explicitly assumes the pricing counterparty to be risk-free, and therefore the pricing is only consistent if this is assumed in the closeout as well.

Risk-free closeout

The recovery rate at counterparty default is for notational purposes given as the fraction $(1 - l^c(s))$ where $l^c(s)$ is the loss-given-default of the counterparty if the counterparty defaults at time s. For completeness, $l^c(s)$ is allowed to be a deterministic function or a stochastic variable; however, in our applications we assume this to be constant. The risk-free closeout at counterparty

[5] Note that all pricing equations are implicitly assuming that $\tau^c > t$, i.e. the counterparty has not defaulted at when the product is priced. Some sources of literature are pointing this out by multiplying all equations by $\mathbb{1}_{\{\tau^c > t\}}$, such that the value of the contract jumps to zero in case of counterparty default. We do not adopt this notation, but are instead assuming that one is only interested in valuing a contract with counterparties that have not defaulted at the present time.

default ($s = \tau^c$) is thereby the fraction $(1 - l^c(\tau^c))$ of the risk-free bond value time τ^c, i.e.

$$R^c(\tau^c) = \mathbb{E}^{\mathbb{Q}}\left[(1 - l^c(\tau^c))e^{-\int_{\tau^c}^{T} r_s \mathrm{d}s} \,\middle|\, \mathcal{F}_{\tau^r}^{c,r}\right].$$

The pricing formula (4) then becomes

$$\begin{aligned}
P_{b,\{c\}}^{\mathrm{UCVA}}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s + \lambda_s^c)\mathrm{d}s} \,\middle|\, \mathcal{F}_t^{c,r}\right] \\
&\quad + \int_t^T \mathbb{E}^{\mathbb{Q}}\left[(1 - l^c(s))\lambda_s^c e^{-\int_t^T r_u \mathrm{d}u} e^{-\int_t^s \lambda_u^c \mathrm{d}u} \,\middle|\, \mathcal{F}_t^{c,r}\right]\mathrm{d}s \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s + \lambda_s^c)\mathrm{d}s} \,\middle|\, \mathcal{F}_t^{c,r}\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \mathrm{d}u} \int_t^T \lambda_s^c e^{-\int_t^s \lambda_u^c \mathrm{d}u} \mathrm{d}s \,\middle|\, \mathcal{F}_t^{c,r}\right] \\
&\quad - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \mathrm{d}u} \int_t^T l^c(s)\lambda_s^c e^{-\int_t^s \lambda_u^c \mathrm{d}u} \mathrm{d}s \,\middle|\, \mathcal{F}_t^{c,r}\right]
\end{aligned}$$

The first two expectations simplify to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s}\left(e^{-\int_t^T \lambda_s^c \mathrm{d}s} + \int_t^T \lambda_s^c e^{-\int_t^s \lambda_u^c \mathrm{d}u} \mathrm{d}s\right) \,\middle|\, \mathcal{F}_t^{c,r}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s}\left(e^{-\int_t^T \lambda_s^c \mathrm{d}s} + \left[-e^{-\int_t^s \lambda_u^c \mathrm{d}u}\right]_{s=t}^{T}\right) \,\middle|\, \mathcal{F}_t^{c,r}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s} e^{-\int_t^t \lambda_u^c \mathrm{d}u} \,\middle|\, \mathcal{F}_t^{c,r}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s} \,\middle|\, \mathcal{F}_t^{c,r}\right],
\end{aligned}$$

which is the price of the risk-free bond. Thereby the UCVA bond price with risk-free closeout can be written as

$$P_{b,\{c\}}^{\mathrm{UCVA}}(t,T) = P_{b,\{c\}}^{\text{Risk-free ZCB}}(t,T) - \mathrm{UCVA},$$

where $P_{b,\{c\}}^{\text{Risk-free ZCB}}(t,T)$ denotes the risk-free bond price and

$$\mathrm{UCVA} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \mathrm{d}u} \int_t^T l^c(s)\lambda_s^c e^{-\int_t^s \lambda_u^c \mathrm{d}u} \mathrm{d}s \,\middle|\, \mathcal{F}_t^{c,r}\right]$$

is the risk compensation demanded by the bank for buying the bond issued by the counterparty, which corresponds to the expected loss from investing in the portfolio compared with investment in a risk-free bond.

UCVA with constant loss-given-default

If the loss-given-default is assumed constant known upon pricing, $l^c(s) = L^c$ for all s, the UCVA is given by

$$\begin{aligned}\text{UCVA} &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \mathrm{d}u} L^c \int_t^T \lambda_s^c e^{-\int_t^s \lambda_u^c \mathrm{d}u} \mathrm{d}s \,\middle|\, \mathcal{F}_t^{c,r}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \mathrm{d}u} L^c \left(-e^{-\int_t^T \lambda_u^c \mathrm{d}u} + 1\right) \,\middle|\, \mathcal{F}_t^{c,r}\right] \\ &= L^c \left(\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \mathrm{d}u} \,\middle|\, \mathcal{F}_t^{c,r}\right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_u + \lambda_u^c) \mathrm{d}u} \,\middle|\, \mathcal{F}_t^{c,r}\right]\right).\end{aligned}$$

This is exactly the fraction lost upon default multiplied by the difference between a risk-free bond and a risky bond issued by the counterparty assuming zero recovery.

If further the default intensity and interest rates are assumed as independent, the price of the bond issued by the counterparty is given by

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s \mathrm{d}s} \,\middle|\, \mathcal{F}_t^{c,r}\right] \left(1 - L^c + L^c \cdot \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \lambda_u^c \mathrm{d}u} \,\middle|\, \mathcal{F}_t^{c,r}\right]\right). \tag{5}$$

Given suitable models for r_s and λ_s^c, Eq. (5) will allow for the bond with UCVA to be computed using standard fixed income pricing methods, by pricing two independent bonds. Furthermore, if both models are assumed to be affine in the state variable(s) X, such that the bond prices have closed-form solutions, the bond price with UCVA also has a closed-form solution. This significantly simplifies both derivatives pricing.

3.2 Valuing a general derivative with UCVA and risk-free closeout

As before the replacement closeout is not feasible since we are under the assumption that the institution pricing the derivative cannot default itself. Hence we are only considering risk-free closeout. Since this framework is for a general contract, we need to define the payoff structure of the derivative. Therefore let $\Pi(u, v)$ be all cash flows generated by the derivative from time u to $v \geq u$ discounted back to time u by the risk-free rate. These cash flows do not include the settlement upon counterparty default. Note that all cash flows are seen from the bank's point of view, and therefore payments by the bank to the counterparty will have negative sign in the $\Pi(u, v)$ function. E.g. for a zero coupon bond the payoff structure is \$1 discounted, if the cash flow time span includes the maturity, and zero otherwise; $\Pi(u, v) = \mathbb{1}_{\{u \leq T \leq v\}} e^{-\int_u^T r_s \mathrm{d}s}$.

The pricing equation will now include four terms, where one corresponds to the counterparty surviving to maturity and the other three for the counterparty defaulting before or at maturity. This can be explained by

$\tau^c > T$ the discounted cash flows are $\Pi(t,T)$.

$\tau^c \leq T$ the discounted cash flows up to the default time τ^c is $\Pi(t,\tau^c)$. At time τ^c there are two (risk-free) closeout possibilities; if the present risk-free value of the derivative is positive, the bank has credit at the counterparty and retrieves the counterparty's recovery rate $(1 - l^c(\tau^c))$ of the risk-free contract value at τ^c. If the risk-free value of the derivative is negative, the bank is in debt to the counterparty and will pay the full debt value to the counterparty's creditors.

Using the notation $f^+ = \max(f,0)$, $f^- = \max(-f,0)$, and the equalities $f = f^+ - f^- \Leftrightarrow f^- = f^+ - f$ for any function f, the derivative value is calculated

$$\begin{aligned}
P^{\text{UCVA}}_{b,\{c\}}(t,T) &= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau^c > T\}}\Pi(t,T) + \mathbb{1}_{\{\tau^c \leq T\}}\Big\{\Pi(t,\tau^c) \\
&\qquad + e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\left((1 - l^c(\tau^c))\Pi(\tau^c,T)^+ - \Pi(\tau^c,T)^-\right)\Big\} \mid \mathcal{F}_t^{c,r}\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau^c > T\}}\Pi(t,T) + \mathbb{1}_{\{\tau^c \leq T\}}\Big\{\Pi(t,\tau^c) + e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\Pi(\tau^c,T) \\
&\qquad - e^{-\int_t^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\Pi(\tau^c,T)^+\Big\} \mid \mathcal{F}_t^{c,r}\Big].
\end{aligned}$$

By definition

$$\Pi(t,\tau^c) + e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\Pi(\tau^c,T) = \Pi(t,T),$$

since both represent all cash flows between t and T discounted to time t. Thereby the first three terms in the expectation simplifies to

$$\mathbb{1}_{\{\tau^c > T\}}\Pi(t,T) + \mathbb{1}_{\{\tau^c \leq T\}}\Pi(t,T) = \Pi(t,T),$$

and hence the pricing equation is

$$\begin{aligned}
P^{\text{UCVA}}_{b,\{c\}}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) - \mathbb{1}_{\{\tau^c \leq T\}} e^{-\int_t^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\Pi(\tau^c,T)^+ \,\middle|\, \mathcal{F}_t^{c,r}\right] \quad (6) \\
&= \mathbb{E}^{\mathbb{Q}}[\Pi(t,T) \mid \mathcal{F}_t^{c,r}] \\
&\quad - \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\lambda_s^c e^{-\int_t^s (r_u + \lambda_u^c)\mathrm{d}u} l^c(s)\Pi(s,T)^+ \,\middle|\, \mathcal{F}_t^{c,r}\right]\mathrm{d}s, \quad (7)
\end{aligned}$$

where the last equality follows from Eq. (3).

It is clear that this general formula also takes the form of a risk-free price minus an always positive UCVA adjustment, which is the risk premium required by the bank due to the default riskiness of the counterparty.

Price using UCVA with the bank being default-risky and the counterparty risk-free

In Sec. 3.5, when considering Bilateral CVA with replacement closeout, we need two different UCVA prices in order to compute the contract value at default. Specifically, we need to derive an expression for the value of the contract from the bank's point of view, with the bank being default-risky and the counterparty assumed to be risk-free. In order to do this, we first generalize the cash flow function

Definition 1 (Generalized cash flow function). $\Pi^i(u,v)$ *denotes all cash flows generated by the contract under consideration between time u and v, with $v \geq u$, discounted to time u by the risk-free rate. The cash flows are seen from entity i's point of view, meaning that a cash flow received by i has positive sign and a cash flow paid by entity i has negative sign.*

Trivially the value of a contract where c is allowed to default but b is not, as seen from b's point of view, is found by replacing the Π functions in (6) with Π^b. Using this pricing equation but reversing the entire situation – pricing from c's perspective with only b as default-risky – the pricing equation can be found by switching all cs with bs in the pricing equation in Eq. (6). This yields

$$P^{\text{UCVA}}_{c,\{b\}}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\Pi^c(t,T) - \mathbb{1}_{\{\tau^b \leq T\}} e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\Pi^c(\tau^b,T)^+ \,\middle|\, \mathcal{F}_t^{b,r}\right]. \quad (8)$$

Assuming that the two counterparties can agree on this price, it holds that $P^{\text{UCVA}}_{b,\{c\}}(t,T) = -P^{\text{UCVA}}_{c,\{b\}}(t,T)$. This means that under the assumption that only b is default-risky, the contract value from b's perspective is exactly the negative of (8). By further using the fact that only two counterparties are present in the contract we have $\Pi^c(u,v) = -\Pi^b(u,v)$, since all cash flows are transferred either from b to c or from c to b. This gives the formula for the value of the derivative

$$\begin{aligned} P^{\text{UCVA}}_{b,\{b\}}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[-\Pi^c(t,T) + \mathbb{1}_{\{\tau^b \leq T\}} e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\Pi^c(\tau^b,T)^+ \,\middle|\, \mathcal{F}_t^{b,r}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\Pi^b(t,T) + \mathbb{1}_{\{\tau^b \leq T\}} e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\Pi^b(\tau^b,T)^- \,\middle|\, \mathcal{F}_t^{b,r}\right], \end{aligned} \quad (9)$$

since

$$\Pi^c(\tau^b,T)^+ = (-\Pi^b(\tau^b,T))^+ = \max(-\Pi^b(\tau^b,T),0) = (\Pi^b(\tau^b,T))^-.$$

3.3 Expected positive exposure and wrong way risk

Consider the pricing equation (7) and the UCVA in this equation;

$$\text{UCVA} = \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\lambda_s^c e^{-\int_t^s (r_u+\lambda_u^c)\mathrm{d}u} l^c(s)\Pi(s,T)^+ \,\middle|\, \mathcal{F}_t^{c,r}\right]\mathrm{d}s.$$

We assume the loss-given-default is constant, $L \equiv l^c(s)$ for all s. Further, we assume that λ is a CIR-process and that the discounting rate is zero for simplicity. The UCVA term then simplify to

$$\text{UCVA} = -L\int_t^T \mathbb{E}^{\mathbb{Q}}\left[\Pi(s,T)^+ \frac{\partial}{\partial s} e^{-\int_t^s \lambda_u^c \mathrm{d}u} \,\middle|\, \mathcal{F}_t^c\right]\mathrm{d}s.$$

We wish to express this in the form of an expected cash flow and the default probability split into two parts while allowing for the presence of *Wrong Way Risk*. The contract is sensitive to WWR if there is a negative correlation between Π and λ, implying that a high default probability tends to correspond to a large (positive) cash flow. Using the approach described in Sec. 8 of Paper C of this thesis, we can use the survival probability $Q(s) \equiv \mathbb{Q}(\tau^c > s \mid \mathcal{F}_t^c)$ to rewrite such an expression by

$$\text{UCVA} = -L\int_t^T \mathbb{E}^{\mathbb{Q}}\left[\Pi(s,T)^+ \mid \mathcal{F}_t^c, \tau^c = s\right]\mathrm{d}Q(s),$$

where the $\tau^c = s$ is present since we do not assume deterministic survival probabilities $Q(\cdot)$, but those governed by the stochastic process.

We define the integrand as the *Expected Positive Exposure* (EPE), such that

$$\begin{aligned}\text{EPE}(s) &= \mathbb{E}^{\mathbb{Q}}\left[\Pi(s,T)^+ \mid \mathcal{F}_t^c, \tau^c = s\right]\\ \text{UCVA} &= -L\int_t^T EPE(s)\mathrm{d}Q(s).\end{aligned}$$

The interpretation of this definition of the EPE is that EPE is the expected value of the contract upon counterparty default (if positive, else zero). Note that if WWR is not present, i.e. the correlation between the Π and λ is zero, the condition $\tau^c = s$ disappears. However, for all contracts with WWR this condition is very important, as shown i Paper B of this thesis.

3.4 Valuing a general derivative with BCVA and risk-free closeout

Valuation with risk-free closeout is a more simple approach than valuation with replacement closeout, and therefore this approach is presented first to explain the general BCVA pricing framework more understandably. The above approach is applied with the difference that now the pricing counterparty, the bank, is assumed to be default-risky. The default time of the bank τ^b is modeled with a Cox-process that is allowed to be correlated with the default time of the counterparty τ^c. Note that we are only concerned with the first of the two possible defaults since the contract is terminated at this point.[6] For completeness the two counterparties are also allowed to default simultaneously, which gives four states of termination: The contract matures before any of the counterparties defaults, the bank defaults before contract maturity and before the counterparty does, the counterparty defaults before contract maturity and before the bank and finally the bank and the counterparty defaults simultaneously prior to contract maturity. The cash flows at each of these four states are as follows.

$$\tau^b > T, \tau^c > T:$$
$$\Pi(t,T)$$

$$\tau^b \le T, \tau^b < \tau^c:$$
$$\begin{aligned}
&\Pi(t,\tau^b) + e^{-\int_t^{\tau^b} r_s \mathrm{d}s}\left(\Pi(\tau^b,T)^+ - \Pi(\tau^b,T)^-\left(1 - l^b(\tau^b)\right)\right)\\
&\quad = \Pi(t,\tau^b) + e^{-\int_t^{\tau^b} r_s \mathrm{d}s}\left(\Pi(\tau^b,T) + l^b(\tau^b)\Pi(\tau^b,T)^-\right)\\
&\quad = \Pi(t,T) + l^b(\tau^b)e^{-\int_t^{\tau^b} r_s \mathrm{d}s}\Pi(\tau^b,T)^-
\end{aligned}$$

$$\tau^c \le T, \tau^c < \tau^b:$$
$$\begin{aligned}
&\Pi(t,\tau^c) + e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\left(\Pi(\tau^c,T)^+\left(1 - l^c(\tau^c)\right) - \Pi(\tau^c,T)^-\right)\\
&\quad = \Pi(t,\tau^c) + e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\left(\Pi(\tau^c,T) - l^c(\tau^c)\Pi(\tau^c,T)^+\right)\\
&\quad = \Pi(t,T) - l^c(\tau^c)e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\Pi(\tau^c,T)^+
\end{aligned}$$

$$\tau^c = \tau^b = \tau \le T:$$
$$\begin{aligned}
&\Pi(t,\tau) + e^{-\int_t^{\tau} r_s \mathrm{d}s}\left(\Pi(\tau,T)^+\left(1 - l^c(\tau)\right) - \Pi(\tau,T)^-\left(1 - l^b(\tau)\right)\right)\\
&\quad = \Pi(t,\tau) + e^{-\int_t^{\tau} r_s \mathrm{d}s}\left(\Pi(\tau,T) - l^c(\tau)\Pi(\tau,T)^+ + l^b(\tau)\Pi(\tau,T)^-\right)\\
&\quad = \Pi(t,T) + e^{-\int_t^{\tau} r_s \mathrm{d}s}\left(l^b(\tau)\Pi(\tau,T)^- - l^c(\tau)\Pi(\tau,T)^+\right)
\end{aligned}$$

The derivative price is then the risk-neutral expectation of the sum of these four cash flows (multiplied by the corresponding indicator function for the state) given $\mathcal{F}_t = \mathcal{F}_t^c \vee \mathcal{F}_t^b \vee \mathcal{G}_t^r$. One thing that is striking when one looks at

[6] As argued in [16], neglecting contract termination at first default will lead to pricing errors. Thus this is an important issue to address when developing pricing formulas.

the four cash flows is that each contains the risk-free discounted cash flows of the entire contract lifespan $\Pi(t,T)$. Since the four states combined corresponds to all possible scenarios, the sum of the four indicators multiplied by $\Pi(t,T)$ becomes $\Pi(t,T)$ itself, since

$$\mathbb{1}_{\{\tau^b>T,\tau^c>T\}} + \mathbb{1}_{\{\tau^b\leq T,\tau^b<\tau^c\}} + \mathbb{1}_{\{\tau^c\leq T,\tau^c<\tau^b\}} + \mathbb{1}_{\{\tau^c=\tau^b=\tau\leq T\}} = 1.$$

When writing out the other terms, it becomes obvious that the fourth state can be incorporated in states two and three by allowing the two states to be equal, i.e. $\tau^b \leq \tau^c$ and $\tau^c \leq \tau^b$, respectively, in the indicators for the states. This yields the pricing formula

$$\begin{aligned} P_b^{BCVA}(t,T) = \mathbb{E}^{\mathbb{Q}}[\Pi(t,T) \mid \mathcal{F}_t] \\ + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^b\leq T,\tau^b\leq\tau^c\}} l^b(\tau^b) e^{-\int_t^{\tau^b} r_s \mathrm{d}s} \Pi(\tau^b,T)^- \,\middle|\, \mathcal{F}_t\right] \\ - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^c\leq T,\tau^c\leq\tau^b\}} l^c(\tau^c) e^{-\int_t^{\tau^c} r_s \mathrm{d}s} \Pi(\tau^c,T)^+ \,\middle|\, \mathcal{F}_t\right]. \end{aligned} \quad (10)$$

The first expectation is the risk-free value of the contract, the second is often referred to as the Debit Value Adjustment (DVA) term and is the risk premium the counterparty requires due to the risk of the bank defaulting, while the third is often referred to as the CVA term which is the value the bank requires due to the default riskiness of its counterparty. Note that this equation is consistent with the UCVA pricing Eq. (6) in the sense that if we let the bank be default free such that it holds almost surely that $\tau^b > T$ and $\tau^b > \tau^c$, then the formula reduces to Eq. (6).

Eq. (10) is consistent with the general BCVA valuation formulas with risk-free closeout presented in [17, Thm. 8.3] and [8, Eq. (12.3)] and bears resemblance to [6, Equation (13.1)]. However, [6] discretizes the possible default time, and at each discrete time step t_j calculates

$$\mathrm{DVA}(t_j) = l^b \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_j} r_s \mathrm{d}s}\Pi(t_j,T)^- \,\middle|\, \mathcal{F}_t\right] \mathbb{Q}(\tau^c > t_j \mid \mathcal{F}_t) \mathbb{Q}\left(t_{j-1} < \tau^b \leq t_j \,\middle|\, \mathcal{F}_t\right)$$

$$\mathrm{CVA}(t_j) = l^c \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{t_j} r_s \mathrm{d}s}\Pi(t_j,T)^+ \,\middle|\, \mathcal{F}_t\right] \mathbb{Q}\left(\tau^b > t_j \,\middle|\, \mathcal{F}_t\right) \mathbb{Q}(t_{j-1} < \tau^c \leq t_j \mid \mathcal{F}_t).$$

This approach assumes that the discretization of default events is feasible, that τ^c and τ^b are independent given $\mathcal{F}^c_{t_j}$ and further that τ^c and τ^b are independent of interest rates and the risk-free discounted cash flow function $\Pi(\cdot,\cdot)$, i.e. there is no wrong way risk or right way risk present.

In [17] and [8] the formula appears in an equivalent representation with (10). This formula marks the endpoint of the BCVA with risk-free closeout for the two books (not considering collateralization, netting, FVA etc.).

3.5 Valuing a general derivative with BCVA and replacement closeout

Here we apply the framework for risk-free closeout in Sec. 3.4, but with a distinction in the closeout value of the derivative contract upon default of $\tau = \inf\{\tau^b, \tau^c\}$ given $\tau \geq T$. With risk-free closeout, this value is simply $\mathbb{E}^{\mathbb{Q}}\left[\Pi(\tau^i, T) \,\middle|\, \mathcal{F}_{\tau^i}\right]$. However, the situation is more complicated with replacement closeout. If the first default event occurs prior to the contract maturity, the closeout value will be a UCVA with the surviving party as the default-risky party in the pricing set-up. Since we are still pricing from b's perspective, the closeout prices are found by

$\tau^b \leq T, \tau^b \leq \tau^c$ UCVA price from b's point of view calculated at time τ^b, where only the surviving party c is default-risky. This corresponds to Eq. (6).

$\tau^c \leq T, \tau^c \leq \tau^b$ UCVA price from b's point of view calculated at time τ^c, where only b itself is default-risky. This corresponds to Eq. (9).

$\tau^c = \tau^b \leq T$ Risk-free closeout applies.

Note that here we are not using the generalized cash flow function presented in Def. 1, and it is implicitly given that $\Pi(\cdot, \cdot)$ is the discounted cash flows seen from b's perspective. The specific values generated in each state is thereby

$$
\begin{aligned}
&\tau^b > T, \tau^c > T: && \Pi(t,T)\\
&\tau^b \leq T, \tau^b < \tau^c: && \Pi(t,\tau^b) + e^{-\int_t^{\tau^b} r_s \mathrm{d}s}\left(P^{\mathrm{UCVA}}_{b,\{c\}}(\tau^b,T) + l^b(\tau^b)\big(P^{\mathrm{UCVA}}_{b,\{c\}}(\tau^b,T)\big)^-\right)\\
&\tau^c \leq T, \tau^c < \tau^b: && \Pi(t,\tau^c) + e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\left(P^{\mathrm{UCVA}}_{b,\{b\}}(\tau^c,T) - l^c(\tau^c)\big(P^{\mathrm{UCVA}}_{b,\{b\}}(\tau^c,T)\big)^+\right)\\
&\tau^c = \tau^b = \tau \leq T: && \Pi(t,T) + e^{-\int_t^{\tau} r_s \mathrm{d}s}\Big(l^b(\tau)\Pi(\tau,T)^- - l^c(\tau)\Pi(\tau,T)^+\Big),
\end{aligned}
$$

where the UCVA prices needed are calculated by

$$
\begin{aligned}
P^{\mathrm{UCVA}}_{b,\{c\}}(\tau^b,T) &= \mathbb{E}^{\mathbb{Q}}\left[\Pi(\tau^b,T) - \mathbb{1}_{\{\tau^c \leq T\}} e^{-\int_{\tau^b}^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\Pi(\tau^c,T)^+ \,\middle|\, \mathcal{F}^{c,r}_{\tau^b}\right]\\
P^{\mathrm{UCVA}}_{b,\{b\}}(\tau^c,T) &= \mathbb{E}^{\mathbb{Q}}\left[\Pi(\tau^c,T) + \mathbb{1}_{\{\tau^b \leq T\}} e^{-\int_{\tau^c}^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\Pi(\tau^b,T)^- \,\middle|\, \mathcal{F}^{b,r}_{\tau^c}\right]
\end{aligned}
$$

The first term of states one and four along with the first two terms of states two and three yields the value of the corresponding risk-free valuation, which is shown in the following equation. In order to show this, we introduce the first event amongst the two possible defaults and the contract maturity as $t^1 = \inf\{\tau^b, \tau^c, T\}$.

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Bigg[\mathbb{1}_{\{\tau^b>T,\tau^c>T\}}\Pi(t,T) + \mathbb{1}_{\{\tau^c=\tau^b=\tau\le T\}}\Pi(t,T) \\
&\qquad + \mathbb{1}_{\{\tau^b\le T,\tau^b<\tau^c\}}\left\{\mathbb{E}^{\mathbb{Q}}\left[\Pi(t,\tau^b) + e^{-\int_t^{\tau^b} r_s \mathrm{d}s}\Pi(\tau^b,T) \,\middle|\, \mathcal{F}^{c,r}_{\tau^b}\right]\right\} \\
&\qquad + \mathbb{1}_{\{\tau^c\le T,\tau^c<\tau^b\}}\left\{\mathbb{E}^{\mathbb{Q}}\left[\Pi(t,\tau^c) + e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\Pi(\tau^c,T) \,\middle|\, \mathcal{F}^{b,r}_{\tau^c}\right]\right\} \,\Bigg|\, \mathcal{F}_t\Bigg] \\
&= \mathbb{E}^{\mathbb{Q}}\Bigg[\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^b>T,\tau^c>T\}}\Pi(t,T) + \mathbb{1}_{\{\tau^c=\tau^b=\tau\le T\}}\Pi(t,T) \,\middle|\, \mathcal{F}_{t^1}\right] \\
&\qquad + \mathbb{1}_{\{\tau^b\le T,\tau^b<\tau^c\}}\mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) \,\middle|\, \mathcal{F}^{c,r}_{\tau^b}\right] \\
&\qquad + \mathbb{1}_{\{\tau^c\le T,\tau^c<\tau^b\}}\mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) \,\middle|\, \mathcal{F}^{b,r}_{\tau^c}\right] \,\Bigg|\, \mathcal{F}_t\Bigg] \\
&\overset{7}{=} \mathbb{E}^{\mathbb{Q}}\Bigg[\left\{\mathbb{1}_{\{\tau^b>T,\tau^c>T\}} + \mathbb{1}_{\{\tau^b\le T,\tau^b<\tau^c\}} + \mathbb{1}_{\{\tau^c\le T,\tau^c<\tau^b\}} + \mathbb{1}_{\{\tau^c=\tau^b=\tau\le T\}}\right\} \\
&\qquad \times \mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) \mid \mathcal{F}_{t^1}\right] \mid \mathcal{F}_t\Bigg] \\
&= \mathbb{E}^{\mathbb{Q}}\Bigg[\mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) \mid \mathcal{F}_{t^1}\right] \mid \mathcal{F}_t\Bigg] = \mathbb{E}^{\mathbb{Q}}[\Pi(t,T) \mid \mathcal{F}_t].
\end{aligned}$$

This yields the price under BCVA with replacement closeout as

$$\begin{aligned}
P_b^{\mathrm{BCVA}}(t,T) = \mathbb{E}^{\mathbb{Q}}\Bigg[&\Pi(t,T) \\
&- \mathbb{1}_{\{\tau^b<\tau^c\le T\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\Pi(\tau^c,T)^+ \,\middle|\, \mathcal{F}^{c,r}_{\tau^b}\right] \\
&+ \mathbb{1}_{\{\tau^c<\tau^b\le T\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\Pi(\tau^b,T)^- \,\middle|\, \mathcal{F}^{b,r}_{\tau^c}\right] \\
&+ \mathbb{1}_{\{\tau^c=\tau^b=\tau\le T\}} e^{-\int_t^{\tau} r_s \mathrm{d}s}\left(l^b(\tau)\Pi(\tau,T)^- - l^c(\tau)\Pi(\tau,T)^+\right) \\
&+ \mathbb{1}_{\{\tau^b\le T,\tau^b<\tau^c\}} e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\left(P^{\mathrm{UCVA}}_{b,\{c\}}(\tau^b,T)\right)^- \\
&- \mathbb{1}_{\{\tau^c\le T,\tau^c<\tau^b\}} e^{-\int_t^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\left(P^{\mathrm{UCVA}}_{b,\{b\}}(\tau^c,T)\right)^+ \,\Bigg|\, \mathcal{F}_t\Bigg].
\end{aligned}$$

[7] Here it is used that $\mathcal{F}_{t^1} \supseteq \mathcal{F}_t$ and remembering that the function $\Pi(t,T)$ only depends on promised payments in the contract and the risk-free rate resulting in the equality $\mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) \,\middle|\, \mathcal{F}^{i,r}_{t^1}\right] = \mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) \,\middle|\, \mathcal{F}_{t^1}\right]$ for $i \in \{b,c\}$. Given the event $\{\tau^b \le T, \tau^b < \tau^c\}$ we have $\mathcal{F}_{t^1} = \mathcal{F}_{\tau^b}$.
Further, since all four default events are known given $\mathcal{F}_{t^1}$ so is the corresponding indicator functions, and so they can be moved outside the expectation.

The second and third term has a conditional expectation on $\mathcal{F}^{c,r}_{\tau^b}$ and $\mathcal{F}^{b,r}_{\tau^c}$ respectively. Both sigma-algebras can be generalized to the full-information sigma-algebra $\mathcal{F}$, since the stochastic variable in the first conditional expectation is independent of all information contained in $\mathcal{F}^{b}_{\tau^b}$ that is not contained in $\mathcal{F}^{c,r}_{\tau^b}$ as well.[8] A similar argument applies to the $\mathcal{F}^{b,r}_{\tau^c}$ conditioned term. The reason for generalizing these sigma-algebras becomes clear in the following.

Considering the term with simultaneous default, one can realize that by splitting this into two different terms and using the law of iterated expectations with $\mathcal{F}_{\tau^i} \supseteq \mathcal{F}_t$, this can be incorporated into terms two and three of the pricing equation, by allowing the defaults to occur simultaneously, i.e.

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^c=\tau^b=\tau\leq T\}} e^{-\int_t^{\tau} r_s \mathrm{d}s}\left(l^b(\tau)\Pi(\tau,T)^- - l^c(\tau)\Pi(\tau,T)^+\right) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau^c=\tau^b\leq T\}} e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\Pi(\tau^b,T)^- \\
&\qquad - \mathbb{1}_{\{\tau^c=\tau^b\leq T\}} e^{-\int_t^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\Pi(\tau^c,T)^+ \,\Big|\, \mathcal{F}_t\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^c=\tau^b\leq T\}} e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\Pi(\tau^b,T)^- \,\middle|\, \mathcal{F}_{\tau^b}\right] \\
&\qquad - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^c=\tau^b\leq T\}} e^{-\int_t^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\Pi(\tau^c,T)^+ \,\middle|\, \mathcal{F}_{\tau^c}\right] \,\Big|\, \mathcal{F}_t\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau^c=\tau^b\leq T\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\Pi(\tau^b,T)^- \,\middle|\, \mathcal{F}_{\tau^b}\right] \\
&\qquad - \mathbb{1}_{\{\tau^c=\tau^b\leq T\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\Pi(\tau^c,T)^+ \,\middle|\, \mathcal{F}_{\tau^c}\right] \,\Big|\, \mathcal{F}_t\Big].
\end{aligned}$$

Thereby the final pricing equation is

$$\begin{aligned}
P_b^{\mathrm{BCVA}}(t,T) = \mathbb{E}^{\mathbb{Q}}\Big[&\Pi(t,T) \\
&- \mathbb{1}_{\{\tau^b\leq\tau^c\leq T\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\Pi(\tau^c,T)^+ \,\middle|\, \mathcal{F}_{\tau^b}\right] \\
&+ \mathbb{1}_{\{\tau^c\leq\tau^b\leq T\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\Pi(\tau^b,T)^- \,\middle|\, \mathcal{F}_{\tau^c}\right] \\
&+ \mathbb{1}_{\{\tau^b\leq T,\tau^b<\tau^c\}} e^{-\int_t^{\tau^b} r_s \mathrm{d}s} l^b(\tau^b)\left(P^{\mathrm{UCVA}}_{b,\{c\}}(\tau^b,T)\right)^- \\
&- \mathbb{1}_{\{\tau^c\leq T,\tau^c<\tau^b\}} e^{-\int_t^{\tau^c} r_s \mathrm{d}s} l^c(\tau^c)\left(P^{\mathrm{UCVA}}_{b,\{b\}}(\tau^c,T)\right)^+ \,\Big|\, \mathcal{F}_t\Big].
\end{aligned}$$

[8] According to [18, Property 9.7 (k)]: for a stochastic variable X and two sigma-algebras $\mathcal{G}$ and $\mathcal{H}$ with $\mathcal{H}$ independent of $\sigma\{\sigma\{X\},\mathcal{G}\}$ it holds that $\mathbb{E}[X \mid \sigma\{\mathcal{G},\mathcal{H}\}] = \mathbb{E}[X \mid \mathcal{G}]$.

4 Scientific contribution of this thesis

We have an emphasis on empirical studies, and provide thorough discussions on the link between theory and applications for: Constant and time-varying copula models, valuation of CVA on contracts subject to WWR and comparison with formulas from the Basel III framework, reduced form model calibration to market data, estimation of market price of risk, and BCVA estimation on CDSs.

In Papers A and C, we provide analysis using market data applied to volumetric risk estimation including correlation structure modeling, and reduced form model calibration to market quotes on CDSs including a discussion of the challenges involved, respectively.

In Paper B, we provide a discussion of the shortcomings of CVA estimation suggested in Basel III in the presence of WWR and present a framework for a closed-form approximation of the value of put and call options under WWR.

References

[1] N. Bingham and R. Kiesel, *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives*, 2nd ed. Springer-Verlag London Limited, 2004.

[2] D. Lando, *Credit Risk Modeling: Theory and Applications*. Princeton University Press, 2004.

[3] D. Duffie and K. J. Singleton, *Credit Risk: Pricing, Measurement, and Management*. Princeton University Press, 2003.

[4] D. Lando, "On Cox Processes and Credit Risky Securities," *Review of Derivatives Research*, vol. 2, no. 2-3, pp. 99–120, Dec. 1998. [Online]. Available: http://dx.doi.org/10.1007/BF01531332

[5] D. Brigo and M. Morini, "Dangers of Bilateral Counterparty Risk: The Fundamental Impact of Closeout Conventions," arXiv.org, Papers 1011.3355, Nov. 2010. [Online]. Available: http://arxiv.org/abs/1011.3355

[6] J. Gregory, *Counterparty Credit Risk and Credit Value Adjustment: A Continuing Challenge for Global Financial Markets*, 2nd ed. John Wiley & Sons, 2012.

[7] D. Brigo and M. Morini, "Close-out convention tensions," *Risk*, vol. 24, no. 12, pp. 74–78, Dec. 2011.

[8] D. Brigo, M. Morini, and A. Pallavicini, *Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for All Asset Classes*. John Wiley & Sons, 2013.

[9] M. Pykhtin, "Unexpected recovery risk," *Risk Magazine*, pp. 74–78, Aug. 2003.

[10] M. Hillebrand, "Modeling and estimating dependent loss given default," *Risk*, vol. 19, no. 9, pp. 120–125, Sep. 2006, retrieved from http://search.proquest.com/docview/201476336?accountid=8144.

[11] J. Frye and M. J. Jr., "Credit loss and systematic loss given default," *The Journal of Credit Risk*, vol. 8, no. 1, pp. 109—140, 2012.

[12] S. Farinelli and M. Shkolnikov, "Two models of stochastic loss given default," *The Journal of Credit Risk*, vol. 8, no. 2, pp. 3—20, 2012.

[13] A. F. R. Koivusalo and R. Schäfer, "Dependence of defaults and recoveries in structural credit risk models," arXiv.org, Papers 1102.3150v2, Mar. 2011. [Online]. Available: http://arxiv.org/abs/1102.3150v2

[14] R. Schäfer and A. F. R. Koivusalo, "Calibration of structural and reduced-form recovery models," arXiv.org, Papers 1102.4864v1, Feb. 2011. [Online]. Available: http://arxiv.org/abs/1102.4864v1

[15] A. Becker, A. F. R. Koivusalo, and R. Schäfer, "Empirical Evidence for the Structural Recovery Model," arXiv.org, Papers 1203.3188v1, Mar. 2012. [Online]. Available: http://arxiv.org/abs/1203.3188v1

[16] D. Brigo, C. Buescu, and M. Morini, "Impact of the first to default time on Bilateral CVA," arXiv.org, Papers 1106.3496, Jun. 2011. [Online]. Available: http://arxiv.org/abs/1106.3496

[17] C. Kenyon and R. Stamm, *Discounting, LIBOR, CVA and Funding: Interest Rate and Credit Pricing*. Palgrave Macmillan, 2012.

[18] D. Williams, *Probability with Martingales*. Cambridge University Press, 1991.

Part II

Papers

Paper A

Joint price and volumetric risk in wind power trading: A copula approach

Anca Pircalabu[1] [2]
Thomas Hvolby[1]
Jesper Jung[2]
Esben Høg[1]

The paper has been published in the
Energy Economics Vol. 62, pp. 139–154, 2017.

[1]Department of Mathematical Sciences, Aalborg University
[2]Quantitative Analytics, Neas Energy A/S

The layout has been revised.

Abstract

This paper examines the dependence between wind power production and electricity prices and discusses its implications for the pricing and the risk distributions associated with contracts that are exposed to joint price and volumetric risk. We propose a copula model for the joint behavior of prices and wind power production, which is estimated to data from the Danish power market. We find that the marginal behavior of the individual variables is best described by ARMA–GARCH models with non-Gaussian error distributions, and the preferred copula model is a time-varying Gaussian copula. As an application of our joint model, we consider the case of an energy trading company entering into longer-term agreements with wind power producers, where the fluctuating future wind power production is bought at a predetermined fixed price. We find that assuming independence between prices and wind power production leads to an underestimation of risk, as the profit distribution becomes left-skewed when the negative dependence that we find in the data is accounted for. By performing a simple static hedge in the forward market, we show that the risk can be significantly reduced. Furthermore, an out-of-sample study shows that the choice of copula influences the price of correlation risk, and that time-varying copulas are superior to the constant ones when comparing actual profits generated with different models.

1 Introduction

Since the European electricity market reforms in the late 1990's, the electricity markets have undergone considerable structural changes. Liberalization has led to extremely volatile electricity prices, and the prioritization of renewable energy sources in order to reduce CO_2 emissions has introduced further challenges in terms of financial risk management. One particular challenge that we study in this paper is related to the production uncertainty associated with wind power generation. Wind power is highly non-dispatchable and therefore fundamentally different from the more traditional thermal power sources in the sense that the production cannot be planned and controlled to the same extent. The dependency on weather variations (wind speed and air density among others) makes the exact future production of a wind turbine or wind-farm very hard to predict. Thus, in addition to facing price volatility, wind power generators are exposed to production uncertainty, often referred to as volumetric risk.

The joint exposure to price and volumetric risk can be further amplified by a high penetration ratio of wind power in the grid. This is due to the mechanism of day-ahead price formation, which is based on finding the equilibrium between supply and demand bids made to the exchange, where the supply curve is built according to merit order stack[3]. Because wind power has a very low marginal cost, a high production for a given hour will, other things being equal, pull the market clearing price downwards. Similarly, if wind power production is low for a given hour, demand will have to be met by either import or turning on more costly generating plants. The latter (and possibly the former) will, again other things being equal, pull the prices upward. This leads to prices and wind power production being negatively correlated, which depending on the strength of this correlation, enhances the joint price and volumetric risk significantly. Empirical evidence regarding this relation between spot electricity prices and wind power production has been demonstrated in the literature, e.g. [1] for the Danish power market, [2] for Spain, and [3] and [4] for Germany.

In practice, it is usually energy trading companies that act on the exchange on behalf of the producers. Due to increasing wind power production in some power markets, some trading companies offer, in addition to the management of production, a predetermined fixed price in exchange for the fluctuating production. Companies offering such insurances against price movements will naturally attempt to cover their exposure, and a typical solution that will eliminate some of the risk is to sell energy on the forward market corresponding to the expected wind power production. The remaining exposure will inevitably cause the energy trading companies to purchase energy on the spot market when being short, and dispose of excess energy on the spot market when expecting less than the realized production. Furthermore, the negative relationship between prices and wind power production adds an additional correlation risk: If being short, chances are that the missing energy will have to be bought at a higher price; similarly, if having to dispose of excess electricity, chances are that this will be sold at times of a lower price. As a result, the negative dependence between price and production introduces a “double” risk that is not straight forward to address or diminish without having a well-specified model for the dependence structure.

The problem of joint price and volumetric risk stems back some decades, and was first discussed in [5] in relation to the classical farmer’s problem – who faces both price and production uncertainty at the time of harvest. In [5], the author considered futures as hedging instruments, and presented an explicit formula for the optimal position in futures contracts (from a minimum-variance perspective); this formula pointed out that the correlation between the two sources of uncertainty is an essential feature of the problem. Later, the work of [6] included options in the hedging portfolio due to the nonlinearity of profit. More recently, energy related work on the subject became available, and some interesting discussions on the hedging of volumetric risk

[3]Supply bids from different power stations are ranked according to their production costs, and the market clearing price corresponds to the highest bid needed to match demand.

associated with consumers' load (demand-side risk) were presented in [7, 8] and [9]. In [7, 8], the authors assumed bivariate lognormality for electricity prices and consumers' demand of electricity with a constant correlation, and focused on hedging strategies that 1) maximize the expected utility of the hedged profit and 2) maximize the expected profit subject to a Value-at-Risk constraint. In [9], the authors propose a structural model that captures the complex dependence structure of electricity price and load dynamics as a base for hedging. While many of the ideas in the existing literature regarding the hedging of volumetric risk can be used in our application, there are some major distinctions between supply-side volumetric risk (associated with wind power production) and demand-side volumetric risk (consumers' load) that pose some challenges when having to specify a joint model for day-ahead electricity prices and wind power production.

One issue of concern when considering a joint model for electricity prices and wind power production is that the price dynamics are very different from the production dynamics, causing us to expect the benchmark bivariate (log) normality assumption to be too restrictive;[4] in fact, the two variables might have univariate marginal distributions from different families, making it very challenging to decide upon a suitable bivariate density. The assumption of constant correlation might also prove too restrictive, and many studies have shown evidence of time-varying dependence between economic time series, see e.g. [10], [11], [12], [13], and [14]. Thus, before addressing issues such as the valuation of correlation risk in the context of fixed price obligations with fluctuating wind power production or the hedging of portfolios containing such obligations, a large part of this paper is concerned with developing a joint model that correctly characterizes the marginal behavior of electricity prices and wind power production and also their dependence structure. For this purpose, we propose the use of copula models.

Copulas are flexible tools that can be used to completely describe the dependence structure between random variables while allowing for arbitrary marginal distributions. They were introduced in the literature by [15], and have found various applications in economics and finance over the past decades: See [16] for the use of copulas in pricing different types of bivariate options, [17] for an application to risk management, and [18] for a thorough review on copula-based models, including methods for estimation, inference and model-selection. Applications of copula models in energy markets are less common, but some examples are [19], [20], [3], and [21].

Specifically, we offer two contributions: Firstly, we propose a flexible joint model that relaxes the assumption of bivariate normality and that accounts for the time variation we observe in the dependence structure. Our empiri-

[4] Alone the fact that electricity prices can go negative rules out the lognormality assumption in some marketplaces.

cal study is based on data from the Danish power market; nonetheless, we expect our results to be generally applicable in all liberalized energy markets with a high penetration of wind power in the grid. By performing statistical tests and Monte Carlo simulation studies, we demonstrate that our proposed empirical model captures the joint distribution accurately, and also its time-varying behavior.

Secondly, we provide applications of our model that are of interest to e.g. an energy trading company managing a large share of wind turbines. We estimate the risk distribution and the price of correlation risk associated with a specific contract exposed to joint price and volumetric risk, i.e. a contract implying that an energy trading company offers wind power producers an insurance against price movements, by purchasing their fluctuating production at a predetermined fixed price. We show that the negative relation between prices and wind power production plays an important role both in relation to the pricing and the risk distribution of such contracts. We find that the price of correlation risk amounts to a significant percentage of the price of a regular fixed price agreement with no volumetric risk (a standard forward contract). Also, the risk distribution becomes left-skewed under the assumption of negative dependence compared to the case of independence. Lastly, we compare the out-of-sample performance of competing models, and show that time-varying copula models outperform the constant copula models.

This paper is organized as follows: Section 2 briefly introduces the notion of copula and the methodology used in building a joint model for electricity prices and wind power production. In Section 3, we apply the theory to data from the Danish power market. In Section 4, we present a simulation study and investigate how different wind scenarios affect the conditional distribution of spot electricity prices. Section 5 presents an application to pricing and risk management, and in Section 6 we conclude.

2 Modeling dependence with copula models

Formally, a d-dimensional copula is a distribution function $C(u_1, \dots, u_d)$ defined on the unit cube $[0,1]^d$ with uniform margins. Since our application is a bivariate one, we shall consider the case where $d = 2$, however copula theory holds for the general multivariate case. The central result when working with copula models is Sklar's theorem, which shows how to decompose a joint distribution function into its univariate marginal distribution functions and a copula.

In our application, we wish to condition on the information generated by past observations of our variables, denoted by $\mathcal{F}_{t-1}$. Thus, we shall consider an extension to Sklar's theorem proposed in [13], which holds for conditional joint distributions. The theorem states that if we let $F(\cdot \mathcal{F}_{t-1})$ be the bivariate conditional distribution function of the random vector $\boldsymbol{Y}_t \equiv (Y_{1,t}, Y_{2,t})'$, with conditional marginal distribution functions $F_1(\cdot \mathcal{F}_{t-1})$ and $F_2(\cdot \mathcal{F}_{t-1})$, then there exists a two dimensional conditional copula $C(\cdot \mathcal{F}_{t-1})$, such that

$$F((y_1, y_2)\mathcal{F}_{t-1}) = C(F_1(y_1\mathcal{F}_{t-1}), F_2(y_2\mathcal{F}_{t-1})\mathcal{F}_{t-1}). \tag{A.1}$$

Furthermore, if the marginal distribution functions are continuous, the copula is unique. The converse also holds, such that given two conditional marginal distributions, we can use the conditional copula to link the variables to form a conditional joint distribution with the specified margins. It is especially this second part of the theorem that is useful here, since it allows us to isolate the description of the dependence structure from the marginal behavior of the individual variables.

Moreover, let us define the probability integral transform variables

$$U_{i,t} \equiv F_i(Y_{i,t}\mathcal{F}_{t-1}), \quad \text{for } i = 1, 2,$$

and let $\boldsymbol{U}_t \equiv (U_{1,t}, U_{2,t})'$. Then $U_{i,t} \sim \text{Unif}(0, 1)$, and note furthermore that the conditional copula in Eq. (A.1) is simply the conditional distribution of $\boldsymbol{U}_t\mathcal{F}_{t-1}$:

$$\boldsymbol{U}_t\mathcal{F}_{t-1} \sim C(\cdot \mathcal{F}_{t-1}).$$

In this paper, we consider different copulas from the elliptical and archimedean families, which are commonly used in the financial literature. For a detailed treatment of these copulas and their properties, we refer to the reference books by [22] and [23].

2.1 Marginal models

As a first step when working with copulas, we need to find proper marginal distribution models. Here, we restrict our attention to marginal models of the ARMA–GARCH type to model the conditional mean and the conditional variance of the individual variables.[5] For example, an ARMA(p, q)–GARCH$(1, 1)$ model for the margins can be written as

$$\begin{aligned}
Y_{i,t} &= \sum_{j=1}^{p} \phi_{i,j} Y_{i,t-j} + \sum_{k=1}^{q} \theta_{i,k} \varepsilon_{i,t-k} + \varepsilon_{i,t}, \\
\varepsilon_{i,t} &= \sigma_{i,t} \eta_{i,t}, \\
\sigma^2_{i,t} &= \omega_i + \alpha_i \varepsilon^2_{i,t-1} + \beta_i \sigma^2_{i,t-1},
\end{aligned}$$

[5] A variety of other parametric specifications can be considered for the conditional mean, such as ARMAX models, long memory models, linear and nonlinear regression models, etc. The same holds for the conditional variance where, among others, different extensions to the ARCH model can be considered; see [24] for a long list of such models.

for $i = 1, 2$, where ω_i, α_i, β_i follow the restrictions posed in e.g. [25], and $\alpha_i + \beta_i < 1$. Furthermore,

$$\eta_{i,t}\mathcal{F}_{t-1}^{(i)} \sim F_i(0,1), \quad \text{for } i = 1, 2 \text{ and all } t.$$

For the marginal distributions we consider the case where F_i does not vary with time and has a parametric form. Also, we relax the normality assumption, allowing for more general distributions. The ARMA–GARCH models function as filters that produce innovation processes $\eta_{1,t}$ and $\eta_{2,t}$ that are serially independent; it is the conditional distributions of $\eta_{1,t}$ and $\eta_{2,t}$ that are then coupled using the conditional copula.

One note of caution has to be made regarding the conditioning set $\mathcal{F}_{t-1}$ emphasizing that this set is generated by $(\boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \dots)$. In our specification for the marginal models however, we do not condition on $\mathcal{F}_{t-1}$, but only a subset $\mathcal{F}_{t-1}^{(i)} \subset \mathcal{F}_{t-1}$. When using such models, the copula is, according to [26], a true copula if and only if

$$Y_{i,t}\mathcal{F}_{t-1} \overset{d}{=} Y_{i,t}\mathcal{F}_{t-1}^{(i)}, \tag{A.2}$$

for $i = 1, 2$ and all t. If the equality in Eq. (A.2) is not satisfied, then the joint conditional distribution of $\boldsymbol{Y}_t\mathcal{F}_{t-1}$ does not have the specified conditional marginal distributions. To study if the equality in Eq. (A.2) holds, we test for cross-equation effects by including lags of one variable in the conditional mean equation of the other variable and vice versa, and perform a standard Wald test for the joint significance of the added explanatory variables, as proposed in [18].

2.2 Estimation procedure for the joint model

To estimate the joint model, we perform maximum likelihood estimation. The joint conditional density is obtained by differentiating the joint conditional distribution function in Eq. (A.1). Thus, the log-likelihood function takes the form

$$\begin{aligned}
\log \mathcal{L} &= \sum_{t=1}^{T} \log f((y_{1,t}, y_{2,t})\mathcal{F}_{t-1}; \Theta) \\
&= \sum_{t=1}^{T} \log f_1(y_{1,t}\mathcal{F}_{t-1}; \Theta_1) + \sum_{t=1}^{T} \log f_2(y_{2,t}\mathcal{F}_{t-1}; \Theta_2) \\
&\quad + \sum_{t=1}^{T} \log c((u_{1,t}, u_{2,t})\mathcal{F}_{t-1}; \gamma),
\end{aligned} \tag{A.3}$$

where f_1 and f_2 are the conditional marginal densities, c is the conditional copula density defined as

$$c((u_{1,t}, u_{2,t})\mathcal{F}_{t-1}) = \frac{\partial^2}{\partial u_1 \partial u_2} C((u_{1,t}, u_{2,t})\mathcal{F}_{t-1}),$$

and

$$u_{i,t} = F_i(y_{i,t}\mathcal{F}_{t-1}; \Theta_i), \quad \text{for } i = 1, 2.$$

In Eq. (A.3), Θ denotes the set of parameters for the entire model, and Θ_1, Θ_2 and γ denote the parameters for the two marginal models and the copula, respectively, and have no common elements. For simplicity, we assume that the copula is completely described by one single parameter γ. We perform multi-stage maximum likelihood estimation, where we consider the two marginal models and the copula model separately. For details on the validity of this procedure, consult [18].

2.3 Time-varying copula models

Since the dependency between electricity prices and wind power production might change through time, extending copula models to allow for time-varying dependence is relevant. Before specifying a parametric model for the copula dependence parameter, it is useful to investigate what type of time variation (if any) we can detect in the data. Here, we employ two tests proposed in [18]: One that tests for the presence of a break in the rank correlation by performing the classical "sup" test, and another that tests for the presence of autocorrelation in a measure of dependence. For a comprehensive description of the two tests the reader is referred to [18].

The Generalized Autoregressive Score model

To model time-varying dependence, we employ the *Generalized Autoregressive Score* (GAS) model of [27]. In order to ease the presentation, we consider the case where the copula has one dependence parameter. For the GAS(1,1) model, a possible updating equation for the transformed copula dependence parameter g_{t+1} is:

$$g_{t+1} = \omega + \alpha g_t + \beta I_t^{-\frac{1}{2}} s_t, \tag{A.4}$$

where

$$\begin{aligned} g_t &= h(\gamma_t), \\ s_t &= \frac{\partial}{\partial \gamma} \log c((u_{1,t}, u_{2,t}); \gamma_t), \\ I_t &= \mathbb{E}_{t-1}\left[s_t^2\right]. \end{aligned}$$

In Eq. (A.4), s_t denotes the score of the copula log-likelihood and I_t is the Fisher information. Moreover, γ_t denotes the time-varying copula dependence parameter, which is usually constrained to lie in a particular range; see Table A.9 in A for details regarding the range of different copula dependence parameters. For estimation purposes, we apply a transformation $h(\cdot)$ to γ_t, to obtain g_t which takes values on the entire real axis. We note that the updating mechanism given in Eq. (A.4) is one of many possible specifications: The GAS model can be extended to include e.g. more lags or exogenous variables. Moreover, the scaling quantity $I_t^{-1/2}$ is simply one convenient choice. GAS models can be generalized to allow for asymmetries or long memory, and to include regime-switching, however such extensions are not considered in the present work.

The parameter estimates from the GAS model can be obtained by maximum likelihood estimation, as proposed by [27]. The only challenge can be finding a closed-form expression for the Fisher information, and thus deriving the updating mechanism in Eq. (A.4). To overcome this issue, the Fisher information is evaluated numerically for most copula specifications by performing the following steps:

1. Given a copula specification, construct a grid of values for the dependence parameter, $[\gamma^{(1)} < \gamma^{(2)} < \cdots < \gamma^{(n)}]$.
2. For each dependence parameter in the grid,
   (a) perform a large number of simulations from the chosen copula model,
   (b) evaluate the score function at each simulation,
   (c) compute the Fisher information, by taking the mean over the evaluated scores squared.
3. Finally, use linear interpolation to get the Fisher information at intermediate points.

2.4 Quantile dependence

As a preliminary study before specifying copula models, one can examine the dependence in the data by considering quantile dependence. For the case of negatively dependent variables, the quantile dependence is defined as:

$$\lambda^q = \begin{cases} \mathbb{P}(U_{1,t} \le q U_{2,t} \ge 1-q), & 0 < q \le 1/2, \\ \mathbb{P}(U_{1,t} > q U_{2,t} < 1-q), & 1/2 < q < 1. \end{cases} \tag{A.5}$$

By computing quantile dependence coefficients at different quantiles q, we obtain a richer description of the dependence structure. This can help narrow down the set of possible parametric copulas to a collection of models that are able to capture some of the characteristics we observe in the data. To obtain standard errors for the quantile dependence coefficients, we use boostrapping; specifically, we follow the procedure proposed in [18], which is based on the stationary block-bootstrap of [28], where the optimal block-length is chosen according to [29] and [30].[6]

2.5 Selection of copula models

To test for whether or not a copula is well specified, we perform two widely used goodness-of-fit tests (GoF): The Kolmogorov-Smirnov (KS) and the Cramer von-Mises (CvM) tests. Under the null that the conditional copula is well specified, we should find that the empirical copula provides a good nonparametric estimate of the null conditional copula. Suppose we have the random sample $\{\boldsymbol{u}_t\} = \{(u_{1,t}, u_{2,t})\}_{t=1}^{T}$ from $\boldsymbol{U}_t$. Then the test statistics can be written as

$$KS^{(C)} = \max_t \left| C(\boldsymbol{u}_t; \hat{\gamma}) - \hat{C}(\boldsymbol{u}_t) \right|, \tag{A.6}$$

$$CvM^{(C)} = \sum_{t=1}^{T} \left\{ C(\boldsymbol{u}_t; \hat{\gamma}) - \hat{C}(\boldsymbol{u}_t) \right\}^2, \tag{A.7}$$

where $C(\boldsymbol{u}_t; \hat{\gamma})$ is an estimator of the null conditional copula. Moreover, $\hat{C}$ denotes the empirical copula defined as

$$\hat{C}(\boldsymbol{z}) \equiv \frac{1}{T+1} \sum_{t=1}^{T} \mathbf{1}\{u_{1,t} \leq z_1, u_{2,t} \leq z_2\},$$

where $\mathbf{1}$ denotes the indicator function and $\boldsymbol{z} = (z_1, z_2) \in [0,1]^2$. The KS and CvM tests described above work solely for the testing of constant copula models. A slight modification will however allow for the additional testing of time-varying copulas: The KS and CvM tests based on the Rosenblatt transform. In our case, the transformation is simply

$$\begin{aligned} V_{1,t} &= U_{1,t} \\ V_{2,t} &= C_{2|1,t}(U_{2,t}|U_{1,t}; \hat{\gamma}_t), \end{aligned}$$

where $C_{2|1,t}$ denotes the conditional copula of the random variable $U_{2,t}U_{1,t}$. Applying the Rosenblatt transform to the data will yield *iid* and $\text{Unif}(0,1)$ variables, and hence we can compare the empirical copula of a random sample $\{\boldsymbol{v}_t\} = \{(v_{1,t}, v_{2,t})\}_{t=1}^{T}$ from $\boldsymbol{V}_t$, against the independence copula, defined as

$$C^{indep}(\boldsymbol{v}_t; \hat{\gamma}_t) \equiv \prod_{i=1}^{2} v_{i,t}.$$

[6] The same bootstrapping procedure can be used to perform inference on other measures of dependence, e.g. linear correlation, rank correlation.

A simulation-based approach is used to obtain p-values for the GoF tests described above, since the test statistics in Eqs. (A.6) and (A.7) depend on estimated parameters. This approach is described in detail in [31], [32] and [18], and will not be elaborated on here.

Another very important issue when dealing with copulas is choosing the best copula model among competing models. Here, we consider pairwise comparisons, where we follow [33] for most in-sample (IS) model comparisons and [34] for out-of-sample (OOS) model comparisons. The IS comparison test can be performed when the models are non-nested; for the case where the models are nested, a likelihood ratio test can usually be used. The OOS model comparison test works for both nested and non-nested models. Also, both tests can be applied regardless of whether the copula is constant or time-varying.

For the IS case, the idea is to compare two models using their joint log-likelihood, and test the null

$$H_0 : \mathbb{E}\left[\boldsymbol{L}^{(1)} - \boldsymbol{L}^{(2)}\right] = 0,$$

against

$$H_1 : \mathbb{E}\left[\boldsymbol{L}^{(1)} - \boldsymbol{L}^{(2)}\right] > 0 \quad \text{and} \quad H_2 : \mathbb{E}\left[\boldsymbol{L}^{(1)} - \boldsymbol{L}^{(2)}\right] < 0,$$

where the superscripts (1) and (2) denote two competing models. The case of comparing joint log-likelihoods reduces in our case to comparing copula log-likelihoods, c.f. Eq. (A.3), since we use the same marginal distribution models. Hence, $\boldsymbol{L}^{(i)} = \log c^{(i)}(\boldsymbol{u}; \gamma^{(i)})$ or $\boldsymbol{L}^{(i)} = \log c^{(i)}(\boldsymbol{u}; \gamma_t^{(i)})$, $i = 1, 2$, depending on whether the copula is constant or time-varying. [33] show that under the null,

$$\frac{\sqrt{T}\left(\bar{L}^{(1)} - \bar{L}^{(2)}\right)}{\sqrt{\hat{\sigma}^2}} \xrightarrow{d} \mathrm{N}(0, 1)$$

where

$$\bar{L}^{(i)} = \frac{1}{T}\sum_{t=1}^{T} \log c^{(i)}\left(\hat{\boldsymbol{u}}_t; \hat{\gamma}_t^{(i)}\right), \quad \text{for } i = 1, 2.$$

As an estimator for the asymptotic variance of $\sqrt{T}\left(\bar{L}^{(1)} - \bar{L}^{(2)}\right)$ we use the Newey-West heteroskedasticity and autocovariance consistent (HAC) estimator.

For OOS comparisons, we consider a fixed estimation window, where the model is estimated using the data from $[1, T]$. We then evaluate the conditional predictive ability of two competing copulas on the OOS period, i.e.

on R observations, where $R = T^* - T$, $T^* > T$. The test for comparing the predictive ability of competing copula models conditional on the estimated parameters proposed by [34] is in fact a special case of the more general framework presented in [35]. The null hypothesis for the OOS case is the same as for the IS case, and a test statistic based on the difference between the sample averages of the copula log-likelihoods can again be used, and is shown by [35] to be asymptotically $\mathrm{N}(0, 1)$ under the null. As an estimator for the asymptotic variance, we use the HAC estimator.

3 Empirical results

A joint model for electricity prices and wind power production is interesting to consider in an area with a high penetration ratio of wind power in the grid. Here, we analyze data from Denmark, which has long been among the top wind power producing countries. According to *Energinet.dk*, the Danish Transmission System Operator, more than a third of the Danish power consumption was covered by wind power in 2013, and in December that year, 57.4% of the consumption came from wind turbines. In 2014, wind turbines produced on average what corresponds to over 39% of the Danish power consumption. Also, in January 2014, 61.7% of the consumption was covered by wind power.

Specifically, we base our analysis on data from one of the two Danish price areas, DK1 (Western Denmark), and a sample period that spans from 01/01/2006 to 31/12/2014. The first time series, Fig. A.1(a), consists of total daily wind power production in DK1 relative to the total installed capacity, and is obtained by performing the normalization

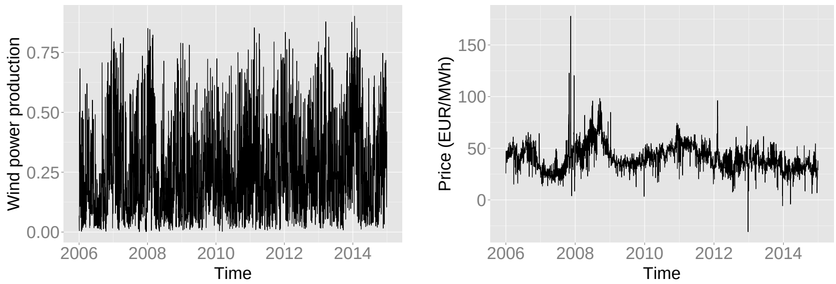
$$\frac{\text{Total daily wind power production (MWh)}}{\text{Installed capacity (MW)} \cdot H}$$

for each day in the sample, where H denotes the total number of hours in the day. We note that we work in UTC time, so $H = 24$ always. The second time series, Fig. A.1(b), represents the daily average of spot electricity prices.[7,8]

Before proceeding to the estimation of a joint model for prices and wind power production, two comments are in order. First, since the production series is bounded, with a lower bound at 0 and an upper bound at 1, we

[7]The data is publicly available on *Energinet.dk* and on the web page of Nord Pool's Elspot market, *nordpoolspot.com*. Elspot is a day-ahead physical delivery market for electricity currently operating in the Nordic and Baltic region.

[8]We note that one observation has been truncated in the price data, corresponding to the date 07/06/13, since this is assessed to be an outlier. On this date, the hourly price reached Nord Pool's cap price due to a combination of low wind, reduced import possibilities caused by planned maintenance on transmission cables and also planned maintenance on central power stations.



(a) Daily wind power production measured relative to the total installed capacity.

(b) Daily spot electricity prices

Fig. A.1: Historical daily observations for the DK1 price area in the period 01/01/2006 to 31/12/2014.

perform a logistic transformation in order to obtain data that can take values on the entire real line. Second, we split our data into an in-sample (IS) period spanning from 01/01/2006 to 31/12/2012, and an out-of-sample (OOS) spanning from 01/01/2013 to 31/12/2014. Estimation of marginal models and copulas is performed on the IS data.

3.1 Marginal specifications for spot electricity prices and wind power production

Prior to modeling the dependence structure of electricity prices and wind power production, we filter out the stylized facts affecting the marginal behavior of the individual variables. As a first step, we demean and correct for deterministic seasonality by performing a regression on a constant and dummy variables. Specifically, we have used the dummy variable *month-of-year* to correct the (transformed) wind power production series for seasonality. For the price series both *day-of-week* and *month-of-year* dummy variables were used as regressors. To model the conditional mean and variance of the variables, we consider ARMA–GARCH models with different specifications for the error distribution. We consider ARMA models up to order (7,7), and GARCH models up to order (2,2). Based on the Bayesian Information Criterion, we find that the optimal model for the wind power production series is an ARMA(1,3)–GARCH(1,1), and use a skewed generalized error distribution for the standardized residuals. For the day-ahead electricity prices, we find the optimal model to be an ARMA(3,1)–GARCH(1,1), and use a skewed t distribution for the standardized residuals. Table A.1 summarizes the estimation results, and Fig. A.10 in B displays the autocorrelation functions, histograms and quantile plots for the standardized residuals resulting from

	Daily wind power production ARMA(1,3) – GARCH(1,1)	Daily spot electricity prices ARMA(3,1) – GARCH(1,1)
	Conditional mean	
$\hat{\phi}_1$	0.8725 (0.0510)	1.4579 (0.0065)
$\hat{\phi}_2$	-	-0.5525 (0.0176)
$\hat{\phi}_3$	-	0.0897 (0.0261)
$\hat{\theta}_1$	-0.3578 (0.0550)	-0.8365 (0.0128)
$\hat{\theta}_2$	-0.2733 (0.0363)	-
$\hat{\theta}_3$	-0.0610 (0.0264)	-
	Conditional variance	
$\hat{\omega}$	0.0803 (0.1269)	2.4433 (0.7388)
$\hat{\alpha}$	0.0251 (0.0199)	0.1657 (0.0312)
$\hat{\beta}$	0.9022 (0.1333)	0.7832 (0.0410)
	Skewed general error dist./Skewed t dist.	
Shape $\hat{\nu}$	2.1348 (0.0967)	4.9967 (0.4711)
Skewness $\hat{\xi}$	0.8024 (0.0269)	0.9583 (0.0222)
	Goodness-of-fit tests	
KS (p-val.)	0.6293	0.7097
CvM (p-val.)	0.5882	0.5996

Table A.1: The first panels display parameter estimates together with their std. errors in parenthesis. The last panel displays the results of GoF tests.

the fitted models. A visual inspection of Fig. A.10 shows that almost no autocorrelation is left in the standardized residuals. The specified distributions provide a reasonable fit, however we observe some deviations in the tails of both distributions. We complement these findings with GoF tests, where we consider the KS and CvM tests. The resulting p-values are given in Table A.1 and indicate that there is not sufficient evidence as to reject the null that the distributional assumptions are well-specified.[9] We note that finding suitable marginal models is of great concern when working with copula models, since the copula takes as input *iid* $\text{Unif}(0,1)$ variables that result from applying the probability integral transform to the standardized residuals. A violation of the assumptions will thus automatically lead to a misspecified copula model.

Because we condition with different information sets when specifying the marginal models, we need to investigate whether or not lagged values of wind power production help explain electricity prices and vice versa. To do this, we consider the specified models for the conditional mean with added

[9] We perform simulation-based GoF tests, that take the parameter estimation errors from the ARMA–GARCH models into account. Specifically, we test for whether or not the probability integral transforms implied by the estimated conditional densities are *iid* $\text{Unif}(0,1)$. The p-values for the tests are based on 999 simulations.

explanatory variables consisting of seven lagged values of the "other" series, and test for the significance of cross-sectional effects by performing a Wald test. For the wind power production, we consider an ARMAX(1,3,7) model, and for the electricity prices, we consider an ARMAX(3,1,7) model. The tests yield a p-value of 0.25 for the wind power production model, and 0.09 for the electricity price model, thus suggesting no cross-equation effects at a 5% significance level.[10]

3.2 Symmetric vs. asymmetric dependence

Having decided upon the marginal models for price and wind power production, the remaining of this section focuses on the modeling of the dependence structure. First, we apply the probability integral transform to the standardized residuals resulting from the marginal models to obtain approximately uniformly distributed variables. To perform this transformation, we use the estimated parametric models for the distribution functions F, i.e. the estimated skewed generalized error distribution and skewed t distribution, see Table A.1. We obtain

$$\hat{U}_{W,t} = F_{skew\ ged}(\hat{\eta}_{W,t}, \hat{\nu}_W, \hat{\xi}_W) \quad \text{(A.8)}$$

$$\hat{U}_{S,t} = F_{skew\ t}(\hat{\eta}_{S,t}, \hat{\nu}_S, \hat{\xi}_S), \quad \text{(A.9)}$$

where $\hat{U}_{W,t}$ and $\hat{U}_{S,t}$ denote the resulting uniforms corresponding to the wind power production time series and the spot price time series, respectively. Standardized residuals are denoted by $\hat{\eta}$, and estimated distribution parameters are denoted by $\hat{\xi}$ (skew parameter) and $\hat{\nu}$ (shape parameter).

As an introductory investigation of the dependence structure, we compute some measures of dependence for $\hat{\boldsymbol{U}}_W$ and $\hat{\boldsymbol{U}}_S$. Table A.2 displays the estimated coefficients for Spearman's ρ, Kendall's τ and linear correlation, implying (not surprisingly) that prices and wind power production are negatively correlated. Based on Eq. (A.5) we also compute quantile dependence measures, and the results, displayed in Fig. A.2, show evidence for a symmetric dependence structure. When considering the farther right and left portions of Fig. A.2(a), the results reveal a slightly larger probability of observing low prices given that the production is high than the opposite. However, according to Fig. A.2(b), this difference is not statistically significant.

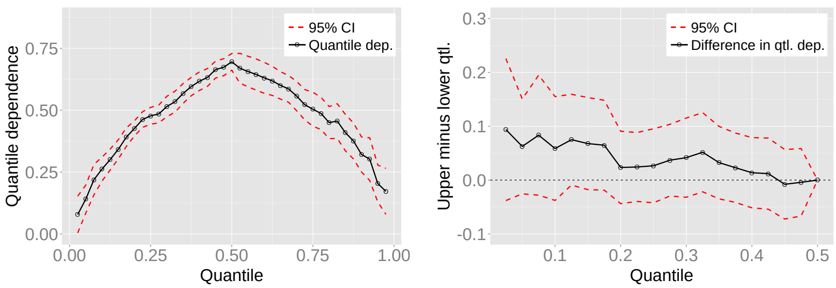
3.3 Constant copula models

Although we anticipate time-variation in the dependence structure, we consider six constant copula models, to have as benchmarks for later compar-

[10]We have tried testing for cross-sectional effects with different other specifications, and none of the results indicate cross-equation effects at a 5% significance level.

	Spearman's ρ	Kendall's τ	Linear correlation
Estimate	-0.5024	-0.3478	-0.5030
95% CI	$(-0.5716, -0.4332)$	$(-0.3987, -0.2969)$	$(-0.5714, -0.4347)$

Table A.2: Estimated dependence measures with 95% confidence intervals based on the block-bootstrap procedure described in Section 2.4 and $M = 999$ bootstrap samples.



(a) Quantile dependence

(b) Difference in quantile dependence

Fig. A.2: Fig. A.2(a) displays estimated quantile dependence for quantile $q \in [0.025, 0.975]$ and a size step of 0.025, along with a 95% confidence interval based on the block-bootstrap procedure described in Section 2.4 and $M = 999$ bootstrap samples. The y-axis provides the probability of $\hat{\boldsymbol{U}}_W$ lying below (above) its q quantile given that $\hat{\boldsymbol{U}}_S$ lies above (below) its $1 - q$ quantile for $q \leq 1/2$ ($q > 1/2$). Fig. A.2(b) shows the difference in corresponding left and right quantile dependence illustrated in Fig. A.2(a) with a corresponding 95% confidence interval.

isons. A brief overview of these copula models is provided in A. The estimation results for the proposed constant copulas are given in Table A.3, together with GoF test results. Among the constant copulas we consider, it is only the Gaussian and Student *t* that allow for negative dependence. To deal with this issue, we have performed suitable rotations of our data when estimating the Clayton, Gumbel, Joe-Frank and Symmetrized Joe-Clayton (SJC) copulas. Furthermore, the Gaussian and the Student *t* copulas are symmetric, the Clayton and Gumbel are asymmetric, and the combinations Joe-Frank and SJC allow for more flexible dependence structures and nest the case of symmetric dependence.

The GoF results in Table A.3 support our earlier findings in Section 3.2. The Gaussian and Student *t* copulas are, according to all tests, a good specification. Clayton is rejected by all tests, while Gumbel is only partly rejected. For the combination copulas, the test results are more surprising: The Joe-Frank specification is accepted by all tests, while the SJC specification is rejected by all tests. We attempt to understand these results by plotting the quantile dependence we observe in our data together with the quantile dependence implied by some of the fitted copulas in Fig. A.3.

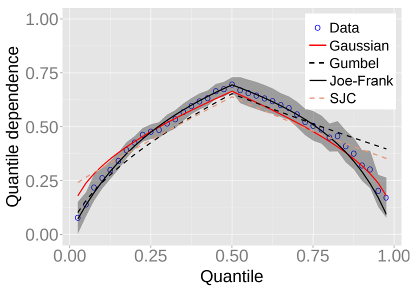


Fig. A.3: Quantile dependence implied by some of the fitted constant copula models in Table A.3.

We observe that the quantile dependence implied by the Gaussian copula provides a reasonable fit to our data. So does the Joe-Frank copula, by providing a fit that generates almost no asymmetry. The Gumbel copula on the other hand is too asymmetric, producing large deviations as we approach one of the tails. Lastly, the SJC, although implying less asymmetry than Gumbel, assigns too much probability to extreme events compared to what we observe in the data, and thus produces large deviations as we approach both tails.[11]

[11] We have omitted the quantile dependence implied by the Student *t* and Clayton copulas in Fig. A.3 for clarity reasons. The Student *t* copula implies quantile dependence coefficients that are almost indistinguishable from the Gaussian ones, which is due to the very high value we estimate for the degree of freedom of this copula. The Clayton copula implies even more asymmetry than Gumbel in the far right side of the quantile plot.

Copula		Parameter estimates	(s.e.)	$\log \mathcal{L}$	GoF tests p-val.			
					Kolmogorov-Smirnov		Cramer-von Mises	
					$KS^{(C)}$	$KS^{(R)}$	$CvM^{(C)}$	$CvM^{(R)}$
Gaussian	$\hat{\rho}$	-0.4923	(0.0134)	355.07	0.4545	0.6096	0.6747	0.4044
Student t	$\hat{\rho}$	-0.4967	(0.0147)	357.15	0.4134	0.7427	0.7227	0.5435
	$\hat{\nu}^{-1}$	0.0318	(0.0170)					
Clayton	$\hat{\theta}$	0.7024	(0.0339)	288.94	*0.0020*	*0.0060*	*0.0000*	*0.0020*
Gumbel	$\hat{\theta}$	1.4455	(0.0220)	331.14	*0.0450*	0.0681	*0.0390*	0.0661
Joe-Frank	$\hat{\theta}$	8.8682	(4.4760)	368.77	0.7778	0.9359	0.7017	0.8278
	$\hat{\delta}$	0.3431	(0.1125)					
SJC	$\hat{\tau}^U$	0.3355	(0.0368)	320.18	*0.0070*	*0.0150*	*0.0230*	*0.0290*
	$\hat{\tau}^L$	0.2012	(0.0408)					

Table A.3: Estimation and GoF test results for constant copula models. The p-values less than 0.05 are given in italics, highlighting that the dependence structure is not well-represented by the proposed copula model. The superscript (C) refers to the tests performed on the empirical copula of the standardized residuals and the superscript (R) refers to the tests performed on the empirical copula of the Rosenblatt transforms. The GoF tests are simulation based (999 bootstraps) and take parameter estimation errors into account.

3.4 Time-varying copula models

To confirm our suspicion that the dependence of spot electricity prices and wind power production is time-varying, we perform the two tests briefly described in Section 2.3. The results are given in Table A.4, showing no evidence of a one-time break in the dependence structure, but strong evidence for the presence of autocorrelation in the rank correlations.

	One-time break	Time-varying dep. of autoreg. type		
		AR(1)	AR(5)	AR(7)
p-value	0.7898	0.0020	0.0110	0.0000

Table A.4: Test results for time-varying dependence. To test for the presence of a one-time break in the rank correlation we use the "sup" test, and test the null of no one-time break. To test for the presence of time-varying dependence of autoregressive type we consider the regression $\hat{U}_{W,t}\hat{U}_{S,t} = \mu + \sum_{i=1}^{p} \phi_i \hat{U}_{W,t-i}\hat{U}_{S,t-i} + \varepsilon_t$, for $p = 1, 5, 7$; the null of a constant copula cannot be rejected if we find that $\phi_i = 0$, for $i = 1, \ldots, p$. For all tests, p-values are obtained by bootstrap testing (based on 999 bootstraps, where bootstrap samples are obtained by randomly drawing rows, with replacement, from $(\hat{\boldsymbol{U}}_W, \hat{\boldsymbol{U}}_S)'$).

In light of these findings, we consider three copula models where the transformed dependence parameter denoted by g evolves according to a GAS(1,1) model, see Eq. (A.4). The transformations applied to the copula dependence parameters, estimation and GoF test results are all displayed in Table A.5. For the Gaussian copula, a closed form expression for the Fisher information can be derived (see e.g. [36]). For the Gumbel and the Joe-Frank copulas, the Fisher information is computed numerically by performing the steps in Section 2.3. The Joe-Frank copula has two dependence parameters, and we consider the case where one parameter evolves according to the GAS specification, while the other is kept constant. It should however be mentioned that letting both parameters vary through time provides very little improvement. Regarding the parameter estimates, α is high in all models, implying a very persistent time-varying correlation process. Also, the intercept parameter ω is not significant in any model. As far as the GoF test results are concerned, the Joe-Frank and Gaussian GAS models are accepted at a 5% level, while the Gumbel GAS model is only partially accepted.

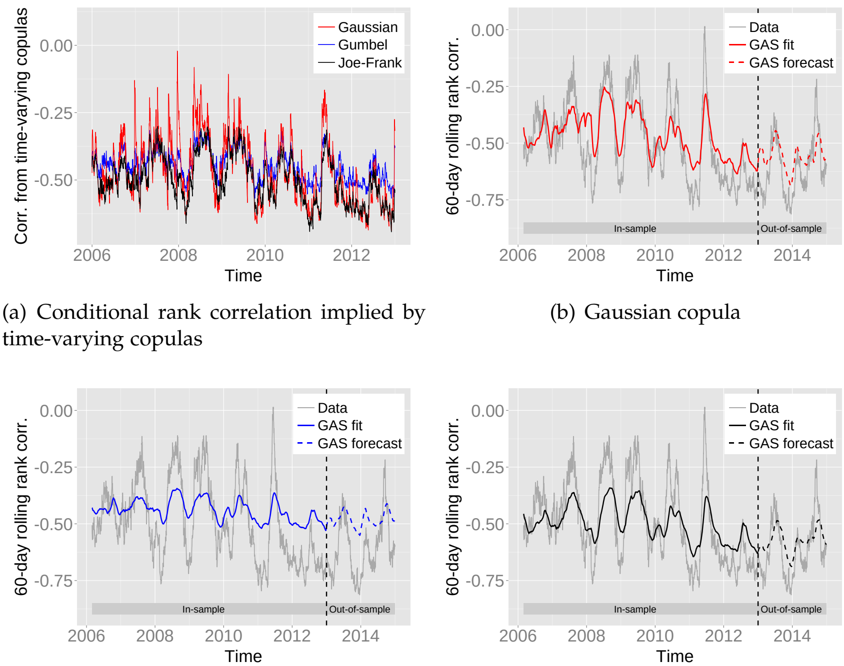
To visualize and compare the fits of the proposed GAS models, we plot the conditional rank correlation implied by the fitted time-varying copula models in Fig. A.4(a). The numbers are obtained by mapping the copula parameter(s) to a rank correlation coefficient[12]. In Figs. A.4(b)-(d) we plot actual 60-day

[12] Specifically, we follow the procedure described in [18]: (1) construct a grid of copula parameters, (2) perform 100,000 simulations from the copula model at each point in the grid, (3) compute the rank correlation of the simulations, and finally (4) use linear interpolation to obtain the correlation at intermediate points. We also mention that the functions mapping the copula

	Transf. h	Parameter estimates (s.e.)				$\log \mathcal{L}$	GoF tests p-val.	
		$\hat{\omega}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$		$KS^{(R)}$	$CvM^{(R)}$
Gaussian	$\log\left(\frac{1+\rho}{1-\rho}\right)$	-0.0196	0.9820	0.0390	-	373.90	0.5354	0.4646
		(0.0268)	(0.0246)	(0.0159)				
Gumbel	$\log(\theta - 1)$	-0.0272	0.9672	0.0420	-	342.90	*0.0404*	0.0707
		(0.1527)	(0.1849)	(0.0334)				
Joe-Frank	$\log(\theta - 1)$	0.0314	0.9860	0.0403	0.2944	388.77	0.7959	0.8571
		(0.0509)	(0.0208)	(0.0127)	(0.0894)			

Table A.5: Estimation and GoF results for time-varying copulas. Due to the high computational time, the GoF tests and standard errors are based on 99 bootstraps. The superscript (R) indicates that the GoF tests are based on the Rosenblatt transform.

rolling rank correlations of the data $(\hat{U}_W, \hat{U}_S)'$ together with the in-sample fit of the proposed time-varying models. To perform the same comparison for the out-of-sample period, we obtain the approx. uniforms $(\hat{U}_W^{OOS}, \hat{U}_S^{OOS})'$ by first applying the estimated function for removing seasonality and then the ARMA–GARCH filters, without re-estimating any parameters, to the out-of-sample wind power production data and the out-of-sample spot electricity price data. The 60-day rolling rank correlations of $(\hat{U}_W^{OOS}, \hat{U}_S^{OOS})'$ are then computed and compared to one-step-ahead forecasts from the time-varying copulas. Due to the elevated computational cost of using a rolling estimation window to produce forecasts, we restrict ourselves to considering a fixed estimation window corresponding to the in-sample period, but enlarge the conditioning set as information becomes available.



(a) Conditional rank correlation implied by time-varying copulas

(b) Gaussian copula

(c) Gumbel copula

(d) Joe-Frank copula

Fig. A.4: Fits and forecasts produced with the three time-varying copulas from Table A.5.

One first and surprising remark regarding Fig. A.4 is related to the data itself and implicitly the fits produced by the GAS models, namely that the correlation is generally stronger during winter than during summer. There are many factors that can help explain this finding since price formation is a

parameters to rank correlation are smooth.

complex process that is not only influenced by supply and demand (which in turn have strong seasonal components), but also transmission capacity. To provide a few facts that can help explain our findings, we mention that the wind power production relative to the consumption in the DK1 price area has been higher for winter periods than summer periods, during the sample period we consider in this paper. Also, we can expect that situations with little wind during summer do not always push the prices upwards. This is (aside from consumption being lower during summer) due to the fact that DK1 is well connected with cables to Norway, Sweden and Germany, which are all heavy producers of renewable energy, and hence electricity could be imported at a lower price compared to the cost of having to turn on the more costly power stations in DK1.

Considering now the fits implied by the proposed time-varying copulas, Fig. A.4 reveals that the Gaussian GAS implies most variation in the correlation and is able to capture periods with weaker dependence the best. The Gumbel GAS specification is the one that least captures the variation that we observe in the data. The Joe-Frank GAS specification is superior at reaching the stronger correlations, but does not produce correlations that are weaker than around -0.3. The plots clearly help establish that the Gumbel GAS specification is the inferior choice. However, it is difficult to choose the better copula when considering the Gaussian GAS against the Joe-Frank GAS.

From fitting not only time-varying copulas but also constant ones, we have so far obtained many different models that are actually well-specified according to the GoF tests. To help choose among all the considered copulas, we perform the pairwise comparison tests described in Section 2.5.[13] The results are summarized in Table A.6. We find that The Joe-Frank GAS specification outperforms all other specifications in-sample, however its superiority over the Gaussian GAS specification is not statistically significant. When considering the out-of-sample results, the situation reverses, with the Gaussian GAS specification performing the best, but not significantly better than the Joe-Frank GAS. Since the Gaussian GAS is the smaller model, we will choose this specification as our preferred one, and continue our investigations using this model to describe the dependence between wind power production and spot electricity prices.

4 A simulation study

Performing simulations from a copula model is straightforward. The basic steps are (1) at time t, generate the pair $(U_{W,t}, U_{S,t})$ from the Gaussian copula with dependence parameter ρ_t, (2) perform the inverse of the transformations

[13]The in-sample pairwise comparison between the Gaussian and the Student t copula is based on a simple t-test, since these models are nested.

	Gaussian	Student t	Clayton	Gumbel	Joe-Frank	SJC	Gaussian^{GAS}	Gumbel^{GAS}	Joe-Frank^{GAS}
In-sample model comparisons									
Gaussian									
Student t	1.88*								
Clayton	−4.61***	−5.04***							
Gumbel	−2.20**	−2.71***	7.07***						
Joe-Frank	1.40	1.28	5.27***	3.10***					
SJC	−3.91***	−4.91***	3.32***	−1.74*	−3.54***				
Gaussian^{GAS}	2.94***	2.54***	5.71***	3.63***	0.44	5.30***			
Gumbel^{GAS}	−1.12	−1.42	8.08***	2.91***	−2.00**	3.07***	−2.91***		
Joe-Frank^{GAS}	3.14***	3.12***	6.37***	4.44***	3.20***	4.88***	1.57	3.66***	
$\log \mathcal{L}$	355.06	357.13	288.93	331.14	368.77	320.17	373.90	342.90	388.77
Out-of-sample model comparisons									
	Gaussian	Student t	Clayton	Gumbel	Joe-Frank	SJC	Gaussian^{GAS}	Gumbel^{GAS}	Joe-Frank^{GAS}
Gaussian									
Student t	1.41								
Clayton	−4.36***	−4.95***							
Gumbel	−1.58	−2.04**	8.18***						
Joe-Frank	0.16	−0.22	3.64***	1.32					
SJC	−2.67***	−3.43***	5.19***	−1.99**	−1.99**				
Gaussian^{GAS}	2.85***	2.76***	5.61***	3.54***	2.04**	4.56***			
Gumbel^{GAS}	−0.67	−0.96	6.53***	2.22**	−0.63	2.53***	−2.75***		
Joe-Frank^{GAS}	1.76*	1.58	4.21***	2.38***	2.46***	2.90***	−0.29	1.82*	
$\log \mathcal{L}$	160.10	161.48	129.55	150.52	160.67	146.11	171.96	155.25	170.56

Table A.6: t-statistics from in-sample and out-of-sample pairwise model comparisons. A positive (negative) value means that the model in the row is superior (inferior) to that in the column. t-statistics are followed by *, ** or *** if one model is significantly better than the other at a 0.1, 0.05 or 0.01 significance level, respectively. The in-sample model comparisons are based on [33] and the out-of-sample model comparisons are based on [34].

given in Eqs. (A.8) and (A.9) to obtain standardized residuals $(\eta_{W,t}, \eta_{S,t})$, (3) insert the standardized residuals in the marginal models from before (see Table A.1) to obtain a deseasonalized pair $(\tilde{Y}_{S,t}, \tilde{Y}_{W,t})$, (4) use the estimated seasonal function to obtain a pair $(Y_{S,t}, Y_{W,t})$ of spot electricity price and wind power production, (5) compute ρ_{t+1} using the Gaussian GAS update equation and (6) repeat steps (1)–(5). Using this procedure one day at a time, we can construct spot electricity price series and wind power production series; and by repeating the process many times, an empirical distribution is produced. Such an empirical distribution is shown in Fig. A.5.

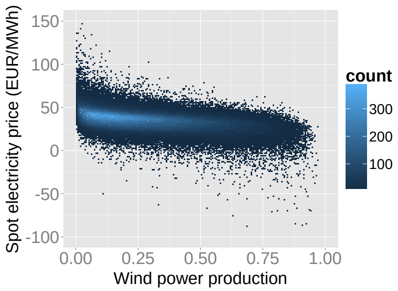


Fig. A.5: Simulated joint distribution for the daily spot electricity prices and the wind power production in December 2013. The results are based on 10,000 simulations (for each day of December) and a Gaussian GAS model for the dependence structure.

Fig. A.5 illustrates the simulated conditional joint distribution for December 2013 obtained by simulating 10,000 random paths for a one-month horizon. Note that although we have chosen a Gaussian copula model for the dependence structure, the marginal distributions were chosen to be a skewed generalized error distribution and a skewed *t* distribution for the wind power production and spot electricity prices, respectively. Therefore, the resulting joint distribution is not bivariate normal; as illustrated in Fig. A.5, the simulated distribution exhibits asymmetry and heavy tails.

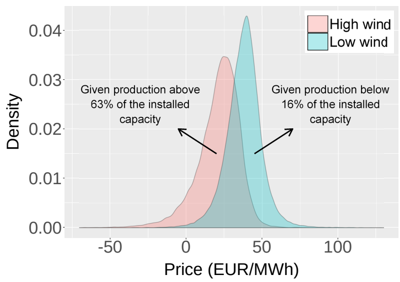
We will now use our model to study how different wind scenarios affect the distribution of prices. To this end, we perform one-month ahead simulations for all OOS months, i.e. a total of 24 months. Due to the elevated computational cost, we do not re-estimate the parameters of our joint model; we do however enlarge the conditioning set one month at the time. In Figs. A.6(a)–(b), we display simulated empirical price distributions conditional on different levels of low/high wind scenarios. The simulations are grouped into winter (Dec., Jan., Feb.) and summer (Jun., Jul., Aug.) months. To define what a low/high wind scenario is during winter, we have considered the 20% and 80% quantiles of our actual OOS wind power production data during the specified winter months; the same procedure was followed for the summer months. For both the winter and the summer period, we observe that the different wind scenarios shift the simulated price distributions. Moreover, the simulated distributions are left-skewed for the high wind scenarios (the skewness parameter is -1.98 for the winter months and -1.03 for the summer months), implying that extreme low prices are more likely than extreme high prices. For the low wind scenarios, the estimated distributions are right-skewed (the skewness is 0.64 and 1.06 for the winter and summer months respectively), thus implying the opposite compared to the high wind cases. We also notice that the low/high wind scenarios push the price distributions further apart for the winter months than the summer months, which we have confirmed by measuring the Kullback-Leibler distance between distributions. This can be explained by the fact that during summer periods, the dependence between electricity prices and wind power production is weaker than during winter periods, as earlier illustrated in Fig. A.4. All these features are present when performing the same calculations on the actual data, which we show in Figs. A.6(c)–(d), confirming that our empirical model captures the dynamics between daily spot electricity prices and wind power production.

5 Application to pricing and risk management

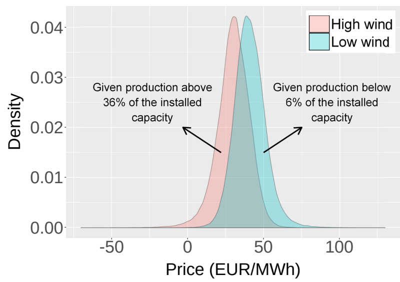
In the following we present applications of the proposed joint model for spot electricity prices and wind power production. We start by consider an energy trading company that enters into agreements with wind power producers, where a predetermined fixed price R is paid for the fluctuating wind power production. Since the production will first become known through the delivery period of the agreements, these products imply a volumetric risk. Furthermore, we assume that the trading company sells the production it receives from the wind power producers on the day-ahead market, at a spot price we denote by S. Hence, the company will also be exposed to price risk. In the remaining of this section, we will refer to such agreements as fixed price for fluctuating wind power production agreements. With such a formulation, we can express the profit of the trading company as

$$\sum_{t=T_1}^{T_2} Q_t(S_t - R), \tag{A.10}$$

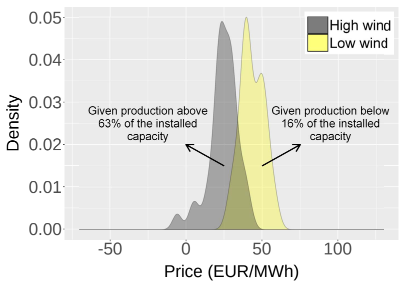
where time is measured in days, Q_t is the wind power production at time period t, S_t is the daily spot electricity price valid at t, and R is a fixed price set at the inception of the contract, which we denote t_0. Furthermore, the contract length spans from T_1 to T_2, where $t_0 < T_1 \leq T_2$. We note that to participate in the day-ahead electricity auction market, buy or sell bids have to be made to the exchange one day before delivery takes place. By working with the payoff in Eq. (A.10), we implicitly assume that the quantity we bid



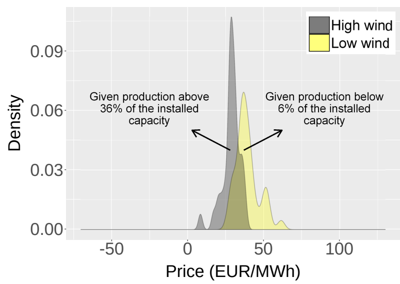
(a) Simulated prices: winter period



(b) Simulated prices: summer period



(c) Actual prices: winter period



(d) Actual prices: summer period

Fig. A.6: Distributions for the daily spot electricity prices during winter and summer months, for the out-of-sample period 01/01/2013 to 31/12/2014, under the assumption of high and low wind power production. The simulated predictive distributions are based on 10,000 one-month-ahead simulations, using a Gassian GAS model for the dependence structure. Fig. A.6(c) and Fig. A.6(d) are based on 37 observations. To define the percentage corresponding to high/low wind scenarios during winter and summer, we used the 0.20 and 0.80 quantiles of the actual out-of-sample wind power production data.

one day before equals the actual wind power production, i.e.

$$Q_t = \mathbb{E}_{t-1}[Q_t],$$

where $\mathbb{E}_{t-1}[Q_t]$ denotes the expectation at time $t-1$ for the production at time t. Thus, we assume no balancing risk.

What differentiates the product described above with payoff given in Eq. (A.10) from a standard forward contract is the production uncertainty associated with the former, and hence the presence of an additional risk due to the correlation between S and Q. If we express the price R in terms of the forward price F, Eq. (A.10) becomes

$$\sum_{t=T_1}^{T_2} Q_t(S_t - (F - c)), \tag{A.11}$$

where $F \equiv F(t_0, T_1, T_2)$ denotes the forward price at time t_0, for the delivery period from T_1 to T_2 and $c \equiv c(t_0, T_1, T_2)$ denotes the compensation that is to be subtracted from the forward price due to the negative correlation between prices and volume. So c can be thought of as the price of correlation risk. The fair value of c can be obtained by the usual practice of setting the discounted conditional expectation of the payoff given in Eq. (A.11) equal to zero.[14] To ease the presentation, we will assume a risk-free rate of zero, thus obtaining:

$$\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\sum_{t=T_1}^{T_2} Q_t(S_t - (F - c))\right] = 0, \tag{A.12}$$

$$c = F - \frac{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\sum_{t=T_1}^{T_2} Q_t S_t\right]}{\mathbb{E}_{t_0}^{\mathbb{Q}}\left[\sum_{t=T_1}^{T_2} Q_t\right]}. \tag{A.13}$$

With our framework, an estimate for c can easily be obtained by performing Monte Carlo simulations from the proposed copula model. However, this estimate will reflect the price of correlation risk under the physical or objective measure $\mathbb{P}$, since the model is fitted to historical spot electricity price and wind power production data. According to Eqs. (A.12) and (A.13), the expectations must be taken under a pricing measure $\mathbb{Q}$, that will reflect the risk premium charged by, in our context, the energy trading company offering the "insurance" to the wind power producer. Following [38], the pricing measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$, but needs not be an equivalent *martingale* measure due to the non-storability of our underlying "assets". Since neither electricity nor

[14]Risk preferences could easily be included by e.g. introducing a simple volumetric risk aversion rule like in [37].

wind can be stored, they are not tradable assets in the classical sense. This implies that the spot–forward relation, for example, cannot be derived based on a buy-and-hold hedging argument. Instead, the usual practice is to simply define the forward price as the conditional expectation of the spot electricity price under the risk-neutral probability measure $\mathbb{Q}$, thereby turning the discounted spot price into a martingale (see [39] and [40]). Indeed, by defining

$$F(t_0, T_1, T_2) = \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\frac{1}{T_2 - T_1 + 1} \sum_{t=T_1}^{T_2} S_t \right],$$

for the case of electricity, one can compute the implied market price of risk by considering the difference between quoted forward prices in the market and forward prices obtained by simulation with our model under $\mathbb{P}$.[15] In theory, the same could be done to estimate the risk premium associated with wind, however forwards with wind index as underlying are not currently traded in most European energy markets – and if they are, they are highly illiquid.

The fact that our setting is a bivariate one complicates the question of measure change even further, since aside from the marginal behavior of spot electricity price and wind power production under $\mathbb{Q}$, implied information regarding the market price of dependency risk must also be provided. A parametrization of this is not straightforward; in fact, the discussion can become quite extensive in the context of copulas and incomplete markets. Such a discussion is outside the scope of this paper, and we refer instead to [45] for more details. Moreover, even if a theoretical procedure to calibrate the market price of dependency risk were to be established, the lack of exchange-traded instruments written on spot times wind would impede applying this in practice.

In light of the above discussion, we turn to the rational expectation hypothesis, which is a valid choice and a common assumption in this context (see e.g. [20], [9], and [7]). This implies that we set $\mathbb{P} = \mathbb{Q}$, i.e. set the market price of risk to zero. Since we suspect a measure change to yield different prices, but not to alter the overall conclusions in our following empirical analysis, we find this assumption to be a reasonable one.

According to the payoff in Eqs. (A.10) or (A.11), it is clear by now that we are dealing with two sources of risk simultaneously: one is related to price uncertainty, and the other is related to production uncertainty; and since the market is incomplete, a perfect hedge cannot be performed. However, the price risk can be hedged. Here, we construct a simple hedging portfolio by taking a short position in a quantity H^* of standard forward power contracts. We assume that the hedge is static and performed at time t_0. The payoff of

[15] For further discussions and empirical studies regarding pricing in electricity markets we refer to [41], [42], [43], and [44].

the hedge for the entire delivery period is given by

$$H^* \left(\mathbb{E}_{t_0}^{\mathbb{Q}} \left[\frac{1}{T_2 - T_1 + 1} \sum_{t=T_1}^{T_2} S_t \right] - \frac{1}{T_2 - T_1 + 1} \sum_{t=T_1}^{T_2} S_t \right),$$

or in a compact form

$$H^*(F - \overline{S}),$$

where F denotes the same forward price as in Eq. (A.11), and $\overline{S}$ denotes the average day-ahead electricity price for the same delivery period. To obtain H^*, we fix c to its value obtained from Eq. (A.13) and follow the standard procedure of minimizing the variance of the portfolio payoff:

$$\min_{H^*} \mathbb{V}_{t_0} \left[\sum_{t=T_1}^{T_2} \tilde{Q}_t (S_t - (F - c)) + H^*(F - \overline{S}) \right]. \tag{A.14}$$

In Eq. (A.14), $\tilde{Q}_t = 24 \cdot Q_t \cdot \Lambda$, with Λ being the total installed capacity under the agreement that pays out a predetermined fixed price in return for the fluctuation wind power production. Since Q_t corresponds to daily wind power production relative to the total installed capacity in the entire DK1 price area, we need to transform this number to daily wind power production measured in MWh corresponding to the total installed capacity that the energy trading company actually has under agreement. By performing this transformation, we imply that our joint model is a good representation on a smaller scale. This is a realistic assumption as long as the energy trading company manages a portfolio of diversified wind turbines in terms of type and location. Solving for H^* in Eq. (A.14) yields

$$H^* = \frac{\mathbb{C}\mathrm{ov}_{t_0} \left[\overline{S}, \sum_{t=T_1}^{T_2} \tilde{Q}_t S_t \right] - (F - c) \mathbb{C}\mathrm{ov}_{t_0} \left[\overline{S}, \sum_{t=T_1}^{T_2} \tilde{Q}_t \right]}{\mathbb{V}_{t_0} \left[\overline{S} \right]}. \tag{A.15}$$

It is clear that by hedging a quantity that is equal to H^*, we are protected on average and not against worst case scenarios, such as the combination of extremely low prices / high wind power production, which is a probable outcome in the DK1 price area. We could remedy the situation to a large extent by adding options to our portfolio, however this is outside the scope of the present paper. Work related to the optimal hedging of volumetric risk associated with wind power production is, to the best of our knowledge, not yet available. However, energy related discussions regarding the hedging of volumetric risk associated with consumers' load are presented in e.g. [8] and [46], where many of the ideas can be transferred to our application. Nonetheless, our simple hedge is actually realistic since the market for options is very illiquid in DK1.

5.1 Example 1

Having developed a joint model for day-ahead electricity prices and wind power production, we can perform Monte Carlo simulations and use Eq. (A.13) to find the fair fixed price/compensation of a contract with any given specifications. Assume that we stand on the last trading day of November 2013, denoted t_0, and wish to find the fixed price for a front month contract, namely a December 2013 contract. Given all information available up to and including the valuation date t_0, we perform 10,000 simulations for price and quantity from our proposed joint model, where for each simulation we keep a path of length 31 (since we work with daily data) corresponding to the number of days in December. We note that we work with a fixed estimation window corresponding to the IS period, but enlarge the filtration, conditioning on the information up to and including the valuation date t_0. The contract specifications and results are summarized in Table A.7, and we see that due to the negative correlation between prices and production, the compensation c that is to be subtracted from the forward price equals 3.24 EUR/MWh.

In addition to calculating the fixed price of a contract with fluctuating wind power production, we can extract information from the performed simulations that can be useful in a risk management context. We assume that agreements corresponding to an installed capacity of 500 MW are entered into on the last trading day of November 2013, with delivery December 2013. The price of a standard forward contract is fixed to its estimated value of 35.26 EUR/MWh, and the price of an agreement with a fluctuating wind power production is set to 32.02 EUR/MWh cf. Table A.7. Given these specifications, we estimate the distribution of the portfolio payoff (see Fig. A.7) and calculate the 5% Value-at-Risk (see Table A.7) in two cases: One where the portfolio includes a price hedge, and one without a price hedge. When covering our price exposure in the forward market by assuming a short position corresponding to a quantity of H^* forwards, we observe that the variance of the profit distribution reduces significantly. In this example, the 5% Value-at-Risk is reduced from approximately EUR 1.1 million to EUR 0.5 million. It is also important to notice that the profit distribution is in both cases asymmetric, with a heavy-tail to the left, translating to the fact that expected losses are greater than expected gains.

Revisiting the issue of pricing and considering the profit distributions in Fig. A.7, alternative approaches to that of performing a measure change can be applied. An example can be to consider an a priori given 5% Value-at-Risk level that is acceptable, and solve for the correlation risk premium that satisfies this level.

To stress the effect of correlation on the profit distribution, we perform additional simulations, where all but the copula model remains unchanged. Specifically, we assume the independence copula and thus a zero compen-

Contract information			
	Time of valuation	t_0	29/11/2013
	Contract length	T_1 to T_2	01/12/2013 to 31/12/2013
Pricing results			
	Simulated forward price	$F \equiv F(t_0, T_1, T_2)$	35.26 EUR/MWh
	Price of correlation risk	$c \equiv c(t_0, T_1, T_2)$ cf. Eq.(A.13)	3.24 EUR/MWh
	Fixed price for fluctuating wind power production	$R \equiv R(t_0, T_1, T_2)$	32.02 EUR/MWh
Risk management results (500 MW installed capacity under agreement)			
	5 % VaR without price hedge	See Fig. A.7	1,099,248 EUR
	5 % VaR with price hedge	See Fig. A.7	465,485 EUR

Table A.7: Simulation results for a December 2013 contract, with valuation date 29 November 2013, i.e. one business day before start delivery of the contract. The results are based on 10,000 simulations and using a Gassian GAS model for the dependence structure.

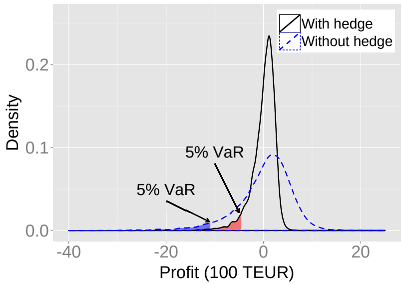
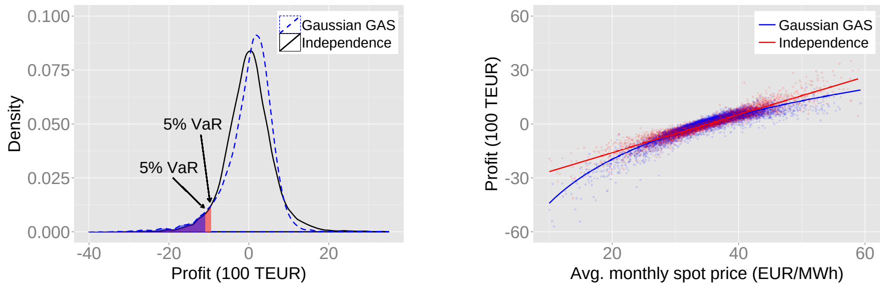


Fig. A.7: Profit distributions for a December 2013 contract. The results are based on 10,000 simulations of price and quantity, using a Gaussian GAS model for the dependence structure. The forward price is fixed to 35.26 EUR/MWh, the compensation is fixed to 3.24 EUR/MWh and the total installed capacity of the portfolio equals 500 MW. The variance minimizing hedge quantity H^* is obtained by performing the calculation in Eq. (A.15).

sation, instead of the Gaussian GAS model which we have established reflects the reality to a much greater extent. Fig. A.8(a) illustrates the estimated profit distributions of the portfolio (with no hedge), and shows that the negative correlation implies a distribution that is more asymmetric. If prices and production were independent, we estimate a 5% Value-at-Risk of EUR 0.93 million corresponding to a reduction of approximately 15% compared to the 5% Value-at-Risk of EUR 1.1 million we obtain with the Gaussian GAS copula. Assuming independence would thus lead to an underestimation of risk. We also display the average spot electricity price for the period of the contract as a function of the estimated profit in Fig. A.8(b). Under independence, we observe that the payoff becomes linear, and hence forwards would suffice as hedging instruments. Under negative dependence, the payoff becomes non-linear, emphasizing the need for options in the hedging portfolio. Furthermore, we observe that a larger profit (smaller loss) can be obtained if prices and production are independent as we move away from the mean average price of 35.26 EUR/MWh. This is also supported by Fig. A.8(a), where we observe that the negative correlation implies that a smaller probability is assigned to large profits, and a higher probability is assigned to large losses.

5.2 Example 2

In Section 3.4, we have established that some of the fitted time-varying copula models are superior to the constant ones, see e.g. Table A.6. Here, we wish to investigate if this also holds when comparing the actual profits or losses generated with different copula models. For this, we consider the OOS period corresponding to the years 2013 and 2014. We assume the following trading strategy: On the last trading day of each month (Dec. 2012 - Nov. 2014), we enter into front month agreements with wind power generators, where



(a) Profit distributions for a portfolio of fixed price December 2013 contracts under different assumptions for the dependence structure.

(b) Estimated payoffs for a portfolio of fixed price December 2013 contracts, as a function of the avg. monthly spot electricity price.

Fig. A.8: Illustration of the importance of correlation in the analysis of profit. The results are based on 10,000 Monte Carlo simulations with a Gaussian GAS copula (c is fixed to 3.24 EUR/MWh) and the independence copula (c is fixed to 0 EUR/MWh), respectively. The total installed capacity of the portfolio is set to 500 MW, and the same marginal models for prices and wind power production are used.

a fixed price is paid for the fluctuating wind power production. The total installed capacity of each monthly portfolio is fixed to 500 MW. We perform 10,000 simulations from joint models with the different copula specifications that we wish to compare against each other (marginal models are kept unchanged), and estimate compensations c and hedge quantities H^* for each month at a time using Eqs. (A.13) and (A.15). For each monthly portfolio, we then calculate the realized profit using the actual daily electricity prices, actual daily wind power production[16] and actual forward prices.

For clarity, let us consider a concrete example: We stand on the last trading day of December 2012, and wish to enter into fixed price agreements with fluctuating wind power production for the January 2013 month. To enter the contract, we first estimate the fixed price that we are willing to pay for the production that we will receive during January. Since we also perform a hedge in the forward market, we estimate the quantity of forwards we are to short. In this example, we will use a constant Clayton copula to describe the dependence between prices and wind power production, and hence we obtain an estimated compensation denoted by $\hat{c}^{\text{Clayton}}_{t_0,\text{Jan}}$ and an estimated hedge quantity $\hat{H}^{*,\text{Clayton}}_{t_0,\text{Jan}}$. On the last trading day of December 2012, we can ob-

[16]The actual daily wind power production is given in % for the entire price area, but only a subset of the existing wind turbines in DK1 is part of our portfolio. Therefore, we note that the realized profit we calculate is an approximation; We obtain the actual production of the wind turbines under agreement by multiplying the actual daily wind power production for the entire price area with the assumed installed capacity of the portfolio of 500 MW and 24 hours.

serve the actual forward price $F_{t_0}^{\text{Obs}}$, and thus the fixed price we offer the wind power producers is

$$\hat{R}_{t_0,\text{Jan}}^{\text{Clayton}} = F_{t_0}^{\text{Obs}} - \hat{c}_{t_0,\text{Jan}}^{\text{Clayton}}.$$

By the end of January 2013, we will also have observed the actual daily spot electricity prices S^{Obs} and the actual daily wind power production Q^{Obs} for the DK1 price area. With this information, we can now approximate the actual profit resulting from the trades we have performed:

$$\text{Actual profit}_{\text{Jan}} = \underbrace{\sum_{t=T_1}^{T_2} \tilde{Q}_t^{\text{Obs}} (S_t^{\text{Obs}} - \hat{R}_{t_0,\text{Jan}}^{\text{Clayton}})}_{\text{Agreement payoff}} + \underbrace{\hat{H}_{t_0,\text{Jan}}^{*,\text{Clayton}} (F_{t_0}^{\text{Obs}} - \overline{S}^{\text{Obs}})}_{\text{Hedge payoff}}$$

where $t_0 = 31/12/2012$, $T_1 = 01/01/2013$, $T_2 = 31/01/2013$ and $\tilde{Q}^{\text{Obs}}$ is the approximation

$$\tilde{Q}_t^{\text{Obs}} = Q_t^{\text{Obs}} \cdot 24 \text{ (h)} \cdot 500 \text{ (MW)}.$$

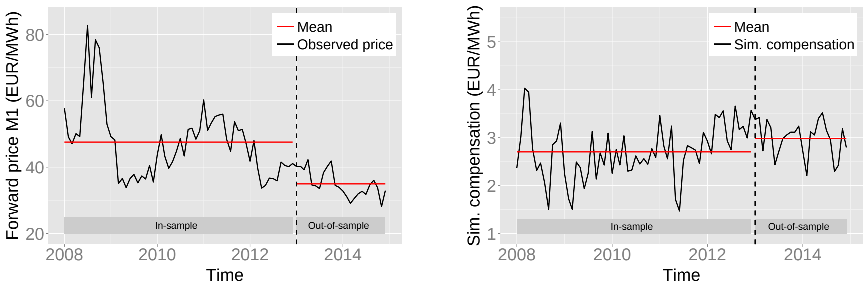
The results obtained by performing the above calculations for all OOS months with different copula specifications are presented in Table A.8. The numbers show that the joint model with a Gaussian GAS copula provides the highest (lowest) monthly profit (loss) in 15 out of the 24 months, corresponding to 62.50%. Considering the second column block of Table A.8, we see that it is indeed the Gaussian GAS and the Joe-Frank GAS that yield the lowest losses in average, which supports the results we obtained in Section 3.4. Hence, allowing for time variation in a suitable copula model is beneficial. The constant Clayton specification performs the poorest, generating the largest average loss. This is again in accordance with earlier findings, where we have established that the constant Clayton specification is not suitable for the dependence of prices and wind power production, and also least suitable among the copula models we consider in Table A.8. The time-varying Gaussian and Joe-Frank copulas outperform the other copulas since they are able to capture the increasingly negative correlation we observe towards the last years of our sample (see Fig. A.4); and thus, they are able to generate larger compensations. For instance, the constant Gaussian copula yields an average compensation for the OOS period of 2.69 EUR/MWh, while the Gaussian GAS copula yields a value of 2.98 EUR/MWh.

Lastly, we illustrate in Fig. A.9 the evolution of actual forward prices and also the evolution of compensations estimated with our proposed joint model for electricity prices and wind power production, i.e. the one with the Gaussian GAS copula specification for the dependence structure.

Overall, compensations amount to an increasing percentage of the forward price during the period of our study. Clearly, this is mainly due to the

	Highest profit (lowest loss) per month for the OOS period		Realized average profit for the OOS period (EUR/MWh)	
	Constant	Time-varying (GAS)	Constant	Time-varying (GAS)
Gaussian	4.16%	**62.50%**	-0.9144	**-0.6103**
Gumbel	16.67%	0.00%	-0.8693	-0.7961
Joe-Frank	0.00%	16.67%	-1.0294	-0.6666
Clayton	0.00%	-	-1.1465	-

Table A.8: OOS model comparisons based on realized monthly profit/loss. In the first column block, we calculate how often each copula model yields the lowest monthly loss or the highest monthly profit. The second column block presents the realized average profit/loss (in EUR/MWh) for selected copula models, obtained by dividing the total realized cash flow for the period by the total realized wind power production. All results are based on the same trading strategy and 10,000 simulations.



(a) Actual forward prices for a front month (M1) contract, valid the last trading day before delivery start. Total of 84 prices, one for each month in our IS and OOS sample.

(b) Simulated compensations for a front month contract, valued the last trading day before delivery start. The results are based on 10,000 simulations, with a Gaussian GAS models for the dependence structure.

Fig. A.9: Evolution of actual forward prices and estimated compensations.

decreasing tendency in forward prices, but also due to the slight increase in compensations if we consider the IS and OOS average compensations. The slight increase in compensations can be justified by the increasing installed capacity of wind power that Denmark has experienced over the past years - and hence the stronger dependence between wind power production and electricity prices. This also explains the decrease in forward prices, but only to a small extent; the major contributing factor here has been the decreasing raw material prices. The reduction in forward price due to the correlation risk amounts to an average of 7%, and can reach as high as 11%. A similar conclusion is reached by [3], where the authors study the market value of wind power at different locations in Germany, and show that this value is reduced compared to the average spot price as a result of increasing wind power penetration.

6 Conclusion

This work concentrates on the dependency between daily spot electricity prices and wind power production, and its role regarding the pricing and the risk distributions associated with contracts exposed to both price and volumetric risk. The analysis is carried out on data from the Danish power market, which is characterized by a high penetration of wind power in the system. We propose a copula approach since we wish to concentrate on the dependence in more detail. We employ marginal models of the ARMA–GARCH type and parametric error distributions for each individual variable, and then link the innovations through various constant and time-varying copulas. Based on statistical tests concerning copula selection, we choose a time-varying Gaussian copula as our preferred specification for the dependence structure. By performing Monte Carlo simulation studies, we are able to visualize the joint empirical distribution implied by our model, and see how this deviates from the Gaussian benchmark. Also, we study the distribution of prices conditional on different levels of wind power penetration, and show that prices decrease (increase), on average, at times of high (low) levels of wind power production; the shape of the conditional distribution of prices is also affected by the different levels of wind power production. These findings confirm what previous studies concerned with the impact of wind power – or predicted wind power penetration – on electricity prices have shown (e.g. [2], [1]).

We apply the developed empirical model in the context of an energy trading company offering wind power producers a predetermined fixed price for their fluctuating wind power production. We find that the correlation risk premium that the energy trading company should charge when entering such agreements is significant, amounting to 7% of the price of a standard forward power contract on average. Furthermore, our results indicate that the choice of copula impacts the price of correlation risk: An out-of-sample study based on comparing realized profits generated by different copulas shows that introducing time-variation in the copula model is beneficial. When considering the profit distribution, we find that under independence, the risk is underestimated. Additionally, we show that a simple hedge in the forward market can reduce e.g. the 5% Value-at-Risk of the profit distribution significantly. However, due to the non-linearity of profit, options should be included in the hedging portfolio in order to reduce the risk even further; this could be an interesting subject for further research.

Finally, although our empirical study concentrates on the Danish power market, the mechanism of spot price formation in e.g. other European electricity markets is also based on matching supply and demand. Further, wind power production has a very low marginal cost, ensuring that it will always be represented in the merit order stack. Due to the physical conditions upon which the day-ahead electricity markets are based, we believe that the proposed modeling framework is relevant and can be applied to other electricity markets that, like Denmark, rely heavily on wind power production. Such extensions are left for future research.

A Properties of selected copula models

A.1 Elliptical copulas

A bivariate elliptical copula is defined as

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)),$$

where $u_1, u_2 \in [0, 1]$. The elliptical copulas we consider in this paper are the Gaussian copula and the Student t copula. In the case of the Gaussian copula, F corresponds to the bivariate standard normal *cdf*, and F_1^{-1} and F_2^{-1} denote the inverse of the univariate standard normal *cdf*. In the case of the Student t copula, F corresponds to the bivariate Student t *cdf*, and F_1^{-1} and F_2^{-1} denote the inverse of the univariate Student t *cdf*. Properties of these copulas are summarized in Table A.9.

A.2 Archimedian copulas

A bivariate Archimedian copula is defined as

$$C(u_1, u_2) = \phi^{-1}(\phi(u_1) + \phi(u_2)),$$

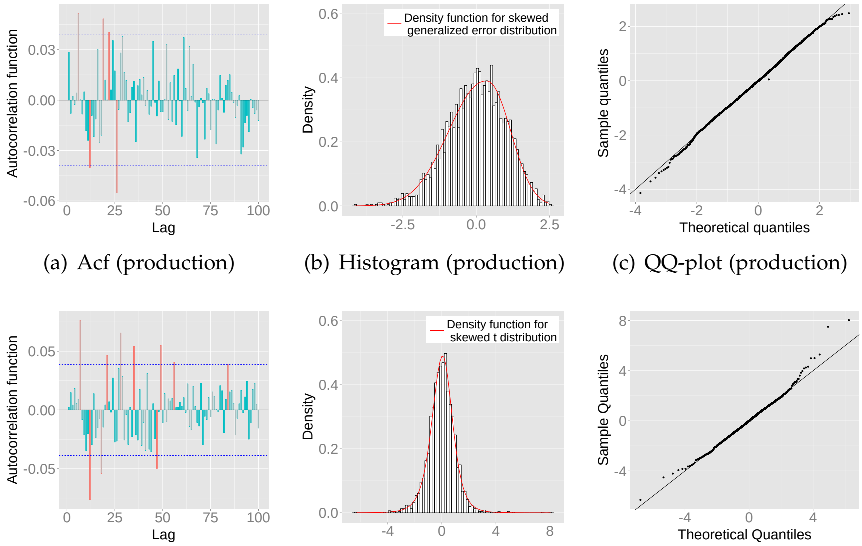
where $u_1, u_2 \in [0, 1]$ and $\phi : [0, 1] \to [0, \infty)$ is a generator function satisfying that ϕ^{-1} is monotone on $[0, \infty)$. The Archimedian copulas we consider in

	Generator function	Parameter range	Symmetric	Neg. dep	Tail dependence
Elliptical family					
Gaussian	-	$\rho \in (-1, 1)$	Yes	Yes	0
Student t	-	$\rho \in (-1, 1)$, $\nu > 2$	Yes	Yes	$2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$
Archimedian family					
Clayton	$\frac{1}{\theta}(u^{-\theta} - 1)$	$\theta > 0$	No	No	$(2^{-\frac{1}{\theta}}, 0)$
Gumbel	$(-\log u)^{\theta}$	$\theta \geq 1$	No	No	$(0, 2 - 2^{\frac{1}{\theta}})$
Joe-Clayton	$(1 - (1-u)^{\theta})^{-\delta} - 1$	$\theta \geq 1, \delta > 0$	No	No	$(2^{-\frac{1}{\delta}}, 2 - 2^{\frac{1}{\theta}})$
Joe-Frank	$-\log\left(\frac{1-(1-\delta u)^{\theta}}{1-(1-\delta)^{\theta}}\right)$	$\theta \geq 1, \delta \in (0, 1]$	No	No	$(0, 0)$

Table A.9: Overview over the properties of selected copulas. Copula features and notation coincide with the R-package **CDVine**. t denotes the univariate Student t pdf with $\nu + 1$ degrees of freedom. We note that we consider the symmetrized Joe-Clayton copula proposed in [13], which is an even more flexible version of the Joe-Clayton copula, since is allows for both symmetry and asymmetry. See [13] for details regarding this copula.

this paper are Clayton and Gumbel, and also the combinations Joe-Frank and a symmetrized version of Joe-Clayton. Properties of these copulas are summarized in Table A.9.

B Additional figures



(a) Acf (production) (b) Histogram (production) (c) QQ-plot (production)

(d) Acf (price) (e) Histogram (price) (f) QQ-plot (price)

Fig. A.10: Diagnostics for marginal models for spot electricity price and wind power production. Figs. A.10(a) - A.10(c) display the autocorrelation function, histogram and quantile plot for the standardized residuals resulting from the marginal model for wind power production. Correspondingly, Figs. A.10(d) - A.10(f) display the same diagnostics for the standardized residuals resulting from the marginal model for spot electricity prices.

Acknowledgments

The authors would like to express their gratitude to Rikke Preisler Vilstrup for her contributions to this paper through a mutual unpublished master thesis project, and Thomas Aalund Fredholm, Christian Sønderup and Jakob Vive Munk at Neas Energy for posing the problem and providing helpful comments and suggestions. The authors also thank the referees for providing constructive criticism and suggestions that enhanced the quality of this paper.

References

[1] T. Jónnson, P. Pinson, and H. Madsen, “On the market impact of wind energy forecasts,” *Energy Economics*, vol. 32, no. 2, pp. 313 – 320, 2010.

[2] L. Gelabert, X. Labandeira, and P. Linares, “An ex-post analysis of the effect of renewables and cogeneration on Spanish electricity prices,” *Energy Economics*, vol. 33, pp. 59 – 65, 2011.

[3] C. Elberg and S. Hagspiel, “Spatial dependencies of wind power and interrelations with spot price dynamics,” *European Journal of Operational Research*, vol. 241, no. 1, pp. 260 – 272, 2015.

[4] F. Paraschiv, D. Erni, and R. Pietsch, “The impact of renewable energies on EEX day-ahead electricity prices,” *Energy Policy*, vol. 73, pp. 196 – 210, 2014.

[5] R. I. McKinnon, “Future markets, buffer stocks, and income stabiliy for primary producers,” *Journal of Political Economy*, vol. 75, no. 6, pp. 844 – 861, 1967.

[6] G. Moschini and H. Lapan, “The hedging role of options and futures under joint price, basis, and production risk,” *International Economic Review*, vol. 36, no. 4, pp. 1025 — 1049, 1995.

[7] Y. Oum and S. S. Oren, “Optimal static hedging of volumetric risk in a competitive wholesale electricity market,” *Decision Analysis*, vol. 7, no. 1, pp. 107 – 122, 2010.

[8] ——, “VaR constrained hedging of fixed price load-following obligations in competitive electricity markets,” *Risk and Decision Anaysis*, vol. 1, no. 1, pp. 43 – 56, 2009.

[9] M. Coulon, W. B. Powell, and R. Sircar, “A model for hedging load and price risk in the Texas electricity market,” *Energy Economics*, vol. 40, pp. 976 – 988, 2013.

[10] K. Avdulaj and J. Barunikl, “Are benefits from oil–stocks diversification gone? New evidence from a dynamic copula and high frequency data,” *Energy Economics*, vol. 51, pp. 31 – 44, 2015.

[11] I. A. De Lira Salvatierra and A. J. Patton, “Dynamic copula models and high frequency data,” June 2013, Economic Research Initiatives at Duke (ERID), Working Paper No. 165. Available at SSRN: http://dx.doi.org/10.2139/ssrn.2284235.

[12] A. Dias and P. Embrechts, "Dynamic copula models for multivariate high-frequency data in finance," 2004, Warwick Business School, Finance Group, Working Paper.

[13] A. J. Patton, "Modelling asymmetric exchange rate dependence," *International Economic Review*, vol. 47, no. 2, pp. 527 – 556, 2006.

[14] X. Wen, Y. Wei, and D. Huang, "Measuring contagion between energy market and stock market during financial crisis: A copula approach," *Energy Economics*, vol. 34, pp. 1435 – 1446, 2012.

[15] A. Sklar, "Fonctions de répartition à n dimensions et leurs marges," *Publications de l'Institut de Statistique de L'Université de Paris*, vol. 8, pp. 229 – 231, 1959.

[16] U. Cherubini and E. Luciano, "Bivariate option pricing with copulas," *Applied Mathematical Finance*, vol. 9, pp. 69 – 85, 2002.

[17] P. Embrechts, A. J. McNeil, and D. Straumann, "Correlation and dependence in risk management: Properties and pitfalls," in *Risk Management: Value at Risk and Beyond*, M. A. H. Dempster, Ed. Cambridge University Press, 1999, pp. 176 – 223.

[18] A. J. Patton, "Copula methods for forecasting multivariate time series," in *Handbook of Economic Forecasting*, G. Elliott and A. Timmermann, Eds. Elsevier B.V., 2013, vol. 2B, pp. 899 – 960.

[19] C. Alexander, "Correlation in crude oil and natural gas markets," in *Managing Energy Price Risk: The New Challenges and Solutions*, V. Kaminski, Ed. Risk Publications, 2004.

[20] F. E. Benth and P. C. Kettler, "Dynamic copula models for the spark spread," *Quantitative Finance*, vol. 11, no. 3, pp. 407 – 421, 2011.

[21] C. González-Pedraz, M. Moreno, and I. Peña, "Portfolio selection with commodities under conditional copulas and skew preferences," *Quantitative Finance*, vol. 15, no. 1, pp. 151 – 170, 2015.

[22] H. Joe, *Multivariate Models and Dependence Concepts*, 1st ed. Chapman and Hall, 1997, ISBN: 978-0412073311.

[23] R. B. Nelsen, *An Introduction to Copulas*. Springer, 1999.

[24] T. Bollerslev, "Glossary to ARCH (GARCH)," School of Economics and Management, University of Aarhus, CREATES Research Papers 2008-49, Sep 2008. [Online]. Available: https://ideas.repec.org/p/aah/create/2008-49.html

[25] D. B. Nelson and C. Q. Cao, "Inequality constraints in the univariate GARCH model," *Journal of Business & Economic Statistics*, vol. 10, no. 2, pp. 229 – 235, 1992.

[26] J.-D. Fermanian and M. Wegkamp, "Time dependent copulas," *Journal of Multivariate Analysis*, vol. 110, pp. 19 – 29, 2012.

[27] D. Creal, S. J. Koopman, and A. Lucas, "Generalized autoregressive score models with applications," *Journal of Applied Econometrics*, vol. 28, no. 5, pp. 777 – 795, 2013.

[28] D. N. Politis and J. P. Romano, "The stationary bootstrap," *Journal of the American Statistical Association*, vol. 89, no. 428, 1994.

[29] D. N. Politis and H. White, "Automatic block-length selection for the dependent bootstrap," *Econometric Reviews*, vol. 23, no. 1, pp. 53 – 70, 2004.

[30] D. N. Politis, H. White, and A. J. Patton, "Correction: Automatic block-length selection for the dependent bootstrap," *Econometric Reviews*, vol. 28, no. 4, pp. 372 – 375, 2009.

[31] D. Berg, "Copula goodness-of-fit testing: An overview and power comparison," *European Journal of Finance*, vol. 15, no. 7-8, pp. 675 – 701, 2009.

[32] C. Genest, B. Remillard, and D. Beaudoin, "Goodness-of-fit tests for copulas: A review and a power study," *Insurance: Mathematics and Economics*, vol. 44, no. 2, pp. 199 – 213, 2009.

[33] D. Rivers and Q. Vuong, "Model selection tests for nonlinear dynamic models," *Econometrics Journal*, vol. 5, pp. 1 – 39, 2002.

[34] C. G. H. Diks, V. Panchenko, and D. J. C. van Dijk, "Out-of-sample comparison of copula specifications in multivariate density forecasts," *Journal of Economic Dynamics and Control*, vol. 34, no. 9, pp. 1596 – 1609, 2010.

[35] R. Giacomini and H. White, "Tests of conditional predictive ability," *Econometrica*, vol. 74, no. 6, pp. 1545 – 1578, 2006.

[36] U. Schepsmeier and J. Stöber, "Derivatives and Fisher information of bivariate copulas," *Statistical Papers*, vol. 55, no. 2, pp. 525 – 542, 2014.

[37] S. P. Kolos and K. Mardanov, "Pricing volumetric risk," *Energy risk*, pp. 54 – 60, 2008.

[38] F. E. Benth, J. Šaltytė Benth, and S. Koekebakker, *Stochastic Modelling of Electricity and Related Markets*. World Scientific, 2008.

[39] F. E. Benth and T. Meyer-Brandis, "The information premium for non-storable commodities," *Journal of Energy Markets*, vol. 2, no. 3, pp. 111 – 140, 2009.

[40] F. E. Benth and J. Šaltytė Benth, *Modelling and Pricing in Financial Markets for Weather Derivatives*. World Scientific, 2012.

[41] F. E. Benth, R. Biegler-König, and R. Kiesel, "An empirical study of the information premium on electricity markets," *Energy Economics*, vol. 36, pp. 55 – 77, 2013.

[42] M. Burger, B. Klar, A. Muller, and G. Schindlmayr, "A spot market model for pricing derivatives in electricity markets," *Quantitative Finance*, vol. 4, no. 1, pp. 109 – 122, 2004.

[43] S. P. Kolos and E. I. Ronn, "Estimating the commodity market price of risk for energy prices," *Energy Economics*, vol. 30, no. 2, pp. 621 – 641, 2008.

[44] J. J. Lucia and E. S. Schwartz, "Electricity prices and power derivatives: Evidence from the Nordic Power Exchange," *Review of Derivatives Research*, vol. 5, pp. 5 – 50, 2000.

[45] U. Cherubini, E. Luciano, and W. Vecchiato, *Copula Methods in Finance*. John Wiley & Sons, 2004.

[46] Y. Oum, S. S. Oren, and S. Deng, "Hedging quantity risks with standard power options in a competitive wholesale electricity market," *Naval Research Logistics*, vol. 53, no. 7, pp. 697 – 712, 2006.

Paper B

Wrong-Way Risk adjusted exposure: Analytical Approximations for Options in Default Intensity Models

Damiano Brigo[1]
Thomas Hvolby[2]
Frédéric Vrins[3]

The paper has been submitted to the
Springer conference proceedings Innovations in Insurance, Risk- and Asset Management.

[1]Department of Mathematics, Imperial College London
[2]Department of Mathematical Sciences, Aalborg University
[3]Louvain School of Management & CORE, Université catholique de Louvain

Abstract

We examine credit value adjustment (CVA) estimation under wrong-way risk (WWR) by computing the expected positive exposure (EPE) under an equivalent measure as suggested in [1], adjusting the drift of the underlying for default risk. We apply this technique to European put and call options and derive the analytic formulas for EPE under WWR obtained with various approximations of the drift adjustment. We give the results of numerical experiments based on 4 parameter sets, and supply figures of the CVA based on both of the suggested proxys, comparing with CVA based on a 2D-Monte Carlo scheme and Gaussian Copula resampling. We also show the CVA obtained by the formulas from Basel III. We observe that the Basel III formula does not account for the credit-market correlation, while the Gaussian Copula resampling method estimates a too large impact of this correlation. The two proxies account for the credit-market correlation, and give results that are mostly similar to the 2D-Monte Carlo results.

1 Introduction

In this paper we aim at computing the credit valuation adjustment (CVA) expressions of European calls and puts under the Black-Scholes-Merton-Cox model, that is when the underlying stock follows GBM dynamics and the default is governed by a totally inaccessible stopping time corresponding to the first jump time of a Cox process. Specifically, we assume that the default intensity follows a CIR-process.

Let us consider a portfolio with maturity T and whose discounted price process is $\tilde{V}$. The CVA associated to such a portfolio traded with a counterparty whose recovery rate is R and default time is τ with survival (risk-neutral) probability curve $G(t) := \mathbb{Q}(\tau > t)$ is given by

$$\text{CVA} = -(1-R)\int_0^T \mathbb{E}^{\mathbb{Q}}\left[\tilde{V}_s^{+}\middle|\,\tau = s\right] dG(s), \tag{B.1}$$

where $x^+ := \max(x, 0)$. It has been shown in [1] that when the default time is modeled as the first jump's time of a Cox process, the "$\tau = s$" condition in the expectation in Eq. (B.1) — associated to market-credit dependency that is, to wrong-way risk — can be absorbed in the drift of the portfolio price process:

$$\text{CVA} = -(1-R)\int_0^T \mathbb{E}^{\mathbb{Q}^{\mathcal{C}_s}}\left[\tilde{V}_s^{+}\right] dG(s).$$

Here, $\mathcal{C}_t$ is a rolling numéraire corresponding to the default leg of a CDS offering protection in a small interval around t, and is not to be confused with the call option price at t, noted C_t. We refer the reader to [1] for more details about this technique.

We define the expected positive exposure (EPE) without taking wrong-way risk into account as the expectation in Eq. (B.1) without the condition. Thus the no-WWR EPE is simply the function $\text{EPE}^{\perp}(s) := \mathbb{E}^{\mathbb{Q}}\left[\tilde{V}_s^+\right]$ for $s \in [0, T]$. The EPE under wrong-way risk (referred to as the WWR EPE) is defined as

$$\text{EPE}(s) = \mathbb{E}^{\mathbb{Q}}\left[\tilde{V}_s^+ \,\middle|\, \tau = s\right] = \mathbb{E}^{\mathbb{Q}^{\mathcal{C}_s}}\left[\tilde{V}_s^+\right].$$

From Girsanov theorem, a $\mathbb{Q}$-Brownian motion on $[0, s]$ will become, under $\mathbb{Q}^{\mathcal{C}_s}$, a Brownian motion plus a drift. In particular, we note $\theta^s_\cdot$ the drift associated to the $\mathbb{Q}$-Brownian motion driving the exposure. Its analytical expression is derived explicitly in [1]. We now show that when this (stochastic) drift is approximated by a deterministic function $\theta(\cdot, s)$, the quantity $\mathbb{E}^{\mathbb{Q}^{\mathcal{C}_s}}\left[\tilde{V}_s^+\right]$ is available in closed form for calls and puts, leading to an analytical approximation for the CVA under wrong-way risk, and compare the effect on CVA of two approximations of this drift to the Monte Carlo set-up.

2 Call and put risk-neutral dynamics

We assume GBM dynamics for the stock under the risk-neutral measure $\mathbb{Q}$, with constant risk-free rate r and volatility $\sigma > 0$. Hence, denote by W a $\mathbb{Q}$-Brownian motion,

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

whose solution is

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t}.$$

Let us note C the price process of a European call option on the stock S with maturity T and strike K. Hence, using the Theta-Delta-Gamma relationship,

$$dC_t = \Theta_t dt + \Delta_t dS_t + \frac{1}{2}\Gamma_t d\langle S\rangle_t = rC_t dt + \sigma S_t \Delta_t dW_t,$$

and it is well-known that

$$\begin{aligned}
C_t &= S_t \Phi(d(T-t)) - Ke^{-r(T-t)}\Phi\left(d(T-t) - \sigma\sqrt{T-t}\right) \\
\Delta_t &= \Phi(d(T-t)) \\
d(s) &:= \frac{1}{\sigma\sqrt{s}}\left(\ln\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)s\right).
\end{aligned}$$

Let us note the time-t discounted value of any process $X = (X_t)_{t\geq 0}$ as $\tilde{X}_t := X_t e^{-rt}$. The discounted call price process $\tilde{C}$ can be written in terms of the discounted stock price process $\tilde{S}$:

$$\tilde{C}_t = \tilde{S}_t \Phi\left(d(t,T)\right) - Ke^{-rT}\Phi\left(d(t,T) - \sigma\sqrt{T-t}\right)$$
$$d(t,T) = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{\tilde{S}_t}{K} + rT + \frac{\sigma^2}{2}(T-t)\right),$$

where we have used that $\ln S_t = \ln \tilde{S}_t + rt$. Using $W_t \overset{(\mathbb{Q})}{\sim} \sqrt{t}Z$ where $Z \overset{(\mathbb{Q})}{\sim} \mathcal{N}(0,1)$, one obtains

$$\tilde{S}_t = S_0 e^{-\frac{\sigma^2}{2}t+\sigma W_t} \sim S_0 e^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}Z}$$
$$\Delta_t = \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{S_0}{K} + \left(r+\frac{\sigma^2}{2}\right)T - \sigma^2 t\right) + \frac{W_t}{\sqrt{T-t}}\right)$$
$$\sim \Phi\left(\underbrace{\frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{S_0}{K} + \left(r+\frac{\sigma^2}{2}\right)T - \sigma^2 t\right)}_{:=\alpha(t)} + \underbrace{\frac{\sqrt{t}}{\sqrt{T-t}}}_{:=\beta(t)}Z\right),$$

so that

$$\tilde{C}_t = S_0 e^{-\frac{\sigma^2}{2}t+\sigma W_t}\Phi\left(\alpha(t) + W_t/\sqrt{T-t}\right)$$
$$- Ke^{-rT}\Phi\left(\alpha(t) - \sigma\sqrt{T-t} + W_t/\sqrt{T-t}\right)$$
$$\sim S_0 e^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}Z}\Phi\left(\alpha(t) + \beta(t)Z\right)$$
$$- Ke^{-rT}\Phi\left(\alpha(t) - \sigma\sqrt{T-t} + \beta(t)Z\right).$$

A similar development yields the dynamics and the marginal distributions of the corresponding put

$$\tilde{P}_t = Ke^{-rT}\Phi\left(\sigma\sqrt{T-t} - \alpha(t) - W_t/\sqrt{T-t}\right)$$
$$- S_0 e^{-\frac{\sigma^2}{2}t+\sigma W_t}\Phi\left(-\alpha(t) - W_t/\sqrt{T-t}\right)$$
$$\sim Ke^{-rT}\Phi\left(\sigma\sqrt{T-t} - \alpha(t) - \beta(t)Z\right)$$
$$- S_0 e^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}Z}\Phi\left(-\alpha(t) - \beta(t)Z\right).$$

3 Expected Positive Exposures under No WWR

As $\tilde{C} \geq 0$ and $\tilde{P} \geq 0$, the expected (discounted) exposure corresponds to the expected *positive* (discounted) exposure. Hence,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\tilde{C}_t\right] &= S_0 e^{-\frac{\sigma^2}{2}t}\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma\sqrt{t}Z}\Phi\left(\alpha(t)+\beta(t)Z\right)\right] \\
&\quad - Ke^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\alpha(t)-\sigma\sqrt{T-t}+\beta(t)Z\right)\right] \\
&= S_0\Phi\left(\frac{\alpha(t)+\beta(t)\sigma\sqrt{t}}{\sqrt{1+\beta^2(t)}}\right) - Ke^{-rT}\Phi\left(\frac{\alpha(t)-\sigma\sqrt{T-t}}{\sqrt{1+\beta^2(t)}}\right) \\
\mathbb{E}^{\mathbb{Q}}\left[\tilde{P}_t\right] &= Ke^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(\sigma\sqrt{T-t}-\alpha(t)-\beta(t)Z\right)\right] \\
&\quad - S_0 e^{-\frac{\sigma^2}{2}t}\mathbb{E}^{\mathbb{Q}}\left[e^{-\sigma\sqrt{t}Z}\Phi\left(-\alpha(t)-\beta(t)Z\right)\right] \\
&= Ke^{-rT}\Phi\left(\frac{\sigma\sqrt{T-t}-\alpha(t)}{\sqrt{1+\beta^2(t)}}\right) - S_0\Phi\left(\frac{-\alpha(t)-\beta(t)\sigma\sqrt{t}}{\sqrt{1+\beta^2(t)}}\right),
\end{aligned}$$

where we have used

$$\mathbb{E}^{\mathbb{Q}}\left[e^{\eta Z}\Phi\left(\mu+\sigma Z\right)\right] = e^{\frac{\eta^2}{2}}\Phi\left(\frac{\mu+\sigma\eta}{\sqrt{1+\sigma^2}}\right).$$

It can be checked that $\mathbb{E}^{\mathbb{Q}}\left[\tilde{C}_t\right] = C_0$ and $\mathbb{E}^{\mathbb{Q}}\left[\tilde{P}_t\right] = P_0$ for all $t \in [0, T]$ as expected from the martingale property of discounted price processes under $\mathbb{Q}$. Nevertheless, because of the drift-adjustment, those expressions will become time-dependent as soon as WWR will enter the picture.

4 Expected Positive Exposures under WWR

Under no WWR (risk-neutral measure $\mathbb{Q}$), $\tilde{C}$ is a martingale,

$$d\tilde{C}_t = \sigma\tilde{S}_t\Delta_t dW_t,$$

whose solution is given by the standard Black-Scholes-Merton equation in Sec. 3. As discussed above, Girsanov theorem yields

$$dW_t = dW_t^s + \theta_t^s dt,$$

where W^s is a $\mathbb{Q}^{\mathcal{C}_s}$-Brownian motion on $[0, s]$. We assume that under $\mathbb{Q}$, the default intensity λ is governed by a CIR process with volatility η, i.e.

$$d\lambda_t = \kappa(\mu-\lambda_t)dt + \eta\sqrt{\lambda_t}dW_t^\lambda,$$

where W_t^λ is a $\mathbb{Q}$-Brownian motion whose correlation with W is ρ. A non-zero value for ρ introduces a dependency between S and λ that controls wrong-way risk. The drift adjustment is given by [1]

$$\theta_t^s = \theta_t^s(\lambda_t) = \rho\eta\sqrt{\lambda_t}\left(\frac{A^\lambda(t,s)B_s^\lambda(t,s)}{A^\lambda(t,s)B_s^\lambda(t,s)\lambda_t - A_s^\lambda(t,s)} - B^\lambda(t,s)\right), \qquad \text{(B.2)}$$

where A^λ, B^λ are known zero-coupon bond functions in affine models [2] :

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s \lambda_u du}\middle| \mathcal{F}_t\right] = A^\lambda(t,s)e^{-B^\lambda(t,s)\lambda_t}.$$

The subscripts refer to the variable with respect to which we compute the derivatives of A^λ and B^λ.

Let us now look at the dynamics of the call for $t \in [0,s]$ under $\mathbb{Q}^{\mathcal{C}_s}$. First, observe that we can write $\tilde{C}$ as a deterministic function of the variables (t, W_t) (instead of the usual (t, S_t) couple):

$$\tilde{C}_t = v(t, W_t),$$

with

$$v(s,x) := S_0 e^{-\frac{\sigma^2}{2}s+\sigma x}\Phi\left(\alpha(s) + x/\sqrt{T-s}\right) - Ke^{-rT}\Phi\left(\alpha(s) + x/\sqrt{T-s} - \sigma\sqrt{T-s}\right).$$

Applying Ito's lemma,

$$d\tilde{C}_t = \left(v_t(t,W_t) + \frac{1}{2}v_{xx}(t,W_t)\right)dt + v_x(t,W_t)dW_t,$$

and we have, for all (s,x) where $s \in [0, T]$ and $x \in \mathbb{R}$ the following relationships for the partial derivatives of v:

$$v_t(s,x) + \frac{1}{2}v_{xx}(s,x) = 0 \qquad \text{(B.3)}$$

$$v_x(s,x) = \sigma S_0 e^{-\frac{\sigma^2}{2}s+\sigma x}\Phi\left(\alpha(s) + x/\sqrt{T-s}\right). \qquad \text{(B.4)}$$

Now let us look at the dynamics of the call as a function of the $\mathbb{Q}^{\mathcal{C}_s}$-Brownian motion W^s on $t \in [0, s]$:

$$\tilde{C}_t := v\left(t, W_t^s + \int_0^t \theta_u^s du\right).$$

Using Ito's lemma and the relationships between v_t, v_x and v_{xx} in Eqs. (B.3) and (B.4), we have

$$\begin{aligned} d\tilde{C}_t = & \left(v_t \left(t, W_t^s + \int_0^t \theta_u^s du \right) + \frac{1}{2} v_{xx} \left(t, W_t^s + \int_0^t \theta_u^s du \right) \right) dt \\ & + v_x \left(t, W_t^s + \int_0^t \theta_u^s du \right) (dW_t^s + \theta_t^s dt) \\ = & \, v_x \left(t, W_t^s + \int_0^t \theta_u^s du \right) \theta_t^s dt + v_x \left(t, W_t^s + \int_0^t \theta_u^s du \right) dW_t^s. \end{aligned}$$

Defining now

$$\begin{aligned} \hat{S}_t &:= \tilde{S}_t e^{\sigma \int_0^t \theta_u^s du} \\ \hat{\Delta}_t &:= \Phi\left(\hat{d}(t,T)\right) \\ \hat{d}(t,T) &:= \frac{1}{\sigma\sqrt{T-t}} \left(\ln \frac{\hat{S}_t}{K} + rT + \frac{\sigma^2}{2}(T-t) \right), \end{aligned}$$

one gets

$$\begin{aligned} d\tilde{C}_t &= \sigma \tilde{S}_t e^{\sigma \int_0^t \theta_u^s du} \hat{\Delta}_t \theta_t^s dt + \sigma \tilde{S}_t e^{\sigma \int_0^t \theta_u^s du} \hat{\Delta}_t dW_t^s \\ &= \sigma \hat{S}_t \hat{\Delta}_t \theta_t^s dt + \sigma \hat{S}_t \hat{\Delta}_t dW_t^s \ . \end{aligned}$$

Clearly, $\tilde{C}$ is a $\mathbb{Q}$-martingale. This is no longer true under the new measure: it features a drift. Moreover, the martingale part is impacted by the drift as well as $\hat{S}$ features θ.

Let us consider the deterministic approximation $\theta_t^s \approx \theta(t,s)$ where λ_t is replaced by a deterministic proxy $\lambda(t)$. By replacing λ_t with $\lambda(t)$ in Eq. (B.2), we have

$$\theta(t,s) := \rho\eta\sqrt{\lambda(t)} \left(\frac{A^\lambda(t,s) B_s^\lambda(t,s)}{A^\lambda(t,s) B_s^\lambda(t,s) \lambda(t) - A_s^\lambda(t,s)} - B^\lambda(t,s) \right) . \tag{B.5}$$

Then, the WWR EPE expression $\text{EPE}(s) = \mathbb{E}^{\mathbb{Q}^{\mathcal{C}_s}}\left[\tilde{C}_s\right]$ is known analytically. To compute $\text{EPE}(t)$, the WWR EPE at time t, we need to evaluate the expectation of $\tilde{C}_t$ under $\mathbb{Q}^{\mathcal{C}_t}$. We thus set $s = t$ and define

$$\begin{aligned} \Theta(t) &:= \int_0^t \theta(u,t) du \\ \hat{\alpha}(t) &:= \alpha(t) + \frac{\Theta(t)}{\sqrt{T-t}} \\ Z^t &\overset{(\mathbb{Q}^{\mathcal{C}_t})}{\sim} \mathcal{N}(0,1). \end{aligned}$$

This yields (up to the approximation of the stochastic drift by its deterministic expression)

$$\begin{aligned}\tilde{C}_t &= v\left(t, W_t^t + \Theta(t)\right)\\ &= S_0 e^{-\frac{\sigma^2}{2}t+\sigma\Theta(t)+\sigma W_t^t}\Phi\left(\hat{\alpha}(t) + W_t^t/\sqrt{T-t}\right)\\ &\quad - Ke^{-rT}\Phi\left(\hat{\alpha}(t) - \sigma\sqrt{T-t} + W_t^t/\sqrt{T-t}\right)\\ &\sim S_0 e^{-\frac{\sigma^2}{2}t+\sigma\Theta(t)+\sigma\sqrt{t}Z^t}\Phi\left(\hat{\alpha}(t) + \beta(t)Z^t\right)\\ &\quad - Ke^{-rT}\Phi\left(\hat{\alpha}(t) - \sigma\sqrt{T-t} + \beta(t)Z^t\right),\end{aligned}$$

showing that the WWR EPE takes a similar form as the No-WWR EPE:

$$\mathbb{E}^{\mathbb{Q}^{\mathcal{C}_t}}\left[\tilde{C}_t\right] \approx S_0 e^{\sigma\Theta(t)}\Phi\left(\frac{\hat{\alpha}(t)+\beta(t)\sigma\sqrt{t}}{\sqrt{1+\beta^2(t)}}\right) - Ke^{-rT}\Phi\left(\frac{\hat{\alpha}(t)-\sigma\sqrt{T-t}}{\sqrt{1+\beta^2(t)}}\right),$$

where the approximation results from the fact that we have replaced the random variable $\int_0^t \theta_u^t du$ by the deterministic quantity $\int_0^t \theta(u,t)du$. As regards to the WWR EPE of the put, one easily gets

$$\mathbb{E}^{\mathbb{Q}^{\mathcal{C}_t}}\left[\tilde{P}_t\right] \approx Ke^{-rT}\Phi\left(\frac{\sigma\sqrt{T-t}-\hat{\alpha}(t)}{\sqrt{1+\beta^2(t)}}\right) - S_0 e^{\sigma\Theta(t)}\Phi\left(\frac{-\hat{\alpha}(t)-\beta(t)\sigma\sqrt{t}}{\sqrt{1+\beta^2(t)}}\right).$$

5 Proxys of θ_t^s

Here we use two different proxys for θ_t^s. As presented in [1], we consider a proxy where the $\mathbb{Q}$-expectation of λ_t is used in the formula for the drift adjustment (B.5). However, here we also present an alternative proxy, by using an approximation of the $\mathbb{Q}^{\mathcal{C}_T}$-expectation of λ_t and compare the impact on CVA in Sec. 7.

5.1 $\mathbb{Q}$-expectation

Here we use $\theta(t,s) = \theta_t^s(\bar{\lambda}_t)$, where $\bar{\lambda}_t := \mathbb{E}^{\mathbb{Q}}[\lambda_t]$. One strength for this proxy is that we have an analytic formula for $\bar{\lambda}_t$ and the proxy $\theta(t,s)$ is straightforward to obtain. The disadvantage is that we are 'operating' under other measures than $\mathbb{Q}$. Specifically when estimating the WWR EPE at time t, we have changed measure to $\mathbb{Q}^{\mathcal{C}_t}$. This changes the dynamics of λ, but it is ignored in this approach.

5.2 Approximation of $\mathbb{Q}^{\mathcal{C}_T}$-expectation

In order to improve the deterministic approximation of θ_t^s, we aim to obtain an approximation for the $\mathbb{Q}^{\mathcal{C}_T}$-expectation of λ_t. Remark that we use the measure for the maturity of the contract for all $t \in [0, T]$. A possible weakness of this proxy is that for calculating WWR EPE at time t we should use the $\mathbb{Q}^{\mathcal{C}_t}$-dynamics not the (terminal) $\mathbb{Q}^{\mathcal{C}_T}$-dynamics. However, using the terminal measure is a more convenient choice, since it is just necessary to obtain one 'term structure' of λ_t, $t \in [0, T]$, whereas using the $\mathbb{Q}^{\mathcal{C}_t}$-dynamics for the WWR EPE at time t will have the effect that it is necessary to compute separate values for λ_u, $u \in [0, t]$, corresponding to each $t \in [0, T]$. This may be computationally heavy, and thus we assume the simpler version with the benefit that only one term structure has to be computed while the effect of the drift-adjustment in λ from the change of measure may be accounted for. A closed-form expectation of λ_t under $\mathbb{Q}^{\mathcal{C}_T}$ can however not be readily found, but in the following we present an approximation of this expectation.

One further remark is that the $\mathbb{Q}^{\mathcal{C}_T}$-dynamics of λ is completely independent of the correlation between the underlying stock and λ, but is solely determined by the parameters of the $\mathbb{Q}$-dynamics of λ as well as the maturity of the contract. This allows for computed λ's to be used for calculating CVA on several contracts with the same counterparty, since the dynamics of the contract and its correlation with the default intensity does not enter any of the expressions.

Firstly, consider the $\mathbb{Q}^{\mathcal{C}_T}$-dynamics of λ_t for $t \in [0, T]$:

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \eta\sqrt{\lambda_t}\left(dW_t^T + \eta\sqrt{\lambda_t}\tilde{\theta}_t^T(\lambda_t)dt\right) \tag{B.6}$$
$$\tilde{\theta}_t^T(x) := \frac{a(t,T)}{a(t,T)x - b(t,T)} - c(t,T),$$

where

$$a(t,T) := A^\lambda(t,s)B_s^\lambda(t,s), \;\; b(t,T) := A_s^\lambda(t,s), \;\text{ and }\; c(t,T) := B^\lambda(t,s)\;.$$

Hence, integrating both sides of the above SDE in Eq. (B.6),

$$\lambda_t = \lambda_0 + \kappa\theta t - \kappa\int_0^t \lambda_s ds + \eta^2\int_0^t \lambda_s\tilde{\theta}_s^T(\lambda_s)ds + \eta\int_0^t \sqrt{\lambda_s}dW_s^T$$

and using Tonelli's theorem,

$$\mathbb{E}^{\mathbb{Q}^{\mathcal{C}_T}}[\lambda_t] = \lambda_0 + \kappa\theta t - \kappa\int_0^t \mathbb{E}^{\mathbb{Q}^{\mathcal{C}_T}}[\lambda_s]ds + \eta^2\int_0^t \mathbb{E}^{\mathbb{Q}^{\mathcal{C}_T}}\left[\lambda_s\tilde{\theta}_s^T(\lambda_s)\right]ds,$$

where the Ito integral has zero expectation, and thus has vanished. We want to simplify the term that includes

$$\lambda_s\tilde{\theta}_s^T(\lambda_s) = \frac{a(t,T)\lambda_s}{a(t,T)\lambda_s - b(t,T)} - c(t,T)\lambda_s,$$

and therefore we use a first-order Taylor-expansion of the function $a(t,T)x/(a(t,T)x - b(t,T))$ around some point $x(t) > 0$. We use a positive function since $\mathbb{Q}^{\mathcal{C}_T}$ is an equivalent measure to $\mathbb{Q}$ and thus the expectation of λ_t is always positive. Expanding the function around zero also turns out to be an undesirable choice that leads to unstable estimates of the expectation close to maturity, as $a(t,T)/b(t,T)$ diverges for $t \to T$. We choose to make the expansion around the $\mathbb{Q}$-expectation of λ_t, since this is indeed a positive function, and the $\mathbb{Q}$-expectation may give some reasonable input to the $\mathbb{Q}^{\mathcal{C}_T}$-expectation. The Taylor expansion looks as follows

$$\begin{aligned}\frac{a(t,T)x}{a(t,T)x - b(t,T)} &= \frac{a(t,T)x(t)}{a(t,T)x(t) - b(t,T)} \\ &\quad - \frac{a(t,T)b(t,T)}{(a(t,T)x(t) - b(t,T))^2}(x - x(t)) + o(x) \\ &= \left(\frac{a(t,T)x(t)}{a(t,T)x(t) - b(t,T)}\right)^2 - \frac{a(t,T)b(t,T)}{(a(t,T)x(t) - b(t,T))^2}x \\ &\quad + o(x).\end{aligned}$$

Setting $g(t) := \mathbb{E}^{\mathbb{Q}^{\mathcal{C}_T}}[\lambda_t]$ we have

$$\begin{aligned}g(t) \approx \lambda_0 + \kappa\theta t - \kappa \int_0^t g(s)ds \\ - \eta^2 \int_0^t \left(\frac{a(s,T)b(s,T)}{(a(s,T)x(s) - b(s,T))^2} + c(s,T)\right) g(s)ds \\ + \eta^2 \int_0^t \left(\frac{a(s,T)x(s)}{a(s,T)x(s) - b(s,T)}\right)^2 ds.\end{aligned}$$

Differentiating both sides we obtain a first-order linear inhomogeneous ODE

$$g'(t) \approx \kappa\theta + \eta^2 \left(\frac{a(t,T)x(t)}{a(t,T)x(t) - b(t,T)}\right)^2 - h(t,T)g(t),$$

where

$$h(t,T) := \kappa + \eta^2 \left(\frac{a(t,T)b(t,T)}{(a(t,T)x(t) - b(t,T))^2} + c(t,T)\right).$$

Disregarding the drift approximation the solution to this SDE is

$$g(t) = e^{-H(t,T)} \left(g(0) + \int_0^t \left(\kappa\theta + \eta^2 G(s,T)\right) e^{H(s,T)} ds\right),$$

where in this context, $g(0) = \lambda_0$, and

$$\begin{aligned}H(s,T) &:= \int_0^s h(u,T)du \\ G(s,T) &:= \left(\frac{a(s,T)x(s)}{a(s,T)x(s) - b(s,T)}\right)^2.\end{aligned}$$

6 Potential Future Exposures (PFE)

We would like to compare the risk-neutral CVA (CVA computed with market-implied default probabilities and WWR EPE) with actuarial CVA, computed with PFE (e.g. 99% quantile of exposures) and historical default probabilities.

From the above, $C_t = H_t(W_t)$ where H_t

$$H_t(x) = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma x}\Phi\left(\alpha(t) + x/\sqrt{T-t}\right)$$
$$- Ke^{-r(T-t)}\Phi\left(\alpha(t) - \sigma\sqrt{T-t} + x/\sqrt{T-t}\right)$$

is the monotonic increasing and invertible function.

The k-PFE is defined as the profile of the exposure's quantile at level k.

All functions H_t being continuous and strictly increasing (with slope given by $\Delta_t(x)$), this means that

$$q(t) = H_t\left(\Phi^{-1}(k)\sqrt{t}\right).$$

This is merely the H_t function (that is, the function that gives the time-t exposure as a function of the time-t value of the Brownian motion) evaluated at the k-quantile of a centered Normal distribution with variance t.

Remark that a similar expression is valid for WWR PFE, by replacing H_t with $\hat{H}_t$ with similar notations as before.

7 Numerical experiments

We use the four parameter sets for the CIR-process of λ used in [1]. Thus in the forthcoming we will regard parameters in Tab. B.1 as set 1–4. Further we use $S_0 = K = 15$, $r = 1\%$, $\mu = 3\%$ and $\sigma = 30\%$, and call options with a time to maturity of 1 year and 5 years, respectively. The corresponding CVA figures are given in Figs. B.1 and B.2, and are compared with a 2D Monte Carlo scheme as well as the Gaussian Copula resampling approach (for more details about this approach, see e.g. [3], [4] or [5]).

We also consider the actuarial CVA; CVA calculated from the 99% PFE, which is described in Sec. 6, and on historical rather than risk-neutral default probabilities. We use the default rates from [6], and we consider parameter set 1, 2 and 4 to have rating *Ba* and set 3 to have rating *A*. Specifically the flat hazard rates used for *A* rating are 2 bps for 1-year contracts and 9.4 bps for 5-year contracts, while for the *Ba* rating we use 110 bps for 1-year contracts and 176 bps for 5-year contracts. The actuarial CVA is shown in Figs. B.3 and B.4.

In the following, we use the terms $\mathbb{Q}^{\mathcal{C}_T}$-proxy and $\mathbb{Q}$-proxy for the proxys of θ_t^s using the λ expectation under $\mathbb{Q}^{\mathcal{C}_T}$ and $\mathbb{Q}$, respectively.

	λ_0 (bps)	κ	θ (bps)	η	Rating used for actuarial CVA
Set 1	300	02%	1610	8%	Ba
Set 2	350	35%	450	15%	Ba
Set 3	100	80%	200	20%	A
Set 4	300	50%	500	50%	Ba

Table B.1: Parameter sets for the dynamics of λ in the numerical experiments.

Consider the CVA on 1-year contracts in Fig. B.1. We observe a pattern that for negative correlations, we tend to estimate a higher CVA compared to the 2D Monte Carlo scheme. Generally this overestimation of CVA is stronger for the $\mathbb{Q}^{\mathcal{C}_T}$-proxy than for the $\mathbb{Q}$-proxy. The exception is parameter set 4, where the $\mathbb{Q}$-proxy which estimates a slightly lower CVA. For positive correlations — when we experience WWR on the call — we observe a very good fit of the $\mathbb{Q}^{\mathcal{C}_T}$-proxy and the Monte Carlo CVAs parameter set 1–3, while the $\mathbb{Q}$-proxy also tends to overestimate the CVA on this end.

Comparing the results for the two proxys, we observe that the $\mathbb{Q}$-proxy tends to suggest a larger WWR-effect, giving a larger compensation than the $\mathbb{Q}^{\mathcal{C}_T}$-proxy for positive correlations, while it suggests a lower compensation in the case of negative correlations. Thus the $\mathbb{Q}$-proxy suggests a higher impact of the "market-credit correlation".

Consider now the CVA on the 5-year contract in Fig. B.2. Firstly, we observe that even for one million sample paths and a time step of 0.01, the Monte Carlo simulations of the CVA does include some bias, since the zero-correlation case does not completely correspond with the analytic formula. This is especially pronounced for parameter set 4. Here we experience a weakness of the 2D Monte Carlo approach; it is computationally heavy, but moreover includes a bias with a very small time-step and large number of paths. Comparing the CVA of the two proxys, we observe similar behavior as in the 1-year case.

For both 1 and 5-year CVA — and for all parameter sets — we observe that the resampling approach is highly sensitive to the market-credit correlation. This allows to model very strong WWR impact. The problem however is that it is unclear how the value of the dependence parameter (the correlation between τ and the exposure V_{t_i} at any given point in time t_i) has to be chosen.

Further, we provide the CVA under Basel, based on [7], [8]. In this context (T being the contract maturity, where we consider 1 and 5 years), the No-WWR figure is given by

$$\mathrm{CVA}^{\mathrm{basel}} = (1 - G(T))\frac{\mathrm{EPE}^{\perp}(0) + \mathrm{EPE}^{\perp}(T)}{2},$$

where $\mathrm{EPE}^{\perp}(t)$ is the (No-WWR) EPE at time t, which is constant and equal to $\tilde{C}_0$ for the call and $\tilde{P}_0$ for the put. The WWR CVA is given by

$$\mathrm{CVA}_{\alpha}^{\mathrm{basel}} = (1 - G(T))\alpha\mathrm{EPE}^{\perp}(T/2), \tag{B.7}$$

where $\alpha\mathrm{EPE}^{\perp}(T/2)$ is called the "exposure at default" (EAD) and the scaling coefficient α is typically set to 1.4. The corresponding levels are indicated on Figs. B.1 and B.2.

The Basel III parameter α cannot be considered as a way to represent market-credit correlation, but in fact capture the "market-credit covariance". This is a crucial point, that complicates drastically the choice of a reasonable value for α. In order for the Basel type formula to be a decent approach to account for WWR, α has to be chosen not only with regards to dependence between portfolio and credit, but also according to both market and credit volatilities. This observation suggests that it is a bit naive to hope that a kind of "universal constant" would be able to account for this effect. The approximation proposed in [1] is therefore, from this perspective, a significant improvement to Basel type formulae. Tab. B.2 show the values of α that make the CVA in the Basel type formulae agree with the ones obtained from Monte Carlo simulation for 1-year contracts. Across parameter sets the value changes quite significantly. Obviously $\alpha \approx 1$ for zero-correlation, and the values are larger (smaller) for positive (negative) correlations. Thus the Basel approach cannot capture right-way risk, which is experienced when the correlation is negative, while the performance of the method is highly dependent both on the correlation and the parameter set used.

ρ	-0.9	-0.6	-0.3	0	0.3	0.6	0.9
Set 1	0.75	0.83	0.92	1.01	1.10	1.20	1.30
Set 2	0.63	0.74	0.87	1.01	1.16	1.31	1.48
Set 3	0.42	0.58	0.78	1.01	1.28	1.59	1.95
Set 4	0.29	0.45	0.70	1.03	1.44	1.95	2.57

Table B.2: The values of α for which Eq. (B.7) agrees with the 2D-MC results for 1-year call options.

The interpretation of the actuarial CVA in Figs. B.3 and B.4 are similar to the interpretation of the risk-neutral CVA results. We generally observe a larger impact of the correlation on the CVA when using the $\mathbb{Q}$-proxy than what suggested by both the Monte Carlo simulations and the $\mathbb{Q}^{\mathcal{C}_T}$-proxy.

Conclusion

From the change-of-measure approach suggested in [1], we have examined the CVA on put and call options under WWR. In the Basel III framework, WWR is treated by a multiplier. But it should not be based on "market-credit correlation", but rather on "market-credit covariance". We find that the Basel III approach is a naive way of estimating the CVA that does not recognize right-way risk and cannot capture the wrong-way risk in a desirable way. However, using the set-up in this paper, one can obtain CVA on put and call options, and capture the effect of the market-credit correlation by analytic formulas. Specifically, we present the formulas for two proxys of the drift-adjustment process, using each proxy, the CVA can be obtained analytically. Further the actuarial CVA based on PFE also has an analytic expression, based on the formulas in the paper.

In the numerical experiments, we examine the estimated CVA — both the risk-neutral and actuarial CVA — from the formulas supplied in the paper, compared with joint (exposure-credit) Monte Carlo simulations, Gaussian Copula resampling and Basel III figures. We observe that the Gaussian Copula resampling approach is very sensitive to the correlation, leading to too high CVA estimates when experiencing WWR. We do not find 2D Monte Carlo to be a desirable method, since it is computationally heavy, and in some cases includes a bias, even for a small time-step and large number of sample paths.

On the other hand, we get very encouraging results from the CVA based on the two proxys, both when calculating the risk-neutral CVA and the actuarial CVA. The simple $\mathbb{Q}$-proxy is performing quite reasonably and captures the general behavior of the CVA. This proxy is very easy to implement and fast to compute since all formulas are analytic (up to the deterministic approximation of the drift adjustment). Using the $\mathbb{Q}^{\mathcal{C}_T}$-proxy requires an additional approximation (to compute the corresponding expectation of λ_t), but gives a slightly more realistic (considering the MC-results to be the true CVA) CVA when experiencing WWR.

References

[1] D. Brigo and F. Vrins, "Disentangling Wrong-Way Risk: Pricing CVA via change of measures and drift adjustment," 2017, preprint available at `https://arxiv.org/abs/1611.02877`.

[2] D. Brigo and F. Mercurio, *Interest Rate Models - Theory and Practice*. Springer, 2006.

[3] A. Sokol, "Modeling and hedging Wrong Way Risk in CVA with exposure sampling," in *RiskMinds USA*. Risk, 2011.

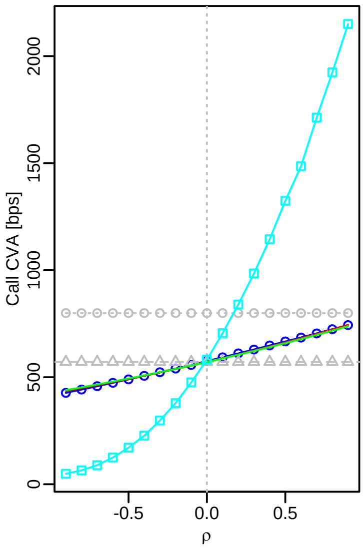
[4] D. Rosen and D. Saunders, "Cva the Wrong Way," *Journal of Risk Management in Financial Institutions*, vol. 5, pp. 252–272, 2012.

[5] F. Vrins, "Wrong-way risk CVA models with analytical EPE profiles under gaussian exposure dynamics," *International Journal of Theoretical and Applied Finance*, vol. 20, 2017.

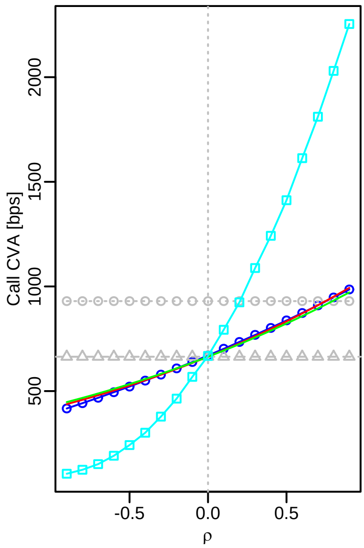
[6] "Corporate One-to-Five-Year Rating Transition Rates," Moody's, Tech. Rep., 2008.

[7] "Basel III: A global regulatory framework for more resilient banks and banking systems," Basel Committee on Banking Supervision, Tech. Rep., 2010, `http://www.bis.org/publ/bcbs189.pdf`.

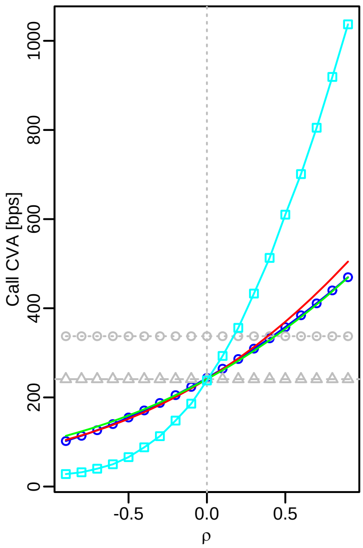
[8] "Basel III framework (annex 4)," Basel Committee on Banking Supervision., Tech. Rep., `https://www.bis.org/publ/bcbs128d.pdf`.



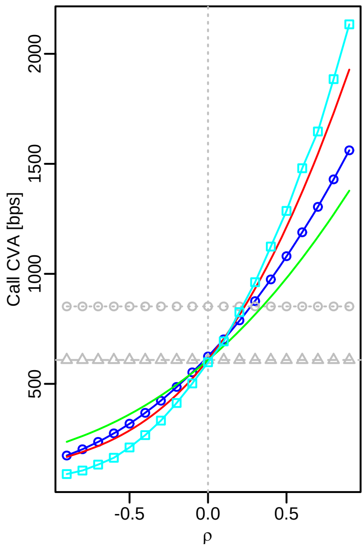
(a) Set 1



(b) Set 2

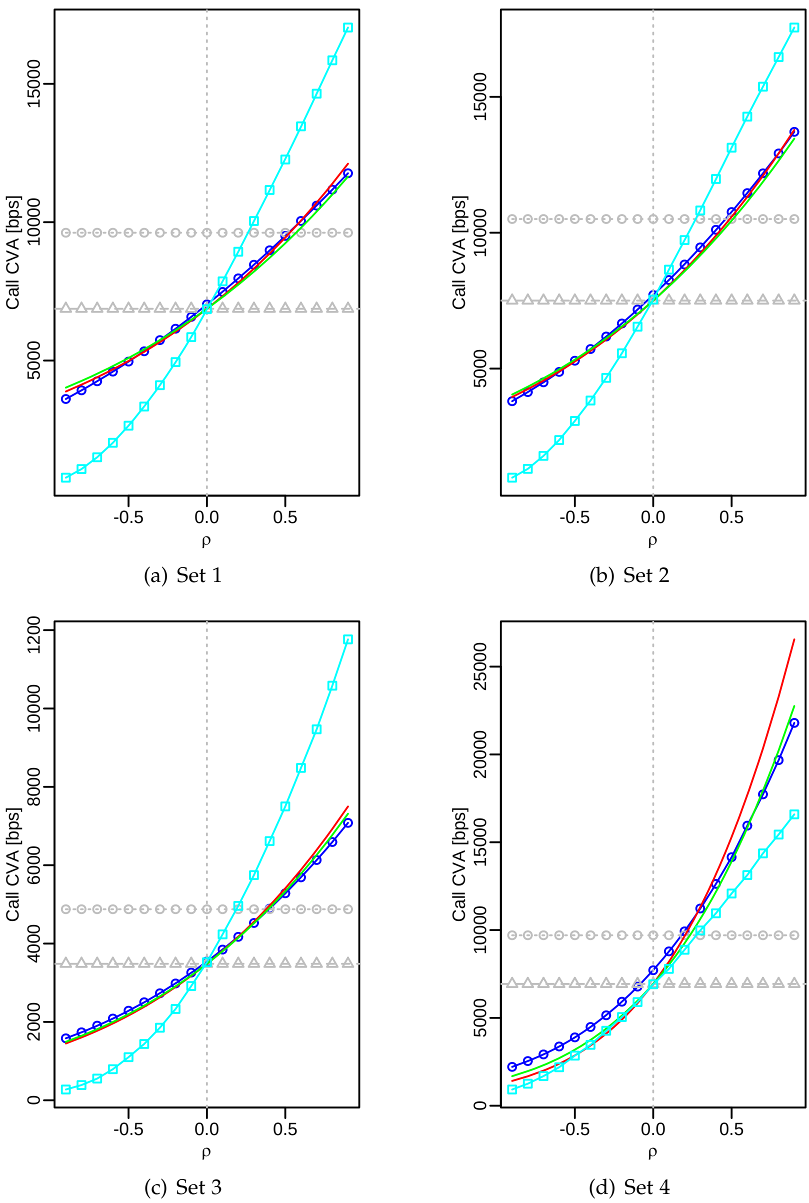


(c) Set 3



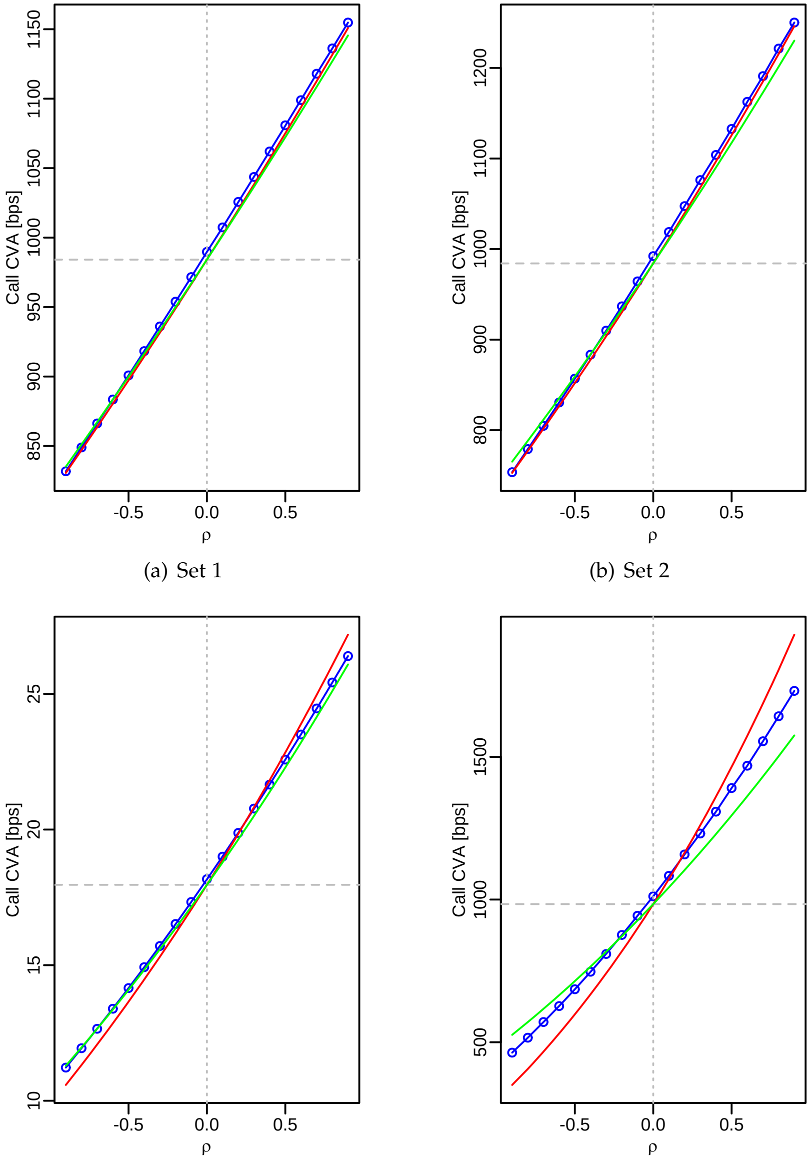
(d) Set 4

Fig. B.1: 1-year call CVA profiles for the four parameter sets using a 2D Monte Carlo scheme (blue) with 10^6 paths and a time step of 0.01, compared with the analytic approximation using the $\mathbb{Q}$-expectation (red) and $\mathbb{Q}^{\mathcal{C}_T}$-approximation (green). CVA based on the Gaussian Copula resampling approach (cyan) and the analytic CVA with zero-correlation (dashed grey line) are included. The Basel no-WWR CVA is indicated by grey dots and the WWR CVA using $\alpha = 1.4$ is indicated by grey triangles.



(a) Set 1 (b) Set 2 (c) Set 3 (d) Set 4

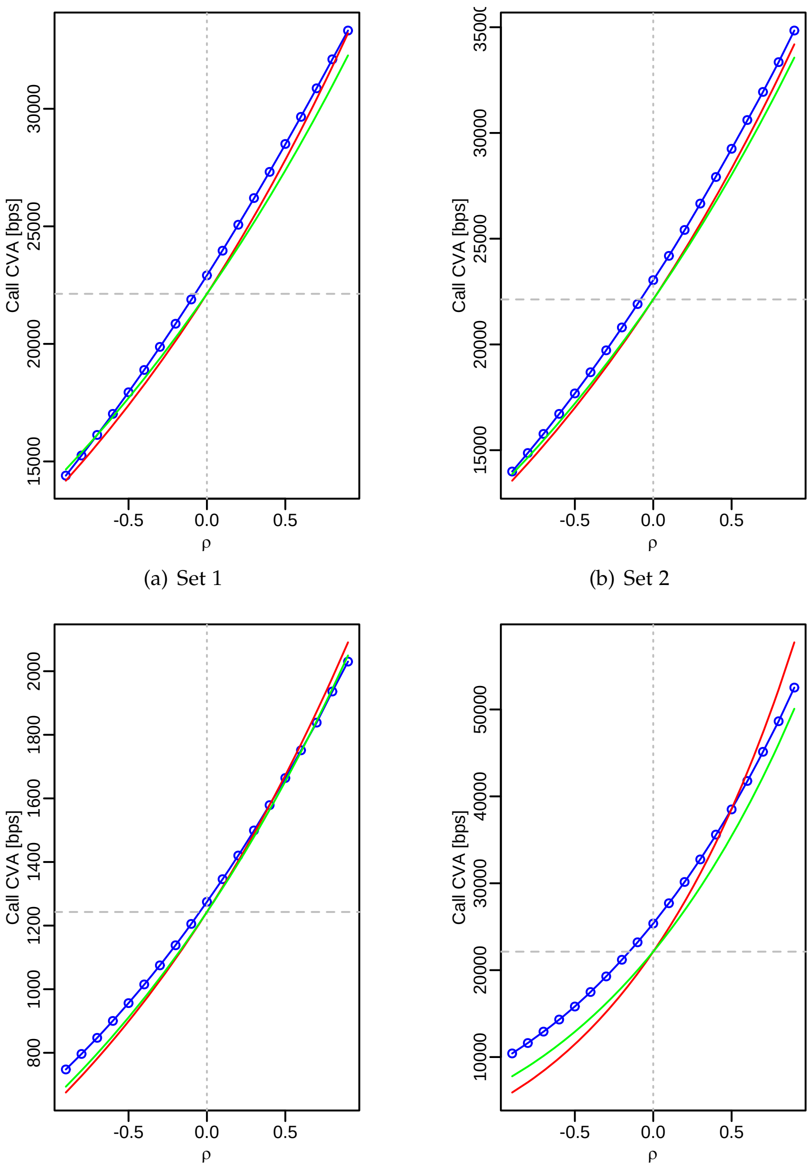
Fig. B.2: 5-year call CVA profiles for the four parameter sets using a 2D Monte Carlo scheme (blue) with 10^6 paths and a time step of 0.01, compared with the analytic approximation using the $\mathbb{Q}$-expectation (red) and $\mathbb{Q}^{\mathcal{C}_T}$-approximation (green). CVA based on the Gaussian Copula resampling approach (cyan) and the analytic CVA with zero-correlation (dashed grey line) are included. The Basel no-WWR CVA is indicated by grey dots and the WWR CVA using $\alpha = 1.4$ is indicated by grey triangles.



(a) Set 1 (b) Set 2

(c) Set 3 (d) Set 4

Fig. B.3: 1-year actuarial call CVA profiles for the four parameter sets using a 2D Monte Carlo scheme (blue) with 10^6 paths and a time step of 0.01, compared with the analytic approximation using the $\mathbb{Q}$-expectation (red) and $\mathbb{Q}^{\mathcal{C}_T}$-approximation (green). The dashed grey line shows the analytic CVA with zero-correlation is included.



(a) Set 1 (b) Set 2

(c) Set 3 (d) Set 4

Fig. B.4: 5-year actuarial call CVA profiles for the four parameter sets using a 2D Monte Carlo scheme (blue) with 10^6 paths and a time step of 0.01, compared with the analytic approximation using the $\mathbb{Q}$-expectation (red) and $\mathbb{Q}^{\mathcal{C}_T}$-approximation (green). The dashed grey line shows the analytic CVA with zero-correlation is included.

Paper C

Calibration of CIR processes to CVA data and applications to estimation of Market Price of Risk

Thomas Hvolby[1]
Bent Jesper Christensen[2]
Esben Høg[1]

Working paper

[1]Department of Mathematical Sciences, Aalborg University
[2]Department of Economics and Business Economics, Aarhus University

The layout has been revised.

Abstract

We present a rigorous framework for CDS valuation and calibration to market quotes on single-name CDSs and methods of calculating the market price of risk (MPR) on such markets. We use the results of MPR for bond markets, and thus the MPR corresponds to the survival probabilities, which is a non-tradable asset. Further, we present a set-up for numerical valuation of triparty CDS agreements, where two default risky parties trade a CDS with a third entity as reference credit.

1 Introduction

Since the recent financial crisis of '07-'08, the necessity of a bilateral framework for valuations of contracts on credit markets has been evident [1–3]. In this paper, we aim at providing a thorough explanation of the method and complications associated with reduced form model calibration to market data on *Credit Default Swaps* (CDSs). Further, we provide frameworks for applying the models to estimate the *Market Price of Risk* (MPR) on CDS markets as well as valuation of *triparty CDS* contracts, which we define as a CDS between two defaultable entities with a third entity as the underlying reference credit.

Specifically, we consider the default of each entity to be governed by a Cox-process as suggested in [4], and consider the CIR++ specification due to [5]. We are focused on numerical experiments and provide the necessary results to conduct the model calibration and include formulas and derivations that are closely related to the implementation itself.

Sec. 2–7 provide the fundamental set-up of reduced form modeling on credit markets with *Bilateral Credit Value Adjustments* (BCVA), including a formulation of the value of CDS contracts that is essential for the model calibration to market data. Sec. 8 provides a description of the Bloomberg data used in our analysis and the assumptions implied by our choice of data. Sec. 9–10 provide the application of the models to estimation of MPR on CDS markets and BCVA estimation for triparty CDS contracts.

2 Modeling default intensities with a Cox-process

.1) We use the modeling framework presented in [4], where default intensities are modeled using a Cox-process. Thus we consider a jump process with stochastic intensity denoted $\lambda_t^i(X_t^i)$ – for shorter notation we use $\lambda_t^i \equiv \lambda_t^i(X_t^i)$ – instantaneously at time t, where X_t^i is the state variable(s) driving the default intensity of firm i observed at time t. Note that in this paper we calibrate the Cox-process to CDS quotes, and thus we consider λ^i to be a process under the pricing measure $\mathbb{Q}$, which is assumed to exist for all considered firms.

Let τ^i be the first jump of the jump process, given that no jump has yet occurred at the current time t:

$$\tau^i = \inf\left\{ s > t : \int_t^s \lambda_u^i \mathrm{d}u \geq E^i \right\},$$

where $E^i \sim \text{Exp}(1)$ is assumed to be independent of the factors driving λ_s^i. Note that τ^i depends on t, but for simplicity this is suppressed in the notation. We define $\mathcal{G}_s^i = \sigma\{X_u^i : u \leq s\}$, $\mathcal{H}_s^i = \sigma\{\mathbb{1}_{\{\tau^i \leq u\}} : u \leq s\}$, and $\mathcal{F}_s^i = \mathcal{G}_s^i \vee \mathcal{H}_s^i$; here $\mathcal{A} \vee \mathcal{B}$ for any two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ represents the minimal σ-algebra containing both $\mathcal{A}$ and $\mathcal{B}$. Thereby E^i is independent of $\mathcal{G}^i$, λ_s^i is a $\mathcal{G}_s^i$ measurable process, and τ^i is $\mathcal{H}_s^i$ measurable.

Now we present a few results for the Cox-process set-up, all of which are necessary for developing CVA pricing formulas. Let f and F denote CDF and PDF, respectively, of $\tau^i | \mathcal{G}_T^i$. For all $t \leq s \leq T$ we have:

$$\begin{aligned}
F_{\tau^i}\left(s \,\middle|\, \mathcal{G}_T^i\right) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^i \leq s\}} \,\middle|\, \mathcal{G}_T^i\right] = 1 - \exp\left\{ - \int_t^s \lambda_u^i \mathrm{d}u \right\} \\
f_{\tau^i}\left(s \,\middle|\, \mathcal{G}_T^i\right) &= \lambda_s^i \exp\left\{ - \int_t^s \lambda_u^i \mathrm{d}u \right\} \qquad \text{(C.1)} \\
\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^i > T\}} \,\middle|\, \mathcal{G}_T^i\right] &= \exp\left\{ - \int_t^T \lambda_u^i \mathrm{d}u \right\} \\
\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^i \leq T\}} g(\tau^i) \,\middle|\, \mathcal{F}_t^i\right] &= \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\lambda_s^i e^{-\int_t^s \lambda_u^i \mathrm{d}u} g(s) \,\middle|\, \mathcal{F}_t^i\right] \mathrm{d}s, \qquad \text{(C.2)}
\end{aligned}$$

where it is assumed that $g(s)$ is a $\mathcal{G}_s^i$-measurable function for all $s \geq t$.

3 Valuing a general derivative with UCVA and risk-free closeout

After the introduction of BCVA, the most widely used recovery assumption was that of risk-free closeout, according to [6]. This resembles the assumption of *recovery of the treasury*, which was commonly used prior to the financial crisis in ’07-’08, see e.g. [7]. Under the risk-free closeout assumption, the value of the contract at default is determined in a completely risk-free environment; meaning both the defaulted and surviving counterparties are seen as risk-free in the closeout valuation. If the value of the contract is positive from the non-defaulted counterparty’s point of view, a fraction of this value is recovered, and if the contract has negative value, the surviving counterparty will give the absolute contract value to the defaulted counterparty’s creditors. Henceforth, we consider estimating UCVA on a derivative from a bank’s point of view (denoted entity b) and denote the bank’s counterparty in the derivative as entity c.

In order to derive UCVA pricing formulas, we need to define the payoff structure. Let $\Pi(u,v)$ denote the sum of all cash flows – not including the settlement upon counterparty default – generated by the derivative from time u to $v \geq u$, each discounted back to time u by the risk-free rate. Note that cash flows received by the bank are positive and payments from the bank to the counterparty are negative. E.g. for a zero coupon bond with a \$1 notional, the payoff structure is \$1 discounted if the cash flow time span includes the maturity and zero otherwise; $\Pi(u,v) = \mathbb{1}_{\{u \leq T \leq v\}} e^{-\int_u^T r_s \mathrm{d}s}$. Further, we define $\mathcal{G}_s^r = \sigma\{X_u^r : t \leq u \leq s\}$ as the sigma-algebra generated by the factors X_u^r driving the risk-free rate, and introduce the sigma-algebra containing all information about counterparty default, default intensity, and the risk-free rate as

$$\mathcal{F}_s^{i,r} = \mathcal{F}_s^i \vee \mathcal{G}_s^r = \mathcal{G}_s^i \vee \mathcal{H}_s^i \vee \mathcal{G}_s^r, \quad i \in \{b,c\}.$$

We denote the (constant) loss-given-default of the counterparty as L^c and of the bank as L^b and introduce the notation $f^+ = \max(f,0)$ and $f^- = \max -f, 0$.

The UCVA pricing formula from the perspective of b with only one default-risky party (c and b, respectively) is calculated by:

$$\begin{aligned}
&P_{b,\{c\}}^{\mathrm{UCVA}}(t,T) \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) - \mathbb{1}_{\{\tau^c \leq T\}} L^c e^{-\int_t^{\tau^c} r_s \mathrm{d}s} \Pi(\tau^c,T)^+ \;\middle|\; \mathcal{F}_t^{c,r}\right] \qquad \text{(C.3)} \\
&\quad = \mathbb{E}^{\mathbb{Q}}[\Pi(t,T) \mid \mathcal{F}_t^{c,r}] - L^c \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\lambda_s^c e^{-\int_t^s (r_u + \lambda_u^c)\mathrm{d}u} \Pi(s,T)^+ \;\middle|\; \mathcal{F}_t^{c,r}\right] \mathrm{d}s \\
&P_{b,\{b\}}^{\mathrm{UCVA}}(t,T) \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) + \mathbb{1}_{\{\tau^b \leq T\}} L^b e^{-\int_t^{\tau^b} r_s \mathrm{d}s} \Pi(\tau^b,T)^- \;\middle|\; \mathcal{F}_t^{b,r}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[\Pi(t,T) \;\middle|\; \mathcal{F}_t^{b,r}\right] + L^b \int_t^T \mathbb{E}^{\mathbb{Q}}\left[\lambda_s^b e^{-\int_t^s (r_u + \lambda_u^b)\mathrm{d}u} \Pi(s,T)^- \;\middle|\; \mathcal{F}_t^{b,r}\right] \mathrm{d}s.
\end{aligned}$$

The second representation of the formulas follows from Eq. (C.2). Note that pricing from b's perspective with only b being default-risky is a constructed scenario that comes into importance when considering BCVA.

It is clear that the pricing formulas with UCVA takes form of a risk-free price adjusted by an always positive value, which we refer to as the CVA value. The CVA value is interpreted as the risk premium required by the bank due to the default riskiness of the counterparty.

4 Valuing a general derivative with BCVA and risk-free closeout

The approach from Sec. 3 is applied with the distinction that both the counterparty and the bank are now considered to be default-risky. The default time of the bank, τ^b, and the default time of the counterparty, τ^c, are both modeled with the Cox-process set-up of Sec. 2 and are allowed to be correlated. As discussed in [8], we are only concerned with the first of the two possible defaults, since we assume the contract is terminated at this point. For completeness, the two counterparties are also allowed to default simultaneously. We denote the "full information" of the risk-free rate as well as the defaults and default-intensity drivers for both the bank and the counterparty as

$$\mathcal{F}_t = \mathcal{F}_t^c \vee \mathcal{F}_t^b \vee \mathcal{G}_t^r.$$

This associated pricing formula from the point of view of b is

$$\begin{aligned} P_b^{\text{BCVA}}(t,T) = \mathbb{E}^{\mathbb{Q}}[\Pi(t,T) \mid \mathcal{F}_t] \\ + L^b \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau^b \le T, \tau^b \le \tau^c\right\}} e^{-\int_t^{\tau^b} r_s \mathrm{d}s} \Pi(\tau^b, T)^- \,\middle|\, \mathcal{F}_t \right] \\ - L^c \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau^c \le T, \tau^c \le \tau^b\right\}} e^{-\int_t^{\tau^c} r_s \mathrm{d}s} \Pi(\tau^c, T)^+ \,\middle|\, \mathcal{F}_t \right]. \end{aligned} \tag{C.4}$$

Here the BCVA pricing formula takes the form of a risk-free contract value, added a positive adjustment from the default-risk of b and subtracted a positive adjustment yielding from c's default-risk. The adjustment from the default-risk of the pricing entity, b, is referred to as the Debit Value Adjustment (DVA) and is interpreted as the risk premium the counterparty requires due to the risk of the bank defaulting. The adjustment from the default-risk of the counterparty is, as in the UCVA case, referred to as the CVA, which is the value the bank requires due to the default riskiness of its counterparty.

Note that Eq. (C.4) is consistent with the UCVA pricing formula in Eq. (C.3) in the sense that if we let the bank be default free such that it holds almost surely that $\tau^b > T$ and $\tau^b > \tau^c$, then the formula reduces to Eq. (C.3).

5 Valuing a general derivative with BCVA and replacement closeout

As stated in Sec. 3 risk-free closeout was the conventional choice after the credit crisis. However, in [6] an alternative closeout type is introduced: The replacement closeout. Essentially replacement closeout assumes that when valuing the closeout after a default event, the defaulted counterparty is seen as risk-free, whereas the surviving party is seen as defaultable. The closeout convention is also described as valuing the contract from the point of view of a risk-free party, taking the position of the defaulted counterparty in the contract. The argument in replacement closeout is that since the new (risk-free) counterparty would not neglect the risk of the surviving party defaulting before the contract matures, this should not be neglected when valuing the contract at closeout.

Here we apply the framework for risk-free closeout in Sec. 4, but with a distinction in the closeout value of the derivative contract upon default at $\tau^1 = \inf\{\tau^b, \tau^c\}$ given $\tau^1 \leq T$. With risk-free closeout, this value is simply $\mathbb{E}^{\mathbb{Q}}\left[\Pi(\tau^i, T) \,\middle|\, \mathcal{F}_{\tau^i}\right]$. However, with replacement closeout, the value is found by applying the formulas from Sec. 3 where the default-risky party is the surviving party, i.e. the party that has not defaulted at τ^1. The cash flows in each possible default-scenario are then given by

$$
\begin{aligned}
&\tau^b > T, \tau^c > T: \quad \Pi(t,T) \\
&\tau^b \leq T, \tau^b < \tau^c: \quad \Pi(t,\tau^b) + e^{-\int_t^{\tau^b} r_s \mathrm{d}s}\left(P_{b,\{c\}}^{\mathrm{UCVA}}(\tau^b,T) + L^b\left(P_{b,\{c\}}^{\mathrm{UCVA}}(\tau^b,T)\right)^-\right) \\
&\tau^c \leq T, \tau^c < \tau^b: \quad \Pi(t,\tau^c) + e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\left(P_{b,\{b\}}^{\mathrm{UCVA}}(\tau^c,T) - L^c\left(P_{b,\{b\}}^{\mathrm{UCVA}}(\tau^c,T)\right)^+\right) \\
&\tau^c = \tau^b = \tau \leq T: \ \Pi(t,T) + e^{-\int_t^{\tau} r_s \mathrm{d}s}\Big(L^b\Pi(\tau,T)^- - L^c\Pi(\tau,T)^+\Big),
\end{aligned}
$$

The BCVA pricing formula is then given by

$$
\begin{aligned}
P_b^{\mathrm{BCVA}}(t,T) = \mathbb{E}^{\mathbb{Q}}\Bigg[&\Pi(t,T) - \mathbb{1}_{\{\tau^b \leq \tau^c \leq T\}} L^c \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\Pi(\tau^c,T)^+ \,\middle|\, \mathcal{F}_{\tau^b}\right] \\
&+ \mathbb{1}_{\{\tau^c \leq \tau^b \leq T\}} L^b \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^b} r_s \mathrm{d}s}\Pi(\tau^b,T)^- \,\middle|\, \mathcal{F}_{\tau^c}\right] \\
&+ \mathbb{1}_{\{\tau^b \leq T, \tau^b < \tau^c\}} L^b e^{-\int_t^{\tau^b} r_s \mathrm{d}s}\left(P_{b,\{c\}}^{\mathrm{UCVA}}(\tau^b,T)\right)^- \\
&- \mathbb{1}_{\{\tau^c \leq T, \tau^c < \tau^b\}} L^c e^{-\int_t^{\tau^c} r_s \mathrm{d}s}\left(P_{b,\{b\}}^{\mathrm{UCVA}}(\tau^c,T)\right)^+ \,\Bigg|\, \mathcal{F}_t\Bigg].
\end{aligned}
$$

In the formula, the value of the corresponding risk-free valuation, $\mathbb{E}^{\mathbb{Q}}[\Pi(t,T) \mid \mathcal{F}_t]$, originates from the first term of states one and four along with the first two terms of states two and three, whereas the remaining terms in the fourth scenario is incorporated through the non-strict inequalities in the case that both counterparties default prior to contract termination.

6 CDS contracts

A *Credit Default Swap* (CDS) is a derivative, which is constructed as an instrument that allows one to recover parts of a possible loss due to counterparty default. Thus a CDS can be seen as an insurance contract against losses due to a default event. A CDS can be specified as with several underlying entities whose default can trigger the CDS recovery payment, however we only consider *single name CDS* which are used to hedge exposure due to the credit riskiness of a single counterparty.

We consider a *receiver* CDS contract on a single-name, which means that there is only one reference credit. Moreover, we will only consider CDSs with an underlying reference credit, and thereby the reference is not specifically a bond or another security, but the issuer itself. Since we consider receiver CDSs, we are approaching the valuation as the party *selling protection* against such a default of a counterparty, and thereby our counterparty is *buying protection*.

In this paper, CDSs are of crucial importance since these are traded assets that give information about the market-expected default probabilities of an entity. Calibration of the parameters associated with a specific Cox-process for the default intensity of an entity is done by matching the default (or equivalently survival) probabilities with those implied by the market, which are extracted from market-quotes on CDS contracts.

6.1 Formulation of the CDS and its cash flows

In this section, we mostly adopt the notation from [9], but we modify some parts to keep consistency in the text. Moreover, we aim to construct a more flexible set-up that allows not only for pricing at initialization and resettlement dates, as is the case in [9], but at any time before or during the lifespan of the CDS. This extension allows for the framework to be used both for pricing CDS options and for hedging purposes. The CDS has an annualized premium S, called the CDS spread, which is the annualization of the amount paid at every resettlement date for the protection. We consider a CDS that is initialized at time T_a and matures at time T_b, and we denote the resettlement dates as $T_1, \dots, T_n$ where $T_1 \geq T_a$ and $T_n \leq T_b$. We denote the distance between two consecutive periods (in yearly terms) by $\alpha_i = T_i - T_{i-1}$ for $i = 1, \dots, n$, with the convention $T_0 = T_a$.

We define two functions that can be used to find the following and previous resettlement date of a CDS at any time. We define $\gamma(t)$ as the index for the first resettlement date after time t provided this exists and $n + 1$ if t is after the last resettlement date. Thus

$$\gamma(t) = \begin{cases} \inf\{i \in \mathbb{N} : t \leq T_i, 1 \leq i \leq n\}, & \text{if } t \leq T_n \\ n+1, & \text{if } t > T_n \end{cases}$$

Moreover, we define $\zeta(t)$ as the index for the last resettlement date before time t or zero if this resettlement date does not exist, i.e. t is before the first resettlement date.

$$\zeta(t) = \begin{cases} \sup\{i \in \mathbb{N} : t \geq T_i, 1 \leq i \leq n\}, & \text{if } t \geq T_1 \\ 0, & \text{if } t < T_1 \end{cases}$$

Lastly, we denote the time of default of the reference credit as τ^r. In the case that the reference credit defaults before the CDS maturity, $\tau^r < T_b$, the protection will be paid. The protection amount depends on the contract, but three common protections are a fraction of the notional, a fraction of the value at τ^r of a risk-free bond maturing at time T_b and the realized loss at default of a corporate bond issued by the defaulted entity. We define the full protection amount paid at default as $P(\tau^r)$.

We now have the necessary definitions to characterize the discounted *risk-free* cash flow structure of the CDS. By risk-free cash flows, we mean the cash flows, disregarding the counterparty credit risk, i.e. the only entity assumed to be defaultable is the reference credit. These cash flows are specifically important for calibration purposes since a CDS that is traded on an exchange (assuming the market is liquid) or through a clearing house can be considered as a contract where we only have a defaultable reference credit. The risk-free cash flows are discounted by the risk-free discount-factor, and thereby this discounted risk-free cash flow structure is equivalent to the Π-function introduced in Sec. 3–5. The cash flow is split up into three terms; the payment of the CDS premium S, the payment of the protection upon default of the reference credit, and payment of the accrued interest of the next premium payment, if the default happens between two resettlement dates. We want to present the function $\Pi(u,v)$ for any $u < T_b$ and $v > T_1$, such that this function can be utilized in the framework presented in Sections 4 and 5. The three terms of the cash flow are analyzed separately in the following.

Premium payment. At each resettlement date, the premium S is paid, provided the reference credit has not defaulted at this time. The first payment occurs at the first resettlement date following u, and the last payment occurs at the last resettlement date prior to v, i.e. $T_{\gamma(u)}$ and $T_{\zeta(v)}$. Each of these payments are discounted to time u, which yields the following cash flow

$$\text{Premium}(u,v) = \sum_{i=\gamma(u)}^{\zeta(v)} S\alpha_i e^{-\int_u^{T_i} r_t \mathrm{d}t} \mathbb{1}_{\{\tau^r > T_i\}}. \tag{C.5}$$

Since the premium S is annualized it is multiplied by α_i, which is the time since the last resettlement date (or since initialization for $i = 1$).

Protection payment. The protection payment is a (potentially) one time transaction. Upon default, the protection seller has promised to pay some specified amount that may be constant, but that may also depend on the realized recovery at default of the reference credit. The protection amount is conventionally the difference between the face value of the underlying defaultable bond and the realized recovery on the corporate bond at default of the reference credit, see [3, p. 21]. The protection payment will only be in the considered u, v time period under certain conditions. In general for the protection to be paid, the default must happen during the CDS's lifetime, i.e. $T_a < \tau^r \leq T_b$. Moreover it must hold that $u \leq \tau^r \leq v$ for the payment to be in the considered time period of cash flows. Thereby, the discounted protection payment is

$$\textsc{Protection}(u, v) = -e^{-\int_u^{\tau^r} r_t \mathrm{d}t} P(\tau^r) \mathbb{1}_{\{\max\{u, T_a\} \leq \tau^r \leq \min\{v, T_b\}\}} \quad \text{(C.6)}$$

Note that this cash flow is negative, as we are taking the role as protection sellers.

Accrued interest. Conventionally it is required of the protection buyer to pay an accrued interest on the next (non-realized) premium payment to the protection seller, if the reference credit defaults between two resettlement dates, see [3, pp. 21-22]. The accrued interest is the most difficult part of the cash flow to construct, and thus we build up by first considering a simplified scenario, and subsequently adjusting for the more complex scenarios. The initial scenario we consider is where $u < T_1$ and $v > T_n$. This means that all resettlement dates are included in the time interval considered. In this case the discounted accrued interest takes the form

$$S e^{-\int_u^{\tau^r} r_t \mathrm{d}t} \sum_{i=1}^{n} \mathbb{1}_{\{T_{i-1} < \tau^r \leq T_i\}} (\tau^r - T_{i-1}), \qquad \text{given } u < T_1, v > T_n.$$

In the above equation, we have used the notation $T_0 = T_a$, such that default of the reference credit between T_a and the first resettlement date T_1 will result in an accrued interest on the T_1-payment of the premium. Note that $\tau^r - T_{i-1}$ should be seen as the fraction (in years) between the default and the previous resettlement date, and since S is annualized, this fraction multiplied by S exactly corresponds to the accrued interest paid at the time of default.

Generalizing the above, we use the definition of $T_{\gamma(t)}$ and $T_{\zeta(t)}$ as the resettlement dates directly before and after time t, provided t is between two resettlement dates. Therefore we can let the sum start at $\gamma(u)$ and end at $\zeta(v)$, which means that if u is below T_1 the sum will begin at one and if v is greater than T_n the sum ends at n. Simultaneously we can have u or v (or both) be between two resettlement dates, and the sum will only consider the accrued interest beginning from the first resettlement date after u and ending with the last resettlement date before v, thus not considering accrued interest on premium payments that are not in the considered time period. The more difficult situations are where $u \leq v < T_1$ or $T_n < u \leq v$. In these two scenarios, the summation will be from 1 to 0 and from $n+1$ to n, respectively. Here we have to use the notion that $\sum_{i=k+1}^{k}(\cdot) = 0$ for any integer k. Note that for any choice of $u = v \in [t, \infty)$ this notion also makes sure that the accrued interest term is zero. The resulting cash flow for the accrued interest between time u and v is

$$\textsc{Accrued}(u,v) = Se^{-\int_u^{\tau^r} r_t \mathrm{d}t} \sum_{i=\gamma(u)}^{\zeta(v)} \mathbb{1}_{\{T_{i-1}<\tau^r \leq T_i\}}(\tau^r - T_{i-1}). \quad \text{(C.7)}$$

Collecting the results in Eqs. (C.5), (C.6), and (C.7), the risk-free discounted cash flows of the CDS are given by

$$\Pi^{\mathrm{CDS}}(u,v) = \sum_{i=\gamma(u)}^{\zeta(v)} S\big(\alpha_i e^{-\int_u^{T_i} r_t \mathrm{d}t} \mathbb{1}_{\{\tau^r > T_i\}} + e^{-\int_u^{\tau^r} r_t \mathrm{d}t} \mathbb{1}_{\{T_{i-1}<\tau^r \leq T_i\}}(\tau^r - T_{i-1})\big)$$
$$- e^{-\int_u^{\tau^r} r_t \mathrm{d}t} P(\tau^r) \mathbb{1}_{\{\max\{u,T_a\} \leq \tau^r \leq \min\{v,T_b\}\}}$$

Note that when pricing a CDS at initialization, i.e. $t = T_a$, this cash flow structure corresponds to [10, Eq. (2.2)] and [3, p. 22]. However, $t \neq T_a$ is an important case when considering model calibration due to the CDS quoting mechanism, explained in Sec. 8.

6.2 Valuing a risk-free CDS

CDS valuation without taking into account the credit risk of the entities entering the CDS serves an important role in all aspects of reduced form credit risk modeling. We call this type of CDS a *risk-free* CDS. When a CDS is either traded on a stock-exchange or cleared through a central clearing house, the two parties entering the CDS can disregard credit risk of their individual counterparty, due to collateralization and guarantee from the central clearing house/stock-exchange. The only defaultable entity we consider in this case is the reference credit, and the cash flows in the CDS are only dependent on the time of the first credit event of the reference credit. When considering CDS quotes from either a stock-exchange or a central clearing house, we thereby allow for calibration of the default intensities separately for each entity, provided this type of data is available. For a discussion of calibration of default intensities to market data, see [3, pp. 23-24].

This section serves as a building block for calibrating default intensities to CDS quotes. In this section no model is assumed for the default intensity of the reference credit, it is only assumed that we are in the Cox-process set-up. Later we exemplify using CIR-models for the default intensities.

Using the payoff-structure of the CDS, the value of a risk-free CDS at time t with initialization T_a, maturity T_b and resettlement dates $\boldsymbol{T} = (T_1, \ldots, T_n)$ can be expressed as

$$\begin{aligned} V^{CDS}(t; T_a, T_b, \boldsymbol{T}) &= \mathbb{E}_t^{\mathbb{Q}}\left[\Pi^{CDS}(t, T_b)\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}[\textsc{Premium}(t, T_b) + \textsc{Protection}(t, T_b) + \textsc{Accrued}(t, T_b)] \end{aligned}$$

where the three terms of the Π-function are given in Eq. (C.5), (C.6) and (C.7). We will now look at each of these terms separately, and obtain equations for their expectations in the Cox-process set-up. Firstly we consider the premium payment.

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}[\textsc{Premium}(t, T_b)] &= \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{i=\gamma(t)}^{\zeta(T_b)} S\alpha_i e^{-\int_t^{T_i} r_u \mathrm{d}u} \mathbb{1}_{\{\tau^r > T_i\}}\right] \\ &= S \sum_{i=\gamma(t)}^{n} \alpha_i \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_i} r_u \mathrm{d}u} \mathbb{1}_{\{\tau^r > T_i\}}\right] \\ &= S \sum_{i=\gamma(t)}^{n} \alpha_i \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_i} (r_u + \lambda_u^r) \mathrm{d}u}\right]. \end{aligned}$$

For the protection payment, we have to go back to the conditional distribution of τ^r found in Eq. (C.1) in Sec. 2. In that section, we find the probability density function for $\tau^i | \mathcal{G}_T^i$, $T > t$, for some entity i, where we recall that $\mathcal{G}^i$ is the σ-algebra generated by all factors driving the λ^i process. The Cox-processes start at time t, and it is assumed that no credit event has happened up to and including t, since a credit event will ruin the motivation for pricing. This means that we have conditioned on knowing $\mathcal{H}_t^i$ and that default of i has yet to happen, where $\mathcal{H}^i$ is the σ-algebra generated by the event-process $\{\tau^i > s\}_{s \geq t}$. To calculate the value of the protection payment, we need the distribution of $\tau^r | \mathcal{G}_{T_b}^{\text{ref}} \vee \mathcal{H}_{\max\{t, T_a\}}^{\text{ref}}$, since knowledge about the default up to time $\max\{t, T_a\}$ is necessary when using the law of iterated expectations on the indicator function $\mathbb{1}_{\{\max\{t, T_a\} \leq \tau^r \leq T_b\}}$. Note the superscript "ref" is used for the σ-algebras that are connected to the reference credit, to avoid confusion with those connected to the risk-free rate. If $t \geq T_a$, we are pricing the CDS when or after it is initialized, and at the time of pricing we have the knowledge that the reference credit has not yet defaulted (otherwise the CDS is a redundant derivative with price zero). In this case, we can carry on, as usual, using the density in Eq. (C.1). If on the other hand $t < T_a$, we aim at pricing a CDS which is initialized at a future time; a relevant scenario when pricing CDS options. In this case we need to adjust for the fact that the only relevant time period for the default to occur is between T_a and T_b. Here there are two scenarios; either the reference credit defaults in the time period between t and T_a in which case the CDS is worthless at initialization or the reference credit defaults in the time period $[T_a, \infty)$. Given $T_a \leq \tau^r$ the conditional stopping time $\tau^r | \mathcal{G}^{\text{ref}}_{T_b} \vee \mathcal{H}^{\text{ref}}_{T_a}$ has the density of Eq. (C.1) with t replaced by T_a and T replaced by T_b, whereas given the opposite, $\tau^r < T_a$, the density is not of interest since the CDS has value zero. This proves itself useful in rewriting the expected protection payment:

$$
\begin{aligned}
&\mathbb{1}_{\{t<T_a\}} \mathbb{E}^{\mathbb{Q}}_t \left[e^{-\int_t^{\tau^r} r_u \mathrm{d}u} P(\tau^r) \mathbb{1}_{\{T_a \leq \tau^r \leq T_b\}} \right] \\
&= \mathbb{1}_{\{t<T_a\}} \mathbb{E}^{\mathbb{Q}}_t \left[\mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{\tau^r} r_u \mathrm{d}u} P(\tau^r) \mathbb{1}_{\{T_a \leq \tau^r \leq T_b\}} \,\middle|\, \mathcal{G}^{r,\text{ref}}_{T_b} \vee \mathcal{H}^{\text{ref}}_{T_a} \right] \right] \\
&= \mathbb{1}_{\{t<T_a\}} \mathbb{E}^{\mathbb{Q}}_t \left[\mathbb{1}_{\{T_a \leq \tau^r\}} \int_{T_a}^{T_b} e^{-\int_t^{v} r_u \mathrm{d}u} P(v) \lambda^r_v e^{-\int_{T_a}^{v} \lambda^r_u \mathrm{d}u} \mathrm{d}v \right] \\
&= \mathbb{1}_{\{t<T_a\}} \mathbb{E}^{\mathbb{Q}}_t \left[\mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\{T_a \leq \tau^r\}} \,\middle|\, \mathcal{G}^{r,\text{ref}}_{T_b} \vee \mathcal{H}^{\text{ref}}_{t} \right] \int_{T_a}^{T_b} e^{-\int_t^{v} r_u \mathrm{d}u} P(v) \lambda^r_v e^{-\int_{T_a}^{v} \lambda^r_u \mathrm{d}u} \mathrm{d}v \right] \\
&= \mathbb{1}_{\{t<T_a\}} \mathbb{E}^{\mathbb{Q}}_t \left[\mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\{T_a \leq \tau^r\}} \,\middle|\, \mathcal{G}^{r,\text{ref}}_{T_a} \vee \mathcal{H}^{\text{ref}}_{t} \right] \int_{T_a}^{T_b} e^{-\int_t^{v} r_u \mathrm{d}u} P(v) \lambda^r_v e^{-\int_{T_a}^{v} \lambda^r_u \mathrm{d}u} \mathrm{d}v \right] \\
&= \mathbb{1}_{\{t<T_a\}} \mathbb{E}^{\mathbb{Q}}_t \left[e^{-\int_t^{T_a} \lambda^r_u \mathrm{d}u} \int_{T_a}^{T_b} e^{-\int_t^{v} r_u \mathrm{d}u} P(v) \lambda^r_v e^{-\int_{T_a}^{v} \lambda^r_u \mathrm{d}u} \mathrm{d}v \right] \\
&= \mathbb{1}_{\{t<T_a\}} \mathbb{E}^{\mathbb{Q}}_t \left[\int_{T_a}^{T_b} \lambda^r_v e^{-\int_t^{v} (r_u + \lambda^r_u) \mathrm{d}u} P(v) \mathrm{d}v \right]
\end{aligned}
$$

The only change in the above formula when $t \geq T_a$ (and $t \leq T_b$) is that the integration is from t to T_b instead of from T_a to T_b. Hereby it is possible to characterize the expected protection payment as

$$
\mathbb{E}^{\mathbb{Q}}_t[\textsc{Protection}(t, T_b)] = -\mathbb{E}^{\mathbb{Q}}_t \left[\int_{\max\{t, T_a\}}^{T_b} \lambda^r_v e^{-\int_t^{v} (r_u + \lambda^r_u) \mathrm{d}u} P(v) \mathrm{d}v \right]
$$

The same trick as in the above is used to calculate the expected accrued interest since this relies on a sum of terms each containing an event of the type $T_{i-1} < \tau^r \leq T_i$.

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}[\textsc{Accrued}(t, T_b)] &= \mathbb{E}_t^{\mathbb{Q}}\left[S e^{-\int_t^{\tau^r} r_u \mathrm{d}u} \sum_{i=\gamma(t)}^{\zeta(T_b)} \mathbb{1}_{\{T_{i-1} < \tau^r \leq T_i\}} (\tau^r - T_{i-1}) \right] \\
&= S \sum_{i=\gamma(t)}^{n} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{\tau^r} r_u \mathrm{d}u} \mathbb{1}_{\{T_{i-1} < \tau^r \leq T_i\}} (\tau^r - T_{i-1}) \right] \\
&= S \sum_{i=\gamma(t)}^{n} \mathbb{E}_t^{\mathbb{Q}}\left[\int_{T_{i-1}}^{T_i} \lambda_v^r e^{-\int_t^v (r_u + \lambda_u^r) \mathrm{d}u} (v - T_{i-1}) \mathrm{d}v \right] \\
&= S \mathbb{E}_t^{\mathbb{Q}}\left[\int_{T_{\gamma(t)-1}}^{T_n} (v - T_{\gamma(v)-1}) \lambda_v^r e^{-\int_t^v (r_u + \lambda_u^r) \mathrm{d}u} \mathrm{d}v \right]
\end{aligned}$$

The following equation sums up the results and provides a general formula for calculating the value of a CDS at any time $t \leq T_b$.

$$\begin{aligned}
V^{CDS}(t; T_a, T_b, \boldsymbol{T}) = S & \sum_{i=\gamma(t)}^{n} \alpha_i \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_i} (r_u + \lambda_u^r) \mathrm{d}u} \right] \\
& - \mathbb{E}_t^{\mathbb{Q}}\left[\int_{\max\{t, T_a\}}^{T_b} \lambda_v^r e^{-\int_t^v (r_u + \lambda_u^r) \mathrm{d}u} P(v) \mathrm{d}v \right] \\
& + S \mathbb{E}_t^{\mathbb{Q}}\left[\int_{T_{\gamma(t)-1}}^{T_n} (v - T_{\gamma(v)-1}) \lambda_v^r e^{-\int_t^v (r_u + \lambda_u^r) \mathrm{d}u} \mathrm{d}v \right] \quad \text{(C.8)}
\end{aligned}$$

Independence and regularity of payments

Here we present CDS pricing under some assumptions usually made when calibrating default intensities to CDS market data from an exchange or a central clearing house. It is assumed that the protection is a constant fraction (corresponding to the expected loss-given-default) of the corporate bond price on the reference credit. In fact according to [3, pp. 23-24], the market spreads are usually found this way using a constant recovery, which is quoted and very irregularly updated. Therefore we want to find market quotes that include a quoted constant recovery, such that the CDS value in Eq. (C.8) is only used to calibrate the λ^r process along with its possible correlation to the risk-free rate. The risk-free rate itself is calibrated to market data on e.g. the overnight interest rate swap (OIS) in the considered currency issued by e.g. a central bank.

Moreover, it is assumed that the CDS contract has equidistant premium payments, i.e. $\alpha_i = \alpha \ \forall i$, that the time between initialization and the first payment is α, i.e. $T_1 = T_a + \alpha$, and that the last payment is at contract termination, $T_b = T_n$. Lastly, a usual assumption is that the risk-free rate and the default intensities are independent, which simplifies the situation considerably. In [11] this assumption is only used in the CDS calibration, and a correlation structure is found afterwards. The argument in [11] for this approach is that when using the CIR++ model for both risk-free rates and default intensities, this correlation does not affect the value of the CDS. A case study of parameters is used to confirm this claim.

Under these assumptions, the CDS price at initialization is given by

$$\begin{aligned} V^{CDS}(T_a;T_a,T_b,\boldsymbol{T}) = S\alpha \sum_{i=1}^{n} D(T_a,T_i)\mathbb{E}^{\mathbb{Q}}_{T_a}\left[e^{-\int_{T_a}^{T_i}\lambda^r_u \mathrm{d}u}\right] \\ - P\int_{T_a}^{T_b} D(T_a,v)\mathbb{E}^{\mathbb{Q}}_{T_a}\left[\lambda^r_v e^{-\int_{T_a}^{v}\lambda^r_u \mathrm{d}u}\right]\mathrm{d}v \\ + S\int_{T_a}^{T_b} (v-T_{\gamma(v)-1})D(T_a,v)\mathbb{E}^{\mathbb{Q}}_{T_a}\left[\lambda^r_v e^{-\int_{T_a}^{v}\lambda^r_u \mathrm{d}u}\right]\mathrm{d}v \end{aligned} \tag{C.9}$$

where $\boldsymbol{T} = (T_a+\alpha,\ldots,T_b-\alpha,T_b)$ and

$$D(u,v) = \mathbb{E}^{\mathbb{Q}}_u\left[e^{-\int_u^v r_u \mathrm{d}u}\right], \quad \text{for } v \geq u,$$

is the zero coupon bond price with the risk-free rate as the underlying. In order to obtain an analytic expression for this value, it is clear that we need analytic expressions for a zero coupon bond with the risk-free rate as the underlying as well as a similar derivative with the default intensity as the underlying. Moreover an analytic expression of

$$\mathbb{E}^{\mathbb{Q}}_{T_a}\left[\lambda^r_v e^{-\int_{T_a}^{v}\lambda^r_u \mathrm{d}u}\right]$$

is necessary in order to obtain an exact analytic pricing formula for the CDS.

In Sec. 7.2 it is shown how to avoid this expectation for calibration purposes.

7 Modeling risk-free interest rates and default intensities

Models for pricing counterparty credit risk in CDS contracts have different properties.

In [11] a Shifted Square Root Diffusion (SSRD) Model is introduced. The SSRD model ensures positivity in the process, which makes the model feasible to treating both the risk-free interest rates and default intensities for different entities.

In [5] the case of CVA with CIR++ processes for the default intensities of both the reference credit and the counterparty is considered. The interest rates and recovery rates are assumed constant, and the pricing entity is assumed default-free (thus we are in the UCVA pricing framework). The correlation between the reference credit and the counterparty is modeled through a Gaussian copula.

In [9] a BCVA extension of [5] is provided. The authors propose a *General bilateral counterparty risk pricing formula* and provides a proof for this formula in the paper's appendix. Further they specify the pricing formula for receiver CDS in which default intensities (for the pricing entity, the counterparty and the reference credit, respectively) are modeled as CIR++ models with jumps. The default intensities are modeled through a trivariate Gaussian copula. The risk-free interest rate is assumed deterministic, as are the recovery rates of the pricing entity and the counterparty. Further the paper provides a numerical example and pseudo code with real data.

We limit our experiments to a CIR++ model, due to simplicity in the calibration and due to the closed-form expressions of the relationship between $\mathbb{Q}$ and $\mathbb{P}$ parameters that are important for the estimation of the market price of risk in Sec. 9.

7.1 Model calibration through implied survival probabilities and implied hazard rates

This section follows the idea of [1, pp. 66-69] and [12, pp. 731-735 and 764-776]. It is rewritten using our notation, and to fit our purpose, with several extra details and comments. Note that the first part of this section is model-independent. In this approach to model calibration, we assume that the model provides survival probabilities in closed form. Specifically, we apply the approach to the CIR-model, however the method's application is not limited to the CIR-model.

Firstly, we express the value of a CDS using the formula in (C.9), rewritten using Theorem 2 on page 108:

$$\begin{aligned} V^{CDS}(T_a; T_a, T_b, \boldsymbol{T}) = {} & S\alpha \sum_{i=1}^{n} D(T_a, T_i) \mathbb{E}_{T_a}^{\mathbb{Q}} \left[e^{-\int_{T_a}^{T_i} \lambda_u^r \mathrm{d}u} \right] \\ & + P \int_{T_a}^{T_b} D(T_a, v) \frac{\partial}{\partial v} \left(\mathbb{E}_{T_a}^{\mathbb{Q}} \left[e^{-\int_{T_a}^{v} \lambda_u^r \mathrm{d}u} \right] \right) \mathrm{d}v \\ & - S \int_{T_a}^{T_b} (v - T_{\gamma(v)-1}) D(T_a, v) \frac{\partial}{\partial v} \left(\mathbb{E}_{T_a}^{\mathbb{Q}} \left[e^{-\int_{T_a}^{v} \lambda_u^r \mathrm{d}u} \right] \right) \mathrm{d}v, \end{aligned} \tag{C.10}$$

where $\boldsymbol{T} = (T_a + \alpha, \ldots, T_b - \alpha, T_b)$. Here we have assumed independence between interest rates and the default intensity of the reference name. The aim is to rewrite each of the three terms in Eq. (C.10) as expressions of the conditional survival probability of the reference credit given the current market information. The first term is trivial:

$$\begin{aligned}\text{Term } 1 &= S\alpha \sum_{i=1}^{n} D(T_a, T_i)\mathbb{E}^{\mathbb{Q}}_{T_a}\left[e^{-\int_{T_a}^{T_i}\lambda^r_u \mathrm{d}u}\right] \\ &= S\alpha \sum_{i=1}^{n} D(T_a, T_i)\mathbb{Q}\big(\tau^r \geq T_i \mid \mathcal{F}_{T_a}\big).\end{aligned}$$

In the second term, the aim is to obtain an integral in the survival probability.

$$\begin{aligned}\text{Term } 2 &= P\int_{T_a}^{T_b} D(T_a, v)\frac{\partial}{\partial v}\left(\mathbb{E}^{\mathbb{Q}}_{T_a}\left[e^{-\int_{T_a}^{v}\lambda^r_u \mathrm{d}u}\right]\right)\mathrm{d}v \\ &= P\int_{T_a}^{T_b} D(T_a, v)\frac{\mathrm{d}}{\mathrm{d}v}\mathbb{Q}\big(\tau^r \geq v \mid \mathcal{F}_{T_a}\big)\mathrm{d}v \\ &= P\int_{T_a}^{T_b} D(T_a, v)\mathrm{d}\mathbb{Q}\big(\tau^r \geq v \mid \mathcal{F}_{T_a}\big).\end{aligned}$$

The third term is rewritten in the same manner as the second.

$$\begin{aligned}\text{Term } 3 &= -S\int_{T_a}^{T_b}(v - T_{\gamma(v)-1})D(T_a, v)\frac{\partial}{\partial v}\left(\mathbb{E}^{\mathbb{Q}}_{T_a}\left[e^{-\int_{T_a}^{v}\lambda^r_u \mathrm{d}u}\right]\right)\mathrm{d}v \\ &= -S\int_{T_a}^{T_b}(v - T_{\gamma(v)-1})D(T_a, v)\mathrm{d}\mathbb{Q}\big(\tau^r \geq v \mid \mathcal{F}_{T_a}\big),\end{aligned}$$

where S is the credit spread, P is the constant loss-given-default. $D(s,t)$ is the discount factor from time s to t for $s \leq t$, which is interpolated from quotes of zero-coupon bonds.

The function to be calibrated to CDS spreads is

$$S = \frac{P\int_{T_a}^{T_b} D(T_a, v)\mathrm{d}\mathbb{Q}^{T_a}(v)}{\alpha\sum_{i=1}^{n} D(T_a, T_i)\mathbb{Q}^{T_a}(v) - \int_{T_a}^{T_b}(v - T_{\gamma(v)-1})D(T_a, v)\mathrm{d}\mathbb{Q}^{T_a}(v)}, \tag{C.11}$$

where $\hat{\mathbb{Q}}^a(v) = \mathbb{Q}\big(\tau^r \geq v \mid \mathcal{F}_{T_a}\big)$.

Using the quoted CDS-spread, we need to extract survival probabilities such that the above formula holds. This is done by considering the integrals as Riemann-Stieltjes integrals in the survival probability and approximate this numerically with Riemann-Stieltjes sums. We consider some discretization

$$\big\{t^{(i)} : i = 0, 1, \ldots, n,\ t^{(i)} < t^{(i+1)},\ t^{(0)} = T_a,\ t^{(n)} = T_b^{\max}\big\},$$

where $T_b^{\max}$ is the largest maturity for all available CDS quotes on the reference name. Then

$$\left\{\mathbb{Q}^{\mathrm{mkt}}\left(\tau^r > t^{(i)} \,\middle|\, \mathcal{F}_{T_a}\right) : t^{(i)} \leq 1y\right\}$$

is calibrated using the 1-year CDS quote. Next

$$\left\{\mathbb{Q}^{\text{mkt}}\left(\tau^r > t^{(i)} \,\middle|\, \mathcal{F}_{T_a}\right) : 1y < t^{(i)} \leq 2y\right\}$$

is calibrated to the 2-year CDS quote, and so forth. Note, we use the notation $\mathbb{Q}^{\text{mkt}}$ for the survival probabilities implied by the market to be able to distinguish these from the model survival probabilities.

Similarly, we can consider the implied hazard rates, using the implied survival probabilities by the formula

$$\mathbb{Q}^{\text{mkt}}\big(\tau^r > s \,\big|\, \mathcal{F}_{T_a}\big) = e^{-\Gamma^{\text{mkt}}(s;T_a)}, \quad \text{for all } s > t.$$

Here $\Gamma^{\text{mkt}}(v;u)$ is the function representing the integrated implied hazard rates from time u to time v for all $v > u \geq T_a$. Specifically

$$\Gamma^{\text{mkt}}(s;T_a) = \int_{T_a}^{s} \lambda_u^{\text{mkt}} \mathrm{d}u, \quad \text{for all } s > T_a. \tag{C.12}$$

Turning to the model survival probabilities, we have assumed that these take on a closed form. We denote the parameter vector for the model $\mathbf{\Theta}$ and use the following notation for the survival probabilities generated by the model:

$$P(T_a, s, \mathbf{\Theta}) = \mathbb{Q}^{\text{model}}\big(\tau^r > s \,\big|\, \mathcal{F}_{T_a}\big) = \mathbb{E}_{T_a}^{\mathbb{Q}}\Big[e^{-\int_{T_a}^{s} \lambda_u^r \mathrm{d}u}\Big].$$

Now, we can express the function that is used for calibration; we wish to calibrate model parameters such that the model and market survival probabilities agree, i.e.

$$\Gamma^{\text{mkt}}(s;T_a) = -\ln\left(P(T_a, s, \mathbf{\Theta})\right) \tag{C.13}$$

Assuming the discretization used for the Riemann–Stieltjes integrals is reasonable and gives a *good* approximation, it is observed that the model CDS spreads will agree with the quoted CDS spreads, if the default intensities for the model agree with those implied from the quotes. Thus we change the problem of calibrating *model CDS spreads to quotes* to the problem of calibrating *model survival probabilities to implied survival probabilities*.

One problem that needs to be addressed: Implied hazard rates from CDS quotes will only yield information on the integrated hazard rates and not the instantaneous ones. This means that one needs to choose to either consider these constant or linear on each interval – both cases considered in [12] – or use interpolation to obtain intermediate values. [3, pp. 23-25] consider only the case of piecewise constant hazard rates, i.e. they assume

$$\lambda_u^{\text{mkt}} = \begin{cases} \lambda_{\gamma(u)}^{\text{mkt}}, & \text{if } u \leq T_n \\ \lambda_{T_n}^{\text{mkt}}, & \text{if } u > T_n. \end{cases}$$

Thus Eq. (C.12) simplify to

$$\Gamma^{\text{mkt}}(s;T_a) = \sum_{i=0}^{\zeta(s)-1} \alpha\lambda^{\text{mkt}}_{T_{i+1}} + (s - T_{\zeta(s)})\lambda^{\text{mkt}}_{T_{\zeta(s)+1}}$$

and the implied survival probabilities can easily be translated into implied hazard rates. The above formulas use the functions γ and ζ as defined in Sec. 6.1.

7.2 The CIR++ model

We consider the CIR++ model, presented in [12], which is a shifted CIR model defined by:

$$\begin{aligned} \lambda_t &= x_t + \phi_t \\ \mathrm{d}x_t &= \kappa(\theta - x_t)\mathrm{d}t + \sigma\sqrt{x_t}\mathrm{d}w_t, \quad x_0 = x \end{aligned} \tag{C.14}$$

where the ϕ_t is a non-negative deterministic function added to ensure that the calibration fits the data exactly. The parameters κ, θ, σ are all required to be non-negative, and the Feller condition $2\kappa\theta > \sigma^2$ must be satisfied in order to secure that the state variable x is positive. The ws are Wiener-processes under the $\mathbb{Q}$-measure.

The SDE in Eq. C.14 has the following analytic expression and conditional mean and variance

$$x_t = e^{-\kappa(t-s)}x_s + \theta\left(1 - e^{-\kappa(t-s)}\right) + \sigma\int_s^t e^{-\kappa(t-u)}\sqrt{x_u}\mathrm{d}w_u,$$

$$\mathbb{E}^{\mathbb{Q}}\left[x_t \,\middle|\, \mathcal{G}^i_s\right] = e^{-\kappa(t-s)}x_s + \theta\left(1 - e^{-\kappa(t-s)}\right)$$

$$\mathbb{V}^{\mathbb{Q}}\left[x_t \,\middle|\, \mathcal{G}^i_s\right] = x_s\frac{\sigma^2}{\kappa}\left(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}\right) + \frac{\sigma^2\theta}{2\kappa}\left(1 - e^{-\kappa(t-s)}\right)^2$$

When calibrating data to CDS spreads, two conditional expectations are of specific interest

$$\mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^T \lambda_u \mathrm{d}u}\right] = e^{-\int_t^T \phi_u \mathrm{d}u}\mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^T x_u \mathrm{d}u}\right] \tag{C.15}$$

$$\mathbb{E}^{\mathbb{Q}}_t\left[\lambda_T e^{-\int_t^T \lambda_u \mathrm{d}u}\right] = \phi_T e^{-\int_t^T \phi_u \mathrm{d}u}\mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^T x_u \mathrm{d}u}\right] + e^{-\int_t^T \phi_u \mathrm{d}u}\mathbb{E}^{\mathbb{Q}}_t\left[x_T e^{-\int_t^T x_u \mathrm{d}u}\right]$$

Therefore we want to analyze the two resulting expectations of the CIR process

$$\mathbb{E}^{\mathbb{Q}}_t\left[e^{-\int_t^T x_u \mathrm{d}u}\right] \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}}_t\left[x_T e^{-\int_t^T x_u \mathrm{d}u}\right]. \tag{C.16}$$

We observe that the first expectation in Eq. (C.16) is equivalent to a bond price when using a CIR model for the short-rate. To obtain the solution to the bond price, we introduce three functions of the time to maturity

$$a(\tau) = 2h\, e^{\frac{1}{2}(\kappa+h)\tau}, \quad b(\tau) = 2\left(e^{h\tau} - 1\right), \quad \text{and} \quad c(\tau) = 2h + (\kappa + h)\left(e^{h\tau} - 1\right).$$

and two constants

$$h = \sqrt{\kappa^2 + 2\sigma^2} \quad \text{and} \quad \alpha = \tfrac{2\kappa\theta}{\sigma^2}.$$

Using the above concepts, the bond price according to [12, p. 66] is

$$P^{\text{CIR}}(t, T, x_t; \kappa, \theta, \sigma) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T x_u \mathrm{d}u}\right] = A(t,T)e^{-B(t,T)x_t}, \tag{C.17}$$

where

$$A(t,T) = \left[\frac{a(T-t)}{c(T-t)}\right]^{\alpha} \quad \text{and} \quad B(t,T) = \frac{b(T-t)}{c(T-t)}.$$

For the calculation of the second mean in (C.16), we need the following two theorems, where Theorem 1 is a subresult of the well known the Fubini-Tonelli theorem[3] and Theorem 2 is a self-constructed theorem by the authors. We remark that this theorem plays an important role in model calibration, and is used to find implied survival probabilities and implied default intensities, as described in Section 7.1.

Theorem 1 (A result from the Fubini-Tonelli theorem). *Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be measure spaces, and μ and ν be finite measures. Let $f : X \times Y \to \mathbb{R}$ be a non-negative function that is measurable wrt. the product σ-algebra $\mathcal{F} \otimes \mathcal{G}$. Then*

$$\int_X \int_Y f(x,y)\mathrm{d}\nu(y)\mathrm{d}\mu(x) = \int_Y \int_X f(x,y)\mathrm{d}\mu(x)\mathrm{d}\nu(y).$$

Theorem 2 (Changing the order of expectation and differentiation). *Let $(\Omega,\ \mathcal{F},\ \mathbb{Q},\ \underline{\mathcal{F}})$ be a probability space with the filtration $\underline{\mathcal{F}} = \{\mathcal{F}_t\}_{t\geq 0}$. Let $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be differentiable in the first variable. Further let $\frac{\partial}{\partial T}F(T,\omega) = f(T,\omega)$ be an $\underline{\mathcal{F}}$-measurable function, i.e. given $\omega \in \Omega$, $f(T,\omega) \in \mathcal{F}_T$ for $T \geq 0$. If $f(T,\omega)$ is non-negative for all $T \in \mathbb{R}_+$ and all $\omega \in \Omega$, then*

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{\partial}{\partial T}F(T,\omega)\right] = \frac{\partial}{\partial T}\mathbb{E}_t^{\mathbb{Q}}[F(T,\omega)]. \tag{C.18}$$

[3] Theorem 1 is based on the Fubini-Tonelli theorem as stated in [13, pp. 77–78] and [14, p. 477].

Proof. The proof takes its initial point in right-hand-side of (C.18) by writing a difference of a finite step of length $h \in (0, \infty)$;

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}[F(T+h,\omega)] - \mathbb{E}_t^{\mathbb{Q}}[F(T,\omega)] &= \mathbb{E}_t^{\mathbb{Q}}\left[\int_T^{T+h} f(s,\omega)\mathrm{d}s\right] \\ &= \int_T^{T+h} \mathbb{E}_t^{\mathbb{Q}}[f(s,\omega)]\mathrm{d}s,\end{aligned}$$

where the last equality is obtained by using Theorem 1 on $f(s,\omega)$, which is allowed due to the assumptions that $f(s,\omega)$ is non-negative and measurable wrt. $\mathcal{F}_s \otimes \mathcal{B}([0,s])$ for all $s \geq 0$, where $\mathcal{B}(\cdot)$ denotes the Borel measure. The proof is completed by dividing with h on both sides of the equality and letting $h \to 0$. □

Using Theorem 2, we get the following result for the second expectation of (C.16).

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}\left[x_T e^{-\int_t^T x_u \mathrm{d}u}\right] &= \mathbb{E}_t^{\mathbb{Q}}\left[-\frac{\partial}{\partial T} e^{-\int_t^T x_u \mathrm{d}u}\right] = -\frac{\partial}{\partial T}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T x_u \mathrm{d}u}\right] \\ &= -\frac{\partial A}{\partial T}(t,T)e^{-B(t,T)x_t} + A(t,T)\frac{\partial B}{\partial T}(t,T)x_t e^{-B(t,T)x_t} \\ &= \left(\alpha\left(\frac{c'(T-t)}{c(T-t)} - \frac{\kappa+h}{2}\right) + \frac{2h\, b'(T-t)x_t}{c(T-t)^2}\right) A(t,T)e^{-B(t,T)x_t}.\end{aligned}$$

Note that here we only present the initial formula used and the result, since rewriting the expressions are rather long and tedious.

The derivative of the bond price is hereby seen to be a known, computable, function of time to maturity and current value of the stochastic process x multiplied by the bond price itself.

8 Calibration to market data

Here, we change the notation such that the CDS is dependent on the quoted CDS spread. We denote the quoted spread of a CDS with initialization T_a and maturity T_b as $S_{a,b}$, and the value of such a CDS at time t as $V^{CDS}(t; T_a, T_b, \boldsymbol{T}, S_{a,b})$.

The loss-given-default of the reference credit is assumed constant equal to 0.4, and will take on the role as the constant protection rate in Sec. 6.2. Note that all data considered in this paper, described in Sec. 8.1, are based on the assumption that loss-given-default is 0.4.

In the market, CDS spreads have formerly been quoted in the following way. At time T_a the contract has value zero, i.e.

$$V^{CDS}(T_a; T_a, T_b, \boldsymbol{T}, S_{a,b}) = 0.$$

Note that the spread $S_{a,b}$ is the mid-spread, and is often found through the bid- and ask quotes obtained from the market. All market quotes are set such that the contract initialization is on the day of the quote and the maturities are a fixed set of tenors e.g. $T_b \in \{1, 2, 3, 4, 5, 7, 10\}$ year(s). The contract length is kept constant in the market such that at time $T_a + x$ day(s) the quoted spread corresponds to a contract maturing at $T_b + x$ day(s). In other words, the quoted 1-year CDS spread will always correspond to a CDS with one year to maturity at the time the spread is quoted. If the reference credit experiences a default event, the CDS spread is no longer quoted.

According to [15, p. 8], the quoting mechanism has changed such that the maturity date stays fixed and the CDS corresponding to the quotes at $T_a + x$ day(s) is x day(s) "closer" to the maturity date than the quotes at T_a. When the quoted contracts reach the maturity date, a new set of tenor dates is determined. Thus it is of great importance to know exactly what kind of data are available.

8.1 Bloomberg CDS data

We consider so-called single-name GCDS data from Bloomberg. A GCDS is a CDX-tranch with a bond as the underlying, and thus a single-name GCDS is equivalent to a CDS on a bond. We use only CBGN-quotes, i.e. the closing quotes from BBG, New York,[4] and consider all available tenors, which are 1, 2, 3, 4, 5, 7, and 10 years. We restrict the data analysis to the following 6 names within the financial sector; the name in parenthesis is used to shorten the notation throughout the analysis:

- Goldman Sachs Group Inc/The (GSachs)
- Citigroup Inc (Citi)
- JPMorgan Chase & Co (JPMorgan)
- Bank of America Corp (BoAmerica)
- Bear Stearns Cos LLC/The (BStearns)
- Lehman Brothers Holdings Inc (Lehman)

The data on all names are from January 27, 2005 to January 28, 2015. The number of observations and the number of NA's in the data for each name and tenor is displayed in Tab. C.1 and C.2. The following is observed

- The number of observations for Lehman Brothers and Bear Stearns is quite low, due to a high number of NA's after the default in the two companies in 2008

[4] 10 sources (exchanges) of GCDS quotes are available in Bloomberg, 4 of which can only be viewed in the Bloomberg terminal and cannot be exported.

- The tenors 5 and 10 years are the most common across names, followed by the tenors 1 and 3 years. The tenors 2 and 4 years have fewer observations, as well as fewer NA's.

Tables C.3, C.4, C.5, and C.6 give some descriptive statistics of the data. We observe that the data tend to be skewed to the right with high standard deviations compared to the spreads' means. Further, the spreads tend to be leptokurtic.

	Tenor						
	I	II	III	IV	V	VII	X
GSachs	2276	1587	2288	1572	2552	1911	2413
Citi	2277	1570	2320	1578	2594	1907	2411
JPMorgan	2236	1569	2296	1584	2599	1871	2226
BoAmerica	2247	1490	2275	1568	2589	1578	2389
BStearns	805	344	889	336	1157	661	782
Lehman	537	28	504	67	889	271	515

Table C.1: Number of observations

	Tenor						
	I	II	III	IV	V	VII	X
GSachs	182	135	266	142	56	369	131
Citi	181	152	234	136	14	373	133
JPMorgan	222	153	258	130	9	409	318
BoAmerica	211	232	279	146	19	702	155
BStearns	1653	1378	1665	1378	1451	1619	1762
Lehman	1921	1694	2050	1647	1719	2009	2029

Table C.2: Number of NA observations

8.2 Assumptions and formulas for calibration to Bloomberg data

In the following, we present our assumptions used to calibrate the model to the GCDS data from Bloomberg. For the *ISDA specifications* of standard corporate CDSs, we refer to [16] for European contracts and to [17] for North American.

Loss-given-default The loss-given-default is assumed to be 40% of the cor-

	Tenor						
	I	II	III	IV	V	VII	X
GSachs	95.01	140.55	111.03	160.77	125.68	153.94	137.57
Citi	97.62	151.27	116.86	175.34	130.22	164.77	143.52
JPMorgan	36.86	60.50	54.73	84.25	74.17	93.22	89.14
BoAmerica	85.14	132.25	104.14	156.44	116.97	173.55	131.77
BStearns	80.68	115.22	76.34	116.20	75.73	95.21	89.49
Lehman	101.97	126.66	94.62	303.64	73.49	131.76	89.60

Table C.3: Mean values

	Tenor						
	I	II	III	IV	V	VII	X
GSachs	113.39	104.62	100.47	88.83	91.37	84.91	84.51
Citi	148.41	141.21	124.60	114.24	112.38	103.21	99.61
JPMorgan	38.41	37.16	40.05	34.76	42.07	39.18	43.39
BoAmerica	113.81	111.66	101.58	92.72	95.19	79.74	89.48
BStearns	109.77	46.95	85.00	38.54	70.89	67.46	59.46
Lehman	191.76	36.18	142.39	134.38	91.69	107.77	82.73

Table C.4: Standard deviations

porate bond price on the reference credit. Thus that the constant protection rate on the CDS, P, is assumed to be 0.4.

Distance between premium payments is assumed to be a quarter of a year, and is assumed to be equal for all consecutive premium payments; thus $\alpha = 0.25$. This assumption coincides with the ISDA specification for both Europe and North America.

Reset dates i.e. the dates on which a new set of maturities are fixed, are assumed to be the ISDA CDS dates; 20^{th} of March, June, September, and December of every year.

Maturity date is assumed to be a reset date.

First premium payment is assumed to be on the pricing date if this date coincides with a reset date. If the pricing date is not a reset date, the first premium payment is the first reset date after the pricing date. The ISDA specifications are that all premium payments are paid on four specific dates; 20^{th} of March, June, September, and December.

Last premium payment is assumed to be on the maturity of the contract.

	Tenor						
	I	II	III	IV	V	VII	X
GSachs	1.79	1.24	1.26	1.09	1.00	0.75	0.81
Citi	2.91	2.12	1.94	1.77	1.47	1.07	0.82
JPMorgan	2.30	1.67	1.10	0.99	0.43	-0.05	0.10
BoAmerica	2.09	1.48	1.51	1.27	1.12	1.29	0.84
BStearns	4.03	1.48	3.44	0.90	2.67	2.51	2.88
Lehman	2.57	0.33	2.12	0.17	2.30	1.18	2.09

Table C.5: Skewness

	Tenor						
	I	II	III	IV	V	VII	X
GSachs	5.86	3.97	4.08	3.51	3.65	3.38	3.38
Citi	12.70	8.17	7.84	7.06	6.30	5.33	4.25
JPMorgan	9.79	6.32	4.20	3.67	2.72	2.73	1.95
BoAmerica	6.86	4.45	5.02	4.17	4.14	4.25	3.56
BStearns	31.38	5.41	25.43	3.66	17.09	17.40	20.43
Lehman	9.10	1.93	6.96	3.35	8.51	4.09	7.55

Table C.6: Kurtosis (Normal distribution: 3)

Time to maturity on the pricing date is assumed to be T years for a T-year CDS if the pricing date coincides with a reset date. If this is not the case, the contract maturity is assumed to be the first reset date, which is more than T years after the pricing date. In this way, a T-year CDS will always have at least T years to maturity when it is traded, and it will always have time to maturity strictly less than $T + \alpha$.

Initialization of the contract is assumed to be on the first reset date prior to the pricing date. If the pricing date coincides with a reset date, then initialization is assumed to be on the pricing date.

Accrued interest on premium payments is assumed to start on the first ISDA CDS date before the pricing date. Thus the period of accrual on the first premium payment starts α before the actual quoting date.

Under these assumptions, we can add further details to Eq. (C.11), which explains the relationship between the default probability of a CDS and the quoted fair spread. For this purpose, we introduce two constant time steps: The time from t to the first premium payment date is denoted δ. We use the convention that δ is zero if the first premium payment occurs at time t; and the time lag between a default occurring and the settlement of the protection and accrued interest, denoted Δ.

Let $S_{t,T}$ denote the quoted CDS spread at time t for a T-year CDS. Eq. (C.11) can then be written as:

$$S_{t,T} = \frac{-f_1\left(\mathbb{Q}(\,\cdot\,); t, T_b, 0.25, 0.4, \Delta, D(t,\,\cdot\,)\right)}{f_2(\mathbb{Q}(\,\cdot\,); t, T, 0.25, D(t,\,\cdot\,)) - f_3(\mathbb{Q}(\,\cdot\,); t, T, T_b, 0.25, \Delta, D(t,\,\cdot\,))} \tag{C.19}$$

where $T_b = t + T + \delta$ by definition and

$$\begin{aligned}
&f_1(\mathbb{Q}(\,\cdot\,); t, T_b, \alpha, P, \Delta, D(t,\,\cdot\,)) \\
&\qquad = P \int_t^{T_b} D(t, v+\Delta) \frac{\partial}{\partial v} \mathbb{Q}(\tau^r \geq v \mid \mathcal{F}_t) \mathrm{d}v \\
&f_2(\mathbb{Q}(\,\cdot\,); t, T, \alpha, D(t,\,\cdot\,)) \\
&\qquad = \alpha \sum_{i=1}^{\frac{T}{\alpha}+1} D(t, T_i) \mathbb{Q}(\tau^r \geq T_i \mid \mathcal{F}_t) \\
&\qquad = \alpha \sum_{i=1}^{\frac{T}{\alpha}+1} D(t, t+\delta+(i-1)\alpha) \mathbb{Q}(\tau^r \geq t+\delta+(i-1)\alpha \mid \mathcal{F}_t) \\
&f_3(\mathbb{Q}(\,\cdot\,); t, T, T_b, \alpha, \Delta, D(t,\,\cdot\,)) \\
&\qquad = \int_t^{T_b} \left(v - T_{\gamma(v)-1}\right) D(t, v+\Delta) \frac{\partial}{\partial v} \mathbb{Q}(\tau^r \geq v \mid \mathcal{F}_t) \mathrm{d}v.
\end{aligned}$$

Several remarks on these formulas are in order. Firstly, a few notes on two of the function inputs: We use the input T as the time span from first to last premium payment in the contract, i.e. $T = 1$ for a 1-year CDS always. The input T_b is used as the maturity date of the contract, and since the first premium payment occurs at time $t + \delta$ and there is a time difference of T between the first and last premium payments, we have $T_b = t + \delta + T$. Note that the number of premium payments is always $T/\alpha + 1$ since the time distance from first to last premium payment is T.

The accrual period in the integral is not from t to the possible default, but starts at the first reset date prior to or at t. This means that if t is not a reset date, a default occurring before the first premium payment date of the contract will result in accrued interest of the missing premium from a time step of $(\alpha - \delta)$ before t until the time of default. This way of defining the accrual period is exactly the ISDA specification of accrual for both European and North American contracts.

Note that the formula for the spread presented in this section is in accordance with the general CDS valuation formula in Eq. (C.8). In order to obtain Eq. (C.8) we use the assumptions that

$$T_a = \begin{cases} t, & \text{if } t \text{ is a reset date} \\ t - (\alpha - \delta), & \text{otherwise} \end{cases}$$

that the maturity date is $T_b = t + T + \delta$, that there are $\frac{T}{\alpha} + 1$ coupon payments occurring at $\boldsymbol{T} = (T_0, \ldots, T_n)$ with $T_i = t + \delta + (i-1)\alpha$ for $i = 0, \ldots, n$, and lastly that the risk-free rate and the default intensity are independent.

8.3 i^{th} tenor

We now consider n CDSs on the same reference credit, quoted on the same date t, with different tenors. In the Bloomberg data, the tenors available are $1, 2, 3, 4, 5, 7$, and 10 years. We denote the contracts in ascending order with respect to the tenors, such that contract 1 has the smallest tenor and contract n has the largest tenor. By assumption, δ is equal for all n contracts, since they all have the same quoting date. Further α and the constant protection fraction P are assumed equal for all contracts, and we assume that $\Delta = 0$, i.e. the settlement of the CDS protection occurs exactly at the (possible) time of default of the reference credit.

Consider the quote for the i^{th} tenor with $1 \leq i \leq n$. Let $T^{(i)}$ be the tenor and $T_b^{(i)} = t + \delta + T^{(i)}$ the maturity of this contract. We assume that – since we are currently considering tenor i – the implied survival probabilities are calibrated (and can be applied in the formulas) up to time $T_b^{(i-1)}$, where we use the convention that $T_b^{(0)} = 0$.

We denote the implied survival probabilities as $\mathbb{Q}_t(s)$ for $s \in [0, T_b^{(i)} - t]$ such that $\mathbb{Q}_t(s)$ is used to approximate $\mathbb{Q}(\tau^r \geq t + s \mid \mathcal{F}_t)$ and the derivative of $\mathbb{Q}_t(s)$ as $\dot{\mathbb{Q}}_t(s)$ for $s \in [0, T_b^{(i)} - t]$ such that $\dot{\mathbb{Q}}_t(s)$ is used to approximate $\frac{\partial}{\partial s}\mathbb{Q}(\tau^r \geq t + s \mid \mathcal{F}_t)$. Further we denote the discount factor to time s as $D_t(s) = D(t, t+s)$, and assume that these are known; for example, we have calibrated the risk-free interest rate process from quoted OIS rates from a central bank and have a closed-form solution of $D_t(s)$ for all $s \in [0, T_b^{(i)} - t]$.

We discretize the integrals in Eq. (C.19), such that there are $N_\alpha \in \mathbb{N}$ discrete time periods between any two consecutive premium payments; thus the discrete time step considered is $\Delta t = \alpha / N_\alpha$. After discretizing the integrals, we approximate each integral on each of these time periods by the trapezoidal rule. Note that this construction does not ensure that the time between the quoting date and the first premium payment can be split into "complete" consecutive discrete time steps each of length Δt. Therefore we force the first step in the discretization to be defined such that all other steps have length Δt. Let

$$N_\delta = \sup\{j \in \mathbb{N}_0 : \delta \geq j\Delta t\}.$$

The first step in the discretization then has length $\delta - N_\delta \Delta t$ and the remaining N_δ steps between t and $t + \delta$ are all "full" periods of length Δt. With this construction, the first discretization step has a length of 0 if $\delta = 0$, i.e. if the quoting date coincides with a reset date. However, this does not have any practical implications on the formulas presented below.

We define the total number of time periods in the discretization for the i^{th} contract as

$$N^{(i)} = N_\delta + 1 + \tfrac{T^{(i)}}{\Delta t} = N_\delta + 1 + \tfrac{T^{(i)}}{\alpha} N_\alpha,$$

and denote the length of each time step in the discretization as

$$d_k = \begin{cases} \delta - N_\delta \Delta t, & \text{for } k = 1 \\ \Delta t, & \text{for } k = 2, \ldots, N^{(i)}. \end{cases}$$

Thus we have the following discretization of the interval $[0,\, T_b^{(i)} - t] = [0,\, \delta + T^{(i)}]$:

$$\begin{aligned} \text{DIS} = \Big(&(0,\, d_1], (d_1,\, d_1 + d_2], (d_1 + d_2,\, d_1 + d_2 + d_3], \ldots, \\ &(\delta + T^{(i)} - d_{N^{(i)}},\, \delta + T^{(i)}] \Big). \end{aligned} \tag{C.20}$$

The objective is now to construct a formula for calibrating implied survival probabilities $\mathbb{Q}_t(s)$ for $s \in (T_b^{(i-1)} - t,\, T_b^{(i)} - t]$ to the quoted spread for the i^{th} tenor. We use the implied survival probability and its derivative obtained from this calibration for smaller tenors (if $i > 1$), i.e. we treat $\mathbb{Q}_t(s)$ and $\dot{\mathbb{Q}}_t(s)$ for $s \in [0,\, T_b^{(i-1)} - t]$ as deterministic and known when considering the quote for tenor i. We use the discretization and assumptions explained above to construct such a formula from Eq. (C.19). Firstly, we need to approximate the integrals in the functions f_1 and f_3 in Eq. (C.19).

Approximation of integrals

We apply the trapezoidal rule to approximate the two integrals in the functions f_1 and f_3 in Eq. (C.19). Thereby we will evaluate the survival probability at discrete time points, specifically the endpoints of the discretization intervals presented in Eq. (C.20). Entry k in the approximation sum will evaluate the survival probability in u_k and v_k which are defined to be the lower and upper endpoint of the k^{th} interval in Eq. (C.20). We have the intervals and their endpoints defined as:

$$\begin{aligned} I_k &= [d_1 + (k-2)d_k,\, d_1 + (k-1)d_k) \\ v_k &= \sup I_k = d_1 + (k-1)d_k \\ u_k &= \inf I_k = d_1 + (k-2)d_k = v_{k-1}, \end{aligned}$$

where we use the convention that $v_0 = 0$. The formula approximating the integral in f_1 is given by

$$\begin{aligned}
f_1 &= P \int_t^{T_b^{(i)}} D(t,v) \frac{\partial}{\partial v} \mathrm{d}\mathbb{Q}(\tau^r \geq v \mid \mathcal{F}_t) \mathrm{d}v \\
&= P \int_0^{T_b^{(i)}-t} D_t(v) \frac{\partial}{\partial v} \mathrm{d}\mathbb{Q}(\tau^r \geq t + v \mid \mathcal{F}_t) \mathrm{d}v \\
&\approx P \frac{d_1}{2} D_t(v_0) \dot{\mathbb{Q}}_t(v_0) + P \sum_{k=1}^{N^{(i)}-1} \frac{d_k + d_{k+1}}{2} D_t(v_k) \dot{\mathbb{Q}}_t(v_k) \\
&\quad + P \frac{d_{N^{(i)}}}{2} D_t(v_{N^{(i)}}) \dot{\mathbb{Q}}_t(v_{N^{(i)}}).
\end{aligned} \tag{C.21}$$

For approximating the integral in f_3 it is necessary to obtain the last premium payment date at any given v_k for $k = 0, 1, \ldots, N^{(i)}$. Specifically, we need to evaluate the function $v - T_{\gamma(v)-1}$ at each of these points. At any time before the first premium payment, the accrued interest on missing payments may extend the time between quoting date and actual default; specifically, this is extended by the time from the quoting date to the last reset date, i.e. by $\alpha - \delta$. Therefore, during the time period prior to the first premium payment, we have:

$$v - T_{\gamma(v)-1} = \alpha - \delta + v - t, \quad \text{for all } v \in [t,\ t + \delta). \tag{C.22}$$

This formula is used to approximate the integrand – by setting $v - t = v_k$ – for all $v_k \in [0,\ \delta)$ or equivalently for $k = 0, 1, \ldots, N_\delta$.

During the next periods, we need to ensure that the accrued interest is only calculated wrt. the time since the previous reset date. We subtract δ in the second period, $\delta + \alpha$ during the third period, $\delta + 2\alpha$ during the fourth period and so forth. Thus for period $j \in \{2, \ldots, \frac{T^{(i)}}{\alpha} + 1\}$, we can write

$$v - T_{\gamma(v)-1} = (2 - j)\alpha - \delta + v - t, \tag{C.23}$$

for $v \in [t + \delta + (j-2)\alpha,\ t + \delta + (j-1)\alpha)$. Eq. (C.23) is used in the approximation for all $v_k \in [\delta + (j-2)\alpha,\ \delta + (j-1)\alpha)$. If we approximate the function f_3 by summing the integrand yielding from Eq. (C.23) using the iterator $k = 1, \ldots, N^{(i)}$, it is necessary to identify the payment period in every entry in the sum, due to the dependence in the accrual on the payment period j. This requires a function to identify j based on k, which does not yield an elegant formulation of the approximated integral. A more elegant formulation is obtained by changing the iterator in the sum from k to k', where in the j^{th} payment period, we define k' such that it satisfies

$$k = \begin{cases} k', & \text{for } j = 1 \\ N_\delta + (j-2)N_\alpha + k', & \text{for } j > 1 \end{cases} \tag{C.24}$$

The benefit of the k' is that in all periods it will initiate at 1, and that it is terminated at N_α in all but the first period. This allows the approximation of the integral to be written as a double sum over the premium payments and the k' s, yielding independence of j in the approximation of the accrued interest itself. The discounting and the implied survival probabilities still depend on j, but the js are known due to the "outer" sum. For all $j > 1$, substituting k with k' yields

$$\begin{aligned}(2-j)\alpha - \delta + v_k &= (2-j)\alpha - \delta + d_1 + (k-1)d_k \\ &= (2-j)\alpha - \delta + (\delta - N_\delta \Delta t) + (N_\delta + (j-2)N_\alpha + k' - 1)\Delta t \\ &= (2-j)(\alpha - N_\alpha \Delta t) + (k'-1)\Delta t \\ &= (k'-1)\Delta t. \end{aligned} \tag{C.25}$$

Here d_1 and d_k are substituted directly from their definition using that since $j > 1$ we have $k > 1$ and thus $d_k = \Delta t$ for all values of k. Further $N_\alpha \Delta t = \alpha$ by the definition of N_α, and thereby the dependency of j disappear.

Lastly, we need to be aware what happens, when the accrued interest is approximated on an interval that contains a premium payment. By construction, a premium payment occurs for $k' = 1$ for any $j > 1$, as is seen in Eq. (C.25). Using the trapezoidal rule, we have a dependency on both k' and $k' - 1$ in the summation. Thus for $k' = 1$, the sum will include a term on the premium payment date and a term just before this date. Since the term just before the premium date does not correspond to using $k' = 0$ in Eq. (C.25), we need to specify what happens in this case. For this purpose we introduce a function for the time of accrual, which corresponds with Eq. (C.22) during the first time period and with Eq. (C.25) for all other periods, but also includes an exception when the discretization overlaps two payment periods. In the latter case, the *lagged* accrual period equals exactly $(N_\alpha - 1)\Delta t$, since this is the accrual time period immediately before a premium payment.

$$\mathrm{TA}(k', j) = \begin{cases} \alpha - \delta, & \text{if } k' = 0, j = 1 \\ (N_\alpha - 1)\Delta t, & \text{if } k' = 0, j > 1 \\ \alpha - \delta + d_1 + (k'-1)d_{k'}, & \text{if } k' \neq 0, j = 1 \\ (k'-1)\Delta t, & \text{if } k' \neq 0, j > 1 \end{cases} \tag{C.26}$$

Note that a consequence of this construction is that the j^{th} premium payment is included in the $j + 1^{\text{st}}$ time period. This is due to the convenience as a consequence of the structure of the function in Eq. (C.23): Considering a contract with premium payments $\boldsymbol{T} = (T_1, T_2, \ldots, T_m)$ for some $m > 1$, the function in Eq. (C.23) is piecewise linear on the interval $[T_{j-1}, T_j)$ for $1 < j < m$. Thus it is convenient to consider intervals that contain a premium payment date as the infimum, and not as the supremum even though this construction may seem more intuitive.

The difference in the expression in Eq. (C.26) between the first and all other payment periods means that we will have two terms in the approximation, one for the initial period and another for all other periods. In this way, we avoid a problem with the termination of the summation over k' as well since k' will terminate at N_δ for $j = 1$, and N_α for all other periods. The approximation formula used for the integral in the f_3 function is:

$$\begin{aligned}
f_3 &= \int_t^{T_b^{(i)}} \left(v - T_{\gamma(v)-1}\right) D(t,v) \frac{\partial}{\partial v} \mathbb{Q}(\tau^r \geq v \mid \mathcal{F}_t) \mathrm{d}v \\
&= \int_0^{T_b^{(i)}-t} \left((t+v) - T_{\gamma(t+v)-1}\right) D_t(v) \frac{\partial}{\partial v} \mathbb{Q}(\tau^r \geq t+v \mid \mathcal{F}_t) \mathrm{d}v \\
&\approx \frac{d_1}{2}(\alpha - \delta) D_t(v_0) \dot{\mathbb{Q}}_t(v_0) \\
&\quad + \sum_{k'=1}^{N_\delta} \frac{d_{k'} + d_{k'+1}}{2} \left(\alpha - \delta + d_1 + (k'-1) d_{k'}\right) D_t(v_k) \dot{\mathbb{Q}}_t(v_k) \\
&\quad + \sum_{j=2}^{\frac{T^{(i)}}{\alpha}+1} \sum_{k'=1}^{N_\alpha} (k'-1)(\Delta t)^2 D_t(v_k) \dot{\mathbb{Q}}_t(v_k), \qquad \text{(C.27)}
\end{aligned}$$

where we obtain the k s by using Eq. (C.24).

Specification of the expression for the spread

We aim to use the approximations presented in Sec. 8.3 for specifying the expression for the fair spread of a CDS, Eq. (C.19), and obtain a sum of known constants multiplied by the implied survival probability to different times. Firstly we rearrange Eq. (C.19). Note that henceforth we do not specify the inputs of the functions f_1, f_2, and f_3; in all occurrences these are as in Eq. (C.19) using the contract specifications for the i^{th} tenor.

$$S_{t,T^{(i)}} = \frac{-f_1}{f_2 - f_3} \qquad \Leftrightarrow \qquad 0 = S_{t,T^{(i)}} f_3 - f_1 - S_{t,T^{(i)}} f_2. \qquad \text{(C.28)}$$

The function f_2 represents the value of the CDS premium payments and thus does not include an integral. This function can be exactly expressed as:

$$\begin{aligned}
f_2 &= \alpha \sum_{j=1}^{\frac{T^{(i)}}{\alpha}+1} D(t, T_j) \mathbb{Q}\left(\tau^r \geq T_j \mid \mathcal{F}_t\right) \\
&= \alpha \sum_{j=1}^{\frac{T^{(i)}}{\alpha}+1} D_t(\delta + (j-1)\alpha) \mathbb{Q}_t(\delta + (j-1)\alpha)
\end{aligned}$$

In order to collect terms in the sums used to approximate the integrals in Eq. (C.28), we rewrite the approximation of f_1 given in Eq. (C.21), such that it includes a double-sum as f_3 does in Eq. (C.27)

$$
\begin{aligned}
f_1 \approx & \frac{d_1}{2} P D_t(v_0)\dot{\mathbb{Q}}_t(v_0) + \sum_{k'=1}^{N_\delta} P \frac{d_{k'} + d_{k'+1}}{2} D_t(v_{k'})\dot{\mathbb{Q}}_t(v_{k'}) \\
& + \sum_{j=2}^{\frac{T^{(i)}}{\alpha}+1} \sum_{k'=1}^{N_\alpha} \Delta t P D_t(v_k)\dot{\mathbb{Q}}_t(v_k) + \frac{\Delta t}{2} P D_t(v_{N^{(i)}})\dot{\mathbb{Q}}_t(v_{N^{(i)}}).
\end{aligned}
$$

Collecting the representations of the three functions, Eq. (C.28) can be expressed as:

$$
\begin{aligned}
0 &= S_{t,T^{(i)}} f_3 - f_1 - S_{t,T^{(i)}} f_2 \qquad \text{(C.29)} \\
&\approx -\alpha S_{t,T^{(i)}} \sum_{j=1}^{\frac{T^{(i)}}{\alpha}+1} D_t(\delta + (j-1)\alpha)\mathbb{Q}_t(\delta + (j-1)\alpha) \\
&+ \sum_{k'=1}^{N_\delta} \frac{d_{k'} + d_{k'+1}}{2} \big(S_{t,T^{(i)}}(\alpha - \delta + d_1 + (k'-1)d_{k'}) - P\big) D_t(v_{k'})\dot{\mathbb{Q}}_t(v_{k'}) \\
&+ \sum_{j=2}^{\frac{T^{(i)}}{\alpha}+1} \sum_{k'=1}^{N_\alpha} \Delta t \big(S_{t,T^{(i)}}(k'-1)\Delta t - P\big) D_t(v_k)\dot{\mathbb{Q}}_t(v_k) \\
&+ \frac{d_1}{2}\big(S_{t,T^{(i)}}(\alpha - \delta) - P\big) D_t(v_0)\dot{\mathbb{Q}}_t(v_0) - \frac{P\Delta t}{2} D_t(v_{N^{(i)}})\dot{\mathbb{Q}}_t(v_{N^{(i)}}),
\end{aligned}
$$

where the ks are obtained from Eq. (C.24), using j and k' from "current values" in the double-sum.

8.4 Specification of the implied survival probabilities

In order to calibrate the implied survival probabilities such that they solve Eq. (C.29), we need a parametric assumption on these. We assume one of two forms, either the default intensities are piecewise constant or piecewise linear on the intervals

$$
\Big((t,\, T_b^{(1)}], (T_b^{(1)},\, T_b^{(2)}], \ldots, (T_b^{(n-1)},\, T_b^{(n)}]\Big). \qquad \text{(C.30)}
$$

For simplicity, we let $t = 0$ in the remainder of this section.

Piecewise constant default intensities

Denoting the constant implied default intensity on the i^{th} interval in Eq. (C.30) as $\lambda^{(i)}$, the implied survival probabilities are specified as

$$\mathbb{Q}_t(v) = \mathbb{Q}_t\big(T_b^{(i-1)}\big)e^{-\lambda^{(i)}v} \quad \text{for all } v \in (T_b^{(i-1)},\, T_b^{(i)}] \tag{C.31}$$

for $i = 1, \ldots, n$ with $T_b^{(0)} = t = 0$. The quoted spread for each tenor on a specific trading date is used to find an implied default intensity $\lambda^{(i)}$, such that Eq. (C.29) holds when using the specification in Eq. (C.31) for the implied survival probabilities. This ensures that the fair spread of the CDS corresponds to the quoted spread for all maturities at the quoting time t, under the assumption that the discretization of the integrals and the specification of the implied default intensities are effective.

Piecewise linear default intensities

For $i = 1$, we assume that the implied default intensity is constant, and thus the implied survival probability is defined as in Eq. (C.31). Thus

$$\mathbb{Q}_t(v) = e^{-\lambda^{(1)}v}, \quad \text{for } v \in (0,\, T_b^{(1)} - t] = (0,\, T_b^{(1)}].$$

For $1 < i \leq n$, the implied default intensity is assumed to be linear on the corresponding interval in Eq. (C.30) and to intersect the implied default intensity from the last interval at its supremum. We define the value of the implied default intensity at these intersections as $\Lambda^{(i-1)}$ for $i = 2, \ldots, n$. Note that for $i = 2$, the intersection is at $\Lambda^{(1)} = \lambda^{(1)}$ since the implied default intensity is constant on the first interval. For $i = 2, \ldots, n$, let $a^{(i)}$ be the slope of the piecewise linear default intensity on the i^{th} interval in Eq. (C.30). Then the intersection at the supremum of the i^{th} interval is

$$\Lambda^{(i)} = \Lambda^{(i-1)} + a^{(i)}\big(T_b^{(i)} - T_b^{(i-1)}\big) \quad \text{for } i = 2, \ldots, n-1.$$

We specify the implied survival probability as

$$\begin{aligned}
\mathbb{Q}_t(v) &= \mathbb{Q}_t\big(T_b^{(i-1)}\big)\exp\left\{-\int_{T_b^{(i-1)}}^{v}\Big(\Lambda^{(i-1)} + a^{(i)}\big(u - T_b^{(i-1)}\big)\Big)\mathrm{d}u\right\} \\
&= \mathbb{Q}_t\big(T_b^{(i-1)}\big)\exp\left\{-\Lambda^{(i-1)}\big(v - T_b^{(i-1)}\big) - \frac{a^{(i)}}{2}\big(v - T_b^{(i-1)}\big)^2\right\} \\
&= \exp\left\{-\Lambda^{(1)}(v-t) - \sum_{j=2}^{i-1} a^{(j)}\big(T_b^{(j)} - T_b^{(j-1)}\big)\big(v - T_b^{(j)}\big)\right\} \\
&\quad \times \exp\left\{-\sum_{j=2}^{i-1}\frac{a^{(j)}}{2}\Big(T_b^{(j)} - T_b^{(j-1)}\Big)^2 - \frac{a^{(i)}}{2}\Big(v - T_b^{(i-1)}\Big)^2\right\}
\end{aligned}$$

for $v \in (T_b^{(i-1)}, T_b^{(i)}]$. The objective is to find $\lambda^{(i)}$ and $a^{(i)}$ for $i = 2, \ldots, n$, such that the fair spread of the CDSs with each available tenor is exactly equal to the quoted spread.

A problem with this specification is that the implied default intensities are not ensured to be non-negative. One can constrain $a^{(i)}$ to be calibrated such that no negative implied survival probabilities occur; however, this specification does not ensure that the fair CDS spread will equal the quoted spread.

8.5 Objective function and calibration of parameters

Using the assumptions of Sec. 8.2 and the formulas derived in Sec. 8.3, we now specify the objective function used to calibrate the model parameters for the default intensity. Note that in Sec. 8.2 and 8.3 no model assumptions on the default intensity process is used. The results of those sections can thus be applied to any process of the default intensity. We have, however, assumed that the default intensity is independent of the risk-free rate, which may be a shortfall.

We now consider the case, when the default intensities follow a CIR model, i.e. the model presented in Sec. 7.2 disregarding the deterministic shifts $\phi(\cdot)$ since these do not influence the parameter calibration. Consider, as in Sec. 8.3, a set of n CDSs on the reference credit quoted at time t with maturities $T_b^{(i)}$ for $i = 1, \ldots, n$. Further, we consider a window of m consecutive quoting dates and denote this window $\boldsymbol{t} = \{t_1, \ldots, t_m\}$. Under the CIR assumption, we have the survival probability of the reference credit specified in Eq. (C.15) for each $t \in \boldsymbol{t}$;

$$\mathbb{Q}\left(\tau^r > T_b^{(i)} \,\middle|\, \mathcal{F}_t^{\text{ref}}\right) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_b^{(i)}} x_u \mathrm{d}u}\right] = P^{\text{CIR}}(t, T_b^{(i)}; x_t, \kappa, \theta, \sigma), \quad \text{(C.32)}$$

where $P^{\text{CIR}}(\cdot)$ is the bond price in the CIR model. The closed-form solution to the CIR bond price in displayed in Eq. (C.17). Note that here we consider x_t as a parameter since this is not observable and needs to be calibrated along with the CIR-parameters κ, θ, and σ. To limit the number of parameters, we do not estimate a term structure of x_t's over the window $\boldsymbol{t}$, but only estimate the initial value x_{t_0} and consider this constant throughout the period.

As "observed data", we consider the implied survival probabilities, introduced in Sec. 8.4. We assume that the implied default intensities are piecewise constant. Using the formulas for piecewise constant default probabilities, we get n *derived quotes*;

$$\left(\mathbb{Q}_t(T_b^{(1)}), \mathbb{Q}_t(T_b^{(2)}), \ldots, \mathbb{Q}_t(T_b^{(n)})\right).$$

Note that we use the term *derived quotes* to stress that these are not in fact market quotes, but implied survival probabilities derived from the market quotes. From the derived quotes of the survival probability and the expression in Eq. (C.32), we can now express the error function, explaining the difference between the i^{th} derived quoted at time $t \in \boldsymbol{t}$ and the model survival probability;

$$E(\boldsymbol{\Theta}; i, t) = \mathbb{Q}_t\big(T_b^{(i)}\big) - P^{\text{CIR}}(t, T_b^{(i)}; \boldsymbol{\Theta}),$$

where $\boldsymbol{\Theta} = (x_{t_0}, \kappa, \theta, \sigma)$ is the parameter vector.

Using mixed cross-sectional and time series approach to parameter estimation and a minimum sum-of-squares objective function, the optimization problem at hand is

$$\min_{\boldsymbol{\Theta}} \sum_{t \in \boldsymbol{t}} \sum_{i=1}^{n} E(\boldsymbol{\Theta}; i, t)^2$$

It is important to notice, that though this is not purely a cross-sectional approach since we estimate the parameters using several quoting dates, we consider these parameters as $\mathbb{Q}$-parameters. For pricing derivatives pricing purposes, we need the $\mathbb{Q}$-parameters, and thus the purely cross-sectional method, where one only considers the derived quotes at time t, is a more natural way of estimating the parameters. However, we have 7 tenors and thus at most 7 observations at time t to calibrate 4 parameters which is problematic. Often several tenors include NA-values and thus one can experience a problem of insufficient data for parameter calibration. Therefore we consider a window; specifically, we consider each calendar month in our data as an estimation window. For comparison and to use as starting values, we also consider calibration with a window spanning the full available data set.

8.6 Calibration routine

The calibration routine is implemented from scratch in C++11.

Assumptions, data, and methods:

1. To evaluate the integrals in order to obtain implied default intensities, we use 2000 equidistant discrete time steps between each tenor, which by assumption is a quarter of a year. To find the implied default intensities that satisfy Eq. (C.13), we use the implementation of the Newton method in the *boost library*. Note that for each tenor the problem at hand is finding the root of an equation in one parameter.

2. We calibrate 4 parameters (including the initial default intensity, x_0).

3. The parameters are bounded such that $x_0 \geq 0.001$, $\kappa \geq 0$, $\theta \in [0, 1]$, and $\sigma \geq 0$.

4. The initial values for the optimizer are found by deterministic search, specifically the DIRECT algorithm, where the maximum number of

evaluations is 1000. We use the *nlopt library* for an implementation of the DIRECT algorithm as well as the algorithms presented in Item 8.

5. Data for all names start on January 27, 2005. For Bear Stearns the data stops at October 14, 2009, for Lehman Brothers at September 12, 2008, and for all other names at Januar 26, 2015.

6. Implied default intensities are assumed to be piecewise constant.

7. NA values in the data are omitted.

8. We use two algorithms to estimate the parameters which are both local derivatives based optimization algorithms. To evaluate the gradient of the objective function, we use algorithmic differentiation with its implementation in the *Eigen library*. The maximum number of evaluations is set at 10000. The algorithms used are:

   - The BFGS algorithm. This algorithm is commonly used and generally supply good results, however non-linear constraints cannot be used in the BFGS implementation in NLOPT. Thus we cannot ensure that the Feller condition is satisfied.
   - The SLSQP algorithm. This algorithm is constructed to handle non-linear constraints; combined with being gradient based this is our choice of an algorithm that ensures that the Feller condition is not violated.

   We estimate the parameters with both BFGS and SLSQP. When the Feller condition is violated in the BFGS estimates, we choose the SLSQP estimates. Otherwise, we choose the estimates that have the lowest sum-of-squares.

8.7 Calibration results

For the 6 names, we calibrate the CIR-parameters, and calculate the mean absolute difference between the model survival probabilities and the implied survival probabilities. The results are displayed in Tab. C.7.

Monthly parameters

We consider the estimation of CIR-parameters for each month in the data. The assumptions are largely the same as in Sec. 8.6 with the exception of the starting values. Here we consider two sets of starting values for the parameters in each month: The estimates for the entire data set, presented in Sec. 8.7, and the estimates from the previous month. The estimation is conducted for each of the two sets of starting values, and we choose the

Name	x_0	κ	θ	σ	Mean abs. diff.
Bank of America	0.001	3.35976	0.02369	0.398938	0.050116
Bear Stearns	0.01421	0.00402	0.03627	0.000004	0.037425
Citi Bank	0.02245	0.18269	0.02783	0.100564	0.058232
Goldman Sachs	0.00854	3.17184	0.02379	0.347606	0.049498
JP Morgan	0.001	0.80846	0.01751	0.155109	0.025138
Lehman Brothers	0.02386	1.4796	0.01397	0.000669	0.058606

Table C.7: Parameter results using all available data as the estimation window

parameter estimates that gives the lowest sum-of-squares. The estimated time series of parameters are shown in Sec. 9.2.

8.8 Challenges in parameter calibration

Standard errors Obtaining standard errors on the CIR parameters includes inversion of a 4 by 4 matrix, which is a source of errors. With reference to [18], we use the following method to estimate standard errors. Consider the Jacobian J of the objective function in Sec. 8.5. The standard errors of $\mathbf{\Theta}$ are estimated by $(J'J)^{-1}\sigma^2_{\text{res}}$, where σ^2_{res} is the sum of squares of the objective function. Here $J'J$ is a 4×4 positive definite matrix which in theory is always invertible. However, in the situation where the columns are close to linearly dependent, the matrix can be numerically singular, see [19] for further discussion on this issue. We exhibit this case in many instances, yielding faulty standard errors. In most cases the estimated standard errors are very large – when the matrix is close to singular – and in some cases, we obtain NA-values, if the matrix is estimated to be (numerically) singular. Characteristics of the standard errors of the parameters based on calendar months is shown in Tab. C.8; it is observed that in the mean standard error will cause a confidence interval that includes the origin for any reasonable parameter estimates.

Parameter bounds By the nature of the CIR-model, all four estimation parameters x_0, κ, θ, and σ are required to be non-negative. An issue that occurs in our estimation is that the parameters – especially the parameter x_0 – tends for longer periods at a time to reach its lower bound, the origin. Due to the approach to measuring the *Market price of risk* presented in Sec. 9, we have imposed a lower bound on x_0 of 0.001, i.e. 10 bps, to secure non-zero denominators. For some names, a similar tendency but with fewer hits are observed for κ and θ, respectively. Moreover, θ tends to reach extreme values for certain names at certain time steps. Thus we have imposed an upper bound on θ at 1. The effect of this bound is that θ tends to fluctuate between high values (near 1) and low values (near 0), which is an undesirable feature. See Figs. C.5, C.6, and C.7 for a visualization of this behavior.

	$\mathrm{se}(x_0)$	$\mathrm{se}(\kappa)$	$\mathrm{se}(\kappa)$	$\mathrm{se}(\sigma)$
Mean	0.71019	431.08	659.5	5553.0
3^{rd} quartile	0.01971	7.44	5.2	99.8
Max	71.60720	75665.70	330680.0	482871.0
#NAs	9	8	9	7

Table C.8: Parameter standard error summary

9 Market price of risk

We consider the model dynamics in Eq. (C.14). For each name $i = 1, \ldots, n$, we have calibrated CIR parameters for each calendar month, as explained in Sec. 8.7. Thus for each name i we have the $\mathbb{Q}$-parameters from the calibration $(\hat{x}^{i,t}, \hat{\kappa}^{i,t}_{\mathbb{Q}}, \hat{\theta}^{i,t}_{\mathbb{Q}}, \hat{\sigma}^{i,t}_{\mathbb{Q}})$ for each month $t = 1, \ldots, T$. Note that $\hat{x}^{i,t}$ is the estimated value of the x-process at the beginning of month t.

We now turn to consider some regressions of x_{t+1} on x_t, i.e. with a distance of one month. Considering the Euler-approximation of Eq. (C.14) with $\mathrm{d}t = 1$ we have

$$x_{t+1} - x_t = \kappa(\theta - x_t) + \sigma x_t^{\frac{1}{2}}(w_{t+1} - w_t). \tag{C.33}$$

Compare this with the exact conditional expectation of x_{t+1}

$$\mathbb{E}[x_{t+1} \mid x_t] = e^{-\kappa} x_t + \theta(1 - e^{-\kappa}), \tag{C.34}$$

and we can improve the Euler-approximation in Eq. (C.33) by replacing the first term in the right-hand side by the exact conditional expectation in Eq. (C.34)

$$x_{t+1} - x_t = e^{-\kappa} x_t + \theta(1 - e^{-\kappa}) + \sigma x_t^{\frac{1}{2}}(w_{t+1} - w_t).$$

Finally, we divide by $\sigma x_t^{\frac{1}{2}}$ and obtain errors that are i.i.d. standard normally distributed:

$$\frac{x_{t+1} - x_t}{\sigma x_t^{\frac{1}{2}}} = \sigma^{-1} e^{-\kappa} x_t^{\frac{1}{2}} + \sigma^{-1}\theta(1 - e^{-\kappa}) x_t^{-\frac{1}{2}} + (w_{t+1} - w_t). \tag{C.35}$$

We want to apply Eq. (C.35) as a non-linear regression using the calibrated values of $\hat{x}^{i,t}$ for each i as data, and thus obtaining parameter values of κ, θ and σ for each month. Since this regression across a month, the parameters we obtain are $\mathbb{P}$-parameters. Moreover since $\hat{x}^{i,t}$ is the estimated value for the process at the beginning of month t we are applying the regression is across month t, and thereby the parameters we obtain from are $\mathbb{P}$-parameters for month t.

In the regression, we also consider the calibrated $\mathbb{Q}$-parameters for each name as data and use the relationship between $\mathbb{P}$ and $\mathbb{Q}$-parameters for a CIR process to estimate the $\mathbb{P}$-parameters for each name as well as a monthly market price of risk for all names.

9.1 $\mathbb{Q}$ and $\mathbb{P}$-parameter relations

Since we are using a CIR-model, we have known formulas for $\mathbb{Q}$ and $\mathbb{P}$ parameters, given by

$$\sigma_{\mathbb{P}} = \sigma_{\mathbb{Q}}$$
$$\kappa_{\mathbb{P}} = \kappa_{\mathbb{Q}} - m$$
$$\theta_{\mathbb{P}} = \frac{\theta_{\mathbb{Q}}\kappa_{\mathbb{Q}}}{\kappa_{\mathbb{Q}} - m}$$

where m is the market price of risk. Applying these relations to Eq. (C.35) we have the equation

$$\frac{x_{t+1} - x_t}{\sigma x_t^{\frac{1}{2}}} = \frac{e^{-\kappa_{\mathbb{Q}}+m}}{\sigma} x_t^{\frac{1}{2}} + \frac{\theta_{\mathbb{Q}}\kappa_{\mathbb{Q}}(1 - e^{-\kappa_{\mathbb{Q}}+m})}{\sigma(\kappa_{\mathbb{Q}} - m)} x_t^{-\frac{1}{2}} + (w_{t+1} - w_t).$$

For a given month, we consider the problem of estimating the market price of risk in CDS markets. We use the $\mathbb{Q}$-parameters estimated in Sec. 8.7 and define the following regression formula for each name $i = 1, \ldots, n$ in order to obtain the market price of risk m_t for each month t:

$$\frac{\hat{x}_{t+1}^{i,t} - \hat{x}_t^{i,t}}{\hat{\sigma}_{\mathbb{Q}}^{i,t}\left(\hat{x}_t^{i,t}\right)^{\frac{1}{2}}} = \frac{e^{-\hat{\kappa}_{\mathbb{Q}}^{i,t}+m_t}}{\hat{\sigma}_{\mathbb{Q}}^{i,t}}\left(\hat{x}_t^{i,t}\right)^{\frac{1}{2}} + \frac{\hat{\theta}_{\mathbb{Q}}^{i,t}\hat{\kappa}_{\mathbb{Q}}^{i,t}(1 - e^{-\hat{\kappa}_{\mathbb{Q}}^{i,t}+m_t})}{\hat{\sigma}_{\mathbb{Q}}^{i,t}(\hat{\kappa}_{\mathbb{Q}}^{i,t} - m_t)}\left(\hat{x}_t^{i,t}\right)^{-\frac{1}{2}} + \epsilon_{i,t}. \quad (C.36)$$

Note that this problem is considering a non-linear regression in n observations. Specifically, when using the estimated parameters in Sec. 8.7 we have $n = 6$ observations. The estimated market price of risk for month t is thus

$$\hat{m}_t = \arg\min_{m_t} \sum_{i=1}^{n} \epsilon_{i,t}^2$$

9.2 Results – monthly price of risk

We apply the method of Sec. 9 with the estimated parameters from Sec. 8.7 to obtain the market price of risk for each month across all names, i.e. a monthly market price of risk in the CDS market.

One problem arises when the $x_{t+1} = x_t$, a common scenario in our calibration since x_t has a tendency to hit its lower bound for several periods in a row. Thus the left-hand side of Eq. (C.36) is zero, in which case low errors can be reached by a large negative value of the market price of risk. The estimated market price of risk is displayed in Fig. C.1 and the corresponding $\mathbb{P}$-parameter estimates of κ and θ are displayed in Figs. C.2–C.7 along with the estimated $\mathbb{Q}$-parameters. Figs. C.2–C.7 also include the $\mathbb{Q}$-estimates of x_0, for which we do not have $\mathbb{P}$-estimates as well as $\mathbb{Q}$-estimates for σ, which is exactly equal to the $\mathbb{P}$-estimates due to the relationships between $\mathbb{P}$ and $\mathbb{Q}$-parameters presented in Sec. 9.1.

The MPR estimates in Fig. C.1 tends to be negative. We note that the MPR relates to survival probabilities since we have used formulas from bond markets. Therefore the MPR is calculated on a non-tradable asset, and we cannot conclude whether MPR for survival probabilities has to be positive or negative based on our understanding of MPR on bond markets. In our context the MPR gives the relation between physical and pricing measures; both are of interest wrt. calculation of survival probabilities. The physical measure is used to calculate e.g. the distance to default of a firm.

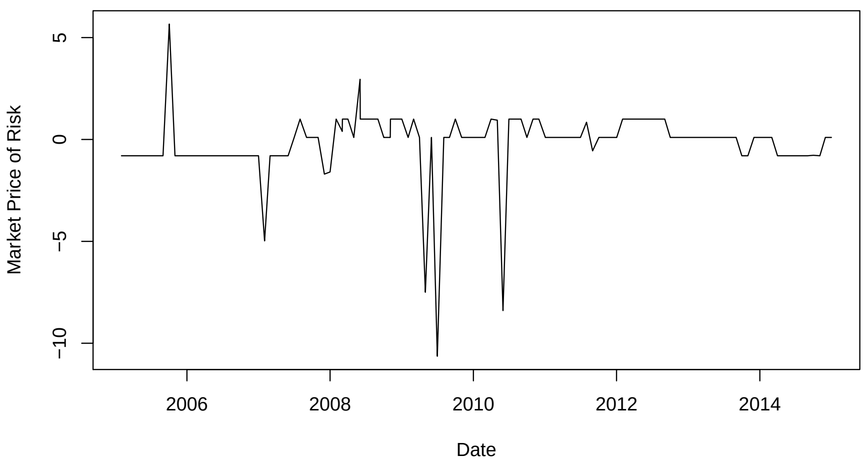


Fig. C.1: Monthly market price of risk based on CIR estimates

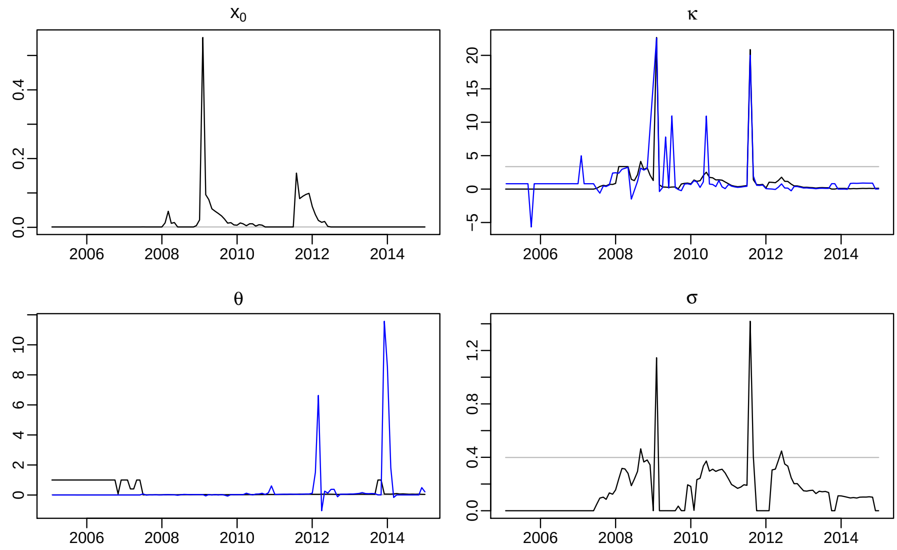


Fig. C.2: Parameter estimates for Bank of America. The black line is the monthly $\mathbb{Q}$-estimates, the grey line is the $\mathbb{Q}$-estimate for the entire data sample and the blue line is the estimated $\mathbb{P}$-parameters.

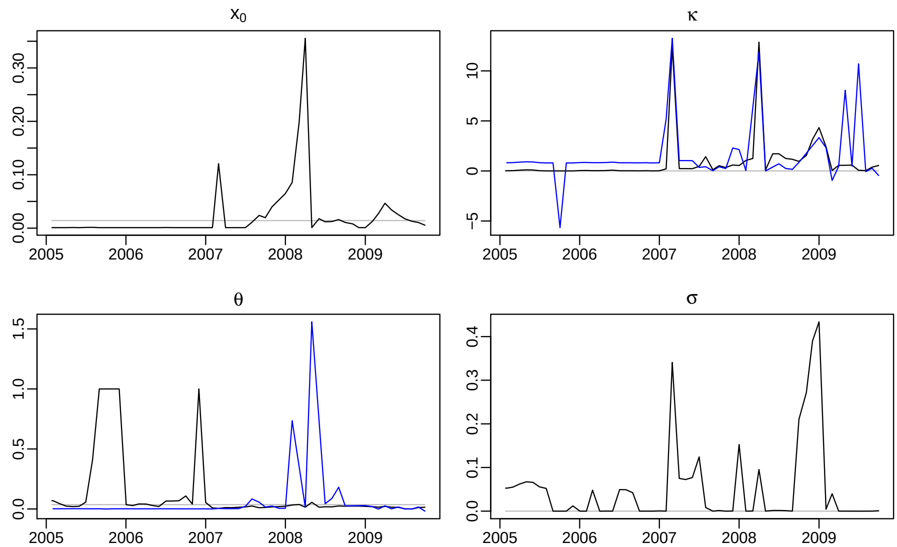


Fig. C.3: Parameter estimates for Bear Stearns. The black line is the monthly $\mathbb{Q}$-estimates, the grey line is the $\mathbb{Q}$-estimate for the entire data sample and the blue line is the estimated $\mathbb{P}$-parameters.

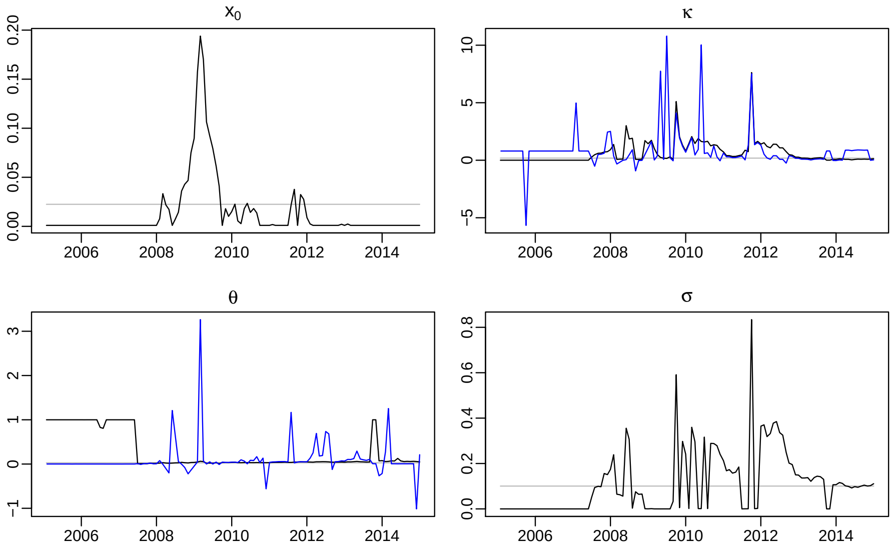


Fig. C.4: Parameter estimates for Citi. The black line is the monthly $\mathbb{Q}$-estimates, the grey line is the $\mathbb{Q}$-estimate for the entire data sample and the blue line is the estimated $\mathbb{P}$-parameters.

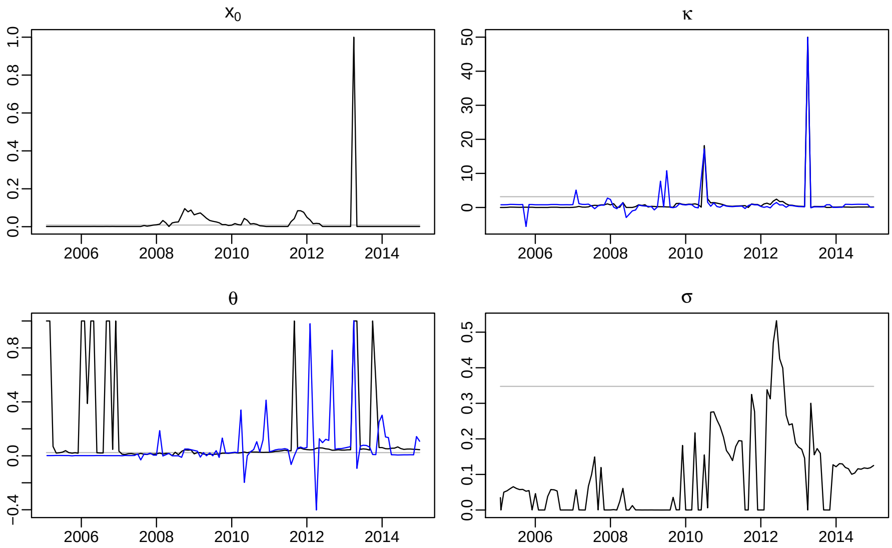


Fig. C.5: Parameter estimates for Goldman Sachs. The black line is the monthly $\mathbb{Q}$-estimates, the grey line is the $\mathbb{Q}$-estimate for the entire data sample and the blue line is the estimated $\mathbb{P}$-parameters.

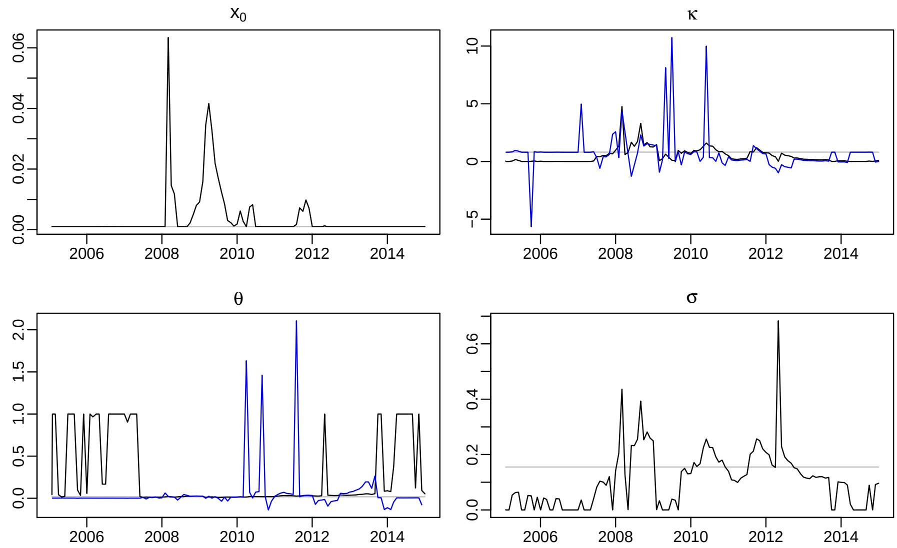


Fig. C.6: Parameter estimates for JP Morgan. The black line is the monthly $\mathbb{Q}$-estimates, the grey line is the $\mathbb{Q}$-estimate for the entire data sample and the blue line is the estimated $\mathbb{P}$-parameters.

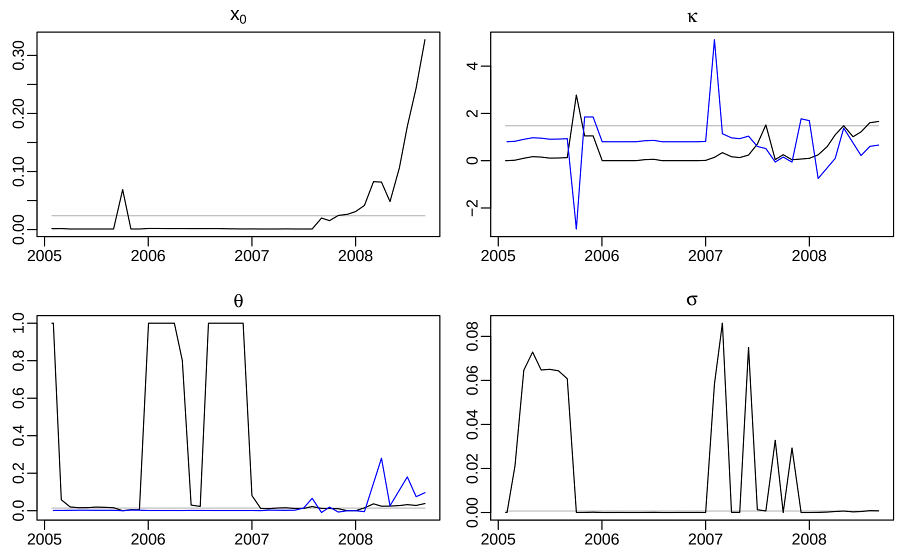


Fig. C.7: Parameter estimates for Lehman Brothers. The black line is the monthly $\mathbb{Q}$-estimates, the grey line is the $\mathbb{Q}$-estimate for the entire data sample and the blue line is the estimated $\mathbb{P}$-parameters.

10 Triparty CDSs with BCVA and risk-free closeout

We now consider the valuation of a CDS with BCVA. Here we have three parties present; the protection buyer (the bank), the protection seller (the counterparty), and the reference credit. We consider all three parties to be subject to credit risk and will use the calibrated parameters from Sec. 8.7 to value such a contract.

In order to obtain pricing formulas, we use the general set-up presented in Sec. 4, where risk-free closeout is considered. We assume risk-free closeout due to convenience with respect to the closeout value at the first default; here either the contract terminates before first default or the reference credit is the first entity to default which leads to the same payoffs as for the risk-free CDS, or contrarily prior to contract termination either the bank or the counterparty experiences the first default in which case the closeout value has to be assessed. In the case of risk-free closeout this value is purely determined by the value of a risk-free CDS on the reference credit with the given spread valued at default with the same maturity date as the initial contract; thus the closeout value has a deterministic solution since the risk-free CDS value can be obtained by the method presented in Sec 6.2. In the case of replacement closeout, the closeout value is the CDS value with UCVA, considering the surviving entity as defaultable; this value requires Monte Carlo simulation and thus this method requires nested Monte Carlo methods. Due to this difference between the two closeout conventions, risk-free closeout is the simpler of the two. It should be mentioned, that replacement closeout is the most used convention, see [1].

We use similar assumptions to those presented in Sec. 8.2 for all involved entities, e.g. we assume a constant loss-given-default at 40 % for all parties, i.e. $L \equiv L^b = L^c = L^{\text{ref}} = 0.4$. We consider valuation at or after initialization T_a with maturity T_b, and assume the default risk of all three parties are independent of the risk-free interest rates. We thus have the pricing formula for all $t \in [T_a, T_b]$

$$V_b^{\text{CDS,BCVA}}(t, T_b) = V^{\text{CDS}}(t, T_b) + \text{DVA}^{\text{CDS}}(t, T_b) - \text{CVA}^{\text{CDS}}(t, T_b), \quad \text{(C.37)}$$

where

$$\begin{aligned} V^{\text{CDS}}(t, T_b) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^r \geq \min\{\tau^b, \tau^c\}\}} \Pi(t, T_b) \,\middle|\, \mathcal{F}_t\right] \\ \text{DVA}^{\text{CDS}}(t, T_b) &= L\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^b \leq T, \tau^b \leq \tau^c, \tau^b < \tau^r\}} D(t, \tau^b) \left(V^{\text{CDS}}(\tau^b, T_b)\right)^- \,\middle|\, \mathcal{F}_t\right] \\ \text{CVA}^{\text{CDS}}(T_a, T_b) &= L\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau^c \leq T, \tau^c \leq \tau^b, \tau^c < \tau^r\}} D(t, \tau^c) \left(V^{\text{CDS}}(\tau^c, T_b)\right)^+ \,\middle|\, \mathcal{F}_t\right] \end{aligned}$$

One CDS spread is calculated and used in all appearances of V^{CDS}; the spread is chosen to ensure that the value of a risk-free CDS is fair at initialization, i.e. it holds that $V^{\text{CDS}}(T_a, T_b) = 0$. We use the risk-free CDS pricing formula presented in Eq. (C.10), this assuming independence between default intensities and risk-free interest rates.

10.1 Simulation of default-intensities and defaults

For $i \in \{b, c, \text{ref}\}$ we update the default intensity for name i using a simulation scheme consisting of the Euler approximation with a positive-parts adjustment to ensure non-negativity, see [20]. Thus the update of λ^i at some time step t_j with equidistant time steps of size Δt is given by

$$\lambda^i(t_j + \Delta t) = \lambda^i(t_j) + \kappa\big(\theta - \big(\lambda^i(t_j)\big)^+\big)\Delta t + \sigma\sqrt{\Delta t\big(\lambda^i(t_j)\big)^+}Z^i_j \quad \text{(C.38)}$$

where $\big(\lambda^i(t_j)\big)^+ = \max\{\lambda^i(t_j), 0\}$. The realizations of the error term Z^i_j are iid. standard normally distributed $j = 1, \ldots, N$; note that these are not i.i.d. across i s and we specifically wish to ensure positive correlation between default intensities of different entities. For each i and each month, the process is initialized at the calibrated x_0 s from Sec. 8.7, i.e. $\lambda^i(t_0) = \hat{x}^{i,t_0}$ where t_0 represents the first day of a calendar month.

Let $\rho^{b,c}$, $\rho^{b,\text{ref}}$, and $\rho^{c,\text{ref}}$ be the correlations between Wiener-processes driving the default intensity of the bank, the counterparty, and the reference credit. We define

$$\xi_1 \equiv \sqrt{1 - \big(\rho^{b,c}\big)^2}, \quad \xi_2 \equiv \frac{\rho^{b,\text{ref}} - \rho^{b,c}\rho^{c,\text{ref}}}{\xi_1}, \quad \xi_3 \equiv \sqrt{1 - \big(\rho^{c,\text{ref}}\big)^2 - \xi_2^2}$$

and simulate three independent error-processes from the standard normal distribution: $\tilde{Z}^1, \tilde{Z}^2$, and $\tilde{Z}^3$. The error processes to be used in Eq. (C.38) are then expressed as

$$Z^b = \tilde{Z}^1, \quad Z^c = \rho^{b,c}\tilde{Z}^1 + \xi_1\tilde{Z}^2, \quad Z^{\text{ref}} = \rho^{b,\text{ref}}\tilde{Z}^1 + \xi_2\tilde{Z}^2 + \xi_3\tilde{Z}^3.$$

With this definition all three error processes are standard normally distributed and simultaneously the Wiener-process correlations are ensured.

10.2 Simulation of CDS cash flows

Here we describe the method used to obtain the value of the triparty CDS agreement by Monte Carlo simulation. Thereby we need to simulate paths of default intensities, simulate corresponding default events, and in each path determine a resulting cash flow to be discounted. If no default event occurs prior to contract termination, we naturally stop the simulation of all processes at this point and start simulating the next paths of the processes. On the contrary, if some default does occur prior to contract termination, we only need to simulate default intensities for the three parties until the first default. Depending on which party defaults first, we may have a contribution to each of the three terms in Eq. (C.37): If the reference credit is the first to default the CDS compensation is paid by the counterparty to the bank which contributes to the first term, if the bank defaults first we have a contribution to the second term if the current value at default of a risk-free CDS is negative, and lastly if the counterparty is the first to default there will be a contribution to the third term if the current value at default of a risk-free CDS is positive. If the situation occurs where both the bank and counterparty defaults simultaneously, the current value of the risk-free CDS will determine which part gets a contribution. Lastly, if another default occurs simultaneously with the reference credit we have a few further adjustments; if $\tau^{\text{ref}} = \tau^b$ and the counterparty survives these defaults the full CDS protection amount is retrieved by the bank's creditors. How if the counterparty and the reference credit defaults simultaneously only a fraction (60% by assumption) is retrieved by the bank or the bank's creditors, depending on whether the bank survives the first defaults or all three parties default simultaneously. These last cases will usually have a rather low probability and thus will not significantly affect the CDS value, however for completeness, we present all options.

For each path $m = 1, \dots, M$ we conduct the following steps:

- Simulate E^b, E^c and E^{ref} as independent draws from the $\text{Exp}(1)$-distribution.

- Define $\Lambda^i(t_0) = 0$ for $i \in \{b, c, \text{ref}\}$, and conduct the following recursion for each time step t_j, $j = 1, \dots, n$, where $t_1 = t_0 + \Delta t$ and $t_n = T_b$

  1. Simulate $\lambda^i(t_j)$ for $i \in \{b, c, \text{ref}\}$, using the simulation scheme from Sec. 10.1. We use equidistant time steps, and thus $\Delta t_j = \Delta t$ and $t_j = t_{j-1} + \Delta t$ for all $j = 1, \dots, n$.
  2. Define $\Lambda^i(t_j) = \Lambda^i(t_{j-1}) + \Delta t \lambda^i(t_j)$, i.e. the integral of the default intensities are obtained by right-point approximation.
  3. If $\Lambda^i(t_j) \geq E^i$ for any i, we have default of entity i by the definition of the default times in Sec. 2. If one or more entities have defaulted, proceed to Item 4, else if contract termination has been reached proceed to Item 5. If neither of these conditions has been met, check whether a CDS payment date has been reached and add the spread to the cash flows if this is the case; afterwards repeat Item 1–3.
  4. Determine which of the default-cases we have and determine the cash flow as described above.

5. The final CDS payment is added to the cash flows and the path is terminated.

11 Premiminary conclusion and further work

We have presented a rigorous framework for CDS valuation and calibration to market quotes on single-name CDSs, and have presented methods of calculating the market price of risk (MPR) on such markets. We use the results of MPR for bond markets, and thus we obtain the MPR corresponding to the survival probabilities, i.e. a non-tradable asset. Further, we have presented a set-up for numerical valuation of a triparty CDS agreement, where two default risky parties trade a CDS with a third entity as reference credit.

Further work includes conducting the numerical experiments based on the framework presented in Sec. 10. To improve the validity of the MPR results, further work also includes expanding the data considered since our current results are based on only six names. Lastly, it is of interest to apply the piecewise linear assumption on the implied survival probabilities, using the theory presented in Sec. 8.4, to our numerical experiments and see the effects of this assumption on both MPR and CVA estimates.

References

[1] D. Brigo, M. Morini, and A. Pallavicini, *Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for All Asset Classes*. John Wiley & Sons, 2013.

[2] J. Gregory, *Counterparty Credit Risk and Credit Value Adjustment: A Continuing Challenge for Global Financial Markets*, 2nd ed. John Wiley & Sons, 2012.

[3] C. Kenyon and R. Stamm, *Discounting, LIBOR, CVA and Funding: Interest Rate and Credit Pricing*. Palgrave Macmillan, 2012.

[4] D. Lando, "On Cox Processes and Credit Risky Securities," *Review of Derivatives Research*, vol. 2, no. 2-3, pp. 99–120, Dec. 1998. [Online]. Available: http://dx.doi.org/10.1007/BF01531332

[5] D. Brigo and K. Chourdakis, "Counterparty Risk For Credit Default Swaps: Impact Of Spread Volatility And Default Correlation," *International Journal of Theoretical and Applied Finance (IJTAF)*, vol. 12, no. 07, pp. 1007–1026, 2009. [Online]. Available: http://ideas.repec.org/a/wsi/ijtafx/v12y2009i07p1007-1026.html

[6] D. Brigo and M. Morini, "Dangers of Bilateral Counterparty Risk: The Fundamental Impact of Closeout Conventions," arXiv.org, Papers 1011.3355, Nov. 2010. [Online]. Available: http://arxiv.org/abs/1011.3355

[7] D. Lando, *Credit Risk Modeling: Theory and Applications*. Princeton University Press, 2004.

[8] D. Brigo, C. Buescu, and M. Morini, "Impact of the first to default time on Bilateral CVA," arXiv.org, Papers 1106.3496, Jun. 2011. [Online]. Available: http://arxiv.org/abs/1106.3496

[9] D. Brigo and A. Capponi, "Bilateral counterparty risk valuation with stochastic dynamical models and application to Credit Default Swaps," arXiv.org, Papers 0812.3705v4, Nov. 2009. [Online]. Available: http://arxiv.org/abs/0812.3705v4

[10] D. Brigo and L. Cousot, "The stochastic intensity ssrd model implied volatility patterns for credit default swap options and the impact of correlation," *International Journal of Theoretical and Applied Finance (IJTAF)*, vol. 09, no. 03, pp. 315—-339, 2006.

[11] D. Brigo and A. Alfonsi, "Credit default swap calibration and derivatives pricing with the SSRD stochastic intensity model," *Finance and Stochastics*, vol. 9, no. 1, pp. 29–42, 2005.

[12] D. Brigo and F. Mercurio, *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*, 2nd ed. Springer-Verlag Berlin Heidelberg, 2006.

[13] D. Williams, *Probability with Martingales*. Cambridge University Press, 1991.

[14] T. Björk, *Arbitrage Theory in Continuous Time*. Oxford University Press, 2009.

[15] D. Brigo, "Candidate Market Models and the Calibrated CIR++ Stochastic Intensity Model for Credit Default Swap Options and Callable Floaters," Banca IMI, Papers, Mar. 2005, short updated version in Risk Magazine, January 2005. [Online]. Available: http://www.damianobrigo.it/cdsmm.pdf

[16] ISDA, *Standard European Corporate CDS – Contract Specification*, May 2009. [Online]. Available: http://www.cdsmodel.com/cdsmodel/assets/cds-model/docs/Standard%20Corporate%20CDS%20Contract%20Specification%20-%20May%2012%2c%202009.pdf

[17] ——, *Standard North American Corporate CDS – Contract Specification*, Mar. 2009. [Online]. Available: http://www.cdsmodel.com/cdsmodel/assets/cds-model/docs/Standard%20CDS%20Contract%20Specification.pdf

[18] A. R. Gallant, *Nonlinear Statistical Models*. John Wiley & Sons, Inc., 1987.

[19] W. H. Greene, *Econometric Analysis*, 7th ed. Prentice Hall, 2012.

[20] A. Alfonsi, "On the discretization schemes for the CIR (and Bessel squared) processes," *Monte Carlo Methods and Applications mcma*, vol. 11, no. 4, pp. 355–384, 2005.

ISSN (online): 2446-1636
ISBN (online): 978-87-7210-138-5

AALBORG **UNIVERSITY PRESS**