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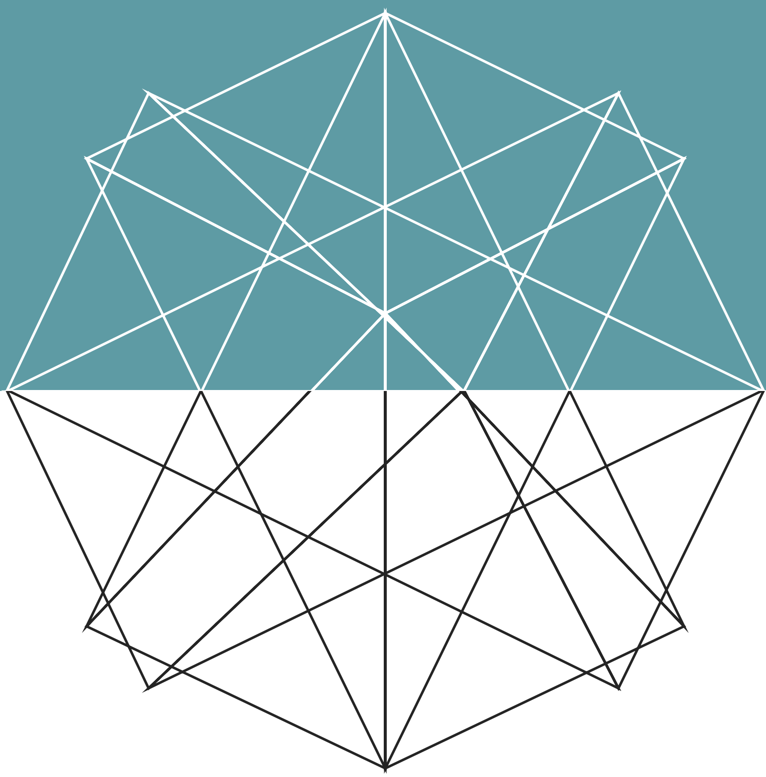

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Editors: Nurettin Cenk Turgay, Elif Özkara Canfes, Joeri Van der Veken, Cornelia-Livia Bejan

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From Editors

Differential geometry is proving to be an increasingly powerful tool that improves its ties to other branches of mathematics such as analysis, topology, algebra, PDEs, and so on, as well as to theoretical physics research.

The growing number of publications in the field of submanifolds was probably the main reason to organize the ”International Workshop on Theory of Submanifolds”, which took place at Istanbul Technical University, Turkey, from June 2 to June 4, 2016. One of the main features of the conference is the originality of its topic, being the only one focussing particularly on submanifold theory in the last few years. This is remarkable since submanifold theory is a very broad and omnipresent topic, going from surface theory in three-space, with applications in engineering and computer vision for example, to very abstract settings with high dimension and codimension, some of them appearing in modern physical theories.

This volume, containing the proceedings of the above mentioned workshop, provides very recent results mainly on the theory of submanifolds, which the reader would be interested in getting acquainted with.

The book is divided into three parts, each of them having a distinct editor. The first part contains surveys on submanifolds with certain properties, in particular on surfaces. Part two is the biggest one, and is devoted to the theory of submanifolds. The last part extends the main subject of the workshop toward some related topics, such as some geometrical structures which are extended from a manifold to the whole space containing the manifold (e.g. the total space of its cotangent bundle).

The experience of the contributors to the Proceedings is illustrated by their publications in this field and the freshness of this conference was given mainly by the presence of many young mathematicians. The workshop was very successful, despite the critical period of this conference, where many participants had to cancel their participation for reasons beyond their control.

All articles included here passed the usual referee process.

Our warm thanks go to all those who contributed to this book by their work, to all participants of the workshop, to the referees of the Proceedings, to the host institution for organizing the conference and last but not least to our sponsors.

Editors: N. C.Turgay, E. Ö. Canfes, J. Van der Veken , C-L. Bejan
September, 2017

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Acknowledgement. In this workshop some of results obtained during the TÜBİTAK project 'Y_EUCL2TIP' (Project Number: 114F199) were also presented.

Foreword

The theory of submanifolds was studied since the invention of calculus and it was started with differential geometry of plane curves. Since then the theory of submanifolds has been developed as an important part of pure and applied mathematics. In recent times, submanifold theory also plays some important roles in computer design, image processing, economic modeling, arts and vision, mathematical physics, relativity theory and cosmology as well as in mathematical biology.

There are two aspects of geometry of submanifolds, namely, intrinsic geometry and extrinsic geometry of submanifolds. Intrinsic differential geometry of submanifolds describes the geometry inside the submanifolds. Extrinsic geometry of submanifolds deals with the shape of submanifolds as subsets of the ambient space.

An important result connecting intrinsic and extrinsic geometry of submanifolds is the 1956 J. F. Nash embedding theorem which states that every Riemannian manifold can be isometrically embedded in a Euclidean space with sufficient high codimension. One important fundamental problem connecting intrinsic geometry and extrinsic geometry of submanifolds is to establish simple optimal relations between the main intrinsic invariants and the main extrinsic invariants of submanifolds as well as to discover their applications.

Since the pioneering work of P. Fermat, L. Euler, G. Monge, and others done in the seventeenth and eighteenth centuries, submanifold theory is still a very active vast research field in pure and applied mathematics. It plays a very important role in the development of modern differential geometry. This branch of mathematics is so far from being exhausted; in fact, only a small portion of an exceedingly fruitful field has been cultivated, much more remains to be discovered in this and coming centuries.

This new series of the Proceedings Book International Workshop on Theory of Submanifolds is a very welcome addition to the literature on the theory of submanifolds. The first volume of this series contains important contribution to the field of submanifold theory. It includes many nice articles on the following contemporary important research topics; submanifolds with parallel mean curvature, biharmonic and biconservative submanifolds, theory of finite type submanifolds, rotational hypersurfaces, curve and surface theory, and quasi-Einstein manifolds.

I expect this new series of Proceedings Book International Workshop on Theory of Submanifolds to play an important role in the future development of geometry of submanifolds for many coming years.

Bang-Yen Chen
April 15, 2017

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Section Editor: **Cornelia-Livia Bejan**

Proceedings Book of International Workshop on Theory of Submanifolds (Volume: 1 (2016)) June 2–4, 2016, Istanbul, Turkey. Editors: Nurettin Cenk Turgay, Elif Özkara Canfes, Joeri Van der Veken and Cornelia-Livia Bejan
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A Survey on Submanifolds with Nonpositive Extrinsic Curvature

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Abstract. We survey on some recent developments on the study of submanifolds with nonpositive extrinsic curvature.
Keywords. Nonpositive extrinsic curvature · Cylindrically bounded submanifolds.
MSC 2010 Classification. Primary: 53C40; Secondary:53C42 · 53A07.

1 Introduction

One of the main problems in submanifold theory is to know whether given complete Riemannian manifolds M^m and N^n, with $m < n$, there exists an isometric immersion $f : M^m \to N^n$. In case the ambient space is the Euclidean space, the Nash embedding theorem says that there is an isometric embedding $f : M^m \to \mathbb{R}^n$ provided the codimension $n - m$ is sufficiently large. For small codimension, the answer in general depends on the geometries of M and N. Isometric immersions $f : M^m \to N^n$ with low codimension and nonpositive extrinsic curvature at any point must satisfy strong geometric conditions. The simplest result along this line is that a surface with nonpositive curvature in $\mathbb{R}^3$ cannot be compact. This is a consequence of the well-know fact that at a point of maximum of a distance function on a compact surface in $\mathbb{R}^3$ the Gaussian curvature must be positive.

In the same direction, the Hilbert-Efimov theorem [4], [5] states that no complete surface M with sectional curvature $K_M \le -\delta^2 < 0$ can be isometrically immersed in $\mathbb{R}^3$. A classical result by Tompkins [17] states that a compact flat m-dimensional Riemannian manifold cannot be isometrically immersed in $\mathbb{R}^{2m-1}$. Tompkins's result was extended in a series of papers by Chern and Kuiper [3], Moore [8], O'Neill [11], Otsuki [12] and Stiel [15], whose results can be summarized as follows:

Theorem 1.1. *Let $f : M^m \to N^n$ be an isometric immersion of a compact Riemannian manifold M into a Cartan-Hadamard manifold N, with $n \leq 2m-1$. Then the sectional curvatures of M and N satisfy*

$$\sup_M K_M > \inf_N K_N.$$

The aim of this paper is to survey on some recent extensions of Theorem 1.1, mostly for the case of complete cylindrically bounded submanifolds.

2 Bounded complete submanifolds with scalar curvature bounded from below

Let $f : M^m \to N^n$ be an isometric immersion. In the statement below and the sequel, ρ stands for the distance function to a given reference point in M^m, $\log^{(j)}$ is the j-th iterate of the logarithm and $t \gg 1$ means that t is sufficiently large. Also $B_N[R]$ denotes the closed geodesic ball with radius $0 < R < \min\left\{\mathrm{inj}_N(o), \pi/2\sqrt{b}\right\}$ centered at a point o of N^n, where $\mathrm{inj}_N(o)$ is the injectivity radius of N^n at o and $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b \leq 0$. Moreover, $K_M(\sigma)$ denotes the sectional curvature of M^m at a point $x \in M^m$ along the plane $\sigma \subset T_xM$, and similarly for N^n,

$$K_f(\sigma) := K_M(\sigma) - K_N(f_*\sigma)$$

is the *extrinsic sectional curvature* of f at x along σ and K_N^{rad} stands for the radial sectional curvature of N^n with respect to o, that is, the sectional curvature of tangent planes to N^n containing the vector $\mathrm{grad}^N r$, where r is the distance function to o in N^n. Finally, let C_b be the real function given by

$$C_b(t) = \begin{cases} \sqrt{b}\cot(\sqrt{b}t) & \text{if } b > 0 \text{ and } 0 < t < \dfrac{\pi}{2\sqrt{b}}, \\ \frac{1}{t} & \text{if } b = 0 \text{ and } t > 0, \\ \sqrt{-b}\coth(\sqrt{-b}t) & \text{if } b < 0 \text{ and } t > 0. \end{cases}$$

Theorem 1.1 was extended by Jorge and Koutrofiotis [6] to bounded complete submanifolds with scalar curvature bounded from below. Pigola, Rigoli and Setti presented in [13] an extension of Theorem 1.1 with scalar curvature satisfying

$$s_M(x) \geq -A^2\rho^2(x)\prod_{j=1}^{J}\left(\log^{(j)}(\rho(x))\right)^2, \ \rho(x) \gg 1, \tag{2.1}$$

for some constant $A > 0$ and some integer $J \geq 1$ (where we use the definition in which the scalar curvature and also the Ricci curvature in Section 5 are divided by $m-1$).

Theorem 2.1 ([13])**.** *Let $f : M^m \to N^n$ be an isometric immersion with codimension $p = n - m < m$ of a complete Riemannian manifold whose scalar curvature satisfies* (2.1)*. Assume that $f(M) \subset B_N[R]$. If $K_N^{\mathrm{rad}} \leq b$ in $B_N[R]$, then*

$$\sup_M K_M \geq C_b^2(R) + \inf_{B_N[R]} K_N. \tag{2.2}$$

Note that if $N^n = \mathbb{Q}_b^n$ is the simply connected space form of constant sectional curvature b and $M = \partial B_{\mathbb{Q}_b^n}[R] \subset \mathbb{Q}_b^n$ is a geodesic sphere of radius R, then equality (2.2) is achieved.

3 Cylindrically bounded submanifolds

In this section we will discuss an extension of Theorem 2.1 due to Alías, Bessa and Montenegro for the case of cylindrically bounded submanifolds. More precisely, in [1] they have provided an estimate for the extrinsic curvatures of complete cylindrically bounded submanifolds of a Riemannian product $P^n \times \mathbb{R}^k$, where *cylindrically bounded* means that there exists a (closed) geodesic ball $B_P[R]$ of P^n, centered at a point $o \in P^n$ with radius satisfying $0 < R < \min\left\{\mathrm{inj}_P(o), \pi/2\sqrt{b}\right\}$ (where $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b \leq 0$), such that

$$f(M) \subset B_P[R] \times \mathbb{R}^k. \tag{3.1}$$

Otherwise, we say that f is *cylindrically unbounded.*

Theorem 3.1 ([1])**.** *Let $f : M^m \to P^n \times \mathbb{R}^k$ be an isometric immersion with codimension $p = n + k - m < m - k$ of a complete Riemannian manifold whose scalar curvature satisfies* (2.1)*. Assume that f is cylindrically bounded and that P^n is complete. If $K_P^{\mathrm{rad}} \leq b$ in $B_P[R]$, then*

$$\sup_M K_f \geq C_b^2(R). \tag{3.2}$$

Moreover,

$$\sup_M K_M \geq C_b^2(R) + \inf_{B_P[R]} K_P. \tag{3.3}$$

We point out that the codimension restriction $p < m - k$ cannot be relaxed. Actually, it implies that $n > 2$ and $m > k + 1$. In particular, in a three-dimensional ambient space N^3, that is, $n + k = 3$, we have that $k = 0$, and therefore $f(M) \subset B_P[R]$. In fact, the flat cylinder $\mathbb{S}^1(R) \times \mathbb{R} \subset B_{\mathbb{R}^2}[R] \times \mathbb{R}$ shows that the restriction $p < m - k$ is necessary.

On the other hand, estimates (3.2) and (3.3) are sharp. Indeed, the function C_b is well-known: the geodesic sphere $\partial B_{\mathbb{Q}_b^m}(R)$ of radius R in the simply connected complete space form $\mathbb{Q}_b^m$ of constant sectional curvature b, with $R < \frac{\pi}{2\sqrt{b}}$ if $b > 0$, is an umbilical hypersurface with principal curvatures being precisely $C_b(R)$. This shows that its extrinsic and intrinsic sectional curvatures are constant and equal to $C_b^2(R)$ and $C_b^2(R) + b$, respectively, the latter following from the former by the Gauss equation. Then, for every $m > 2$ and $k \geq 0$ we can consider $M^{m-1+k} = \partial B_{\mathbb{Q}_b^m}(R) \times \mathbb{R}^k$ and take $f : M^{m-1+k} \to B_{\mathbb{Q}_b^m}[R] \times \mathbb{R}^k$ to be the canonical isometric embedding. Therefore $\sup_M K_f$ and $\sup_M K_M$ are the constant extrinsic and intrinsic sectional curvatures $C_b^2(R)$ and $C_b^2(R) + b$ of $\partial B_{\mathbb{Q}_b^m}(R)$, respectively.

Remark 3.2. The geometry of the Euclidean factor $\mathbb{R}^k$ plays essentially no role in the proof of Theorem 3.1. Indeed, estimate (3.3) remains true if the former is replaced by any Riemannian manifold Q^k, which need not be even complete, whereas for (3.2) the only requirement is that K_Q be bounded from above. In the next section we will discuss a more accurate conclusion than the one of Theorem 3.1 (see Theorem 4.1 and comment below).

As a consequence of Theorem 3.1, the following results about extrinsic radius were obtained.

Theorem 3.3. *[1] Let $f : M^m \to P^n \times \mathbb{R}^l$ be an isometric immersion of a compact Riemannian M^m with codimension $p = n + l - m < m - l$. Assume that P^n is a complete Riemannian manifolds with a pole and radial sectional curvature $K_P^{rad} \leq b \leq 0$. Then, the extrinsic radius satisfies*

$$R_f \geq C_b^{-1}\left(\sqrt{\sup K_M - \inf K_N}\right).$$

In particular, if $P^n = \mathbb{R}^n$ we have that

$$R_f \geq \frac{1}{\sqrt{\sup K_M}}.$$

Theorem 3.4. *[1] Let $f : M^m \to \mathbb{S}^n \times \mathbb{R}^l$ be an isometric immersion of a compact Riemannian M^m with codimension $p = n+l-m < m-l$. If $\sup K_M \leq 1$, then*

$$R_f \geq \frac{\pi}{2}.$$

4 Cylindrically bounded submanifolds: a more general setting

The purpose of this section is to discuss a more accurate conclusion than the one of Theorem 3.1. More precisely, the authors [2] understood how much extrinsic (respectively, intrinsic) sectional curvature satisfying estimate (3.2) (respectively (3.3)) appears depending on how low the codimension is. The idea is that the lower the codimension is, the more extrinsic (respectively, intrinsic) sectional curvature satisfying (3.2) (respectively (3.3)) will appear.

In the same way as in (3.1), an isometric immersion $f : M^m \to P^n \times Q^k$ is said to be *cylindrically bounded* if there exists a (closed) geodesic ball $B_P[R]$ of P^n, centered at a point $o \in P^n$ with radius $R > 0$, such that

$$f(M) \subset B_P[R] \times Q^k, \tag{4.1}$$

with $0 < R < \min\left\{\mathrm{inj}_P(o), \frac{\pi}{2\sqrt{b}}\right\}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$.

Theorem 4.1 ([2])**.** *Let $f : M^m \to P^n \times Q^k$ be an isometric immersion with codimension $p = n + k - m < m - k$ of a complete Riemannian manifold whose radial sectional curvature $K_M^{\mathrm{rad}}(x)$ satisfies*

$$K_M^{\mathrm{rad}}(x) \geq -A^2 \rho^2(x) \prod_{j=1}^{J} \left(\log^{(j)}(\rho(x))\right)^2, \ \rho(x) \gg 1. \tag{4.2}$$

Assume that f is cylindrically bounded. If $K_P^{\mathrm{rad}} \leq b$ in $B_P[R]$, then

$$\sup_M \min\left\{\max_{\sigma \subset W} K_f(\sigma) : \dim W > p + k\right\} \geq C_b^2(R). \tag{4.3}$$

Moreover,

$$\sup_M \min\left\{\max_{\sigma \subset W} K_M(\sigma) : \dim W > p + k\right\} \geq C_b^2(R) + \inf_{B_P[R]} K_P. \tag{4.4}$$

The estimates of Theorem 4.1 are clearly better than the ones of Theorem 3.1. Actually, (4.3) and (4.4) reduce to (3.2) and (3.3), respectively, only in the case of the highest allowed codimension $p = m - 1 - k$. On the other hand, although one has a stronger assumption on the curvature of M^m, if (2.1) holds but (4.2) does not, then, since the scalar curvature is an average of sectional curvatures, we have that $\sup_M K_M = +\infty$, and hence (3.3) is trivially satisfied. Moreover, K_P is clearly bounded in $B_P[R]$, thus if also K_Q is bounded from above, we conclude that $\sup_M K_f = +\infty$ by the Gauss equation, so that (3.2) also holds trivially in this case. Finally, note that the same example considered below Theorem 3.1 shows that our estimates (4.3) and (4.4) are also sharp.

5 Applications

In this section we will discuss some applications of Theorem 4.1. Denote by R_f the *extrinsic radius* of a cylindrically bounded isometric immersion f, that is, the smallest R for which (4.1) holds. A first application of Theorem 4.1 are the following versions of Theorem 3.3 and 3.4.

Corollary 5.1 ([2])**.** *Let $f : M^m \to P^n \times Q^k$ be an isometric immersion with codimension $p = n + k - m < m - k$ of a complete Riemannian manifold whose radial sectional curvature satisfies* (4.2)*. Assume that P^n is a complete Riemannian manifold with a pole and radial sectional curvatures $K_P^{\mathrm{rad}} \leq b \leq 0$. If f is cylindrically bounded, then*

$$\sup_M \min\left\{\max_{\sigma \subset W} K_f(\sigma) : \dim W > p + k\right\} > -b$$

and the extrinsic radius satisfies

$$R_f \geq C_b^{-1}\left(\sqrt{\sup_M \min\left\{\max_{\sigma\subset W} K_f(\sigma) : \dim W > p+k\right\}}\right). \tag{5.1}$$

In particular, if

$$\sup_M \min\left\{\max_{\sigma\subset W} K_f(\sigma) : \dim W > p+k\right\} \leq -b,$$

then f is cylindrically unbounded.

Corollary 5.2 ([2])**.** *Let $f : M^m \to \mathbb{S}^n \times Q^k$ be an isometric immersion with codimension $p = n+k-m < m-k$ of a complete Riemannian manifold whose radial sectional curvature satisfies* (4.2)*. If*

$$\sup_M \min\left\{\max_{\sigma\subset W} K_M(\sigma) : \dim W > p+k\right\} \leq 1,$$

then

$$R_f \geq \frac{\pi}{2}. \tag{5.2}$$

On the other hand, a sharp lower bound for the Ricci curvature of bounded complete Euclidean hypersurfaces was obtained by Leung [7] and extended by Veeravalli [18] to nonflat ambient space forms. For simplicity of notation we shall denote by $\sup_M Ric(M)$ the $\sup_{X\in UM} Ric(X,X)$, where UM is the unitary tangent bundle.

Theorem 5.3 ([18])**.** *Let $f : M^m \to \mathbb{Q}_b^{m+1}$ be a complete hypersurface with sectional curvature bounded away from $-\infty$ such that $f(M) \subset B_{\mathbb{Q}_b^{m+1}}[R]$, with $R < \frac{\pi}{2\sqrt{b}}$ if $b > 0$. Then*

$$\sup_M \mathrm{Ric}(M) \geq C_b^2(R) + b. \tag{5.3}$$

Theorem 4.1 also gives an improvement of the above result, where we consider hypersurfaces of much more general ambient spaces and obtain that estimate (5.3) actually holds for the scalar curvature. This shows the unifying character of Theorem 4.1.

Corollary 5.4 ([2])**.** *Let $f : M^m \to P^{m+1}$ be a complete hypersurface whose radial sectional curvatures satisfy* (4.2)*. Assume that $f(M) \subset B_P[R]$, with R as in Theorem 4.1. If $K_P^{\mathrm{rad}} \leq b$ in $B_P[R]$, then*

$$\sup_M s_M \geq C_b^2(R) + \inf_{B_P[R]} K_P.$$

Again observe that for the geodesic sphere $M^m = \partial B_{\mathbb{Q}_b^{m+1}}(R)$ of radius R in $\mathbb{Q}_b^{m+1}$ the above inequality is in fact an equality. Corollary 5.5 leads to similar extrinsic radius results to Corollaries 5.1 and 5.2 and, in particular, a criterion of unboundedness:

Corollary 5.5 ([2]). *Let $f : M^m \to P^{m+1}$ be a complete hypersurface whose radial sectional curvatures satisfy* (4.2). *Assume that P^{m+1} is a complete Riemannian manifold with a pole and sectional curvatures $K_P \geq c$ and $K_P^{\mathrm{rad}} \leq b \leq 0$. If $f(M)$ is bounded, then $\sup_M s_M > c - b$ and*

$$R_f \geq C_b^{-1}\left(\sqrt{\sup_M s_M - c}\right).$$

In particular, if $\sup_M s_M \leq c - b$, then $f(M)$ is unbounded.

Corollary 5.6 ([2]). *Let $f : M^m \to \mathbb{S}^{m+1}$ be a complete hypersurface whose radial sectional curvature satisfies* (4.2). *If $\sup_M s_M \leq 1$, then*

$$R_f \geq \frac{\pi}{2}.$$

Remark 5.7. One of the main tools to prove this kind of result, in particular Theorem 4.1, is an algebraic lemma due to Otsuki [12], about symmetric bilinear forms. On the other hand, a key ingredient to handle the noncompact case is a maximum principle due to Omori [10] and generalized by Pigola-Rigoli-Setti [13].

6 Conjecture

One of the most important open problems in the area of geometry of submanifolds is an old conjecture on the higher-dimensional extension of Hilbert's classical theorem asserting that the complete hyperbolic plane $\mathbb{H}^2$ cannot be isometrically immersed into three-dimensional Euclidean space $\mathbb{R}^3$. Hilbert's theorem was proven at the turn of the last century in [5] and was one of the first *global* theorems from the Riemannian geometry of surfaces. It is quite natural to explore whether this result could be extended to higher dimensions. It follows from Otsuki's lemma that there are no m-dimensional submanifolds of constant negative curvature in $\mathbb{R}^{2m-2}$. In $\mathbb{R}^{2m-1}$, Moore [9] showed that the existence of an isometric immersion $f : \mathbb{H}^m \to \mathbb{R}^{2m-1}$ implies the existence of a Chebyshev net on $\mathbb{H}^m$, thereby extending the main step in the standard proof of Hilbert's theorem to m dimensions. However, despite the effort of many geometers such as Tenenblat and Terng [16], Xavier [19], and Aminov, it is remarkable that the conjectured extension of Hilbert's theorem has not been solved yet even in the next case $m = 3$. Most of the attempts were made by trying to face the problem directly, exploring the fairly complete understanding of the structure of m-dimensional submanifolds of constant curvature in $\mathbb{R}^{2m-1}$ provided by the study of the fundamental equations to reduce the question to a problem of global analysis generalizing the sine-Gordon equation. But as it often happens in mathematics, the answer for a conjecture may arise out of the solution of a more general problem. Hilbert's own theorem illustrates this point, since it is just the special constant curvature case of Efimov's much stronger statement that a complete surface with sectional curvature $K \leq -c < 0$ cannot be immersed isometrically in $\mathbb{R}^3$. Generalizations to higher dimensions of this stronger result have been in the direction of hypersurfaces [14] rather than to codimension $m-1$. Nevertheless, we point out that Theorem 4.1 leads to a conjecture that goes right into the latter direction. Indeed, it is a natural question to ask whether Theorem 4.1 is still true in the limiting case, that is, when $R = \mathrm{inj}_P(o) = \frac{\pi}{2\sqrt{b}}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$, which motivates the following:

Conjecture 6.1. *Let $f : M^m \to N^{n+l} = P^n \times Q^l$ be an isometric immersion with codimension $p = n + l - m < m - l$ of a complete Riemannian manifold. Assume that $R = \mathrm{inj}_P(o) = \frac{\pi}{2\sqrt{b}}$, where $\frac{\pi}{2\sqrt{b}}$ is replaced by $+\infty$ if $b \leq 0$. If $K_P^{\mathrm{rad}} \leq b$ in $B_P[R]$, then*

$$\sup_M \min \left\{ \max_{\sigma \subset W} K_f(\sigma) : \dim W > p + l \right\} \geq \max\{-b, 0\}.$$

Moreover,

$$\sup_M \min \left\{ \max_{\sigma \subset W} K_M(\sigma) : \dim W > p + l \right\} \geq \max\{-b, 0\} + \inf_{B_P[R]} K_P.$$

It is not clear the extent to which the above conjecture is true, but an affirmative answer at least in the most important case $P^n = \mathbb{R}^n$, $l = 0$, $p = m-1$ would provide the extension of Efimov's theorem to codimension $m-1$ and consequently settle the problem of isometric immersions $f : \mathbb{H}^m \to \mathbb{R}^{2m-1}$.

Remark 6.2. We said that Conjecture 6.1 was the limiting case of Theorem 4.1. However, we do not add hypothesis (4.2). Indeed, (4.2) is important only to ensure that the Omori-Yau maximum principle for the Hessian holds on M^m. This latter principle is one of our main tools to build the proof of Theorem 4.1, but the above conjecture seems to be inaccessible to techniques using it. Moreover, removing (4.2) allows us to include the aforementioned extension of Efimov's theorem as an important particular case of the conjecture.

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A Short Survey on Surfaces Endowed with a Canonical Principal Direction

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Abstract. In this paper, we would like to give a short survey of recent results on hypersurfaces with canonical principal direction relative to a fixed direction in a (semi-)Riemannian manifold. We also present some of our first results that we have recently obtained in this direction.
Keywords. Minkowski space · Lorentzian surfaces · canonical principal direction · Generalized constant ratio submanifolds.
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1 Introduction

Let $\hat{M}$ be a (semi-)Riemannian manifold, M a hypersurface of $\hat{M}$ and X a vector field tangent to $\hat{M}$. M is said to have a canonical principal direction relative to X if the tangential projection of X to M gives a principal direction. For example, a rotational hypersurface in Euclidean spaces has a canonical principal direction relative to a vector field parallel to its rotation axis, [12]. It turns out that when $\hat{M}$ is a product space $\tilde{M}\times\mathbb{R}$ or a semi-Euclidean space, some common interesting geometrical properties of hypersurfaces endowed with a canonical principal direction relative to X occur if X is chosen to be a fixed direction k (See Theorem 3.6, Theorem 3.13, Theorem 3.15, Theorem 4.1 and Theorem 4.6).

Let $M^n(c)$, $c=\pm1$ denote the Riemannian space-form given by

$$M^n(c)=\begin{cases}\mathbb{S}^n & \text{if } c=1,\\ \mathbb{H}^n & \text{if } c=-1.\end{cases}$$

We would like to note the following important property which relates constant angle surfaces to surfaces with a canonical principal direction. The projection U of the unit vector field T tangent to the second factor $\mathbb{R}$ to the tangential bundle of the surface is a principal direction for M with the corresponding principal curvature equal to zero. Therefore, a constant angle surface in $M^2(c)\times\mathbb{R}$ is endowed with canonical principal direction relative to T. There are many classification results obtained so far, in different ambient spaces, [1, 3, 5, 6, 13, 15, 18].

A recent natural problem is that appears in the context of constant angle surfaces is to study those surfaces for which U remains a principal direction but the corresponding principal curvature is different from zero. This problem was studied in $\mathbb{S}^2 \times \mathbb{R}$ [4] and $\mathbb{H}^2 \times \mathbb{R}$ [7]. Further, this problem has been recently studied in Euclidean spaces and semi-Euclidean spaces, (see in [10, 19, 20]) where T is replaced by a constant direction k.

On the other hand, in [8, 9, 11, 23] authors study generalized constant ratio surfaces. A hypersurface M in a semi-Euclidean space $\mathbb{E}_t^{n+1}$ is said to be a generalized constant ratio surfaces if the tangential component of its position vector is a principal direction of M. It is well-known that planes and complete hypersurfaces of $\mathbb{E}_t^{n+1}$ with constant sectional curvatures are trivial examples of generalized constant ratio surfaces.

This paper is organized as follows. In Sect. 2, we mention the notation that we use in this paper. In Sect. 3 and Sect.4, we present a short survey of recent results on surfaces endowed with a canonical principle curvatures. In Sect. 5, we show some of the results that we have recently obtained. In Sect. 6 we present classifications of generalized constant ratio hypersurfaces in Minkowski spaces.

2 Preliminaries

In this section, we would like to give a brief summary of basic results on Lorentzian surfaces, (see for detail, [2, 21]).

Let $\mathbb{E}_t^m$ denote the semi-Euclidean m-space with the canonical semi-Euclidean metric tensor of index t given by

$$\tilde{g} = \sum_{i=1}^{m-t} dx_i^2 - \sum_{j=m-t+1}^{m} dx_j^2,$$

where $x_1, x_2, \ldots, x_m$ are rectangular coordinates of the points of $\mathbb{E}_t^m$.

Let $\mathbb{S}_t^n(r^2)$ and $\mathbb{H}_{t-1}^n(-r^2)$ denote the de Sitter space-time and the hyperbolic space of dimension $n > 2$ defined by

$$\begin{aligned} \mathbb{S}_t^n(1/r^2) &= \{x \in \mathbb{E}_t^{n+1} : \langle x, x \rangle = r^{-2}\}, \\ \mathbb{H}_{t-1}^n(-1/r^2) &= \{x \in \mathbb{E}_t^{n+1} : \langle x, x \rangle = -r^{-2}\}. \end{aligned}$$

For a short notation, we put $\mathbb{H}_0^n(-1) = \mathbb{H}^n$ and $\mathbb{S}_0^n(1) = \mathbb{S}^n$.

We would like to note that all further notations, basic definitions and basic facts that we will use in this paper are described in [8, 23]. We also would like to refer to [4, 7, 19, 20] for detailed information of definition and geometrical interpretation of surfaces endowed with canonical principal direction.

3 Surface endowed with canonical principal direction in product spaces

In recent years, a lot of research has been done about $\tilde{M}^2 \times \mathbb{R}$ by considering the unit vector field T tangent to the second factor, parallel along $\tilde{M}^2 \times \mathbb{R}$. A special case is when $\tilde{M}$ is a 2-dimensional Riemanmian space form, i.e., $\tilde{M} = M^2(c),\ c = \pm 1$. A surface M in $M^2(c) \times \mathbb{R}$ is said to be endowed with canonical principal direction (in short, CPD) if the projection of T, i.e. the canonical unit vector tangent to the $\mathbb{R}-$direction, onto the tangent space of M, is a principal direction. In this case, T can be decomposed as

$$T = \sin\theta U + \cos\theta N$$

where N is the unit normal vector field of surface M. Here, $SU = k_1 U$ for a smooth function k_1 where S is the shape operator of M in $\mathbb{M}^2 \times \mathbb{R}$, respectively. Note that we consider the case $\theta \notin \left\{0, \frac{\pi}{2}\right\}$ to eliminate trivial cases.

In this section, we would like to present a survey of classification results recently obtained. However, before we proceed, we would like to note that a further generalization of this notion is isometric immersions which belongs to the class $\mathcal{A}$. An isometric immersion $f : M \to \mathbb{S}^n \times \mathbb{R}$ is said to have this property if U is an eigenvector of all shape operators of f, where M is an m-dimensional submanifold of $\mathbb{S}^n \times \mathbb{R}$. This class was introduced in [22], where a complete description was given for hypersurfaces, and extended to submanifolds of $\mathbb{S}^n \times \mathbb{R}$ in [17].

3.1 Surfaces in $\mathbb{S}^2 \times \mathbb{R}$

We may note that the study of CPD surfaces in $\mathbb{S}^2 \times \mathbb{R}$ was investigated in [4]. The following results were obtained in that paper.

Let M be a surface endowed with canonical principal direction in $\mathbb{S}^2 \times \mathbb{R}$. By choosing an appropriate local coordinate system on M, one can see that the induced metric g of M becomes

$$g = dx^2 + \beta^2(x, y)dy^2.$$

Moreover, the shape operator S with respect to the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is given by

$$S = \begin{pmatrix} \theta_x & 0 \\ 0 & \frac{\beta_x \tan\theta}{\beta} \end{pmatrix}.$$

(See [4].)

Remark 3.1. An analogous result for CPD surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is obtained in [7].

First, we would like to give the following characterization for CPD surfaces in $\mathbb{S}^2 \times \mathbb{R}$.

Theorem 3.2. *[5] Let M be an immersed in $\mathbb{S}^2\times\mathbb{R}$ and p a point of M for which $\theta(p)\notin\{0,\pi/2\}$. Then, U is a principal direction if and only if M considered as a surface in $\mathbb{E}^4$ is normally flat.*

The following classification result is obtained in [4].

Proposition 3.3. *[4] A surface M immersed in $\mathbb{S}^2\times\mathbb{R}$ is a surface for which U is a principal direction if and only if the immersion F is in the neighborhood of a point p where $\theta(p)\notin\left\{0,\frac{\pi}{2}\right\}$ given by*

$$\begin{array}{rcl} F: M & \longrightarrow & \mathbb{S}^2\times\mathbb{R} \\ (x,y) & \longmapsto & (F_1(x,y),F_2(x,y),F_3(x,y),F_4(x)), \end{array}$$

where

$$F_j(x,y)=\int_{y_0}^{y}\alpha_j(v)\sin(\psi(x)+\phi(v))dv$$

for $j=1,2,3$, $\psi'(x)=\cos(\theta(x))$, $F_4'(x)=\sin(\theta(x))$ and $(\alpha_1,\alpha_2,\alpha_3)$ is a curve in $\mathbb{S}^2$ and $F_1^2+F_2^2+F_3^2=1$. Moreover $\alpha_1,\alpha_2,\alpha_3,\psi$ and ϕ are functions on M related by

$$\begin{aligned} \alpha_j'(y)= & -\cos(\psi(x)+\phi(y))\int_{y_0}^{y}\alpha_j(v)\cos(\psi(x)+\phi(v))dv \\ & -\sin(\psi(x)+\phi(y))\int_{y_0}^{y}\alpha_j(v)\sin(\psi(x)+\phi(v))dv. \end{aligned}$$

A direct consequence of this proposition is

Corollary 3.4. *[4] A surface M immersed in $\mathbb{S}^2\times\mathbb{R}$ is a minimal surface with U a principal direction if and only if the immersion F is (up to isometries of $\mathbb{S}^2\times\mathbb{R}$) in the neighborhood of a point p where $\theta\notin\left\{0,\frac{\pi}{2}\right\}$ given by*

$$\begin{aligned} & F: M \longrightarrow \mathbb{S}^2\times\mathbb{R}, \\ & (x,y)\longmapsto\left(\frac{\sin x}{\sqrt{1+c^2}},\frac{\sqrt{\cos^2 x+c^2}\cos y}{\sqrt{1+c^2}},\frac{\sqrt{\cos^2 x+c^2}\sin y}{\sqrt{1+c^2}},F_4(x)\right) \end{aligned}$$

with

$$F_4(x)=\int_0^x\frac{c}{\sqrt{\cos^2(u)+c^2}}du.$$

Corollary 3.5. *[4] A surface M immersed in $\mathbb{S}^2\times\mathbb{R}$ is a flat surface with U a principal direction if and only if the immersion F is (up to isometries of $\mathbb{S}^2\times\mathbb{R}$) in the neighborhood of a point p where $\theta\notin\left\{0,\frac{\pi}{2}\right\}$ given by*

$$\begin{aligned} & F: M \rightarrow \mathbb{S}^2\times\mathbb{R}, \\ & (x,y)\longmapsto\left(\frac{\sqrt{1+d-x^2}}{\sqrt{1+d}},\frac{x\cos y}{\sqrt{1+d}},\frac{x\sin y}{\sqrt{1+d}},F_4(x)\right) \end{aligned}$$

with

$$F_4(x)=\int_0^x\frac{\sqrt{d-u^2}}{\sqrt{1+d-u^2}}du.$$

3.2 Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

In [7], the authors studied CPD surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

Note that we have the following characterization.

Theorem 3.6. *[7] Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$ such that $\theta \notin 0$. U is a principal direction if and only if M is normally flat in $\mathbb{R}^3_1 \times \mathbb{R}$.*

Theorem 3.7. *[7] If $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion with $\theta \notin \left\{0, \frac{\pi}{2}\right\}$, then U is a principal direction if and only if F is given by*

$$F(x, y) = (F_1(x, y), F_2(x, y), F_3(x, y), F_4(x)),$$

with $F_j(x, y) = A_j(y) \sinh \phi(x) + B_j(y) \cosh \phi(x)$, for $j = 1, 2, 3$ and $F_4(x) = \int_0^x \sin(\theta(\tau)d\tau)$, where $\phi'(x) = \cos(\theta)$. The six functions A_j and B_j are found in one of the following three cases.

- *Case 1.*

$$\begin{aligned} A_j(y) =& \int_0^y H_j(\tau) \cosh \psi(\tau) d\tau + c_{1j}, \\ B_j(y) =& \int_0^y H_j(\tau) \sinh \psi(\tau) d\tau + c_{2j}, \\ H'_j(y) =& B_j(y) \sinh \psi(y) - A_j(y) \cosh \psi(y); \end{aligned}$$

- *Case 2.*

$$\begin{aligned} A_j(y) =& \int_0^y H_j(\tau) \sinh \psi(\tau) d\tau + c_{1j}, \\ B_j(y) =& \int_0^y H_j(\tau) \cosh \psi(\tau) d\tau + c_{2j}, \\ H'_j(y) =& - A_j(y) \sinh \psi(y) + B_j(y) \cosh \psi(y); \end{aligned}$$

- *Case 3.*

$$\begin{aligned} A_j(y) =& \pm \int_0^y H_j(\tau) d\tau + c_{1j}, \\ B_j(y) =& \int_0^y H_j(\tau) d\tau + c_{2j}, \\ H'_j(y) =& c_{2j} \mp c_{1j}; \end{aligned}$$

where $H = (H_1, H_2, H_3)$ is a curve on the de Sitter space $\mathbb{S}^2_1, \psi$ is a smooth function on M and $c_1 = (c_{11}, c_{12}, c_{13}), c_2 = (c_{21}, c_{22}, c_{23})$ are constant vectors.

Remark 3.8. [7] In order to obtain a unified description, we note that in all cases F is given by

$$F(x, y) = \left(A(y) \sinh \phi(x) + B(y) \cosh \phi(x), \int_0^x \sin \theta(\tau) d\tau \right),$$

where A is a curve in $\mathbb{S}^2_1$ and B is a curve in $\mathbb{H}^2$ orthogonal to A such that the two speeds A' and B' are parallel. Denoting by H the unit vector of their common direction, one has $H = A \otimes B$ and moreover

- H is a space-like curve in the first case,
- H is a time-like curve in the second case,
- H is a light-like curve in the last case.

Theorem 3.9. *[7] If $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion with angle function $\theta \notin \left\{0, \frac{\pi}{2}\right\}$, then U is a principal direction if and only if F is locally given by*

$$F(x, y) = (A(y) \sinh \phi(x) + B(y) \cosh \phi(x), \chi(x)),$$

where $A(y)$ is a curve in $\mathbb{S}^2_1$ and B is a curve in $\mathbb{H}^2$, such that $\langle A, B\rangle = 0, A' \parallel B'$ and where $(\phi(x), \chi(x))$ is a regular curve in $\mathbb{R}^2$. The angle function θ of M depends only on x and coincides with the angle function of the curve (ϕ, χ). In particular, we can arc length reparametrize (ϕ, χ); then (x, y) are canonical cordinates and $\theta'(x) = \kappa(x)$, the curvature of (ϕ, χ).

Theorem 3.10. *[7] Let $F : M \to \mathbb{H}^2 \times \mathbb{R}$ is an isometric immersion with $\theta \notin \left\{0, \frac{\pi}{2}\right\}$. Then M has U as a principal direction if and only if F is given by*

$$F(x, y) = (f(y) \cosh \phi(x) + N_f(y) \sinh \phi(x), \chi(x)),$$

where $f(y)$ is a regular curve in $\mathbb{H}^2$ and $N_f(y) = \frac{f(y)\otimes f'(y)}{\sqrt{\langle f'(y), f'(y)\rangle}}$ represents the normal of f. Moreover, (ϕ, χ) is a regular curve in $\mathbb{R}^2$ and the angle function θ of this curve is the same as the angle function of the surface parametrized by F.

Consequently, authors obtained the following classification results by considering minimal and flat surfaces.

Corollary 3.11. *[7] Let M be a surface isometrically immersed in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \notin \left\{0, \frac{\pi}{2}\right\}$. Then M is minimal with U a principal direction if and only if the immersion is, up to isometries of the ambient space, locally given by one of the next cases*

- $F(x, y) = \left(\frac{b(x)}{\sqrt{1+c_1^2-c_2^2}}, \sinh y \frac{\sqrt{a^2(x)+1}}{\sqrt{1+c_1^2-c_2^2}}, \cosh y \frac{\sqrt{a^2(x)+1}}{\sqrt{1+c_1^2-c_2^2}}, \chi(x)\right),$
- $F(x, y) = \left(\cos y \frac{\sqrt{a^2(x)+1}}{\sqrt{-1-c_1^2+c_2^2}}, \sin y \frac{\sqrt{a^2(x)+1}}{\sqrt{-1-c_1^2+c_2^2}}, \frac{b(x)}{\sqrt{-1-c_1^2+c_2^2}}, \chi(x)\right),$
- $F(x, y) = \left(b(x)y, \frac{b(x)}{2}(1 - y^2) - \frac{1}{2b(x)}, \frac{b(x)}{2}(1 + y^2) + \frac{1}{2b(x)}, \chi(x)\right),$

where $\chi(x) = \int_0^x \frac{1}{\sqrt{a^2(\tau)+1}} d\tau$, with $a(x) = c_1 \cosh x + c_2 \sinh x, b(x) = a'(x)$ and c_1, c_2 are constants.

Theorem 3.12. *[7] Let M be a surface in $\mathbb{H}^2 \times \mathbb{R}$, with $\theta \notin \left\{0, \frac{\pi}{2}\right\}$. Then M is flat with U a principal direction if and only if the immersion F is, up to isometries of the ambient space, given by*

- $F(x, y) = \left(\frac{x}{\sqrt{1+c}} \cos y, \frac{x}{\sqrt{1+c}} \sin y, \frac{\sqrt{x^2+c+1}}{\sqrt{1+c}}, \chi(x)\right)$,
- $F(x, y) = \left(\frac{\sqrt{x^2+c+1}}{\sqrt{-1-c}}, \frac{x}{\sqrt{-1-c}} \sinh y, \frac{x}{\sqrt{-1-c}} \cosh y, \chi(x)\right)$,
- $F(x, y) = \left(xy, \frac{x}{2}(1-y^2) - \frac{1}{2x}, \frac{x}{2}(1+y^2) + \frac{1}{2x}, \chi(x)\right)$,

where $\chi(x) = \int_0^x \frac{\sqrt{\tau^2+c}}{\sqrt{\tau^2+c+1}} d\tau, c \in \mathbb{R}$.

3.3 Surfaces in $M^2(c) \times \mathbb{R}_1$

In [10], Fu and Nistor gave a partial classification of CPD surfaces by assuming that the fixed vector is time-like. In this case, the fixed vector is $k = (0, 0, 1)$ which is time-like.

Similar to previous case, let U stand for the unit tangent vector on the direction of k^T.

Theorem 3.13. *[10] Let M be a space-like surface in Lorentzian product spaces $M^2(c) \times \mathbb{R}_1$. Then, U is a principal direction if and only if M is normally flat in $\mathbb{R}^3_1$ for $c = 0$, $\mathbb{R}^4_1$ for $c = 1$, $\mathbb{R}^4_2$ for $c = -1$.*

Next, we would like to mention the following theorem obtained in [10] where authors assume $k = (0, 0, 1)$.

Theorem 3.14. *[10, 20] Let $L : M \to M^2(c) \times \mathbb{R}_1$ be a space-like surface. Then, U is a canonical principal direction for M if and only if M is parametrized as:*

- *If $c = 1$, then $L : M \to \mathbb{S}^2 \times \mathbb{R}_1$,*

$$L(x, y) = (\cos \phi(x) f(y) + \sin \phi(x) N_f(y), \chi(x)),$$

  where $f(y)$ is a regular curve on $\mathbb{S}^2$ and

$$N_f(y) = \frac{f(y) \otimes f'(y)}{\sqrt{\langle f'(y), f'(y)\rangle}}$$

  represents the normal of f.

- *If $c = -1$, then $L : M \to \mathbb{H}^2 \times \mathbb{R}_1$,*

$$L(x, y) = (\cosh \phi(x) f(y) + \sinh \phi(x) N_f(y), \chi(x)),$$

  where $f(y)$ is a regular curve in $\mathbb{S}^2$ and

$$N_f(y) = \frac{f(y) \otimes f'(y)}{\sqrt{\langle f'(y), f'(y)\rangle}}$$

  represents the normal of f.

- *If $c = 0$, then we have $L : M \to \mathbb{R}^3_1$ is congruent to one of the following two surfaces.*

  1. $L(x, y) = (\cos y, \sin y, 0)\,\phi(x) - (0, 0, 1)\chi(x) + \gamma(v)$, *where*

  $$\gamma(v) = \left(\int_0^y \psi(\tau)\sin(\tau)d\tau, -\int_0^y \psi(\tau)\cos(\tau)d\tau, 0\right), \psi \in C^\infty(M).$$

  2. $L(x, y) = (\cos y_0, \sin(y_0), 0)\,\phi(x) - (0, 0, 1)\chi(x) + \gamma_0(y)$, *where*

  $$\gamma_0(y) = (-(\sin y_0)y, (\cos y_0)y, 0)$$

  and y_0 is a real constant.

In all three cases $\phi(x) = \int_{x_0}^x \cosh\theta(\tau)d\tau$ and $\chi(x) = \int_{x_0}^x \sinh\theta(\tau)d\tau$.

Now, we give the following results obtained in [10] for Lorentzian surfaces with canonical principal direction. We note that they gave the partial classification of those surfaces in that paper.

Theorem 3.15. *[10] Let M be a Lorentzian surface in Lorentzian product spaces $M^2(c) \times \mathbb{R}_1$, and let θ be the hyperbolic angle function. Then, U is a principal direction if and only if M is normally flat in $\mathbb{R}^3_1$ for $c = 0$, $\mathbb{R}^4_1$ for $c = 1$, $\mathbb{R}^4_2$ for $c = -1$.*

Theorem 3.16. *[10, 7] Let $L : M \to M^2(c) \times \mathbb{R}_1$ be a Lorentzian surface and let $\theta \notin 0$ be the hyperbolic angle function. Then, U is a canonical principal direction for M if and only if M is parametrized as:*

- *If $c = 1$, then $L : M \to \mathbb{S}^2 \times \mathbb{R}_1$ is*

  $$L(x, y) = (\cos\chi(x)f(y) + \sin\chi(x)N_f(y), \phi(x)),$$

  where $f(y)$ is a regular curve on $\mathbb{S}^2$ and

  $$N_f(y) = \frac{f(y) \otimes f'(y)}{\sqrt{\langle f'(y), f'(y)\rangle}}$$

  represents the normal of f.

- *If $c = -1$, then $L : M \to \mathbb{H}^2 \times \mathbb{R}_1$ is*

  $$L(x, y) = (\cosh\chi(x)f(y) + \sinh\chi(x)N_f(y), \phi(x)),$$

  where $f(y)$ is a regular curve in $\mathbb{S}^2$ and

  $$N_f(y) = \frac{f(y) \otimes f'(y)}{\sqrt{\langle f'(y), f'(y)\rangle}}$$

  represents the normal of f.

- *If $c = 0$, then $L : M \to \mathbb{R}^3_1$*
  1. $L(x, y) = (\chi(x) \cos y, \chi(x) \sin y, \phi(x)) + \gamma(y)$ *where*
  $$\gamma(y) = \left(-\int_0^y \psi(\tau) \sin(\tau) d\tau, \int_0^y \psi(\tau) \cos(\tau) d\tau, 0 \right), \psi \in C^\infty(M).$$
  2. $L(x, y) = (\chi(x) \cos y_0, \chi(x) \sin y_0, \phi(x)) + \gamma_0 y$, *where*
  $$\gamma_0 = (-\sin y_0, \cos y_0, 0)$$
  and y_0 is a real constant.

In all these cases $\phi(x) = \int_{x_0}^x \cosh \theta(\tau) d\tau$ and $\chi(x) = \int_{x_0}^x \sinh \theta(\tau) d\tau$.

We have the following corollaries of the previvous theorem.

Corollary 3.17. *[10] The only flat Lorentz surfaces M immersed in $\mathbb{E}^3_1$ endowed with a canonical principal direction are given by the cylindirical surfaces parametrized in the last case of Theorem 3.16.*

Corollary 3.18. *[10] The only minimal Lorentz surfaces M immersed in $\mathbb{E}^3_1$ endowed with a canonical principal direction are given by the catenoids of the 3rd kind parametrized as:*

$$L(x, y) = \left(m \cos \frac{t}{m} \cos y, m \cos \frac{t}{m} \sin y, x \right), m \in \mathbb{R} \ \{0\}.$$

4 Surfaces endowed with canonical principal direction in Euclidean and semi-Euclidean spaces

A surface in a semi-Euclidean space $\mathbb{E}^3_r$ is said to be endowed with canonical principal direction (CPD) if there exists a fixed direction k such that $S(k^T) = k_1 k^T$, where k^T denote the tangential component of k. In [19], Munteanu and Nistor studied surfaces with CPD in $\mathbb{E}^3$, while some classifications of such surfaces in the Minkowski space $\mathbb{E}^3_1$ is obtained in [20] for some cases.

4.1 Surfaces in $\mathbb{E}^3$

Let M be a surface with CPD in $\mathbb{E}^3$. Note that by choosing an appropriate rotation in $\mathbb{E}^3$, we may assume $k = (0, 0, 1)$ and we denote $U = k^T / \|k^T\|$. We define θ by $k = \sin \theta U + \cos \theta N$. To eliminate trivial cases we consider a point $p \in M$ with $\theta(p) \notin \left\{0, \frac{\pi}{2}\right\}$.

Note that if U is a principal direction, then we can choose a local coordinate system (x, y) in a neighborhood of p so that ∂_x is in the direction of U and the metric g has the form $g = dx^2 + \beta^2(x,y)dy^2$. Further, the shape operator S is given by

$$S = \begin{pmatrix} \theta_x & 0 \\ 0 & \frac{\beta_x \tan\theta}{\beta} \end{pmatrix}.$$

Moreover, θ and β are related by $\frac{\beta_x}{\cos\theta}$ is independent of x and $\theta_y = 0$, [19].

In [19] the following characterization theorem obtained.

Theorem 4.1. *[19] Let M be a surface in $\mathbb{E}^3$ and $\theta \notin 0$ be the angle function. Let (x,y) be local coordinates on M such that ∂_x is the direction of U. Then, U is a principal direction if and only if $\theta_y = 0$.*

Further, the classification of surfaces with CPD in $\mathbb{E}^3$ was given as following.

Theorem 4.2. *[19] Let M be a surface isometrically immersed in $\mathbb{E}^3$ and let $\theta \notin 0, \frac{\pi}{2}$ be as before. Then, U is a canonical principal direction if and only if M is given, up to isometries of $\mathbb{E}^3$, by one of the following cases:*

- $r : M \to \mathbb{E}^3$,

  $$r(x,y) = \left(\phi(x)(\cos y, \sin y) + \gamma(y), \int_0^x \sin\theta(\tau)d\tau\right)$$

  with

  $$\gamma(y) = \left(-\int_0^y \psi(\tau)\sin(\tau)d\tau, \int_0^y \psi(\tau)\cos(\tau)d\tau\right),$$

  where ψ is a smooth function on a certain interval I.

- $r : M \to \mathbb{E}^3, r(x,y) = \left(\phi(x)\cos(y_0), \phi(x)\sin(y_0), \int_0^y \sin\theta(\tau)d\tau + y(v_0)\right)$ *with $v_0 = (-\sin(y_0), \cos(y_0), 0), y_0 \in \mathbb{R}$. Notice that these surfaces are cylinders. In both cases $\phi(x)$ denotes a primitive of $\cos\theta$.*

Similar to Sect. 3, the classifications of minimal and flat surfaces follows from the previous theorem.

Corollary 4.3. *[19] Let M be a surface isometrically immersed in $\mathbb{E}^3$. Then M is minimal surface with U a principal direction if and only if the immersion is, up to isometries of the ambient space, given by*

$$r(x,y) = \left(\sqrt{x^2+c^2}\cos y, \sqrt{x^2+c^2}\sin y, c\log(x + \sqrt{x^2+c^2})\right), c \in \mathbb{R}.$$

Remark 4.4. [19] We notice that this surface can be obtained by rotating the catenary around the z-axis. Hence, the only minimal surface in Euclidean 3-space with canonical principal direction is the catenoid.

Corollary 4.5. *[19] Let M be a surface isometrically immersed in $\mathbb{E}^3$ and let $\theta \notin 0, \frac{\pi}{2}$ be the angle function. Then M is a flat surface with U a principal direction if and only if the immersion is, up to isometries of the ambient space, given by*

$$r(x,y) = \left(\phi(x)\cos(y_0), \phi(x)\sin(y_0), \int_0^x \sin\theta(\tau)d\tau\right) + yv_0$$

where $v_0 = (-\sin y_0, \cos y_0, 0), y_0 \in \mathbb{R}$, and $\phi(x)$ represents a primitive of $\cos\theta$.

4.2 Surfaces in $\mathbb{E}_1^3$

On the other hand, some classification results for surfaces endowed with canonical principal direction in $\mathbb{E}_1^3$ were obtained in [20], where Nistor studied space-like surfaces. In that paper, the author gave a classification of those surfaces by assuming that the fixed direction is time-like and the fixed vector k is considered to be $k = (0, 0, 1)$.

Theorem 4.6. *[20] Let M be a space-like surface in $\mathbb{E}_1^3$ and $\theta \notin 0$ be the hyperbolic angle function. Let (x, y) be local coordinates on M such that ∂_x is the direction of U. Then, U is a principal direction if and only if $\theta_y = 0$.*

Theorem 4.7. *[20] Let M immersed in $\mathbb{E}_1^3$ be a space-like surface and $\theta \notin 0$ be the hyperbolic angle function. Then, M has a principal direction if and only if M is parametrized in the last case of Theorem 3.14.*

Consequently, we mention following two theorems related with minimality and flatness.

Theorem 4.8. *[20] The only maximal space-like surfaces in $\mathbb{E}_1^3$ with a canonical principal direction are catenoids of the 1st kind, parametrized in local coordinates (x, y) as*

$$(x, y) \mapsto \left(\sqrt{x^2 - c^2}\cos y, \sqrt{x^2 - c^2}\sin y, c\ln(x + \sqrt{x^2 - c^2})\right), c \in \mathbb{R} \ \{0\}.$$

Theorem 4.9. *[20] The only flat space-like surfaces in $\mathbb{E}_1^3$ with a canonical principal direction are generalized cylinders, parametrized in local coordinates (x, y) as*

$$(x, y) \mapsto \sigma(x) + v_0 y,$$

where $\sigma(x) = \left(\cos y_0 \int_0^x \cosh\theta(\tau)d\tau, \sin y_0 \int_0^x \cosh\theta(\tau)d\tau, -\int_0^x \sinh\theta(\tau)d\tau\right), v_0 = (-\sin y_0, \cos y_0, 0), y_0 \in \mathbb{R}$*, and $\theta \notin 0$ denotes the hyperbolic angle function.*

Remark 4.10. [20] The flat space-like surfaces endowed with a canonical principal direction classified in Theorem 4.9 are given by the generalized cylinders from the last case of Theorem 3.14. More precisely, these surfaces are cylinders over space-like curves with space-like rulings orthogonal to $k = (0, 0, 1)$.

5 New examples of surfaces in $\mathbb{E}_1^3$

In this section we would like to present some new examples of Lorentzian surface endowed with CPD in the Minkowski 3-space. Before we proceed, we would like to note that if M is space-like, then its shape operator S is diagonalizable, i.e., there exists a local orthonormal frame field $\{e_1, e_2; N\}$ such that $Se_i = k_i e_i, \quad i = 1, 2, \ldots, n$. In this case, the vector field e_i and smooth function k_i are called a principal direction and a principal curvature of M.

On the other hand, if M is Lorentzian, then its shape operator can be non-diagonalizable. In this case, if all of the eigenvalues of S are real at any point of M, then the matrix representation of S with respect to a suitable pseudo-orthonormal frame field $\{f_1, f_2; N\}$ such that

$$\langle f_i, f_j \rangle = \delta_{ij} - 1, \quad i, j = 1, 2$$

of the tangent bundle of M, the shape operator of a Lorentzian surface in $\mathbb{E}^3_1$ can be assumed to be one of canonical forms given by

$$\begin{gathered}\text{Case 1.} \quad S = \operatorname{diag}(k_1, k_2), \quad \text{Case 2.} \quad S = \begin{pmatrix} k_1 & \mu \\ 0 & k_1 \end{pmatrix} \\ \text{Case 3.} \quad S = \begin{pmatrix} k_1 & \mu \\ -\mu & k_1 \end{pmatrix}\end{gathered} \tag{5.1}$$

for some smooth functions k_1, k_2 and a non-vanishing function μ, where the frame field is chosen to be orthonormal in Case 1 and Case 3 and pseudo-orthonormal in Case 2 (See for example [16]). We note that if the shape operator of M is as given in Case 3 of (5.1), then S has no eigenvalue. So, we will consider surfaces whose shape operator is as given in Case 1 or Case 2 of (5.1).

A null curve $\beta(s)$ in $\mathbb{E}^3_1$ is said to have a Cartan frame if there exists vector fields $\{A, B, C\}$ on β such that $\langle A, A \rangle = \langle B, B \rangle = 0$, $\langle A, B \rangle = -1$, $\langle A, C \rangle = \langle B, C \rangle = 0$ and $\langle C, C \rangle = 1$ with $\beta' = A$, $A' = k_1(s)C$ and $B' = k_2 C$ for a constant k_2 and a smooth function k_1 which is vanishing only on a subset $\mathcal{U}$ with $\operatorname{int} U = \emptyset$. Then, the surface M given by

$$f(s, t) = \beta(s) + tB(s) \tag{5.2}$$

is said to be a B-scroll. Note that in [16], M. Magid have proved that a surface in $\mathbb{E}^3_1$ with non-diagonalizable shape operator is isoparametric if and only if it is a B-scroll.

Example 5.1. [14] Consider the B-scroll given by

$$c(\hat{s}, t) = \left(\frac{\hat{s}^2}{2} + t, \frac{(2\hat{s} - 1)^{3/2}}{3}, \frac{\hat{s}^2}{2} - \hat{s} + t \right). \tag{5.3}$$

It turns out that the shape operator of this surface with respect to the pseudo-orthonormal frame field $\{\partial_t, \partial_s\}$ is

$$S = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}$$

Moreover, it is a surface endowed with a canonical principal direction relative to $k = (1, 0, 0)$.

Further, we have recently obtained the following result.

Proposition 5.2. *A flat minimal surfaces in* $\mathbb{E}^3_1$ *endowed with a canonical principal direction relative to a fixed direction is either an open part of a plane or congruent to the surface given in* (5.3).

Remark 5.3. By considering the above example, the problem of considering surfaces with shape operator given in Case 2 of (5.1) in terms of having canonical principal direction arises. Authors would like to announce that they have recently completed the classification of such surfaces with a canonical principal direction relative to a fixed direction in $\mathbb{E}^3_1$.

Example 5.4. [14] Consider the rotational surface with a light-like rotational axis in $\mathbb{E}^3_1$ given by

$$x(s,t) = \left(\frac{1}{2}st^2 + s + \phi(s), st, \frac{1}{2}st^2 + \phi(s)\right) \tag{5.4}$$

for a smooth function ϕ. It is well-kown that the principal directions of M are

$$e_1 = \frac{1}{\sqrt{\varepsilon_1(-2\phi' - 1)}}\partial_s, \quad e_2 = \frac{1}{s}\partial_t.$$

Further, we have

$$(1,0,1) = \psi(e_1 - N)$$

for a smooth function ψ. Hence, the surface given by (5.4) is endowed with a canonical principal direction relative to $k = (1,0,1)$.

Remark 5.5. A direct computation yields that the surface given by (5.4) is minimal if and only if

$$\frac{\phi''}{(2\phi' + 1)} = \frac{1}{s}$$

On the other hand, the surface given by (5.4) is flat if and only if ϕ is linear.

Remark 5.6. Authors also would like to announce that they have recently completed the classification of surfaces with a canonical principal direction relative to a fixed light-like direction in $\mathbb{E}^3_1$.

6 Generalized constant ratio surfaces in $\mathbb{E}^3_1$

Generalized constant ratio surfaces in Euclidean spaces are firstly investigated in [9, 23]. By definition, let M be a surface in the ambient space, x its position vector and θ denote the angle function between x and the unit normal vector field N of M. If the tangential part of x is one of its principal directions, then M is said to be a generalized constant ratio (in short, GCR surfaces). Note that, we would like to remember two following definition. The time-like cone of $\mathbb{E}^3_1$ is defined as the set of all time-like vectors of $\mathbb{E}^3_1$, that is,

$$T = \left\{x \in \mathbb{E}^3_1 : \langle x, x\rangle < 0\right\}.$$

The space-like cone of $\mathbb{E}^3_1$ is defined as the set of all space-like vectors of $\mathbb{E}^3_1$, that is,

$$S = \left\{x \in \mathbb{E}^3_1 : \langle x, x\rangle > 0\right\}.$$

In this section, we just would like to present classification of GCR surfaces in Minkowski 3-space obtained in [9, 8, 24]. Note that in [8], authors studied this surface independently from the paper at [24]. Fistly, we would like to give results for this surface obtained in [9].

6.1 Lorentzian surfaces in $\mathbb{E}_1^3$

Most recently, Lorentzian GCR surfaces in the 3-dimensional Minkowski space investigated by Fu and Yang in [11].

Theorem 6.1. *[11] Let $x : M \to \mathbb{E}_1^3$ be a surface immersed in the 3-dimensional Minkowski space $\mathbb{E}_1^3$. If the immersion x lies in the space-like cone, then M is a GCR surface if and only if the immersion $x(M)$ is given by one of the following eight statements holds:*

- $x(s,t) = s\left(\cos u(s), \sin u(s)\cosh t, \sin u(s)\sinh t\right)$, *where* $u(s) = \int \frac{\cot\theta(s)}{s}ds$;
- $x(s,t) = s\left(\sin u(s), \cos u(s)\cosh t, \cos u(s)\sinh t\right)$, *where* $u(s) = \int \frac{\cot\theta(s)}{s}ds$;
- $x(s,t) = s\left(\cos u(s) f(t) + \sin u(s) f(t) \times f'(t)\right)$, *where f is a time-like unit speed curve on $\mathbb{S}_1^2$ satisfying* $(f, f', f'') \notin 0$, $u(s) = \int \frac{\cot\theta(s)}{s}ds$;
- $x(s,t) = \frac{s}{2}\left(-e^{-u(s)} + e^u(s)(t^2-1), 2e^u(s)t, -e^{-u(s)} + e^{u(s)}(t^2+1)\right)$, *where* $u(s) = \int \frac{\coth\theta(s)}{s}ds$;
- $x(s,t) = \frac{s}{2}\left(-e^{u(s)} + e^{-u(s)}(t^2-1), 2e^{-u(s)}t, -e^{u(s)} + e^{-u(s)}(t^2+1)\right)$, *where* $u(s) = \int \frac{\coth\theta(s)}{s}ds$;
- $x(s,t) = s\left(\cosh u(s)\cos t, \sinh u(s)\sin t, \sinh u(s)\right)$, *where*
$$u(s) = \int \frac{\coth\theta(s)}{s}ds;$$
- $x(s,t) = s\left(\cosh u(s), \sinh u(s)\sinh t, \sinh u(s)\cosh t\right)$, *where*
$$u(s) = \int \frac{\coth\theta(s)}{s}ds;$$
- $x(s,t) = s\left(\cosh u(s) f(t) + \sinh u(s) f(t) \times f'(t)\right)$, *where f is a time-like unit speed curve on $\mathbb{S}_1^2$ satisfying* $(f, f', f'') \notin 0$, $u(s) = \int \frac{\coth\theta(s)}{s}ds$.

Further, if x lies in the time-like cone, the following classification theorem was obtained.

Theorem 6.2. *[11] Let $x : M \to \mathbb{E}_1^3$ be a surface immersed in the 3-dimensional Minkowski space. If the immersion x lies in the timelike cone, then M is a GCR surface if and only if the immersion $x(M)$ is given by one of the following five statements holds:*

- $x(s,t) = \frac{s}{2}\left(e^{-u(s)} + e^{u(s)}(t^2-1), 2e^{u(s)}t, e^{-u(s)} + e^{u(s)}(t^2+1)\right)$, *where*

$$u(s) = \int \frac{\tanh\theta(s)}{s} ds;$$

- $x(s,t) = \frac{s}{2}\left(e^{u(s)} + e^{-u(s)}(t^2-1), 2e^{-u(s)}t, e^{u(s)} + e^{-u(s)}(t^2+1)\right)$, *where* $u(s) = \int \frac{\tanh\theta(s)}{s} ds$*;*
- $x(s,t) = s\left(\sinh u(s), \cosh u(s)\sinh t, \cosh u(s)\cosh t\right)$, *where*

$$u(s) = \int \frac{\tanh\theta(s)}{s} ds;$$

- $x(s,t) = s\left(\sinh u(s)\sin t, \sinh u(s)\cos t, \cosh u(s)\right)$, *where*

$$u(s) = \int \frac{\tanh\theta(s)}{s} ds;$$

- $x(s,t) = s\left(\cosh u(s) f(t) + \sinh u(s) f(t) \times f'(t)\right)$, *where* f *is a unit speed curve on* $\mathbb{H}^2$ *satisfying* $\langle f'', f''\rangle \notin -\langle f, f''\rangle^2$, $u(s) = \int \frac{\tanh\theta(s)}{s} ds$.

We would like to also note the following consequences of the previous theorems.

Corollary 6.3. *[11] A flat Lorentz GCR surface in* $\mathbb{E}^3_1$ *is an open part of a plane or of a cylinder.*

Corollary 6.4. *[11] A Lorentzian GCR surface in* $\mathbb{E}^3_1$ *with constant mean curvature is a surface of revolution.*

6.2 Space-like GCR Surfaces in Minkowski 3-Space

In [8] and [24], the authors independently studied the space-like GCR surface in Minkowski spaces. After, they independently obtained the complete classification of GCR surfaces in the Minkowski 3-space. All the following results obtained for space-like GCR surfaces in Minkowski spaces were given in [8, 24].

Theorem 6.5. *[8] Let* M *be a non-degenerated hypersurface in* $\mathbb{E}^{n+1}_1$ *with position vector* x*. If* M *is GCR, then the tangential part of* x *is either space-like or time-like.*

Proposition 6.6. *[8] Let* M *be an oriented hypersurface in the Minkowski space* $\mathbb{E}^{n+1}_1$ *and* x *its position vector. Consider a unit tangent vector field* e_1 *in the direction of* x^T*. Then,* M *is a GCR hypersurface if and only if a curve* α *is a geodesic of* M *whenever it is an integral curve of* e_1*.*

Proposition 6.7. *[8, 24] Let* M *be a space-like hypersurface in the Minkowski space* $\mathbb{E}^{n+1}_1$ *and* $x : M \to \mathbb{E}^{n+1}_1$ *the position vector with the tangential component* x^T*. Then* M *is GCR hypersurface if and only if* $Y(\theta) = 0$*, whenever* $\langle Y, x^T\rangle = 0$*, where* θ *is the angle function.*

First we assume that the surface is contained in the time-like cone.

Theorem 6.8. *[8, 24] Let $x : M \to \mathbb{E}^3_1$ be a space-like surface immersed in the 3-dimensional Minkowski space. Also, assume that M is lying in the time-like cone of $\mathbb{E}^3_1$. Then, M is GCR if and only if it can be parametrized by*

$$x(s,t) = s\left(\cosh u(s)\varphi(t) + \sinh u(s)\varphi(t) \wedge \varphi'(t)\right), \tag{6.1}$$

where $\varphi = \varphi(t)$ is an arc-length parametrized curve lying on $\mathbb{H}^2(-1)$ and $u = u(s)$ is a smooth function. In this case, x can be decomposed as

$$x = -s\left(\sinh\theta e_1 + \cosh\theta N\right) \tag{6.2}$$

for the function θ given by

$$\coth\theta = su' \tag{6.3}$$

Now, we will give the classification of space-like GCR surfaces in case the image of the immersion x lies in the space-like cone.

Theorem 6.9. *[8, 24] Let $x : M \to \mathbb{E}^3_1$ be a space-like surface immersed in the 3-dimensional Minkowski space. Also, assume that M is lying in the space-like cone of $\mathbb{E}^3_1$. Then, M is GCR if and only if it can be parametrized by*

$$x(s,t) = s\left(\cosh u(s)\varphi(t) + \sinh u(s)\varphi(t) \wedge \varphi'(t)\right) \tag{6.4}$$

where $\varphi = \varphi(t)$ is an arclength parametrized curve lying on $\mathbb{S}^2_1(1)$ and $u = u(s)$ is a smooth function. In this case, x can be decomposed as

$$x = s\left(\cosh\theta e_1 + \sinh\theta N\right) \tag{6.5}$$

for the function θ given by

$$\tanh\theta = su'. \tag{6.6}$$

As a direct corollary of the previous theorems, we have

Corollary 6.10. *[8] A space-like rotational surface given by*

$$x(s,t) = (s\cosh u\cosh t, s\cosh u\sinh t, s\sinh u) \tag{6.7}$$

or

$$x(s,t) = (s\cosh u\sinh t, s\cosh u\cosh t, s\sinh u) \tag{6.8}$$

is a GCR surface, where $u = u(s)$ is a non-vanishing smooth function.

Theorem 6.11. *[8, 24] The flat space-like GCR surfaces immersed in $\mathbb{E}^3_1$ are an open parts of a plane or of a cylinder.*

Proposition 6.12. *[24] The space-like GCR surfaces with constant mean curvature immersed in $\mathbb{E}^3_1$ are surfaces of revolution.*

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Global Properties of Biconservative Surfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$

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Abstract. We survey some recent results on biconservative surfaces in 3-dimensional space forms $N^3(c)$ with a special emphasis on the $c = 0$ and $c = 1$ cases. We study the local and global properties of such surfaces, from extrinsic and intrinsic point of view. We obtain all non-CMC complete biconservative surfaces in $\mathbb{R}^3$ and $\mathbb{S}^3$.
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1 Introduction

The study of submanifolds with constant mean curvature, i.e., CMC submanifolds, and, in particular, that of CMC surfaces in 3-dimensional spaces, represents a very active research topic in Differential Geometry for more than 50 years.

There are several ways to generalize these submanifolds. For example, keeping the CMC hypothesis and adding other geometric hypotheses to the submanifold or, by contrast, in the particular case of hypersurfaces in space forms, studying the hypersurfaces which are "highly non-CMC".

The biconservative submanifolds seem to be an interesting generalization of CMC submanifolds. Biconservative submanifolds in arbitrary manifolds (and in particular, biconservative surfaces) which are also CMC have some remarkable properties (see, for example [10, 18, 22, 28]). CMC hypersurfaces in space forms are trivially biconservative, so more interesting is the study of biconservative hypersurfaces which are non-CMC; recent results in non-CMC biconservative hypersurfaces were obtained in [12, 19, 21, 29, 30].

The biconservative submanifolds are closely related to the biharmonic submanifolds. More precisely, let us consider the *bienergy functional* defined for all smooth maps between two Riemannian manifolds (M^m, g) and (N^n, h) and given by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2\ v_g, \qquad \varphi \in C^\infty(M,N),$$

where $\tau(\varphi)$ is the tension field of φ. A critical point of E_2 is called a *biharmonic map* and is characterized by the vanishing of the *bitension field* $\tau_2(\varphi)$ (see [15]).

A Riemannian immersion $\varphi : M^m \to (N^n, h)$ or, simply, a submanifold M of N, is called *biharmonic* if φ is a biharmonic map.

Now, if $\varphi : M \to (N, h)$ is a fixed map, then E_2 can be thought as a functional defined on the set of all Riemannian metrics on M. This new functional's critical points are Riemannian metrics determined by the vanishing of the *stress-bienergy tensor* S_2. This tensor field satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi \rangle.$$

If $\operatorname{div} S_2 = 0$ for a submanifold M in N, then M is called a biconservative submanifold and it is characterized by the fact that the tangent part of its bitension field vanishes. Thus we can expect that the class of biconservative submanifolds to be much larger than the class of biharmonic submanifolds.

The paper is organized as follows. After a section where we recall some notions and general results about biconservative submanifolds, we present in *Section* 3 the local, intrinsic characterization of biconservative surfaces. The local, intrinsic characterization theorem provides the necessary and sufficient conditions for an abstract surface $\left(M^2, g\right)$ to admit, locally, a biconservative embedding with positive mean curvature function f and $\operatorname{grad} f \neq 0$ at any point.

Our main goal is to extend the *local* classification results for biconservative surfaces in $N^3(c)$, with $c = 0$ and $c = 1$, to *global* results, i.e., we ask that biconservative surfaces to be *complete*, with $f > 0$ everywhere and $|\operatorname{grad} f| > 0$ on an open dense subset.

In *Section* 4 we consider the global problem and construct complete biconservative surfaces in $\mathbb{R}^3$ with $f > 0$ on M and $\operatorname{grad} f \neq 0$ at any point of an open dense subset of M. We determine such surfaces in two ways. One way is to use the local, extrinsic characterization of biconservative surfaces in $\mathbb{R}^3$ and "glue" two pieces together in order to obtain a complete biconservative surface. The other way is more analytic and consists in using the local, intrinsic characterization theorem in order to obtain a biconservative immersion from $\left(\mathbb{R}^2, g_{C_0}\right)$ in $\mathbb{R}^3$ with $f > 0$ on $\mathbb{R}^2$ and $|\operatorname{grad} f| > 0$ on an open dense subset of $\mathbb{R}^2$ (the immersion has to be unique); here, C_0 is a positive constant and therefore we obtain a one-parameter family of solutions. It is worth mentioning that, by a simple transformation of the metric g_{C_0}, $\left(\mathbb{R}^2, \sqrt{-K_{C_0}} g_{C_0}\right)$ is (intrinsically) isometric to a helicoid.

In the *last section* we consider the global problem of biconservative surfaces in $\mathbb{S}^3$ with $f > 0$ on M and $\operatorname{grad} f \neq 0$ at any point of an open dense subset of M. As in the $\mathbb{R}^3$ case, we use the local, extrinsic classification of biconservative surfaces in $\mathbb{S}^3$, but now the "gluing" process is not as clear as in $\mathbb{R}^3$.

Further, we change the point of view and use the local, intrinsic characterization of biconservative surfaces in $\mathbb{S}^3$. We determine the complete Riemannian surfaces $\left(\mathbb{R}^2, g_{C_1,C_1^*}\right)$ which admit a biconservative immersion in $\mathbb{S}^3$ with $f>0$ everywhere and $|\operatorname{grad} f|>0$ on an open dense subset of $\mathbb{R}^2$ and we show that, up to isometries, there exists only a one-parameter family of such Riemannian surfaces indexed by C_1.

We end the paper with some figures, obtained for particular choices of the constants, which represent the non-CMC complete biconservative surfaces in $\mathbb{R}^3$ and the way how these surfaces can be obtained in $\mathbb{S}^3$.

2 Biconservative submanifolds; general properties

Throughout this work, all manifolds, metrics, maps are assumed to be smooth, i.e. in the C^∞ category, and we will often indicate the various Riemannian metrics by the same symbol $\langle , \rangle$. All surfaces are assumed to be connected and oriented.

A *harmonic map* $\varphi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds is a critical point of the *energy functional*

$$E : C^\infty(M,N) \to \mathbb{R}, \quad E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \, v_g,$$

and it is characterized by the vanishing of its *tension field*

$$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi.$$

The idea of the stress-energy tensor associated to a functional comes from D. Hilbert ([14]). Given a functional E, one can associate to it a symmetric 2-covariant tensor field S such that $\operatorname{div} S = 0$ at the critical points of E. When E is the energy functional, P. Baird and J. Eells ([1]), and A. Sanini ([27]), defined the tensor field

$$S = e(\varphi)g - \varphi^* h = \frac{1}{2}|d\varphi|^2 g - \varphi^* h,$$

and proved that

$$\operatorname{div} S = -\langle \tau(\varphi), d\varphi \rangle.$$

Thus, S can be chosen as the stress-energy tensor of the energy functional. It is worth mentioning that S has a variational meaning. Indeed, we can fix a map $\varphi : M^m \to (N^n, h)$ and think E as being defined on the set of all Riemannian metrics on M. The critical points of this new functional are Riemannian metrics determined by the vanishing of their stress-energy tensor S.

More precisely, we assume that M is compact and denote

$$\mathcal{G} = \{g \;:\; g \text{ is a Riemannian metric on } M\}.$$

For a deformation $\{g_t\}$ of g we consider $\omega = \frac{d}{dt}\big|_{t=0} g_t \in T_g\mathcal{G} = C\left(\odot^2 T^*M\right)$. We define the new functional

$$\mathcal{F} : \mathcal{G} \to \mathbb{R}, \quad \mathcal{F}(g) = E(\varphi)$$

and we have the following result.

Theorem 2.1 ([1, 27])**.** *Let $\varphi : M^m \to (N^n, h)$ and assume that M is compact. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(g_t) = \frac{1}{2}\int_M \langle \omega, e(\varphi)g - \varphi^* h\rangle \ v_g.$$

Therefore g is a critical point of $\mathcal{F}$ if and only if its stress-energy tensor S vanishes.

We mention here that, if $\varphi : (M^m, g) \to (N^n, h)$ is an arbitrary isometric immersion, then $\operatorname{div} S = 0$.

A natural generalization of harmonic maps is given by biharmonic maps. A *biharmonic map* $\varphi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds is a critical point of the *bienergy functional*

$$E_2 : C^\infty(M, N) \to \mathbb{R}, \quad E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \ v_g,$$

and it is characterized by the vanishing of its *bitension field*

$$\tau_2(\varphi) = -\Delta^\varphi \tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi,$$

where

$$\Delta^\varphi = -\operatorname{trace}_g (\nabla^\varphi \nabla^\varphi - \nabla^\varphi_\nabla)$$

is the rough Laplacian of $\varphi^{-1}TN$ and the curvature tensor field is

$$R^N(X, Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z, \quad \forall X, Y, Z \in C(TM).$$

We remark that the *biharmonic equation* $\tau_2(\varphi) = 0$ is a fourth-order non-linear elliptic equation and that any harmonic map is biharmonic. A non-harmonic biharmonic map is called proper biharmonic.

In [16], G. Y. Jiang defined the stress-energy tensor S_2 of the bienergy (also called *stress-bienergy tensor*) by

$$\begin{aligned} S_2(X, Y) =& \frac{1}{2}|\tau(\varphi)|^2 \langle X, Y\rangle + \langle d\varphi, \nabla\tau(\varphi)\rangle\langle X, Y\rangle \\ & - \langle d\varphi(X), \nabla_Y \tau(\varphi)\rangle - \langle d\varphi(Y), \nabla_X \tau(\varphi)\rangle, \end{aligned}$$

as it satisfies

$$\operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi\rangle.$$

The tensor field S_2 has a variational meaning, as in the harmonic case. We fix a map $\varphi : M^m \to (N^n, h)$ and define a new functional

$$\mathcal{F}_2 : \mathcal{G} \to \mathbb{R}, \quad \mathcal{F}_2(g) = E_2(\varphi).$$

Then we have the following result.

Theorem 2.2 ([17]). *Let $\varphi : M^m \to (N^n, h)$ and assume that M is compact. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_2\left(g_t\right) = -\frac{1}{2}\int_M \langle \omega, S_2\rangle \ v_g,$$

so g is a critical point of $\mathcal{F}_2$ if and only if $S_2 = 0$.

We mention that, if $\varphi : (M^m, g) \to (N^n, h)$ is an isometric immersion then $\operatorname{div} S_2$ does not necessarily vanish.

A submanifold of a given Riemannian manifold (N^n, h) is a pair (M^m, φ), where M^m is a manifold and $\varphi : M \to N$ is an immersion. We always consider on M the induced metric $g = \varphi^* h$, thus $\varphi : (M, g) \to (N, h)$ is an isometric immersion; for simplicity we will write $\varphi : M \to N$ without mentioning the metrics. Also, we will write $\varphi : M \to N$, or even M, instead of (M, φ).

A submanifold $\varphi : M^m \to N^n$ is called *biharmonic* if the isometric immersion φ is a biharmonic map from (M^m, g) to (N^n, h).

Even if the notion of biharmonicity may be more appropriate for maps than for submanifolds, as the domain and the codomain metrics are fixed and the variation is made only through the maps, the biharmonic submanifolds proved to be an interesting notion (see, for example, [24]).

In order to fix the notations, we recall here only the fundamental equations of first order of a submanifold in a Riemannian manifold. These equations define the second fundamental form, the shape operator and the connection in the normal bundle. Let $\varphi : M^m \to N^n$ be an isometric immersion. For each $p \in M$, $T_{\varphi(p)}N$ splits as an orthogonal direct sum

$$T_{\varphi(p)}N = d\varphi(T_pM) \oplus d\varphi(T_pM)^\perp, \tag{2.1}$$

and $NM = \bigcup_{p\in M} d\varphi(T_pM)^\perp$ is referred to as the normal bundle of φ, or of M, in N.

Denote by ∇ and ∇^N the Levi-Civita connections on M and N, respectively, and by ∇^φ the induced connection in the pull-back bundle

$$\varphi^{-1}(TN) = \bigcup_{p\in M} T_{\varphi(p)}N.$$

Taking into account the decomposition in (2.1), one has

$$\nabla^\varphi_X d\varphi(Y) = d\varphi(\nabla_X Y) + B(X, Y), \qquad \forall\, X, Y \in C(TM),$$

where $B \in C(\odot^2 T^*M \otimes NM)$ is called the second fundamental form of M in N. Here T^*M denotes the cotangent bundle of M. The mean curvature vector field of M in N is defined by $H = (\operatorname{trace} B)/m \in C(NM)$, where the trace is considered with respect to the metric g.

Furthermore, if $\eta \in C(NM)$, then

$$\nabla^\varphi_X \eta = -d\varphi(A_\eta(X)) + \nabla^\perp_X \eta, \quad \forall\, X \in C(TM),$$

where $A_\eta \in C(T^*M \otimes TM)$ is called the shape operator of M in N in the direction of η, and $\nabla^\perp$ is the induced connection in the normal bundle. Moreover, $\langle B(X,Y),\eta\rangle = \langle A_\eta(X),Y\rangle$, for all $X,Y \in C(TM)$, $\eta \in C(NM)$. In the case of hypersurfaces, we denote $f = \operatorname{trace} A$, where $A = A_\eta$ and η is the unit normal vector field, and we have $H = (f/m)\eta$; f is the *(m times) mean curvature function.*

A submanifold M of N is called PMC if H is parallel in the normal bundle, and CMC if $|H|$ is constant.

When confusion is unlikely we identify, locally, M with its image through φ, X with $d\varphi(X)$ and $\nabla^\varphi_X d\varphi(Y)$ with $\nabla^N_X Y$. With these identifications in mind, we write

$$\nabla^N_X Y = \nabla_X Y + B(X,Y),$$

and

$$\nabla^N_X \eta = -A_\eta(X) + \nabla^\perp_X \eta.$$

If $\operatorname{div} S_2 = 0$ for a submanifold M in N, then M is called *biconservative.* Thus, M is biconservative if and only if the tangent part of its bitension field vanishes.

We have the following characterization theorem of biharmonic submanifolds, obtained by splitting the bitension field in the tangent and normal part.

Theorem 2.3. *A submanifold M^m of a Riemannian manifold N^n is biharmonic if and only if*

$$\operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + \operatorname{trace} \nabla A_H + \operatorname{trace} \left(R^N(\cdot,H)\cdot\right)^T = 0$$

and

$$\Delta^\perp H + \operatorname{trace} B\left(\cdot, A_H(\cdot)\right) + \operatorname{trace}\left(R^N(\cdot,H)\cdot\right)^\perp = 0,$$

where $\Delta^\perp$ is the Laplacian in the normal bundle.

Various forms of the above result were obtained in [7, 17, 23]. From here we deduce some characterization formulas for the biconservativity.

Corollary 2.4. *Let M^m be a submanifold of a Riemannian manifold N^n. Then M is a biconservative submanifold if and only if:*

1. $\operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + \operatorname{trace} \nabla A_H + \operatorname{trace}\left(R^N(\cdot,H)\cdot\right)^T = 0$;
2. $\frac{m}{2} \operatorname{grad}\left(|H|^2\right) + 2\operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + 2\operatorname{trace}\left(R^N(\cdot,H)\cdot\right)^T = 0$;
3. $2\operatorname{trace}\nabla A_H - \frac{m}{2}\operatorname{grad}\left(|H|^2\right) = 0$.

The following properties are immediate.

Proposition 2.5. *Let M^m be a submanifold of a Riemannian manifold N^n. If $\nabla A_H = 0$ then M is biconservative.*

Proposition 2.6. *Let M^m be a submanifold of a Riemannian manifold N^n. Assume that N is a space form, i.e., it has constant sectional curvature, and M is PMC. Then M is biconservative.*

Proposition 2.7 ([2]). *Let M^m be a submanifold of a Riemannian manifold N^n. Assume that M is pseudo-umbilical, i.e., $A_H = |H|^2 I$, and $m \neq 4$. Then M is CMC.*

If we consider the particular case of hypersurfaces, then Theorem 2.3 becomes

Theorem 2.8 ([2, 25]). *If M^m is a hypersurface in a Riemannian manifold N^{m+1}, then M is biharmonic if and only if*

$$2A(\operatorname{grad} f) + f \operatorname{grad} f - 2f \left(\operatorname{Ricci}^N(\eta)\right)^T = 0,$$

and

$$\Delta f + f|A|^2 - f \operatorname{Ricci}^N(\eta, \eta) = 0,$$

where η is the unit normal vector field of M in N.

Corollary 2.9. *A hypersurface M^m in a space form $N^{m+1}(c)$ is biconservative if and only if*

$$A(\operatorname{grad} f) = -\frac{f}{2} \operatorname{grad} f.$$

Corollary 2.10. *Any CMC hypersurface in $N^{m+1}(c)$ is biconservative.*

Therefore, the biconservative hypersurfaces may be seen as the next research topic after that of CMC surfaces.

3 Intrinsic characterization of biconservative surfaces

We are interested to study biconservative surfaces which are non-CMC. We will first look at them from a local, extrinsic point of view and then from a global point of view. While by "local" we will mean the biconservative surfaces $\varphi : M^2 \to N^3(c)$ with $f > 0$ and $\operatorname{grad} f \neq 0$ at any point of M, by "global" we will mean the *complete* biconservative surfaces $\varphi : M^2 \to N^3(c)$ with $f > 0$ at any point of M and $\operatorname{grad} f \neq 0$ at any point of an open and dense subset of M.

In this section, we consider the local problem, i.e., we take $\varphi : M^2 \to N^3(c)$ a biconservative surface and assume that $f > 0$ and $\operatorname{grad} f \neq 0$ at any point of M. Let $X_1 = (\operatorname{grad} f)/|\operatorname{grad} f|$ and X_2 two vector fields such that $\{X_1(p), X_2(p)\}$ is a positively oriented orthonormal basis at any point $p \in M$. In particular, we obtain that M is parallelizable. If we denote by $\lambda_1 \leq \lambda_2$ the eigenvalues functions of the shape operator A, since $A(X_1) = -(f/2)X_1$ and $\operatorname{trace} A = f$, we get $\lambda_1 = -f/2$ and $\lambda_2 = 3f/2$. Thus the matrix of A with respect to the (global) orthonormal frame field $\{X_1, X_2\}$ is

$$A = \begin{pmatrix} -\frac{f}{2} & 0 \\ 0 & \frac{3f}{2} \end{pmatrix}.$$

We denote by K the Gaussian curvature and, from the Gauss equation, $K = c + \det A$, we obtain

$$f^2 = \frac{4}{3}(c - K). \tag{3.1}$$

Thus $c - K > 0$ on M.

From the definitions of X_1 and X_2, we find that

$$\operatorname{grad} f = (X_1 f)\, X_1 \quad \text{and} \quad X_2 f = 0.$$

Using the connection 1-forms, the Codazzi equation and then the extrinsic and intrinsic expression for the Gaussian curvature, we obtain the next result which shows that the mean curvature function of a non-CMC biconservative surface must satisfy a second-order partial differential equation. More precisely, we have the following theorem.

Theorem 3.1 ([5])**.** *Let $\varphi : M^2 \to N^3(c)$ a biconservative surface with $f > 0$ and* $\operatorname{grad} f \neq 0$ *at any point of M. Then we have*

$$f\Delta f + |\operatorname{grad} f|^2 + \frac{4}{3}cf^2 - f^4 = 0, \tag{3.2}$$

where Δ is the Laplace-Beltrami operator on M.

In fact, we can see that around any point of M there exists $(U; u, v)$ local coordinates such that $f = f(u, v) = f(u)$ and (3.2) is equivalent to

$$ff'' - \frac{7}{4}\left(f'\right)^2 - \frac{4}{3}cf^2 + f^4 = 0, \tag{3.3}$$

i.e., f must satisfy a *second-order ordinary differential equation.*

Indeed, let $p_0 \in M$ be an arbitrary fixed point of M and let $\gamma = \gamma(u)$ be an integral curve of X_1 with $\gamma(0) = p_0$. Let ϕ the flow of X_2 and $(U; u, v)$ local coordinates with $p_0 \in U$ such that

$$X(u, v) = \phi_{\gamma(u)}(v) = \phi(\gamma(u), v).$$

We have

$$X_u(u, 0) = \gamma'(u) = X_1(\gamma(u)) = X_1(u, 0)$$

and

$$X_v(u, v) = \phi'_{\gamma(u)}(v) = X_2\left(\phi_{\gamma(u)}(v)\right) = X_2(u, v).$$

If we write the Riemannian metric g on M in local coordinates as

$$g = g_{11}du^2 + 2g_{12}du dv + g_{22}dv^2,$$

we get $g_{22} = |X_v|^2 = |X_2|^2 = 1$, and X_1 can be expressed with respect to X_u and X_v as

$$X_1 = \frac{1}{\sigma}\left(X_u - g_{12}X_v\right) = \sigma \operatorname{grad} u,$$

where $\sigma = \sqrt{g_{11} - g_{12}^2} > 0$, $\sigma = \sigma(u, v)$.

Let $f \circ X = f(u, v)$. Since $X_2 f = 0$, we find that

$$f(u, v) = f(u, 0) = f(u), \quad \forall (u, v) \in U.$$

It can be proved that

$$[X_1, X_2] = \frac{3\,(X_1 f)}{4f} X_2,$$

and thus $X_2 X_1 f = X_1 X_2 f - [X_1, X_2]\, f = 0$.

On the other hand we have

$$\begin{array}{rl} X_2 X_1 f = & X_v \left(\frac{1}{\sigma} f'\right) = X_v \left(\frac{1}{\sigma}\right) f' \\ = & 0 \end{array} . \tag{3.4}$$

We recall that

$$\operatorname{grad} f = (X_1 f)\, X_1 = \left(\frac{1}{\sigma} f'\right) X_1 \neq 0$$

at any point of U, and then $f' \neq 0$ at any point of U. Therefore, from (3.4), $X_v\,(1/\sigma) = 0$, i.e., $\sigma = \sigma(u)$. Since $g_{11}(u, 0) = 1$, and $g_{12}(u, 0) = 0$, we have $\sigma = 1$, i.e.,

$$X_1 = X_u - g_{12} X_v = \operatorname{grad} u. \tag{3.5}$$

In [5] it was found an equivalent expression for (3.2), i.e.,

$$(X_1 X_1 f)\, f = \frac{7}{4}\,(X_1 f)^2 + \frac{4c}{3} f^2 - f^4.$$

Therefore, using (3.5), relation (3.2) is equivalent to (3.3).

Remark 3.2. If $\varphi : M^2 \to N^3(c)$ is a non-CMC biharmonic surface, then, there exists an open subset U such that $f > 0$, $\operatorname{grad} f \neq 0$ at any point of U, and f satisfies the following system

$$\left\{ \begin{array}{l} \Delta f = f\left(2c - |A|^2\right) \\ \\ A(\operatorname{grad} f) = -\frac{f}{2} \operatorname{grad} f \end{array} \right. .$$

As we have seen, this system implies

$$\left\{ \begin{array}{l} \Delta f = f\left(2c - |A|^2\right) \\ \\ f\Delta f + |\operatorname{grad} f|^2 + \frac{4}{3} c f^2 - f^4 = 0 \end{array} \right. .$$

which, in fact, is a ODE system. We get

$$\begin{cases} ff'' - \frac{3}{4}(f')^2 + 2cf^2 - \frac{5}{2}f^4 = 0 \\ ff'' - \frac{7}{4}(f')^2 - \frac{4}{3}cf^2 + f^4 = 0 \end{cases}. \tag{3.6}$$

As an immediate consequence we obtain

$$(f')^2 + \frac{10}{3}cf^2 - \frac{7}{2}f^4 = 0,$$

and combining it with the first integral

$$(f')^2 = 2f^4 - 8cf^2 + \alpha f^{3/2}$$

of the first equation from (3.6), where $\alpha \in \mathbb{R}$ is a constant, we obtain

$$\frac{3}{2}f^{5/2} + \frac{14}{3}cf^{1/2} - \alpha = 0.$$

If we denote $\tilde{f} = f^{1/2}$, we get $3\tilde{f}^5/2 + 14c\tilde{f}/3 - \alpha = 0$. Thus, $\tilde{f}$ satisfies a polynomial equation with constant coefficients, so $\tilde{f}$ has to be a constant and then, f is a constant, i.e., $\operatorname{grad} f = 0$ on U (in fact, f has to be zero). Therefore, we have a contradiction (see [6, 8] for $c = 0$ and [3, 4], for $c = \pm 1$).

We can also note that relation (3.2), which is an extrinsic relation, together with (3.1), allows us to find an *intrinsic relation* that (M, g) must satisfy. More precisely, the Gaussian curvature of M has to satisfy

$$(c - K)\Delta K - |\operatorname{grad} K|^2 - \frac{8}{3}K(c - K)^2 = 0, \tag{3.7}$$

and the conditions $c - K > 0$ and $\operatorname{grad} K \neq 0$.

Formula (3.7) is very similar to the Ricci condition. Further, we will briefly recall the Ricci problem. Given an abstract surface $\left(M^2, g\right)$, we want to find the conditions that have to be satisfied by M such that, locally, it admits a minimal embedding in $N^3(c)$. It was proved (see [20, 26]) that if $\left(M^2, g\right)$ is an abstract surface such that $c - K > 0$ at any point of M, where $c \in \mathbb{R}$ is a constant, then, locally, it admits a minimal embedding in $N^3(c)$ if and only if

$$(c - K)\Delta K - |\operatorname{grad} K|^2 - 4K(c - K)^2 = 0. \tag{3.8}$$

Condition (3.8) is called the *Ricci condition with respect to* c, or simply the *Ricci condition*. If (3.8) holds, then, locally, M admits a one-parameter family of minimal embeddings in $N^3(c)$.

We can see that relations (3.7) and (3.8) are very similar and, in [9], the authors studied the link between them. Thus, for $c = 0$, it was proved that if we consider a surface $\left(M^2, g\right)$ which satisfies (3.7) and $K < 0$, then there exists a very simple conformal transformation of the metric g such that $\left(M^2, \sqrt{-K}g\right)$

satisfies (3.8). A similar result was also proved for $c \neq 0$, but in this case, the conformal factor has a complicated expression (and it is not enough to impose that $\left(M^2, g\right)$ satisfy (3.7), but we need the stronger hypothesis of it to admit a non-CMC biconservative immersion in $N^3(c)$).

Unfortunately, condition (3.7) does not imply, locally, the existence of a biconservative immersion in $N^3(c)$, as in the minimal case. We need a stronger condition. It was obtained the following local, intrinsic characterization theorem.

Theorem 3.3 ([9]). *Let $\left(M^2, g\right)$ be an abstract surface and $c \in \mathbb{R}$ a constant. Then, locally, M can be isometrically embedded in a space form $N^3(c)$ as a biconservative surface with positive mean curvature having the gradient different from zero at any point if and only if the Gaussian curvature K satisfies $c - K(p) > 0$, $(\operatorname{grad} K)(p) \neq 0$, for any point $p \in M$, and its level curves are circles in M with constant curvature*

$$\kappa = \frac{3|\operatorname{grad} K|}{8(c-K)}.$$

Remark 3.4. If the surface M in Theorem 3.3 is simply connected, then the theorem holds globally, but, in this case, instead of a local isometric embedding we have a global isometric immersion.

We remark that unlike in the minimal immersions case, if M satisfies the hypotheses from Theorem 3.3, then there exists a *unique* biconservative immersion in $N^3(c)$ (up to an isometry of $N^3(c)$), and not a one-parameter family.

The characterization theorem can be equivalently rewritten as below.

Theorem 3.5. *Let $\left(M^2, g\right)$ be an abstract surface with Gaussian curvature K satisfying $c - K(p) > 0$ and $(\operatorname{grad} K)(p) \neq 0$ at any point $p \in M$, where $c \in \mathbb{R}$ is a constant. Let $X_1 = (\operatorname{grad} K)/|\operatorname{grad} K|$ and $X_2 \in C(TM)$ be two vector fields on M such that $\{X_1(p), X_2(p)\}$ is a positively oriented basis at any point of $p \in M$. Then, the following conditions are equivalent:*

(a) the level curves of K are circles in M with constant curvature

$$\kappa = \frac{3|\operatorname{grad} K|}{8(c-K)} = \frac{3X_1 K}{8(c-K)};$$

(b)

$$X_2\left(X_1 K\right) = 0 \quad \text{and} \quad \nabla_{X_2} X_2 = \frac{-3X_1 K}{8(c-K)} X_1;$$

(c) locally, the metric g can be written as $g = (c-K)^{-3/4}\left(du^2 + dv^2\right)$, where (u, v) are local coordinates positively oriented, $K = K(u)$, and $K' > 0$;

(d) locally, the metric g can be written as $g = e^{2\varphi}\left(du^2 + dv^2\right)$, where (u, v) are local coordinates positively oriented, and $\varphi = \varphi(u)$ satisfies the equation

$$\varphi'' = e^{-2\varphi/3} - ce^{2\varphi} \tag{3.9}$$

and the condition $\varphi' > 0$; moreover, the solutions of the above equation, $u = u(\varphi)$, are

$$u = \int_{\varphi_0}^{\varphi} \frac{d\tau}{\sqrt{-3e^{-2\tau/3} - ce^{2\tau} + a}} + u_0,$$

where φ is in some open interval I and $a, u_0 \in \mathbb{R}$ are constants;

(e) *locally, the metric g can be written as $g = e^{2\varphi}\left(du^2 + dv^2\right)$, where (u, v) are local coordinates positively oriented, and $\varphi = \varphi(u)$ satisfies the equation*

$$3\varphi''' + 2\varphi'\varphi'' + 8ce^{2\varphi}\varphi' = 0 \tag{3.10}$$

and the conditions $\varphi' > 0$ and $c + e^{-2\varphi}\varphi'' > 0$; moreover, the solutions of the above equation, $u = u(\varphi)$, are

$$u = \int_{\varphi_0}^{\varphi} \frac{d\tau}{\sqrt{-3be^{-2\tau/3} - ce^{2\tau} + a}} + u_0,$$

where φ is in some open interval I and $a, b, u_0 \in \mathbb{R}$ are constants, $b > 0$.

The proof follows by direct computations and by using Remark 4.3 in [9] and Proposition 3.4 in [21].

Remark 3.6. From the above theorem we have the following remarks.

(i) If condition (a) is satisfied, i.e., the integral curves of X_2 are circles in M with a precise constant curvature, then the integral curves of X_1 are geodesics of M.

(ii) If condition (c) is satisfied, then K has to be a solution of the equation

$$3K''(c-K) + 3\left(K'\right)^2 + 8K(c-K)^{5/4} = 0.$$

(iii) If condition (c) is satisfied and $c > 0$, then $\left(M^2, (c-K)^{3/4}g\right)$ is a flat surface and, trivially, a Ricci surface with respect to c.

(iv) Let $\varphi = \varphi(u)$ be a solution of equation (3.10). We consider the change of coordinates

$$(u, v) = \left(\alpha\tilde{u} + \beta, \alpha\tilde{v} + \beta\right),$$

where $\alpha \in \mathbb{R}$ is a positive constant and $\beta \in \mathbb{R}$, and define

$$\phi = \varphi\left(\alpha\tilde{u} + \beta\right) + \log\alpha.$$

Then $g = e^{2\phi}\left(d\tilde{u}^2 + d\tilde{v}^2\right)$ and ϕ also satisfies equation (3.10). If $\varphi = \varphi(u)$ satisfies the first integral

$$\varphi'' = be^{-2\varphi/3} - ce^{2\varphi},$$

where $b>0$, then, for $\alpha = b^{-3/8}$, $\phi = \phi(\tilde{u})$ satisfies

$$\phi'' = e^{-2\phi/3} - ce^{2\phi}.$$

From here, as the classification is done up to isometries, we note that the parameter b in the solution of (3.10) is not essential and only the parameter a counts. Thus we have a one-parameter family of solutions.

(v) If φ is a solution of (3.10), for some c, then $\varphi + \alpha$, where α is a real constant, is a solution of (3.10) for $ce^{2\alpha}$.

(vi) If $c = 0$, we note that if φ is a solution of (3.10), then also $\varphi +$ constant is a solution of the same equation, i.e, condition (a) from Theorem 3.5 is invariant under the homothetic tranformations of the metric g. Then, we see that equation (3.10) is invariant under the affine change of parameter $u = \alpha\tilde{u} + \beta$, where $\alpha > 0$. Therefore, we must solve equation (3.10) up to this change of parameter and an additive constant of the solution φ. The additive constant will be the parameter that counts.

In the $c = 0$ case, the solutions of equation (3.10), are explicitly determined in the next proposition.

Proposition 3.7 ([21]). *The solutions of the equation*

$$3\varphi''' + 2\varphi'\varphi'' = 0$$

which satisfy the conditions $\varphi' > 0$ and $\varphi'' > 0$, up to affine transformations of the parameter with $\alpha > 0$, are given by

$$\varphi(u) = 3\log(\cosh u) + \text{constant}, \qquad u > 0.$$

We note that, when $c = 0$, we have a one-parameter family of solutions of equation (3.10), i.e., $g_{C_0} = C_0(\cosh u)^6\left(du^2 + dv^2\right)$, C_0 being a positive constant.

If $c \neq 0$, then we can not determine explicitly $\varphi = \varphi(u)$. Another way to see that in the $c \neq 0$ case we have only a one-parameter family of solutions of equation (3.10) is to rewrite the metric g in certain non-isothermal coordinates.

Further, we will consider only the $c = 1$ case and we have the next result.

Proposition 3.8 ([21]). *Let $\left(M^2, g\right)$ be an abstract surface with $g = e^{2\varphi(u)}(du^2 + dv^2)$, where $u = u(\varphi)$ satisfies*

$$u = \int_{\varphi_0}^{\varphi} \frac{d\tau}{\sqrt{-3be^{-2\tau/3} - e^{2\tau} + a}} + u_0,$$

where φ is in some open interval I, $a, b \in \mathbb{R}$ are positive constants, and $u_0 \in \mathbb{R}$ is a constant. Then $\left(M^2, g\right)$ is isometric to

$$\left(D_{C_1}, g_{C_1} = \frac{3}{\xi^2\left(-\xi^{8/3} + 3C_1\xi^2 - 3\right)}d\xi^2 + \frac{1}{\xi^2}d\theta^2\right),$$

where $D_{C_1} = (\xi_{01}, \xi_{02}) \times \mathbb{R}$, $C_1 \in \left(4/\left(3^{3/2}\right), \infty\right)$ *is a positive constant, and* ξ_{01} *and* ξ_{02} *are the positive vanishing points of* $-\xi^{8/3} + 3C_1\xi^2 - 3$*, with* $0 < \xi_{01} < \xi_{02}$.

Remark 3.9. Let us consider

$$\left(D_{C_1}, g_{C_1} = \frac{3}{\xi^2\left(-\xi^{8/3} + 3C_1\xi^2 - 3\right)} d\xi^2 + \frac{1}{\xi^2} d\theta^2\right)$$

and

$$\left(D_{C_1'}, g_{C_1'} = \frac{3}{\tilde{\xi}^2\left(-\tilde{\xi}^{8/3} + 3C_1'\tilde{\xi}^2 - 3\right)} d\tilde{\xi}^2 + \frac{1}{\tilde{\xi}^2} d\tilde{\theta}^2\right).$$

The surfaces (D_{C_1}, g_{C_1}) and $\left(D_{C_1'}, g_{C_1'}\right)$ are isometric if and only if $C_1 = C_1'$ and the isometry is $\Theta(\xi, \theta) = (\xi, \pm\theta + \text{constant})$. Therefore, we have a one-parameter family of surfaces.

Remark 3.10. We note that the expression of the Gaussian curvature of (D_{C_1}, g_{C_1}) does not depend on C_1. More precisely,

$$K_{C_1}(\xi, \theta) = -\frac{1}{9}\xi^{8/3} + 1.$$

But, if we change further the coordinates $(\xi, \theta) = \left(\xi_{01} + \tilde{\xi}\left(\xi_{02} - \xi_{01}\right), \tilde{\theta}\right)$, then we "fix" the domain, i.e., (D_{C_1}, g_{C_1}) is isometric to $((0,1), \tilde{g}_{C_1})$ and C_1 appears in the expression of $K_{C_1}\left(\tilde{\xi}, \tilde{\theta}\right)$.

4 Complete biconservative surfaces in $\mathbb{R}^3$

In this section we consider the global problem and construct complete biconservative surfaces in $\mathbb{R}^3$ with $f > 0$ everywhere and $\operatorname{grad} f \neq 0$ at any point of an open dense subset. Or, from intrinsic point of view, we construct a complete abstract surface $\left(M^2, g\right)$ with $K < 0$ everywhere and $\operatorname{grad} K \neq 0$ at any point of an open dense subset of M, that admits a biconservative immersion in $\mathbb{R}^3$, defined on the whole M, with $f > 0$ on M and $|\operatorname{grad} f| > 0$ on the open dense subset.

First, we recall a local extrinsic result which provides a characterization of biconservative surfaces in $\mathbb{R}^3$.

Theorem 4.1 ([13])**.** *Let* M^2 *be a surface in* $\mathbb{R}^3$ *with* $f(p) > 0$ *and* $(\operatorname{grad} f)(p) \neq 0$ *for any* $p \in M$*. Then,* M *is biconservative if and only if, locally, it is a surface of revolution, and the curvature* $\kappa = \kappa(u)$ *of the profile curve* $\sigma = \sigma(u)$, $|\sigma'(u)| = 1$*, is a positive solution of the following ODE*

$$\kappa''\kappa = \frac{7}{4}\left(\kappa'\right)^2 - 4\kappa^4.$$

In [5] there was found the local explicit parametric equation of a biconservative surface in $\mathbb{R}^3$.

Theorem 4.2 ([5])**.** *Let M^2 be a biconservative surface in $\mathbb{R}^3$ with $f(p) > 0$ and $(\operatorname{grad} f)(p) \neq 0$ for any $p \in M$. Then, locally, the surface can be parametrized by*

$$X_{\tilde{C}_0}(\rho, v) = \left(\rho \cos v, \rho \sin v, u_{\tilde{C}_0}(\rho)\right),$$

where

$$u_{\tilde{C}_0}(\rho) = \frac{3}{2\tilde{C}_0}\left(\rho^{1/3}\sqrt{\tilde{C}_0\rho^{2/3}-1} + \frac{1}{\sqrt{\tilde{C}_0}}\log\left(\sqrt{\tilde{C}_0}\rho^{1/3} + \sqrt{\tilde{C}_0\rho^{2/3}-1}\right)\right)$$

with $\tilde{C}_0$ a positive constant and $\rho \in \left(\tilde{C}_0^{-3/2}, \infty\right)$.

We denote by $S_{\tilde{C}_0}$ the image $X_{\tilde{C}_0}\left(\left(\tilde{C}_0^{-3/2}, \infty\right) \times \mathbb{R}\right)$. We note that any two such surfaces are not locally isometric, so we have a one-parameter family of biconservative surfaces in $\mathbb{R}^3$. These surfaces are not complete.

Remark 4.3. If $\varphi : M^2 \to \mathbb{R}^3$ is a biconservative surface with $f > 0$ and $\operatorname{grad} f \neq 0$ at any point, then there exists a unique $\tilde{C}_0$ such that $\varphi(M) \subset S_{\tilde{C}_0}$. Indeed, any point admits an open neighborhood which is an open subset of some $S_{\tilde{C}_0}$. Let us consider $p_0 \in M$. Then, there exists a unique $\tilde{C}_0$ such that $\varphi(U) \subset S_{\tilde{C}_0}$, where U is an open neighborhood of p_0. If A denotes the set of all points of M such that they admit open neighborhoods which are open subsets of that $S_{\tilde{C}_0}$, then the set A is non-empty, open and closed in M. Thus, as M is connected, it follows that $A = M$.

The "boundary" of $S_{\tilde{C}_0}$, i.e., $\overline{S}_{\tilde{C}_0} \setminus S_{\tilde{C}_0}$, is the circle

$$\left(\tilde{C}_0^{-3/2}\cos v, \tilde{C}_0^{-3/2}\sin v, 0\right),$$

which lies in the Oxy plane. At a boundary point, the tangent plane to the closure $\overline{S}_{\tilde{C}_0}$ of $S_{\tilde{C}_0}$ is parallel to Oz. Moreover, along the boundary, the mean curvature function is constant $f_{\tilde{C}_0} = \left(2\tilde{C}_0^{3/2}\right)/3$ and $\operatorname{grad} f_{\tilde{C}_0} = 0$.

Thus, in order to obtain a complete biconservative surface in $\mathbb{R}^3$, we can expect to "glue" along the boundary two biconservative surfaces of type $S_{\tilde{C}_0}$ corresponding to the same $\tilde{C}_0$ (the two constants have to be the same) and symmetric to each other, at the level of C^∞ smoothness.

In fact, it was proved that we can glue two biconservative surfaces $S_{\tilde{C}_0}$ and $S_{\tilde{C}_0'}$, at the level of C^∞ smoothness, only along the boundary and, in this case, $\tilde{C}_0 = \tilde{C}_0'$.

Proposition 4.4 ([19, 21])**.** *If we consider the symmetry of* $\operatorname{Graf} u_C$*, with respect to the $O\rho(= Ox)$ axis, we get a smooth, complete, biconservative surface $\tilde{S}_{\tilde{C}_0}$ in $\mathbb{R}^3$. Moreover, its mean curvature function $\tilde{f}_{\tilde{C}_0}$ is positive and* $\operatorname{grad} \tilde{f}_{\tilde{C}_0}$ *is different from zero at any point of an open dense subset of $\tilde{S}_{\tilde{C}_0}$.*

Remark 4.5. The profile curve $\sigma_{\tilde{C}_0} = \left(\rho, 0, u_{\tilde{C}_0}(\rho)\right) \equiv \left(\rho, u_{\tilde{C}_0}(\rho)\right)$ can be reparametrized as

$$\begin{aligned}\sigma_{\tilde{C}_0}(\theta) = & \left(\sigma^1_{\tilde{C}_0}(\theta), \sigma^2_{\tilde{C}_0}(\theta)\right) \\ = & \tilde{C}_0^{-3/2}\left((\theta+1)^{3/2}, \tfrac{3}{2}\left(\sqrt{\theta^2+\theta} + \log\left(\sqrt{\theta} + \sqrt{\theta+1}\right)\right)\right), \qquad \theta > 0,\end{aligned} \tag{4.1}$$

and now $X_{\tilde{C}_0} = X_{\tilde{C}_0}(\theta, v)$.

Proposition 4.6. *The homothety of* $\mathbb{R}^3$, $(x,y,z) \to \tilde{C}_0(x,y,z)$, *renders* $\tilde{S}_1$ *onto* $\tilde{S}_{\tilde{C}_0^{-2/3}}$.

In [21], there were also found the complete biconservative surfaces in $\mathbb{R}^3$ with $f > 0$ at any point and $\operatorname{grad} f \neq 0$ at any point of an open dense subset, but there, the idea was to use the intrinsic characterization of the biconservative surfaces. More precisely, we have the next global result.

Theorem 4.7 ([21])**.** *Let* $\left(\mathbb{R}^2, g_{C_0} = C_0 (\cosh u)^6 \left(du^2 + dv^2\right)\right)$ *be a surface, where* $C_0 \in \mathbb{R}$ *is a positive constant. Then we have:*

(a) the metric on $\mathbb{R}^2$ *is complete;*

(b) the Gaussian curvature is given by

$$K_{C_0}(u,v) = K_{C_0}(u) = -\frac{3}{C_0 (\cosh u)^8} < 0, \quad K'_{C_0}(u) = \frac{24 \sinh u}{C_0 (\cosh u)^9},$$

and therefore $\operatorname{grad} K_{C_0} \neq 0$ *at any point of* $\mathbb{R}^2 \setminus Ov$*;*

(c) the immersion $\varphi_{C_0} : \left(\mathbb{R}^2, g_{C_0}\right) \to \mathbb{R}^3$ *given by*

$$\varphi_{C_0}(u,v) = \left(\sigma^1_{C_0}(u)\cos(3v), \sigma^1_{C_0}(u)\sin(3v), \sigma^2_{C_0}(u)\right)$$

is biconservative in $\mathbb{R}^3$*, where*

$$\sigma^1_{C_0}(u) = \frac{\sqrt{C_0}}{3}(\cosh u)^3, \quad \sigma^2_{C_0}(u) = \frac{\sqrt{C_0}}{2}\left(\frac{1}{2}\sinh(2u) + u\right), \qquad u \in \mathbb{R}.$$

Sketch of the proof. The first two items follow by standard arguments. For the last part, we note that choosing $\tilde{C}_0 = (9/C_0)^{1/3}$ in (4.1) and using the change of coordinates $(\theta, v) = \left((\sinh u)^2, 3v\right)$, where $u > 0$, the metric induced by $X_{(9/C_0)^{1/3}}$ coincides with g_{C_0}. Then, we define φ_{C_0} as: for $u > 0$, $\varphi_{C_0}(u,v)$ is obtained by rotating the profile curve

$$\sigma^+_{\left(\frac{9}{C_0}\right)^{1/3}}(u) = \sigma_{\left(\frac{9}{C_0}\right)^{1/3}}(u) = \left(\sigma^1_{\left(\frac{9}{C_0}\right)^{1/3}}(u), \sigma^2_{\left(\frac{9}{C_0}\right)^{1/3}}(u)\right),$$

and for $u < 0$, $\varphi_{C_0}(u,v)$ is obtained by rotating the profile curve

$$\sigma^-_{\left(\frac{9}{C_0}\right)^{1/3}}(u) = \left(\sigma^1_{\left(\frac{9}{C_0}\right)^{1/3}}(-u), -\sigma^2_{\left(\frac{9}{C_0}\right)^{1/3}}(-u)\right).$$

□

By simple transformations of the metric, $\left(\mathbb{R}^2, g_{C_0}\right)$ becomes a Ricci surface or a surface with constant Gaussian curvature.

Theorem 4.8. *Consider the surface $\left(\mathbb{R}^2, g_{C_0}\right)$. Then $\left(\mathbb{R}^2, \sqrt{-K_{C_0}} g_{C_0}\right)$ is complete, satisfies the Ricci condition and can be minimally immersed in $\mathbb{R}^3$ as a helicoid or a catenoid.*

Proposition 4.9. *Consider the surface $\left(\mathbb{R}^2, g_{C_0}\right)$. Then $\left(\mathbb{R}^2, -K_{C_0} g_{C_0}\right)$ has constant Gaussian curvature $1/3$ and it is not complete. Moreover, $\left(\mathbb{R}^2, -K_{C_0} g_{C_0}\right)$ is the universal cover of the surface of revolution in $\mathbb{R}^3$ given by*

$$Z(u,v) = \left(\alpha(u) \cosh\left(\frac{\sqrt{3}}{a} v\right), \alpha(u) \sinh\left(\frac{\sqrt{3}}{a} v\right), \beta(u)\right), \qquad (u,v) \in \mathbb{R}^2,$$

where $a \in (0, \sqrt{3}]$ and

$$\alpha(u) = \frac{a}{\cosh u}, \ \beta(u) = \int_0^u \frac{\sqrt{(3-a^2)\cosh^2 \tau + a^2}}{\cosh^2 \tau} \, d\tau.$$

Remark 4.10. When $a = \sqrt{3}$, the immersion Z has only umbilical points and the image $Z\left(\mathbb{R}^2\right)$ is the round sphere of radius $\sqrt{3}$, without the North and the South poles. Moreover, if $a \in (0, \sqrt{3})$, then Z has no umbilical points.

Concerning the biharmonic surfaces in $\mathbb{R}^3$ we have the following non-existence result.

Theorem 4.11 ([6, 8])**.** *There exists no proper biharmonic surface in $\mathbb{R}^3$.*

5 Complete biconservative surfaces in $\mathbb{S}^3$

As in the previous section, we consider the global problem for biconservative surfaces in $\mathbb{S}^3$, i.e., our aim is to construct complete biconservative surfaces in $\mathbb{S}^3$ with $f > 0$ everywhere and $\operatorname{grad} f \neq 0$ at any point of an open and dense subset.

We start with the following local, extrinsic result.

Theorem 5.1 ([5]). *Let M^2 be a biconservative surface in $\mathbb{S}^3$ with $f(p) > 0$ and $(\operatorname{grad} f)(p) \neq 0$ at any point $p \in M$. Then, locally, the surface, viewed in $\mathbb{R}^4$, can be parametrized by*

$$Y_{\tilde{C}_1}(u,v) = \sigma(u) + \frac{4\kappa(u)^{-3/4}}{3\sqrt{\tilde{C}_1}} \left(\overline{f}_1(\cos v - 1) + \overline{f}_2 \sin v\right),$$

where $\tilde{C}_1 \in \left(64/\left(3^{5/4}\right), \infty\right)$ is a positive constant; $\overline{f}_1, \overline{f}_2 \in \mathbb{R}^4$ are two constant orthonormal vectors; $\sigma(u)$ is a curve parametrized by arclength that satisfies

$$\langle \sigma(u), \overline{f}_1 \rangle = \frac{4\kappa(u)^{-3/4}}{3\sqrt{\tilde{C}_1}}, \qquad \langle \sigma(u), \overline{f}_2 \rangle = 0,$$

and, as a curve in $\mathbb{S}^2$, its curvature $\kappa = \kappa(u)$ is a positive non constant solution of the following ODE

$$\kappa''\kappa = \frac{7}{4}\left(\kappa'\right)^2 + \frac{4}{3}\kappa^2 - 4\kappa^4$$

such that

$$\left(\kappa'\right)^2 = -\frac{16}{9}\kappa^2 - 16\kappa^4 + \tilde{C}_1\kappa^{7/2}.$$

Remark 5.2. The constant $\tilde{C}_1$ determines uniquely the curvature κ, up to a translation of u, and then κ, $\overline{f}_1$ and $\overline{f}_2$ determines uniquely the curve σ.

We consider $\overline{f}_1 = \overline{e}_3$ and $\overline{f}_2 = \overline{e}_4$ and change the coordinates (u,v) in (κ, v). Then, we get

$$\begin{aligned} Y_{\tilde{C}_1}(\kappa,v) = \ & \left(\sqrt{1 - \left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa^{-3/4}\right)^2} \cos\mu(\kappa), \sqrt{1 - \left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa^{-3/4}\right)^2} \sin\mu(\kappa), \right. \\ & \left. \tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa^{-3/4}\cos v, \tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa^{-3/4}\sin v \right), \end{aligned} \tag{5.1}$$

where $(\kappa, v) \in (\kappa_{01}, \kappa_{02}) \times \mathbb{R}$, κ_{01} and κ_{02} are positive solutions of

$$-\frac{16}{9}\kappa^2 - 16\kappa^4 + \tilde{C}_1\kappa^{7/2} = 0$$

and

$$\mu(\kappa) = \pm 108 \int_{\kappa_0}^{\kappa} \frac{\sqrt{\tilde{C}_1}\tau^{3/4}}{\left(-16 + 9\tilde{C}_1\tau^{3/2}\right)\sqrt{9\tilde{C}_1\tau^{3/2} - 16\left(1 + 9\tau^2\right)}}\, d\tau + c_0,$$

with $c_0 \in \mathbb{R}$ a constant and $\kappa_0 \in (\kappa_{01}, \kappa_{02})$. We note that an alternative expression for $Y_{\tilde{C}_1}$ was given in [11].

Remark 5.3. The limits $\lim_{\kappa \searrow \kappa_{01}} \mu(\kappa) = \mu(\kappa_{01})$ and $\lim_{\kappa \nearrow \kappa_{02}} \mu(\kappa) = \mu(\kappa_{02})$ are finite.

Remark 5.4. For simplicity, we choose $\kappa_0 = (3\tilde{C}_1/64)^2$.

If we denote $S_{\tilde{C}_1}$ the image of $Y_{\tilde{C}_1}$, then we note that the boundary of $S_{\tilde{C}_1}$ is made up from two circles and along the boundary, the mean curvature function is constant (two different constants) and its gradient vanishes. More precisely, the boundary of $S_{\tilde{C}_1}$ is given by the curves

$$\left(\sqrt{1-\left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{01}^{-3/4}\right)^2}\cos\mu\left(\kappa_{01}\right), \sqrt{1-\left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{01}^{-3/4}\right)^2}\sin\mu\left(\kappa_{01}\right), \right.$$
$$\left. \tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{01}^{-3/4}\cos v, \tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{01}^{-3/4}\sin v\right)$$

and

$$\left(\sqrt{1-\left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{02}^{-3/4}\right)^2}\cos\mu\left(\kappa_{02}\right), \sqrt{1-\left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{02}^{-3/4}\right)^2}\sin\mu\left(\kappa_{02}\right), \right.$$
$$\left. \tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{02}^{-3/4}\cos v, \tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{02}^{-3/4}\sin v\right).$$

These curves are circles in affine planes in $\mathbb{R}^4$ parallel to the Ox^3x^4 plane and their radii are $\left(4\kappa_{01}^{-3/4}\right)/\left(3\sqrt{\tilde{C}_1}\right)$ and $\left(4\kappa_{02}^{-3/4}\right)/\left(3\sqrt{\tilde{C}_1}\right)$, respectively.

At a boundary point, using the coordinates (μ, v), we get that the tangent plane to the closure of $S_{\tilde{C}_1}$ is spanned by a vector which is tangent to the corresponding circle and by

$$\left(-\sqrt{1-\left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{0i}^{-3/4}\right)^2}\sin\mu\left(\kappa_{0i}\right), \sqrt{1-\left(\tfrac{4}{3\sqrt{\tilde{C}_1}}\kappa_{0i}^{-3/4}\right)^2}\cos\mu\left(\kappa_{0i}\right), 0, 0\right),$$

where $i = 1$ or $i = 2$.

Thus, in order to construct a complete biconservative surface in $\mathbb{S}^3$, we can expect to glue along the boundary two biconservative surfaces of type $S_{\tilde{C}_1}$, corresponding to the same $\tilde{C}_1$. In fact, if we want to glue two surfaces corresponding to $\tilde{C}_1$ and $\tilde{C}_1'$ along the boundary, then these constants have to coincide and there is no ambiguity concerning along which circle of the boundary we should glue the two pieces. But this process is not as clear as in $\mathbb{R}^3$ since we should repeat it infinitely many times.

Further, as in the $\mathbb{R}^3$ case, we change the point of view and use the intrinsic characterization of the biconservative surfaces in $\mathbb{S}^3$.

The surface (D_{C_1}, g_{C_1}) defined in Section 3 is not complete but it has the following properties.

Theorem 5.5 ([21]). *Consider (D_{C_1}, g_{C_1}). Then, we have*

(a) $K_{C_1}(\xi,\theta) = K(\xi,\theta)$,

$$1 - K(\xi,\theta) = \frac{1}{9}\xi^{8/3} > 0, \quad K'(\xi) = -\frac{8}{27}\xi^{5/3}$$

and $\operatorname{grad} K \neq 0$ *at any point of* D_{C_1}*;*

(b) the immersion $\phi_{C_1} : (D_{C_1}, g_{C_1}) \to \mathbb{S}^3$ *given by*

$$\phi_{C_1}(\xi,\theta) = \left(\sqrt{1-\frac{1}{C_1\xi^2}}\cos\zeta(\xi), \sqrt{1-\frac{1}{C_1\xi^2}}\sin\zeta(\xi), \frac{\cos(\sqrt{C_1}\theta)}{\sqrt{C_1}\xi}, \frac{\sin(\sqrt{C_1}\theta)}{\sqrt{C_1}\xi}\right),$$

is biconservative in $\mathbb{S}^3$*, where*

$$\zeta(\xi) = \pm\int_{\xi_{00}}^{\xi} \frac{\sqrt{C_1}\tau^{4/3}}{(-1+C_1\tau^2)\sqrt{-\tau^{8/3}+3C_1\tau^2-3}}\, d\tau + c_1,$$

with $c_1 \in \mathbb{R}$ *a constant and* $\xi_{00} \in (\xi_{01}, \xi_{02})$*.*

Sketch of the proof. The first item follows by standard arguments. For the second item, we note that choosing $\tilde{C}_1 = 3^{1/4} \cdot 16C_1$ in (5.1) and using the change of coordinates $(\kappa, v) = \left(3^{-3/2}\xi^{4/3}, \left(3^{-1/8}\sqrt{C_1}\theta\right)/4\right)$, the metric induced by $Y_{3^{1/4}\cdot 16C_1}$ coincides with g_{C_1}.

Then, we define ϕ_{C_1} as

$$\phi_{C_1}(\xi,\theta) = Y_{3^{1/4}\cdot 16C_1}\left(3^{-3/2}\xi^{4/3}, \frac{3^{-1/8}\sqrt{C_1}\theta}{4}\right).$$

□

Remark 5.6. The limits $\lim_{\xi\searrow\xi_{01}}\zeta(\xi) = \zeta(\xi_{01})$ and $\lim_{\xi\nearrow\xi_{02}}\zeta(\xi) = \zeta(\xi_{02})$ are finite.

Remark 5.7. For simplicity, we choose $\xi_{00} = (9C_1/4)^{3/2}$.

Remark 5.8. The immersion ϕ_{C_1} depends on the sign $\pm$ and on the constant c_1 in the expression of ζ. As the classification is up to isometries of $\mathbb{S}^3$, the sign and the constant are not important, but they will play an important role in the gluing process.

The construction of complete biconservative surfaces in $\mathbb{S}^3$ consists in two steps, and the key idea is to notice that (D_{C_1}, g_{C_1}) is, locally and intrinsically, isometric to a surface of revolution in $\mathbb{R}^3$.

The *first step* is to construct a complete surface of revolution in $\mathbb{R}^3$ which on an open dense subset is locally isometric to (D_{C_1}, g_{C_1}). We start with the next result.

Theorem 5.9 ([21])**.** *Let us consider* (D_{C_1}, g_{C_1}) *as above. Then* (D_{C_1}, g_{C_1}) *is the universal cover of the surface of revolution in* $\mathbb{R}^3$ *given by*

$$\psi_{C_1,C_1^*}(\xi,\theta) = \left(\chi(\xi)\cos\frac{\theta}{C_1^*}, \chi(\xi)\sin\frac{\theta}{C_1^*}, \nu(\xi)\right), \tag{5.2}$$

where $\chi(\xi) = C_1^*/\xi$,

$$\nu(\xi) = \pm \int_{\xi_{00}}^{\xi} \sqrt{\frac{3\tau^2 - (C_1^*)^2 \left(-\tau^{8/3} + 3C_1\tau^2 - 3\right)}{\tau^4 \left(-\tau^{8/3} + 3C_1\tau^2 - 3\right)}}\, d\tau + c_1^*, \qquad (5.3)$$

$C_1^* \in \left(0, \left(C_1 - 4/3^{3/2}\right)^{-1/2}\right)$ *is a positive constant and* $c_1^* \in \mathbb{R}$ *is constant.*

Remark 5.10. The immersion ψ_{C_1,C_1^*} depends on the sign $\pm$ and on the constant c_1^* in the expression of ν. We denote by $S^{\pm}_{C_1,C_1^*,c_1^*}$ the image of ψ_{C_1,C_1^*}.

Remark 5.11. The limits $\lim_{\xi \searrow \xi_{01}} \nu(\xi) = \nu(\xi_{01})$ and $\lim_{\xi \nearrow \xi_{02}} \nu(\xi) = \nu(\xi_{02})$ are finite.

We note that the boundary of $S^{\pm}_{C_1,C_1^*,c_1^*}$ is given by the curves

$$\left(\frac{C_1^*}{\xi_{01}} \cos\frac{\theta}{C_1^*}, \frac{C_1^*}{\xi_{01}} \sin\frac{\theta}{C_1^*}, \nu(\xi_{01})\right)$$

and

$$\left(\frac{C_1^*}{\xi_{02}} \cos\frac{\theta}{C_1^*}, \frac{C_1^*}{\xi_{02}} \sin\frac{\theta}{C_1^*}, \nu(\xi_{02})\right)$$

These curves are circles in affine planes in $\mathbb{R}^3$ parallel to the Oxy plane and their radii are C_1^*/ξ_{01} and C_1^*/ξ_{02}, respectively.

At a boundary point, using the coordinates (ν, θ), we get that the tangent plane to the closure of $S^{\pm}_{C_1,C_1^*,c_1^*}$ is spanned by a vector which is tangent to the corresponding circle and by the vector $(0,0,1)$. Thus, the tangent plane is parallel to the rotational axis Oz.

Geometrically, we start with a piece of type $S^{\pm}_{C_1,C_1^*,c_1^*}$ and by symmetry to the planes where the boundary lie, we get our complete surface $\tilde{S}_{C_1,C_1^*}$; the process is periodic and we perform it along the whole Oz axis.

Analytically, we fix C_1 and C_1^*, and alternating the sign and with appropriate choices of the constant c_1^*, we can construct a complete surface of revolution $\tilde{S}_{C_1,C_1^*}$ in $\mathbb{R}^3$ which on an open subset is locally isometric to (D_{C_1}, g_{C_1}). In fact, these choices of $+$ and $-$, and of the constants c_1^* are uniquely determined by the "first" choice of $+$, or of $-$, and of the constant c_1^*. We start with $+$ and $c_1^* = 0$.

The profile curve of $S^{\pm}_{C_1,C_1^*,c_1^*}$ can be seen as the graph of a function depending on ν and this allows us to obtain a function F such that the profile curve of $\tilde{S}_{C_1,C_1^*}$ to be the graph of the function $\chi \circ F$ depending on ν and defined on the whole Oz (or $O\nu$). The function $F : \mathbb{R} \to [\xi_{01}, \xi_{02}]$ is periodic and at least of class C^3.

Theorem 5.12 ([21])**.** *The surface of revolution given by*

$$\Psi_{C_1,C_1^*}(\nu,\theta) = \left((\chi \circ F)(\nu) \cos\frac{\theta}{C_1^*}, (\chi \circ F)(\nu) \sin\frac{\theta}{C_1^*}, \nu\right), \qquad (\nu,\theta) \in \mathbb{R}^2,$$

is complete and, on an open dense subset, it is locally isometric to (D_{C_1}, g_{C_1}). *The induced metric is given by*

$$g_{C_1,C_1^*}(\nu,\theta) = \frac{3F^2(\nu)}{3F^2(\nu) - (C_1^*)^2\left(-F^{8/3}(\nu) + 3C_1F^2(\nu) - 3\right)}d\nu^2 + \frac{1}{F^2(\nu)}d\theta^2,$$

$(\nu,\theta) \in \mathbb{R}^2$. *Moreover,* $\operatorname{grad} K \neq 0$ *at any point of that open dense subset, and* $1 - K > 0$ *everywhere.*

From Theorem 5.12 we easily get the following result.

Proposition 5.13 ([21])**.** *The universal cover of the surface of revolution given by* Ψ_{C_1,C_1^*} *is* $\mathbb{R}^2$ *endowed with the metric* g_{C_1,C_1^*}. *It is complete,* $1-K > 0$ *on* $\mathbb{R}^2$ *and, on an open dense subset, it is locally isometric to* (D_{C_1}, g_{C_1}) *and* $\operatorname{grad} K \neq 0$ *at any point. Moreover any two surfaces* $\left(\mathbb{R}^2, g_{C_1,C_1^*}\right)$ *and* $\left(\mathbb{R}^2, g_{C_1,C_1^{*\prime}}\right)$ *are isometric.*

The *second step* is to construct effectively the biconservative immersion from $\left(\mathbb{R}^2, g_{C_1,C_1^*}\right)$ in $\mathbb{S}^3$, or from $\tilde{S}_{C_1,C_1^*}$ in $\mathbb{S}^3$. The geometric ideea of the construction is the following: from each piece $S^{\pm}_{C_1,C_1^*,c_1^*}$ of $\tilde{S}_{C_1,C_1^*}$ we "go back" to (D_{C_1}, g_{C_1}) and then, using ϕ_{C_1} and a specific choice of $+$ or $-$ and of the constant c_1, we get our biconservative immersion Φ_{C_1,C_1^*}. Again, the choices of $+$ and $-$, and of the constant c_1 are uniquely determined (modulo 2π, for c_1) by the "first" choice of $+$, or of $-$, and of the constant c_1 (see [21] for all details).

Some *numerical experiments* suggest that Φ_{C_1,C_1^*} is not periodic and it has self-intersections along circles parallel to Ox^3x^4.

The projection of Φ_{C_1,C_1^*} on the Ox^1x^2 plane is a curve which lies in the annulus of radii $\sqrt{1 - 1/\left(C_1\xi_{01}^2\right)}$ and $\sqrt{1 - 1/\left(C_1\xi_{02}^2\right)}$. It has self-intersections and is dense in the annulus.

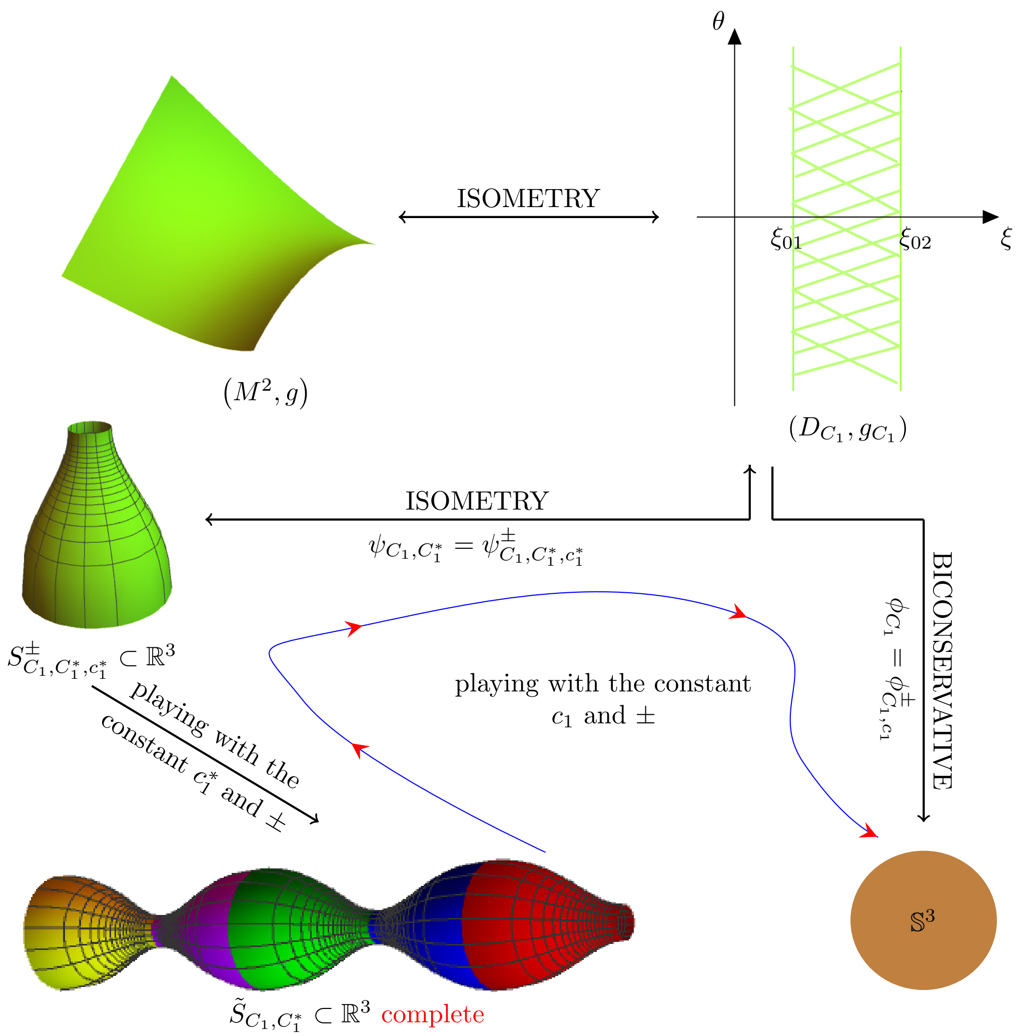
Concerning the biharmonic surfaces in $\mathbb{S}^3$ we have the following classification result.

Theorem 5.14 ([4])**.** *Let* $\varphi : M^2 \to \mathbb{S}^3$ *be a proper biharmonic surface. Then* $\varphi(M)$ *is an open part of the small hypersphere* $\mathbb{S}^2(1/\sqrt{2})$.

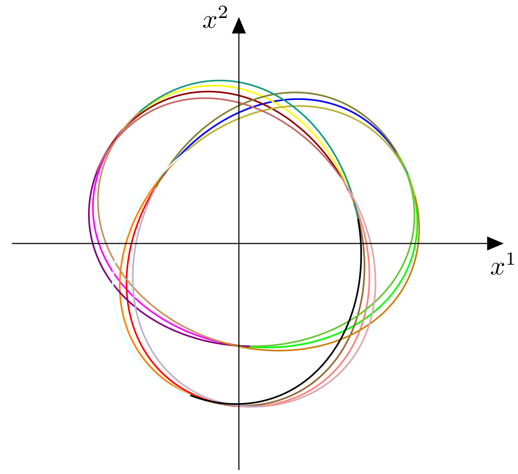
Appendix

In the $c = 0$ case, the idea was to construct, by symmetry, a complete biconservative surface in $\mathbb{R}^3$ starting with a piece of a biconservative surface. We illustrate this in the following figure obtained for $C_0 = 1$.

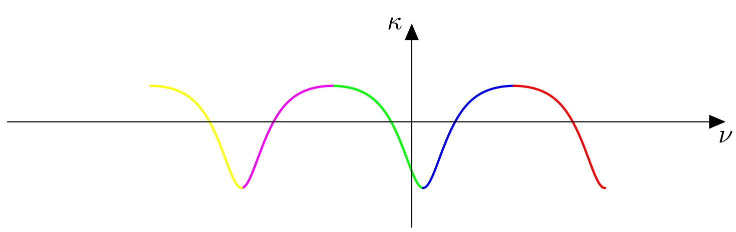
In the $c = 1$ case, the construction of a complete biconservative surface in $\mathbb{S}^3$ can be summarized in the next diagram, obtained for $C_1 = C_1^* = 1$, $c_1^* = 0$ and we started with $+$ in the expression of ν.

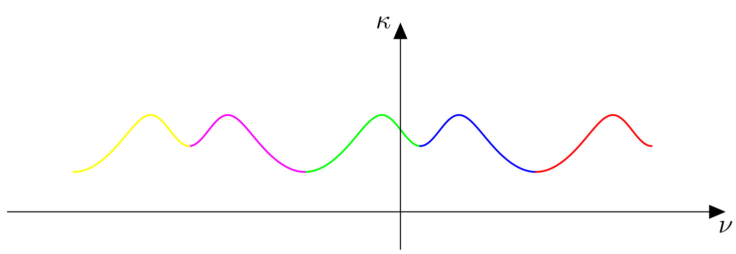


The projection of $\Phi_{1,1}$ on the Ox^1x^2 plane is represented in the next figure ($c_1 = 0$).



The last two figures represent the signed curvature of the profile curve of $\tilde{S}_{C_1,C_1^*}$ and the signed curvature of the curve obtained projecting $\Phi_{1,1}$ on the Ox^1x^2 plane.





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Parallel Mean Curvature Surfaces in Four-Dimensional Homogeneous Spaces

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Abstract. We survey different classification results for surfaces with parallel mean curvature immersed into some Riemannian homogeneous four-manifolds, including real and complex space forms and product spaces. We provide a common framework for this problem, with special attention to the existence of holomorphic quadratic differentials on such surfaces. The case of spheres with parallel mean curvature is also explained in detail, as well as the state-of-the-art advances in the general problem.
Keywords. parallel mean curvature · constant mean curvature · holomorphic quadratic differentials · Thurston geometry.
MSC 2010 Classification. Primary: 53C42; Secondary:53C30.

1 Introduction

Surface Theory in three-dimensional manifolds is a classical topic in Differential Geometry. Although the most extensive investigation has been carried out in ambient three-manifolds with constant curvature, the so-called *space forms*, there has been a growing interest in considering the broader family of homogeneous three-manifolds. Among the different geometrically distinguished families of surfaces, we will focus on those which have constant mean curvature (cmc in the sequel). When the codimension is bigger than one, a natural generalization of cmc surfaces are those whose mean curvature vector is not constant but parallel in the normal bundle. These surfaces are called parallel mean curvature surfaces (pmc from now on, see Definition 2.1), and enjoy some of the properties of cmc surfaces in codimension one.

The aim of this work is to gather some results on the classification of pmc surfaces when the codimension is two and the ambient space is homogeneous, sketching a parallelism with CMC surfaces in homogeneous three-manifolds. The connection between these two theories principally comes from the fact that CMC surfaces in totally umbilical CMC hypersurfaces of a four-manifold become PMC surfaces (cf. Proposition 3.1). When the four-manifold is homogeneous typically such a hypersurface is also homogeneous. Nonetheless, there could be PMC surfaces not factoring through a hypersurface in this sense, as we will discuss below. The interested reader can refer to [DHM09] and [MP12] for an introduction to CMC surfaces in homogeneous three-manifolds. Another approach that covers both CMC and PMC surfaces as critical points of extended area functionals can be found in [Sal10].

In the seventies, Ferus [Fer71] proved that an immersed PMC sphere in a space form is a round sphere (cf. Theorem 4.3), and afterwards Chen [Che73] and Yau [Yau74] classified PMC surfaces in space forms, showing that they are CMC surfaces in three-dimensional totally umbilical hypersurfaces (cf. Theorem 5.1). It is also important to mention the contribution of Hoffman [Hof73], who classified PMC surfaces of $\mathbb{R}^4$ and $\mathbb{S}^4$ in terms of analytical functions assuming their Gauss curvature does not change sign.

Almost thirty years later, Kenmotsu and Zhou [KZ00] undertook the classification of PMC surfaces in the complex space forms $\mathbb{CP}^2$ and $\mathbb{CH}^2$, based on a result of Ogata [Oga95]. However, soon thereafter Hirakawa [Hir06] pointed out a mistake in Ogata's equation, but gave a classification of the PMC spheres in $\mathbb{CP}^2$ and $\mathbb{CH}^2$. The mistake was corrected in [KO15] but the classification was still incomplete. Finally, Kenmotsu has recently published a correction [Ken16] that closes the classification problem. The complete classification then follows from both [Hir06] and [Ken16].

The classification of PMC surfaces in four-dimensional manifolds has also been treated in $\mathbb{M}^3(c)\times\mathbb{R}$, where $\mathbb{M}^n(c)$ denotes the n-dimensional space form of constant sectional curvature c. On the one hand, de Lira and Vitório [dLV10] classified the PMC spheres (cf. Theorem 4.12). On the other hand, Alencar, do Carmo and Tribuzy [ACT10] proved reduction of codimension for PMC surfaces in $\mathbb{M}^n(c)\times\mathbb{R}$ (cf. Theorem 5.3) also classifying PMC spheres (cf. Theorem 4.13). Mendonça and Tojeiro [MT14] improved Alencar, do Carmo and Tribuzy's result under some additional conditions (see Section 5.1).

A few years ago, the second author and Urbano [TU12] classified the PMC spheres in the product four-manifolds $\mathbb{S}^2\times\mathbb{S}^2$ and $\mathbb{H}^2\times\mathbb{H}^2$, as well as a large family of PMC surfaces that satisfy an extra condition on the extrinsic normal curvature (cf. Theorem 5.8). Fetcu and Rosenberg also tackled the problem in other ambient manifolds obtaining several partial results, namely, in $\mathbb{S}^3\times\mathbb{R}$ and $\mathbb{H}^3\times\mathbb{R}$ [FR12], in $\mathbb{M}^n(c)\times\mathbb{R}$ [FR13], in $\mathbb{CP}^n\times\mathbb{R}$ and $\mathbb{CH}^n\times\mathbb{R}$ [FR14] and also in Sasakian space forms [FR15], including the Heisenberg space of any odd dimension.

An interesting family where to study the classification problem for PMC surfaces in is that of the four-dimensional Thurston geometries, i.e., homogeneous four-manifolds whose isometry group acts transitively and effectively on them, and the stabilizer subgroup at each point is compact. Usually the isometry group is required to be maximal in the sense that it cannot be enlarged to another subgroup. Under these assumptions, there are 19 types of Thurston geometries in dimension 4, listed in Table 1 below. We will emphasize the product geometries, which might be the first spaces where PMC surface should be understood:

- The product spaces $\mathbb{M}^3(c)\times\mathbb{R}$ (see Sections 4.4 and 5.2). The classification of the PMC spheres was done by de Lira and Vitório [dLV10], but the general classification remains open.
- The product spaces $\mathbb{M}^2(c_1)\times\mathbb{M}^2(c_2)$. The classification of spheres is known when $c_1 = c_2$ (see Sections 4.3 and 5.4), but the general case remains still open, although there are some partial results (see Section 4.5).
- The product spaces $\mathrm{Nil}_3\times\mathbb{R}$, $\widetilde{\mathrm{Sl}}_2(\mathbb{R})\times\mathbb{R}$ and $\mathrm{Sol}^3\times\mathbb{R}$ (the latter is included in the family $\mathrm{Sol}^4_{m,n}$ in Table 1).

Dropping the condition on the maximality of the isometry group, a simply connected homogeneous four-dimensional product manifold is either of the form $\mathbb{M}^2(c_1)\times\mathbb{M}^2(c_2)$ or $G\times\mathbb{R}$, where G is a Lie group endowed with a left-invariant metric (see [MP12]). In the latter family, it is worth highligthing the family $\mathbb{E}(\kappa,\tau)\times\mathbb{R}$ where $\mathbb{E}(\kappa,\tau)$, $\kappa-4\tau^2\neq 0$, denotes the two-parameter family of simply connected three-manifolds with isometry group of dimension four (see [VdV08], [DHM09] and the references therein).

The existence of holomorphic quadratic differentials for PMC surfaces has been central in their classification. Note that CMC surfaces in $\mathbb{E}(\kappa,\tau)$-spaces admit a holomorphic quadratic differential called the Abresch-Rosenberg differential [AR05]. This fact plays a key role in the definition of holomorphic quadratic differentials for PMC surfaces in $\mathbb{H}^3\times\mathbb{R}$ and $\mathbb{S}^3\times\mathbb{R}$ (see Section 5.2).

Geometry	Isotropy	dim(Iso)	Kähler
$\mathbb{S}^4$, $\mathbb{R}^4$, $\mathbb{H}^4$	SO_4	10	No, except $\mathbb{R}^4$
$\mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{H}^2$	U_2	9	Yes
$\mathbb{S}^3\times\mathbb{R}$, $\mathbb{H}^3\times\mathbb{R}$	SO_3	7	No
$\mathbb{S}^2\times\mathbb{S}^2$, $\mathbb{H}^2\times\mathbb{H}^2$, $\mathbb{S}^2\times\mathbb{R}^2$, $\mathbb{S}^2\times\mathbb{H}^2$, $\mathbb{H}^2\times\mathbb{R}^2$	$\mathrm{SO}_2\times\mathrm{SO}_2$	6	Yes
$\widetilde{\mathrm{Sl}}_2(\mathbb{R})\times\mathbb{R}$, $\mathrm{Nil}_3\times\mathbb{R}$, Sol^4_0	SO_2	5	No
F^4	$(\mathrm{S}^1)_{1,2}$	5	Yes
Nil_4, $\mathrm{Sol}^4_{m,n}$, Sol^4_1	$\{1\}$	4	No

Table 1: List of Thurston four-dimensional geometries, their isotropy group (cf. [Wal86, §1] and [Mai98]), the dimension of their isometry group and whether they admit a Kähler structure compatible with the geometric structure (cf. [Wal86, Theorem 1.1]). The spaces $\widetilde{\mathrm{Sl}}_2(\mathbb{R})\times\mathbb{R}$, $\mathrm{Nil}_3\times\mathbb{R}$, Sol^4_0, and Sol^4_1 are not Kähler but do admit complex structures. Here, $(\mathrm{S}^1)_{m,n}$ denotes the image of the unit circle $\mathbb{S}^1\subset\mathbb{C}$ in U_2 by $z\mapsto(z^m,z^n)$.

It is worth mentioning that PMC surfaces have been also studied in pseudo-Riemannian manifolds. A classification was achieved for non-degenerate PMC surfaces in the four-dimensional Lorentzian space forms [CV09]. It turns out that, as in the Riemannian case, all PMC surfaces lie in three-dimensional submanifolds. This classification was afterwards extended to any codimension and any signature of the metric (see [Che09] for the spacelike case and [Che10, FH10] for the timelike case). In the sequel we will restrict ourselves to the Riemannian case.

2 Definitions and first properties

Let M be an n-dimensional orientable Riemannian manifold with metric $\langle\,,\rangle$ and Levi-Civita connection $\overline{\nabla}$, and let $\phi:\Sigma\to M$ be an isometric immersion of an orientable Riemannian surface Σ. The tangent space $T_p\Sigma$ will be identified with $\mathrm{d}\phi(T_p\Sigma)\subset T_{\phi(p)}M$ in the sequel, so the metric on Σ will also be denoted by $\langle\,,\,\rangle$ since the immersion is isometric. Therefore $T_{\phi(p)}M$ admits an orthogonal decomposition $T_{\phi(p)}M=T_p\Sigma\oplus T_p^\perp\Sigma$, where $T^\perp\Sigma$ is the so-called *normal bundle* of the immersion. This leads to considering the space $\mathfrak{X}(\Sigma)$ of (tangent) vector fields, i.e., smooth sections of $T\Sigma$, and the space $\mathfrak{X}^\perp(\Sigma)$ of normal vector fields, i.e., smooth sections of $T^\perp\Sigma$. We will denote by $u^\top\in T_p\Sigma$ and $u^\perp\in T_p^\perp\Sigma$ the components of a vector $u\in T_{\phi(p)}M$ with respect to this decomposition.

Given a normal vector field $\eta\in\mathfrak{X}^\perp(\Sigma)$, we can define the shape operator associated with η as the self-adjoint endomorphism $A_\eta:\mathfrak{X}(\Sigma)\to\mathfrak{X}(\Sigma)$ given by $A_\eta(X)=-(\overline{\nabla}_X\eta)^\top$. Then the second fundamental form $\sigma:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}^\perp(\Sigma)$ satisfies $\langle\sigma(X,Y),\eta\rangle=\langle A_\eta(X),Y\rangle$ for all $X,Y\in\mathfrak{X}(\Sigma)$ and $\eta\in\mathfrak{X}^\perp(\Sigma)$. The mean curvature vector H of the immersion at $p\in\Sigma$ is defined as $H(p)=\frac{1}{2}(\sigma(e_1,e_1)+\sigma(e_2,e_2))$, where $\{e_1,e_2\}$ is an orthonormal basis of $T_p\Sigma$.

The normal bundle $T^\perp\Sigma$ can also be endowed with a connection $\nabla^\perp:\mathfrak{X}(\Sigma)\times\mathfrak{X}^\perp(\Sigma)\to\mathfrak{X}^\perp(\Sigma)$ defined as $\nabla^\perp_X\eta=(\overline{\nabla}_X\eta)^\perp$ for all $X\in\mathfrak{X}(\Sigma)$ and $\eta\in\mathfrak{X}^\perp(\Sigma)$. This connection is called the *normal connection*, and gives rise to a *curvature tensor* $R^\perp:\mathfrak{X}(\Sigma)\times\mathfrak{X}(\Sigma)\times\mathfrak{X}^\perp(\Sigma)\to\mathfrak{X}^\perp(\Sigma)$, given by

$$R^\perp(X,Y)\eta=\nabla^\perp_X\nabla^\perp_Y\eta-\nabla^\perp_Y\nabla^\perp_X\eta-\nabla^\perp_{[X,Y]}\eta,\quad X,Y\in\mathfrak{X}(\Sigma),\ \eta\in\mathfrak{X}^\perp(\Sigma).\tag{2.1}$$

Definition 2.1 (Parallel mean curvature immersion)**.** An isometric immersion $\phi:\Sigma\to M$ is said to have parallel mean curvature (PMC for short) if its mean curvature vector $H\in\mathfrak{X}^\perp(\Sigma)$ is parallel in the normal bundle, i.e., $\nabla^\perp H=0$, but not identically zero.

Remark 2.2. The minimal case $H=0$ has been excluded from Definition 2.1 due to several reasons that will become clear after Lemma 2.4. Essentially, it is not possible to define a natural orthonormal frame in the normal bundle if $H=0$, which is crucial for some of the arguments below.

Although several results in higher codimension will be mentioned hereinafter, let us assume now that the codimension is 2, where PMC surfaces enjoy additional properties. In the first place, we will furnish the normal bundle with a natural orientation provided that both M and Σ are oriented, and define a notion of curvature in the normal bundle.

Definition 2.3. A basis $\{\eta, \nu\}$ in $T_p^\perp\Sigma$ is said to be *positively oriented* if and only if $\{e_1, e_2, \eta, \nu\}$ is positively oriented in $T_{\phi(p)}M$ whenever $\{e_1, e_2\}$ is a positively oriented basis of $T_p\Sigma$.

The *normal curvature* of ϕ is the smooth function $K^\perp \in C^\infty(\Sigma)$ defined by

$$K^\perp(p) = \langle R^\perp(e_1, e_2)e_3, e_4\rangle, \tag{2.2}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of T_pM such that $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are positively oriented bases in $T_p\Sigma$ and $T_p^\perp\Sigma$, respectively.

Lemma 2.4. *Let $\phi : \Sigma \to M$ be a* PMC *immersion. Then the mean curvature vector H has constant length (in particular, H never vanishes). If additionally the codimension is 2, then:*

(i) There exists a unique parallel normal field $\widetilde{H} \in \mathfrak{X}^\perp(\Sigma)$ such that the global frame $\{\widetilde{H}/|H|, H/|H|\}$ is positively oriented and orthonormal in $T^\perp\Sigma$.

(ii) $K^\perp$ is identically zero, i.e., the normal bundle is flat.

(iii) The Ricci equation for the Riemann curvature tensor $\overline{R}$ of M reads

$$\langle \overline{R}(X, Y)H, \eta\rangle = \langle [A_\eta, A_H]X, Y\rangle, \qquad X, Y \in \mathfrak{X}(\Sigma),\ \eta \in \mathfrak{X}^\perp(\Sigma). \tag{2.3}$$

Remark 2.5. Flatness of the normal bundle of a PMC surface is a typical property in codimension 2. If the codimension is bigger than 2, it is possible to define likewise the normal sectional curvature of the normal bundle, but it does not necessarily vanish for PMC surfaces.

Proof. Since H is parallel in the normal bundle we have

$$X(|H|^2) = 2\langle \overline{\nabla}_X H, H\rangle = 2\langle \nabla^\perp_X H, H\rangle = 0,$$

for all $X \in \mathfrak{X}(\Sigma)$, so $|H|$ is constant on Σ. As for (i), the normal bundle is orientable in the sense of Definition 2.3, so we can define a rotation R_p of angle $\pi/2$ in $T_p^\perp\Sigma$ such that $\{\eta, R_p\eta\}$ is positively oriented for all $\eta \in T_p^\perp\Sigma$. This rotation leaves the normal bundle of Σ invariant, and should not be confused with a possible complex structure on M.

Hence $\widetilde{H} = -RH$ is such that $\{\widetilde{H}/|H|, H/|H|\}$ is a positively oriented global orthonormal frame of the normal bundle. Moreover, $\widetilde{H}$ is also parallel since it has constant length and is orthogonal to the parallel vector field H. Given a positively oriented orthonormal frame $\{e_1, e_2\}$ in $T\Sigma$, we can consider $e_3 = \widetilde{H}/|H|$ and $e_4 = H/|H|$, so Equation (2.1) and the fact that $\widetilde{H}$ is parallel yield

$$\begin{aligned} |H|R^\perp(e_1, e_2)e_3 &= \nabla^\perp_{e_1}\nabla^\perp_{e_2}\widetilde{H} - \nabla^\perp_{e_2}\nabla^\perp_{e_1}\widetilde{H} - \nabla^\perp_{[e_1,e_2]}\widetilde{H} = 0,\\ |H|R^\perp(e_1, e_2)e_4 &= \nabla^\perp_{e_1}\nabla^\perp_{e_2}H - \nabla^\perp_{e_2}\nabla^\perp_{e_1}H - \nabla^\perp_{[e_1,e_2]}H = 0. \end{aligned} \tag{2.4}$$

From (2.2) and the first equation in (2.4), we get that $K^\perp \equiv 0$, so (ii) is proved. Finally, given $X, Y \in \mathfrak{X}(\Sigma)$ and $\xi, \eta \in \mathfrak{X}^\perp(\Sigma)$, the Ricci equation reads $\overline{R}(X, Y, \xi, \eta) = R^\perp(X, Y, \xi, \eta) - \langle [A_\xi, A_\eta]X, Y\rangle$, so (iii) is a consequence of taking $\xi = H$ in the Ricci equation and of the second identity in (2.4). □

3 The relation between CMC and PMC surfaces

Parallel mean curvature surfaces are often considered the natural generalization to higher codimension of CMC surfaces in three-manifolds, so a leading idea in the study of PMC surfaces in four-manifolds is to reduce the codimension and rely on results for CMC surfaces. Our first approach to this idea will consist in finding natural assumptions on a hypersurface N of a four-manifold M guaranteeing that any CMC surface immersed in N has parallel mean curvature vector in M. This is evident if N is totally geodesic, but this condition can be relaxed as the following result ensures.

Proposition 3.1. *Let N be a totally umbilical* CMC *hypersurface of a four-manifold M. Then every* CMC *surface immersed in N is either* PMC *or minimal in M.*

Proof. Let $\phi : \Sigma \to N$ be a CMC immersion with second fundamental form $\widetilde{\sigma}$ and mean curvature vector $\widetilde{H}$. The immersion ϕ can also be regarded as an immersion into M, so let us denote by σ and H the second fundamental form and the mean curvature vector of the immersion $\phi : \Sigma \to M$, respectively. We will also define $\widehat{\sigma}$ and $\widehat{H}$ as the second fundamental form and the mean curvature vector of N as a hypersurface of M, respectively.

Since $\sigma = \widetilde{\sigma} + \widehat{\sigma}$, taking the trace on Σ we get that $2H = 2\widetilde{H} + 3\widehat{H} - \widehat{\sigma}(\eta, \eta)$, where η is a unit normal vector field to $\phi(\Sigma)$ tangent to N. Taking into account that N is totally umbilical, i.e., $\widehat{\sigma}(x, y) = \langle x, y\rangle \widehat{H}$ for all $x, y \in TN$, we finally get that $H = \widetilde{H} + \widehat{H}$. Taking the derivative of this last equation with respect to a tangent vector field $V \in \mathfrak{X}(\Sigma)$ gives

$$\begin{aligned}
\nabla^{\perp}_V H &= \big(\overline{\nabla}_V(\widetilde{H} + \widehat{H})\big)^{\perp} = (\overline{\nabla}_V \widetilde{H})^{\perp} + (\overline{\nabla}_V \widehat{H})^{\perp} \\
&= \big(\nabla^N_V \widetilde{H} + \widehat{\sigma}(V, \widetilde{H})\big)^{\perp} + (\overline{\nabla}_V \widehat{H})^{\perp} \\
&= (\nabla^N_V \widetilde{H})^{\perp} + (\langle V, \widetilde{H}\rangle \widetilde{H})^{\perp} + (\overline{\nabla}_V \widehat{H})^{\perp} = (\overline{\nabla}_V \widehat{H})^{\perp},
\end{aligned}$$

where ∇^N is the Levi-Civita connection of N and we have taken into account that $(\nabla^N_V \widetilde{H})^{\perp} = 0$ since $\widetilde{H}$ has constant length. We distinguish two cases:

- If $\widehat{H} = 0$ (N is a totally geodesic hypersurface of M), then $\nabla^{\perp}_V H = (\overline{\nabla}_V \widehat{H})^{\perp} = 0$ and H is parallel, so we are done.
- Assume now that $\widehat{H} \neq 0$. Observe that $\langle \overline{\nabla}_V \widehat{H}, \widehat{H}\rangle = 0$ since $\widehat{H}$ has constant length, so $(\overline{\nabla}_V \widehat{H})^{\perp}$ is proportional to a unit vector field η, normal to Σ, but tangent to N. Hence

$$\begin{aligned}
(\overline{\nabla}_V \widehat{H})^{\perp} &= \langle \overline{\nabla}_V \widehat{H}, \eta\rangle \eta = -\langle \widehat{H}, \overline{\nabla}_V \eta\rangle \eta = -\langle \widehat{H}, \widehat{\sigma}(V, \eta)\rangle \eta \\
&= -\langle \widehat{H}, \langle V, \eta\rangle \widehat{H}\rangle \eta = 0,
\end{aligned}$$

where we used again that N is a totally umbilical hypersurface in M. □

Remark 3.2. Under the assumptions of Proposition 3.1, the mean curvature vector H of Σ in M is just the sum of the mean curvature vector $\widetilde{H}$ of Σ in N and the mean curvature vector $\widehat{H}$ of N in M, i.e., we have the orthogonal decomposition $H = \widetilde{H} + \widehat{H}$. Hence Σ is PMC if and only if $\nabla^{\perp} H = 0$ and Σ is not minimal in N or N is not totally geodesic in M.

Let us analyse how Proposition 3.1 can be applied in different four-manifolds where totally umbilical surfaces are classified in order to construct PMC surfaces.

1. In the space forms $\mathbb{R}^4$, $\mathbb{S}^4$ and $\mathbb{H}^4$, totally umbilical hypersurfaces have constant sectional curvature and constant mean curvature. Hence, the PMC surfaces provided by Proposition 3.1 are CMC surfaces in the three-dimensional space forms $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ embedded totally umbilically in the four-dimensional space form.

2. There are no totally umbilical hypersurfaces in the complex space forms $\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{H}^2$ [TT63]. This is one of the difficulties when trying to produce examples of PMC immersions. In fact, there are no PMC spheres in $\mathbb{C}\mathrm{P}^2$ or in $\mathbb{C}\mathrm{H}^2$ (cf. Theorem 4.6).

3. In $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$ there are plenty of totally umbilical hypersurfaces since both spaces are locally conformally flat, but only the totally geodesic ones have constant mean curvature [MT14]. Since totally geodesic submanifolds in a product are the product or totally geodesic submanifolds, we conclude that such totally geodesic hypersurfaces are locally congruent to $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, or $\mathbb{H}^2 \times \mathbb{R}$.

4. In a Riemannian product $\mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$ of two surfaces of constant Gaussian curvatures c_1 and c_2, with $(c_1, c_2) \neq (0, 0)$, the only totally umbilical hypersurfaces with constant mean curvature are totally geodesic. Hence, they are open subsets of products of one surface and a geodesic in the other surface. This was proven for $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$, where both factors have the same curvature, in [TU12], but the proof can easily be extended to the other cases.

5. Consider $\mathbb{E}(\kappa, \tau) \times \mathbb{R}$, the Riemannian product of a homogeneous three-space with the Euclidean line. If $\kappa - 4\tau^2 = 0$, the first factor has constant sectional curvature and the classification of totally umbilical hypersurfaces with constant mean curvature has been treated in item 3. If $\tau = 0$ (and $\kappa \neq 0$), the first factor is either $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, so the space under consideration is either $\mathbb{S}^2 \times \mathbb{R}^2$ or $\mathbb{H}^2 \times \mathbb{R}^2$, which have been treated in item 4. In all other cases, if was proven in [ST09, VdV08] that there are no totally umbilical surfaces in $\mathbb{E}(\kappa, \tau)$, so the only totally umbilical hypersurfaces of $\mathbb{E}(\kappa, \tau) \times \mathbb{R}$ are open parts of the slices $\mathbb{E}(\kappa, \tau) \times \{t_0\}$. A more general result in $G \times \mathbb{R}$, where G is a simply connected three-dimensional Lie group endowed with a left-invariant metric, follows from the classification of totally umbilical surfaces in G (see [MS15]).

4 QUADRATIC DIFFERENTIALS AND THE CLASSIFICATION OF PMC SPHERES

As in the theory of CMC surfaces in homogeneous Riemannian three-manifolds, the existence of quadratic differentials that are holomorphic for PMC immersions comes in handy in some ambient four-manifolds. For instance, in symmetric four-manifolds such as

- the space forms $\mathbb{R}^4$, $\mathbb{S}^4$ and $\mathbb{H}^4$,
- the complex hyperbolic and projective spaces $\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{H}^2$,
- the Riemannian products $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$,

it is possible to define two holomorphic quadratic differentials for PMC surfaces. It is also hitherto possible to define one holomophic quadratic differential in a few other cases, such as in $\mathbb{M}^3(c) \times \mathbb{R}$ and $\mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$ (de Lira and Vitório [dLV10] and Kowalczyk [Kow11]), or in Sasakian space forms (Rosenberg and Fetcu [FR15]). This is instrumental, for instance, in the classification of PMC spheres in the aforementioned spaces, for the fact that a non-trivial holomorphic differential vanishes often gives precious information.

Throughout this section, we will consider a PMC immersion $\phi : \Sigma \to M$ of an oriented surface Σ into a four-manifold M with second fundamental form σ. As in the previous section, ∇ and $\overline{\nabla}$ will denote the Levi-Civita connections in Σ and M, respectively, and $\overline{R}$ will stand for the Riemann curvature tensor of M. Also, $z = x + iy$ will be a conformal parameter on Σ with conformal factor e^{2u}, giving rise to the usual basic vectors $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

Lemma 4.1. *Under the previous assumptions, the following formulae hold:*

(i) $\langle \partial_z, \partial_{\bar{z}} \rangle = \frac{1}{2}e^{2u}$ *and* $\langle \partial_z, \partial_z \rangle = 0$.

(ii) $\nabla_{\partial_z}\partial_{\bar{z}} = \nabla_{\partial_{\bar{z}}}\partial_z = 0$ *and* $\nabla_{\partial_z}\partial_z = 2u_z\partial_z$.

(iii) $2\sigma(\partial_{\bar{z}}, \partial_z) = e^{2u}H$.

(iv) $\langle \sigma(\partial_z, \partial_z), \eta \rangle_{\bar{z}} = \langle \overline{R}(\partial_{\bar{z}}, \partial_z)\partial_z, \eta \rangle$ *for any parallel normal section* η.

Proof. (i) is a consequence of z being a conformal parameter, (ii) is a direct computation using Koszul's formula and (iii) is straightforward from the definition of ∂_z and $\partial_{\bar{z}}$. We prove (iv):

$$\begin{aligned}
\langle \sigma(\partial_z, \partial_z), \eta \rangle_{\bar{z}} &= \langle \nabla^{\perp}_{\partial_{\bar{z}}}\sigma(\partial_z, \partial_z), \eta \rangle + \langle \sigma(\partial_z, \partial_z), \nabla^{\perp}_{\partial_{\bar{z}}}\eta \rangle \\
&= \langle (\overline{\nabla}_{\partial_{\bar{z}}}\sigma)(\partial_z, \partial_z) + 2\sigma(\nabla_{\partial_{\bar{z}}}\partial_z, \partial_z), \eta \rangle \\
&= \langle (\overline{\nabla}_{\partial_z}\sigma)(\partial_{\bar{z}}, \partial_z) + \overline{R}(\partial_{\bar{z}}, \partial_z)\partial_z, \eta \rangle \\
&= \langle \nabla^{\perp}_{\partial_z}\sigma(\partial_{\bar{z}}, \partial_z) - \sigma(\nabla_{\partial_z}\partial_{\bar{z}}, \partial_z) - \sigma(\partial_{\bar{z}}, \nabla_{\partial_z}\partial_z) + \overline{R}(\partial_{\bar{z}}, \partial_z)\partial_z, \eta \rangle \\
&= \langle \nabla^{\perp}_{\partial_z}(\tfrac{1}{2}e^{2u}H) - u_z e^{2u}H + \overline{R}(\partial_{\bar{z}}, \partial_z)\partial_z, \eta \rangle = \langle \overline{R}(\partial_{\bar{z}}, \partial_z)\partial_z, \eta \rangle,
\end{aligned}$$

where we have taken into account (ii), (iii), the fact that η is parallel, the definition of the covariant derivative of σ and the Codazzi equation $(\overline{\nabla}_X\sigma)(Y,Z) - (\overline{\nabla}_Y\sigma)(X,Z) = (\overline{R}(X,Y)Z)^\perp$. □

From now on $z = x + iy$ will denote a conformal parameter on Σ compatible with the orientation and $\widetilde{H}$ is given in Lemma 2.4.

4.1 Space forms

Let $M = \mathbb{M}^4(c)$ be the space form of constant sectional curvature $c \in \mathbb{R}$, and define in conformal coordinates the quadratic differentials

$$\begin{aligned} \Theta(z) &= \langle \sigma(\partial_z, \partial_z), H\rangle \mathrm{d}z \otimes \mathrm{d}z, \\ \widetilde{\Theta}(z) &= \langle \sigma(\partial_z, \partial_z), \widetilde{H}\rangle \mathrm{d}z \otimes \mathrm{d}z. \end{aligned} \tag{4.1}$$

Equation (4.1) defines globally Θ and $\widetilde{\Theta}$, i.e., their expressions do not depend upon the choice of the conformal parameter.

Proposition 4.2. *Let $\phi : \Sigma \to \mathbb{M}^4(c)$ be a parallel mean curvature immersion of an oriented surface Σ. Then Θ and $\widetilde{\Theta}$ defined by* (4.1) *are holomorphic quadratic differentials.*

Proof. Taking into account that $\langle \overline{R}(\partial_{\bar{z}}, \partial_z)\partial_z, \eta\rangle$ is zero for any normal vector field η in a space form, the statement follows from Lemma 4.1. □

Theorem 4.3 (Ferus [Fer71], see also [Hof73, Theorem 2.2])**.** *Let $\phi : S \to \mathbb{M}^4(c)$ be a* PMC *immersion of a sphere S in a space form. Then $\phi(S)$ is contained in a totally umbilical hypersurface of $\mathbb{M}^4(c)$ as a minimal surface.*

Proof. For illustration purposes, we will prove the case $c = 0$, that is, $\mathbb{M}^4(0) = \mathbb{R}^4$. Since S is a sphere and ϕ is a PMC immersion, both Θ and $\widetilde{\Theta}$ defined in (4.1) vanish. Since $\Theta = 0$, we obtain that $A_H = |H|^2\mathrm{Id}$ (i.e., ϕ is *pseudo-umbilical*).

Arguing as in the classical proof that complete and connected totally umbilical surfaces in $\mathbb{R}^3$ are spheres or planes, we consider the function $f : S \to \mathbb{R}^4$ given by $f(p) = H_p + |H_p|^2\phi(p)$. For any tangent vector field $V \in \mathfrak{X}(S)$ we get

$$V(f) = V(H + |H|^2\phi) = \overline{\nabla}_V H + |H|^2 V = -A_H V + \nabla^\perp_V H + |H|^2 V = 0,$$

by using the pseudo-umbilicity, and identifying T_pS with its image by $\mathrm{d}\phi$ in $T_{\phi(p)}\mathbb{M}^4(c)$. Hence f is constant $a \in \mathbb{R}^4$, so the immersion satisfies

$$\left|\phi - \frac{a}{|H|^2}\right|^2 = \frac{1}{|H|^2}.$$

This means that $\phi(S)$ is contained in a sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ of radius $1/|H|$, which is totally umbilical in $\mathbb{R}^4$ with mean curvature $|\widehat{H}| = |H|$. Thus the mean curvature $\widetilde{H} = H - \widehat{H}$ of S as a surface of $\mathbb{S}^3$ is zero (observe that H and $\widehat{H}$ have the same length, and $\widetilde{H}$ and $\widehat{H}$ are orthogonal, see Remark 3.2). □

Remark 4.4. In the proof of Theorem 4.3 we have only used one of the holomorphic differentials associated to the PMC immersion to get the result. Nevertheless, both holomorphic differentials will be needed to get a complete classification of PMC immersions in space forms (cf. Theorem 5.1) as well as to classify PMC spheres in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$ (cf. Theorem 4.9).

Besides, [ACT10] showed that the spheres are not the only PMC surfaces in space forms for which the quadratic differential Θ vanishes identically: there is also a complete non-flat example in $\mathbb{H}^n$ with non-negative Gaussian curvature (cf. Remark 4.10).

4.2 Complex hyperbolic and projective spaces

Let us consider $M = \mathbb{C}\mathrm{M}^2(c)$, i.e., the complex projective or hyperbolic space of constant holomorphic curvature c, also including $\mathbb{C}^2 = \mathbb{C}\mathrm{M}^2(0)$. The situation in the complex space forms is quite similar to that of real space forms, due to the fact that Fetcu [Fet12] defined a couple of holomorphic quadratic differentials associated with PMC immersions in $\mathbb{C}\mathrm{M}^2(c)$.

The Riemann tensor of these spaces reads

$$\overline{R}(X,Y)Z = \frac{c}{4}\Big\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\langle X,JY\rangle JZ\Big\}, \tag{4.2}$$

where $J : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the complex structure, which satisfies:

1. $J^2 = -\mathrm{Id}$.
2. J is an isometry, i.e., $\langle JX, JY\rangle = \langle X, Y\rangle$.
3. J is parallel, i.e., $\overline{\nabla}_X JY = J\overline{\nabla}_X Y$, being $\overline{\nabla}$ the Levi-Civita connection of $\mathbb{C}\mathrm{M}^2(c)$.

Proposition 4.5 ([Fet12, Proposition 2.3 and Section 3.1])**.** *Let $\phi : \Sigma \to \mathbb{C}\mathrm{M}^2(c)$ be a* PMC *immersion of an oriented suface Σ, and let $z = x + iy$ be a conformal parameter on Σ. Then*

$$\begin{aligned} \Theta(z) &= \big(8|H|^2\langle\sigma(\partial_z,\partial_z),H\rangle + 3c\langle J\phi_z,H\rangle^2\big)\mathrm{d}z\otimes\mathrm{d}z,\\ \widetilde{\Theta}(z) &= \big(8i|H|^2\langle\sigma(\partial_z,\partial_z),\widetilde{H}\rangle + 3c\langle J\phi_z,\widetilde{H}\rangle^2\big)\mathrm{d}z\otimes\mathrm{d}z, \end{aligned} \tag{4.3}$$

define two quadratic holomorphic differentials on Σ.

On the one hand, if $c = 0$, then these differentials reduce to the corresponding diffentials in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. On the other hand, the appearance of the new extra term $\langle J\Phi_z, H\rangle$ can be motivated by the fact that the Codazzi equation in $\mathbb{C}\mathrm{M}^2(c)$ is not as simple as in the case of $\mathbb{M}^4(c)$.

Proof. The holomorphicity follows easily from Lemma 4.1, from the expression of the Riemann tensor (4.2) and from the following equalities:

$$\langle JH,\widetilde{H}\rangle = 2i|H|^2e^{-2u}\langle J\Phi_z,\Phi_{\bar z}\rangle, \qquad (J\Phi_z)^\top = 2e^{-2u}\Phi_z.$$

Let us justify the first one, by showing that if $\{e_1, e_2, e_3, e_4\}$ is an oriented orthonormal basis, then $\langle Je_1, e_2\rangle = \langle Je_3, e_4\rangle$. Let $C = \langle Je_1, e_2\rangle$, which satisfies $C^2 \leq 1$ by Cauchy-Schwarz inequality. If $C^2 = 1$, then $Je_1 = \pm e_2$, so $Je_3 = \pm e_4$ and we are done. If $C^2 < 1$, let us define $\widetilde{e}_3 = (1 - C^2)^{-1/2}(Ce_1 + Je_2)$ and $\widetilde{e}_4 = (1 - C^2)^{-1/2}(Je_1 - Ce_2)$. Then $\{\widetilde{e}_3, \widetilde{e}_4\}$ is an oriented orthonormal basis spanning the same plane as $\{e_3, e_4\}$, so they differ in a rotation of angle θ, and it is easy to check that $\langle Je_3, e_4\rangle = \langle J\widetilde{e}_3, \widetilde{e}_4\rangle = C$. □

Although there exist two holomorphic quadratic differentials, there is no direct proof of the classification of the PMC spheres in $\mathbb{C}\mathrm{M}^2(c)$. All the known proofs use the structure equations for PMC surfaces in $\mathbb{C}\mathrm{M}^2(c)$ provided by Ogata [Oga95]. The proof given by Fetcu in [Fet12, Corollary 3.2] uses the two holomorphic differentials to show that such a sphere must have constant Gauss curvature, so the result follows from [Hir06, Theorem 1.1].

Theorem 4.6 ([Hir06, Corollary 1.2] and also [Fet12, Corollary 3.2])**.** *Let $\phi : S \to \mathbb{C}\mathrm{M}^2(c)$ a PMC immersion of a sphere S. Then $c = 0$ and S is a round sphere in a hyperplane of $\mathbb{C}^2$.*

This non-existence result of PMC spheres in $\mathbb{C}\mathrm{H}^2$ and $\mathbb{C}\mathrm{P}^2$ contrasts with the rest of the symmetric spaces, where there do exist PMC spheres (cf. Theorem 4.3 and Theorem 4.9). In other Thurston four-geometries like $\mathbb{M}^3(c) \times \mathbb{R}$, $\mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$, $\mathbb{E}(\kappa, \tau) \times \mathbb{R}$ or $\mathrm{Sol}_3 \times \mathbb{R}$, there always exist PMC spheres, since $\mathbb{H}^3$ and the $\mathbb{E}(\kappa, \tau)$-spaces or Sol_3 do admit CMC spheres (see the comments below Proposition 3.1).

4.3 The Riemannian products $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$

Now let $M = \mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$, where $\mathbb{M}^2(\epsilon)$ stands for the 2-sphere $\mathbb{S}^2$ ($\epsilon = 1$) or the hyperbolic plane $\mathbb{H}^2$ ($\epsilon = -1$). Since both $\mathbb{S}^2$ and $\mathbb{H}^2$ admit a complex structure J, we can define on M two different (but equivalent) complex structures $J_1 = (J, J)$ and $J_2 = (J, -J)$ (see [TU12, Section 3]). Moreover, we can define a *product structure* $P : TM \to TM$ as $P(u, v) = (u, -v)$, which enjoys the following properties:

1. P is a self-adjoint linear involutive isometry of every tangent plane of M.
2. $J_2 = PJ_1 = J_1P$
3. P is parallel, i.e., $\overline{\nabla}_X PY = P\overline{\nabla}_X Y$ for all $X, Y \in \mathfrak{X}(M)$.

The operator P allows us to write the Riemann tensor of $\mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$ as

$$\overline{R}(X, Y)Z = \frac{\epsilon}{2}\big[\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle Y, PZ\rangle PX - \langle X, PZ\rangle PY\big]. \tag{4.4}$$

In particular, $\mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$ is an Einstein manifold of constant scalar curvature 4ϵ (this is no longer true in the general case $\mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$). The existence of two holomorphic differential was shown in [TU12].

Proposition 4.7 ([TU12, Proposition 3]). *Let $\phi : \Sigma \to \mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$ be a* PMC *immersion of an oriented suface Σ, and let $z = x + iy$ be a conformal parameter on Σ. Then*

$$\begin{aligned}\Theta_1(z) &= \left(2\langle\sigma(\partial_z,\partial_z), H + i\widetilde{H}\rangle + \frac{\epsilon}{4|H|^2}\langle J_1\phi_z, H + i\widetilde{H}\rangle^2\right)\mathrm{d}z\otimes\mathrm{d}z,\\ \Theta_2(z) &= \left(2\langle\sigma(\partial_z,\partial_z), H - i\widetilde{H}\rangle + \frac{\epsilon}{4|H|^2}\langle J_2\phi_z, H - i\widetilde{H}\rangle^2\right)\mathrm{d}z\otimes\mathrm{d}z,\end{aligned} \tag{4.5}$$

are two holomorphic quadratic differentials.

Proof. It also follows from Lemma 4.1 after some manipulations, as in the previous cases. □

Remark 4.8. The differentials Θ_1 and Θ_2 can be chosen in different ways, since any linear combination of them is also holomorphic. As a particular case and taking into account that $\langle J_1\phi_z, H\rangle = i\langle J_1\phi_z, \widetilde{H}\rangle$, $\langle J_2\phi_z, H\rangle = -i\langle J_2\phi_z, \widetilde{H}\rangle$, and $J_2 = PJ_1$, we can define the following two holomorphic quadratic differentials (cf. equation (4.3)):

$$\begin{aligned}\Theta &= \Big(4|H|^2\langle\sigma(\partial_z,\partial_z), H\rangle + \epsilon\big[\langle J_1\phi_z, H\rangle^2 + \langle J_1\phi_z, PH\rangle^2\big]\Big)\mathrm{d}z\otimes\mathrm{d}z\\ \widetilde{\Theta} &= \Big(4i|H|^2\langle\sigma(\partial_z,\partial_z), \widetilde{H}\rangle - \epsilon\big[\langle J_1\phi_z, \widetilde{H}\rangle^2 - \langle J_1\phi_z, P\widetilde{H}\rangle^2\big]\Big)\mathrm{d}z\otimes\mathrm{d}z\end{aligned} \tag{4.6}$$

It is easy to show that $\Theta = |H|^2(\Theta_1 + \Theta_2)$ and $\widetilde{\Theta} = |H|^2(\Theta_1 - \Theta_2)$, so these expressions make it clear that Θ_1 and Θ_2 extend the classical differentials in $\mathbb{R}^4$ given by (4.1).

Using that these two differentials vanish on spheres, it is shown in [TU12] that the extrinsic normal curvature of an immersed PMC sphere has to be zero. Then the following classification is a consequence of Theorem 5.8.

Theorem 4.9 ([TU12, Corollary 1]). *Let $\phi : S \to \mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$, $\epsilon^2 = 1$, be a* PMC *immersion of a sphere S. Then ϕ is a* CMC *sphere in a totally geodesic hypersurface of $\mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$.*

Remark 4.10. It is interesting to highlight that PMC spheres are not the only surfaces with vanishing holomorphic differentials. Indeed, the product of two hypercycles in $\mathbb{H}^2 \times \mathbb{H}^2$ with curvatures satisfying $k_1^2 + k_2^2 = 1$ and a special embedding of the hyperbolic plane in $\mathbb{H}^2 \times \mathbb{H}^2$ also satisfy that condition (see [TU12, Theorem 4]).

4.4 The Riemannian products $\mathbb{M}^3(c) \times \mathbb{R}$

The study of PMC surfaces in $M = \mathbb{M}^3(c) \times \mathbb{R}$ was tackled by de Lira and Vitório [dLV10], as well as by Alencar, Do Carmo and Tribuzy [ACT10]. As in all the previous cases these authors found a holomorphic quadratic differential. In spite of their claim that there are two holomorphic differentials Q^h and Q^v, a deeper analysis shows that Q^h and Q^v coincide.

The Riemann tensor of $\mathbb{M}^3(c) \times \mathbb{R}$ is given by:

$$\begin{aligned}\overline{R}(X,Y)Z &= \frac{c}{4}\big(\langle Y + PY, Z\rangle(X + PX) - \langle X + PX, Z\rangle(Y + PY)\big)\\ &= c\Big(\langle Y, Z\rangle X - \langle X, Z\rangle Y - \langle Y, \zeta\rangle\langle Z, \zeta\rangle X + \langle X, \zeta\rangle\langle Z, \zeta\rangle Y + \\ &\quad + \langle X, Z\rangle\langle Y, \zeta\rangle\zeta - \langle Y, Z\rangle\langle X, \zeta\rangle\zeta\Big),\end{aligned} \tag{4.7}$$

where P is the product structure in $T(\mathbb{M}^3(c) \times \mathbb{R}) \equiv T\mathbb{M}^3(c) \times \mathbb{R}$ given by $P(u,t) = (u, -t)$ for all $u \in T\mathbb{M}^3(c)$, and $t \in \mathbb{R}$, and ζ is a unit tangent vector to the factor $\mathbb{R}$. The second expression in (4.7) follows from the identity $PX = X - 2\langle X, \zeta\rangle\zeta$ for all $X \in \mathfrak{X}(M)$.

Proposition 4.11. *Let $\phi : \Sigma \to \mathbb{M}^3(c) \times \mathbb{R}$ be a* PMC *immersion of an oriented surface Σ and let $z = x + iy$ be a conformal parameter. Then*

$$\begin{aligned}\Theta(z) &= \big(2\langle\sigma(\partial_z, \partial_z), H\rangle - c\langle\phi_z, \zeta\rangle^2\big)\mathrm{d}z \otimes \mathrm{d}z\\ &= \big(2\langle\sigma(\partial_z, \partial_z), H\rangle + \tfrac{c}{2}\langle\phi_z, P\phi_z\rangle\big)\mathrm{d}z \otimes \mathrm{d}z\end{aligned} \tag{4.8}$$

is a holomorphic quadratic differential in Σ.

Proof. Both expressions for Θ coincide, which follows from the equality $P\phi_z = \phi_z - 2\langle\phi_z, \zeta\rangle\zeta$ and the fact that z is a conformal parameter, i.e., $\langle\phi_z, \phi_z\rangle = 0$. Using now Lemma 4.1 and the second equality in (4.7), we deduce that $\langle\sigma(\partial_z, \partial_z), H\rangle_{\bar{z}} = \frac{c}{2}e^{2u}\langle\phi_z, \zeta\rangle\langle H, \zeta\rangle$, and also

$$(\langle\phi_z, \zeta\rangle^2)_{\bar{z}} = 2\langle\phi_z, \zeta\rangle\langle\overline{\nabla}_{\partial_{\bar{z}}}\phi_z, \zeta\rangle = e^{2u}\langle\phi_z, \zeta\rangle\langle H, \zeta\rangle,$$

where we have taken into account that ζ is a parallel vector field. Consequently, the differential is holomorphic. □

De Lira and Vitório use this quadratic differential Θ to classify the PMC spheres in $\mathbb{M}^3(c) \times \mathbb{R}$ by showing that there is a principal frame $\{e_1, e_2\}$ on the surface such that the the associated curvature lines to e_1 lie in horizontal slices. Then an analysis of these curvature lines leads to the following result:

Theorem 4.12 ([dLV10, Theorem 3.2])**.** *The only* PMC *spheres immersed in $\mathbb{M}^3(c) \times \mathbb{R}$ are the rotationally invariant* CMC *surfaces embedded in totally geodesic cylinders $\mathbb{M}^2(c) \times \mathbb{R}$ or in totally geodesic slices $\mathbb{M}^3(c) \times \{t_0\}$, $t_0 \in \mathbb{R}$.*

A result of the same kind is obtained by Alencar, do Carmo and Tribuzy in $\mathbb{M}^4(c) \times \mathbb{R}$ (codimension 3), as we show next. One expects that a PMC sphere in $\mathbb{M}^4(c) \times \mathbb{R}$ lies either in a slice $\mathbb{M}^4(c) \times \{t_0\}$ or in some $\mathbb{M}^2(c) \times \mathbb{R}$ as a CMC sphere (hence rotationally invariant). Unfortunately, a further reduction of the codimension still remains an open problem, which would give the complete classification of PMC spheres in $\mathbb{M}^n(c) \times \mathbb{R}$ for all $n \geq 4$ (see Theorem 5.3).

Theorem 4.13 ([ACT10, Theorem 2])**.** *Let $\phi : S \to \mathbb{M}^4(c) \times \mathbb{R}$ be a* PMC *immersion of a sphere S. Then one of the following assertions holds:*

(i) $\phi(S)$ *is contained in a totally umbilical hypersurface of* $\mathbb{M}^4(c) \times \{t_0\}$ *as a* CMC *surface.*

(ii) *Considering* $\mathbb{M}^4(c) \times \mathbb{R}$ *isometrically embedded in* $\mathbb{R}^6$ *(*$c = 1$*) or* $\mathbb{R}^6_1$ *(*$c = -1$*), there is a plane* Π *such that* $\phi(S)$ *is invariant under rotations which fix* $\Pi^\perp$*, and the level curves of the height function* $p \mapsto \langle \phi(p), \zeta \rangle$ *are circles lying in planes parallel to* Π*.*

Remark 4.14. Mendonça and Tojeiro [MT14] improve item (ii) in the previous result by showing that, in general codimention, $\phi(\Sigma)$ is a rotationally surface in a totally geodesic $\mathbb{M}^m(c) \times \mathbb{R}$, $m \leq 4$, over a curve in a totally geodesic $\mathbb{M}^s(c) \times \mathbb{R}$, $s \leq 3$.

4.5 The Riemannian products $\mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$.

Let us finally consider $M = \mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$. Following the notation introduced in Section 4.3, the Riemann tensor of M can be expressed as

$$\overline{R}(X, Y)Z = c_1 R_0(P_1 X, P_1 Y)Z + c_2 R_0(P_2 X, P_2 Y)Z,$$

where $R_0(X, Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$, $P_1 = \frac{1}{2}(I + P)$ and $P_2 = \frac{1}{2}(I - P)$ are the projections to the factors, i.e., $P_1(u, v) = (u, 0)$ and $P_2(u, v) = (0, v)$.

De Lira and Vitório [dLV10] defined a holomorphic quadratic differential for PMC surfaces in $\mathbb{S}^2 \times \mathbb{H}^2$ (where the constant Gauss curvatures of the factors are exactly opposite) and the holomorphicity of this differential also follows from the ideas in [TU12]. Kowalczyk [Kow11] extended this by defining a quadratic differential in the general case of $\mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$, cf. the next proposition. In contrast to the previous cases, the classification of PMC spheres in $\mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$ is still an open problem, even in $\mathbb{S}^2 \times \mathbb{H}^2$. The natural candidates are those given by Proposition 3.1, i.e., CMC spheres immersed in totally geodesic hypersurfaces of $\mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$.

Proposition 4.15. *Let* $\phi : \Sigma \to \mathbb{M}^2(c_1) \times \mathbb{M}^2(c_2)$ *be a* PMC *immersion of an oriented surface* Σ *and* $z = x + iy$ *a conformal parameter. Then*

$$\begin{aligned}\Theta(z) = \Big(2|H|^2 \langle \sigma(\partial_z, \partial_z), H \rangle &+ c_1 \langle R_0(P_1 \phi_z, P_1 H)H, \phi_z \rangle \\ &- c_2 \langle R_0(P_2 \phi_z, P_2 H)H, \phi_z \rangle \Big) \mathrm{d}z \otimes \mathrm{d}z\end{aligned} \tag{4.9}$$

is a holomorphic quadratic differential on Σ*.*

In the case $c_1 = c_2 = \pm 1$, the holomorphic differential given by (4.9) is a linear combination of the two holomorphic differentials in Proposition 4.7.

5 The general non-spherical case

Proposition 3.1 reveals that the ambient spaces considered above are plentiful of PMC immersions in general: any CMC immersion into a totally umbilical

CMC hypersurface is PMC. Nonetheless, this description does not give all PMC surfaces in general, as examples in complex space forms or in product manifolds $\mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$ below show. On account of the fact that listing all PMC surfaces is not reasonable, instead local classification results have been considered so far, based either on reducing the codimension to the CMC case (space form cases and $\mathbb{M}^n(c) \times \mathbb{R}$), or on associating some analytic data with the immersion (complex hyperbolic and projective spaces, see also Hoffman's examples [Hof73, Theorem 5.1] in $\mathbb{R}^4$ at the end of Section 5.1). We will present as well results with extra conditions on the immersion.

5.1 PMC surfaces in space forms

Chen [Che73] classified PMC surfaces in Euclidean space $\mathbb{R}^4$, and Yau [Yau74] gave an independent classification in an arbitrary space form $\mathbb{M}^4(c)$.

Theorem 5.1 ([Yau74, Theorem 4]). *Let $\phi : \Sigma \to \mathbb{M}^4(c)$ be a PMC immersion of an oriented surface Σ. Then Σ is contained in a totally umbilical hypersurface of $\mathbb{M}^4(c)$ as a CMC surface.*

Remark 5.2. Although Theorem 5.1 is stated in dimension four, Chen and Yau proved this result in arbitrary dimension, showing, more precisely that either ϕ is minimal in a totally umbilical hypersurface of $\mathbb{M}^n(c)$, or ϕ is a CMC immersion into a totally umbilical three-dimensional submanifold of $\mathbb{M}^n(c)$.

Proof. The idea is to use both differentials defined by (4.1) to show the existence of a parallel normal section ξ such that $A_\xi = \lambda\,\mathrm{Id}$, and the same argument as in the proof of Theorem 4.3 will ensure that Σ satisfies the desired conditions To illustrate this, let us assume $c = 0$.

If $\Theta = 0$, then $A_H = |H|^2\,\mathrm{Id}$ and we can reason as in the proof of Theorem 4.3. Likewise, if $\widetilde{\Theta} = 0$, then $A_{\widetilde{H}} = \lambda\,\mathrm{Id}$ with $\lambda = \langle H, \widetilde{H}\rangle = 0$ so $p \mapsto \widetilde{H}_p$ is constant in $\mathbb{R}^4$ since $V(\widetilde{H}) = -A_{\widetilde{H}}V + \nabla^\perp_V \widetilde{H} = 0$ for all $V \in \mathfrak{X}(\Sigma)$. The function $f : \Sigma \to \mathbb{R}$ defined as $f(p) = \langle \phi(p) - \phi(p_0), \widetilde{H}\rangle$ for some $p_0 \in \Sigma$ satisfies

$$V(f) = \langle V, \widetilde{H}\rangle + \langle \phi(p) - \phi(p_0), \overline{\nabla}_V \widetilde{H}\rangle = 0, \quad \text{for all } V \in \mathfrak{X}(\Sigma),$$

so f is constant and $\phi(\Sigma)$ lies in a hyperplane of $\mathbb{R}^4$. Moreover, $\phi(\Sigma)$ has constant mean curvature in this hyperplane

Hence we can assume that Θ and $\widetilde{\Theta}$ are not identically zero. It is not hard to prove that the imaginary part of the meromorphic function $g : \Sigma \to \mathbb{C}$, $g(p) = \Theta(p)/\widetilde{\Theta}(p)$, coincides with the commutator $[A_H, A_{\widetilde{H}}]$, which is zero by the Ricci equation (2.3). Hence the imaginary part of g identically vanishes, whence $g \equiv \tan(\alpha)$ for some constant $|\alpha| < \frac{\pi}{2}$. The normal vector field $\xi = \cos(\alpha)H - \sin(\alpha)\widetilde{H}$ is parallel and satisfies $A_\xi = \cos\alpha|H|^2\,\mathrm{Id}$, so we can again continue as in the proof of Theorem 4.3, considering the function $f(p) = \xi_p + \cos\alpha|H|^2\phi(p)$. □

For illustrative purposes, let us consider Lawson's minimal examples [Law70, Theorem 2] in $\mathbb{S}^3$ as a PMC surfaces in $\mathbb{R}^n$, $n \geq 4$. Hence, any compact orientable surface of genus g can be embedded as a PMC surface in $\mathbb{R}^n$, $n \geq 4$. Hoffman gave more examples of PMC surfaces in the space forms not lying in any hypersphere as minimal surfaces [Hof73, Theorem 5.1]. More particularly, he showed that, given any holomorphic function $\varphi : U \to \mathbb{C}$ on an open domain $U \subseteq \mathbb{C}$, and constants $H > 0$ and $\alpha \in \mathbb{R}$ there exists a PMC immersion in $\mathbb{M}^4(c)$ such that the length of the mean curvature vector is H, $\Theta = \varphi(\mathrm{d}z)^2$ and $\widetilde{\Theta} = \alpha\varphi(\mathrm{d}z)^2$.

5.2 PMC surfaces in $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$

Alencar, Do Carmo and Tribuzy [ACT10] studied PMC immersions in $\mathbb{M}^n(c) \times \mathbb{R}$, $c \neq 0$, for arbitrary n. They realized that the quadratic differential (4.8) introduced by de Lira and Vitório [dLV10] is holomorphic for any $n \geq 2$ (for $n = 2$ it is actually the Abresch-Rosenberg differential [AR05]). They showed that for a PMC immersion in $\mathbb{M}^n(c) \times \mathbb{R}$ either H is an umbilical direction, i.e., $A_H = |H|^2\mathrm{Id}$ (so $\phi(\Sigma)$ lies in a slice $\mathbb{M}^n(c) \times \mathbb{R}$, see items (i) and (ii) in Theorem 5.3), or one can reduce the codimension to three.

Theorem 5.3 ([ACT10, Theorem 1]). *Let $\phi : \Sigma \to \mathbb{M}^n(c) \times \mathbb{R}$ be a* PMC *immersion of an oriented surface Σ. Then, one of the following assertions holds:*

(i) *$\phi(\Sigma)$ is minimal in a totally umbilical hypersurface of $\mathbb{M}^n(c) \times \{t_0\}$, $t_0 \in \mathbb{R}$.*

(ii) *$\phi(\Sigma)$ is* CMC *in a three-dimensional totally umbilical submanifold of $\mathbb{M}^n(c) \times \{t_0\}$, $t_0 \in \mathbb{R}$.*

(iii) *If $n \geq 4$, then $\phi(\Sigma)$ lies in a totally geodesic $\mathbb{M}^4(c) \times \mathbb{R}$.*

Remark 5.4. Notice that Theorem 5.3 does not provide a classification of PMC surfaces in $\mathbb{M}^3(c) \times \mathbb{R}$ or in $\mathbb{M}^4(c) \times \mathbb{R}$. Therefore the final classification of PMC surfaces in $\mathbb{M}^n(c) \times \mathbb{R}$ depends upon the cases $n = 3$ and $n = 4$, which remain open. Moreover, PMC spheres have only been classified for $n = 3$ (cf. Theorem 4.12), though it is proven that they must be *rotationally invariant* for $n = 4$ [ACT10, Theorem 2].

Mendonça and Tojeiro have also discussed PMC immersions in $\mathbb{M}^n(c) \times \mathbb{R}$ in [MT14]. They obtained more information adding an extra hypothesis. They show that, if $\phi(\Sigma)$ is not contained in a slice (cases (i) and (ii) in Theorem 5.3) and $\Theta \equiv 0$ then $\phi(\Sigma)$ is rotationally invariant in the sense exposed in Remark 4.14. In particular, this condition is fulfilled if either Σ is diffeomorphic to a sphere or Σ is a complete non-flat surface in $\mathbb{H}^n \times \mathbb{R}$ with non-negative Gaussian curvature (cp. Remarks 4.4 and 4.10).

In order to prove the latter assertion, observe that if $\Theta \not\equiv 0$, then $\Delta \log|\Theta| = 4K \geq 0$, i.e., $\log|\Theta|$ is a superharmonic function bounded from below in Σ. Since $K \geq 0$ it follows that Σ has quadratic area growth, so $\log|\Theta|$ must be constant in view of [CY75, Corollary 1]. From the fact that $|\Theta|$ is constant, it follows that K is also constantly zero.

This idea was previously developed by Hoffman [Hof73] for PMC surfaces in space forms. It is worth pointing out that Hoffman was able to deal with the cases $K \geq 0$ and $K \leq 0$ in both $\mathbb{S}^4$ and $\mathbb{H}^4$, by finding suitable superharmonic functions bounded from below and reducing to the constant Gauss curvature case. On the contrary, Alencar, do Carmo and Tribuzy only treated the case $K \geq 0$ in $\mathbb{H}^n(c) \times \mathbb{R}$. This result has been extended to $\mathbb{S}^n(c) \times \mathbb{R}$ by Fetcu and Rosenberg [FR11, Theorem 1.2] by using a Simon-type equation.

5.3 PMC surfaces in $\mathbb{C}\mathrm{H}^2$ and $\mathbb{C}\mathrm{P}^2$

The classification of PMC surfaces in $\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{H}^2$ appeared first in a paper of Kenmotsu and Zhou [KZ00]. Unfortunately, their result depended upon the structure equations for PMC surfaces given by Ogata [Oga95], which turned out to be incorrect (see [KO15] for the correction). However, Hirakawa [Hir06, Theorem 2.1], who spotted Ogata's mistake, gave a partial solution to the problem, recently completed by Kenmotsu [Ken16] in a non-explicit way.

Given an immersion of an oriented surface Σ in $\mathbb{C}\mathrm{M}^2(c)$ (or more generally, in any complex manifold), the Kähler function $C : \Sigma \to [-1, 1]$ is defined by $C(p) = \langle Je_1, e_2\rangle$, where $\{e_1, e_2\}$ is an oriented orthonormal basis of $T_p\Sigma$ and J is the complex structure (some authors define $\theta = \arccos C$ as the *Kähler angle* of the immersion instead). The points $p \in \Sigma$ where $C^2(p) = 1$ are called *complex*, that is, they are the points where $T_p\Sigma$ is complex. Likewise, the points p where $C(p) = 0$ are the points where $T_p\Sigma$ is totally real. In particular, if C is constant zero, then the immersion is Lagrangian.

The main goal of [Hir06] was to study PMC surfaces with constant Gauss curvature (in particular, constant Kähler angle PMC surfaces, see the following paragraph), but Hirakawa also dealt with PMC surfaces satisfying a technical condition in Ogata's equation, namely $a = \bar{a}$, where $a = \langle J\nabla C, H + i\widetilde{H}\rangle$ (see Remark 5.6). This condition implies geometrically the existence of special coordinates in Σ such that the C only depends on one coordinate, see [KO15]. He also pointed out some examples that were missing in Kenmotsu and Zhou's paper. Hirakawa found, among others, PMC spheres and Delaunay CMC surfaces in $\mathbb{R}^3 \subset \mathbb{R}^4$, and gave four different types of solutions in $\mathbb{C}^2$, one type in $\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{H}^2$ with $H^2 \geq 2$, and two special types in $\mathbb{C}\mathrm{H}^2$ for $H^2 = 4/3$. Kenmotsu described the rest of PMC examples in $\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{H}^2$, that is, those with $a \neq \bar{a}$ (in particular with non-constant Kähler angle) in terms of a real-valued harmonic function and five real constants (cp. [Hof73, Theorem 5.1]).

Theorem 5.5 ([Hir06, Theorem 2.1] and [Ken16])**.** *Let Σ be a* PMC *surface immersed in $\mathbb{C}\mathrm{M}^2(c)$ and $a : \Sigma \to \mathbb{C}$ given by $a = \langle J\nabla C, H + i\widetilde{H}\rangle$, where ∇C is the gradient of the Kähler function and $\widetilde{H}$ is defined in Lemma 2.4.*

1. *If a is a real-valued function, then one of the following assertions holds:*
   - *(i) $|H|^2 \geq -c/2$ and the immersion is Lagrangian, or*
   - *(ii) $|H|^2 = -c/3$ and either the Kähler function is constant $1/3$ or it is a special solution (see item (iii)-2 in [Hir06, Theorem 2.1]).*

2. *If $a \neq \bar{a}$, then the solution depends on one real-valued harmonic function and five real constants.*

Remark 5.6. Our definition of a differs from the definition in [Hir06, Ken16] in a multiplicative real function plus a constant term $-\frac{1}{2}|H|$, which is irrelevant for the statement. Actually,

$$a = \frac{1}{2|H|(1-C^2)}\langle \nabla C, J(H - i\widetilde{H})\rangle - \frac{1}{2}|H|,$$

which is defined in the open dense set $\Sigma \setminus \{p \in \Sigma : C(p)^2 = 1\}$ (observe that the interior of the set $\{p \in \Sigma : C(p)^2 = 1\}$ is empty since otherwise the interior will be a complex surface, hence minimal, and we are supposing that Σ is PMC).

Remark 5.7. Among the solutions given by Theorem 5.5, the following are those with constant Gauss curvature (see [Hir06, Theorem 1.1]):

- Either $K = -H^2/2$ and $\Sigma \subset \mathbb{C}\mathrm{M}^2(-3|H|^2)$ is (an open piece of):
  - (i) the slant surface found by Chen in [Che98], or
  - (ii) one of the examples described in [Hir06, Examples, p. 230].
- Or $K = 0$ and the immersion is Lagrangian and Σ is (an open piece of):
  - (i) the product of two circles in $\mathbb{C}\mathrm{P}^2(c)$, $c > 0$, [DT95, Theorem 2], or
  - (ii) a plane, a cylinder, or a product of two circles in $\mathbb{C}\mathrm{H}^2(c)$, $c < 0$ with $|H|^2 \geq -c/2$, [Hir04, Theorem 1].

In Theorem 5.5 we omitted the case of $\mathbb{C}^2$ on purpose. Nevertheless, Hoffman [Hof73, Proposition 3.4] proved that a PMC flat surface in $\mathbb{C}^2$ is part of a cylinder or a product of two circles (see also [Che90, Theorem 7.1]). Hirakawa also studied PMC surfaces with constant Gauss curvature in $\mathbb{C}^2$ (see items (2)-(b) and (3) in [Hir06, Theorem 1.1] and item (ii) in [Hir06, Theorem 2.1]).

5.4 PMC surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$

The case $\mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$, $\epsilon^2 = 1$, is of different nature to the other cases we have presented so far. The classification is still incomplete, being only known under an extra assumption on the *extrinsic normal curvature*. This curvature is defined in the same fashion as the normal curvature $K^\perp$, but using the ambient Riemannian curvature tensor $\overline{R}$ in Equation (2.2) rather than the curvature tensor $R^\perp$.

Theorem 5.8 ([TU12, Theorems 2 and 3])**.** *Let $\phi : \Sigma \to \mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$ be a* PMC *immersion of an oriented surface Σ with vanishing extrinsic normal curvature. Then ϕ is locally congruent to*

1. *a* CMC *surface in a totally geodesic $\mathbb{M}^2(\epsilon) \times \mathbb{M}^1(\epsilon)$, or*
2. *a specific example given in [TU12, Example 1 and Proposition 5].*

Moreover, if ϕ is Lagrangian (not necessarily with vanishing extrinsic normal curvature), then $\phi(\Sigma)$ is an open set of the examples given in [TU12, Example 1].

Remark 5.9. Among the examples described in [TU12], there are PMC surfaces not lying in a totally geodesic hypersurface of $\mathbb{M}^2(\varepsilon) \times \mathbb{M}^2(\varepsilon)$.

The proof, which will not be sketched here, heavily relies upon the complex structure of $\mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$, not only on the product structure as in other cases. It is worth mentioning that there is also a local correspondence between pairs of CMC immersions in $\mathbb{M}^2(\epsilon) \times \mathbb{R}$ and PMC immersions in $\mathbb{M}^2(\epsilon) \times \mathbb{M}^2(\epsilon)$ [TU12, Theorem 1]. This relation provides a weak rigidity result for CMC surfaces in $\mathbb{M}^2(\epsilon) \times \mathbb{R}$. It is conjectured that the condition on the extrinsic normal curvature can be dropped, but probably that problem needs a different approach. If this conjecture were true, it would also imply a strong rigidity result for CMC surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ (cf. [TU12, Corollary 3]).

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Section 2: **THEORY OF SUBMANIFOLDS**

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Homothetic Motion and Surfaces with Pointwise 1-Type Gauss Map in $\mathbb{E}^4$

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Abstract. In this paper, we determine a surface M by means of homothetic motion in $\mathbb{E}^4$ and we give necessary and sufficient conditions for flat surface M with flat normal bundle to have pointwise 1-type Gauss map. Also, we show that flat surface M with flat normal bundle which have pointwise 1-type Gauss map of the first kind is a Clifford Torus. Morever, we obtain a characterization of minimal surface M with pointwise 1-type Gauss map.
Keywords. Homothetic motion · submanifolds · Gauss map · Pointwise 1-type Gauss map.
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1 Introduction

A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to be of finite type if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + x_1 + ... + x_k$, where x_0 is a constant map, $x_1, ..., x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are all different, then M is said to be of $k-$type. This definition was similarly extended to differentiable maps, in particular, to Gauss maps of submanifolds [3].

If a submanifold M of a Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface M of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if M is a hypersphere in $\mathbb{E}^{n+1}$ [3].

Hovewer, the Laplacian of the Gauss map of some typical well known surfaces such as a helicoid, a catenoid and a right cone in Euclidean 3-space $\mathbb{E}^3$ take a some what different form, namely,

$$\Delta G = f (G + C) \tag{1.1}$$

for some smooth function f on M and some constant vector C. A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for some smooth function f on M and some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind. A pointwise 1-type Gauss map is called proper if the function f given by (1.1) is non-constant. Non-proper pointwise 1-type Gauss map is just usual 1-type Gauss map.

Surfaces in Euclidean space with pointwise 1-type Gauss map were recently studied in [4], [5], [6]. Also Dursun and Turgay in [7] gave all general rotational surfaces in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan et al. in [1] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan at el. in [2] gave necessary and sufficent conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [8] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus.

In this paper, we determine a surface M by means of homothetic motion in $\mathbb{E}^4$ and we give necessary and sufficient conditions for flat surface M with flat normal bundle to have pointwise 1-type Gauss map. We show that flat surface with flat normal bundle which has pointwise 1-type Gauss map of the first kind is a Clifford Torus. Morever we obtain a characterization of minimal surface M with pointwise 1-type Gauss map.

2 Preliminaries

Let M be an oriented $n-$dimensional submanifold in $m-$dimensional Euclidean space $\mathbb{E}^m$. Let $e_1,...,e_n, e_{n+1},...,e_m$ be an oriented local orthonormal frame in $\mathbb{E}^m$ such that $e_1,...,e_n$ are tangent to M and $e_{n+1},...,e_m$ normal to M. We use the following convention on the ranges of indices: $1 \leq i, j, k,... \leq n$, $n+1 \leq r, s, t,... \leq m$, $1 \leq A, B, C,... \leq m$.

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^m$ and ∇ the induced connection on M. Let ω_A be the dual-1 form of e_A defined by $\omega_A(e_B) = \delta_{AB}$. Also, the connection forms ω_{AB} are defined by

$$de_A = \sum_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0.$$

Then we have

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{m} h_{ik}^r e_r$$

and

$$\tilde{\nabla}_{e_k} e_s = -A_s(e_k) + D_{e_k} e_s, \quad D_{e_k} e_s = \sum_{r=n+1}^{m} \omega_{sr}(e_k) e_r,$$

where D is the normal connection, h^r_{ik} the coefficients of the second fundamental form h and A_s the Weingarten map in the direction e_s.

For any real function f on M, the Laplacian of f is defined by

$$\Delta f = -\sum_i \left(\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla^{e_i}_{e_i}} f \right). \tag{2.1}$$

The mean curvature vector H and Gaussian curvature K are defined by

$$H = \frac{1}{n} \sum_{r,i} h^r_{ii} e_r \tag{2.2}$$

and

$$K = \sum_{s=n+1}^{m} \left(h^s_{11} h^s_{22} - h^s_{12} h^s_{21} \right). \tag{2.3}$$

Also normal curvature tensor R^D of M in $\mathbb{E}^m$ is given by

$$R^D(e_j, e_k; e_r, e_s) = \sum_{i=1}^{n} \left(h^r_{ik} h^s_{ij} - h^r_{ij} h^s_{ik} \right). \tag{2.4}$$

Let us now define the Gauss map G of a submanifold M into $G(n,m)$ in $\wedge^n \mathbb{E}^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented $n-$planes through the origin of $\mathbb{E}^m$ and $\wedge^n \mathbb{E}^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n \mathbb{E}^m$ with some Euclidean space $\mathbb{E}^N$ where $N = \begin{pmatrix} m \\ n \end{pmatrix}$. The map $G : M \rightarrow G(n,m) \subset E^N$ defined by $G(p) = (e_1 \wedge ... \wedge e_n)(p)$ is called the Gauss map of M, that is, a smooth map which carries a point p in M into the oriented $n-$plane through the origin of $\mathbb{E}^m$ obtained from parallel translation of the tangent space of M at p in $\mathbb{E}^m$.

The Laplacian of the Gauss map G for an $n-$dimensional submanifold M of Euclidean space $\mathbb{E}^m$ was given by

Lemma 2.1. (*See [3]*) *Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of an oriented n-dimensional Riemannian manifold M into $\mathbb{E}^m$. Then the Laplacian of the Gauss map $G : M \rightarrow G(n,m) \subset \wedge^n \mathbb{E}^m$ is given by*

$$\begin{aligned} \Delta G &= -n \sum_i e_1 \wedge ... \wedge D_{e_i} H \wedge ... \wedge e_n \\ &\quad + R^D(e_j, e_k; e_r, e_s)\, e_1 \wedge ... \wedge e_s^{k\ th} \wedge ... \wedge e_r^{j\ th} \wedge ... \wedge e_n + \|h\|^2 G. \end{aligned} \tag{2.5}$$

3 Homothetic Motion and Surfaces with Pointwise 1-Type Gauss Map

In this section, we define a surface by using the homothetic motion as follows:

$$f\left(t,s\right)=h(t)\begin{pmatrix}\cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & \cos t & -\sin t\\ 0 & 0 & \sin t & \cos t\end{pmatrix}\begin{pmatrix}\alpha_1(s)\\ \alpha_2(s)\\ \alpha_3(s)\\ \alpha_4(s)\end{pmatrix}+\begin{pmatrix}C_1\left(t\right)\\ C_2\left(t\right)\\ C_3\left(t\right)\\ C_4\left(t\right)\end{pmatrix}, \tag{3.1}$$

where $h(t)$ is the homothetic scale of the motion, $C\left(t\right)=\left(C_1\left(t\right),C_2\left(t\right),C_3\left(t\right),C_4\left(t\right)\right)$ is the translation vector and $\alpha\left(s\right)=\left(\alpha_1\left(s\right),\alpha_2\left(s\right),\alpha_3\left(s\right),\alpha_4\left(s\right)\right)$ is a profile curve. If we choose the profile curve α as $\alpha\left(s\right)=(u(s)\cos s,0,u(s)\sin s,0)$ and the translation vector $C\left(t\right)=\vec{0}$ in (3.1), we obtain the surface M as follows:

$$f\left(s,t\right)=(u(s)h(t)\cos s\cos t,u(s)h(t)\cos s\sin t,u(s)h(t)\sin s\cos t,u(s)h(t)\sin s\sin t) \tag{3.2}$$

Let M be a surface in $\mathbb{E}^4$ given by the parametrization (3.2). The tangent vectors of $f\left(s,t\right)$ can be easily computed as

$$\begin{aligned}\vec{v}_1 &= \frac{\partial f}{\partial t}=\left(A_1B_1',A_1B_2',A_2B_1',A_2B_2'\right),\\ \vec{v}_2 &= \frac{\partial f}{\partial s}=\left(\dot{A}_1B_1,\dot{A}_1B_2,\dot{A}_2B_1,\dot{A}_2B_2\right)\end{aligned}$$

and a basis of the normal space of $f\left(s,t\right)$ can be given as follows:

$$\begin{aligned}\vec{v}_3 &= \left(-A_2B_2,A_2B_1,A_1B_2,-A_1B_1\right),\\ \vec{v}_4 &= \left(-\dot{A}_2B_2',\dot{A}_2B_1',\dot{A}_1B_2',-\dot{A}_1B_1'\right),\end{aligned}$$

where

$$\begin{aligned}A_1 &= u(s)\cos s,\quad A_2= u(s)\sin s\\ B_1 &= h(t)\cos t,\quad B_2=h(t)\sin t\end{aligned}$$

and $\dot{A}_i=\frac{\partial A_i}{\partial s}$ for $i=1,2$ and $B_j'=\frac{\partial B_j}{\partial t}$ $j=1,2$. By using Gramm-Schmidth orthonormalization, the orthonormal vectors of tangent and normal spaces of M are obtained, respectively, by

$$\begin{aligned}e_1 &= \frac{1}{\sqrt{v_{11}}}\vec{v}_1,\\ e_2 &= \frac{1}{\sqrt{\left|v_{11}\left(v_{11}v_{22}-v_{12}^2\right)\right|}}\left(v_{11}\vec{v}_2-v_{12}\vec{v}_1\right)\end{aligned}$$

and

$$\begin{aligned} e_3 &= \frac{1}{\sqrt{v_{33}}}\vec{v}_3, \\ e_4 &= \frac{1}{\sqrt{\left|v_{33}\left(v_{33}v_{44}-v_{34}^2\right)\right|}}\left(v_{33}\vec{v}_4 - v_{34}\vec{v}_3\right), \end{aligned}$$

where

$$\begin{aligned} v_{11} &= \langle \vec{v}_1, \vec{v}_1 \rangle = u^2(s)\left(h^2(t) + (h'(t))^2\right), \\ v_{12} &= \langle \vec{v}_1, \vec{v}_2 \rangle = u(s)\,\dot{u}(s)\,h(t)\,h'(t), \\ v_{22} &= \langle \vec{v}_2, \vec{v}_2 \rangle = \left(u^2(s) + (\dot{u}(s))^2\right)h^2(t), \\ v_{33} &= \langle \vec{v}_3, \vec{v}_3 \rangle = u^2(s)\,h^2(t), \\ v_{34} &= \langle \vec{v}_3, \vec{v}_4 \rangle = u(s)\,\dot{u}(s)\,h(t)\,h'(t), \\ v_{44} &= \langle \vec{v}_4, \vec{v}_4 \rangle = \left(u^2(s) + (\dot{u}(s))^2\right)\left(h^2(t) + (h'(t))^2\right). \end{aligned}$$

Hence, $\{e_1, e_2, e_3, e_4\}$ is orthonormal moving frame on M. Then we have the dual 1-forms as:

$$\begin{aligned} \omega_1 &= \frac{\dot{u}hh'}{\left(h^2+(h')^2\right)^{\frac{1}{2}}}ds + \frac{u\left(h^2+(h')^2\right)}{\left(h^2+(h')^2\right)^{\frac{1}{2}}}dt \\ \omega_2 &= \frac{h\left(u^2h^2 + u^2(h')^2 + (\dot{u})^2h^2\right)^{\frac{1}{2}}}{\left(h^2+(h')^2\right)^{\frac{1}{2}}}ds \end{aligned}$$

By a direct computation we have components of the second fundamental form and the connection forms as:

$$h_{11}^3 = 0, \quad h_{12}^3 = -\frac{1}{W^{\frac{1}{2}}}, \quad h_{22}^3 = 2\frac{\dot{u}h'}{W} \tag{3.3}$$

$$\begin{aligned} h_{11}^4 &= \frac{\left(2(h')^2 - hh'' + h^2\right)}{\left(h^2+(h')^2\right)W^{\frac{1}{2}}}, \\ h_{12}^4 &= \frac{\dot{u}h'\left(hh'' - (h')^2\right)}{\left(h^2+(h')^2\right)W}, \\ h_{22}^4 &= \frac{\left(2(\dot{u})^2 - u\ddot{u} + u^2\right)\left(h^2+(h')^2\right)^2 - (\dot{u})^2(h')^2\left(hh''+h^2\right)}{\left(h^2+(h')^2\right)W^{\frac{3}{2}}} \end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
\omega_{12} &= -\frac{\dot{u}h\left(2\left(h'\right)^2 - hh'' + h^2\right)}{u\left(h^2+\left(h'\right)^2\right)^{\frac{3}{2}} W^{\frac{1}{2}}}\omega_1 \\
&\quad + \frac{u^2h'\left(h^2+\left(h'\right)^2\right)^2 + \left(\dot{u}\right)^2 h^2 h'\left(2\left(h'\right)^2 - hh'' + h^2\right)}{uh\left(h^2+\left(h'\right)^2\right)^{\frac{3}{2}} W}\omega_2, \qquad (3.5) \\
\omega_{34} &= \frac{\dot{u}h}{u\left(h^2+\left(h'\right)^2\right)^{\frac{1}{2}} W^{\frac{1}{2}}}\omega_1 + \frac{h'\left(u^2h^2 + u^2\left(h'\right)^2 - \left(\dot{u}\right)^2 h^2\right)}{uh\left(h^2+\left(h'\right)^2\right)^{\frac{1}{2}} W}\omega_2,
\end{aligned}
$$

where $W = u^2h^2 + u^2\left(h'\right)^2 + \left(\dot{u}\right)^2 h^2$.

Proposition 3.1. *Let M be the surface given by the parameterization (3.2). The Gaussian curvature and the normal bundle curvature of M are given, respectively, by*

$$
K = \frac{\left(2\left(h'\right)^2 - hh'' + h^2\right)\left(2\left(\dot{u}\right)^2 - u\ddot{u} + u^2\right) - \left(h^2+\left(h'\right)^2\right)\left(u^2+\left(\dot{u}\right)^2\right)}{W^2} \tag{3.6}
$$

and

$$
R^D = \frac{\left(2\left(\dot{u}\right)^2 - u\ddot{u} + u^2\right)\left(h^2+\left(h'\right)^2\right) - \left(2\left(h'\right)^2 - hh'' + h^2\right)\left(u^2+\left(\dot{u}\right)^2\right)}{W^2} \tag{3.7}
$$

Proof. By using (2.3), (2.4), (3.3) and (3.4), we obtain (3.6) and (3.7). □

Corollary 3.2. *Let M be the surface given by the parameterization (3.2). M is a flat surface with flat normal bundle if and only if it is parameterized by*

$$
f(t,s) = a_1a_2e^{k_1t+k_2s}\left(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t\right) \tag{3.8}
$$

or

$$
f(t,s) = \frac{c_1c_2}{\sqrt{\left|\cos\left(2t+b_1\right)\right|}\sqrt{\left|\cos\left(2s+b_2\right)\right|}}\left(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t\right) \tag{3.9}
$$

Proof. Let M be a flat surface with flat normal bundle. Then both $K=0$ and $R^D=0$. From (3.6), we have

$$
\frac{2\left(h'\right)^2 - hh'' + h^2}{h^2+\left(h'\right)^2}.\frac{2\left(\dot{u}\right)^2 - u\ddot{u} + u^2}{u^2+\left(\dot{u}\right)^2} = 1 \tag{3.10}
$$

and from (3.7), we get

$$\frac{2\left(h'\right)^2-hh''+h^2}{h^2+\left(h'\right)^2}=\frac{2\left(\dot{u}\right)^2-u\ddot{u}+u^2}{u^2\left(s\right)+\left(\dot{u}\right)^2}. \tag{3.11}$$

By combining (3.10) and (3.11) and solving these differential equations we obtain

$$h(t)=a_1e^{k_1t} \text{ and } u\left(s\right)=a_2e^{k_2s}$$

or

$$h(t)=\frac{c_1}{\sqrt{\left|\cos\left(2t+b_1\right)\right|}} \text{ and } u\left(s\right)=\frac{c_2}{\sqrt{\left|\cos\left(2s+b_2\right)\right|}},$$

where $a_1, a_2, b_1, b_2, c_1, c_2, k_1$ and k_2 are real constants. □

Remark 3.3. The surface M given by the parameterization (3.2) can be considered as the tensor product surface of two Euclidean planar curves, that is, let $\alpha:\mathbb{R}\to\mathbb{R}^2$, $\alpha(s)=(\alpha_1(s),\alpha_2(s))$ and $\beta:\mathbb{R}\to\mathbb{R}^2$, $\beta(t)=(\beta_1(t),\beta_2(t))$ be two Euclidean planar curves. The tensor product surface $f\left(t,s\right)$ is defined by

$$f=\alpha\otimes\beta:\mathbb{R}^2\to\mathbb{R}^4,$$

$$f\left(t,s\right)=\left(\alpha_1(s)\beta_1(t),\alpha_1(s)\beta_2(t),\alpha_2(s)\beta_1(t),\alpha_2(s)\beta_2(t)\right).$$

In particular, for the curves $\alpha(s)=\left(u\left(s\right)\cos s,u\left(s\right)\sin s\right)$ and $\beta(t)=\left(h(t)\cos t,h(t)\sin t\right)$ the tensor product of them gives the surface M given by the parameterization (3.2).

Theorem 3.4. *(See [9]). A regular tensor product surface $x(s,t)=\alpha(s)\otimes\beta(t)$ of two curves $\alpha:\mathbb{R}\to\mathbb{R}^2$, $\alpha(s)=\left(u\left(s\right)\cos s,u\left(s\right)\sin s\right)$ or $\beta:\mathbb{R}\to\mathbb{R}^2$, $\beta(t)=\left(h(t)\cos t,h(t)\sin t\right)$ is flat if and only if either*

1. α or β is a straight line through the origin.

2. α and β are sinusoidal spirals, that is, the curves α and β are parameterized by

$$\begin{aligned}\alpha(s) &= c_1\left|\cos\left(\left(a+1\right)s+b_1\right)\right|^{-\frac{1}{a+1}}\left(\cos s,\sin s\right)\\ \beta(t) &= c_2\left|\cos\left(\left(\frac{1}{a}+1\right)t+b_2\right)\right|^{-\frac{1}{\frac{1}{a}+1}}\left(\cos t,\sin t\right)\end{aligned}$$

3. α and β are logarithmic spirals, that is, the curves α and β are parameterized by

$$\alpha(s)=a_1e^{k_1s}\left(\cos s,\sin s\right) \text{ and } \beta(t)=a_2e^{k_2t}\left(\cos t,\sin t\right)$$

with $a_1, a_2, b_1, b_2, c_1, c_2, k_1$ and k_2 are real constants, $a_1, a_2, c_1, c_2>0$ and $a\neq -1$.

Remark 3.5. In [8] Yoon studied Vranceanu surface parameterized by

$$f(s,t) = (u(s)\cos s\cos t, u(s)\cos s\sin t, u(s)\sin s\cos t, u(s)\sin s\sin t).$$

He proved that flat Vranceanu surface in E^4 has pointwise 1-type Gauss map if and only if it is a Clifford torus. Also the normal bundle of flat Vranceanu surface is flat, too.

Now we investigate flat surface M with flat normal bundle with pointwise 1-type Gauss map.

Theorem 3.6. *Let M be flat surface with flat normal bundle given by the parameterization (3.2). Then M has pointwise 1-type Gauss map if and only if either*

1. M is a Clifford torus, that is, the product of two plane circles with same radius

2. It is the product of two logarithmic spirals which is parameterized by

$$f(t,s) = e^{k(t\pm s)}(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t)$$

where k is non zero real constant.

Proof. Firstly, we assume that the flat surface M with flat normal bundle given by the parameterization (3.8) has pointwise 1-type Gauss map. If necessary, by an appropriate homothetic transformation we may assume that $a_1 = a_2 = 1$. Then we have $h(t) = e^{k_1 t}$ and $u(s) = e^{k_2 s}$. By using (3.3), (3.4) and (3.5) we have components of the second fundamental form and the connection forms as:

$$h_{11}^3 = 0, \quad h_{12}^3 = -\alpha(s,t), \quad h_{22}^3 = a\alpha(s,t)$$
$$h_{11}^4 = \alpha(s,t), \quad h_{12}^4 = 0, \quad h_{22}^4 = \alpha(s,t)$$

and

$$\begin{aligned}\omega_{12} &= b\alpha(s,t)\,\omega_1 + c\alpha(s,t)\,\omega_2, \quad \omega_{13} = -\alpha(s,t)\,\omega_2, \quad \omega_{14} = \alpha(s,t)\,\omega_1\\ \omega_{23} &= -\alpha(s,t)\,\omega_1 + a\alpha(s,t)\,\omega_2 \quad \omega_{24} = \alpha(s,t)\,\omega_2, \quad \omega_{34} = -b\alpha(s,t)\,\omega_1 + d\alpha(s,t)\,\omega_2,\end{aligned}$$

By covariant differentiation with respect to e_1 and e_2 a straightforward calculation gives:

$$\begin{aligned}\tilde{\nabla}_{e_1}e_1 &= b\alpha e_2 + \alpha e_4,\\ \tilde{\nabla}_{e_2}e_1 &= c\alpha e_2 - \alpha e_3,\\ \tilde{\nabla}_{e_1}e_2 &= -b\alpha e_1 - \alpha e_3,\\ \tilde{\nabla}_{e_2}e_2 &= -c\alpha e_1 + a\alpha e_3 + \alpha e_4,\\ \tilde{\nabla}_{e_1}e_3 &= \alpha e_2 - b\alpha e_4,\\ \tilde{\nabla}_{e_2}e_3 &= \alpha e_1 - a\alpha e_2 + d\alpha e_4\\ \tilde{\nabla}_{e_1}e_4 &= -\alpha e_1 + b\alpha e_3,\\ \tilde{\nabla}_{e_2}e_4 &= -\alpha e_2 - d\alpha e_3,\end{aligned} \tag{3.12}$$

where

$$\alpha(s,t) = \frac{1}{u(s)h(t)\left(1+k_1^2+k_2^2\right)^{\frac{1}{2}}},\ a = \frac{2k_1k_2}{\left(1+k_1^2+k_2^2\right)^{\frac{1}{2}}},\ b = -\frac{k_2}{\left(1+k_1^2\right)^{\frac{1}{2}}},$$
$$c = \frac{k_1\left(1+k_1^2+k_2^2\right)^{\frac{1}{2}}}{\left(1+k_1^2\right)^{\frac{1}{2}}},\ d = \frac{k_1\left(1+k_1^2-k_2^2\right)}{\left(1+k_1^2\right)^{\frac{1}{2}}\left(1+k_1^2+k_2^2\right)^{\frac{1}{2}}} \tag{3.13}$$

By using (2.1) and (3.12) and after straight-forward computations, the Laplacian ΔG of the Gauss map G can be expressed as

$$\Delta G = \left(4+a^2\right)\alpha^2 e_1\wedge e_2 + (c+d)\alpha^2 e_1\wedge e_3 - (2b+ad)\alpha^2 e_1\wedge e_4 + (2b-ac)\alpha^2 e_2\wedge e_3 - (c+d)\alpha^2 e_2\wedge e_4. \tag{3.14}$$

We suppose that the flat surface M with flat normal bundle has pointwise 1-type Gauss map. From (1.1) and (3.14), we get

$$\left(4+a^2\right)\alpha^2 = f + f\langle C, e_1\wedge e_2\rangle \tag{3.15}$$

$$(c+d)\alpha^2 = f\langle C, e_1\wedge e_3\rangle \tag{3.16}$$

$$(-2b-ad)\alpha^2 = f\langle C, e_1\wedge e_4\rangle \tag{3.17}$$

$$(2b-ac)\alpha^2 = f\langle C, e_2\wedge e_3\rangle \tag{3.18}$$

$$-(c+d)\alpha^2 = f\langle C, e_2\wedge e_4\rangle \tag{3.19}$$

Then, we have

$$\langle C, e_3\wedge e_4\rangle = 0 \tag{3.20}$$

By differentiating (3.20) with respect to e_1, we get

$$\langle C, e_1\wedge e_3\rangle + \langle C, e_2\wedge e_4\rangle = 0 \tag{3.21}$$

When we take the derivative of (3.20) with respect to e_2, we have

$$\langle C, e_1\wedge e_4\rangle + \langle C, e_2\wedge e_3\rangle - a\langle C, e_2\wedge e_4\rangle = 0 \tag{3.22}$$

If we evaluate the derivative of (3.22) with respect to e_2 again, we get

$$2\langle C, e_1\wedge e_2\rangle = -(c+d)\langle C, e_1\wedge e_3\rangle + ac\langle C, e_1\wedge e_4\rangle + ad\langle C, e_2\wedge e_3\rangle + (c+d)\langle C, e_2\wedge e_4\rangle \tag{3.23}$$

By using (3.15), (3.16), (3.17), (3.18), (3.19), (3.21) and (3.23) we then have

$$f = \left(4+a^2+(c+d)^2+abc+a^2cd-abd\right)\alpha^2 = A\alpha^2 \tag{3.24}$$

that is, a smooth function f depends on s and t. Differentiating (3.24) with respect to e_1, we have

$$e_1(f) = -2cA\alpha^3. \tag{3.25}$$

On the other hand, by differentiating (3.19) with respect to e_1 and by using (3.12), (3.15), (3.17), (3.18), (3.19), (3.24) and (3.25) we obtain

$$4b^2 + 2abd - 2abc - a^2cd - (c+d)^2 = 0. \tag{3.26}$$

By substituting (3.13) into (3.26) we get

$$\left(k_1^2 - k_2^2\right)\left(1 + k_1^2 + k_2^2 + k_1^2k_2^2\right) = 0 \tag{3.27}$$

and from (3.27) we obtain that $k_1 = \pm k_2$. In particular, if we take as $k_1 = k_2 = 0$, we obtain Clifford torus. For the other cases, we obtain the tensor product surface of two logarithmic spirals.

Conversely, we assume that $k_1^2 = k_2^2$. In that case the flat surface M with flat normal bundle is given by the parametrization (3.8) has pointwise 1-type Gauss map for the function

$$f(s,t) = \left(4 + a^2 + (c+d)^2 + abc + a^2cd - abd\right)\alpha^2 = A\alpha^2$$

and the constant vector

$$\begin{aligned} C =& \frac{1}{A}\left(\left(4 + a^2 - A\right)e_1 \wedge e_2 + (c+d)\,e_1 \wedge e_3 - (2b + ad)\,e_1 \wedge e_4\right) \\ &+ \frac{1}{A}\left((2b - ac)\,e_2 \wedge e_3 - (c+d)\,e_2 \wedge e_4\right). \end{aligned}$$

Now, we assume that the flat surface M with flat normal bundle is given by the parametrization (3.9). We research whether this surface has pointwise 1-type Gauss map. We can write as

$$u(s) = c_1\left(\varepsilon \cos(2s)\right)^{-\frac{1}{2}},$$

where if $\cos(2s) > 0$ (resp. < 0), then $\varepsilon = 1$ (resp. $= -1$). Analogously, we can write as

$$h(t) = c_2\left(\delta \cos(2t)\right)^{-\frac{1}{2}},$$

where if $\cos(2t) > 0$ (< 0, respectively) then $\delta = 1(-1$,respectively). By using (3.3), (3.4) and (3.5) we have components of the second fundamental form and the connection forms as:

$$h_{11}^3 = 0, \quad h_{12}^3 = -\lambda(s,t), \quad h_{22}^3 = \varkappa(s,t)\lambda(s,t)$$

$$h_{11}^4 = -\lambda(s,t), \quad h_{12}^4 = \varkappa(s,t)\lambda(s,t), \quad h_{22}^4 = -\left(1 + \varkappa^2(s,t)\right)\lambda(s,t)$$

and

$$\begin{aligned} \omega_{12} &= \tau(s,t)\,\lambda(s,t)\,\omega_1 + \beta(s,t)\,\lambda^2(s,t)\,\omega_2 \\ \omega_{34} &= \tau(s,t)\,\lambda(s,t)\,\omega_1 + \beta(s,t)\,\lambda^2(s,t)\,\omega_2. \end{aligned}$$

By covariant differentiation with respect to e_1 and e_2, we get

$$\begin{aligned}
\tilde{\nabla}_{e_1} e_1 &= \tau\lambda e_2 - \lambda e_4, \\
\tilde{\nabla}_{e_2} e_1 &= \beta\lambda^2 e_2 - \lambda e_3 + \varkappa\lambda e_4, \\
\tilde{\nabla}_{e_1} e_2 &= -\tau\lambda e_1 - \lambda e_3 + \varkappa\lambda e_4, \\
\tilde{\nabla}_{e_2} e_2 &= -\beta\lambda^2 e_1 + \varkappa\lambda e_3 - \left(1+\varkappa^2\right)\lambda e_4, \\
\tilde{\nabla}_{e_1} e_3 &= \lambda e_2 + \tau\lambda e_4, \\
\tilde{\nabla}_{e_2} e_3 &= \lambda e_1 - \varkappa\lambda e_2 + \beta\lambda^2 e_4 \\
\tilde{\nabla}_{e_1} e_4 &= \lambda e_1 - \varkappa\lambda e_2 - \tau\lambda e_3, \\
\tilde{\nabla}_{e_2} e_4 &= -\varkappa\lambda e_1 + \left(1+\varkappa^2\right)\lambda e_2 - \beta\lambda^2 e_3,
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
\varkappa(s,t) &= \frac{2\left(\varepsilon\sin(2s)\right)\left(\delta\sin(2t)\right)}{\left(1-\left(\varepsilon\sin(2s)\right)^2\left(\delta\sin(2t)\right)^2\right)^{\frac{1}{2}}}, \\
\tau(s,t) &= \frac{\left(\varepsilon\sin(2s)\right)\left(\delta\cos(2t)\right)}{\left(\varepsilon\cos(2s)\right)}, \\
\beta(s,t) &= \frac{c_1c_2\left(\delta\sin(2t)\right)\left(\left(\varepsilon\cos(2s)\right)^2-\left(\varepsilon\sin(2s)\right)^2\left(\delta\cos(2t)\right)^2\right)}{\left(\varepsilon\cos(2s)\right)^{\frac{5}{2}}\left(\delta\cos(2t)\right)^{\frac{5}{2}}}, \\
\lambda(s,t) &= \frac{1}{W^{\frac{1}{2}}} = \frac{\left(\varepsilon\cos(2s)\right)^{\frac{3}{2}}\left(\delta\cos(2t)\right)^{\frac{3}{2}}}{c_1c_2\left(1-\left(\varepsilon\sin(2s)\right)^2\left(\delta\sin(2t)\right)^2\right)^{\frac{1}{2}}}.
\end{aligned}$$

By using (2.1), straight-forward computation the Laplacian ΔG of the Gauss map G can be expressed as

$$\begin{aligned}
\Delta G = {} & \left(4+5\varkappa^2+\varkappa^4\right)\lambda^2 e_1\wedge e_2 + \left(-e_2\left(\varkappa\lambda\right) - \beta\left(2+\varkappa^2\right)\lambda^3\right)e_1\wedge e_3 \\
& + \left(e_2\left(\left(2+\varkappa^2\right)\lambda\right) - \beta\varkappa\lambda^3\right)e_1\wedge e_4 \\
& + \left(e_1\left(\varkappa\lambda\right) + \tau\left(2+\varkappa^2\right)\lambda^2\right)e_2\wedge e_3 + \left(-e_1\left(\left(2+\varkappa^2\right)\lambda\right) + \tau\varkappa\lambda^2\right)e_2\wedge e_4.
\end{aligned} \tag{3.29}$$

We suppose that the flat surface M with flat normal bundle has pointwise 1-type Gauss map. From (1.1) and (3.29), we get

$$\left(4+5\varkappa^2+\varkappa^4\right)\lambda^2 = f + f\left\langle C, e_1\wedge e_2\right\rangle, \tag{3.30}$$

$$-e_2\left(\varkappa\lambda\right) - \beta\left(2+\varkappa^2\right)\lambda^3 = f\left\langle C, e_1\wedge e_3\right\rangle, \tag{3.31}$$

$$e_2\left(\left(2+\varkappa^2\right)\lambda\right) - \beta\varkappa\lambda^3 = f\left\langle C, e_1\wedge e_4\right\rangle, \tag{3.32}$$

$$e_1\left(\varkappa\lambda\right) + \tau\left(2+\varkappa^2\right)\lambda^2 = f\left\langle C, e_2\wedge e_3\right\rangle, \tag{3.33}$$

$$-e_1\left(\left(2+\varkappa^2\right)\lambda\right) + \tau\varkappa\lambda^2 = f\left\langle C, e_2\wedge e_4\right\rangle. \tag{3.34}$$

Then we have

$$\left\langle C, e_3\wedge e_4\right\rangle = 0. \tag{3.35}$$

By differentiating (3.35) with respect to e_1, we get

$$\langle C, e_2 \wedge e_4\rangle - \langle C, e_1 \wedge e_3\rangle + \varkappa \langle C, e_2 \wedge e_3\rangle = 0. \tag{3.36}$$

By considering together with (3.31), (3.33), (3.34) and (3.36), we have

$$\begin{aligned}-e_1\left(\left(2+\varkappa^2\right)\lambda\right) + \tau\varkappa\lambda^2 + \varkappa\left(e_1\left(\varkappa\lambda\right) + \tau\left(2+\varkappa^2\right)\lambda^2\right) \\ + e_2\left(\varkappa\lambda\right) + \beta\left(2+\varkappa^2\right)\lambda^3 = 0.\end{aligned} \tag{3.37}$$

On the other hand, after some long computations we have

$$e_1\left(\varkappa\right) = \frac{4\left(\varepsilon\sin\left(2s\right)\right)\left(\varepsilon\cos\left(2s\right)\right)^{\frac{1}{2}}\left(\delta\cos\left(2t\right)\right)^{\frac{5}{2}}}{c_1c_2\left(1-\left(\varepsilon\sin\left(2s\right)\right)^2\left(\delta\sin\left(2t\right)\right)^2\right)^{\frac{3}{2}}}, \tag{3.38}$$

$$\begin{aligned}e_2\left(\varkappa\right) =& \frac{4\lambda\left(\varepsilon\cos\left(2s\right)\right)\left(\delta\sin\left(2t\right)\right)\left(\delta\cos\left(2t\right)\right)^{-1}}{\left(1-\left(\varepsilon\sin\left(2s\right)\right)^2\left(\delta\sin\left(2t\right)\right)^2\right)^{\frac{3}{2}}} \\ &- \frac{4\lambda\left(\varepsilon\sin\left(2s\right)\right)^2\left(\varepsilon\cos\left(2s\right)\right)^{-1}\left(\delta\sin\left(2t\right)\right)\left(\delta\cos\left(2t\right)\right)}{\left(1-\left(\varepsilon\sin\left(2s\right)\right)^2\left(\delta\sin\left(2t\right)\right)^2\right)^{\frac{3}{2}}},\end{aligned} \tag{3.39}$$

$$\begin{aligned}e_1(\lambda) = \frac{((\varepsilon\cos(2s))^2(\delta\sin(2t))(\delta\cos(2t))^2)}{c_1^2c_2^2(1-(\varepsilon\sin(2s))^2(\delta\sin(2t))^2)^{\frac{3}{2}}}\Big(-3+2(\varepsilon\sin(2s))^2 \\ +(\varepsilon\sin(2s))^2(\delta\sin(2t))^2\Big)\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}e_2(\lambda) =& \zeta\Big((-3+2(\delta\sin(2t))^2+(\varepsilon\sin(2s))^2(\delta\sin(2t))^2) \\ &-(\delta\sin(2t))^2(-3+2(\varepsilon\sin(2s))^2+(\varepsilon\sin(2s))^2(\delta\sin(2t))^2)\Big), \\ \zeta =& \frac{\lambda(\varepsilon\sin(2s)(\varepsilon\cos(2s))^{\frac{1}{2}}(\delta\cos(2t))^{\frac{1}{2}})}{c_1c_2(1-(\varepsilon\sin(2s))^2(\delta\sin(2t))^2)^{\frac{3}{2}}}.\end{aligned} \tag{3.41}$$

By combining (3.38), (3.39), (3.40) and (3.41) with (3.37), we obtain that this equation is not satisfied. So, there is no flat surface with flat normal bundle given by the parameterization (3.9) which has pointwise 1-type Gauss map. □

Corollary 3.7. *Let M be flat surface with flat normal bundle given by the parameterization (3.2). M has pointwise 1-type Gauss map of the first kind if and only if it is a Clifford Torus.*

Proof. From Theorem 3.6 the flat surface M with flat normal bundle is given by the parameterization (3.2) has pointwise 1-type Gauss map for the function

$$f\left(s,t\right) = A\alpha^2$$

and the constant vector

$$C = \frac{1}{A}\Big((4+a^2-A)e_1\wedge e_2 + (c+d)e_1\wedge e_3 - (2b+ad)e_1\wedge e_4\Big) + \frac{1}{A}\Big((2b-ac)e_2\wedge e_3 - (c+d)e_2\wedge e_4\Big)$$

with $k_1^2 = k_2^2$, where

$$A = \Big(4+a^2+(c+d)^2+abc+a^2cd-abd\Big).$$

We assume that the surface M has pointwise 1-type Gauss map of the first kind. Then, we obtain $C=0$, that is, all components of C is zero. Then, we get $k_1 = k_2 = 0$. This completes the proof. □

Theorem 3.8. *An oriented minimal surface M in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M has a flat normal bundle [6].*

Theorem 3.9. *There exists no minimal surface given by the parameterization (3.2) with pointwise 1-type Gauss map of the first kind.*

Proof. We suppose that the surface M given by the parameterization (3.2) is minimal surface with pointwise 1-type Gauss map of the first kind. From Theorem 3.8 we have $R^D = 0$. Since the surface M is minimal and its normal bundle is flat then (2.2) and (2.4) imply, respectively

$$h_{11}^3 + h_{22}^3 = 0 \text{ and } h_{11}^4 + h_{22}^4 = 0 \tag{3.42}$$

$$h_{12}^3\left(h_{11}^4 - h_{22}^4\right) + h_{12}^4\left(h_{22}^3 - h_{11}^3\right) = 0. \tag{3.43}$$

By combining (3.3), (3.4), (3.42) and (3.43) we have

$$h_{22}^3 = h_{11}^4 = h_{22}^4 = 0. \tag{3.44}$$

The equation (3.44) conflicts with the regularity of the surface. □

Theorem 3.10. *(See [6]). A non-planar minimal oriented surface M in the Euclidean space E^4 has pointwise 1-type Gauss map of the second kind if and only if, with respect to some suitable local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on M, the shape operators of M are given by*

$$A_3 = \begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix} \text{ and } A_4 = \begin{pmatrix} 0 & \varepsilon\rho \\ \varepsilon\rho & 0 \end{pmatrix},$$

where $\varepsilon = \pm 1$ and ρ is a smooth non-zero function on M.

Theorem 3.11. *Let M be minimal surface given by the parameterization (3.2). Then M has pointwise 1-type Gauss map of the second kind if and only if it is parametrized by*

$$f(t,s) = \frac{bd}{\sqrt{|\cos(2s+c)|}}(\cos s\cos t, \cos s\sin t, \sin s\cos t, \sin s\sin t)$$

or

$$f(t,s) = \frac{bd}{\sqrt{|\cos(2t+c)|}} (\cos s \cos t, \cos s \sin t, \sin s \cos t, \sin s \sin t)$$

where b, d and c are real constants.

Proof. We assume that M is a minimal surface with pointwise 1-type Gauss map of second kind. In that case the mean curvature of M is zero and we have

$$h_{22}^3 = 0 \tag{3.45}$$

and

$$h_{11}^4 + h_{22}^4 = 0. \tag{3.46}$$

By using (2.5), the Laplacian ΔG of the Gauss map G is written as

$$\Delta G = \|h\|^2 G + 2R^D e_3 \wedge e_4, \tag{3.47}$$

where $R^D \neq 0$. In the opposite case, M has pointwise 1-type Gauss map of the first kind. By using (2.4), (3.45) and (3.46) we get

$$R^D = 2h_{12}^3 h_{11}^4 \neq 0. \tag{3.48}$$

Since M has pointwise 1-type Gauss map of the second kind, from (1.1) and (3.47) we have

$$\|h\|^2 G + 2R^D e_3 \wedge e_4 = fG + fC \tag{3.49}$$

for some smooth non-zero function f on M and some constant vector C. Since the vector C is a linear combination of $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, $e_2 \wedge e_3$, $e_2 \wedge e_4$, $e_3 \wedge e_4$. From (3.49) we get

$$\|h\|^2 = f(1 + \langle C, e_1 \wedge e_2 \rangle) \tag{3.50}$$

$$2R^D = f \langle C, e_3 \wedge e_4 \rangle \neq 0 \tag{3.51}$$

and

$$\langle C, e_1 \wedge e_3 \rangle = \langle C, e_1 \wedge e_4 \rangle = \langle C, e_2 \wedge e_3 \rangle = \langle C, e_2 \wedge e_4 \rangle = 0$$

Since h_{12}^3 is not equal to zero on M, it follows that$\|h\| \neq 0$ or $\langle C, e_1 \wedge e_2 \rangle \neq -1$. Differentiating $\langle C, e_1 \wedge e_3 \rangle = 0$ with respect to e_1 and e_2, we get

$$h_{12}^3 \langle C, e_1 \wedge e_2 \rangle + h_{11}^4 \langle C, e_3 \wedge e_4 \rangle = 0 \tag{3.52}$$

and

$$h_{12}^4 \langle C, e_3 \wedge e_4 \rangle = 0, \tag{3.53}$$

respectively. On the other hand, differentiating $\langle C, e_1 \wedge e_4 \rangle = 0$ with respect to e_1 and e_2, we have

$$h_{12}^4 \langle C, e_1 \wedge e_2 \rangle = 0 \tag{3.54}$$

and

$$h_{12}^3 \langle C, e_3 \wedge e_4 \rangle + h_{11}^4 \langle C, e_1 \wedge e_2 \rangle = 0, \tag{3.55}$$

respectively. The equation (3.54) implies that $h_{12}^4 = 0$ or $\langle C, e_1 \wedge e_2 \rangle = 0$. If $\langle C, e_1 \wedge e_2 \rangle = 0$ then from (3.55) we get $h_{12}^3 \langle C, e_3 \wedge e_4 \rangle = 0$. h_{12}^3 is not equal to zero on M. Hence we have $\langle C, e_3 \wedge e_4 \rangle = 0$ and (3.51) implies that $R^D = 0$. This is a contradiction. So $\langle C, e_1 \wedge e_2 \rangle \neq 0$ and $h_{12}^4 = 0$. By using (3.52) and (3.55) we obtain

$$\left(h_{12}^3\right)^2 = \left(h_{11}^4\right)^2. \tag{3.56}$$

From (3.45) and (3.3) we get $\dot{u} = 0$ or $h' = 0$. Firstly we assume that $h' \neq 0$. Then we have $u = d =$constant. By considering together with (3.3), (3.4), (3.46) and (3.56) we obtain

$$h(t) = \frac{c}{\sqrt{|\cos(2t+b)|}}.$$

Now we assume that $\dot{u} \neq 0$. Then we have $h = d =$ constant. By using (3.3) and (3.4) with $h = d$, we can see that (3.56) is satisfied directly. So, if we consider (3.4), (3.46) for $h = d$ we obtain

$$u(s) = \frac{c}{\sqrt{|\cos(2s+b)|}},$$

where b, c and d are real constants.

If we consider as both $\dot{u} = 0$ and $h' = 0$, then the surface M is not minimal surface.

On the other hand by using (3.47), (3.48), (3.50), (3.51), (3.55) and (3.56) (or see the proof of Theorem 5 in [6]) we can find the function f and the constant vector C as

$$f(s) = 8\left(h_{12}^3\right)^2 \tag{3.57}$$

and

$$C = -\frac{e_1 \wedge e_2}{2} + \varepsilon \frac{e_3 \wedge e_4}{2}. \tag{3.58}$$

Hence the minimal surface M has pointwise 1-type Gauss map of the second kind for the function f and the constant vector C given by (3.57) and (3.58), respectively. This completes the proof. □

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Rotational Surfaces with Pointwise 1-Type Gauss Map in Pseudo Euclidean Space $\mathbb{E}_2^4$

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Abstract. In this paper, we study rotational surfaces of elliptic, hyperbolic and parabolic type with pointwise 1-type Gauss map which have spacelike profile curve in four dimensional pseudo Euclidean space $\mathbb{E}_2^4$ and obtain some characterizations for these rotational surfaces to have pointwise 1-type Gauss map.
Keywords. Pseudo-Euclidean space · Rotational surfaces of elliptic, hyperbolic and parabolic type · Gauss map · Pointwise 1-type Gauss map.
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1 Introduction

The Gauss map G of a submanifold M into $G(n,m)$ in $\wedge^n \mathbb{E}_s^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented $n-$planes through the origin of $\mathbb{E}_s^m$ and $\wedge^n \mathbb{E}_s^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}_s^m$ is a smooth map which carries a point p in M into the oriented $n-$plane in $\mathbb{E}_s^m$ obtained from parallel translation of the tangent space of M at p in $\mathbb{E}_s^m$. Since the vector space $\wedge^n \mathbb{E}_s^m$ identify with a semi-Euclidean space $\mathbb{E}_t^N$ for some positive integer t, where $N = \begin{pmatrix} m \\ n \end{pmatrix}$, the Gauss map is defined by $G : M \to G(n,m) \subset \mathbb{E}_t^N$, $G(p) = (e_1 \wedge ... \wedge e_n)(p)$. The notion of submanifolds with finite type Gauss map was introduced by B. Y.Chen and P.Piccinni in 1987 [6] and after then many works were done about this topic, especially 1-type Gauss map and 2- type Gauss map.

If a submanifold M of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map G, then G satisfies

$$\Delta G = \lambda (G + C)$$

for some $\lambda \in \mathbb{R}$ and some constant vector C.

On the other hand the Laplacian of the Gauss map of some typical well-known surfaces satisfies the form

$$\Delta G = f\,(G + C) \tag{1.1}$$

for some smooth function f on M and some constant vector C. A submanifold of a Euclidean space or pseudo-Euclidean space is said to have pointwise 1-type Gauss map, if its Gauss map satisfies (1.1) for some smooth function f on M and some constant vector C. If the vector C in (1.1) is zero, a submanifold with pointwise 1-type Gauss map is said to be of the first kind, otherwise it is said to be of the second kind.

A lot of papers were recently published about rotational surfaces with pointwise 1-type Gauss map in four dimensional Euclidean and pseudo Euclidean space in [1],[3],[4], [8], [9] [11].Timelike and spacelike rotational surfaces of elliptic, hyperbolic and parabolic types in Minkowski space $\mathbb{E}^4_1$ with pointwise 1-type Gauss map were studied in [5, 7]. Aksoyak and Yaylı in [2] studied boost invariant surfaces (rotational surfaces of hyperbolic type) with pointwise 1-type Gauss map in Minkowski space $\mathbb{E}^4_1$. They gave a characterization for flat boost invariant surfaces with pointwise 1-type Gauss map. Also they obtain some results for boost invariant marginally trapped surfaces with pointwise 1-type Gauss map. Ganchev and Milousheva in [10] defined three types of rotational surfaces with two dimensional axis rotational surfaces of elliptic, hyperbolic and parabolic type in pseudo Euclidean space $\mathbb{E}^4_2$. They classify all rotational marginally trapped surfaces of elliptic, hyperbolic and parabolic type, respectively.

In this paper, we study rotational surfaces of elliptic, hyperbolic and parabolic type with pointwise 1-type Gauss map which have spacelike profile curve in four dimensional pseudo Euclidean space and give all classifications of flat rotational surfaces of elliptic, hyperbolic and parabolic type with pointwise 1-type Gauss map.

2 Preliminaries

Let $\mathbb{E}^m_s$ be the $m-$dimensional pseudo-Euclidean space with signature $(s, m-s)$. Then the metric tensor g in $\mathbb{E}^m_s$ has the form

$$g = \sum_{i=1}^{m-s} (dx_i)^2 - \sum_{i=m-s+1}^{m} (dx_i)^2$$

where $(x_1, \ldots, x_m)$ is a standard rectangular coordinate system in $\mathbb{E}^m_s$.

A vector v is called spacelike (resp., timelike) if $\langle v, v\rangle > 0$ (resp., $\langle v, v\rangle < 0$). Avector v is called lightlike if it $v \neq 0$ and $\langle v, v\rangle = 0$, where $\langle , \rangle$ is indefinite inner scalar product with respect to g.

Let M be an $n-$dimensional pseudo-Riemannian submanifold of a $m-$dimensional pseudo-Euclidean space $\mathbb{E}_s^m$ and denote by $\tilde{\nabla}$ and ∇ Levi-Civita connections of $\mathbb{E}_s^m$ and M , respectively. We choose local orthonormal frame $\{e_1, \ldots, e_n, \ \ e_{n+1}, \ldots, e_m\}$ on M with $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$ such that $e_1, \ldots e_n$ are tangent to M and $e_{n+1}, \ldots, e_m$ are normal to M. We use the following convention on the ranges of indices: $1 \leq i, j, k, \ldots \leq n$, $n+1 \leq r, s, t, \ldots \leq m$, $1 \leq A, B, C, \ldots \leq m$.

Denote by ω_A the dual-1 form of e_A such that $\omega_A(X) = \langle e_A, X \rangle$ and ω_{AB} the connection forms defined by

$$de_A = \sum_B \varepsilon_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0.$$

Then the formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{m} \varepsilon_r h_{ik}^r e_r$$

and

$$\tilde{\nabla}_{e_k} e_s = -\sum_{j=1}^{n} \varepsilon_j h_{kj}^s e_j + D_{e_k} e_s, \quad D_{e_k} e_s = \sum_{r=n+1}^{m} \varepsilon_r \omega_{sr}(e_k) e_r,$$

where D is the normal connection, h_{ik}^r the coefficients of the second fundamental form h.

For any real function f on M, the Laplacian operator of M with respect to induced metric is given by

$$\Delta f = -\varepsilon_i \sum_i \left(\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla_{e_i} e_i} f \right). \tag{2.1}$$

The mean curvature vector H and the Gaussian curvature $K$of M in $\mathbb{E}_s^m$ are defined by

$$H = \frac{1}{n} \sum_{s=n+1}^{m} \sum_{i=1}^{n} \varepsilon_i \varepsilon_s h_{ii}^s e_s \tag{2.2}$$

and

$$K = \sum_{s=n+1}^{m} \varepsilon_s \left(h_{11}^s h_{22}^s - h_{12}^s h_{21}^s \right), \tag{2.3}$$

respectively. We recall that a surface M is called minimal if its mean curvature vector vanishes identically, i.e. $H = 0$. If the mean curvature vector satisfies $DH = 0$, then the surface M is said to have parallel mean curvature vector. Also if Gaussian curvature of M vanishes identically, i.e. $K = 0$, the surface M is called flat.

3 Rotational Surfaces with Pointwise 1-Type Gauss Map in $\mathbb{E}_2^4$

In this section, we consider rotational surfaces of elliptic, hyperbolic and parabolic type in four dimensional pseudo-Euclidean space $\mathbb{E}_2^4$ which are defined by Ganchev and Milousheva in [10] and investigate these rotational surfaces with pointwise 1-type Gauss map.

We denote the standart orthonormal basis of $\mathbb{E}_2^4$ by $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ where $\epsilon_1 = (1,0,0,0)$, $\epsilon_2 = (0,1,0,0)$, $\epsilon_3 = (0,0,1,0)$ and $\epsilon_4 = (0,0,0,1)$, and $\langle \epsilon_1, \epsilon_1 \rangle = \langle \epsilon_2, \epsilon_2 \rangle = 1$, $\langle \epsilon_3, \epsilon_3 \rangle = \langle \epsilon_4, \epsilon_4 \rangle = -1$.

3.1 Rotational surfaces of elliptic type with pointwise 1-type Gauss map in $\mathbb{E}_2^4$

In this subsection, first we consider the rotational surfaces of elliptic type with harmonic Gauss map.Then, we give a characterization of the flat rotational surfaces of elliptic type with pointwise 1-type Gauss map and obtain a relationship for non-minimal these surfaces with parallel mean curvature vector and pointwise 1-type Gauss map of the first kind.

Rotational surface of elliptic type M_1 is defined by

$$\varphi(t,s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \\ 0 \end{pmatrix}$$

$$M_1 : \ \varphi(t,s) = (x_1(s), x_2(s), x_3(s)\cos t, x_3(s)\sin t), \tag{3.1}$$

where the surface M_1 is obtained by the rotation of the curve

$$x(s) = (x_1(s), x_2(s), x_3(s), 0)$$

about the two dimensional Euclidean plane span$\{\epsilon_1, \epsilon_2\}$. Let the profile curve of M_1 be unit speed spacelike curve. In that case, $(x_1{}'(s))^2 + (x_2{}'(s))^2 - (x_3{}'(s))^2 = 1$. We suppose that $x_3(s) > 0$. The moving frame field $\{e_1, e_2, e_3, e_4\}$ on M_1 is determined as follows:

$$\begin{aligned}
e_1 &= (x_1'(s), x_2'(s), x_3'(s)\cos t, x_3'(s)\sin t), \\
e_2 &= (0, 0, -\sin t, \cos t), \\
e_3 &= \frac{1}{\sqrt{1 + x_3'(s)^2}} (-x_2'(s), x_1'(s), 0, 0), \\
e_4 &= \frac{1}{\sqrt{1 + x_3'(s)^2}} \Big(x_3'(s)x_1'(s), x_3'(s)x_2'(s), (1 + x_3'(s)^2)\cos t, \\
&\quad (1 + x_3'(s)^2)\sin t \Big),
\end{aligned}$$

where e_1, e_2 and e_3, e_4 are tangent vector fields and normal vector fields to M_1, respectively.Then it is easily seen that

$$\langle e_1, e_1\rangle = \langle e_3, e_3\rangle = 1, \ \ \langle e_2, e_2\rangle = \langle e_4, e_4\rangle = -1.$$

We have the dual 1-forms as:

$$\omega_1 = ds \qquad \text{and} \qquad \omega_2 = -x_3(s)dt.$$

After some computations, the components of the second fundamental form and the connection forms are given as follows:

$$\begin{aligned} h_{11}^3 &= -\, d(s),\ h_{12}^3 = 0,\ h_{22}^3 = 0, \\ h_{11}^4 &= -\, c(s), \ \ h_{12}^4 = 0, \ \ h_{22}^4 = b(s) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \omega_{12} =& a(s)b(s)\omega_2, \quad \omega_{13} = -d(s)\omega_1, \quad \omega_{14} = -c(s)\omega_1, \\ \omega_{23} =& 0, \quad \omega_{24} = -b(s)\omega_2, \quad \omega_{34} = a(s)d(s)\omega_1. \end{aligned}$$

By taking the covariant derivative with respect to e_1 and e_2 we have

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= -d(s)e_3 + c(s)e_4, \\ \tilde{\nabla}_{e_2} e_1 &= a(s)b(s)e_2, \\ \tilde{\nabla}_{e_1} e_2 &= 0, \\ \tilde{\nabla}_{e_2} e_2 &= a(s)b(s)e_1 - b(s)e_4, \\ \tilde{\nabla}_{e_1} e_3 &= d(s)e_1 - a(s)d(s)e_4, \\ \tilde{\nabla}_{e_2} e_3 &= 0, \\ \tilde{\nabla}_{e_1} e_4 &= c(s)e_1 - a(s)d(s)e_3, \\ \tilde{\nabla}_{e_2} e_4 &= b(s)e_2, \end{aligned} \tag{3.3}$$

where

$$a(s) = \frac{x_3'(s)}{\sqrt{1 + ({x_3}')^2}}, \tag{3.4}$$

$$b(s) = \frac{\sqrt{1 + ({x_3}')^2}}{x_3(s)}, \tag{3.5}$$

$$c(s) = \frac{x_3''(s)}{\sqrt{1 + ({x_3}')^2}}, \tag{3.6}$$

$$d(s) = \frac{x_1''(s)x_2'(s) - x_2''(s)x_1'(s)}{\sqrt{1 + ({x_3}')^2}}. \tag{3.7}$$

By using (2.2), (2.3) and (3.2), the mean curvature vector and Gaussian curvature of the surface M_1 are obtained as:

$$H = \frac{1}{2}\left(-d(s)e_3 + \left(c(s) + b\left(s\right)\right)e_4\right) \tag{3.8}$$

and

$$K = c(s)b\,(s)\,, \tag{3.9}$$

respectively.

By using (2.1) and (3.3), we find the Laplacian of the Gauss map of M_1 as :

$$\Delta G = L(s)\,(e_1 \wedge e_2) + M(s)\,(e_2 \wedge e_3) + N(s)\,(e_2 \wedge e_4)\,, \tag{3.10}$$

where

$$L(s) = d^2(s) - b^2\,(s) - c^2\,(s)\,, \tag{3.11}$$

$$M(s) = d'\,(s) + a(s)d(s)(b(s) + c(s)), \tag{3.12}$$

$$N(s) = b'(s) + c'(s) + a(s)d^2(s). \tag{3.13}$$

Theorem 3.1. *Let M_1 be rotation surface of elliptic type given by the parametrization (3.1). If M_1 has harmonic Gauss map then it has constant Gaussian curvature.*

Proof. Let the Gauss map of M_1 be harmonic, i.e., $\Delta G = 0$. So, from (3.10), (3.11), (3.12) and (3.13) we have

$$\begin{aligned} d^2(s) - b^2\,(s) - c^2\,(s) &= 0, \\ d'\,(s) + a(s)d(s)(b(s) + c(s)) &= 0, \\ b'(s) + c'(s) + a(s)d^2(s) &= 0. \end{aligned} \tag{3.14}$$

By multiplying both sides of the second equation of (3.14) with $d(s)$ and using the third equation of (3.14) we have

$$d(s)d'\,(s) - b(s)b'\,(s) - c(s)c'\,(s) = (b(s)c(s))'. \tag{3.15}$$

By differeniating the first equation of (3.14) with respect to s and using (3.15), we have that $b(s)c(s)$ =constant.Hence, from (3.9) we get $K = K_0$ =constant. □

Theorem 3.2. *Let M_1 be the flat rotational surface of elliptic type given by the parametrization (3.1). Then M_1 has a pointwise 1-type Gauss map if and only if the profile curve of M_1 is characterized by one of the following way:*

i)

$$\begin{aligned} x_1(s) &= -\frac{1}{\delta_1}\sin\left(-\delta_1 s + \delta_2\right) + \delta_4, \\ x_2(s) &= \frac{1}{\delta_1}\cos\left(-\delta_1 s + \delta_2\right) + \delta_4, \\ x_3(s) &= \delta_3, \end{aligned} \tag{3.16}$$

where δ_1, δ_2, δ_3 and δ_4 are real constants and the Gauss map of M_1 satisfies (1.1) for $f = \delta_1^2 - \frac{1}{\delta_3^2}$ and $C = 0$. If $\delta_1\delta_3 = \pm 1$ then the function f becomes zero and it implies that the Gauss map is harmonic.

ii)

$$
\begin{aligned}
x_1(s) =& \int \left(1+\lambda_1^2\right)^{\frac{1}{2}} \cos\left(-\frac{\lambda_3}{\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}}}\ln(\lambda_1 s+\lambda_2)+\lambda_4\right) ds,\\
x_2(s) =& \int \left(1+\lambda_1^2\right)^{\frac{1}{2}} \sin\left(-\frac{\lambda_3}{\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}}}\ln(\lambda_1 s+\lambda_2)+\lambda_4\right) ds,\\
x_3(s) =& \lambda_1 s+\lambda_2,
\end{aligned} \tag{3.17}
$$

where λ_1, λ_2, λ_3 and λ_4 are real constants and the Gauss map of M_1 satisfies (1.1) for $f(s)=\frac{1}{(\lambda_1 s+\lambda_2)^2}\left(\frac{\lambda_3^2}{1+\lambda_1^2}-1\right)$ and $C=\lambda_1^2 e_1\wedge e_2+\lambda_1\left(1+\lambda_1^2\right)^{\frac{1}{2}} e_2\wedge e_4$.

Proof. We suppose that M_1 has pointwise 1-type Gauss map. By using (1.1) and (3.10), we get

$$
\begin{aligned}
-f+f\langle C,e_1\wedge e_2\rangle &= -L(s),\\
f\langle C,e_2\wedge e_3\rangle &= -M(s),\\
f\langle C,e_2\wedge e_4\rangle &= N(s)
\end{aligned} \tag{3.18}
$$

and

$$
\langle C,e_1\wedge e_3\rangle=\langle C,e_1\wedge e_4\rangle=\langle C,e_3\wedge e_4\rangle=0. \tag{3.19}
$$

By taking the derivatives of all equations in (3.19) with respect to e_2 and using (3.18) we obtain

$$
\begin{aligned}
a(s)N(s)-L(s)+f &= 0,\\
a(s)M(s) &= 0,\\
M(s) &= 0,
\end{aligned} \tag{3.20}
$$

respectively. From above equations, we have two cases. One of them is $a(s)=0$, $M(s)=0$ and the other is $a(s)\neq 0$, $M(s)=0$. Firstly, we suppose that $a(s)=0$ and $M(s)=0$. By using (3.4), we have that $x_3(s)=\delta_3$=constant. It implies that $c(s)=0$, $b(s)=\frac{1}{\delta_3}$ and M_1 is flat. Since the profile curve x is spacelike curve which is parameterized by arc-length, we can put

$$
\begin{aligned}
x_1'(s) &= \cos\delta(s) \text{ (or resp. } \sin\delta(s)),\\
x_2'(s) &= \sin\delta(s) \text{ (or resp. } \cos\delta(s)),
\end{aligned} \tag{3.21}
$$

where δ is smooth angle function. Without loss of generality we assume that

$$
x_1'(s)=\cos\delta(s) \text{ and } x_2'(s)=\sin\delta(s)
$$

We can do similar computations for the another case, too. By using third equation of (3.20) and (3.12) we obtain that

$$
d(s)=\delta_1,\ \delta_1 \text{ is non zero constant.} \tag{3.22}
$$

On the other hand by using (3.7), (3.21) and (3.22) we get

$$\delta(s) = -\delta_1 s + \delta_2, \tag{3.23}$$

where δ_1, δ_2 are real constants. Then by substituting (3.23) into (3.21) and taking the integral we have the equation (3.16). Also the Laplacian of the Gauss map of M_1 with the equations $a(s) = 0$, $b(s) = \frac{1}{\delta_3}$, $c(s) = 0$ and $d(s) = \delta_1$ is found as $\Delta G = \left(\delta_1^2 - \frac{1}{\delta_3^2}\right) G$

Now we suppose that $a(s) \neq 0$ and $M(s) = 0$. Since the surface M_1 is flat, i.e., $K = 0$. By using (3.9) we have that $c(s) = 0$. From (3.6) we get

$$x_3(s) = \lambda_1 s + \lambda_2 \tag{3.24}$$

for some constants $\lambda_1 \neq 0$ and λ_2. In that case by using (3.4), (3.5) and (3.24) we have

$$a(s) = \frac{\lambda_1}{(1+\lambda_1^2)^{\frac{1}{2}}} \tag{3.25}$$

and

$$b(s) = \frac{\left(1+\lambda_1^2\right)^{\frac{1}{2}}}{\lambda_1 s + \lambda_2}. \tag{3.26}$$

Let consider that $M(s) = 0$ with $c(s) = 0$. In that case from (3.12), we obtain that

$$d'(s) + a(s)b(s)d(s) = 0 \tag{3.27}$$

By using (3.25), (3.26) and (3.27) we have

$$d(s) = \frac{\lambda_3}{\lambda_1 s + \lambda_2}, \tag{3.28}$$

where λ_3 is constant of integration. On the other hand, Since the profile curve x is spacelike curve which is parameterized by arc-length, we can put

$$\begin{aligned} x_1'(s) &= \left(1+\lambda_1^2\right)^{\frac{1}{2}} \cos\lambda(s), \\ x_2'(s) &= \left(1+\lambda_1^2\right)^{\frac{1}{2}} \sin\lambda(s), \end{aligned} \tag{3.29}$$

where λ is smooth angle function. By differentiating (3.29). we obtain

$$\begin{aligned} x_1''(s) &= -\left(1+\lambda_1^2\right)^{\frac{1}{2}} \sin\lambda(s)\,\lambda'(s), \\ x_2''(s) &= \left(1+\lambda_1^2\right)^{\frac{1}{2}} \cos\lambda(s)\,\lambda'(s). \end{aligned} \tag{3.30}$$

By using (3.7), (3.24), (3.29) and (3.30), we get

$$d(s) = -\left(1+\lambda_1^2\right)^{\frac{1}{2}} \lambda'(s). \tag{3.31}$$

By combining (3.28) and (3.31) we obtain

$$\lambda(s) = -\frac{\lambda_3}{\lambda_1 (1+\lambda_1^2)^{\frac{1}{2}}} \ln(\lambda_1 s + \lambda_2) + \lambda_4. \tag{3.32}$$

So by substituting (3.32) into (3.29), we get

$$\begin{aligned} x_1(s) &= \int \left(1+\lambda_1^2\right)^{\frac{1}{2}} \cos\left(-\frac{\lambda_3}{\lambda_1 \left(1+\lambda_1^2\right)^{\frac{1}{2}}} \ln(\lambda_1 s+\lambda_2)+\lambda_4\right) ds, \\ x_2(s) &= \int \left(1+\lambda_1^2\right)^{\frac{1}{2}} \sin\left(-\frac{\lambda_3}{\lambda_1 \left(1+\lambda_1^2\right)^{\frac{1}{2}}} \ln(\lambda_1 s+\lambda_2)+\lambda_4\right) ds, \end{aligned}$$

Conversely, the surface M_1 whose the profil curve given by (3.17) is pointwise 1-type Gauss map for

$$f(s) = \frac{1}{\left(\lambda_1 s+\lambda_2\right)^2}\left(\frac{\lambda_3^2}{1+\lambda_1^2}-1\right)$$

and

$$C = \lambda_1^2 e_1 \wedge e_2 + \lambda_1 \left(1+\lambda_1^2\right)^{\frac{1}{2}} e_2 \wedge e_4.$$

□

Theorem 3.3. *A non- minimal rotational surfaces of elliptic type M_1 defined by (3.1) has pointwise 1-type Gauss map of the first kind if and only if the mean curvature vector of M_1 is parallel .*

Proof. From (3.8) we have that $H = \frac{1}{2}\left(-d(s)e_3 + (c(s)+b(s))\,e_4\right)$. Let the mean curvature vector of M_1 be parallel, i.e., $DH = 0$. Then we get

$$D_{e_1} H = \frac{1}{2}\left(-M(s)e_3 + N(s)e_4\right) = 0.$$

In this case we obtain that $M(s) = N(s) = 0$. From (3.10), we have that $\Delta G = L(s)e_1 \wedge e_2$.

Conversely, if M_1 has pointwise 1-type Gauss map of the first kind then from (3.10) we get $M(s) = N(s) = 0$ and it implies that M_1 has parallel mean curvature vector. □

Corollary 3.4. *If rotational surfaces of elliptic type M_1 given by (3.1) is minimal then it has pointwise 1-type Gauss map of the first kind.*

3.2 Rotational surfaces of hyperbolic type with pointwise 1-type Gauss map in $\mathbb{E}_2^4$

In this subsection, first we consider rotational surfaces of hyperbolic type with harmonic Gauss map. Moreover, we obtain a characterization of flat rotational surfaces of hyperbolic type with pointwise 1-type Gauss map and give a relationship for non-minimal these surfaces with parallel mean curvature vector and pointwise 1-type Gauss map of the first kind. The proofs of theorems in this subsection are similar the proofs of theorems in previous section so we give the theorems as without proof.

Rotational surface of hyperbolic type M_2 is defined by

$$\varphi(t,s) = \begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & 1 & 0 & 0 \\ \sinh t & 0 & \cosh t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \\ 0 \\ x_4(s) \end{pmatrix}$$

$$M_2 : \ \varphi(t,s) = (x_1(s)\cosh t, x_2(s), x_1(s)\sinh t, x_4(s)), \tag{3.33}$$

where the surface M_2 is obtained by the rotation of the curve

$$x(s) = (x_1(s), x_2(s), 0, x_4(s))$$

about the two dimensional Euclidean plane spanned by ϵ_2 and ϵ_4. Let the profile curve of M_2 be unit speed spacelike curve. In that case $(x_1{}'(s))^2 + (x_2{}'(s))^2 - (x_4{}'(s))^2 = 1$. We assume that $x_1(s) > 0$. The moving frame field $\{e_1, e_2, e_3, e_4\}$ on M_2 is choosen as follows:

$$\begin{aligned} e_1 &= (x_1'(s)\cosh t, x_2'(s), x_1'(s)\sinh t, x_4'(s)), \\ e_2 &= (\sinh t, 0, \cosh t, 0), \\ e_3 &= \frac{1}{\sqrt{\varepsilon\,(x_1'(s)^2 - 1)}}\,(0, x_4'(s), 0, x_2'(s)), \\ e_4 &= \frac{1}{\sqrt{\varepsilon(x_1'(s)^2 - 1)}}\Big(\left(x_1'(s)^2 - 1\right)\cosh t, -x_1'(s)x_2'(s), \left(x_1'(s)^2 - 1\right)\sinh t, \\ & -x_1'(s)x_4'(s)\Big), \end{aligned}$$

where e_1, e_2 and e_3, e_4 are tangent vector fields and normal vector fields to M_2, respectively and ε is signature of $(x_1{}')^2 - 1$. If $(x_1{}')^2 - 1$ is positive (resp. negative) then $\varepsilon = 1$ (resp. $\varepsilon = -1$). It is easily seen that

$$\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = 1, \ \ \langle e_3, e_3\rangle = -\langle e_4, e_4\rangle = \varepsilon.$$

we have the dual 1-forms as:

$$\omega_1 = ds \qquad \text{and} \qquad \omega_2 = -x_1(s)dt.$$

After some computations, components of the second fundamental form and the connection forms are obtained by:

$$\begin{aligned} h_{11}^3 &= d(s), \ h_{12}^3 = 0, \ h_{22}^3 = 0, \\ h_{11}^4 &= c(s), \ \ h_{12}^4 = 0, \ \ h_{22}^4 = -\varepsilon b(s) \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \omega_{12} &= a(s)b(s)\omega_2, \quad \omega_{13} = d(s)\omega_1, \quad \omega_{14} = c(s)\omega_1, \\ \omega_{23} &= 0, \quad \omega_{24} = \varepsilon b(s)\omega_2, \quad \omega_{34} = a(s)d(s)\omega_1. \end{aligned}$$

Differentiating covariantly with respect to e_1 and e_2 we get

$$\begin{aligned}
\tilde{\nabla}_{e_1}e_1 &= \varepsilon d(s)e_3 - \varepsilon c(s)e_4 \\
\tilde{\nabla}_{e_2}e_1 &= a(s)b(s)e_2 \\
\tilde{\nabla}_{e_1}e_2 &= 0 \\
\tilde{\nabla}_{e_2}e_2 &= a(s)b(s)e_1 + b(s)e_4 \\
\tilde{\nabla}_{e_1}e_3 &= -d(s)e_1 - \varepsilon a(s)d(s)e_4 \\
\tilde{\nabla}_{e_2}e_3 &= 0 \\
\tilde{\nabla}_{e_1}e_4 &= -c(s)e_1 - \varepsilon a(s)d(s)e_3 \\
\tilde{\nabla}_{e_2}e_4 &= -\varepsilon b(s)e_2
\end{aligned} \tag{3.35}$$

where

$$a(s) = \frac{x_1'(s)}{\sqrt{\varepsilon\left(({x_1}')^2 - 1\right)}},$$

$$b(s) = \frac{\sqrt{\varepsilon\left(({x_1}')^2 - 1\right)}}{x_1(s)},$$

$$c(s) = \frac{x_1''(s)}{\sqrt{\varepsilon\left(({x_1}')^2 - 1\right)}},$$

$$d(s) = \frac{x_2''(s)x_4'(s) - x_4''(s)x_2'(s)}{\sqrt{\varepsilon\left(({x_1}')^2 - 1\right)}}.$$

By using (2.2), (2.3) and (3.34), the mean curvature vector and Gaussian curvature of the surface M_2 are obtained as follows:

$$H = \frac{1}{2}\left(\varepsilon d(s)e_3 - \varepsilon\left(c(s) + \varepsilon b\left(s\right)\right)e_4\right)$$

and

$$K = c(s)b\left(s\right),$$

respectively.

By using (2.1) and (3.35) , we find the Laplacian of the Gauss map of M_2 as:

$$\Delta G = L(s)\left(e_1 \wedge e_2\right) + M(s)\left(e_2 \wedge e_3\right) + N(s)\left(e_2 \wedge e_4\right),$$

where

$$L(s) = \varepsilon\left(d^2(s) - c^2\left(s\right) - b^2\left(s\right)\right),$$

$$M(s) = \varepsilon\left(d'\left(s\right) + \varepsilon a(s)d(s)\left(c(s) + \varepsilon b\left(s\right)\right)\right),$$

$$N(s) = -\varepsilon\left(c'(s) + \varepsilon b'(s) + \varepsilon a(s)d^2(s)\right).$$

Theorem 3.5. *Let M_2 be rotation surface of hyperbolic type given by the parameterization (3.33). If M_2 has Gauss map harmonic then it has constant Gaussian curvatrure.*

Theorem 3.6. *Let M_2 be flat rotation surface of hyperbolic type given by the parameterization (3.33). Then M_2 has pointwise 1-type Gauss map if and only if the profile curve of M_2 is characterized in one of the following way:*

i)

$$\begin{aligned} x_1(s) &= \delta_1, \\ x_2(s) &= -\frac{1}{\delta_2}\sinh\left(-\delta_2 s + \delta_3\right) + \delta_4, \\ x_4(s) &= -\frac{1}{\delta_2}\cosh\left(-\delta_2 s + \delta_3\right) + \delta_4, \end{aligned}$$

where δ_1, δ_2, δ_3 and δ_4 are real constants and the Gauss map G satisfies (1.1) for $f = \frac{1}{\delta_1^2} - \delta_2^2$ and $C = 0$. If $\delta_1\delta_2 = \pm 1$ then the function f becomes zero and it implies that the Gauss map is harmonic.

ii)

$$\begin{aligned} x_1(s) &= \lambda_1 s + \lambda_2, \\ x_2(s) &= \int \left(\lambda_1^2 - 1\right)^{\frac{1}{2}} \sinh\left(\frac{\lambda_3}{\lambda_1\left(\lambda_1^2 - 1\right)^{\frac{1}{2}}}\ln(\lambda_1 s + \lambda_2) + \lambda_4\right) ds, \\ x_4(s) &= \int \left(\lambda_1^2 - 1\right)^{\frac{1}{2}} \cosh\left(\frac{\lambda_3}{\lambda_1\left(\lambda_1^2 - 1\right)^{\frac{1}{2}}}\ln(\lambda_1 s + \lambda_2) + \lambda_4\right) ds, \end{aligned}$$

where λ_1, λ_2, λ_3 and λ_4 are real constants and without loss of generality we suppose that $\lambda_1^2 - 1 > 0$. Morever the Gauss map G satisfies (1.1) for the function $f(s) = \frac{1}{(\lambda_1 s + \lambda_2)^2}\left(1 - \frac{\lambda_3^2}{\lambda_1^2 - 1}\right)$ and $C = -\lambda_1^2 e_1 \wedge e_2 + \lambda_1\left(\lambda_1^2 - 1\right)^{\frac{1}{2}} e_2 \wedge e_4$.

Theorem 3.7. *A non- minimal rotational surfaces of hyperbolic type M_2 defined by (3.33) has pointwise 1-type Gauss map of the first kind if and only if M_2 has parallel mean curvature vector*

Corollary 3.8. *If rotational surfaces of hyperbolic type M_2 given by (3.33) is minimal then it has pointwise 1-type Gauss map of the first kind.*

3.3 Rotational surfaces of parabolic type with pointwise 1-type Gauss map in $\mathbb{E}_2^4$

In this subsection, we study rotational surfaces of parabolic type with pointwise 1-type Gauss map. We show that flat rotational surface of parabolic type has pointwise 1-type Gauss map if and only if its Gauss map is harmonic. Also we conclude that flat rotational surface of parabolic type has harmonic Gauss map if and only if it has parallel mean curvature vector.

We consider the pseudo-orthonormal base $\{\epsilon_1, \xi_2, \xi_3, \epsilon_4\}$ of $\mathbb{E}^4_2$ such that $\xi_2 = \frac{\epsilon_2+\epsilon_3}{\sqrt{2}}$, $\xi_3 = \frac{-\epsilon_2+\epsilon_3}{\sqrt{2}}$ $\langle \xi_2, \xi_2 \rangle = \langle \xi_3, \xi_3 \rangle = 0$ and $\langle \xi_2, \xi_3 \rangle = -1$. Let consider α spacelike curve is given by

$$x\,(s) = x_1(s)\epsilon_1 + x_2(s)\epsilon_2 + x_3(s)\epsilon_3$$

or we can express x according to pseudo-orthonormal base $\{\epsilon_1, \xi_2, \xi_3, \epsilon_4\}$ as follows:

$$x\,(s) = x_1(s)\epsilon_1 + p(s)\xi_2 + q(s)\xi_3,$$

where $p(s) = \frac{x_2(s)+x_3(s)}{\sqrt{2}}$ and $q(s) = \frac{-x_2(s)+x_3(s)}{\sqrt{2}}$. The rotational surface of parabolic type M_3 is defined by

$$M_3 : \varphi\,(t,s) = x_1(s)\epsilon_1 + p(s)\xi_2 + (-t^2 p(s) + q(s))\xi_3 + \sqrt{2}tp(s)\epsilon_4, \qquad (3.36)$$

We suppose that x is parameterized by arc-length, that is, $({x_1}'(s))^2 - 2p'(s)q'(s) = 1$. Now we can give a moving orthonormal frame $\{e_1, e_2, e_3, e_4\}$ for M_3 as follows:

$$\begin{aligned}
e_1 &= {x_1}'(s)\epsilon_1 + p'(s)\xi_2 + (-t^2 p'(s) + q'(s))\xi_3 + \sqrt{2}tp'(s)\epsilon_4, \\
e_2 &= -\sqrt{2}t\xi_3 + \epsilon_4, \\
e_3 &= \epsilon_1 + \frac{{x_1}'(s)}{p'(s)}\xi_3, \\
e_4 &= {x_1}'(s)\epsilon_1 + p'(s)\xi_2 + (\frac{1}{p'(s)} + q'(s) - t^2 p'(s))\xi_3 + \sqrt{2}tp'(s)\epsilon_4,
\end{aligned}$$

where $p'(s)$ is non zero. Then it is easily seen that

$$\langle e_1, e_1 \rangle = \langle e_3, e_3 \rangle = 1, \;\; \langle e_2, e_2 \rangle = \langle e_4, e_4 \rangle = -1.$$

We have the dual 1-forms as:

$$\omega_1 = ds \qquad \text{and} \qquad \omega_2 = -\sqrt{2}p\,(s)\,dt.$$

Also we obtain components of the second fundamental form and the connection forms as:

$$\begin{aligned}
h^3_{11} =& c(s),\;\; h^3_{12} = 0,\;\; h^3_{22} = 0, \\
h^4_{11} =& -b(s), \quad h^4_{12} = 0, \quad h^4_{22} = a(s)
\end{aligned} \qquad (3.37)$$

and

$$\begin{aligned}
\omega_{12} =& a(s)\omega_2, \quad \omega_{13} = c(s)\omega_1, \quad \omega_{14} = -b(s)\omega_1, \\
\omega_{23} =& 0, \quad \omega_{24} = -a(s)\omega_2, \quad \omega_{34} = -c(s)\omega_1.
\end{aligned}$$

Then, by taking the covariant derivatives with respect to e_1 and $e_2, weget$ as follows:

$$\begin{aligned}
\tilde{\nabla}_{e_1}e_1 &= c(s)e_3 + b(s)e_4, \\
\tilde{\nabla}_{e_2}e_1 &= a(s)e_2, \\
\tilde{\nabla}_{e_1}e_2 &= 0, \\
\tilde{\nabla}_{e_2}e_2 &= a(s)e_1 - a(s)e_4, \\
\tilde{\nabla}_{e_1}e_3 &= -c(s)e_1 + c(s)e_4, \\
\tilde{\nabla}_{e_2}e_3 &= 0, \\
\tilde{\nabla}_{e_1}e_4 &= b(s)e_1 + c(s)e_3, \\
\tilde{\nabla}_{e_2}e_4 &= a(s)e_2,
\end{aligned} \tag{3.38}$$

where

$$a(s) = \frac{p'(s)}{p(s)}, \tag{3.39}$$

$$b(s) = \frac{p''(s)}{p'(s)}, \tag{3.40}$$

$$c(s) = \frac{x_1''(s)p'(s) - p''(s)x_1'(s)}{p'(s)}. \tag{3.41}$$

By using (2.2), (2.3) and (3.37), the mean curvature vector and Gaussian curvature of the surface M_3 are obtained as follows:

$$H = \frac{1}{2}\left(c(s)e_3 + (a(s) + b\,(s))\,e_4\right) \tag{3.42}$$

and

$$K = a(s)b\,(s)\,, \tag{3.43}$$

respectively.

By using (2.1) and (3.38), we find the Laplacian of the Gauss map of M_3 by

$$\Delta G = L(s)\,(e_1 \wedge e_2) + M(s)\,(e_2 \wedge e_3) + N(s)\,(e_2 \wedge e_4)\,, \tag{3.44}$$

where

$$L(s) = c^2(s) - a^2\,(s) - b^2\,(s)\,, \tag{3.45}$$

$$M(s) = c'\,(s) + c(s)(a(s) + b(s)), \tag{3.46}$$

$$N(s) = c^2(s) + a'\,(s) + b'\,(s)\,. \tag{3.47}$$

Theorem 3.9. *Let M_3 be flat rotation surface of parabolic type given by the parameterization (3.36). Then M_3 has pointwise 1-type Gauss map if and only if the profile curve of M_3 is given by*

$$\begin{aligned}
x_1(s) &= \frac{\varepsilon}{\mu_1}\left(\ln(\mu_1 s + \mu_2)(\mu_1 s + \mu_2)\right) + (\mu_4 - \varepsilon)\,s + \mu_5, \\
p(s) &= \mu_1 s + \mu_2, \\
q(s) &= \frac{1}{2\mu_1}\int \left(\left(\varepsilon \ln\left(\mu_1 s + \mu_2\right) + \mu_4\right)^2 - 1\right) ds,
\end{aligned}$$

where μ_1, μ_2, μ_4, μ_5 real constants. Morever the surface M_3 has harmonic Gauss map for $f = 0$.

Proof. We suppose that M_3 has pointwise 1-type Gauss map. In that case the Gauss map of M_3 satisfies (1.1). By using (1.1) and (3.44), we get

$$\begin{aligned} -f + f\langle C, e_1 \wedge e_2\rangle &= -L(s), \\ f\langle C, e_2 \wedge e_3\rangle &= -M(s), \\ f\langle C, e_2 \wedge e_4\rangle &= N(s) \end{aligned} \tag{3.48}$$

and

$$\langle C, e_1 \wedge e_3\rangle = \langle C, e_1 \wedge e_4\rangle = \langle C, e_3 \wedge e_4\rangle = 0. \tag{3.49}$$

By taking the derivatives of all equations in (3.49) with respect to e_2 and using (3.48) we obtain

$$\begin{aligned} L(s) - N(s) &= f, \\ M(s) &= 0, \end{aligned} \tag{3.50}$$

respectively. Since the surface M_3 is flat, i.e., $K = 0$ from (3.43) we have that $b(s) = 0$. From (3.40) we obtain that

$$p(s) = \mu_1 s + \mu_2 \tag{3.51}$$

for some constants $\mu_1 \neq 0$ and μ_2. By using (3.39) and (3.51) we have that

$$a(s) = \frac{\mu_1}{\mu_1 s + \mu_2}. \tag{3.52}$$

If we consider $M(s) = 0$ with the equations $b(s) = 0$ and $a(s) = \frac{\mu_1}{\mu_1 s+\mu_2}$, from (3.46) we get

$$c(s) = \frac{\mu_3}{\mu_1 s + \mu_2}. \tag{3.53}$$

On the other hand, by using the first equation of (3.50), (3.45), (3.47), (3.52) and (3.53) we obtain that $f = 0$. It means that $L(s) = N(s) = 0$ and we have

$$\mu_3 = \varepsilon\mu_1,\ \varepsilon = \pm 1.$$

If we consider (3.41), (3.51) and (3.53) we get

$$x_1(s) = \frac{\varepsilon}{\mu_1}\left(\ln(\mu_1 s + \mu_2)(\mu_1 s + \mu_2)\right) + (\mu_4 - \varepsilon)\, s + \mu_5, \tag{3.54}$$

where μ_4, μ_5 are constants of integration. Since x is unit speed spacelike curve we get

$$q'(s) = \frac{\left(x_1{}'(s)\right)^2 - 1}{2p'(s)}. \tag{3.55}$$

By substituting (3.51) and (3.54) into (3.55) we obtain

$$q(s) = \frac{1}{2\mu_1}\int \left(\left(\varepsilon \ln\left(\mu_1 s + \mu_2\right) + \mu_4\right)^2 - 1\right) ds.$$

This completes the proof. □

Theorem 3.10. *Let M_3 be flat rotational surfaces of parabolic type given by (3.36). M_3 has harmonic Gauss map if and only if its mean curvature vector is parallel.*

Proof. We suppose that M_3 has parallel mean curvature vector, i.e., $DH = 0$. From (3.42) we have that

$$D_{e_1}H = \frac{1}{2}\left(M(s)e_3 + N(s)e_4\right) = 0.$$

In this case we obtain that $M(s) = N(s) = 0$. Since M_3 is a flat surface, from the previous theorem we have

$$b(s) = 0 \text{ and } a(s) = \frac{\mu_1}{\mu_1 s + \mu_2}.$$

By considering the equation $M(s) = 0$ with above equations and using (3.46) we get

$$c(s) = \frac{\mu_3}{\mu_1 s + \mu_2},$$

where μ_3 is the constant of integration. It implies that $L(s) = 0$. Hence we obtain that Gauss map of M_3 is harmonic .

Conversely, if M_3 is harmonic then it is easily seen that $DH = 0$. □

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On the Solutions to the $H_R = H_L$ Hypersurface Equation

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Abstract. Spacelike hypersurfaces in the Lorentz-Minkowski (n+1)-dimensional space $\mathbb{L}^{n+1}$ can be endowed with another Riemannian metric, the one induced by the Euclidean space $\mathbb{R}^{n+1}$. The hypersurfaces with the same mean curvature with respect to both metrics can be locally determined by a smooth function u satisfying $|Du| < 1$, and being the solution to a certain partial differential equation. We call this equation the $H_R = H_L$ hypersurface equation. In the particular case in which $n = 2$ and both curvatures vanish, Kobayashi proved that the graphs determined by the solutions of such equation are open pieces of spacelike planes or helicoids, in the region where they are spacelike. In this manuscript we prove the existence of a family of solutions whose graphs have non-zero mean curvature, and we present an inequality relating the mean curvature to the width of the domain of certain solutions, those without critical points.
Keywords. Mean curvature · spacelike hypersurfaces · rotational hypersurfaces · elliptic partial differential equations.
MSC 2010 Classification. Primary: 53C42; Secondary: 35J93 · 53C50.

1 Introduction and background

Let us consider the differential operator given by

$$Q(u) = \operatorname{div}\left(\left(\frac{1}{\sqrt{1-|Du|^2}} - \frac{1}{\sqrt{1+|Du|^2}}\right) Du\right),$$

where $u \in \mathcal{C}^2(\mathbb{R}^n)$, and D, div and $|\cdot|$ stand for the gradient, the divergence and the Euclidean norm on $\mathbb{R}^n$, respectively. We are interested in studying the equation

$$Q(u) = 0, \text{ with } |Du| < 1. \tag{1.1}$$

The above divergence-type partial differential equation is not an arbitrary one, it has a geometrical meaning.

A hypersurface in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ is said to be spacelike if its induced metric is a Riemannian one. Therefore, spacelike hypersurfaces in $\mathbb{L}^{n+1}$ can be endowed with two different Riemannian metrics, the metric induced by the Euclidean space $\mathbb{R}^{n+1}$ and the metric inherited from $\mathbb{L}^{n+1}$. Consequently, we can consider two different mean curvature functions on a spacelike hypersurface related to both metrics, H_R and H_L respectively.

On the other hand, it is well known that any spacelike hypersurface Σ in $\mathbb{L}^{n+1}$ can be locally described as a spacelike graph over an open subset of a spacelike hyperplane, which without loss of generality can be supposed to be the hyperplane $x_{n+1} = 0$ (see [4, Proposition 3.3]). Let u be the function that describes such a graph, then the spatiality condition becomes $|Du| < 1$. The functions H_R and H_L can be written in terms of the function u and its partial derivatives obtaining the expressions

$$H_R = \frac{1}{n}\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \qquad \text{and} \qquad H_L = \frac{1}{n}\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right). \tag{1.2}$$

Therefore, a spacelike graph determined by u satisfies $H_R = H_L$ if and only if u is a solution of (1.1). For this reason (1.1) is called the $H_R = H_L$ *hypersurface equation*. This equation is a quasilinear elliptic partial differential equation, everywhere except at those points at which Du vanishes, where the equation is parabolic, see [1].

As a particular case, we can consider the situation where the graph is simultaneously minimal and maximal, that is $H_R = H_L = 0$. The geometry of minimal and maximal graphs has been widely studied. One of the main results on minimal graphs is the well-known Bernstein theorem, proved by Bernstein [5] in 1915, which states that the only entire minimal graphs in $\mathbb{R}^3$ are the planes. Some decades later, in 1970, Calabi [7] proved its analogous version for spacelike graphs in the Lorentz-Minkowski space, the Calabi-Bernstein theorem, which states that the only entire maximal graphs in $\mathbb{L}^3$ are the spacelike planes. An important difference between both results is that the Bernstein theorem can be extended to minimal graphs in $\mathbb{R}^{n+1}$ up to dimension $n = 7$, as it was proved by Bombieri, di Giorgi and Giusti [6], but it is no longer true for higher dimensions. However, the Calabi-Bernstein theorem holds true for any dimension as it was proved by Calabi [7] for dimension $n \leq 4$, and by Cheng and Yau [8] for arbitrary dimension.

As an immediate consequence of the above results, we conclude that the only entire graphs that are simultaneously minimal in $\mathbb{R}^{n+1}$ and maximal in $\mathbb{L}^{n+1}$ are the spacelike hyperplanes.

Going a step further, we can consider spacelike graphs with the same constant mean curvature functions H_R and H_L. Heinz [11], Chern [9] and Flanders [10] proved that the only entire graphs with constant mean curvature H_R in $\mathbb{R}^{n+1}$ are the minimal graphs. The Lorentzian version of this fact is not true, since there are examples of entire spacelike graphs with constant mean curvature H_L in $\mathbb{L}^{n+1}$ which are not maximal, for instance the hyperbolic spaces. However, taking into account the Calabi-Bernstein theorem, we conclude again that the only complete spacelike hypersurfaces in $\mathbb{L}^{n+1}$ with the same constant mean curvature functions H_R and H_L are the spacelike hyperplanes.

Kobayashi [12] studied the same problem without assuming any global hypothesis. He showed that the graphs of the solutions to (1.1) with $H_R = H_L = 0$ are open pieces of a spacelike plane or of a helicoid, in the region where the helicoid is a spacelike surface. Recently, Albujer, Caballero and Sánchez [2, 3] have continued with the study of spacelike surfaces with the same mean curvature in $\mathbb{R}^3$ and in $\mathbb{L}^3$, not necessarily constant. On one hand, they have shown that the Gaussian curvature in $\mathbb{R}^3$ of those surfaces is always non-positive and have obtained several interesting consequences about the geometry of such surfaces. On the other hand, they have obtained results on the solutions to the $H_R = H_L$ surface equation, which are not derived from the sign of the Gaussian curvature.

In general dimension, Lee and Lee [13] have recently presented non-planar examples of simultaneously minimal and maximal spacelike graphs in the Lorentz-Minkowski space. Their examples can be seen as generalized ruled hypersurfaces, in fact they are a natural generalization of helicoids. However, there is no known classification of such hypersurfaces similar to Kobayashi's result. In [1] the authors have shown that those hypersurfaces do not have elliptic points and have obtained several interesting consequences about the geometry of such hypersurfaces, generalizing some results in [2].

In this manuscript we prove the existence of a solution to the $H_R = H_L$ hypersurface equation which constitutes the first evidence of the existence of examples with non-zero mean curvature, following the ideas of the example obtained in [3] in dimension 2. Finally, we generalize some results on the graphs of the solutions which are not a consequence of the non-existence of elliptic points, specifically Lemma 7, Theorem 8 and Corollary 1 from [2].

2 Preliminaries

Let $\mathbb{L}^{n+1}$ be the $(n+1)$-dimensional Lorentz-Minkowski space, that is, $\mathbb{R}^{n+1}$ endowed with the metric

$$\langle \cdot, \cdot \rangle_L = dx_1^2 + ... + dx_n^2 - dx_{n+1}^2,$$

where $(x_1, ..., x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$, and let $|\cdot|_L$ denote its norm. It is easy to see that the Levi-Civita connections of the Euclidean space $\mathbb{R}^{n+1}$ and the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ coincide, so we will just denote it by $\overline{\nabla}$.

A (connected) hypersurface Σ^n in $\mathbb{L}^{n+1}$ is said to be a spacelike hypersurface if $\mathbb{L}^{n+1}$ induces a Riemannian metric on Σ, which is also denoted by $\langle \cdot, \cdot \rangle_L$. Given a spacelike hypersurface Σ, we can choose a unique future-directed unit normal vector field N_L on Σ. The mean curvature function of Σ with respect to N_L is defined by

$$H_L = -\frac{1}{n}(k_1^L + ... + k_n^L),$$

where k_i^L, $i = 1, ..., n$, stand for the principal curvatures of $(\Sigma, \langle \cdot, \cdot \rangle_L)$.

The same topological hypersurface can also be considered as a hypersurface of the Euclidean space, that is $\mathbb{R}^{n+1}$ with its usual Euclidean metric. For simplicity, we will just denote the Euclidean space by $\mathbb{R}^{n+1}$, the Euclidean metric and the induced metric on Σ by $\langle \cdot, \cdot \rangle_R$, and its norm by $| \cdot |_R$. In such a case, Σ admits a unique upwards directed unit normal vector field, N_R. The mean curvature function of Σ with respect to N_R is defined by

$$H_R = \frac{1}{n}(k_1^R + ... + k_n^R),$$

where k_i^R, $i = 1, ..., n$, stand for the principal curvatures of $(\Sigma, \langle \cdot, \cdot \rangle_R)$.

It is interesting to observe that the mean curvature functions have an expression in terms of the normal curvatures of any set of orthogonal directions. Specifically,

$$H_L = -\frac{1}{n}(\kappa_{w_1}^L + \ldots + \kappa_{w_n}^L) \qquad \text{and} \qquad H_R = \frac{1}{n}(\kappa_{v_1}^R + \ldots + \kappa_{v_n}^R), \tag{2.1}$$

where $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are orthonormal basis of $T_p\Sigma$ with respect to $\langle \cdot, \cdot \rangle_R$ and $\langle \cdot, \cdot \rangle_L$, respectively.

If our spacelike hypersurface is the graph of a smooth function $u \in \mathcal{C}^\infty(\Omega)$,

$$\Sigma_u = \{(x_1, ..., x_n, u(x_1, ..., x_n)) : (x_1, ..., x_n) \in \Omega\},$$

Ω being an open subset of the hyperplane $x_{n+1} = 0$, which can be identified with $\mathbb{R}^n$, it is easy to check that the spatiality condition is written as $|Du| < 1$, where D and $| \cdot |$ stand for the gradient operator and the norm in the Euclidean space $\mathbb{R}^n$, respectively. In this case, it is possible to get expressions for the normal vector fields N_L and N_R, as well as for the mean curvature functions H_L and H_R, in terms of u. Specifically, with a straightforward computation we get

$$N_L = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}} \qquad \text{and} \qquad N_R = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}. \tag{2.2}$$

And for the mean curvature functions we have

$$H_L = \frac{1}{n} \operatorname{div}\left(\frac{Du}{\sqrt{1 - |Du|^2}}\right) \qquad \text{and} \qquad H_R = \frac{1}{n} \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right), \tag{2.3}$$

where div denotes the divergence operator in $\mathbb{R}^n$.

Let us observe that

$$\cosh \psi = \frac{1}{\sqrt{1 - |Du|^2}} \qquad \text{and} \qquad \cos \theta = \frac{1}{\sqrt{1 + |Du|^2}},$$

where ψ and θ denote the hyperbolic angle between N_L and $e_{n+1} = (0, ..., 0, 1)$ and the angle between N_R and e_{n+1}, respectively.

The following result can be found in [2], and will be used in Section 4.

Lemma 2.1. *[2, Lemma 2] Let Σ be a spacelike hypersurface in $\mathbb{L}^{n+1}$. Given $p \in \Sigma$ and $v \in T_p\Sigma$, let $\kappa_v^L(p)$ and $\kappa_v^R(p)$ denote the normal curvatures at p in the direction of v with respect to $\langle\cdot,\cdot\rangle_L$ and $\langle\cdot,\cdot\rangle_R$, respectively. Then*

$$\frac{|v|_R^2}{\cos\theta(p)}\,\kappa_v^R(p) = -\frac{|v|_L^2}{\cosh\psi(p)}\,\kappa_v^L(p).$$

3 A solution with non-zero mean curvature

Let us consider rotationally invariant spacelike graphs with respect to a vertical axis. Therefore, we can assume without loss of generality that the graph Σ_u^* is determined by a function

$$u(x_1, ..., x_n) = f(r), \quad r = x_1^2 + ... + x_n^2, \tag{3.1}$$

being $f \in \mathcal{C}^\infty(I)$ for certain $I \subseteq [0, +\infty)$. In this case, $|Du| < 1$ reads $4\,(f'(r))^2\, r < 1$ and the $H_R = H_L$ hypersurface equation yields

$$\frac{2f''r + f'n + 4(f')^3 r(n-1)}{(1+4(f')^2 r)^{3/2}} = \frac{2f''r + f'n - 4(f')^3 r(n-1)}{(1-4(f')^2 r)^{3/2}}. \tag{3.2}$$

It can be checked that, given any set of initial conditions $(r_0, f(r_0), f'(r_0))$ such that $r_0 > 0$, $f'(r_0) \neq 0$ and $4\,(f'(r_0))^2\, r_0 < 1$, there exists a local solution of (3.2) by the Picard-Lindelöf theorem.

It is interesting to observe that these examples cannot be entire because of the following theorem which can be found in [1].

Theorem 3.1. *The only entire spacelike graphs Σ_u determined by a function u given by* (3.1) *such that $H_R = H_L$ are the horizontal hyperplanes.*

4 On the width of the domain of the solutions

We define the *width* of a set in $\mathbb{R}^n$ as the supremum of the diameter of the closed balls contained in it. This is an intuitive definition which is a generalization of the classical concept of width for a convex body, see [15].

Let u be a solution to (1.1) over an open set $\Omega \subseteq \mathbb{R}^n$, Σ_u its graph and $\pi : \Sigma_u \longrightarrow \Omega$ the canonical projection. We define Σ_u^* as the graph of u over the following open set

$$\Omega^* = \{(x_1, \ldots, x_n) \in \Omega : Du(x_1, \ldots, x_n) \neq 0\}. \tag{4.1}$$

The goal of this section is to give an upper bound for the width of the set Ω^*. Before stating our main result, we get some previous local computations involving the Riemannian and Lorentzian normal curvatures of Σ_u^* in some privileged directions. As well as a lemma relating the mean curvature of Σ_u to that of its level hypersurfaces.

Given $p \in \Sigma_u^*$, we consider its corresponding level hypersurface contained in $\mathbb{R}^n$, $\widetilde{S_c}$, and its lifting to Σ_u, S_c. We will work in a neighborhood of p, hence we can assume that S_c lies on Σ_u. Since $Du \neq 0$ in Ω^*, this distribution is integrable, so we can consider the integral curve through $\pi(p)$. We denote by α its lifting to Σ_u^*. Notice that $\alpha' = (Du, |Du|^2) \circ \pi$.

Therefore, we have two submanifolds of Σ_u^*, namely S_c and α, defined on a neighborhood of p which are orthogonal at p for both $\langle \cdot, \cdot \rangle_R$ and $\langle \cdot, \cdot \rangle_L$. Now, let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis of $T_{\pi(p)}\widetilde{S_c}$. The vectors $\{(e_1, 0), \ldots, (e_{n-1}, 0)\}$ constitute an orthonormal basis of T_pS_c in both $\mathbb{R}^{n+1}$ and $\mathbb{L}^{n+1}$, and are orthogonal to α' for both metrics. Then, Lemma 2.1 gives us the following relationships, where we have omitted the point p on behalf of simplicity

$$\kappa^R_{(e_i,0)} = -\frac{\cos\theta}{\cosh\psi}\,\kappa^L_{(e_i,0)} = -\sqrt{\frac{1-|Du|^2}{1+|Du|^2}}\,\kappa^L_{(e_i,0)}, \qquad i = 1, \ldots, n-1 \qquad \text{and}$$

$$\kappa^R_{\alpha'} = -\frac{|\alpha'|^2_L}{|\alpha'|^2_R}\frac{\cos\theta}{\cosh\psi}\,\kappa^L_{\alpha'} = -\left(\frac{1-|Du|^2}{1+|Du|^2}\right)^{\frac{3}{2}}\,\kappa^L_{\alpha'}.$$

By denoting $A = \sqrt{\frac{1-|Du|^2}{1+|Du|^2}}$, we rewrite the previous expressions as

$$\kappa^R_{(e_i,0)} = -A\,\kappa^L_{(e_i,0)}, \qquad i = 1, \ldots, n-1 \qquad \text{and} \qquad \kappa^R_{\alpha'} = -A^3\,\kappa^L_{\alpha'}. \tag{4.2}$$

As we are dealing with orthogonal directions at p for both $\langle \cdot, \cdot \rangle_R$ and $\langle \cdot, \cdot \rangle_L$, and u is a solution of the $H_R = H_L$ hypersurface equation, from (2.1) we get

$$-\kappa^L_{(e_1,0)} - \ldots - \kappa^L_{(e_{n-1},0)} - \kappa^L_{\alpha'} = \kappa^R_{(e_1,0)} + \ldots + \kappa^R_{(e_{n-1},0)} + \kappa^R_{\alpha'},$$

which jointly with (4.2) implies

$$-\kappa^L_{\alpha'} = \frac{1}{A^2+A+1}(\kappa^L_{(e_1,0)} + \ldots + \kappa^L_{(e_{n-1},0)}). \tag{4.3}$$

Lemma 4.1. *Let Σ_u be a spacelike graph in $\mathbb{L}^{n+1}$ over a domain $\Omega \subseteq \mathbb{R}^n$ such that $H_R = H_L$. If $\widetilde{S_c}$ denotes the level hypersurface $u \equiv c$ in Ω^* and H_c is its mean curvature, then*

$$|H_L| \leq \frac{n-1}{n\sqrt{2}}|H_c| \circ \pi \tag{4.4}$$

and the equality is hold if and only if $H_L = 0$.

Proof. We work at a point $p \in S_c$ and we follow the notation introduced at the beginning of this section. For each $i = 1, \ldots, n$ we take a curve in $\widetilde{S_c}$, $\widetilde{\alpha}_i$, with $\widetilde{\alpha}_i(0) = p$ and $\widetilde{\alpha}_i{}'(0) = e_i$. Let α_i be its lifting to S_c. Notice that $\alpha_i' = (\widetilde{\alpha}_i', 0)$.

It is possible to relate the Lorentzian normal curvature $\kappa^L_{(e_i,0)}$ of Σ_u at p in the direction of α_i' with the normal curvature $\kappa^c_{e_i}$ of $\widetilde{S_c}$ at $\pi(p)$ in the direction of $\widetilde{\alpha_i}'$:

$$\kappa^L_{(e_i,0)} = \langle \overline{\nabla}_{t_i} t_i, N_L\rangle_L = \frac{|Du|}{\sqrt{1-|Du|^2}} \left\langle D_{\widetilde{t_i}}\, \widetilde{t_i}, \frac{Du}{|Du|} \right\rangle_{\mathbb{R}^n} \circ\pi = \frac{|Du|}{\sqrt{1-|Du|^2}} \kappa^c_{e_i} \circ \pi.$$

Here D and $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ stand for the Levi-Civita connection and the usual metric of the Euclidean space $\mathbb{R}^n$, respectively, $t_i = \dfrac{\alpha_i'}{|\alpha_i'|_L}$, $\widetilde{t_i} = \dfrac{\widetilde{\alpha_i}'}{|\widetilde{\alpha_i}'|}$ and $\dfrac{Du}{|Du|}$ is the unitary normal vector field to $\widetilde{S_c}$ in $\mathbb{R}^n$.

Therefore, from (4.3) we get

$$n|H_L| = (n-1)\frac{A+1}{A^2+A+1}\frac{|Du|}{\sqrt{1+|Du|^2}}|H_c|\circ\pi \le (n-1)f(|Du|)\,|H_c|\circ\pi,$$

where $f(x) = \frac{x}{\sqrt{1+x^2}}$. Since f is increasing and $|Du| < 1$, we get (4.4). □

Theorem 4.2. *Let u be a solution to the $H_R = H_L$ hypersurface equation defined on an open set $\Omega \subseteq \mathbb{R}^n$. Then*

$$\text{width}(\Omega^*) \le \frac{\sqrt{2}\,(n-1)}{n\,\inf_{\Omega^*}|H_L|}. \tag{4.5}$$

Proof. If $\inf_{\Omega^*}|H_L| = 0$, there is nothing to prove.

Otherwise, we have $|H_L| \ge \inf_{\Omega^*}|H_L| = C > 0$ in Σ_u^*. And, as a consequence of (4.4), we get

$$|H_c| > \frac{nC\sqrt{2}}{n-1} > 0 \quad \text{in} \quad \Omega^*. \tag{4.6}$$

First of all, let us notice that Ω^* is an open set of $\mathbb{R}^n$. We consider all the level hypersurfaces in Ω^*, we order them by the value of u on each of them and we orient them in a way such that its normal vectors point to the direction on which u decreases.

We proceed by reductio ad absurdum assuming that the width of Ω^* is bigger than $\dfrac{(n-1)\sqrt{2}}{nC}$. Then, there exists a point $q \in \Omega^*$ such that $\bar{B}_q = \bar{B}_q\left((n-1)/n\sqrt{2}\,C\right) \subset \Omega^*$. Since $\bar{B}_q$ is compact, u attains a maximum in it. Even more, Du does not vanish in $B_q = B_q\left((n-1)/n\sqrt{2}\,C\right)$, and so this extremal value is only attained on the boundary of the ball.

We pick a point p at which a maximum is attained. The level hypersurface through p lies in $\Omega^* \setminus B_q$. And so, it is tangent to the boundary of the ball at p. The normal vector to the hypersurface at p points to the interior of the ball, see Figure 1. Consequently, using the tangency principle (see [14, Theorem 3.2.4]

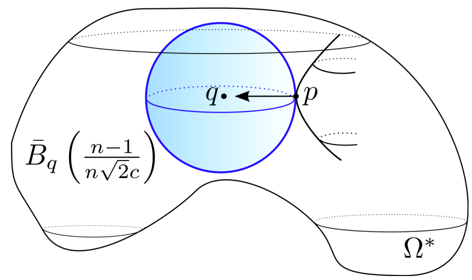

Figure 1: Level hypersurface at p.

for the 2-dimensional case), inequality (4.6) implies that $H_c \leq -n\sqrt{2}C/(n-1)$ at p. Analogously, we get that $H_c \geq n\sqrt{2}C/(n-1)$ at $\bar{p}$, $\bar{p}$ being a point at which u attains a minimum in the ball. By a continuity argument, there is a point in the ball at which H_c vanishes, which is a contradiction.

□

As a direct consequence of Theorem 4.2, we get the following results.

Corollary 4.3. *Let u be a solution to the $H_R = H_L$ hypersurface equation defined on an open set $\Omega \subseteq \mathbb{R}^n$ and assume that Ω^* is a set of infinite width. Then* $\inf_{\Sigma_u} |H_L| = 0$.

Equivalently, there do not exist spacelike graphs satisfying $H_R = H_L$, $|H_L| \geq C$ for a certain constant $C > 0$ and $\text{width}(\Omega^*) = \infty$.

Corollary 4.4. *Let u be a solution to the $H_R = H_L$ hypersurface equation defined on an open set $\Omega \subseteq \mathbb{R}^n$ with constant mean curvature. Then*

$$\text{width}(\Omega^*) \leq \frac{\sqrt{2}\,(n-1)}{n\,|H_L|}.$$

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On Pseudo-Umbilical Rotational Surfaces with Pointwise 1-Type Gauss Map in $\mathbb{E}^4_2$

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Abstract. In this work, we study two families of rotational surfaces in the pseudo–Euclidean space $\mathbb{E}^4_2$ with profile curves lying in 2–dimensional planes. First, we obtain a classification of pseudo–umbilical spacelike surfaces and timelike surfaces in these families. Then, we show that in this classification there exists no a pseudo–umbilical rotational surface in $\mathbb{E}^4_2$ with pointwise 1–type Gauss map of second kind. Finally, we determine such pseudo–umbilical rotational surfaces in $\mathbb{E}^4_2$ having pointwise 1–type Gauss map of first kind.
Keywords. Rotational Surfaces · Pseudo–Umbilical Surfaces · Gauss Map · Pointwise 1–Type Gauss Map.
MSC 2010 Classification. Primary: 53B25; Secondary:53C50.

1 Introduction

In late 1970, the theory of finite type submanifolds of Euclidean submanifolds was introduced by B.-Y. Chen, [7]. Since then, many mathematicians have characterized or classified submanifolds of Euclidean space or pseudo–Euclidean space in terms of finite type. Later, B.-Y.Chen and P. Piccinni extended the notion of finite type of submanifolds to Gauss map of submanifolds, [8]. The report [9] and the second edition of above mentioned book [10] are useful references to understand recent developments and open problems of this area.

A smooth map $\phi : M \longrightarrow \mathbb{E}^m_s$ from a (pseudo)–Riemannian manifold into a (pseudo)–Euclidean space is called *of finite type* if it has a finite spectral decomposition

$$\phi = \phi_0 + \sum_{i=1}^{k} \phi_i, \tag{1.1}$$

where ϕ_0 is a constant map, and each non–constant maps ϕ_i satisfies $\Delta\phi_i = \lambda_i\phi_i$ for some constant $\lambda_i \in \mathbb{R}$. If the spectral decomposition (1.1) contains exactly k terms with different values for λ_i, then the map ϕ is called *of k-type*. Thus, a (pseudo)–Riemannian submanifold M of a (pseudo)–Euclidean space has 1–type Gauss map ν if and only if $\Delta\nu = \lambda(\nu + C)$ for some $\lambda \in \mathbb{R}$ and for some constant vector C.

On the other hand, it was observed that the Gauss map of some submanifolds such as helicoid, catenoid, right cones in $\mathbb{E}^3$ and Enneper's hypersurfaces in $\mathbb{E}^{n+1}_1$ satisfies

$$\Delta\nu = f(\nu + C) \tag{1.2}$$

for some smooth function f on M and some constant vector C, [13, 16]. This gives a new terminology, namely that, a submanifold of a (pseudo)–Euclidean space is said to have pointwise 1–type Gauss map if it satisfies (1.2). In particular, if C is zero, it is said to be of *the first kind*. Otherwise, it is said to be of *the second kind*.

Also, rotational surfaces in a (pseudo)–Euclidean space which are the main focus of the present paper are another active research field in differential geometry. In 1919, C. L. Moore introduced generel rotational surfaces in the four dimensional Euclidean space, [19]. A rotational surface in $\mathbb{E}^4$ is a surface left invariant by a rotation in $\mathbb{E}^4$ which is defined as a linear transformation of positive determinant preserving distance and leaving one point fixed. Moreover, F. Cole studied the general theory of rotation in $\mathbb{E}^4$, [12].

The rotational surfaces in the pseudo–Euclidean space $\mathbb{E}^4_2$, called Vranceanu rotational surfaces which is a particular case of the rotational surfaces studied in this article were studied for different purposes. The complete classification of Vranceanu rotational surfaces in $\mathbb{E}^4_2$ with zero mean curvature was obtained in [15]. It was proved that a flat rotational surface in $\mathbb{E}^4_2$ with pointwise 1–type Gauss map is either the product of two plane hyperbolas or the product of a plane circle and a plane hyperbola, [17].

In [1], F. K. Aksoyak and Y. Yaylı gave a classification of flat general rotational surfaces with pointwise 1–type Gauss map in the pseudo–Euclidean space $\mathbb{E}^4_2$ which includes similar results given in [17].

Recently, Y. Aleksieva, V. Milousheva and N. C. Turgay studied general rotational surfaces in the pseudo–Euclidean space $\mathbb{E}^4_2$ with zero mean curvature vector in [2] and then the first author, E. Canfes and U. Dursun classified such rotational surfaces with pointwise 1–type Gauss map in [4].

Moreover, there are many studies about the rotational surfaces in the pseudo–Euclidean space and different spaces with pointwise 1–type Gauss map, [3, 11, 18].

On the other hand, pseudo–umbilical submanifolds are also well–known and have been studied in many articles, [6, 14, 5].

In this article, we consider two families of rotational surfaces in the pseudo–Euclidean space $\mathbb{E}^4_2$ with profile curves lying in 2–dimensional planes. First, we determine the pseudo–umbilical rotational surfaces in these families. Then, we show that there exists no a non–planar pseudo–umbilical rotational surface in these families with pointwise 1–type Gauss map of the second kind. Finally, we give a classification of all such pseudo–umbilical surfaces in $\mathbb{E}^4_2$ with pointwise 1–type Gauss map of the first kind.

2 Preliminaries

2.1 Basics of Submanifold Theory

Let $\mathbb{E}^m_t$ be the m–dimensional pseudo–Euclidean space with the canonical metric given by

$$\tilde{g} = \sum_{i=1}^{m-t}(dx_i)^2 - \sum_{i=m-t+1}^{m}(dx_i)^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a standard rectangular coordinate system in $\mathbb{E}^m_t$.

For a point $\mathbf{x_0} \in \mathbb{E}^m_t$ and $c \neq 0$, we put

$$\mathbb{S}^{m-1}_t(\mathbf{x_0}, c) = \left\{\mathbf{x} \in \mathbb{E}^m_t \mid \langle \mathbf{x} - \mathbf{x_0}, \mathbf{x} - \mathbf{x_0}\rangle = c^{-1}\right\} \text{ if } c > 0,$$
$$\mathbb{H}^{m-1}_t(\mathbf{x_0}, c) = \left\{\mathbf{x} \in \mathbb{E}^m_{t+1} \mid \langle \mathbf{x} - \mathbf{x_0}, \mathbf{x} - \mathbf{x_0}\rangle = c^{-1}\right\} \text{ if } c < 0,$$

where $\langle , \rangle$ denotes the indefinite inner product associated to $\tilde{g}$. $\mathbb{S}^{m-1}_t(\mathbf{x_0}, c)$ and $\mathbb{H}^{m-1}_t(\mathbf{x_0}, c)$ are called, respectively, a pseudo–sphere and a pseudo–hyperbolic space. When $\mathbf{x}_0$ is the origin, we simply denote $\mathbb{S}^{m-1}_t(\mathbf{0}, c)$ and $\mathbb{H}^{m-1}_t(\mathbf{0}, c)$ by $\mathbb{S}^{m-1}_t(c)$ and $\mathbb{H}^{m-1}_t(c)$.

A vector $v \in \mathbb{E}^m_t$ is called spacelike (resp., timelike) if $\langle v, v\rangle > 0$ or $v = 0$ (resp., $\langle v, v\rangle < 0$). A vector v is called lightlike if it is non–zero and it satisfies $\langle v, v\rangle = 0$.

From now on, we use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+2, \;\; 1 \leq i, j, k, \ldots \leq n, \;\; n+1 \leq r, s, t, \ldots \leq n+2.$$

Let M be an oriented n–dimensional submanifold in an $(n+2)$–dimensional pseudo–Euclidean space $\mathbb{E}^{n+2}_2$. We denote the Levi–Civita connections of $\mathbb{E}^{n+2}_2$ and M respectively, by $\widetilde{\nabla}$ and ∇. Then, we choose an oriented local orthonormal frame $\{e_1, \ldots, e_{n+2}\}$ on M with $\varepsilon_A = \langle e_A, e_A\rangle = \pm 1$ such that $e_1, \ldots, e_n$ are tangent to M and e_{n+1}, e_{n+2} are normal to M in $\mathbb{E}^{n+2}_2$. Denote the dual frame and connection forms associated to $\{e_1, \ldots, e_{n+2}\}$ by $\{\omega^1, \ldots, \omega^{n+2}\}$ and ω_{AB}, respectively.

The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{n+2} \varepsilon_r h^r_{ik} e_r,$$
$$\widetilde{\nabla}_{e_k} e_r = -A_r(e_k) + \sum_{s=n+1}^{n+2} \varepsilon_s \omega_{rs}(e_k) e_s,$$

where h^r_{ij} is the coefficients of the second fundamental form h, and A_r the Weingarten map in the direction e_r.

The mean curvature vector H and the scalar curvature S of M in $\mathbb{E}^{n+2}_2$ are defined, respectively, by

$$H = \frac{1}{n}\sum_{r=n+1}^{n+2}\varepsilon_r \mathrm{tr} A_r e_r, \tag{2.1}$$

$$S = n^2\langle H, H\rangle - \|h\|^2, \tag{2.2}$$

where $\|h\|^2 = \sum_{i,j=1}^{n}\sum_{r=n+1}^{n+2}\varepsilon_i\varepsilon_j\varepsilon_r(h^r_{ij})^2$. A submanifold M is called minimal if H vanishes identically and a non–minimal submanifold is called pseudo–umbilical if there exist a smooth function ρ such that $A_H = \rho\mathrm{I}$, where I is an identity $n\times n$ matrix and ρ is a smooth function on M. In particular, the Gaussian curvature K which is also defined by $K = \varepsilon_3\det A_3 + \varepsilon_4 \det A_4$ is half of the scalar curvature S for $n = 2$. If K vanishes identically, the surface M is called flat.

The Codazzi equations of M in $\mathbb{E}^{n+2}_2$ are given by

$$\begin{aligned} &h^r_{ij,k} = h^r_{jk,i},\\ &h^r_{jk,i} = e_i(h^r_{jk}) + \sum_{s=n+1}^{n+2}\varepsilon_s h^s_{jk}\omega_{sr}(e_i) - \sum_{\ell=1}^{n}\varepsilon_\ell\left(\omega_{j\ell}(e_i)h^r_{\ell k} + \omega_{k\ell}(e_i)h^r_{\ell j}\right).\end{aligned} \tag{2.3}$$

Also, from the Ricci equation of M in $\mathbb{E}^{n+2}_2$, we have

$$R^D(e_j, e_k; e_r, e_s) = \langle [A_{e_r}, A_{e_s}](e_j), e_k\rangle = \sum_{i=1}^{n}\varepsilon_i\left(h^r_{ik}h^s_{ij} - h^r_{ij}h^s_{ik}\right), \tag{2.4}$$

where R^D is the normal curvature tensor.

The gradient of a smooth function f on M is defined by $\nabla f = \sum\limits_{i=1}^{n}\varepsilon_i e_i(f)e_i$, and the Laplace operator acting on M is $\Delta = \sum\limits_{i=1}^{n}\varepsilon_i(\nabla_{e_i}e_i - e_ie_i)$.

2.2 Gauss Map

Let $G(m-n, m)$ be the Grassmannian manifold consisting of all oriented $(m-n)$–planes through the origin of a pseudo–Euclidean space $\mathbb{E}^m_t$ with index t, and let $\bigwedge^{m-n}\mathbb{E}^m_t$ be the vector space obtained by the exterior product of $m-n$ vectors in $\mathbb{E}^m_t$. Let $f_{i_1}\wedge\cdots\wedge f_{i_{m-n}}$ and $g_{i_1}\wedge\cdots\wedge g_{i_{m-n}}$ be two vectors in $\bigwedge^{m-n}\mathbb{E}^m_t$, where $\{f_1, f_2, \ldots, f_m\}$ and $\{g_1, g_2, \ldots, g_m\}$ are two orthonormal bases of $\mathbb{E}^m_t$. Define an indefinite inner product $\langle\langle, \rangle\rangle$ on $\bigwedge^{m-n}\mathbb{E}^m_t$ by

$$\langle\langle f_{i_1}\wedge\cdots\wedge f_{i_{m-n}}, g_{i_1}\wedge\cdots\wedge g_{i_{m-n}}\rangle\rangle = \det(\langle f_{i_\ell}, g_{j_k}\rangle). \tag{2.5}$$

Therefore, for some positive integer s, we may identify $\bigwedge^{m-n}\mathbb{E}^m_t$ with some pseudo–Euclidean space $\mathbb{E}^N_s$, where $N = \binom{m}{m-n}$. The map $\nu : M \to G(m-$

$n, m) \subset \mathbb{E}_s^N$ from an oriented pseudo–Riemannian submanifold M into $G(m-n, m)$ defined by

$$\nu(p) = (e_{n+1} \wedge e_{n+2} \wedge \cdots \wedge e_m)(p) \tag{2.6}$$

is called the *Gauss map* of M which assigns to a point p in M the oriented $(m-n)$–plane through the origin of $\mathbb{E}_t^m$ and parallel to the normal space of M at p, [17].

We put $\varepsilon = \langle\langle \nu, \nu \rangle\rangle = \varepsilon_{n+1}\varepsilon_{n+2}\cdots\varepsilon_m = \pm 1$ and

$$\widetilde{M}_s^{N-1}(\varepsilon) = \begin{cases} \mathbb{S}_s^{N-1}(1) & \text{in } \mathbb{E}_s^N \quad \text{if } \varepsilon = 1 \\ \mathbb{H}_{s-1}^{N-1}(-1) & \text{in } \mathbb{E}_s^N \quad \text{if } \varepsilon = -1. \end{cases}$$

Then the Gauss image $\nu(M)$ can be viewed as $\nu(M) \subset \widetilde{M}_s^{N-1}(\varepsilon)$.

Lemma 2.1. *Let M be an n–dimensional submanifold of a pseudo–Euclidean space $\mathbb{E}_t^{n+2}$. Then, the Laplacian of the Gauss map $\nu = e_{n+1} \wedge e_{n+2}$ is given by*

$$\begin{aligned} \Delta\nu =& ||h||^2\nu + 2\sum_{j<k} \varepsilon_j\varepsilon_k R^D(e_j, e_k; e_{n+1}, e_{n+2}) e_j \wedge e_k \\ &+ \nabla(\mathrm{tr}A_{n+1}) \wedge e_{n+2} + e_{n+1} \wedge \nabla(\mathrm{tr}A_{n+2}) \\ &+ n\sum_{j=1}^{n} \varepsilon_j \omega_{(n+1)(n+2)}(e_j) H \wedge e_j, \end{aligned} \tag{2.7}$$

where $||h||^2$ is the squared length of the second fundamental form, R^D the normal curvature tensor, and $\nabla(\mathrm{tr}A_r)$ the gradient of $\mathrm{tr}A_r$.

Let M be a surface in the pseudo–Euclidean space $\mathbb{E}_2^4$. We choose a local orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ on M such that e_1, e_2 are tangent to M, and e_3, e_4 are normal to M. Let C be a vector field in $\bigwedge^2 \mathbb{E}_2^4 \equiv \mathbb{E}_4^6$. Since the set $\{e_A \wedge e_B | 1 \leq A < B \leq 4\}$ is an orthonormal basis for $\mathbb{E}_4^6$, the vector C can be expressed as

$$C = \sum_{1 \leq A < B \leq 4} \varepsilon_A \varepsilon_B C_{AB}\, e_A \wedge e_B, \tag{2.8}$$

where $C_{AB} = \langle\langle C, e_A \wedge e_B \rangle\rangle$.

Lemma 2.2. *A vector C in $\Lambda^2\mathbb{E}_2^4 \equiv \mathbb{E}_4^6$ written by (2.8) is constant if and only if the following equations are satisfied for $i = 1, 2$*

$$e_i(C_{12}) = \varepsilon_3 h_{i2}^3 C_{13} + \varepsilon_4 h_{i2}^4 C_{14} - \varepsilon_3 h_{i1}^3 C_{23} - \varepsilon_4 h_{i1}^4 C_{24}, \tag{2.9}$$

$$e_i(C_{13}) = -\varepsilon_2 h_{i2}^3 C_{12} + \varepsilon_4 \omega_{34}(e_i) C_{14} + \varepsilon_2 \omega_{12}(e_i) C_{23} - \varepsilon_4 h_{i1}^4 C_{34}, \tag{2.10}$$

$$e_i(C_{14}) = -\varepsilon_2 h_{i2}^4 C_{12} - \varepsilon_3 \omega_{34}(e_i) C_{13} + \varepsilon_2 \omega_{12}(e_i) C_{24} + \varepsilon_3 h_{i1}^3 C_{34}, \tag{2.11}$$

$$e_i(C_{23}) = \varepsilon_1 h_{i1}^3 C_{12} - \varepsilon_1 \omega_{12}(e_i) C_{13} + \varepsilon_4 \omega_{34}(e_i) C_{24} - \varepsilon_4 h_{i2}^4 C_{34}, \tag{2.12}$$

$$e_i(C_{24}) = \varepsilon_1 h_{i1}^4 C_{12} - \varepsilon_1 \omega_{12}(e_i) C_{14} - \varepsilon_3 \omega_{34}(e_i) C_{23} + \varepsilon_3 h_{i2}^3 C_{34}, \tag{2.13}$$

$$e_i(C_{34}) = \varepsilon_1 h_{i1}^4 C_{13} - \varepsilon_1 h_{i1}^3 C_{14} + \varepsilon_2 h_{i2}^4 C_{23} - \varepsilon_2 h_{i2}^3 C_{24}. \tag{2.14}$$

3 Rotational Surfaces in $\mathbb{E}^4_2$

In this section, we focus on rotational surfaces in $\mathbb{E}^4_2$ with profile curves which lie in 2–dimensional planes, and we obtain some geometric quantities about these surfaces.

Let $M_1(b)$ and $M_2(b)$ be rotational surfaces in the pseudo–Euclidean space $\mathbb{E}^4_2$ whose profile curves lie in 2–planes. These rotational surfaces defined below are invariant under some rotation subgroup of rotation group in $\mathbb{E}^4_2$. We can choose a profile curve α of $M_1(b)$ in the yw–plane as $\alpha(u) = (0, y(u), 0, w(u))$, defined on an open subset I of $\mathbb{R}$ and thus the parametrization of $M_1(b)$ is given by

$$M_1(b) : r_1(u, v) = (w(u) \sinh v, y(u) \cosh(bv), y(u) \sinh(bv), w(u) \cosh v) \quad (3.1)$$

with some constant $b > 0$, where $u \in I$ is an open subset of $\mathbb{R}$ and $v \in \mathbb{R}$.

We consider the following orthonormal moving frame field e_1, e_2, e_3, e_4 on $M_1(b)$ in $\mathbb{E}^4_2$ such that e_1, e_2 are tangent to $M_1(b)$, and e_3, e_4 are normal to $M_1(b)$:

$$e_1 = \frac{1}{q}\frac{\partial}{\partial v}, \quad e_2 = \frac{1}{A}\frac{\partial}{\partial u}, \quad (3.2)$$

$$e_3 = \frac{1}{A}(y'(u) \sinh v, w'(u) \cosh(bv), w'(u) \sinh(bv), y'(u) \cosh v), \quad (3.3)$$

$$e_4 = -\frac{\varepsilon\varepsilon^*}{q}(by(u) \cosh v, w(u) \sinh(bv), w(u) \cosh(bv), by(u) \sinh v), \quad (3.4)$$

where $A = \sqrt{\varepsilon({y'}^2(u) - {w'}^2(u))} \neq 0$, $q = \sqrt{\varepsilon^*(w^2(u) - b^2y^2(u))} \neq 0$, and $\varepsilon = \operatorname{sgn}({y'}^2(u) - {w'}^2(u))$, $\varepsilon^* = \operatorname{sgn}(w^2(u) - b^2y^2(u))$. Then $\varepsilon_1 = -\varepsilon_4 = \varepsilon^*$, $\varepsilon_2 = -\varepsilon_3 = \varepsilon$.

By a direct calculation, we have the components of the second fundamental form and the connection forms as

$$h^3_{11} = \frac{1}{Aq^2}(b^2y(u)w'(u) - w(u)y'(u)), \; h^3_{22} = \frac{1}{A^3}(w'(u)y''(u) - y'(u)w''(u)), \quad (3.5)$$

$$h^4_{12} = \frac{\varepsilon\varepsilon^* b}{Aq^2}(w(u)y'(u) - y(u)w'(u)), \quad h^3_{12} = h^4_{11} = h^4_{22} = 0, \quad (3.6)$$

$$\omega_{12}(e_1) = \frac{1}{Aq^2}(b^2y(u)y'(u) - w(u)w'(u)), \quad \omega_{12}(e_2) = 0, \quad (3.7)$$

$$\omega_{34}(e_1) = \frac{\varepsilon\varepsilon^* b}{Aq^2}(w(u)w'(u) - y(u)y'(u)), \quad \omega_{34}(e_2) = 0. \quad (3.8)$$

For the rotational surface $M_2(b)$, we can choose a profile curve β in the xz–plane as $\beta(u) = (x(u), 0, z(u), 0)$ defined on an open subset I of $\mathbb{R}$, and thus the parametrization of $M_2(b)$ is given by

$$M_2(b) : r_2(u, v) = (x(u) \cos v, x(u) \sin v, z(u) \cos(bv), z(u) \sin(bv)) \quad (3.9)$$

with some constant $b > 0$, where $u \in I$ is an open subset of $\mathbb{R}$ and $v \in (0, 2\pi)$.

We consider the following orthonormal moving frame fields e_1, e_2, e_3, e_4 on $M_2(b)$ in $\mathbb{E}^4_2$ such that e_1, e_2 are tangent to $M_2(b)$, and e_3, e_4 are normal to $M_2(b)$:

$$e_1 = \frac{1}{\bar{q}}\frac{\partial}{\partial v}, \quad e_2 = \frac{1}{\bar{A}}\frac{\partial}{\partial u}, \tag{3.10}$$

$$e_3 = \frac{1}{\bar{A}}(z'(u)\cos v, z'(u)\sin v, x'(u)\cos(bv), x'(u)\sin(bv)), \tag{3.11}$$

$$e_4 = -\frac{\varepsilon\varepsilon^*}{\bar{q}}(bz(u)\sin v, -bz(u)\cos v, x(u)\sin(bv), -x(u)\cos(bv)), \tag{3.12}$$

where $\bar{A} = \sqrt{\varepsilon(x'^2(u) - z'^2(u))} \neq 0$, $\bar{q} = \sqrt{\varepsilon^*(x^2(u) - b^2z^2(u))} \neq 0$, and $\varepsilon = \operatorname{sgn}(x'^2(u) - z'^2(u))$, $\varepsilon^* = \operatorname{sgn}(x^2(u) - b^2z^2(u))$. Then $\varepsilon_1 = -\varepsilon_4 = \varepsilon^*$, $\varepsilon_2 = -\varepsilon_3 = \varepsilon$.

By a direct computation, we have the components of the second fundamental form and the connection forms as

$$h^3_{11} = \frac{1}{\bar{A}\bar{q}^2}(b^2z(u)x'(u) - x(u)z'(u)),\ h^3_{22} = \frac{1}{\bar{A}^3}(z'(u)x''(u) - x'(u)z''(u)), \tag{3.13}$$

$$h^4_{12} = \frac{\varepsilon\varepsilon^* b}{\bar{A}\bar{q}^2}(z(u)x'(u) - x(u)z'(u)), \quad h^3_{12} = h^4_{11} = h^4_{22} = 0, \tag{3.14}$$

$$\omega_{12}(e_1) = \frac{1}{\bar{A}\bar{q}^2}(b^2z(u)z'(u) - x(u)x'(u)), \quad \omega_{12}(e_2) = 0, \tag{3.15}$$

$$\omega_{34}(e_1) = \frac{\varepsilon\varepsilon^* b}{\bar{A}\bar{q}^2}(z(u)z'(u) - x(u)x'(u)), \quad \omega_{34}(e_2) = 0. \tag{3.16}$$

Therefore, we have the mean curvature vector H, Gaussian curvature K and normal curvature R^D for the rotational surfaces for $M_1(b)$ and $M_2(b)$ as follows

$$H = -\frac{1}{2}(\varepsilon\varepsilon^* h^3_{11} + h^3_{22})e_3, \tag{3.17}$$

$$K = \varepsilon^*(h^4_{12})^2 - \varepsilon h^3_{11}h^3_{22}, \tag{3.18}$$

$$R^D(e_1, e_2; e_3, e_4) = h^4_{12}(\varepsilon h^3_{22} - \varepsilon^* h^3_{11}). \tag{3.19}$$

On the other hand, by using the Codazzi equation (2.3) we obtain

$$e_2(h^3_{11}) = \varepsilon^* h^4_{12}\omega_{34}(e_1) + \omega_{12}(e_1)(\varepsilon^* h^3_{11} - \varepsilon h^3_{22}), \tag{3.20}$$

$$e_2(h^4_{12}) = -\varepsilon h^3_{22}\omega_{34}(e_1) + 2\varepsilon^* h^4_{12}\omega_{12}(e_1). \tag{3.21}$$

The rotational surfaces $M_1(b)$ and $M_2(b)$ defined by (3.1) and (3.9) for $b = 1$, $x(u) = y(u) = f(u)\sinh u$ and $z(u) = w(u) = f(u)\cosh u$ are also known as Vranceanu rotational surface, where $f(u)$ is a smooth function, [15, 17].

4 Pseudo–Umbilical Rotational Surfaces in $\mathbb{E}^4_2$

In this section, we obtain all pseudo–umbilical rotational surfaces $M_1(b)$ and $M_2(b)$ in $\mathbb{E}^4_2$ defined by (3.1) and (3.9).

By the definition of pseudo–umbilical surface and (3.17), the rotational surfaces $M_1(b)$ and $M_2(b)$ are pseudo–umbilical if and only if $\varepsilon^* h^3_{11} = \varepsilon h^3_{22}$.

Hence, from (3.5) the surface $M_1(b)$ is pseudo–umbilical if and only if the component functions $y(u)$ and $w(u)$ of the profile curve α satisfy the following differential equation

$$w'(u)y''(u) - y'(u)w''(u) - ({y'}^2(u) - {w'}^2(u))\frac{b^2 y(u)w'(u) - w(u)y'(u)}{w^2(u) - b^2 y^2(u)} = 0. \quad (4.1)$$

By a simple computation, it can be shown that a non–planar rotational surface $M_1(b)$ in $\mathbb{E}^4_2$ defined by (3.1) for $b = 1$ is pseudo–umbilical if and only if its profile curve is given by

$$w(u) + y(u) = \lambda_0 (w(u) - y(u))^{\mu_0} \quad (4.2)$$

for some constants $\lambda_0 \neq 0$ and μ_0 such that $(w(u) - y(u))^{\mu_0}$ is real valued.

If $\mu_0 = 1$ and $\lambda_0^2 \neq 1$, from (4.2) we have $y(u) = \frac{\lambda_0 - 1}{\lambda_0 + 1} w(u)$, that is, the profile curve α is a part of line passing through the origin. It can be shown easily that $M_1(1)$ is an open part of a timelike plane in $\mathbb{E}^4_2$.

If $\mu_0 = -1$, then (4.2) implies that $w^2(u) - y^2(u) = \lambda_0$ which gives (a–5) and (a–6) in Theorem 4.2 for $b = 1$.

From equations (3.5), (3.6) we obtain $h^4_{12} = -\varepsilon\varepsilon^* h^3_{11}$ in the case $b = 1$. Also, we know that such a relation $\varepsilon^* h^3_{11} = \varepsilon h^3_{22}$ exists. Hence, from the equation (3.18) we conclude:

Proposition 4.1. *Let $M_1(1)$ be a rotational surface in $\mathbb{E}^4_2$ given by* (3.1)*. Then, $M_1(1)$ is pseudo–umbilical if and only if $M_1(1)$ is flat.*

In [17], flat Vranceanu surfaces which are pseudo–umbilical surfaces $M_1(1)$ were studied for different purposes. It was proven that the Vranceanu rotational surface is flat if $f(u) = \lambda e^{\mu u}$, where λ and μ are real numbers. Then, we have $\varepsilon^* = \mathrm{sgn}(\lambda^2 e^{2\mu u}) = 1$ and $\varepsilon = \mathrm{sgn}(\lambda^2(1-\mu^2)e^{2\mu u}) = 1$ for $|\mu| < 1$ and $\varepsilon = -1$ for $|\mu| > 1$. Thus, the Vranceanu rotational surface is spacelike pseudo–umbilical for $|\mu| < 1$ and timelike pseudo–umbilical for $|\mu| > 1$.

For $c_0 \neq 0$ and $\theta > 0$, we define the following functions

$$\Phi(\theta, b, \varepsilon, \varepsilon^*) = \int_0^\theta \sqrt{\frac{\varepsilon^* c_0^2(\sinh^2\eta - b^2\cosh^2\eta)}{\varepsilon^* c_0^2(\sinh^2\eta - b^2\cosh^2\eta) - \varepsilon}}\, d\eta \quad (4.3)$$

and

$$\Omega(\theta, b, \varepsilon, \varepsilon^*) = \int_0^\theta \sqrt{\frac{\varepsilon^* c_0^2(\cosh^2\eta - b^2\sinh^2\eta)}{\varepsilon^* c_0^2(\cosh^2\eta - b^2\sinh^2\eta) + \varepsilon}}\, d\eta. \quad (4.4)$$

such that the integrands are real valued functions.

Theorem 4.2. *Let $M_1(b)$ be a non–planar rotational surface in the pseudo–Euclidean space $\mathbb{E}^4_2$ given by (3.1). Then,*

(a) $M_1(b)$ is a spacelike pseudo–umbilical surface in $\mathbb{E}^4_2$ if and only if the component functions of the unit speed profile curve α of $M_1(b)$ are given by one of the followings:

(a-1)

$$y(\theta) = ce^{\psi(\theta)} \cosh\theta \ \text{ and } \ w(\theta) = ce^{\psi(\theta)} \sinh\theta,$$

where $\psi(\theta) = \Phi(\theta, b, 1, 1)$, $0 < b < 1$ and $c_0^2(\sinh^2\theta - b^2\cosh^2\theta) > 1$ for some $c_0 \in \mathbb{R}$ and $c \in \mathbb{R}_+$;

(a-2)

$$y(\theta) = ce^{\psi(\theta)} \cosh\theta \ \text{ and } \ w(\theta) = ce^{\psi(\theta)} \sinh\theta,$$

where $\psi(\theta) = \Phi(\theta, b, -1, -1)$, $b \geq 1$ and $c \in \mathbb{R}_+$. In this case, the surface $M_1(b)$ has negative definite metric;

(a-3)

$$y(\theta) = ce^{\varphi(\theta)} \sinh\theta \ \text{ and } \ w(\theta) = ce^{\varphi(\theta)} \cosh\theta,$$

where $\varphi(\theta) = \Omega(\theta, b, 1, 1)$, $0 < b \leq 1$ and $c \in \mathbb{R}_+$;

(a-4)

$$y(\theta) = ce^{\varphi(\theta)} \sinh\theta \ \text{ and } \ w(\theta) = ce^{\varphi(\theta)} \cosh\theta,$$

where $\varphi(\theta) = \Omega(\theta, b, -1, -1)$, $b > 1$ and $c_0^2(b^2\sinh^2\theta - \cosh^2\theta) > 1$ for some $c_0 \in \mathbb{R}$ and $c \in \mathbb{R}_+$. In this case, the surface $M_1(b)$ has negative definite metric;

(a-5)

$$y(\theta) = r_0 \sinh\theta \ \text{ and } \ w(\theta) = r_0 \cosh\theta,$$

where r_0 is non–zero constant and $0 < b \leq 1$. In this case, the surface $M_1(b)$ lies in $\mathbb{H}^3_1(-r_0^{-2}) \subset \mathbb{E}^4_2$;

(a-6)

$$y(\theta) = r_0 \cosh\theta \ \text{ and } \ w(\theta) = r_0 \sinh\theta,$$

where r_0 is non–zero constant and $b \geq 1$. In this case, the surface $M_1(b)$ has negative definite metric and is lying in $\mathbb{S}^3_2(r_0^{-2}) \subset \mathbb{E}^4_2$.

(b) $M_1(b)$ is a timelike pseudo–umbilical surface in $\mathbb{E}^4_2$ if and only if the component functions of the unit speed profile curve α of $M_1(b)$ are given by one of the followings:

(b-1)

$$y(\theta) = ce^{\psi(\theta)} \cosh\theta \ \text{ and } \ w(\theta) = ce^{\psi(\theta)} \sinh\theta,$$

where $\psi(\theta) = \Phi(\theta, b, 1, -1)$, $b \geq 1$ and $c_0^2(b^2\cosh^2\theta - \sinh^2\theta) > 1$ for some $c_0 \in \mathbb{R}$ and $c \in \mathbb{R}_+$;

(b-2)

$$y(\theta) = ce^{\psi(\theta)} \cosh\theta \ \text{ and } \ w(\theta) = ce^{\psi(\theta)} \sinh\theta,$$

where $\psi(\theta) = \Phi(\theta, b, -1, 1)$, $0 < b < 1$ *and* $c \in \mathbb{R}_+$*;*

(b-3)

$$y(\theta) = ce^{\varphi(\theta)} \sinh\theta \ \ \textit{and} \ \ w(\theta) = ce^{\varphi(\theta)} \cosh\theta,$$

where $\varphi(\theta) = \Omega(\theta, b, 1, -1)$, $b > 1$ *and* $c \in \mathbb{R}_+$*;*

(b-4)

$$y(\theta) = ce^{\varphi(\theta)} \sinh\theta \ \ \textit{and} \ \ w(\theta) = ce^{\varphi(\theta)} \cosh\theta,$$

where $\varphi(\theta) = \Omega(\theta, b, -1, 1)$, $0 < b \leq 1$ *and* $c_0^2(\cosh^2\theta - b^2\sinh^2\theta) > 1$ *for some* $c_0 \in \mathbb{R}$ *and* $c \in \mathbb{R}_+$*;*

(b-5)

$$y(\theta) = r_0 \sinh\theta \ \ \textit{and} \ \ w(\theta) = r_0 \cosh\theta,$$

where r_0 *is non–zero constant and* $b > 1$*. In this case, the surface* $M_1(b)$ *lies in* $\mathbb{H}_1^3(-r_0^{-2}) \subset \mathbb{E}_2^4$*;*

(b-6)

$$y(\theta) = r_0 \cosh\theta \ \ \textit{and} \ \ w(\theta) = r_0 \sinh\theta,$$

where r_0 *is non–zero constant and* $0 < b < 1$*. In this case, the surface* $M_1(b)$ *lies in* $\mathbb{S}_2^3(r_0^{-2}) \subset \mathbb{E}_2^4$*.*

Proof. Let $M_1(b)$ be a rotational surface in the pseudo–Euclidean space $\mathbb{E}_2^4$ given by (3.1). From (3.7) and (3.8), it is seen that $\omega_{12}(e_1)$ and $\omega_{34}(e_1)$ are functions of u, and $\omega_{12}(e_2) = \omega_{34}(e_2) = 0$. By using these facts and (3.19), we have

$$-e_2(\omega_{34}(e_1)) + \varepsilon^*\omega_{12}(e_1)\omega_{34}(e_1) = h_{12}^4(\varepsilon h_{22}^3 - \varepsilon^* h_{11}^3). \tag{4.5}$$

Now, assume that $M_1(b)$ is pseudo-umbilical surface, i.e., $\varepsilon^* h_{11}^3 = \varepsilon h_{22}^3$. Then (4.5) implies

$$e_2(\omega_{34}(e_1)) - \varepsilon^*\omega_{12}(e_1)\omega_{34}(e_1) = 0. \tag{4.6}$$

This equation together with the second equation in (3.2) and the first equation in (3.7) gives

$$\frac{d}{du}(\omega_{34}(e_1)) = -\frac{w(u)w'(u) - b^2 y(u)y'(u)}{w^2(u) - b^2 y^2(u)}\omega_{34}(e_1). \tag{4.7}$$

It is clear that $\omega_{34}(e_1) = 0$ is a solution of (4.7). In this case, from (3.8) we have $w(u)w'(u) - y(u)y'(u) = 0$ which implies that $w^2(u) - y^2(u) = \lambda_0$, for non–zero constant λ_0.

For $\lambda_0 = r_0^2 > 0$, we put $y(u) = r_0 \sinh\theta(u)$ and $w(u) = r_0 \cosh\theta(u)$, where $\theta(u)$ is a smooth function with $\theta'(u) \neq 0$. So, $\varepsilon = \operatorname{sgn}(r_0^2\theta'^2(u)) = 1$ and $\varepsilon^* = \operatorname{sgn}(r_0^2(\cosh^2\theta(u) - b^2\sinh^2\theta(u))) = 1$ for $0 < b \leq 1$ and $\varepsilon^* = -1$ for $b > 1$. Therefore, for $0 < b \leq 1$, $M_1(b)$ is a spacelike pseudo-umbilical surface which gives (a–5), and for $b > 1$, $M_1(b)$ is a timelike pseudo-umbilical surface which gives (b–5). Moreover, $M_1(b)$ lies in $\mathbb{H}_1^3(-{r_0}^{-2}) \subset \mathbb{E}_2^4$.

For $\lambda_0 = -r_0^2 < 0$, we put $y(u) = r_0 \cosh\theta(u)$ and $w(u) = r_0 \sinh\theta(u)$, where $\theta(u)$ is a smooth function with $\theta'(u) \neq 0$. So $\varepsilon = \operatorname{sgn}(-r_0^2\theta'^2(u)) = -1$, and $\varepsilon^* = \operatorname{sgn}(r_0^2(\sinh^2\theta(u) - b^2\cosh^2\theta(u))) = -1$ for $b \geq 1$ and $\varepsilon^* = 1$ for $0 < b < 1$. Then, for $b \geq 1$, $M_1(b)$ is a spacelike pseudo-umbilical surface with negative definite metric which gives (a–6), and for $0 < b < 1$, $M_1(b)$ is a timelike pseudo-umbilical surface which gives (b–6). Moreover, $M_1(b)$ lies in $\mathbb{S}_2^3({r_0}^{-2}) \subset \mathbb{E}_2^4$.

Let $\omega_{34} \neq 0$ on M_1. By combining (3.8) and (4.7) we have

$$\frac{\varepsilon\varepsilon^* b(w(u)w'(u) - y(u)y'(u))}{\sqrt{\varepsilon^*(w^2(u) - b^2y^2(u))}\sqrt{\varepsilon({y'}^2(u) - {w'}^2(u))}} = b_0 \tag{4.8}$$

for some constant $b_0 \neq 0$.

Now we suppose that the profile curve α is a unit speed curve, that is, ${y'}^2(u) - {w'}^2(u) = \varepsilon$. Thus equation (4.8) becomes

$$\frac{w(u)w'(u) - y(u)y'(u)}{\sqrt{\varepsilon^*(w^2(u) - b^2y^2(u))}} = c_0 \tag{4.9}$$

for some constant $c_0 \neq 0$.

Without loss of generality, firstly we choose $y(u) = r(u)\cosh\theta(u)$ and $w(u) = r(u)\sinh\theta(u)$. Then, from ${y'}^2(u) - {w'}^2(u) = \varepsilon$ and (4.9) we have, respectively,

$$\varepsilon du^2 = dr^2 - r^2 d\theta^2 \quad \text{and} \quad du = -\frac{dr}{c_0\sqrt{\varepsilon^*(\sinh^2\theta - b^2\cosh^2\theta)}}$$

from which we obtain that

$$\frac{dr}{r} = \sqrt{\frac{\varepsilon^* c_0^2(\sinh^2\theta - b^2\cosh^2\theta)}{\varepsilon^* c_0^2(\sinh^2\theta - b^2\cosh^2\theta) - \varepsilon}}\, d\theta, \tag{4.10}$$

where $\varepsilon^* c_0^2(\sinh^2\theta - b^2\cosh^2\theta) > \varepsilon$. The integration of (4.10) gives

$$r(\theta) = ce^{\Phi(\theta,b,\varepsilon,\varepsilon^*)}, \tag{4.11}$$

where $\Phi(\theta,b,\varepsilon,\varepsilon^*)$ is defined by (4.3) and $c \in \mathbb{R}_+$. From $\varepsilon^* = \operatorname{sgn}(r^2(u)(\sinh^2\theta(u) - b^2\cosh^2\theta(u)))$, we get $\varepsilon^* = 1$ for $0 < b < 1$, and $\varepsilon^* = -1$ for $b \geq 1$. Now, by (4.11) and (4.3) we have (a–1) if $\varepsilon = \varepsilon^* = 1$, and the integrand in (4.11) is defined for $c_0^2(\sinh^2\theta - b^2\cosh^2\theta) > 1$ for some $c_0 \in \mathbb{R}$; (a–2) if $\varepsilon = \varepsilon^* = -1$; (b–1) if $\varepsilon = -\varepsilon^* = 1$, and the integrand in (4.3) is defined for $c_0^2(b^2\cosh^2\theta - \sinh^2\theta) > 1$ for some $c_0 \in \mathbb{R}$; (b–2) if $\varepsilon^* = -\varepsilon = 1$..

Secondly, let $y(u) = r(u)\sinh\theta(u)$ and $w(u) = r(u)\cosh\theta(u)$. By a similar calculation we obtain that

$$r(\theta) = ce^{\Omega(\theta,b,\varepsilon,\varepsilon^*)}, \tag{4.12}$$

where $\Omega(\theta,b,\varepsilon,\varepsilon^*)$ is defined by (4.4) and $c \in \mathbb{R}_+$. From $\varepsilon^* = \operatorname{sgn}(r^2(u)(\cosh^2\theta(u) - b^2\sinh^2\theta(u)))$ we get $\varepsilon^* = 1$ for $0 < b \leq 1$, and $\varepsilon^* = -1$ for $b > 1$. Now by considering (4.12) and (4.4) we have (a–3) if $\varepsilon = \varepsilon^* = 1$, and (a–4) if $\varepsilon = \varepsilon^* = -1$

and the integrand in (4.4) is defined for $c_0^2(b^2\sinh^2\theta - \cosh^2\theta) > 1$ for some $c_0 \in \mathbb{R}$; (b–3) if $\varepsilon = -\varepsilon^* = 1$, and (b–4) if $\varepsilon^* = -\varepsilon = 1$ and the integrand in (4.4) is defined for $c_0^2(\cosh^2\theta - b^2\sinh^2\theta) > 1$ for some $c_0 \in \mathbb{R}$.

Conversely, we assume that $y(\theta)$ and $w(\theta)$ are given by $y(\theta) = ce^{\Phi(\theta,b,\varepsilon,\varepsilon^*)}\cosh\theta$ and $w(\theta) = ce^{\Phi(\theta,b,\varepsilon,\varepsilon^*)}\sinh\theta$ for the function Φ defined by (4.3). Since $y(\theta)$ and $w(\theta)$ satisfy (4.6), equation (4.5) implies that either $h_{12}^4 = 0$ or $\varepsilon^* h_{11}^3 = \varepsilon h_{22}^3$. From the first equation in (3.6) we have $h_{12}^4 \neq 0$ as $\frac{d\theta}{du} \neq 0$, and thus $\varepsilon^* h_{11}^3 = \varepsilon h_{22}^3$. In the case, the profile curve α given by $y(\theta) = ce^{\Omega(\theta,b,\varepsilon,\varepsilon^*)}\sinh\theta$ and $w(\theta) = ce^{\Omega(\theta,b,\varepsilon,\varepsilon^*)}\cosh\theta$ for the function Ω defined by (4.4), by a similar argument it can be seen that $\varepsilon^* h_{11}^3 = \varepsilon h_{22}^3$. Therefore $M_1(b)$ is a pseudo–umbilical surface in the pseudo–Euclidean space $\mathbb{E}_2^4$. □

Similarly, we determine pseudo–umbilical rotational surface $M_2(b)$ in $\mathbb{E}_2^4$ given by (3.9). From (3.13), the surface $M_2(b)$ is pseudo–umbilical if and only if the component functions $x(u)$ and $z(u)$ of the profile curve β satisfy the differential equation

$$z'(u)x''(u) - x'(u)z''(u) - ({x'}^2(u) - {z'}^2(u))\frac{b^2 z(u)x'(u) - x(u)z'(u)}{x^2(u) - b^2 z^2(u)} = 0. \quad (4.13)$$

By a simple computation, it can be shown that a non–planar rotational surface $M_2(b)$ in $\mathbb{E}_2^4$ defined by (3.9) for $b = 1$ is pseudo–umbilical if and only if its profile curve is given by

$$z(u) - x(u) = \lambda_0(z(u) + x(u))^{\mu_0} \quad (4.14)$$

for some constants $\lambda_0 \neq 0$ and μ_0 such that $(z(u) + x(u))^{\mu_0}$ is real valued.

If $\mu_0 = 1$ and $\lambda_0^2 \neq 1$, from (4.14) we have $x(u) = \frac{1-\lambda_0}{1+\lambda_0}z(u)$, that is the profile curve β is a part of a line passing through the origin. It can be shown easily that $M_2(1)$ is an open part of a spacelike plane in $\mathbb{E}_2^4$.

If $\mu_0 = -1$, then (4.14) implies that $z^2(u) - x^2(u) = \lambda_0$ which gives (b–5) and (b–6) in Theorem 4.4 for $b = 1$.

Because of the similar reason for the rotational surface $M_1(b)$, we have the following:

Proposition 4.3. *Let $M_2(1)$ be a rotational surface in $\mathbb{E}_2^4$ given by* (3.9)*. Then, $M_2(1)$ is pseudo–umbilical if and only if $M_2(1)$ is flat.*

In [17], it was shown that the Vranceanu rotational surface is flat if $f(u) = \lambda e^{\mu u}$, where λ and μ are real numbers. For the function $f(u)$, the component function $x(u)$ and $z(u)$ satisfy the solution (4.14). Moreover, $\varepsilon^* = \mathrm{sgn}(-\lambda^2 e^{2\mu u}) = -1$ and $\varepsilon = \mathrm{sgn}(\lambda^2(1-\mu^2)e^{2\mu u}) = 1$ for $|\mu| < 1$, and $\varepsilon = -1$ for $|\mu| > 1$. Thus, the Vranceanu rotational surface is timelike pseudo–umbilical for $|\mu| < 1$ and it is spacelike pseudo–umbilical with negative definite metric for $|\mu| > 1$.

We omit the proof of the next theorem because it is similar to the proof of Theorem 4.4.

For $\bar{c_0} \neq 0$ and $\theta > 0$ let us define the following functions

$$\bar{\Phi}(\theta, b, \varepsilon, \varepsilon^*) = \int_0^\theta \sqrt{\frac{\varepsilon^* \bar{c_0}^2 (\cosh^2 \eta - b^2 \sinh^2 \eta)}{\varepsilon^* \bar{c_0}^2 (\cosh^2 \eta - b^2 \sinh^2 \eta) - \varepsilon}} d\eta \tag{4.15}$$

and

$$\bar{\Omega}(\theta, b, \varepsilon, \varepsilon^*) = \int_0^\theta \sqrt{\frac{\varepsilon^* \bar{c_0}^2 (\sinh^2 \eta - b^2 \cosh^2 \eta)}{\varepsilon^* \bar{c_0}^2 (\sinh^2 \eta - b^2 \cosh^2 \eta) + \varepsilon}} d\eta. \tag{4.16}$$

such that the integrands are real valued functions.

Theorem 4.4. *Let $M_2(b)$ be a non–planar rotational surface in the pseudo–Euclidean space $\mathbb{E}^4_2$ given by (3.9). Then,*

(a) $M_2(b)$ is a spacelike pseudo–umbilical surface in $\mathbb{E}^4_2$ if and only if the component functions of the unit speed profile curve β of $M_2(b)$ are given by one of the followings:

(a-1)

$$x(\theta) = \bar{c} e^{\psi(\theta)} \cosh \theta \ \ \textit{and} \ \ z(\theta) = \bar{c} e^{\psi(\theta)} \sinh \theta,$$

where $\psi(\theta) = \bar{\Phi}(\theta, b, 1, 1)$, $0 < b \leq 1$ and $\bar{c_0}^2(\cosh^2 \theta - b^2 \sinh^2 \theta) > 1$ for some $\bar{c_0} \in \mathbb{R}$ and $\bar{c} \in \mathbb{R}_+$;

(a-2)

$$x(\theta) = \bar{c} e^{\psi(\theta)} \cosh \theta \ \ \textit{and} \ \ z(\theta) = \bar{c} e^{\psi(\theta)} \sinh \theta,$$

where $\psi(\theta) = \bar{\Phi}(\theta, b, -1, -1)$, $b > 1$ and $\bar{c} \in \mathbb{R}_+$. In this case, the surface $M_2(b)$ has negative definite metric;

(a-3)

$$x(\theta) = \bar{c} e^{\varphi(\theta)} \sinh \theta \ \ \textit{and} \ \ z(\theta) = \bar{c} e^{\varphi(\theta)} \cosh \theta,$$

where $\varphi(\theta) = \bar{\Omega}(\theta, b, 1, 1)$, $0 < b < 1$ and $\bar{c} \in \mathbb{R}_+$;

(a-4)

$$x(\theta) = \bar{c} e^{\varphi(\theta)} \sinh \theta \ \ \textit{and} \ \ z(\theta) = \bar{c} e^{\varphi(\theta)} \cosh \theta,$$

where $\varphi(\theta) = \bar{\Omega}(\theta, b, -1, -1)$, $b \geq 1$ and $\bar{c_0}^2(b^2 \cosh^2 \theta - \sinh^2 \theta) > 1$ for some $\bar{c_0} \in \mathbb{R}$ and $\bar{c} \in \mathbb{R}_+$. In this case, the surface $M_2(b)$ has negative definite metric;

(a-5)

$$x(\theta) = r_0 \sinh \theta \ \ \textit{and} \ \ z(\theta) = r_0 \cosh \theta,$$

where r_0 is non–zero constant and $0 < b < 1$. In this case, the surface $M_2(b)$ lies in $\mathbb{H}^3_1(-r_0^{-2}) \subset \mathbb{E}^4_2$;

(a-6)

$$x(\theta) = r_0 \cosh \theta \ \ \textit{and} \ \ z(\theta) = r_0 \sinh \theta,$$

where r_0 is non–zero constant and $b > 1$. In this case, the surface $M_2(b)$ is lying in $\mathbb{S}^3_2(r_0^{-2}) \subset \mathbb{E}^4_2$ with negative definite metric.

(b) $M_2(b)$ *is a timelike pseudo–umbilical surface in* $\mathbb{E}^4_2$ *if and only if the component functions of the unit speed profile curve* β *of* $M_2(b)$ *are given by one of the followings:*

(b-1)

$$x(\theta) = \bar{c}e^{\psi(\theta)}\cosh\theta \ \ \textit{and} \ \ z(\theta) = \bar{c}e^{\psi(\theta)}\sinh\theta,$$

where $\psi(\theta) = \bar{\Phi}(\theta, b, 1, -1)$*,* $b > 1$ *and* $\bar{c_0}^2(b^2\sinh^2\theta - \cosh^2\theta) > 1$ *for some* $\bar{c_0} \in \mathbb{R}$ *and* $\bar{c} \in \mathbb{R}_+$*;*

(b-2)

$$x(\theta) = \bar{c}e^{\psi(\theta)}\cosh\theta \ \ \textit{and} \ \ z(\theta) = \bar{c}e^{\psi(\theta)}\sinh\theta,$$

where $\psi(\theta) = \bar{\Phi}(\theta, b, -1, 1)$*,* $0 < b \leq 1$ *and* $\bar{c} \in \mathbb{R}_+$*;*

(b-3)

$$x(\theta) = \bar{c}e^{\varphi(\theta)}\sinh\theta \ \ \textit{and} \ \ z(\theta) = \bar{c}e^{\varphi(\theta)}\cosh\theta,$$

where $\varphi(\theta) = \bar{\Omega}(\theta, b, 1, -1)$*,* $b \geq 1$ *and* $\bar{c} \in \mathbb{R}_+$*;*

(b-4)

$$x(\theta) = \bar{c}e^{\varphi(\theta)}\sinh\theta \ \ \textit{and} \ \ z(\theta) = \bar{c}e^{\varphi(\theta)}\cosh\theta,$$

where $\varphi(\theta) = \bar{\Omega}(\theta, b, -1, 1)$*,* $0 < b < 1$ *and* $\bar{c_0}^2(\sinh^2\theta - b^2\cosh^2\theta) > 1$ *for some* $\bar{c_0} \in \mathbb{R}$ *and* $\bar{c} \in \mathbb{R}_+$*;*

(b-5)

$$x(\theta) = r_0\sinh\theta \ \ \textit{and} \ \ z(\theta) = r_0\cosh\theta,$$

where r_0 *is non–zero constant and* $b \geq 1$*. In this case, the surface* $M_2(b)$ *lies in* $\mathbb{H}^3_1(-r_0^{-2}) \subset \mathbb{E}^4_2$*;*

(b-6)

$$x(\theta) = r_0\cosh\theta \ \ \textit{and} \ \ z(\theta) = r_0\sinh\theta,$$

where r_0 *is non–zero constant and* $0 < b \leq 1$*. In this case, the surface* $M_2(b)$ *lies in* $\mathbb{S}^3_2(r_0^{-2}) \subset \mathbb{E}^4_2$*.*

5 Pseudo–Umbilical Rotational Surfaces with Pointwise 1–Type Gauss Map

In this section, we determine pseudo–umbilical rotational surfaces in $\mathbb{E}^4_2$ with pointwise 1–type Gauss map of first kind and second kind.

Theorem 5.1. *There exists no pseudo–umbilical rotational surface defined by* (3.1) *in* $\mathbb{E}^4_2$ *with pointwise 1–type Gauss map of the second kind.*

Proof. Assume that $M_1(b)$ is a non–planar regular pseudo–umbilical rotational surface in $\mathbb{E}^4_2$ defined by (3.1). From equation (2.7), the Laplacian of the Gauss map of the rotational surface $M_1(b)$ is given by

$$\begin{aligned}\Delta\nu =&||h||^2\nu + 2h^4_{12}(\varepsilon^* h^3_{22} - \varepsilon h^3_{11})e_1 \wedge e_2\\ &+ \omega_{34}(e_1)(\varepsilon h^3_{11} + \varepsilon^* h^3_{22})e_1 \wedge e_3 + (\varepsilon\varepsilon^* e_2(h^3_{11}) + e_2(h^3_{22}))e_2 \wedge e_4. \qquad (5.1)\end{aligned}$$

Since $\varepsilon^* h^3_{11} = \varepsilon h^3_{22}$, equation (5.1) becomes

$$\Delta\nu = \|h\|^2\nu + 2\varepsilon h^3_{11}\omega_{34}(e_1)e_1 \wedge e_3 + 2\varepsilon h^4_{12}\omega_{34}(e_1)e_2 \wedge e_4. \qquad (5.2)$$

Suppose that $M_1(b)$ has pointwise 1–type Gauss map of second kind. Comparing (1.2) and (5.2), we get

$$f(1 + \varepsilon\varepsilon^* C_{34}) = ||h||^2, \qquad (5.3)$$

$$fC_{13} = -2\varepsilon^* h^3_{11}\omega_{34}(e_1), \qquad (5.4)$$

$$fC_{24} = -2\varepsilon^* h^4_{12}\omega_{34}(e_1), \qquad (5.5)$$

$$C_{12} = C_{14} = C_{23} = 0. \qquad (5.6)$$

From (5.4) and (5.5), we have

$$h^4_{12}C_{13} - h^3_{11}C_{24} = 0. \qquad (5.7)$$

When we write the equation (2.9) for $i = 2$, we obtain

$$h^3_{11}C_{13} - h^4_{12}C_{24} = 0. \qquad (5.8)$$

Since the Gauss map ν is of the second kind, equations (5.7) and (5.8) must have non–zero solution which implies $(h^3_{11})^2 - (h^4_{12})^2 = 0$. Considering the first equations in (3.5) and (3.6) we have $(b^2-1)(b^2y^2(u)w'^2(u)-w^2(u)y'^2(u)) = 0$. If $b^2y^2(u)w'^2(u)-w^2(u)y'^2(u) = 0$, by solving this equation and $y'^2(u)-w'^2(u) = \varepsilon A^2$ together, we get

$$y'^2(u) = -\varepsilon\varepsilon^* b^2\frac{A^2}{q^2}y^2(u) \text{ and } w'^2(u) = -\varepsilon\varepsilon^*\frac{A^2}{q^2}w^2(u) \qquad (5.9)$$

where $A = \sqrt{\varepsilon(y'^2(u) - w'^2(u))} \neq 0$ and $q = \sqrt{\varepsilon^*(w^2(u) - b^2y^2(u))} \neq 0$. Differentiating equations in (5.9) with respect to u, we obtain

$$\begin{aligned}2y'(u)y''(u) &= -\varepsilon\varepsilon^*\left(\frac{A^2}{q^2}\right)' y^2(u) - 2\varepsilon\varepsilon^* b^2\frac{A^2}{q^2}y(u)y'(u),\\ 2w'(u)w''(u) &= -\varepsilon\varepsilon^*\left(\frac{A^2}{q^2}\right)' w^2(u) - 2\varepsilon\varepsilon^* b^2\frac{A^2}{q^2}w(u)w'(u). \qquad (5.10)\end{aligned}$$

If we multiply these equations by $-w'^2(u)$ and $y'^2(u)$, respectively, add them and also consider $b^2y^2(u)w'^2(u) - w^2(u)y'^2(u) = 0$, we get

$$y'(u)w'(u)\left(y'(u)w''(u) - w'(u)y''(u) - \varepsilon\varepsilon^*\left(\frac{A^2}{q^2}\right)(y(u)w'(u) - w(u)y'(u))\right) = 0.$$

If $y = y_0 = constant$ or $w = w_0 = constant$ on an open subinterval of I, then $M_1(b)$ is a planar rotational surface. So, there is an open subinterval $J \subset I$ on which $y'(u)w'(u) \neq 0$, that is,

$$y'(u)w''(u) - w'(u)y''(u) - \varepsilon\varepsilon^* \frac{A^2}{q^2}(b^2 y(u)w'(u) - w(u)y'(u)) = 0.$$

Using (3.5) in the equation given above, we get $\varepsilon\varepsilon^* h^3_{11} + h^3_{22} = 0$. On the other hand, from the equation (3.17), $M_1(b)$ has zero mean curvature vector in $\mathbb{E}^4_2$. That is contradiction to the definition of pseudo–umbilical surface. Thus, $b^2 y^2(u) w'^2(u) - w^2(u) y'^2(u) \neq 0$, that is, $b = 1$. In this case, $h^4_{12} = -\varepsilon\varepsilon^* h^3_{11}$ and $\omega_{34}(e_1) = -\varepsilon\varepsilon^* \omega_{12}(e_1)$. Thus, from the equation (5.7) $C_{13} = -\varepsilon\varepsilon^* C_{24}$. Also, C_{34} is zero due to equations (2.10) and (2.13) for $i = 2$. On the other hand, from (2.11) for $i = 1$ we have $\omega_{34}(e_1) = 0$ which is a contradiction. Thus, ν is not of pointwise 1–type of second kind. □

Note that if ν were pointwise 1–type of first kind, it would happen that from (5.4) and (5.5) $h^3_{11} = h^4_{12} = 0$ or $\omega_{34}(e_1) = 0$.

In the case $h^3_{11} = h^4_{12} = 0$, $M_1(b)$ lies in the 3–dimensional Euclidean or pseudo–Euclidean space. Thus, we omit this case.

For $\omega_{34}(e_1) = 0$, we obtained some class of rotational surfaces as seen in the proof of Theorem 4.2. Thus, we conclude the following results:

Corollary 5.2. *Let $M_1(b)$ be a non–planar pseudo–umbilical rotational surface in the pseudo–Euclidean space $\mathbb{E}^4_2$ given by (3.1). Then, M_1 has pointwise 1–type Gauss map of the first kind if and only if the component functions of the unit speed profile curve α of $M_1(b)$ are given by one of the followings:*

i.

$$y(\theta) = r_0 \sinh\theta \;\; and \;\; w(\theta) = r_0 \cosh\theta,$$

where r_0 is non–zero constant and $0 < b \leq 1$. In this case, $M_1(b)$ is a spacelike surface in $\mathbb{H}^3_1(-r_0^{-2}) \subset \mathbb{E}^4_2$;

ii.

$$y(\theta) = r_0 \cosh\theta \;\; and \;\; w(\theta) = r_0 \sinh\theta,$$

where r_0 is non–zero constant and $b \geq 1$. In this case, $M_1(b)$ is a spacalike surface with negative definite metric in $\mathbb{S}^3_2(r_0^{-2}) \subset \mathbb{E}^4_2$;

iii.

$$y(\theta) = r_0 \sinh\theta \;\; and \;\; w(\theta) = r_0 \cosh\theta,$$

where r_0 is non–zero constant and $b > 1$. In this case, $M_1(b)$ is a timelike surface in $\mathbb{H}^3_1(-r_0^{-2}) \subset \mathbb{E}^4_2$;

iv.

$$y(\theta) = r_0 \cosh\theta \;\; and \;\; w(\theta) = r_0 \sinh\theta,$$

where r_0 is non–zero constant and $0 < b < 1$. In this case, $M_1(b)$ is a timelike surface in $\mathbb{S}^3_2(r_0^{-2}) \subset \mathbb{E}^4_2$.

Similarly, we can give same results for the rotational surface $M_2(b)$ given by (3.9).

Theorem 5.3. *There exists no pseudo–umbilical rotational surface defined by* (3.9) *in* $\mathbb{E}^4_2$ *with pointwise 1–type Gauss map of the second kind.*

Corollary 5.4. *Let* $M_2(b)$ *be a non–planar pseudo–umbilical rotational surface in the pseudo–Euclidean space* $\mathbb{E}^4_2$ *given by* (3.9)*. Then,* M_2 *has pointwise 1–type Gauss map of the first kind if and only if the component functions of the unit speed profile curve* β *of* $M_2(b)$ *are given by one of the followings:*

i.
$$x(\theta) = r_0 \sinh\theta \ \text{ and } \ z(\theta) = r_0 \cosh\theta,$$

where r_0 *is non–zero constant and* $0 < b < 1$*. In this case,* $M_2(b)$ *is a spacelike surface lying in* $\mathbb{H}^3_1(-r_0^{-2}) \subset \mathbb{E}^4_2$*;*

ii.
$$x(\theta) = r_0 \cosh\theta \ \text{ and } \ z(\theta) = r_0 \sinh\theta,$$

where r_0 *is non–zero constant and* $b > 1$*. In this case,* $M_2(b)$ *is a spacelike surface lying in* $\mathbb{S}^3_2(r_0^{-2}) \subset \mathbb{E}^4_2$ *with negative definite metric;*

iii.
$$x(\theta) = r_0 \sinh\theta \ \text{ and } \ z(\theta) = r_0 \cosh\theta,$$

where r_0 *is non–zero constant and* $b \geq 1$*. In this case,* $M_2(b)$ *is a timelike surface lying in* $\mathbb{H}^3_1(-r_0^{-2}) \subset \mathbb{E}^4_2$

iv.
$$x(\theta) = r_0 \cosh\theta \ \text{ and } \ z(\theta) = r_0 \sinh\theta,$$

where r_0 *is non–zero constant and* $0 < b \leq 1$*. In this case,* $M_2(b)$ *is a timelike surface lying in* $\mathbb{S}^3_2(r_0^{-2}) \subset \mathbb{E}^4_2$*.*

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Meridian Surfaces on Rotational Hypersurfaces with Lightlike Axis in $\mathbb{E}^4_2$

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Abstract. We construct a special class of Lorentz surfaces in the pseudo-Euclidean 4-space with neutral metric which are one-parameter systems of meridians of rotational hypersurfaces with lightlike axis and call them meridian surfaces. We give the complete classification of the meridian surfaces with constant Gauss curvature and prove that there are no meridian surfaces with parallel mean curvature vector field other than CMC surfaces lying in a hyperplane. We also classify the meridian surfaces with parallel normalized mean curvature vector field. We show that in the family of the meridian surfaces there exist Lorentz surfaces which have parallel normalized mean curvature vector field but not parallel mean curvature vector.
Keywords. Meridian surfaces · constant Gauss curvature · parallel mean curvature vector · parallel normalized mean curvature vector.
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1 Introduction

A fundamental problem of the contemporary differential geometry of surfaces and hypersurfaces in standard model spaces such as the Euclidean space $\mathbb{E}^n$ and the pseudo-Euclidean space $\mathbb{E}^n_k$ is the investigation of the basic invariants characterizing the surfaces. Curvature invariants are the number one Riemannian invariants and the most natural ones. The basic intrinsic curvature invariant of a surface in 4-dimensional Euclidean or pseudo-Euclidean space is the Gauss curvature and one basic extrinsic invariant is the curvature of the normal connection. The most important normal vector field of a surface is the mean curvature vector field. So, a fundamental question is to investigate various important classes of surfaces characterized by conditions on the Gauss curvature, the normal curvature, or the mean curvature vector field, and to find examples of surfaces belonging to these classes.

Rotational surfaces and hypersurfaces are basic source of examples of many geometric classes of surfaces in Riemannian and pseudo-Riemannian geometry. The main purpose of this paper is to provide a comprehensive survey on a special class of surfaces (called meridian surfaces) in 4-dimensional Euclidean or pseudo-Euclidean spaces which are one-parameter systems of meridians of rotational hypersurfaces. We present briefly recent results on meridian surfaces in the Euclidean space $\mathbb{E}^4$ and the Minkowski space $\mathbb{E}^4_1$.

In the present paper, the new contribution to the theory of meridian surfaces is the construction of 2-dimensional Lorentz surfaces in the pseudo-Euclidean space $\mathbb{E}^4_2$ which are one-parameter systems of meridians of a rotational hypersurface with lightlike axis. They are analogous to the meridian surfaces lying on rotational hypersurfaces with spacelike or timelike axis in $\mathbb{E}^4_2$ which have been studied in [3] and [4]. We show that all meridian surfaces are surfaces with flat normal connection and classify completely the meridian surfaces with constant Gauss curvature (Theorem 4.1 and Theorem 4.2). In Theorem 5.1 we give the classification of the meridian surfaces with parallel mean curvature vector field H. Theorem 6.1 describes all meridian surfaces which have parallel normalized mean curvature vector field but not parallel H.

2 Preliminaries

Let $\mathbb{E}^4_2$ be the 4-dimensional pseudo-Euclidean space with the canonical pseudo-Euclidean metric of index 2 given in local coordinates by

$$\widetilde{g} = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}^4_2$. Denote by $\langle ., . \rangle$ the indefinite inner scalar product associated with $\widetilde{g}$. Since $\widetilde{g}$ is an indefinite metric, a vector $v \in \mathbb{E}^4_2$ can have one of the three casual characters: *spacelike* if $\langle v, v \rangle > 0$ or $v = 0$, *timelike* if $\langle v, v \rangle < 0$, and *lightlike* if $\langle v, v \rangle = 0$ and $v \neq 0$. This terminology is inspired by general relativity and the Minkowski 4-space $\mathbb{E}^4_1$.

We use the following standard denotations:

$$\mathbb{S}^3_2(1) = \left\{ V \in \mathbb{E}^4_2 : \langle V, V \rangle = 1 \right\};$$
$$\mathbb{H}^3_1(-1) = \left\{ V \in \mathbb{E}^4_2 : \langle V, V \rangle = -1 \right\}.$$

The space $\mathbb{S}^3_2(1)$ is known as the de Sitter space, and the space $\mathbb{H}^3_1(-1)$ is the anti-de Sitter space [22].

A surface M in $\mathbb{E}^4_2$ is called *Lorentz*, if $\langle ., . \rangle$ induces a Lorentzian metric g on M, i.e. at each point $p \in M$ we have the following decomposition

$$\mathbb{E}^4_2 = T_pM \oplus N_pM$$

with the property that the restriction of the metric onto the tangent space T_pM is of signature $(1, 1)$, and the restriction of the metric onto the normal space N_pM is of signature $(1, 1)$.

Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\mathbb{E}^4_2$, respectively. For any tangent vector fields X, Y and any normal vector field ξ, the Gauss formula and the Weingarten formula are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\overline{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where h is the second fundamental form of M, D is the normal connection on the normal bundle, and A_ξ is the shape operator with respect to ξ.

The mean curvature vector field H of M in $\mathbb{E}^4_2$ is defined as $H = \dfrac{1}{2}\operatorname{tr} h$. A surface M is called *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$. A natural extension of minimal surfaces are quasi-minimal surfaces. A surface M is called *quasi-minimal* (or *pseudo-minimal*) if its mean curvature vector is lightlike at each point, i.e. $H \neq 0$ and $\langle H, H \rangle = 0$. In the Minkowski space $\mathbb{E}^4_1$ the quasi-minimal surfaces are also called *marginally trapped*. This notion is borrowed from general relativity. A surface M is said to have constant mean curvature if $\langle H, H \rangle = const$. We shall consider Lorentz surfaces in $\mathbb{E}^4_2$ for which $\langle H, H \rangle = const \neq 0$. Such surfaces we call *CMC surfaces*.

A normal vector field ξ on M is called *parallel in the normal bundle* (or simply *parallel*) if $D\xi = 0$ holds identically [7]. A surface M is said to have *parallel mean curvature vector field* if its mean curvature vector H satisfies $DH = 0$.

Surfaces for which the mean curvature vector field H is non-zero, $\langle H, H \rangle \neq 0$, and there exists a unit vector field H_0 in the direction of the mean curvature vector H, such that H_0 is parallel in the normal bundle, are called surfaces with *parallel normalized mean curvature vector field* [6]. It is easy to see that if M is a surface with non-zero parallel mean curvature vector field H (i.e. $DH = 0$), then M is a surface with parallel normalized mean curvature vector field, but the converse is not true in general. It is true only for surfaces with $\|H\| = const$.

3 Construction of Meridian Surfaces in Pseudo-Euclidean 4-Space

Meridian surfaces in the Euclidean 4-space $\mathbb{E}^4$ we defined in [15] as one-parameter systems of meridians of the standard rotational hypersurface in $\mathbb{E}^4$. The classification of meridian surfaces with constant Gauss curvature, with constant mean curvature, Chen meridian surfaces and meridian surfaces with parallel normal bundle is given in [15] and [17]. The meridian surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map are classified in [1]. The idea from the Euclidean space is used in [16], [18], and [19] for the construction of meridian spacelike surfaces lying on rotational hypersurfaces in $\mathbb{E}^4_1$ with timelike, spacelike, or lightlike axis. The classification of marginally trapped meridian surfaces is given in [16] and [19]. Meridian surfaces in $\mathbb{E}^4_1$ with pointwise 1-type Gauss map are classified in

[2]. The classification of meridian surfaces with constant Gauss curvature, with constant mean curvature, Chen meridian surfaces and meridian surfaces with parallel normal bundle is given in [18] and [20].

Following the idea from the Euclidean and Minkowski spaces, in [3] and [4] we constructed Lorentz meridian surfaces in the pseudo-Euclidean 4-space $\mathbb{E}^4_2$ as one-parameter systems of meridians of rotational hypersurfaces with timelike or spacelike axis. We gave the classification of quasi-minimal meridian surfaces and meridian surfaces with constant mean curvature [3]. The classification of meridian surfaces with parallel mean curvature vector field and the classification of meridian surfaces with parallel normalized mean curvature vector is given in [4].

In the present paper we construct Lorentz meridian surfaces in $\mathbb{E}^4_2$ which are one-parameter systems of meridians of rotational hypersurfaces with lightlike axis.

Let $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system in $\mathbb{E}^4_2$, i.e. $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1, \langle e_3, e_3\rangle = \langle e_4, e_4\rangle = -1$. We denote $\xi_1 = \dfrac{e_2 + e_4}{\sqrt{2}}$, $\xi_2 = \dfrac{-e_2 + e_4}{\sqrt{2}}$ and consider the pseudo-orthonormal base $\{e_1, e_3, \xi_1, \xi_2\}$ of $\mathbb{E}^4_2$. Note that $\langle \xi_1, \xi_1\rangle = 0$, $\langle \xi_2, \xi_2\rangle = 0$, $\langle \xi_1, \xi_2\rangle = -1$.

A rotational hypersurface with lightlike axis in $\mathbb{E}^4_2$ can be parametrized by

$$\mathcal{M} : Z(u, w^1, w^2) = f(u)w^1(\cosh w^2 e_1 + \sinh w^2 e_3) + (f(u)\frac{(w^1)^2}{2} + g(u))\xi_1 + f(u)\xi_2,$$

where $f = f(u)$, $g = g(u)$ are smooth functions, defined in an interval $I \subset \mathbb{R}$ and $f(u) > 0,\ u \in I$.

Let $w^1 = w^1(v)$, $w^2 = w^2(v)$, $v \in J$, $J \subset \mathbb{R}$ and consider the surface $\mathcal{M}_m$ in $\mathbb{E}^4_2$ given by

$$\mathcal{M}_m : z(u, v) = Z(u, w^1(v), w^2(v)), \tag{3.1}$$

where $u \in I$, $v \in J$. The surface $\mathcal{M}_m$, defined by (3.1), is a one-parameter system of meridians of the rotational hypersurface $\mathcal{M}$. So, we call $\mathcal{M}_m$ a *meridian surface on* $\mathcal{M}$.

Without loss of generality we can assume that $w^1 = \varphi(v)$, $w^2 = v$. Then the meridian surface $\mathcal{M}_m$ is parametrized as follows:

$$\mathcal{M}_m : z(u, v) = f(u)(\varphi(v)\cosh v\, e_1 + \varphi(v)\sinh v\, e_3 + \frac{\varphi^2(v)}{2}\,\xi_1 + \xi_2) + g(u)\,\xi_1. \tag{3.2}$$

If we denote $l(v) = \varphi(v)\cosh v\, e_1 + \varphi(v)\sinh v\, e_3 + \dfrac{\varphi^2(v)}{2}\,\xi_1 + \xi_2$, then the parametrization (3.2) is written as

$$\mathcal{M}_m : z(u, v) = f(u)\, l(v) + g(u)\,\xi_1.$$

Now we shall find the coefficients of the first fundamental form of $\mathcal{M}_m$. The tangent vector fields z_u and z_v are

$$
\begin{aligned}
z_u &= f'\varphi\cosh v\, e_1 + f'\varphi\sinh v\, e_3 + \left(f'\frac{\varphi^2}{2} + g'\right)\xi_1 + f'\,\xi_2;\\
z_v &= f(\dot\varphi\cosh v + \varphi\sinh v)\, e_1 + f(\dot\varphi\sinh v + \varphi\cosh v)\, e_3 + f\varphi\dot\varphi\,\xi_1,
\end{aligned}
\tag{3.3}
$$

where $\dot\varphi$ denotes the derivative of φ with respect to v. So, the coefficients of the first fundamental form are

$$
E = -2f'(u)g'(u);\quad F = 0;\quad G = f^2(u)(\dot\varphi^2(v) - \varphi^2(v)).
$$

Since we are studying Lorentz surfaces, in the case $\dot\varphi^2(v)-\varphi^2(v) > 0$ we assume that $f'(u)g'(u) > 0$; in the case $\dot\varphi^2(v)-\varphi^2(v) < 0$ we assume that $f'(u)g'(u) < 0$.

We shall consider the tangent frame field defined by $X = \dfrac{z_u}{\sqrt{2\varepsilon f'g'}}$, $Y = \dfrac{z_v}{f\sqrt{\varepsilon(\dot\varphi^2-\varphi^2)}}$, where $\varepsilon = 1$ in the case $\dot\varphi^2 - \varphi^2 > 0$, $f'g' > 0$, and $\varepsilon = -1$ in the case $\dot\varphi^2 - \varphi^2 < 0$, $f'g' < 0$. Thus we have $\langle X,X\rangle = -\varepsilon$, $\langle Y,Y\rangle = \varepsilon$, $\langle X,Y\rangle = 0$. Let us choose the following normal frame field:

$$
\begin{aligned}
n_1 &= \sqrt{\frac{\varepsilon f'}{2g'}}\left(\varphi\cosh v\, e_1 + \varphi\sinh v\, e_3 + \frac{f'\varphi^2 - 2g'}{2f'}\,\xi_1 + \xi_2\right);\\
n_2 &= \frac{1}{\sqrt{\varepsilon(\dot\varphi^2-\varphi^2)}}\big((\dot\varphi\sinh v + \varphi\cosh v)\, e_1 + (\dot\varphi\cosh v + \varphi\sinh v)\, e_3 + \varphi^2\,\xi_1\big),
\end{aligned}
\tag{3.4}
$$

which satisfies $\langle n_1,n_1\rangle = \varepsilon$, $\langle n_2,n_2\rangle = -\varepsilon$, $\langle n_1,n_2\rangle = 0$. Taking into account (3.3), we calculate the second partial derivatives of $z(u,v)$:

$$
\begin{aligned}
z_{uu} = {}& f''\varphi\cosh v\, e_1 + f''\varphi\sinh v\, e_3 + \left(f''\frac{\varphi^2}{2} + g''\right)\xi_1 + f''\,\xi_2;\\
z_{uv} = {}& f'(\dot\varphi\cosh v + \varphi\sinh v)\, e_1 + f'(\dot\varphi\sinh v + \varphi\cosh v)\, e_3 + f'\varphi\dot\varphi\,\xi_1;\\
z_{vv} = {}& f\left((\ddot\varphi + \varphi)\cosh v + 2\dot\varphi\sinh v\right) e_1 + f\left((\ddot\varphi + \varphi)\sinh v + 2\dot\varphi\cosh v\right) e_3\\
& + f\left(\dot\varphi^2 + \varphi\ddot\varphi\right)\xi_1.
\end{aligned}
$$

The last equalities together with (3.4) imply

$$
\begin{aligned}
&\langle z_{uu}, n_1\rangle = \frac{f''g' - g''f'}{\sqrt{2\varepsilon f'g'}}; && \langle z_{uu}, n_2\rangle = 0;\\
&\langle z_{uv}, n_1\rangle = 0; && \langle z_{uv}, n_2\rangle = 0;\\
&\langle z_{vv}, n_1\rangle = -f\sqrt{\frac{\varepsilon f'}{2g'}}\,(\dot\varphi^2 - \varphi^2); && \langle z_{vv}, n_2\rangle = f\,\frac{\varphi\ddot\varphi - 2\dot\varphi^2 + \varphi^2}{\sqrt{\varepsilon(\dot\varphi^2-\varphi^2)}}.
\end{aligned}
$$

Hence, we obtain

$$\begin{aligned} h(X,X) &= \varepsilon \frac{f''g' - g''f'}{(2\varepsilon f'g')^{\frac{3}{2}}}\, n_1; \\ h(X,Y) &= 0; \\ h(Y,Y) &= -\frac{1}{f}\sqrt{\frac{\varepsilon f'}{2g'}}\, n_1 - \varepsilon \frac{\varphi\ddot{\varphi} - 2\dot{\varphi}^2 + \varphi^2}{f(\varepsilon(\dot{\varphi}^2 - \varphi^2))^{\frac{3}{2}}}\, n_2. \end{aligned} \tag{3.5}$$

Now, we shall consider the parametric lines of the meridian surface $\mathcal{M}_m$. The parametric u-line $v = v_0 = const$ is given by

$$c_u : z(u) = c\alpha f(u)\, e_1 + c\beta f(u)\, e_3 + \left(\frac{c^2}{2} f(u) + g(u)\right) \xi_1 + f(u)\, \xi_2,$$

where $\alpha = \cosh v_0$, $\beta = \sinh v_0$, $c = \varphi(v_0)$. So, the unit tangent vector field t_{c_u} of c_u is:

$$t_{c_u} = \frac{1}{\sqrt{2\varepsilon f'g'}} \left(c\alpha f'\, e_1 + c\beta f'\, e_3 + \left(\frac{c^2}{2} f' + g'\right) \xi_1 + f'\, \xi_2 \right).$$

We denote by s the arc-length of c_u and calculate the derivative

$$\frac{dt_{c_u}}{ds} = \frac{t'_{c_u}}{s'} = \frac{\varepsilon(f''g' - g''f')}{(2\varepsilon f'g')^2} \left(c\alpha f'\, e_1 + c\beta f'\, e_3 + \left(\frac{c^2}{2} f' - g'\right) \xi_1 + f'\, \xi_2 \right).$$

Thus we obtain that the curvature of c_u is $\dfrac{\varepsilon(f''g' - g''f')}{(2\varepsilon f'g')^{\frac{3}{2}}}$. Finally, for each $v = const$ the parametric lines c_u are congruent in $\mathbb{E}^4_2$. These curves are the meridians of $\mathcal{M}_m$. We denote $\kappa_m(u) = \dfrac{\varepsilon(f''g' - g''f')}{(2\varepsilon f'g')^{\frac{3}{2}}}$.

Now, we shall consider the parametric v-lines of $\mathcal{M}_m$. Let $u = u_0 = const$ and denote $a = f(u_0)$, $b = g(u_0)$. The corresponding parametric v-line is given by

$$c_v : z(v) = a\varphi(v) \cosh v\, e_1 + a\varphi(v) \sinh v\, e_3 + \left(a\frac{\varphi^2(v)}{2} + b \right) \xi_1 + a\, \xi_2.$$

The unit tangent vector field t_{c_v} of c_v is

$$t_{c_v} = \frac{1}{\sqrt{\varepsilon(\dot{\varphi}^2 - \varphi^2)}} \left((\dot{\varphi}\cosh v + \varphi \sinh v)\, e_1 + (\dot{\varphi}\sinh v + \varphi\cosh v)\, e_3 + \varphi\dot{\varphi}\, \xi_1 \right).$$

Knowing the tangent vector field t_{c_v} we calculate the curvature $\varkappa_{c_v}$ of c_v and obtain that $\varkappa_{c_v} = \dfrac{\varphi\ddot{\varphi} - 2\dot{\varphi}^2 + \varphi^2}{a(\varepsilon(\dot{\varphi}^2 - \varphi^2))^{\frac{3}{2}}}$. We denote $\kappa(v) = \dfrac{\varphi\ddot{\varphi} - 2\dot{\varphi}^2 + \varphi^2}{(\varepsilon(\dot{\varphi}^2 - \varphi^2))^{\frac{3}{2}}}$. Then, for each $u = u_0 = const$ the curvature of the corresponding parametric v-line is expressed as $\varkappa_{c_v} = \dfrac{1}{a}\,\kappa(v)$, where $a = f(u_0)$. Actually, $\kappa(v)$ is the curvature of the curve

$$c : l = l(v) = \varphi(v)\cosh v\, e_1 + \varphi(v)\sinh v\, e_3 + \frac{\varphi^2(v)}{2}\,\xi_1 + \xi_2.$$

Consequently, formulas (3.5) take the form

$$\begin{aligned} &h(X,X) = \kappa_m\, n_1;\\ &h(X,Y) = 0;\\ &h(Y,Y) = -\frac{1}{f}\sqrt{\frac{\varepsilon f'}{2g'}}\, n_1 - \varepsilon\frac{\kappa}{f}\, n_2. \end{aligned} \tag{3.6}$$

It follows from (3.6) that the Gauss curvature K of the meridian surface $\mathcal{M}_m$ is expressed as

$$K = \varepsilon\frac{\kappa_m}{f}\sqrt{\frac{\varepsilon f'}{2g'}}$$

and the mean curvature vector field H is given by

$$H = -\frac{\varepsilon}{2}\left(\kappa_m + \frac{1}{f}\sqrt{\frac{\varepsilon f'}{2g'}}\right) n_1 - \frac{\kappa}{2f}\, n_2.$$

Without loss of generality we can assume that $2\varepsilon f'g' = 1$, which implies $\kappa_m = \dfrac{f''}{f'}$. Hence,

$$K = \varepsilon\frac{f''}{f}, \tag{3.7}$$

$$H = -\frac{\varepsilon(ff'' + (f')^2)}{2ff'}\, n_1 - \frac{\kappa}{2f}\, n_2. \tag{3.8}$$

Now, using (3.4) and (3.6) we obtain that

$$\begin{aligned} &\overline{\nabla}_X n_1 = \kappa_m\, X; \qquad && \overline{\nabla}_X n_2 = 0;\\ &\overline{\nabla}_Y n_1 = \frac{1}{f}\sqrt{\frac{\varepsilon f'}{2g'}}\, Y; \qquad && \overline{\nabla}_Y n_2 = -\varepsilon\frac{\kappa}{f}\, Y. \end{aligned} \tag{3.9}$$

Hence,

$$\begin{aligned} &D_X n_1 = 0; \qquad && D_X n_2 = 0;\\ &D_Y n_1 = 0; \qquad && D_Y n_2 = 0, \end{aligned} \tag{3.10}$$

where D is the normal connection of the surface. The last equalities imply that the curvature of the normal connection of $\mathcal{M}_m$ is zero. So, we obtain the following statement.

Proposition 3.1. *The meridian surface $\mathcal{M}_m$, defined by* (3.2)*, is a surface with flat normal connection.*

In the next sections we will give the classification of the meridian surfaces with constant Gauss curvature, with parallel mean curvature vector field and with parallel normalized mean curvature vector field.

4 Meridian Surfaces with constant Gauss curvature

The study of surfaces with constant Gauss curvature is one of the essential topics in differential geometry. Surfaces with constant Gauss curvature in Minkowski space have drawn the interest of many geometers, see for example [14], [21], and the references therein.

Let $\mathcal{M}_m$ be a meridian surface, defined by (3.2). Then the Gauss curvature of $\mathcal{M}_m$ depends only on the meridian curve m and is expressed by formula (3.7). First, we shall describe the meridian surfaces with zero Gauss curvature.

Theorem 4.1. *Let $\mathcal{M}_m$ be a meridian surface, defined by* (3.2)*. Then $\mathcal{M}_m$ is flat if and only if the meridian curve m is given by*

$$f(u) = au + b; \quad g(u) = \frac{\varepsilon}{2a} u + c,$$

where $a = const \neq 0$, $b = const$, $c = const$. In this case $\mathcal{M}_m$ is a developable ruled surface.

Proof. It follows from (3.7) that $K = 0$ if and only if $f(u) = au + b$, $a = const \neq 0$, $b = const$. Using that $2\varepsilon f' g' = 1$, we obtain $g(u) = \dfrac{\varepsilon}{2a} u + c$, $c = const$. Since in this case $\kappa_m = 0$, then the meridian curve m is part of a straight line, i.e. $\mathcal{M}_m$ lies on a ruled surface. Moreover, it follows from (3.9) that $\overline{\nabla}_X n_1 = 0$; $\overline{\nabla}_X n_2 = 0$, which implies that the normal space is constant at the points of a fixed straight line, and hence the tangent space is one and the same at the points of a fixed line. Consequently, $\mathcal{M}_m$ is part of a developable ruled surface.

□

The following theorem describes the meridian surfaces with constant non-zero Gauss curvature.

Theorem 4.2. *Let $\mathcal{M}_m$ be a meridian surface, defined by* (3.2)*. Then $\mathcal{M}_m$ has constant non-zero Gauss curvature K if and only if the meridian curve m is given by*

$$\begin{aligned} f(u) &= \alpha \cosh \sqrt{\varepsilon K} u + \beta \sinh \sqrt{\varepsilon K} u, \quad \textit{if} \quad \varepsilon K > 0; \\ f(u) &= \alpha \cos \sqrt{-\varepsilon K} u + \beta \sin \sqrt{-\varepsilon K} u, \quad \textit{if} \quad \varepsilon K < 0, \end{aligned} \tag{4.1}$$

where α and β are constants, $g(u)$ is defined by $g'(u) = \dfrac{\varepsilon}{2f'(u)}$.

Proof. Using that the Gauss curvature is expressed by (3.7), we obtain that $K = const \neq 0$ if and only if the function $f(u)$ satisfies the following differential equation

$$f''(u) - \varepsilon K f(u) = 0.$$

The general solution of this equation is given by (4.1), where α and β are constants. Since we assume that $2\varepsilon f'g' = 1$, then the function $g(u)$ is determined by $g'(u) = \dfrac{\varepsilon}{2f'(u)}$.

□

5 Meridian surfaces with parallel mean curvature vector field

Another basic class of surfaces in Riemannian and pseudo-Riemannian geometry are surfaces with parallel mean curvature vector field, since they are critical points of some functionals and play important role in differential geometry, the theory of harmonic maps, as well as in physics. The classification of surfaces with parallel mean curvature vector field in Riemannian space forms was given by Chen [5] and Yau [23]. Recently, spacelike surfaces with parallel mean curvature vector field in pseudo-Euclidean spaces with arbitrary codimension were classified in [8] and [9]. The classification of quasi-minimal surfaces with parallel mean curvature vector in $\mathbb{E}^4_2$ is given in [12]. Lorentz surfaces with parallel mean curvature vector field in arbitrary pseudo-Euclidean space $\mathbb{E}^m_s$ are studied in [10] and [13]. A nice survey on classical and recent results on submanifolds with parallel mean curvature vector in Riemannian manifolds as well as in pseudo-Riemannian manifolds is presented in [11].

In this section we shall describe the meridian surfaces with non-zero parallel mean curvature vector field, i.e. $H \neq 0$ and $DH = 0$.

Under the assumption $2\varepsilon f'g' = 1$ the mean curvature vector field H of the meridian surface $\mathcal{M}_m$ is given by formula (3.8). Using that $D_X n_1 = D_Y n_1 = D_X n_2 = D_Y n_2 = 0$, and $X = z_u$, $Y = \dfrac{z_v}{f\sqrt{\varepsilon(\dot{\varphi}^2 - \varphi^2)}}$, we get

$$\begin{aligned} D_X H &= -\frac{\varepsilon}{2}\left(\frac{ff'' + (f')^2}{ff'}\right)' n_1 + \frac{\kappa f'}{2f^2}\, n_2; \\ D_Y H &= -\frac{\kappa'}{2f^2\sqrt{\varepsilon(\dot{\varphi}^2 - \varphi^2)}}\, n_2. \end{aligned} \tag{5.1}$$

Theorem 5.1. *Let $\mathcal{M}_m$ be a meridian surface, defined by* (3.2)*. Then $\mathcal{M}_m$ has parallel mean curvature vector field if and only if the curvature of c is $\kappa = 0$*

and the meridian curve m is determined by $f' = \phi(f)$ where

$$\phi(t) = \frac{a\,t^2 + b}{2t}, \quad a = const \neq 0, \quad b = const,$$

$g(u)$ is defined by $g'(u) = \dfrac{\varepsilon}{2f'(u)}$. In this case $\mathcal{M}_m$ is a non-flat CMC surface lying in a hyperplane of $\mathbb{E}^4_2$.

Proof. Using formulas (5.1) we get that $\mathcal{M}_m$ has parallel mean curvature vector field if and only if the following conditions hold

$$\begin{aligned} &\left(\frac{ff'' + (f')^2}{ff'}\right)' = 0;\\ &\kappa f' = 0;\\ &\kappa' = 0. \end{aligned} \tag{5.2}$$

Since $f' \neq 0$, the equalities (5.2) imply that $\kappa = 0$ and $\dfrac{ff'' + (f')^2}{ff'} = a = const$. If $a = 0$, then $H = 0$, i.e. $\mathcal{M}_m$ is minimal. Since we consider non-minimal surfaces, we assume that $a \neq 0$. In this case the meridian curve m is determined by the following differential equation:

$$ff'' + (f')^2 = aff', \qquad a = const \neq 0. \tag{5.3}$$

The solutions of the last differential equation can be found as follows. Setting $f' = \phi(f)$ in equation (5.3), we obtain that the function $\phi = \phi(t)$ is a solution of the equation

$$\phi' + \frac{1}{t}\,\phi = a. \tag{5.4}$$

The general solution of equation (5.4) is given by

$$\phi(t) = \frac{a\,t^2 + b}{2t}, \quad b = const. \tag{5.5}$$

In this case, the mean curvature vector field H is given by $H = -\dfrac{\varepsilon a}{2}\,n_1$, and thus $\langle H, H\rangle = \dfrac{\varepsilon a^2}{4} = const$. Hence, the surface $\mathcal{M}_m$ is a CMC surface. Moreover, since $\kappa = 0$, from (3.9) it follows that $\overline{\nabla}_X n_2 = 0$, $\overline{\nabla}_Y n_2 = 0$. Hence, $\mathcal{M}_m$ lies in a 3-dimensional constant hyperplane parallel to $span\{X, Y, n_1\}$. The Gauss curvature $K \neq 0$, so $\mathcal{M}_m$ is a non-flat CMC surface lying in a hyperplane of $\mathbb{E}^4_2$.

Conversely, if the meridian curve m is determined by (5.5), then by direct computation we get that $D_X H = D_Y H = 0$, i.e. the surface has parallel mean curvature vector field.

□

Theorem 5.1 shows that each meridian surface with parallel mean curvature vector field is a CMC surface and lies in a hyperplane of $\mathbb{E}^4_2$. So, we have the following result.

Corollary 5.2. *There are no Lorentz meridian surfaces with parallel mean curvature vector field other than CMC surfaces lying in a hyperplane of* $\mathbb{E}^4_2$.

Remark. The same result holds true for meridian surfaces lying on rotational hypersurfaces with spacelike or timelike axis [4].

6 Meridian surfaces with parallel normalized mean curvature vector field

The class of surfaces with parallel mean curvature vector field is naturally extended to the class of surfaces with parallel normalized mean curvature vector field. A submanifold in a Riemannian manifold is said to have parallel normalized mean curvature vector field if the mean curvature vector is non-zero and the unit vector in the direction of the mean curvature vector is parallel in the normal bundle [6]. It is well known that submanifolds with non-zero parallel mean curvature vector field have parallel normalized mean curvature vector field. But the condition to have parallel normalized mean curvature vector field is much weaker than the condition to have parallel mean curvature vector field. For example, every surface in the Euclidean 3-space has parallel normalized mean curvature vector field but in the 4-dimensional Euclidean space, there exist abundant examples of surfaces which lie fully in $\mathbb{E}^4$ with parallel normalized mean curvature vector field, but not with parallel mean curvature vector field. In the pseudo-Euclidean space with neutral metric $\mathbb{E}^4_2$ the study of Lorentz surfaces with parallel normalized mean curvature vector field, but not parallel mean curvature vector field, is still an open problem.

In this section we give the classification of all meridian surfaces which have parallel normalized mean curvature vector field but not parallel H.

Let $\mathcal{M}_m$ be a meridian surface, defined by (3.2). The mean curvature vector field H is given by formula (3.8). We assume that $\langle H, H\rangle \neq 0$, i.e. $(ff'' + (f')^2)^2 - \kappa^2 f'^2 \neq 0$.

If $\kappa = 0$, then the normalized mean curvature vector field is $H_0 = n_1$ and in view of (3.10) we have $D_X H_0 = D_Y H_0 = 0$, i.e. H_0 is parallel in the normal bundle. We consider this case as trivial, since under the assumption $\kappa = 0$ the surface $\mathcal{M}_m$ lies in a 3-dimensional hyperplane of $\mathbb{E}^4_2$ and every surface in 3-dimensional space has parallel normalized mean curvature vector field. So, further we assume that $\kappa \neq 0$.

A unit normal vector field in the direction of H is

$$H_0 = \frac{-1}{\sqrt{|(ff'' + (f')^2)^2 - \kappa^2 f'^2|}} \left((ff'' + (f')^2)\, n_1 + \kappa f'\, n_2\right). \tag{6.1}$$

For simplicity we denote

$$A = \frac{-(ff'' + (f')^2)}{\sqrt{|(ff'' + (f')^2)^2 - \kappa^2 f'^2|}}, \quad B = \frac{-\kappa f'}{\sqrt{|(ff'' + (f')^2)^2 - \kappa^2 f'^2|}},$$

so, the normalized mean curvature vector field is expressed as $H_0 = A\, n_1 + B\, n_2$. Then from equalities (6.1) and (3.10) we get

$$\begin{aligned} D_X H_0 &= X(A)\, n_1 + X(B)\, n_2; \\ D_Y H_0 &= Y(A)\, n_1 + Y(B)\, n_2. \end{aligned} \tag{6.2}$$

Theorem 6.1. *Let $\mathcal{M}_m$ be a meridian surface, defined by* (3.2)*. Then $\mathcal{M}_m$ has parallel normalized mean curvature vector field but not parallel mean curvature vector if and only if one of the following cases holds:*

(i) $\kappa \neq 0$ and the meridian curve m is defined by

$$f(u) = \sqrt{au + b}, \quad g(u) = \frac{2}{3a^2}(au + b)^{\frac{3}{2}} + c,$$

where $a = const \neq 0$, $b = const$, $c = const$.

(ii) $\kappa = const \neq 0$ and the meridian curve m is determined by $f' = \phi(f)$ where

$$\phi(t) = \frac{c\,t + b}{t}, \quad c = const \neq 0,\ c^2 \neq \kappa^2,\ b = const,$$

$g(u)$ is defined by $g'(u) = \dfrac{\varepsilon}{2f'(u)}$.

Proof. Let $\mathcal{M}_m$ be a surface with parallel normalized mean curvature vector field, i.e. $D_X H_0 = 0$, $D_Y H_0 = 0$. Then from (6.2) it follows that $A = const$, $B = const$. Hence,

$$\begin{aligned} \frac{-(ff'' + (f')^2)}{\sqrt{|(ff'' + (f')^2)^2 - \kappa^2 f'^2|}} &= \alpha = const; \\ \frac{-\kappa f'}{\sqrt{|(ff'' + (f')^2)^2 - \kappa^2 f'^2|}} &= \beta = const. \end{aligned} \tag{6.3}$$

We have the following two cases.

Case (i): $ff'' + (f')^2 = 0$. In this case, from (3.8) we get that the mean curvature vector field is $H = -\dfrac{\kappa}{2f}\, n_2$ and the normalized mean curvature vector field is $H_0 = n_2$. Since we study surfaces with $\langle H, H \rangle \neq 0$, we get $\kappa \neq 0$. The solution of the differential equation $ff'' + (f')^2 = 0$ is given by the formula $f(u) = \sqrt{au + b}$, where $a = const \neq 0$, $b = const$. Using that $g'(u) = \dfrac{\varepsilon}{2f'(u)}$, we obtain $g(u) = \dfrac{2}{3a^2}(au + b)^{\frac{3}{2}} + c$, where $c = const$.

Case (ii): $ff'' + (f')^2 \neq 0$ in an interval $\tilde{I} \subset I \subset \mathbb{R}$. Then, from (6.3) we get

$$\frac{\alpha}{\beta}\,\kappa = \frac{ff'' + (f')^2}{f'}, \quad \alpha \neq 0,\ \beta \neq 0. \tag{6.4}$$

Since the left-hand side of equality (6.4) is a function of v, the right-hand side of (6.4) is a function of u, we obtain that

$$\frac{ff'' + (f')^2}{f'} = c, \quad c = const \neq 0;$$
$$\kappa = \frac{\beta}{\alpha}\, c.$$

In this case we have $\langle H, H \rangle = \dfrac{\varepsilon(c^2 - \kappa^2)}{4f^2}$. Since we study surfaces with $\langle H, H \rangle \neq 0$, we get $c^2 \neq \kappa^2$. The meridian curve m is determined by the following differential equation:

$$ff'' + (f')^2 = cf'. \tag{6.5}$$

Setting $f' = \phi(f)$ in equation (6.5), we obtain that the function $\phi = \phi(t)$ satisfies

$$\phi' + \frac{1}{t}\,\phi = \frac{c}{t},$$

whose general solution is $\phi(t) = \dfrac{ct + b}{t}$, $b = const$.

Conversely, if one of the cases (i) or (ii) stated in the theorem holds true, then by direct computation we get that $D_X H_0 = D_Y H_0 = 0$, i.e. the surface has parallel normalized mean curvature vector field. Moreover, in case (i) we have

$$D_X H = \frac{\kappa f'}{2f^2}\, n_2; \qquad D_Y H = -\frac{\kappa'}{2f^2\sqrt{\varepsilon(\dot{\varphi}^2 - \varphi^2)}}\, n_2,$$

which implies that H is not parallel in the normal bundle, since $\kappa \neq 0$, $f' \neq 0$. In case (ii) we get

$$D_X H = \frac{\varepsilon c f'}{2f^2}\, n_1 + \frac{\kappa f'}{2f^2}\, n_2; \qquad D_Y H = 0,$$

and again we have that H is not parallel in the normal bundle. □

Remark. Theorem 6.1 gives examples of Lorentz surfaces in the pseudo-Euclidean space $\mathbb{E}^4_2$ which have parallel normalized mean curvature vector field but not parallel mean curvature vector field.

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On Slant Curves with Pseudo-Hermitian C-parallel Mean Curvature Vector Fields

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Abstract. We study pseudo-Hermitian C-parallel and C-proper slant curves in contact metric 3-manifolds. As an application, we give two examples of pseudo-Hermitian Legendre circle and pseudo-Hermitian slant helix in Sasakian Heisenberg group.
Keywords. C-parallel mean curvature vector · slant curve · Heisenberg group.
MSC 2010 Classification. Primary: 53C25; Secondary:53A05 · 53C40.

1 Introduction

In [7], Chen defined biharmonic submanifold as a Riemannian submanifold with vanishing Laplacian of mean curvature vector field ΔH. Curves in a Euclidean space satisfying the condition $\Delta^{\perp} H = \lambda H$ were classified in [2], by Barros and Garay, where $\Delta^{\perp}$ denotes the Laplacian of the curve in the normal bundle and λ is a real valued function. In the real space form, the classification of curves satisfying $\Delta H = \lambda H$ and $\Delta^{\perp} H = \lambda H$ were given in [1], by Arroyo, Barros and Garay.

A curve in a contact metric manifold is said to be *slant* [9], if its tangent vector field has a constant angle with the Reeb vector field. In particular, if the contact angle is equal to $\frac{\pi}{2}$, then the curve is called a *Legendre curve*. In [8], Cho and Lee studied slant curves in pseudo-Hermitian contact 3-manifolds. Legendre curves with pseudo-Hermitian parallel mean curvature vector field, pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian proper mean curvature vector field in the normal bundle in contact pseudo-Hermitian 3-manifolds were studied by Lee in [12]. In [14], the present author and Güvenç studied slant curves with pseudo-Hermitian parallel mean curvature vector field, pseudo-Hermitian proper mean curvature vector field and pseudo-Hermitian proper mean curvature vector field in the normal bundle in contact pseudo-Hermitian 3-manifolds. The notions of C-parallel and C-proper curves in the tangent and normal bundles were introduced by Lee, Suh and Lee in [13]. A curve in an almost contact metric manifold is defined to be *C-parallel* if $\nabla_T H = \lambda\xi$, *C-proper* if $\Delta H = \lambda\xi$, *C-parallel in the normal bundle* if $\nabla^{\perp}_T H = \lambda\xi$, *$C$-proper in the normal bundle* if $\Delta^{\perp} H = \lambda\xi$, where T is the unit tangent vector field of the curve and λ is a differentiable function along the curve. In [13], Lee, Suh and Lee studied C-parallel and C-proper slant curves in Sasakian 3-manifolds. C-parallel and C-proper slant curves in trans-Sasakian manifolds were studied in [15], by Güvenç and the present author. On the other hand, slant and Legendre curves in Bianchi-Cartan-Vranceanu geometry were studied by Călin and Crasmareanu in [6]. Slant curves in normal almost contact geometry were studied in [5].

Motivated by the above studies, in the present paper, we study pseudo-Hermitian C-parallel and C-proper slant curves in contact metric 3-manifolds. We give two examples of pseudo-Hermitian Legendre circle and pseudo-Hermitian slant helix in Sasakian Heisenberg group.

2 Preliminaries

Let M be a $(2n+1)$-dimensional manifold. M is called a *contact manifold* [3] if there exists a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Given a contact form η, there exists a unique vector field ξ, the *characteristic vector field*, which satisfies $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for any vector field X on M. There exists an associated Riemannian metric g and a $(1,1)$-type tensor field φ satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$. From (2.1), it is easy to see that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

A Riemannian manifold equipped with the structure tensors (φ, ξ, η, g) satisfying (2.1) is called a *contact metric manifold*. It is denoted by $M = \{M, \varphi, \xi, \eta, g\}$. The operator h is defined by $h = \frac{1}{2} L_\xi \varphi$, where L_ξ is the Lie differentiation operator in the characteristic direction ξ. From the definition of h, it is easy to see that h is symmetric and satisfies the following equations (see [3], page 67):

$$h\xi = 0, \quad h\varphi = -\varphi h, \quad \nabla_X \xi = -\varphi X - \varphi h X, \tag{2.3}$$

where ∇ denotes the Levi-Civita connection.

For a $(2n+1)$-dimensional contact metric manifold $M = \{M, \varphi, \xi, \eta, g\}$, the almost complex structure J on $M \times \mathbb{R}$ is defined by

$$J(X, f\tfrac{d}{dt}) = (\varphi X - f\xi, \eta(X)\tfrac{d}{dt}), \tag{2.4}$$

where X is a vector field tangent to M, t is the coordinate function of $\mathbb{R}$ and f is a C^∞ function on $M \times \mathbb{R}$. If J is integrable then the contact metric manifold M is called a *Sasakian manifold* [3].

For a $(2n+1)$-dimensional contact metric manifold $M = \{M, \varphi, \xi, \eta, g\}$ provides a splitting of the tangent bundle

$$TM = Ker(\varphi) \oplus \mathrm{Im}(\varphi)$$

and the restriction $J = \varphi\mid_D$ defines an almost complex structure on $D = \mathrm{Im}(\varphi)$. There is a well-known concept of almost CR-structure as follows: Let M be a $(2n+s)$-dimensional smooth manifold. Let $\mathcal{D}$ be a smooth distribution on M of real dimension $2n$ and J a $(1,1)$-tensor field on M such that

$$J^2X = -X, \quad X \in \mathcal{D}.$$

Then $(\mathcal{D}, J)$ is called almost complex distribution (or an almost CR-structure). Then M is an *almost CR-manifold* (or *a contact strongly pseudo-convex pseudo-Hermitian manifold*) [3].

The Tanaka-Webster connection $\widehat{\nabla}$ (or the pseudo-Hermitian connection) ([16], [18]) on a contact strongly pseudo-convex pseudo-Hermitian manifold M is defined by

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all $X, Y \in \chi(M)$. By the use of (2.3), $\widehat{\nabla}$ can be rewritten as

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Y)\xi. \quad (2.5)$$

From (2.5), the torsion of the Tanaka-Webster connection $\widehat{\nabla}$ is

$$\widehat{T}(X,Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY. \quad (2.6)$$

If M is a Sasakian manifold, since $h = 0$, then the equations (2.5) and (2.6) turn into

$$\begin{gathered} \widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi, \\ \widehat{T}(X,Y) = 2g(X, \varphi Y)\xi, \end{gathered} \quad (2.7)$$

respectively.

3 Slant curves in contact pseudo-Hermitian geometry

Let $M = \{M, \varphi, \xi, \eta, g\}$ be a contact metric 3-manifold and $\gamma : I \to M$ a curve parametrized by arc-length in M. The Frenet frame field $\{T, N, B\}$ along γ for the pseudo-Hermitian connection $\widehat{\nabla}$ can be defined by

$$\begin{aligned} \widehat{\nabla}_T T &= \widehat{\kappa} N, \\ \widehat{\nabla}_T N &= -\widehat{\kappa} T + \widehat{\tau} B, \\ \widehat{\nabla}_T B &= -\widehat{\tau} N, \end{aligned} \quad (3.1)$$

where $\widehat{\kappa} = \left\|\widehat{\nabla}_T T\right\|$ is the *pseudo-Hermitian curvature* of γ and $\widehat{\tau}$ its *pseudo-Hermitian torsion* [8]. Similar to the general curve theory, a curve, whose pseudo-Hermitian curvature and pseudo-Hermitian torsion are non-zero constants, is called a *pseudo-Hermitian helix.* Curves with constant non-zero pseudo-Hermitian curvature and zero pseudo-Hermitian torsion are called *pseudo-Hermitian circles*. *Pseudo-Hermitian geodesics* are curves whose pseudo-Hermitian curvature and pseudo-Hermitian torsion are zero [8].

Let $\gamma : I \to M$ be a Frenet curve parametrized by arc-length parameter s in a contact metric 3-manifold M. The contact angle $\alpha(s)$ is a function defined by $cos[\alpha(s)] = g(T(s), \xi)$. If the contact angle $\alpha(s)$ is a constant, then γ is called a *slant curve* [9]. Slant curves with contact angle $\pi/2$ are traditionally called *Legendre curves* [3].

Throughout the present paper, we assume that all curves are non-geodesic Frenet curves, that is, $\widehat{\kappa} \neq 0$.

In [8], Cho and Lee proved the following three propositions:

Proposition 3.1. [*8*] *A curve γ for $\widehat{\nabla}$ is a slant curve if and only if it satisfies $\eta(N) = 0$.*

Proposition 3.2. [*8*] *Let γ be a slant curve for $\widehat{\nabla}$ in a 3-dimensional contact metric manifold M. Then the ratio of $\widehat{\tau}$ and $\widehat{\kappa}$ is a constant.*

Note that

$$\frac{\widehat{\tau}}{\widehat{\kappa}} = \cot \alpha_0, \tag{3.2}$$

where α_0 is the contact angle of γ [14].

Proposition 3.3. [*8*] *If a curve in a 3-dimensional contact metric manifold for Tanaka-Webster connection $\widehat{\nabla}$ is a Legendre curve, then $\widehat{\tau} = 0$.*

In [14], the present author and Güvenç showed that the converse statement of the above proposition is also true. They gave the following result:

Corollary 3.4. [*14*] *Let γ be a slant curve for Tanaka-Webster connection $\widehat{\nabla}$ with contact angle α_0 in a 3-dimensional contact metric manifold M. Then γ is a Legendre curve if and only if $\widehat{\tau} = 0$.*

4 Pseudo-Hermitian mean curvature vector field

The *pseudo-Hermitian mean curvature vector field* $\widehat{H}$ of a curve γ in a 3-dimensional contact metric manifold is defined by

$$\widehat{H} = \widehat{\nabla}_T T = \widehat{\kappa} N, \tag{4.1}$$

(see [12]). From (4.1), it is easy to see that

$$\widehat{\nabla}_T \widehat{H} = -\widehat{\kappa}^2 T + \widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B, \tag{4.2}$$

$$\widehat{\nabla}^{\perp}_{T}\widehat{H} = \widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B, \tag{4.3}$$

where $\widehat{H}$ is the pseudo-Hermitian mean curvature vector field of γ [12].

Definition 4.1. Let H be the mean curvature vector field of a curve γ in a 3-dimensional contact metric manifold M. The mean curvature vector field H is said to be *pseudo-Hermitian C-parallel* if $\widehat{\nabla}_T\widehat{H} = \lambda\xi$. The vector field H is said to be *pseudo Hermitian C-proper mean curvature vector field* if $\widehat{\triangle}\widehat{H} = \lambda\xi$. Similarly, H is said to be *pseudo-Hermitian C-parallel vector field in the normal bundle* if $\widehat{\nabla}^{\perp}_{T}\widehat{H} = \lambda\xi$, and H is said to be *pseudo-Hermitian C-proper mean curvature vector field in the normal bundle* if $\widehat{\triangle}^{\perp}\widehat{H} = \lambda\xi$, where λ is a differentiable function along the curve.

Lemma 4.2. [14]*Let γ be a curve in a 3-dimensional contact metric manifold M. Then*

$$\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T = -3\widehat{\kappa}\widehat{\kappa}' T + (\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B, \tag{4.4}$$

$$\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T} T = (\widehat{\kappa}'' - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B \tag{4.5}$$

and

$$\begin{aligned} \widehat{\triangle}\widehat{H} &= -\widehat{\nabla}_T\widehat{\nabla}_T\widehat{\nabla}_T T, \\ \widehat{\triangle}^{\perp}\widehat{H} &= -\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T}\widehat{\nabla}^{\perp}_{T} T. \end{aligned} \tag{4.6}$$

Using Lemma 4.2, we have the following theorem:

Theorem 4.3. *A slant curve γ in a 3-dimensional contact metric manifold M has pseudo-Hermitian C-parallel mean curvature vector field if and only if it is a pseudo-Hermitian helix satisfying*

$$\widehat{\kappa} = \mp\sqrt{-\lambda\cos\alpha_0} \quad \textit{and} \quad \widehat{\tau} = \mp\frac{\lambda\sin\alpha_0}{\sqrt{-\lambda\cos\alpha_0}},$$

where $\lambda\cos\alpha_0 < 0$.

Proof. Assume that a slant curve γ has pseudo-Hermitian C-parallel mean curvature vector field. Then from (4.2), the condition $\widehat{\nabla}_T\widehat{H} = \lambda\xi$ gives

$$-\widehat{\kappa}^2 T + \widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B = \lambda\xi. \tag{4.7}$$

Since γ is a slant curve we can write

$$\xi = \cos\alpha_0 T + \sin\alpha_0 B. \tag{4.8}$$

So using (4.7) and (4.8) we can write

$$-\widehat{\kappa}^2 T + \widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B = \lambda\left(\cos\alpha_0 T + \sin\alpha_0 B\right). \tag{4.9}$$

Taking the inner product of (4.9) with N and using $\eta(N) = 0$ we find $\widehat{\kappa}' = 0$, which implies that $\widehat{\kappa}$ is a constant. Hence from the equation (4.9), it follows that $\widehat{\kappa} = \mp\sqrt{-\lambda\cos\alpha_0}$. Since $\frac{\widehat{\tau}}{\widehat{\kappa}} = \cot\alpha_0$ we obtain $\widehat{\tau} = \mp\frac{\lambda\sin\alpha_0}{\sqrt{-\lambda\cos\alpha_0}}$, where $\lambda\cos\alpha_0 < 0$.

The converse statement is trivial. □

Theorem 4.4. *A slant curve γ in a 3-dimensional contact metric manifold M has pseudo-Hermitian C-parallel mean curvature vector field in the normal bundle if and only if it is a pseudo-Hermitian Legendre circle.*

Proof. Assume that a slant curve γ has pseudo-Hermitian C-parallel mean curvature vector field in the normal bundle. Then from (4.2), the condition $\widehat{\nabla}^{\perp}_{T}\widehat{H} = \lambda\xi$ gives

$$\widehat{\kappa}' N + \widehat{\kappa}\widehat{\tau} B = \lambda \left(\cos\alpha_0 T + \sin\alpha_0 B\right). \tag{4.10}$$

So we have

$$\widehat{\kappa}' = 0, \tag{4.11}$$

$$\lambda \cos\alpha_0 = 0, \tag{4.12}$$

$$\widehat{\kappa}\widehat{\tau} = \lambda \sin\alpha_0. \tag{4.13}$$

Then $\widehat{\kappa}$ is a constant. From (4.12), if $\cos\alpha_0 = 0$, then $\alpha_0 = \pi/2$. So it is a Legendre curve. Then from Proposition 3.4, $\widehat{\tau} = 0$, which implies γ is a pseudo-Hermitian Legendre circle. Moreover, from (4.13) we have $\lambda = 0$.

The converse statement is trivial. □

Theorem 4.5. *There does not exist non-geodesic slant curve in a 3-dimensional contact metric manifold M with pseudo Hermitian C-proper mean curvature.*

Proof. Assume that γ is a non-geodesic slant curve with contact angle α_0 and has pseudo Hermitian C-proper mean curvature field. Then by definition, $\widehat{\triangle}\widehat{H} = \lambda\xi$. Using (4.6) and (4.8), we get

$$\begin{aligned} 3\widehat{\kappa}\widehat{\kappa}' T - (\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2) N - (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}') B \\ = \lambda \left(\cos\alpha_0 T + \sin\alpha_0 B\right). \end{aligned} \tag{4.14}$$

Hence we have

$$3\widehat{\kappa}\widehat{\kappa}' = \lambda\cos\alpha_0,$$

$$\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2 = 0,$$

$$2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}' = -\lambda\sin\alpha_0.$$

So using $\frac{\widehat{\tau}}{\widehat{\kappa}} = \cot\alpha_0$, we find $\lambda = 0$. Then using Theorem 4.4. in [14], we find $\widehat{\kappa} = 0$. Since γ is not a geodesic, it can not have pseudo Hermitian C-proper mean curvature.

This completes the proof. □

Theorem 4.6. *A slant curve γ in a 3-dimensional contact metric manifold M has pseudo-Hermitian C-proper mean curvature field in the normal bundle if and only if either it is a Legendre curve with pseudo-Hermitian curvature $\widehat{\kappa}(s) = as + b$, where a and b are real constants or it is a pseudo-Hermitian Legendre circle.*

Proof. Assume that γ is a non-geodesic slant curve with contact angle α_0 and has pseudo Hermitian C-proper mean curvature vector field in the normal bundle. Then by definition, $\widehat{\triangle}^{\perp}\widehat{H} = \lambda\xi$. Using (4.6) and (4.8), we get

$$-(\widehat{\kappa}'' - \widehat{\kappa}\widehat{\tau}^2)N - (2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}')B = \lambda\left(\cos\alpha_0 T + \sin\alpha_0 B\right).$$

Then we have

$$\widehat{\kappa}'' - \widehat{\kappa}\widehat{\tau}^2 = 0, \tag{4.15}$$

$$-(2\widehat{\kappa}'\widehat{\tau} + \widehat{\kappa}\widehat{\tau}') = \lambda\sin\alpha_0, \tag{4.16}$$

$$\lambda\cos\alpha_0 = 0. \tag{4.17}$$

From (4.17), if $\cos\alpha_0 = 0$, then $\alpha_0 = \pi/2$. So it is a Legendre curve. Then from Proposition 3.4, $\widehat{\tau} = 0$. Thus the equations (4.15) and (4.16) give us $\lambda = 0$. Then by Theorem 4.7 in [14], it follows that γ is a Legendre curve with pseudo-Hermitian curvature $\widehat{\kappa}(s) = as + b$, where a and b are real constants. If $\cos\alpha_0 \neq 0$ and $\lambda = 0$ then in view of Theorem 4.7 in [14], it follows that γ is a pseudo-Hermitian Legendre circle.

The converse statement is trivial. □

5 Slant curves of Sasakian Heisenberg Group with Pseudo-Hermitian Connection

The Heisenberg group H_3 can be viewed as $\mathbb{R}^3$ equipped with Riemannian metric

$$g = dx^2 + dy^2 + \eta \otimes \eta,$$

where (x, y, z) are standard coordinates in $\mathbb{R}^3$ and

$$\eta = dz + ydx - xdy.$$

The 1-form η satisfies $d\eta \wedge \eta = -\lambda dx \wedge dy \wedge dz$. Hence η is a contact form. In [10], Inoguchi obtained the Levi-Civita connection ∇ of the metric g with respect to the left-invariant orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z}.$$

He obtained

$$\begin{array}{lll} \nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = e_3, & \nabla_{e_1}e_3 = -e_2, \\ \nabla_{e_2}e_1 = -e_3, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = e_1, \\ \nabla_{e_3}e_1 = -e_2, & \nabla_{e_3}e_2 = e_1, & \nabla_{e_3}e_3 = 0. \end{array} \tag{5.1}$$

We also have the Heisenberg brackets

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = [e_3, e_1] = 0.$$

Let φ be the $(1,1)$-tensor field defined by $\varphi(e_1) = e_2$, $\varphi(e_2) = -e_1$ and $\varphi(e_3) = 0$. Then using the linearity of φ and g we have

$$\eta(e_3) = 1, \quad \varphi^2(X) = -X + \eta(X)e_3, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y).$$

We also have

$$d\eta(X,Y) = g(X, \varphi Y)$$

for all $X, Y \in \chi(M)$. Thus for $\xi = e_3$, (φ, ξ, η, g) is a contact metric structure and the Heisenberg group H_3 is a Sasakian space form of constant holomorphic sectional curvature -3 [10].

Now, let $\gamma : I \to H_3$ be a slant curve with contact angle α_0. Assume that γ is parametrized by arc length s and $\{T, N, B\}$ denote the Frenet frame of γ. Then we can write

$$T = \sin\alpha_0 \cos\beta e_1 + \sin\alpha_0 \sin\beta e_2 + \cos\alpha_0 e_3, \tag{5.2}$$

where $\beta = \beta(s)$. Using (5.1) we have

$$\begin{aligned} \nabla_T T &= \left(-\sin\alpha_0 \sin\beta \left(\beta' - 2\cos\alpha_0\right)\right) e_1 \\ &\quad + \left(\sin\alpha_0 \cos\beta \left(\beta' - 2\cos\alpha_0\right)\right) e_2. \end{aligned} \tag{5.3}$$

On the other hand by the use of (5.2) it follows that

$$\varphi T = -\sin\alpha_0 \sin\beta e_1 + \sin\alpha_0 \cos\beta e_2. \tag{5.4}$$

By the use of (2.7) we find

$$\widehat{\nabla}_T T = -\beta' \sin\alpha_0 \sin\beta e_1 + \beta' \sin\alpha_0 \cos\beta e_2. \tag{5.5}$$

Since $\widehat{\nabla}_T T = \widehat{\kappa} N$, the equation (5.5) gives us

$$\widehat{\kappa} = |\beta'| \sin\alpha_0. \tag{5.6}$$

Hence the principal normal vector field N of γ can be written as

$$N = sgn(\beta') \left(-\sin\beta e_1 + \cos\beta e_2\right).$$

Since $B = T \times N$, we find

$$B = sgn(\beta') \left(-\cos\alpha_0 \cos\beta e_1 - \cos\alpha_0 \sin\beta e_2 + \sin\alpha_0 e_3\right).$$

Then it is easy to see that

$$B' = sgn(\beta') \left(\beta' \cos\alpha_0 - \cos 2\alpha_0\right) \left(\sin\beta e_1 - \cos\beta e_2\right),$$

which gives us

$$\widehat{\tau} = \beta' \cos\alpha_0 - \cos 2\alpha_0. \tag{5.7}$$

Now assume that $\widehat{\tau} = 0$. Then from Proposition 3.4, γ is a Legendre curve. Using (5.7), we obtain

$$\beta(s) = \frac{\cos 2\alpha_0}{\cos \alpha_0} s + c,$$

where c is a real constant. Hence from (5.6), $\widehat{\kappa}$ is a constant.

Let $\gamma(s) = (x(s), y(s), z(s))$. To find the explicit equations, we should integrate the system $\frac{d\gamma}{ds} = T$. Then

$$\frac{dx}{ds} = \sin \alpha_0 \cos \left(\frac{\cos 2\alpha_0}{\cos \alpha_0} s + c \right),$$

$$\frac{dy}{ds} = \sin \alpha_0 \sin \left(\frac{\cos 2\alpha_0}{\cos \alpha_0} s + c \right),$$

$$\frac{dz}{ds} = \cos \alpha_0 + \frac{1}{2} \sin \alpha_0 \left(\sin \left(\frac{\cos 2\alpha_0}{\cos \alpha_0} s + c \right) x(s) - \cos \left(\frac{\cos 2\alpha_0}{\cos \alpha_0} s + c \right) y(s) \right).$$

So using the method given in [4], the integration of above system gives the following example:

Example 5.1. Let $\gamma : I \to H_3$ be a curve with the following parametric equations.

$$x(s) = \frac{\cos \alpha_0}{\cos 2\alpha_0} \sin \alpha_0 \sin(\frac{\cos 2\alpha_0}{\cos \alpha_0} s + c) + d_1,$$

$$y(s) = -\frac{\cos \alpha_0}{\cos 2\alpha_0} \sin \alpha_0 \cos(\frac{\cos 2\alpha_0}{\cos \alpha_0} s + c) + d_2,$$

$$\begin{aligned} z(s) \quad = \quad & \left(\cos \alpha_0 + \frac{\cos \alpha_0}{\cos 2\alpha_0} \sin^2 \alpha_0 \right) s - d_1 \frac{\cos \alpha_0}{\cos 2\alpha_0} \sin \alpha_0 \cos(\frac{\cos 2\alpha_0}{\cos \alpha_0} s + c) \\ & - d_2 \frac{\cos \alpha_0}{\cos 2\alpha_0} \sin \alpha_0 \sin(\frac{\cos 2\alpha_0}{\cos \alpha_0} s + c) + d_3. \end{aligned}$$

Then γ is a pseudo-Hermitian Legendre circle with *pseudo-Hermitian curvature* $\widehat{\kappa} = \left| \frac{\cos 2\alpha_0}{\cos \alpha_0} \right| \sin \alpha_0$, where c, d_1, d_2 and d_3 are some real constants.

Now assume that $\widehat{\tau} \neq 0$ and $\widehat{\kappa}$ is a constant. Then from (5.6), β' is a constant. Then we can write $\beta(s) = as + b$, where a and b are real constants. By the use of equation (5.7), we find $\widehat{\tau} = a \cos \alpha_0 - \cos 2\alpha_0$. Hence $\widehat{\tau}$ is a constant. Similar to the method using in previous example, let $\gamma(s) = (x(s), y(s), z(s))$. To find the explicit equations, we should integrate the system $\frac{d\gamma}{ds} = T$. Then

$$\frac{dx}{ds} = \sin \alpha_0 \cos (as + b),$$

$$\frac{dy}{ds} = \sin \alpha_0 \sin (as + b),$$

$$\frac{dz}{ds} = \cos \alpha_0 + \frac{1}{2} \sin \alpha_0 (\sin (as + b) x(s) - \cos (as + b) y(s)).$$

Similarly, using the method given in [4], the integration of above system gives the following example:

Example 5.2. Let $\gamma : I \to H_3$ be a curve with the following parametric equations.

$$x(s) = \frac{1}{a}\sin\alpha_0 \sin(as+b) + c_1,$$

$$y(s) = -\frac{1}{a}\sin\alpha_0 \cos(as+b) + c_2,$$

$$\begin{aligned} z(s) &= \left(\cos\alpha_0 + \frac{1}{a}\sin^2\alpha_0\right)s - \frac{c_1}{a}\sin\alpha_0\cos(as+b) \\ &\quad -\frac{c_2}{a}\sin\alpha_0\sin(as+b) + c_3. \end{aligned}$$

Then γ is a pseudo-Hermitian slant helix with *pseudo-Hermitian curvature* $\widehat{\kappa} = |a|\sin\alpha_0$ and *pseudo-Hermitian torsion* $\widehat{\tau} = a\cos\alpha_0 - \cos 2\alpha_0$, where a, b, c_1, c_2 and c_3 are some real constants such that $a \neq \frac{\cos 2\alpha_0}{\cos\alpha_0}$.

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On the Shape Operator of Biconservative Hypersurfaces in $\mathbb{E}^5_2$

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Abstract. In this paper, we present a short survey about recent results on biconservative hypersurfaces in pseudo-Riemannian space forms and discuss some open problems on this topic. Further, we consider canonical forms of the shape operator of biconservative hypersurface of index 2 in $\mathbb{E}^5_2$ and by choosing an appropriate base field, we obtain that there are six possible canonical forms of the shape operator satisfying $\langle \nabla H, \nabla H\rangle = 0$, i.e., ∇H is light-like vector.

Keywords. Biconservative hypersurface, pseudo-Euclidean space, biharmonic submanifold, shape operator.
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1 Introduction

In 1981, Nomizu introduced isoparametric hypersurfaces in Lorentzian space forms. A hypersurface is called isoparametric if the minimal polynomial of shape operator is constant. It is well-known that the shape operator of a Riemannian submanifold is always diagonalizable, but this is not the case for the shape operator of a Lorentzian submanifold. This makes the isoparametric theory in pseudo-Riemannian space forms different from that in Riemannian space forms. In [5], Magid classified Lorentzian isoparametric hypersurfaces and obtained that the shape operator of a Lorentzian hypersurface in a Minkowski space can have four possible canonical forms by choosing an appropriate frame field. He obtained this result by Petrov's consideration in [6] , i.e., a symmetric endomorphism of a vector space with a Lorentzian inner product can be put into one of four possible canonical forms. By considering recent results obtained by Turgay in [7] and Deepika in [1], one can conclude that there is only two different families of biconservative hypersurfaces in $\mathbb{E}^4_1$ by considering the canonical forms of their shape operator (see Theorem 3.3).

Now, there arise a natural question: What will be the canonical forms of the shape operator if we increase the dimension of the pseudo-Euclidean space as well as index of the hypersurface? Thus, one can ask a general question which is still open to all, i.e., "Does there exist any specific formula from which one can get all possible canonical forms of the shape operator of a hypersurface with variable index of general ambient pseudo-Euclidean space $\mathbb{E}^n_s$" ? So it is natural to start index 2 hypersurfaces in $\mathbb{E}^5_2$. During study, it is observed that if one consider index 2 hypersurfaces in $\mathbb{E}^5_2$ then the number of canonical forms of the shape operator increases to 9 whereas it is 4 in case of $\mathbb{E}^4_1$.

The paper is organized as follows. In Sect. 2, we give some basic definitions and formulas which we used in other sections of the paper. In Sect. 3, we present a short survey about recent papers on biconservative hypersurfaces and try to point out problems which left open in these papers. In Sect. 4, we study existance of all possible canonical forms of the shape operator of biconservative hypersurfaces of index 2 with an additional condition, i.e., ∇H is a lightlike vector whereas H is mean curvature vector field of the hypersurface, and further, we obtain our main result.

2 Preliminaries

In this section we recall some basic definitions and formulas that we will use in other part of the paper.

2.1 Hypersurfaces of $\mathbb{E}^5_2$

Let $\mathbb{E}^5_2$ denote the 5-dimensional real vector space $\mathbb{R}^5$ with the canonical inner product of signature $(2,3)$ given by

$$\widetilde{g}(x,y) = \langle x,y\rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5.$$

We consider an oriented hypersurface M of $\mathbb{E}^5_2$ with index 2. Let N be its unit normal vector associated with the orientation of M. We define the shape operator S of M by the Weingarten formula

$$\widetilde{\nabla}_X N = -SX,$$

where X is a vector field tangent to M and $\widetilde{\nabla}$ denotes the Levi-Civita connection of $\mathbb{E}^5_2$. Let ∇ stands for the Levi-Civita connection of M with respect to the induced metric on M, then the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

where h is the second fundamental form of M. Note that h and S are related with the equation

$$\langle SX,Y\rangle = \langle h(X,Y),N\rangle. \tag{2.1}$$

The eigenvalues of S are called principal curvatures of M. Corresponding to every principal curvature k, we have algebraic multiplicity and geometric multiplicity. Algebraic multiplicity ν is the exponent of $(x--k)$ in the characteristic polynomial and geometric multiplicity μ is the dimension of the eigenspace

$$T_k = \{X \in T_xM : SX = kX\}.$$

A principal curvature k is called diagonalizable if $\nu = \mu$. The hypersurface M is called biconservative if it satisfies

$$S(\nabla H) = -2H\nabla H. \tag{BC}$$

Remark 2.1. If M has constant mean curvature, then (BC) is satisfied trivially. Hence, in the remaining part of the paper, we will assume that ∇H does not vanish on M, i.e., there exists a vector field X on M such that $X(H) \neq 0$.

On the other hand, since the ambient space $\mathbb{E}^5_2$ is a flat space, its curvature tensor $\tilde{R}$ vanishes identically. Thus, Gauss and Codazzi equations become

$$\left(\tilde{R}(X,Y)Z\right)^T = 0, \quad \text{and} \quad \left(\tilde{R}(X,Y)Z\right)^\perp = 0,$$

respectively, for any vector fields X, Y, Z tangent to M.

2.2 Shape operator of hypersurfaces with index 2 in $\mathbb{E}^5_2$

In this section, we consider canonical forms of the shape operator index 2 hypersurfaces in the pseudo-Euclidean space $\mathbb{E}^5_2$.

By a frame field in a hypersurface M of $\mathbb{E}^5_2$, we mean a base field $\{e_1, e_2, e_3, e_4\}$ of the tangent bundle of M. We put $\varepsilon_i = \langle e_i, e_i\rangle \in \{-1, 0, 1\}$. Furthermore, we define the connection forms ω_{ij} corresponding to a given frame field by

$$\omega_{ij}(e_k) = \langle \nabla_{e_k} e_i, e_j\rangle.$$

Now, it is well-known that shape operator S is symmetric because of (2.1). On the other hand, there is a well-known theorem that a symmetric operator in a positive definite vector space is diagonalizable over R. The situation is more complicated if the metric is not definite, and the complete answer is given by the following theorem of Petrov [6].

Theorem 2.2. ***(Principal Axis Theorem for a Tensor [6])*** *In a vector space V^n with a non- degenerate metric g every symmetric operator S can be put into the following form:*

$$S = \begin{pmatrix} B_{p_1} & & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & B_{p_k} & & & & \\ & & & & & C_{2t_1} & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \\ & & & & & & & & & C_{2t_m} \end{pmatrix}$$

where B_{p_i} is $p_i \times p_i$, C_{2t_j} is $2t_j \times 2t_j$ and $\sum p_i + 2\sum t_j = n$. This is with respect to a basis for which the inner product has the form

$$\begin{pmatrix} g_{p_1} & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & g_{p_k} & & & \\ & & & & & g_{2t_1} & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & & g_{2t_m} \end{pmatrix}$$

$$B_{p_i} = \begin{pmatrix} l_i\lambda_i & l_i & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & \cdot & \\ & & & & & & \cdot & \cdot \\ & & & & & & & \cdot & l_i \\ & & & & & & & & l_i\lambda_i \end{pmatrix}$$

$$C_{2j} = \begin{pmatrix} \alpha_j & \beta_j & 1 & 0 & & & & & & \\ -\beta_j & \alpha_j & 0 & 1 & & & & & & \\ & & \alpha_j & \beta_j & 1 & 0 & & & & \\ & & -\beta_j & \alpha_j & 0 & 1 & & & & \\ & & & & \cdot & & & & & \\ & & & & & \cdot & & & & \\ & & & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \alpha_j & \beta_j \\ & & & & & & & & -\beta_j & \alpha_j \end{pmatrix}$$

and

$$g_{p_i} = \begin{pmatrix} & & & l_i \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ l_i & & & \end{pmatrix}, \quad g_{2t_j} = \begin{pmatrix} & & & & & 1 & 0 \\ & & & & & 0 & -1 \\ & & & & \cdot & & \\ & & & \cdot & & & \\ & & \cdot & & & & \\ 1 & 0 & & & & & \\ 0 & -1 & & & & & \end{pmatrix}.$$

Now, with the help of Theorem 2.2 and by choosing an appropriate base field $\{e_1, e_2, e_3, e_4\}$ of the tangent bundle of M, we obtain that the possible canonical forms of the shape operator S of $\mathbb{E}^5_2$ can have one of the following forms. Note that in each cases below, g denotes the induced metric tensor of M, i.e., $g_{ij} = \langle e_i, e_j \rangle$.

$$\text{Case I. } S = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case II. } S = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case III. } S = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & 1 \\ 0 & 0 & 0 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$

$$\text{Case IV. } S = \begin{pmatrix} k_1 & 1 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & \beta_1 \\ 0 & 0 & -\beta_1 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case V. } S = \begin{pmatrix} k_1 & 0 & 1 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & -1 & k_1 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case VI. } S = \begin{pmatrix} k_1 & \beta_1 & 0 & 0 \\ -\beta_1 & k_1 & 0 & 0 \\ 0 & 0 & k_3 & \beta_2 \\ 0 & 0 & -\beta_2 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case VII. } S = \begin{pmatrix} k_1 & \beta_1 & 1 & 0 \\ -\beta_1 & k_1 & 0 & 1 \\ 0 & 0 & k_1 & \beta_1 \\ 0 & 0 & -\beta_1 & k_1 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

$$\text{Case VIII. } S = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & \beta_1 \\ 0 & 0 & -\beta_1 & k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\text{Case IX. } S = \begin{pmatrix} k_1 & 0 & 1 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_1 & 1 \\ 0 & 1 & 0 & k_1 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

for some smooth functions $k_1, k_2, k_3, k_4, \beta_1, \beta_2$.

3 Recent Results about Biconservative Hypersurfaces

3.1 Shape operator of biconservative hypersurfaces in Minkowski spaces.

The second named author obtained the following results by considering the shape operator of biconservative hypersurfaces in a Minkowski space of arbitrary dimension (See [7, Theorem 4.1]).

Theorem 3.1. *[7] Let M be a hypersurface in the Minkowski space $\mathbb{E}^4_1$, S its shape operator and H its mean curvature. Assume that ∇H is light-like and S has the minimal polynomial*

$$P(\lambda) = \prod_{i=1}^{t} (\lambda - k_1)^2 (\lambda - k_2)(\lambda - k_3) \cdots (\lambda - k_t)$$

for some t. If $t \leq 5$, then M is not biconservative.

On the other hand, in [1], Deepika considered hypersurface with complex principle curvature in an arbitrary Minkowski space and obtained the following result.

Theorem 3.2. *[1] Let M^n_1 in $\mathbb{E}^{n+1}_1$ be a biconservative Lorentz hypersurface having non diagonal shape operator with complex eigenvalues and with at most five distinct principal curvatures. Then M^n_1 has constant mean curvature.*

By combining these results, we would like to state the following result on the shape operator of biconservative hypersurface in $\mathbb{E}^4_1$.

Theorem 3.3. *Let M be a hypersurface in $\mathbb{E}^4_1$ and H its mean curvature. Then by choosing an appropriated frame field $\{e_1, e_2, e_3\}$ the matrix representation of the shape operator S of M can have one of the following two canonical forms*

$$\begin{aligned}
&\textit{Case 1.}\quad S = \begin{pmatrix} -\frac{3\varepsilon}{2}H & & \\ & k_2 & \\ & & \frac{3}{2}(2+\varepsilon)H - k_2 \end{pmatrix} \quad \textit{for a function } k_2,\\
&\textit{Case 2.}\quad S = \begin{pmatrix} -\frac{3H}{2} & & \\ & \frac{9H}{4} & 1 \\ & & \frac{9H}{4} \end{pmatrix},
\end{aligned} \tag{3.1}$$

where e_1 is proportional to ∇H and ε is the signature of the normal of M, i.e.,

$$\varepsilon = \begin{cases} -1 & \textit{if } M \textit{ is Riemannian} \\ 1 & \textit{if } M \textit{ is Lorentzian.} \end{cases}$$

At this instant, we would like to mention that the complete classification of biconservative hypersurfaces, given in Case 1 of (3.1), is obtained by Yu Fu and the second named author in [4] (See Sect. 3.2). However, the following problem is still open.

Problem 1. Classify all biconservative hypersurfaces in $\mathbb{E}^4_1$ with the shape operator given in the Case 2 of (3.1).

3.2 Biconservative hypersurfaces in Minkowski spaces.

In [2], Yu Fu obtained the following results.

Proposition 3.4. *[2] Let M be a nondegenerate biconservative surface immersed in the 3-dimensional Minkowski space $\mathbb{E}^3_1$. Then the immersed surface M is either a CMC surface or locally given by one of the following eight surfaces.*

1. *A timelike surface of revolution with spacelike axis, given by*

$$x(s,t) = (f(s), s\cosh t, s\sinh t) \tag{3.2}$$

where $s \in (27, +\infty)$ and

$$f(s) = \frac{9}{2}(s^{\frac{1}{3}}\sqrt{s^{\frac{2}{3}} - 9} + 9In(s^{\frac{1}{3}} + \sqrt{s^{\frac{2}{3}} - 9})).$$

2. *A spacelike surface of revolution with spacelike axis, given by*

$$x(s,t) = (f(s), s\sinh t, s\cosh t) \tag{3.3}$$

where $s \in (0, 27)$ and

$$f(s) = \frac{81}{2}arcsin\frac{1}{3}s^{\frac{1}{3}} - \frac{9}{2}s^{\frac{1}{3}}\sqrt{9 - s^{\frac{2}{3}}}.$$

3. *A spacelike surface of revolution with timelike axis, given by*

$$x(s,t) = (s\cos t, s\sin t, f(s)), \tag{3.4}$$

where $s \in (0, +\infty)$ and

$$f(s) = \frac{9}{2}(s^{\frac{1}{3}}\sqrt{s^{\frac{2}{3}} + 9} - 9In(s^{\frac{1}{3}} + \sqrt{s^{\frac{2}{3}} + 9})).$$

4. *A spacelike surface of revolution with lightlike axis, given by*

$$x(s,t) = (\frac{1}{2}st^2 - \frac{1}{30}s^{\frac{5}{3}} - \frac{1}{2}s, st, \frac{1}{2}st^2 - \frac{1}{30}s^{\frac{5}{3}} + \frac{1}{2}s), \tag{3.5}$$

where $s \in (0, +\infty)$.

5. *A timelike surface of revolution with spacelike axis, given by*

$$x(s,t) = (f(s), s\sinh t, s\cosh t), \tag{3.6}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \frac{9}{2}(s^{\frac{1}{3}}\sqrt{s^{\frac{2}{3}}+9} - 9In(s^{\frac{1}{3}} + \sqrt{s^{\frac{2}{3}}+9})).$$

6. *A timelike surface of revolution with timelike axis, given by*

$$x(s,t) = (s\cos t, s\sin t, f(s)), \tag{3.7}$$

where $s \in (0, 27)$ *and*

$$f(s) = \frac{81}{2}arcsin\frac{1}{3}s^{\frac{1}{3}} - \frac{9}{2}s^{\frac{1}{3}}\sqrt{9 - s^{\frac{2}{3}}}.$$

7. *A timelike surface of revolution with lightlike axis, given by*

$$x(s,t) = (\frac{1}{2}st^2 + \frac{1}{30}s^{\frac{5}{3}} - \frac{1}{2}s, st, \frac{1}{2}st^2 + \frac{1}{30}s^{\frac{5}{3}} + \frac{1}{2}s), \tag{3.8}$$

where $s \in (0, +\infty)$.

8. *A null sroll with non-constant mean curvature.*

In [3], Yu Fu give a complete explicit classification of biconservative surfaces in de Sitter 3-spaces and anti-de Sitter 3-spaces. He obtained the following results.

Proposition 3.5. *[3] Let M be a nondegenerate bi-conservative surface immersed in the 3-dimensional de Sitter space $\mathbb{S}^3_1(1) \in \mathbb{E}^4_1$. Then the immersed surface M is either a CMC surface or locally given by one of the following nine surfaces.*

1. *A timelike rotational surface, given by*

$$x(s,t) = (s\sinh t, s\cosh t, \sqrt{1-s^2}\cos f, \sqrt{1-s^2}\sin f), \tag{3.9}$$

where $s \in (0, 1)$ *and*

$$f(s) = \pm\int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{1 - 9s^{-\frac{2}{3}} - s^2}}ds.$$

2. *A spacelike rotational surface, given by*

$$x(s,t) = (s\cosh t, s\sinh t, \sqrt{1+s^2}cosf, \sqrt{1+s^2}sinf), \tag{3.10}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \pm\int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{9s^{-\frac{2}{3}} - s^2 - 1}}ds.$$

3. A spacelike rotational surface, given by

$$x(s,t) = (\sqrt{1-s^2}\sinh f, \sqrt{1-s^2}\cosh f, scost, ssint), \tag{3.11}$$

where $s \in (0,1)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{1+9s^{-\frac{2}{3}}-s^2}} ds.$$

4. A spacelike rotational surface, given by

$$x(s,t) = (\sqrt{s^2-1}\cosh f, \sqrt{s^2-1}\sinh f, scost, ssint), \tag{3.12}$$

where $s \in (1,+\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{1+9s^{-\frac{2}{3}}-s^2}} ds.$$

5. A spacelike rotational surface, given by

$$x(s,t) = (\frac{1}{2}(st^2+sf^2-\frac{1}{s}+s), \frac{1}{2}(st^2+sf^2-\frac{1}{s}-s), sf, st) \tag{3.13}$$

where $s \in (0, 3^{\frac{3}{4}})$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{9-s^{-\frac{8}{3}}}} ds.$$

6. A timelike rotational surface, given by

$$x(s,t) = (s\cosh t, s\sinh t, \sqrt{1+s^2}cosf, \sqrt{1+s^2}sinf), \tag{3.14}$$

where $s \in (0,+\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{9s^{-\frac{2}{3}}+s^2+1}} ds.$$

7. A timelike rotational surface, given by

$$x(s,t) = (\sqrt{1-s^2}\sinh f, \sqrt{1-s^2}\cosh f, scost, ssint), \tag{3.15}$$

where $s \in (0,1)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{9s^{-\frac{2}{3}}+s^2-1}} ds.$$

8. *A timelike rotational surface, given by*

$$x(s,t) = (\sqrt{s^2-1}\cosh f, \sqrt{s^2-1}\sinh f, scost, ssint), \tag{3.16}$$

where $s \in (1, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{9s^{-\frac{2}{3}}+s^2-1}} ds.$$

9. *A timelike rotational surface, given by*

$$x(s,t) = (\frac{1}{2}(st^2 + sf^2 - \frac{1}{s} + s), \frac{1}{2}(st^2 + sf^2 - \frac{1}{s} - s), sf, st) \tag{3.17}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{9+s^{-\frac{8}{3}}}} ds.$$

Proposition 3.6. *[3] Let M be a nondegenerate bi-conservative surface immersed in the 3-dimensional anti-de Sitter space $\mathbb{H}^3_1(-1) \in \mathbb{E}^4_2$. Then the immersed surface M is either a CMC surface or locally given by one of the following eleven surfaces.*

1. *A timelike rotational surface, given by*

$$x(s,t) = (s\sinh t, \sqrt{1+s^2}\cosh f, s\cosh t, \sqrt{1+s^2}\sinh f), \tag{3.18}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{1-9s^{-\frac{2}{3}}+s^2}} ds.$$

2. *A spacelike rotational surface, given by*

$$x(s,t) = (s\cosh t, \sqrt{1-s^2}\cosh f, s\sinh t, \sqrt{1-s^2}\sinh f), \tag{3.19}$$

where $s \in (0, 1)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{9s^{-\frac{2}{3}}+s^2-1}} ds.$$

3. *A spacelike rotational surface, given by*

$$x(s,t) = (s\cosh t, \sqrt{s^2-1}\sinh f, s\sinh t, \sqrt{s^2-1}\cosh f), \tag{3.20}$$

where $s \in (1, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{9s^{-\frac{2}{3}}+s^2-1}} ds.$$

4. A spacelike rotational surface, given by

$$x(s,t) = (\sqrt{1+s^2}cosf, \sqrt{1+s^2}sinf, scost, ssint), \tag{3.21}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{1+9s^{-\frac{2}{3}}-s^2}}ds.$$

5. A spacelike rotational surface, given by

$$x(s,t) = (scost, ssint, \sqrt{s^2-1}cosf, \sqrt{s^2-1}sinf), \tag{3.22}$$

where $s \in (1, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{s^2-9s^{-\frac{2}{3}}-1}}ds.$$

6. A spacelike rotational surface, given by

$$x(s,t) = (\frac{1}{2}(st^2 - sf^2 + \frac{1}{s} + s), sf, \frac{1}{2}(st^2 - sf^2 + \frac{1}{s} - s), st) \tag{3.23}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{9+s^{\frac{8}{3}}}}ds.$$

7. A timelike rotational surface, given by

$$x(s,t) = (\frac{1}{2}(st^2 + sf^2 + \frac{1}{s} + s), sf, \frac{1}{2}(st^2 + sf^2 + \frac{1}{s} - s), sf), \tag{3.24}$$

where $s \in (3^{\frac{3}{4}}, +\infty)$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{s^{\frac{8}{3}}-9}}ds.$$

8. A timelike rotational surface, given by

$$x(s,t) = (s\cosh t, \sqrt{1-s^2}\cosh f, s\sinh t, \sqrt{1-s^2}\sinh f), \tag{3.25}$$

where $s \in (0, 1)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1-s^2)\sqrt{9s^{-\frac{2}{3}}-s^2+1}}ds.$$

9. A timelike rotational surface, given by

$$x(s,t) = (s\cosh t, \sqrt{s^2-1}\sinh f, s\sinh t, \sqrt{s^2-1}\cosh f), \tag{3.26}$$

where $s \in (1, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(s^2-1)\sqrt{9s^{-\frac{2}{3}} - s^2 + 1}} ds.$$

10. A timelike rotational surface, given by

$$x(s,t) = (\sqrt{1+s^2}cosf, \sqrt{1+s^2}sinf, scost, ssint), \tag{3.27}$$

where $s \in (0, +\infty)$ *and*

$$f(s) = \pm \int \frac{3s^{-\frac{1}{3}}}{(1+s^2)\sqrt{9s^{-\frac{2}{3}} - s^2 - 1}} ds.$$

11. A timelike rotational surface, given by

$$x(s,t) = (\frac{1}{2}(st^2 - sf^2 + \frac{1}{s} + s), sf, \frac{1}{2}(st^2 - sf^2 + \frac{1}{s} - s), st) \tag{3.28}$$

where $s \in (0, 3^{\frac{3}{4}})$ *and*

$$f(s) = \int \frac{3}{s^2\sqrt{9 - s^{\frac{8}{3}}}} ds.$$

Further, in [4], the author and Yu Fu considered biconservative hypersurfaces in the Minkowski 4-space with diagonalizable shape operator. They obtained the following results.

Proposition 3.7. *[4] Let M be a hypersurface in* $\mathbb{E}^4_1$ *given by*

$$\begin{aligned} x(s,t,u) = \Bigg(&\frac{1}{2}s(t^2+u^2) + au^2 + s + \phi(s), st, (s+2a)u, \\ &\frac{1}{2}s(t^2+u^2) + au^2 + \phi(s)\Bigg), \quad a \neq 0. \end{aligned} \tag{3.29}$$

Then, M is biconservative if and only if either M is Riemannian and

$$\phi(s) = c_1\left(\ln(s+2a) - \ln s - \frac{a}{s} - \frac{a}{s+2a}\right) - \frac{s}{2}$$

or it is Lorentzian and

$$\phi(s) = c_1 \int_{s_0}^{s} \left(\xi(\xi+2a)\right)^{2/3} d\xi - \frac{s}{2},$$

where $c_1 \neq 0$ *and* s_0 *are some constants.*

Theorem 3.8. *[4] Let M be a hypersurface in $\mathbb{E}^4_1$ with diagonalizable shape operator and three distinct principal curvatures. Then M is biconservative if and only if it is congruent to one of hypersurfaces*

1. *A generalized cylinder $M^2_0 \times \mathbb{E}^1_1$ where M is a biconservative surface in $\mathbb{E}^3$;*
2. *A generalized cylinder $M^2_0 \times \mathbb{E}^1$ where M is a biconservative Riemannian surface in $\mathbb{E}^3_1$;*
3. *A generalized cylinder $M^2_1 \times \mathbb{E}^1$, where M is a biconservative Lorentzian surface in $\mathbb{E}^3_1$;*
4. *A Rimannian surface given by*
$$x(s,t,u) = (s\,\cosh t, s\,\sinh t, f_1(s)\cos u, f_1(s)\sin u) \tag{3.30}$$
for a function f_1 satisfying
$$\frac{f_1''}{f_1'^2 - 1} = \frac{f_1 f_1' + s}{s f_1};$$
5. *A Lorentzian surface with the parametrization given in* (3.30) *for a function f_1 satisfying*
$$\frac{-3f_1''}{f_1'^2 - 1} = \frac{f_1 f_1' + s}{s f_1};$$
6. *A Rimannian surface given by*
$$x(s,t,u) = (s\,\sinh t, s\,\cosh t, f_2(s)\cos u, f_2(s)\sin u) \tag{3.31}$$
for a function f_2 satisfying
$$\frac{f_2''}{f_2'^2 + 1} = \frac{f_2 f_2' + s}{s f_2};$$
7. *A surface given in Proposition 3.7.*

3.3 Biconservative Hypersurfaces in $\mathbb{E}^5_2$

In [8], we study biconservative hypersurfaces of index 2 in $\mathbb{E}^5_2$ and obtain the complete classification of biconservative hypersurfaces with diagonalizable shape operator at exactly three distinct principal curvatures. The results are following.

Theorem 3.9. *[8] Let M be an oriented biconservative hypersurface of index 2 in the pseudo-Euclidean space $\mathbb{E}^5_2$. Assume that its shape operator has the form*
$$S = \mathrm{diag}(k_1, 0, 0, k_4), \quad k_4 \neq 0.$$
Then, it is congruent to one of the following eight type of generalized cylinders over surfaces for some smooth functions $\phi = \phi(s)$ and $\psi = \psi(s)$.

(i). $x(s,t,u,v) = (t, u, \phi\cos v, \phi\sin v, \psi), \quad \phi'^2 + \psi'^2 = 1;$

(ii). $x(s,t,u,v) = (\phi\sinh v, t, u, \phi\cosh v, \psi), \quad \phi'^2 + \psi'^2 = 1;$

(iii). $x(s,t,u,v) = (\psi, t, u, \phi\cos v, \phi\sin v), \quad \phi'^2 - \psi'^2 = -1;$

(iv). $x(s,t,u,v) = (\phi\cosh v, t, u, \phi\sinh v, \psi), \quad \phi'^2 - \psi'^2 = 1;$

(v). $x(s,t,u,v) = \left(\frac{v^2 s}{2} + \psi + s, t, u, vs, \frac{v^2 s}{2} + \psi\right), \ 1 - 2\psi' < 0;$

(vi). $x(s,t,u,v) = (\phi\cos v, \phi\sin v, t, u, \psi), \quad \phi'^2 - \psi'^2 = 1;$

(vii). $x(s,t,u,v) = (\phi\sinh v, \psi, t, u, \phi\cosh v), \quad \phi'^2 - \psi'^2 = -1;$

(viii). $x(s,t,u,v) = \left(\frac{sv^2}{2} + \psi, sv, t, u, \frac{sv^2}{2} + \psi + s\right), \ 1 + 2\psi' < 0.$

Theorem 3.10. *[8] Let M be an oriented hypersurface of index 2 in the pseudo-Euclidean space $\mathbb{E}^5_2$. Assume that its shape operator has the form*

$$S = \mathrm{diag}(k_1, k_2, k_2, 0), \quad k_2 \neq 0.$$

Then, it is congruent to one of the following eight type of cylinders for some smooth functions $\phi = \phi(s)$ and $\psi = \psi(s)$.

(i). $x(s,t,u,v) = (v, \phi\cosh t, \phi\sinh t\cos u, \phi\sinh t\sin u, \psi), \quad \phi'^2 - \psi'^2 = 1;$

(ii). $x(s,t,u,v) = (v, \psi, \phi\cos t, \phi\sin t\cos u, \phi\sin t\sin u), \quad \phi'^2 - \psi'^2 = -1;$

(iii). $x(s,t,u,v) = (\phi\cosh t\sin u, \phi\cosh t\cos u, \phi\sinh t, \psi, v), \quad \phi'^2 - \psi'^2 = 1;$

(iv). $x(s,t,u,v) = (\psi, \phi\sinh t, \phi\cosh t\cos u, \phi\cosh t\sin u, v), \quad \phi'^2 - \psi'^2 = -1;$

(v). $x(s,t,u,v) = (v, \phi\sinh t, \phi\cosh t\cos u, \phi\cosh t\sin u, \psi), \quad \phi'^2 + \psi'^2 = 1;$

(vi). $x(s,t,u,v) = (\phi\sinh t cosu, \phi\sinh t\sin u, \phi\cosh u, \psi, v), \quad \phi'^2 + \psi'^2 = 1;$

(vii). $x(s,t,u,v) = \left(\frac{s(t^2+u^2)}{2} + \psi, v, st, su, \frac{s(t^2+u^2)}{2} + \psi - s\right), \quad 1 - 2\psi' < 0;$

(viii). $x(s,t,u,v) = \left(\frac{s(t^2-u^2)}{2} + \psi, st, su, v, \frac{s(t^2-u^2)}{2} + \psi + s\right), \quad 1 + 2\psi' < 0.$

Theorem 3.11. *[8] Let M be an oriented hypersurface of index 2 in the pseudo-Euclidean space $\mathbb{E}^5_2$. Assume that its shape operator has the form*

$$S = \mathrm{diag}(k_1, k_2, k_2, k_4), \quad k_4 \neq k_2$$

for some non-vanishing smooth functions k_1, k_2, k_4. Then, it is congruent to one of the following eight type of hypersurfaces for some smooth functions $\phi_1 = \phi_1(s)$ and $\phi_2 = \phi_2(s)$.

(i). $x(s,t,u,v) = (\phi_2 \sinh v, \phi_1 \cosh t, \phi_1 \sinh t \cos u, \phi_1 \sinh t \sin u, \phi_2 \cosh v)$, $\phi_1'^2 - \phi_2'^2 = 1$;

(ii). $x(s,t,u,v) = (\phi_2 \cos v, \phi_2 \sin v, \phi_1 \cos t, \phi_1 \sin t \cos u, \phi_1 \sin t \sin u)$, $\phi_1'^2 - \phi_2'^2 = -1$;

(iii). $x(s,t,u,v) = (\phi_1 \cosh t \sin u, \phi_1 \cosh t \cos u, \phi_1 \sinh t, \phi_2 \cos v, \phi_2 \sin v)$, $\phi_1'^2 - \phi_2'^2 = 1$;

(iv). $x(s,t,u,v) = (\phi_2 \sinh v, \phi_1 \sinh t, \phi_1 \cosh t \cos u, \phi_1 \cosh t \sin u, \phi_2 \cosh v)$, $\phi_1'^2 + \phi_2'^2 = 1$;

(v). $x(s,t,u,v) = (\phi_2 \cosh v, \phi_1 \sinh t, \phi_1 \cosh t \cos u, \phi_1 \cosh t \sin u, \phi_2 \sinh v)$, $\phi_1'^2 - \phi_2'^2 = -1$;

(vi). $x(s,t,u,v) = (\phi_1 \sinh t cosu, \phi_1 \sinh t \sin u, \phi_1 \cosh u, \phi_2 \cos v, \phi_2 \sin v)$, $\phi_1'^2 + \phi_2'^2 = 1$;

(vii). A hypersurface given by

$$x(s,t,u,v) = \Big(\frac{s}{2}\left(t^2+u^2-v^2\right) - av^2 + \psi, v(2a+s), st, su, \frac{s}{2}\left(t^2+u^2-v^2\right) - av^2 + \psi - s\Big) \tag{3.32}$$

for a non-zero constants a and a smooth function $\psi = \psi(s)$ *such that* $1 - 2\psi' < 0$*;*

(viii). A hypersurface given by

$$x(s,t,u,v) = \left(\frac{s\left(t^2-u^2-v^2\right)}{2} + av^2 + \psi, st, su, v(s-2a), \frac{s\left(t^2-u^2-v^2\right)}{2} + av^2 + \psi + s\right) \tag{3.33}$$

for a non-zero constants a and a smooth function $\psi = \psi(\tilde{s})$ *such that* $1 + 2\psi' < 0$.

4 Shape operator of biconservative hypersurfaces of index 2 in $\mathbb{E}^5_2$

In this section, we only consider hypersurfaces with non-constant mean curvature. Before we proceed, we would like to mention that in [8], authors considered hypersurfaces of index 2 in $\mathbb{E}^5_2$. It is proved that if ∇H is assumed not to be a light-like vector, then the shape operator of a biconservative hypersurface has one of the four possible canonical forms given below.

Lemma 4.1. *[8] Let M be a hypersurface of index 2 in $\mathbb{E}^5_2$ with H as its (first) mean curvature. Assume that ∇H is not light-like. If M is biconservative, then with respect to a suitable frame field $\{e_1 = \frac{\nabla H}{\|\nabla H\|}, e_2, e_3, e_4\}$, its shape operator S has one of the following forms:*

$$\begin{aligned}
&\textit{Case I. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case II. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 1 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case III. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & -\nu & 0 \\ 0 & \nu & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case IV. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & 2H & 0 & 0 \\ 0 & 0 & 2H & -1 \\ 0 & 1 & 0 & 2H \end{pmatrix},
\end{aligned} \tag{4.1}$$

for some smooth functions k_2, k_3, k_4, ν. In Cases I and III, the induced metric $g_{ij} = g(e_i, e_j) = \langle e_i, e_j \rangle$ of M is $g_{ij} = \varepsilon_i \delta_{ij} \in \{-1, 0, 1\}$, while in Cases II and IV, it is given by

$$g = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_1 \end{pmatrix}.$$

4.1 Main Results

In this subsection, we consider the shape operator of a biconservative hypersurface in $\mathbb{E}^5_2$ with an additional hypothesis of being light-like of gradient of its mean curvature. Our aim is to investigate possible canonical forms of the shape operator S of M under the following assumption.

Assumption. ∇H is light-like, where H is the mean curvature of the biconservative hypersurface M with index 2.

By the above assumption, (BC) implies that ∇H is an eigenvector of S with corresponding eigenvalue $-2H$. It is very easy to observe that the matrix representation of S with respect to a suitable frame field $\{e_1, e_2, e_3, e_4\}$ can not be one of Case VI, Case VII or Case IX given in Sect. 2.2.

First, we obtain the following result.

Proposition 4.2. *The subspace* $\ker(S - 2HI)$ *is degenerate, where I is the identity operator acting on the space of tangent vector fields of M.*

Proof. Let us consider, the subspace $\ker(S-2HI)$ is non-degenerate. Since $\nabla H \in \Omega = \ker(S-2H)$ and it is light-like then the index of Ω should be at least 1. Thus, there exists two unit vector fields X, Y such that $SX = -2HX$, $SY = -2HY$, $\nabla H = \tau(X-Y)$ for a smooth function τ and $\langle X, X\rangle = -\langle Y, Y\rangle = 1$. Furthermore, we have $X(H) \neq 0$ and $Y(H) \neq 0$. However, this contradicts with the Codazzi equation $\left(\widetilde{R}(X,Y)X\right)^{\perp} = 0$ which yields $X(H) = 0$. □

By using this result, we conclude that the matrix representation of S with respect to a suitable frame field $\{e_1, e_2, e_3, e_4\}$ can not be one of Case I, Case VIII or Case II with $k_3 = k_4 = -2H$. Hence, we have the following result.

Lemma 4.3. *The matrix representation of S with respect to a suitable frame field $\{e_1, e_2, e_3, e_4\}$ is one of the following four forms, where we assume e_1 to be proportional to ∇H and g denotes the induced metric tensor of M, i.e., $g_{ij} = \langle e_i, e_j\rangle$.*

Case I. $S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & 8H-k_3 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
for a smooth function k_3;

Case II. $S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & 4H & 1 \\ 0 & 0 & 0 & 4H \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix};$

Case III. $S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & 4H & \beta_1 \\ 0 & 0 & -\beta_1 & 4H \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
for a smooth function β_1;

Case IV. $S = \begin{pmatrix} -2H & 0 & 1 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & -1 & -2H & 0 \\ 0 & 0 & 0 & 10H \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$

Now, since e_1 is proportional to ∇H, we have

$$e_1(H) = e_3(H) = e_4(H) = 0. \tag{4.2}$$

Proposition 4.4. *There exists no hypersurfaces of index 2 in $\mathbb{E}^5_2$ with shape operator given by Case II of Lemma 4.3.*

Proof. Assume that the shape operator of M is as given in Case II of Lemma 4.3. Then, the second fundamental form of M satisfies

$$h(e_1,e_2) = 2HN, \quad h(e_2,e_2) = -N, \qquad h(e_3,e_4) = -4HN, \quad h(e_4,e_4) = -N$$

and for all other cases, we have $h(e_i, e_j) = 0$.

Note that we have

$$\begin{aligned}\nabla_{e_k} e_1 =& -\omega_{12}(e_k)e_1 - \omega_{14}(e_k)e_3 - \omega_{13}(e_k)e_4,\\ \nabla_{e_k} e_2 =& \omega_{12}(e_k)e_2 - \omega_{24}(e_k)e_3 - \omega_{23}(e_k)e_4,\\ \nabla_{e_k} e_3 =& \omega_{23}(e_k)e_1 + \omega_{13}(e_k)e_2 - \omega_{34}(e_k)e_3,\\ \nabla_{e_k} e_4 =& \omega_{24}(e_k)e_1 + \omega_{14}(e_k)e_2 + \omega_{34}(e_k)e_4.\end{aligned} \tag{4.3}$$

Moreover, because of (4.2), we have $[e_1, e_3](H) = [e_1, e_4](H) = [e_3, e_4](H) = 0$ which give

$$\omega_{13}(e_1) = \omega_{14}(e_1) = 0, \qquad \omega_{14}(e_3) = \omega_{13}(e_4). \tag{4.4}$$

We apply the Codazzi equation $\left(\widetilde{R}(e_i, e_j)e_k\right)^{\perp} = 0$ for each triplet (i, j, k) in the set $\{(3,1,2)$, $(3,2,1)$, $(4,1,2)$, $(4,2,1)$, $(1,4,3)$, $(1,3,4)$, $(3,2,3)$, $(4,3,4)\}$ and combine equations obtained with (4.4) and (4.3) to get

$$\begin{aligned}\omega_{23}(e_1) = \omega_{13}(e_2) = \omega_{24}(e_1) = \omega_{14}(e_2) =& 0,\\ \omega_{34}(e_3) = \omega_{13}(e_4) = \omega_{13}(e_3) = \omega_{23}(e_3) =& 0.\end{aligned}$$

Therefore, from (4.3) we have

$$\begin{aligned}\omega_{13} = 0, \qquad &\nabla_{e_4} e_1 = -\omega_{12}(e_4)e_1 - \omega_{14}(e_4)e_3, \quad \nabla_{e_2} e_1 = -\omega_{12}(e_2)e_1,\\ &\nabla_{e_2} e_3 = \omega_{23}(e_2)e_1 - \omega_{34}(e_2)e_3.\end{aligned} \tag{4.5}$$

However, the Gauss equation $\left(\tilde{R}(e_2, e_4)e_1\right)^T = 0$ implies $H \equiv 0$ on M which yields a contradiction. □

Similarly, we have

Proposition 4.5. *There exists no hypersurfaces of index 2 in $\mathbb{E}^5_2$ with shape operator given by Case IV of Lemma 4.3.*

Proof. Assume that the shape operator of M is as given in Case III of Lemma 4.3. Then, the second fundamental form of M satisfies

$$h(e_1, e_2) = 2HN, \quad h(e_2, e_3) = -N, \qquad h(e_3, e_3) = -2HN, \quad h(e_4, e_4) = -10HN$$

and for all other cases, we have $h(e_i, e_j) = 0$. Similar to proof of Proposition 4.4 we have (4.4).

Note that we have

$$\begin{aligned}\nabla_{e_k} e_1 =& -\omega_{12}(e_k)e_1 + \omega_{13}(e_k)e_3 - \omega_{14}(e_k)e_4,\\ \nabla_{e_k} e_2 =& \omega_{12}(e_k)e_2 + \omega_{23}(e_k)e_3 - \omega_{24}(e_k)e_4,\\ \nabla_{e_k} e_3 =& \omega_{23}(e_k)e_1 + \omega_{13}(e_k)e_2 - \omega_{34}(e_k)e_4,\\ \nabla_{e_k} e_4 =& \omega_{24}(e_k)e_1 + \omega_{14}(e_k)e_2 - \omega_{34}(e_k)e_3.\end{aligned} \tag{4.6}$$

We apply the Codazzi equation $\left(\widetilde{R}(e_i, e_j)e_k\right)^{\perp} = 0$ for each triplet (i, j, k) in the set $\{(3,4,3)$, $(4,1,4)$, $(4,3,4)$, $(1,2,3)$, $(1,3,2)$, $(1,2,4)$, $(1,4,2)$, $(1,3,4)$, $(1,4,3)\}$ and combine equations obtained with (4.4) and (4.6) to get

$$\omega_{12}(e_1) = \omega_{13}(e_3) = \omega_{34}(e_1) = \omega_{13}(e_4) = \omega_{34}(e_3) = 0,$$
$$\omega_{14}(e_4) = \omega_{34}(e_4) = \omega_{14}(e_2) = \omega_{24}(e_1) = 0.$$

By combining these equations with (4.6), we obtain

$$\begin{aligned}
\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_1 = \omega_{13}(e_2)e_3 - \omega_{12}(e_2)e_1, &\quad \nabla_{e_i} e_1 = -\omega_{12}(e_i)e_1,\\
\nabla_{e_1} e_2 = \omega_{23}(e_1)e_3, &\quad \nabla_{e_j} e_2 = \omega_{12}(e_j)e_2 + \omega_{23}(e_j)e_3 - \omega_{24}(e_j)e_4,\\
\nabla_{e_k} e_3 = \omega_{23}(e_k)e_1, &\quad \nabla_{e_2} e_3 = \omega_{23}(e_2)e_1 + \omega_{13}(e_2)e_2 - \omega_{34}(e_2)e_4,\\
\nabla_{e_1} e_4 = 0, \nabla_{e_2} e_4 = \omega_{24}(e_2)e_1 - \omega_{34}(e_2)e_3, &\quad \nabla_{e_i} e_4 = \omega_{24}(e_i)e_1.
\end{aligned}$$

for $i = 3, 4$, $j = 2, 3, 4$ and $k = 1, 3, 4$.

However, the Gauss equations $R(e_3, e_4, e_4, e_3) = 20H^2$ implies $H = 0$ on M which yields a contradiction. □

Thus, by combining Lemma 4.1, Lemma 4.3, Proposition 4.4 and Proposition 4.5, we obtain the following result.

Theorem 4.6. *Let M be a hypersurface of index 2 in $\mathbb{E}^5_2$ with H as its (first) mean curvature. If M is biconservative and ∇H is a lightlike vector, then with respect to a suitable frame field $\{e_1, e_2, e_3, e_4\}$, its shape operator S has one of* *the following six forms, where e_1 is proportional to ∇H*

$$
\begin{aligned}
&\textit{Case I. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case II. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & 1 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case III. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & k_2 & -\nu & 0 \\ 0 & \nu & k_2 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix}, \\
&\textit{Case IV. } S = \begin{pmatrix} -2H & 0 & 0 & 0 \\ 0 & 2H & 0 & 0 \\ 0 & 0 & 2H & -1 \\ 0 & 1 & 0 & 2H \end{pmatrix}, \\
&\textit{Case V. } S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & 8H - k_3 \end{pmatrix}, \\
&\textit{Case VI. } S = \begin{pmatrix} -2H & 1 & 0 & 0 \\ 0 & -2H & 0 & 0 \\ 0 & 0 & 4H & \beta_1 \\ 0 & 0 & -\beta_1 & 4H \end{pmatrix}
\end{aligned}
\tag{4.7}
$$

for some smooth functions k_2, k_3, k_4, ν. In Cases I and III, the induced metric $g_{ij} = g(e_i, e_j) = \langle e_i, e_j \rangle$ of M is given by $g_{ij} = \varepsilon_i \delta_{ij} \in \{-1, 1\}$, in Cases II and IV, it is given by

$$
g = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_1 \end{pmatrix}
$$

for $\varepsilon = \pm 1$, whereas in Cases V and VI, it takes the form

$$
g = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
$$

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Section 3: **RELATED TOPICS**

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On Some Geometric Structures on the Cotangent Bundle of a Manifold

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Abstract. Let (M, ∇) be a manifold with a symmetric linear connection. The natural Riemann extension $\overline{g}$ (defined by Kowalski-Sekizawa) generalizes the Riemann extension (introduced by Patterson-Walker). The harmonic morphisms form a special class of harmonic maps, with many applications [1]. On $(T^*M, \overline{g})$ we obtain here a para-Hermitian structure, we construct a harmonic morphism and we generalize a result of [2].
Keywords. Cotangent bundle · Harmonicity · Riemann extension.
MSC 2010 Classification. Primary: 53C15; Secondary:53C43.

Dedicated to the memory of Professor Ioan Gottlieb and to his wife, Professor Cleopatra Mociutchi

1 Introduction

The affine geometry of a manifold (M, ∇), endowed with a symmetric linear connection, induces a (semi-)Riemannian geometry on the total space of the cotangent bundle T^*M, given by the Riemann extension $\overline{g}$ introduced by Patterson-Walker [8].

Osserman problem, Walker manifolds, almost para-Hermitian manifolds, non-Lorentzian geometry and so on, are related to the Riemann extension. For some other applications, see [5].

The Riemann extension is a metric of signature (n, n) on T^*M which was generalized by Kowalski and Sekizawa to the natural Riemann extension (see [7], [9] and for the notion of naturality see [6]). Another generalization is the deformed Riemann extension (see [4]).

In [2], the harmonic functions were characterized with respect to both natural Riemann extension and (classical) Riemann extension on the phase space T^*M.

A special class of harmonic maps is given by harmonic morphisms, see [1]. A harmonic morphism between (semi-)Riemannian manifolds is defined as a smooth map between (semi-)Riemannian manifolds which pulls back (local) harmonic functions from the target manifold to (local) harmonic functions on the domain manifold.

The present paper gives some applications of the deformed and natural Riemann extensions on T^*M. Our main goal is to provide some harmonic morphisms with respect to natural Riemann extension on T^*M.

Note: This paper is an announcement of the forthcoming paper [3]. The geometric structures, induced from the base manifold M to the total space of its cotangent bundle T^*M, can also be seen as some extensions of several geometric objects from a submanifold to the whole space.

2 Preliminaries

The technique of lifting several geometric objects from a base n-dimensional manifold M to its cotangent bundle T^*M goes back to the last century, for which we cite [10]. The natural projection $p : T^*M \to M$, $p(x, w) = x$ associates to each local chart $(U, x^1, \dots, x^n)$ around $x \in M$ the corresponding local chart $(p^{-1}(\mathcal{U}), x^1, \dots, x^n, x^{1*}, \dots, x^{n*})$ around $(x, w) \in T^*M$. On the cotangent space $(T^*M)_{(x,w)}$ at (x, w) of T^*M one has a canonical basis:

$$\{(\partial_1)_{(x,w)}, \dots, (\partial_n)_{(x,w)}, (\partial_{1*})_{(x,w)}, \dots, (\partial_{n*})_{(x,w)}\},$$

where $\partial_i = \partial/\partial x^i$ and $\partial_{i*} = \partial/\partial w_i$, $i = \overline{1, n}$. In local coordinates, the global defined vertical vector field:

$$W = \sum_{i=1}^{n} w_i \partial_{i*}$$

is of Liouville type.

The vertical lift $f^v \in \mathcal{F}(T^*M)$ of any function $f \in \mathcal{F}(M)$ is defined by $f^v = f \circ p$. Then the vertical lift X^v is a function on T^*M associated to the vector field $X \in \mathcal{X}(M)$ and defined by:

$$X^v(x, w) = w(X_x)$$

at any point $(x, w) \in T^*M$. When X is written in local coordinates as $X = \sum_{i=1}^n \xi^i \partial_i$ then X^v can be written in local coordinates as $X^v(x, w) = \sum_{i=1}^n w_i \xi^i(x)$ at any point $(x, w) \in T^*M$.

We note that X^v is not a vector field on T^*M but X^v is a function preserved by the action of the canonical vector field W, that is:

$$W(X^v) = X^v, \quad \forall X \in \mathcal{X}(M). \tag{2.1}$$

Proposition 2.1. *([11]) If X and Y are vector fields on T^*M such that $X(Z^v) = Y(Z^v)$, $\forall Z \in \mathcal{X}(M)$ then $X = Y$.*

We recall from [10] that the vertical lift α^v is a vector field tangent to T^*M, associated to any 1-form $\alpha \in \Omega^1(M)$ and defined by $\alpha^v(Z^v) = (\alpha(Z))^v$, $\forall Z \in \mathcal{X}(M)$. When α is written in local coordinates as $\alpha = \sum_{i=1}^n \alpha_i dx_i$ then α^v can be written in local coordinates as $\alpha^v = \sum_{i=1}^n \alpha_i \partial_{i*}$ where we identify $f^v = f \circ p \in \mathcal{F}(T^*M)$ for any $f \in \mathcal{F}(M)$. Hence $\alpha^v(f^v) = 0$, $\forall f \in \mathcal{F}(M)$. The complete lift of a vector field $X \in \mathcal{X}(M)$ is a vector field $X^c \in \mathcal{X}(T^*M)$ defined by:

$$X^c(Z^v) = [X, Z]^v, \ \ \forall Z \in \mathcal{X}(M).$$

When X is written in local coordinates by $X = \sum_{i=1}^n \xi^i \partial_i$ then X^c can be written in local coordinates as:

$$X^c_{(x,w)} = \sum_{i=1}^n \xi^i(x)(\partial_i)_{(x,w)} - \sum_{h,i=1}^n w_h(\partial_i \xi^h)(x)(\partial_{i*})_{(x,w)},$$

at each point $(x, w) \in T^*M$.

Then $X^c(f^v) = (Xf)^v$, $\forall f \in \mathcal{F}(M)$ and on T^*M the Lie bracket satisfies:

$$[X^c, Y^c] = [X, Y]^c, \ \ [X^c, \alpha^v] = (\mathcal{L}_X \alpha)^v,$$
$$[\alpha^v, \beta^v] = 0 = [X^c, W], [\alpha^v, W] = \alpha^v, \forall X, Y \in \mathcal{X}(M), \alpha, \beta \in \Omega^1(M),$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

3 Deformed Riemann extension

If (M, ∇) is an n-dimensional manifold endowed with a symmetric linear connection, then the deformed Riemann extension is a semi-Riemannian metric $\overline{g}$ of signature (n, n) on the total space of T^*M defined at any $(x, w) \in T^*M$ by

$$\overline{g}_{(x,w)}(X^c, Y^c) = -aw(\nabla_X Y + \nabla_Y X) + \Phi(X, Y) \tag{3.1}$$
$$\overline{g}(X^c, \alpha^v) = a\alpha(X), \ \overline{g}(\alpha^v, \beta^v) = 0, \tag{3.2}$$

for any vector fields X, Y and any differential 1-forms α, β on M, where $a \in \mathbb{R}^*$ and the real function $\Phi(X, Y)$ on T^*M is symmetric in X and Y. We assume $a > 0$.

Remark that the deformed Riemann extension generalizes both the Riemann extension introduced by Patterson, Walker in [8] (when $\Phi = 0$) and also the natural Riemann extension (see [7] and [9]) when $\Phi(X, Y) = bw(X)w(Y)$, $\forall X, Y \in \mathcal{X}(M)$, where $b \in \mathbb{R}$.

Proposition 3.1. *Let (M, ∇) be a manifold endowed with a symmetric linear connection. Then the total space of its cotangent bundle carries a para-Hermitian structure $(G_a, \bar{P})$ where G_a is the deformed Riemann extension with $\Phi_a(X, Y) = a(\nabla_X Y + \nabla_Y X)^v$ and $\bar{P}$ is defined by:*

$$\bar{P}X^c = X^c, \bar{P}\alpha^v = -\alpha^v, \ \forall X \in \mathcal{X}(M), \alpha \in \Omega^1(M). \tag{3.3}$$

Proof. One can see that $\bar{P}^2 = I$ and $\bar{P} \neq \pm I$ where I is the identity. We note that:

$$G_a(X^c, Y^c) = G_a(\alpha^v, \beta^v) = 0, \ G_a(X^c, \alpha^v) = a\alpha(X),$$
$$\forall X, Y \in \mathcal{X}(M), \alpha, \beta \in \Omega^1(M). \tag{3.4}$$

Moreover, $\bar{P}$ is skew-symmetric with respect to G_a:

$$G_a(\bar{P}U, \bar{P}V) = -G_a(U, V), \ \ \forall U, V \in \mathcal{X}(T^*M). \tag{3.5}$$

Hence $(G_a, \bar{P})$ is an almost para-Hermitian structure. Since the Nijenhuis tensor field of $\bar{P}$ vanishes identically it follows that $\bar{P}$ is integrable and therefore $(G_a, \bar{P})$ is a para-Hermitian structure which complete the proof. □

By using a similar but longer computation, we generalize Theorem 5.1 obtained in [2]:

Theorem 3.2. *If $X^v, Z^v \in \mathcal{F}(T^*M)$ are the vertical lifts of the vector field $X, Z \in \mathcal{X}(M)$ then:*

$$((grad\, Z^v)X^v)_{(x,w)} = \frac{1}{a}\{(\nabla_X Z + \nabla_Z X)^v - \frac{1}{a}\Phi(Z, X)\}_{(x,w)}, \tag{3.6}$$

where we used the characterization for $grad\, Z^v$ given by:

$$\overline{g}(grad\, Z^v, U) = UZ^v, \ \forall U \in \mathcal{X}(T^*M).$$

4 Harmonic maps and morphisms

We recall that a map $\varphi : (N, h) \to (\widetilde{N}, \widetilde{h})$ between (semi-)Riemannian manifolds is a harmonic map if the Euler-Lagrange operator $\tau(\varphi)$ defined as the trace (with respect to h) of the second fundamental form $\nabla d\varphi$ of φ vanishes identically, that is

$$\tau(\varphi) = \text{trace}_h \nabla d\varphi = 0.$$

Definition 4.1. A map $\varphi : (N, h) \to (\widetilde{N}, \widetilde{h})$ between (semi-)Riemannian manifolds is:

(i) a harmonic morphism if for any harmonic function f defined (locally) on $\widetilde{N}$, its pull-back $f \circ \varphi$ is a (locally) harmonic function on N.

(ii) horizontally weakly conformal if there exists a function $\Lambda : N \to \mathbb{R}$ (called square dilatation) such that in any local coordinates $(y^{\widetilde{1}}, \ldots, y^{\widetilde{n}})$ on $\widetilde{N}$ one has:

$$h(\text{grad}\varphi^\alpha, \text{grad}\varphi^\beta) = \Lambda \widetilde{h}^{\alpha\beta}, \ \alpha, \beta = \overline{1, \widetilde{n}},$$

see [1].

Theorem 4.2. *Let (M,g) be a Riemannian manifold and let $(T^*M,\overline{g})$ be the total space of its cotangent bundle endowed with the natural Riemannian extension constructed with the Levi-Civita connection ∇ of g. For any map $\varphi=(\varphi^1,\dots,\varphi^k):(M,g)\to\mathbb{R}^k$, its associated map is defined at any point $(x,w)\in T^*M$ by:*

$$\widetilde{\varphi}(x,w)=((grad\,\varphi^1)_{(x,w)},\dots,(grad\varphi^k)_{(x,w)}). \tag{4.1}$$

Then $\widetilde{\varphi}$ is a harmonic morphism if and only if $\widetilde{\varphi}$ is an eigenmap of the vertical lift of the Laplacian, i.e.

$$(\Delta\varphi)^v=\frac{b(n+1)}{2a}\widetilde{\varphi}, \tag{4.2}$$

and $(grad\,\varphi^1)^c,\dots,(grad\,\varphi^k)^c$ are mutually orthogonal and of the name length.

Proof. Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{g}$. We take $\alpha\in\Omega^1(M)$ such that $\alpha_x=w$, (but the proof is independent of the choice of α that satisfies this condition). From the relation

$$(\mathrm{grad}\,Z^v)_{(x,w)}=\frac{1}{a}\{Z^c-2\overline{\nabla}_{\alpha^v}Z^c+\frac{b}{a}w(Z)\alpha^v\}_{(x,w)}$$

and (2.1) at any $(x,w)\in T^*M$, we obtain for any set of vector fields $\{Z_i\}_{i=\overline{1,k}}$:

$$\begin{aligned}\bar{g}\left(\mathrm{grad}\,Z_i^v,\mathrm{grad}\,Z_j^v\right)_{(x,w)}&=\frac{1}{a^2}\bar{g}(Z_i^c-2\bar{\nabla}_{\alpha^v}Z_i^c+cZ_i^v\alpha^v,Z_j^c-2\bar{\nabla}_{\alpha^v}Z_j^c\\&\quad+cZ_j^v\alpha^v)_{(x,w)}\\&=\frac{1}{a^2}\{\bar{g}\left(Z_i^c,Z_j^c\right)-2\bar{g}\left(Z_i^c,\bar{\nabla}_{\alpha^v}Z_j^c\right)\\&\quad+cZ_j^v\bar{g}\left(Z_i^c,\alpha^v\right)-2\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,Z_j^c\right)\\&\quad+4\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,\bar{\nabla}_{\alpha^v}Z_j^c\right)-2cZ_j^v\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,\alpha^v\right)\\&\quad+cZ_i^v\bar{g}\left(\alpha^v,Z_j^c\right)-2cZ_i^v\bar{g}\left(\alpha^v,\bar{\nabla}_{\alpha^v}Z_j^c\right)\\&\quad+c^2Z_i^vZ_j^v\bar{g}\left(\alpha^v,\alpha^v\right)\}_{(x,w)}\end{aligned} \tag{4.3}$$

Using local coordinates, we can easily check:

$$\bar{g}(\mathbf{W},\beta^v)=\bar{g}(\mathbf{W},\mathbf{W})=0,\quad\forall\beta\in\Omega^1(M). \tag{4.4}$$

By using the definition of natural Riemann extension, the definition relation of $\overline{\nabla}$ and (4.4), we express some of the terms involved in (4.3):

$$\begin{aligned}\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,\bar{\nabla}_{\alpha^v}Z_j^c\right)_{(x,w)}&=\bar{g}\left(\bar{\nabla}_{\alpha^v}Z_i^c,\alpha^v\right)_{(x,w)}=\bar{g}\left(\alpha^v,\bar{\nabla}_{\alpha^v}Z_j^c\right)_{(x,w)}\\&=\bar{g}\left(\alpha^v,\alpha^v\right)_{(x,w)}=0;\\\bar{g}\left(Z_i^c,Z_j^c\right)_{(x,w)}&=-aw\left(\nabla_{Z_i}Z_j+\nabla_{Z_j}Z_i\right)_x+bw\left(Z_i\right)_xw\left(Z_j\right)_x.\end{aligned}$$

Using the definition formula of $\overline{\nabla}$, we obtain

$$\bar{g}\left(Z_i^c, \bar{\nabla}_{\alpha^v} Z_j^c\right)_{(x,w)} = -a\alpha_x\left(\nabla_{Z_i} Z_j\right)_x + bw\left(Z_i\right)_x w\left(Z_j\right)_x$$

and $\bar{g}\left(Z_i^c, \alpha^v\right)_{(x,w)} = a\alpha_x\left(Z_i\right)_x$ where $i = \overline{1,k}$.

By substituting the previous relations (and also the above relations in which i and j replace each other) into (4.3), we obtain:

$$\begin{aligned}
&\bar{g}\left(\operatorname{grad} Z_i^v, \operatorname{grad} Z_j^v\right)_{(x,w)} \\
&= \frac{1}{a^2}\{-aw\left(\nabla_{Z_i} Z_j + \nabla_{Z_j} Z_i\right)_x + bw\left(Z_i\right)_x w\left(Z_j\right)_x \\
&+ 2aw\left(\nabla_{Z_i} Z_j\right)_x - 2bw\left(Z_i\right)_x w\left(Z_j\right)_x + 2aw\left(\nabla_{Z_j} Z_i\right)_x\} \\
&= \frac{1}{a^2}\left\{aw\left(\nabla_{Z_i} Z_j + \nabla_{Z_j} Z_i\right)_x - bw\left(Z_i\right)_x w\left(Z_j\right)_x\right\} \\
&= -\frac{1}{a^2}\bar{g}\left(Z_i^c, Z_j^c\right)_{(x,w)}, \quad i,j = \overline{1,k}.
\end{aligned} \tag{4.5}$$

Now, $Z_1^c, \ldots, Z_k^c$ are mutually orthogonal and of the same length on $(T^*M, \bar{g})$ if and only if there exists a real function $\Lambda : T^*M \to \mathbb{R}$ such that $\bar{g}\left(Z_i^c, Z_j^c\right) = \Lambda\delta_{ij}$, $i,j = \overline{1,k}$.

From (4.5), by taking $Z_i = \operatorname{grad}\varphi^i$, $i = \overline{1,k}$, we obtain that $\overline{\varphi}$ is horizontally weakly conformal if and only if $(\operatorname{grad}\varphi^1)^c, \ldots, (\operatorname{grad}\varphi^k)^c$ are mutually orthogonal and of the same length.

We recall from ([2], Corollary 4.2) that the vertical lift Y^v of a vector field $Y \in \mathcal{X}(M)$ is a harmonic function (with respect to a natural Riemann extension $\overline{g}$) on T^*M if and only if

$$(\operatorname{div} Y)^v = \frac{b(n+1)}{2a} Y^v.$$

Hence $\overline{\varphi}$ is a harmonic map with respect to $\overline{g}$ if and only if $\overline{\varphi}$ is an eigenmap of the vertical lift of the Laplacian

$$(\Delta\varphi)^v = ((\Delta\varphi^1)^v, \ldots, (\Delta\varphi^k)^v),$$

i.e. (4.2) is satisfied.

We complete the proof since any map between (semi-)Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal see [1]. □

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Quarter Symmetric Connections On Complex Weyl Manifolds

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Abstract. In this work, we study quarter symmetric linear connections on Kähler Weyl manifolds and almost contact Weyl manifolds.
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1 Introduction

A gauge invariant theory which unifies the gravity and the electromagnetic fields was first introduced by Weyl in 1918 [13]. For physical reasons, his theory was not accepted but it remained both as a part of physics and mathematics. Weyl manifold is a differentiable manifold with a torsion free connection which is non-metric.

In 1924, Friedmann and Schouten introduced a semi-symmetric linear connection in a differentiable manifold [4]. After that, in 1932, Hayden introduced the notion of metric connection with torsion in a Riemannian manifold [6]. Moreover, Yano studied semi-symmetric metric connection in a Riemannian manifold and obtained a result about conformally flatness [14].

The notion of semi symmetric connection was generalized to quarter symmetric connection by Golab in 1975 [5]. There are many papers about quarter symmetric connection not only in Riemannian manifolds but also in Hermitian, Kähler, Kenmotsu manifolds (see Mishra and Pandey [9], Dwivedi [3], Pusic [12], Yano and Imai [15]).

In this work, we consider a quarter symmetric connection on Kähler Weyl manifolds and almost contact Weyl manifolds and examine the properties of this connection.

2 Preliminaries

A Weyl manifold is a differentiable manifold M of dimension n with a conformal metric tensor g and a symmetric connection D which satisfies, called the compatibility condition,

$$D_k g_{ij} = 2\omega_k g_{ij}, \tag{2.1}$$

where ω is a 1-form. Such a Weyl manifold is denoted by $M_n(g,\omega)$. If ω is a closed form, $M_n(g,\omega)$ is conformal to a Riemannian manifold.

Under the conformal change of the metric tensor g,

$$\breve{g}_{ij} = \lambda^2 g_{ij}, \quad \lambda > 0, \tag{2.2}$$

the 1-form ω changes by the law

$$\breve{\omega}_k = \omega_k + D_k \ln \lambda. \tag{2.3}$$

A quantity S is called a satellite of g with weight r if it admits a transformation of the form

$$\breve{S} = \lambda^r S, \tag{2.4}$$

under the change (2.2) of the metric tensor g.

The prolonged (extended) covariant derivative of a satellite S of weight r is defined by

$$\dot{D}_k S = D_k S - r w_k S, \tag{2.5}$$

from which it follows that $\dot{D}_k g_{ij} = 0$ (see [16], [7], [10]).

It is easy to see from (2.1) that

$$\Gamma^i_{kl} = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} - g^{im}(g_{mk}\omega_l + g_{ml}\omega_k - g_{kl}\omega_m), \tag{2.6}$$

where Γ^i_{jk} are the coefficients of the Weyl connection D and $\left\{ \begin{smallmatrix} i \\ kl \end{smallmatrix} \right\}$ are the connection coefficients of the Levi-Civita connection.

The mixed curvature tensor, the covariant curvature tensor, the Ricci tensor and the scalar curvature for $M_n(g,\omega)$ are respectively given by [11]:

$$v^j W^p_{jkl} = (D_k D_l - D_l D_k) v^p, \tag{2.7}$$

$$W_{hjkl} = g_{hp} W^p_{jkl}, \tag{2.8}$$

$$W_{ij} = W^p_{ijp} = g^{hk} W_{hijk}, \tag{2.9}$$

$$s = g^{ij} W_{ij}. \tag{2.10}$$

By considering (2.7), the explicit form of the mixed curvature tensor W^p_{jkl} for $M_n(g,\omega)$ is

$$W^p_{jkl} = \partial_k \Gamma^p_{jl} - \partial_l \Gamma^p_{jk} + \Gamma^p_{hk}\Gamma^h_{jl} - \Gamma^p_{hl}\Gamma^h_{jk}. \tag{2.11}$$

The mixed curvature tensor, the covariant curvature tensor and the Ricci tensor of $M_n(g,\omega)$ satisfy the following properties [11]:

$$W_{ijkl} + W_{ijlk} = 0, \quad W^i_{ikl} = n(D_l\omega_k - D_k\omega_l) = 2nD_{[l}\omega_{k]}, \tag{2.12}$$

$$W_{ijkl} + W_{jikl} = 4g_{ij} D_{[l}\omega_{k]}, \quad W_{[ij]} = nD_{[i}\omega_{j]}. \tag{2.13}$$

Let N be an n-dimensional Riemannian manifold endowed with a linear connection ∇. Then ∇ is said to be quarter symmetric if the torsion tensor T^i_{jk} of ∇ satisfies

$$T^i_{jk} = p_j A^i_k - p_k A^i_j, \tag{2.14}$$

where p_k is a 1-form and A^j_i is any $(1,1)$ tensor field. If A_{jk} is defined as $A^i_j g_{jk} = A_{jk}$, then

$$A_{jk} = U_{jk} + V_{jk}, \tag{2.15}$$

where U_{jk} and V_{jk} are respectively symmetric and anti symmetric part of A_{jk} [15].

3 Kähler Weyl Manifolds

A Weyl manifold of dimension $2n$ is called Kähler Weyl if

$$F^i_j F^h_i = -\delta^h_j, \tag{3.1}$$

$$F^t_j F^s_i g_{ts} = g_{ji}, \tag{3.2}$$

and

$$\dot{D}_j F^k_i = 0, \quad \forall i, j, k, \tag{3.3}$$

where F^j_i is a $(1,1)$ tensor field of weight zero and called an almost complex structure, g_{ij} is Hermitian metric. Such a manifold will be denoted by $KM_{2n}(g, \omega)$ [2].

The $(0,2)$ tensor field F_{ij} of weight 2 and $(2,0)$ tensor field F^{ij} of weight -2 are, respectively, given by

$$F_{ij} = F^k_i g_{kj} = -F_{ji} \tag{3.4}$$

and

$$F^{ij} = F^j_k g^{ik} = -F^{ji}. \tag{3.5}$$

The contraction of (3.4) on the indices k and i gives $F^i_i = 0$.

Suppose that a Kähler Weyl manifold admits a quarter symmetric linear connection $\bar{D}$ with the torsion tensor $\bar{T}$ and satisfies the following compatibility condition

$$\bar{D}_k g_{ij} = 2\omega_k g_{ij}. \tag{3.6}$$

If we take $U_{jk} = g_{jk}$ and $V_{jk} = F_{jk}$ in (2.15), then we find that $A^k_i = \delta^k_i + F^k_i$. Therefore, the torsion tensor $\bar{T}$ takes the form

$$\bar{T}^i_{jk} = p_j \left(\delta^i_k + F^i_k\right) - p_k \left(\delta^i_j + F^i_j\right). \tag{3.7}$$

We note that the 1-form p_k is of zero weight.

Theorem 3.1. *On every Kähler Weyl manifold there exists a unique quarter symmetric linear connection associated to every 1-form p and $(1,1)$ tensor field F.*

Proof. Assume that the relation between $\bar{\Gamma}^j_{mk}$ and Γ^j_{mk} is given by

$$\bar{\Gamma}^j_{mk} = \Gamma^j_{mk} + U^j_{mk}, \tag{3.8}$$

where $\bar{\Gamma}^j_{mk}$ and Γ^j_{mk} are respectively the coefficients of $\bar{D}$ and D, and U^j_{mk} is any $(1, 2)$ tensor field. Then from (2.1), (3.6) and (3.8), we find that

$$U^h_{ik} g_{hj} + U^h_{jk} g_{hi} = 0. \tag{3.9}$$

After permuting the indices i, j and k in the above equation cyclicly and using some algebraic operations, we obtain

$$\begin{aligned}(U^h_{ik} + U^h_{ki}) g_{hj} &= (U^h_{ij} - U^h_{ji}) g_{hk} + (U^h_{kj} - U^h_{jk}) g_{hi} \\ &= \bar{T}^h_{ij} g_{hk} + \bar{T}^h_{kj} g_{hi}.\end{aligned} \tag{3.10}$$

From (3.7) and after some simplifications, we have

$$U^t_{ik} + U^t_{ki} = p_i \left(\delta^t_k - F^t_k\right) + p_k \left(\delta^t_i - F^t_i\right) - 2p^t g_{ik}, \tag{3.11}$$

where $p^t = p_k g^{tk}$. Since $\bar{T}^t_{ik} = U^t_{ik} - U^t_{ki}$, by considering the above equation, we obtain

$$U^t_{ik} = p_i \delta^t_k - p_k F^t_i - p^t g_{ik}. \tag{3.12}$$

Hence, we find that $\bar{\Gamma}^j_{mk} = \Gamma^j_{mk} + p_m \delta^j_k - p_k F^j_m - p^j g_{mk}$ which completes the proof. □

The mixed curvature tensor $\bar{W}^i_{jkl}$ for $\bar{D}$ is given by

$$\bar{W}^i_{jkl} = \partial_k \bar{\Gamma}^i_{jl} - \partial_l \bar{\Gamma}^i_{jk} + \bar{\Gamma}^m_{jl} \bar{\Gamma}^i_{mk} - \bar{\Gamma}^m_{jk} \bar{\Gamma}^i_{mk}. \tag{3.13}$$

Hence, by considering the definition of $\bar{\Gamma}^i_{jl}$ and after a long straightforward calculations, we obtain

$$\begin{aligned}\bar{W}^i_{jkl} &= W^i_{jkl} + \delta^i_l \alpha_{jk} - \delta^i_k \alpha_{jl} + g_{jk} g^{it} \alpha_{tl} - g_{jl} g^{it} \alpha_{tk} - 2F^i_j \dot{D}_{[k} p_{l]} \\ &+ p^i \left(F_{jk} p_l - F_{jl} p_k\right) + p_j \left(F^i_k p_l - F^i_l p_k\right),\end{aligned} \tag{3.14}$$

where $\alpha_{jk} = \dot{D}_k p_j - p_j p_k + F^m_j p_m p_k + \frac{1}{2} g_{jk} p^m p_m$.

Therefore, the covariant curvature tensor $\bar{W}_{ijkl}$, the Ricci tensor $\bar{W}_{jk}$ and the scalar curvature $\bar{s}$ are respectively given by

$$\begin{aligned}\bar{W}_{ijkl} &= W_{ijkl} + g_{il} \alpha_{jk} - g_{ik} \alpha_{jl} + g_{jk} \alpha_{il} - g_{jl} \alpha_{ik} + 2F_{ij} \dot{D}_{[k} p_{l]} \\ &+ p_i \left(F_{jk} p_l - F_{jl} p_k\right) + p_j \left(F_{il} p_k - F_{ik} p_l\right),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\bar{W}_{jk} &= W_{jk} + (n-2)\alpha_{jk} + g_{jk} g^{il} \alpha_{il} - g_{jl} \alpha_{ik} + 2g^{il} F_{ij} \dot{D}_{[k} p_{l]} \\ &+ F_{jk} p_l p^l - F_{jl} p^l p_k - F_{lk} p^l p_j,\end{aligned} \tag{3.16}$$

and

$$\bar{s} = s + 2(n-1) g^{jk} \alpha_{jk} + 2F^{lk} \dot{D}_{[k} p_{l]}. \tag{3.17}$$

Proposition 3.2. *The mixed curvature tensor* $\bar{W}^i_{ikl}$ *and the covariant curvature tensor* $\bar{W}_{ijkl}$ *of a Kähler Weyl manifold with a quarter symmetric linear connection satisfy the following relations:*

(i) $\bar{W}_{ijkl} + \bar{W}_{ijlk} = 0$

(ii) $\bar{W}_{ijkl} + \bar{W}_{jikl} = 4g_{ij}D_{[l}\omega_{k]}$

(iii) $\bar{W}^i_{ikl} = W^i_{ikl} = 2nD_{[l}\omega_{k]}$.

Proof. (i) The covariant curvature tensor $\bar{W}_{ijkl}$ of a Kähler Weyl manifold endowed with a quarter symmetric linear connection $\bar{D}$ is given by (3.15). By changing the indices k and l and then taking the sum of the equations obtained gives $\bar{W}_{ijkl} + \bar{W}_{ijlk} = W_{ijkl} + W_{ijlk} = 0$.

(ii) Similarly, if we change the indices k and l in (3.15) and sum up the obtained equations, then we get $\bar{W}_{ijkl} + \bar{W}_{jikl} = W_{ijkl} + W_{jikl} = 4g_{ij}D_{[l}\omega_{k]}$.

(iii) Since $F^i_i = 0$, the result follows easily from (3.14). □

Theorem 3.3. *If the curvature tensor of a Kähler Weyl manifold with a quarter symmetric linear connection vanishes and the* 1*-form* p_k *is locally a gradient, then the connection reduces to the Weyl connection.*

Proof. If $\bar{W}_{ijkl} = 0$ and the 1-form p_k is locally a gradient, then (3.15) takes the form

$$\begin{aligned} W_{ijkl} &= -g_{il}\alpha_{jk} + g_{ik}\alpha_{jl} - g_{jk}\alpha_{il} + g_{jl}\alpha_{ik} \\ &- p_i\left(F_{jk}p_l - F_{jl}p_k\right) - p_j\left(F_{il}p_k - F_{ik}p_l\right). \end{aligned} \tag{3.18}$$

If we permute the indices j, k and l in (3.18) cyclicly, then we obtain two more equations. Now, by taking the sum of the three equations and taking into account of the 1^{st} Bianchi Identity for Weyl manifolds, we obtain

$$0 = g_{il}\alpha_{[jk]} + g_{ij}\alpha_{[kl]} + g_{ik}\alpha_{[lj]} + F_{jk}p_ip_l + F_{kl}p_ip_j + F_{lj}p_ip_k. \tag{3.19}$$

By contracting the above equation with $F^{jl}g^{ik}$, we get for $n \neq 2$

$$F^{jl}\alpha_{[jl]} = -p^kp_k. \tag{3.20}$$

Since $\alpha_{[jl]} = \frac{1}{2}\left(F_{jm}p^mp_l - F_{lm}p^mp_j\right)$,

$$\begin{aligned} -p^kp_k &= \frac{1}{2}F^{jl}\left(F_{jm}p^mp_l - F_{lm}p^mp_j\right) \\ &= \frac{1}{2}\left(\delta^l_mp^mp_l + \delta^j_mp^mp_j\right) \\ &= p^kp_k, \end{aligned} \tag{3.21}$$

from which we find that $p_k = 0$ for positive definite metric tensors belonging to the conformal class. Now since $p_k = 0$, $\bar{\Gamma}^j_{mk} = \Gamma^j_{mk} + p_m\delta^j_k - p_kF^j_m - p^jg_{mk}$ takes the form $\bar{\Gamma}^j_{mk} = \Gamma^j_{mk}$ which completes the proof. □

4 Almost Contact Weyl Manifolds

Let M_{2n+1} be a differentiable manifold of dimension $2n+1$. An almost contact structure (ϕ, ξ, η) on M_{2n+1} is a triple satisfying the following relations

$$\phi_i^j \phi_j^k = -\delta_i^k + \eta_i \xi^k, \tag{4.1}$$
$$\eta_i \xi^i = 1, \tag{4.2}$$
$$\phi_i^j \xi^i = 0, \tag{4.3}$$
$$\eta_i \phi_j^i = 0, \tag{4.4}$$

where ϕ_i^j is a tensor field of type $(1,1)$, ξ^i is a vector field and η_i is a 1-form. Moreover, if there is given a Riemannian metric g_{ij} such that

$$g_{ij}\phi_t^i \phi_s^j = g_{ts} - \eta_t \eta_s, \tag{4.5}$$
$$g_{ij}\xi^j = \eta_i, \tag{4.6}$$

then (ϕ, ξ, η, g) is called an almost contact metric structure on M. A differentiable manifold M_{2n+1} with almost contact metric structure (ϕ, ξ, η, g) is called almost contact metric manifold [1].

It is easy to see that the tensor ϕ_{ij}, which is defined by $\phi_i^k g_{jk} = \phi_{ij}$, is anti symmetric and contraction of ϕ_i^j gives $\phi_i^i = 0$.

It follows immediately from the equations (4.1), (4.2) and (4.5) that the $(1,1)$ tensor field ϕ_i^j , the 1-form η_i and the vector field ξ^i are weight of 0, 1 and -1, respectively.

Let $M_{2n+1}(g,\omega)$ be a Weyl manifold with the connection D. Then $M_{2n+1}(g,\omega)$ has an almost contact structure if the following conditions are satisfied in addition to the conditions (4.1)-(4.6) [8]:

$$\dot{D}_k g_{ij} = 0, \quad \dot{D}_k \phi_i^j = 0, \quad \dot{D}_k \eta_i = 0, \quad \dot{D}_k \xi^i = 0. \tag{4.7}$$

Such a manifold is called almost contact Weyl manifold and will be denoted by $ACM_{2n+1}(g,\omega)$.

Now, we consider the manifold $ACM_{2n+1}(g,\omega)$ with a quarter symmetric linear connection $\widetilde{D}$ and the torsion tensor $\widetilde{T}$ is of the form

$$\widetilde{T}_{jk}^i = q_j \phi_k^i - q_k \phi_j^i, \tag{4.8}$$

where the $(1,1)$ tensor field $A_i^j = \phi_i^j$ and the 1-form $q_j = f\eta_j$, where f is any function of weight -1. Here, we also have

$$\widetilde{D}_k g_{ij} = 2\omega_k g_{ij}. \tag{4.9}$$

Theorem 4.1. *On every almost contact Weyl manifold there exists a unique quarter symmetric linear connection associated to every 1-form q and $(1,1)$ tensor field ϕ.*

Proof. Suppose that the relation between $\widetilde{\Gamma}^j_{mk}$ and Γ^j_{mk} be

$$\widetilde{\Gamma}^j_{mk} = \Gamma^j_{mk} + U^j_{mk}, \tag{4.10}$$

where $\widetilde{\Gamma}^j_{mk}$ and Γ^j_{mk} are the coefficients of the connections $\widetilde{D}$ and D, respectively and U^j_{mk} is any tensor field of type $(1,2)$.

From (2.1), (4.9) and (4.10) we have

$$U^h_{ik}g_{hj} + U^h_{jk}g_{hi} = 0. \tag{4.11}$$

Permuting cyclicly the indices i, j, k and after some modifications, we obtain

$$\begin{aligned}(U^h_{ik} + U^h_{ki})g_{hj} &= (U^h_{ij} - U^h_{ji})g_{hk} + (U^h_{kj} - U^h_{jk})g_{hi} \\ &= \widetilde{T}^h_{ij}g_{hk} + \widetilde{T}^h_{kj}g_{hi}\end{aligned} \tag{4.12}$$

Using the definition of $\widetilde{T}^h_{ij}$ and after straightforward calculations, we find that

$$(U^h_{ik} + U^h_{ki})g_{hj} = -q_i\phi_{kj} - q_k\phi_{ij}. \tag{4.13}$$

Multiplying the last equation by g^{jt}, we have

$$U^t_{ik} + U^t_{ki} = -q_i\phi^t_k - q_k\phi^t_i. \tag{4.14}$$

We conclude from the above equation that

$$\widetilde{T}^h_{ij} = U^t_{ik} - U^t_{ki} = q_i\phi^t_k - q_k\phi^t_i. \tag{4.15}$$

Hence, we obtain

$$U^t_{ik} = -q_k\phi^t_i, \tag{4.16}$$

and therefore

$$\widetilde{\Gamma}^j_{mk} = \Gamma^j_{mk} - q_k\phi^j_m. \tag{4.17}$$

□

The mixed curvature tensor for an almost contact Weyl manifold endowed with a quarter symmetric linear connection $\widetilde{D}$ is of the form

$$\widetilde{W}^i_{jkl} = \partial_k\widetilde{\Gamma}^i_{jl} - \partial_l\widetilde{\Gamma}^i_{jk} + \widetilde{\Gamma}^m_{jl}\widetilde{\Gamma}^i_{mk} - \widetilde{\Gamma}^m_{jk}\widetilde{\Gamma}^i_{mk}. \tag{4.18}$$

By using (4.17) and (4.18) we have

$$\begin{aligned}\widetilde{W}^i_{jkl} &= \partial_k\left(\Gamma^i_{jl} - q_l\phi^i_j\right) - \partial_l\left(\Gamma^i_{jk} - q_k\phi^i_k\right) \\ &+ \left(\Gamma^m_{jl} - q_l\phi^m_j\right)\left(\Gamma^i_{mk} - q_k\phi^i_m\right) - \left(\Gamma^m_{jk} - q_k\phi^m_j\right)\left(\Gamma^i_{ml} - q_l\phi^i_m\right) \\ &= W^i_{jkl} - \partial_k\left(q_l\phi^i_j\right) + \partial_l\left(q_k\phi^i_j\right) - q_k\Gamma^m_{jl}\phi^i_m - q_l\Gamma^i_{mk}\phi^m_j \\ &+ q_lq_k\phi^m_j\phi^i_m + q_l\Gamma^m_{jk}\phi^i_m + q_k\Gamma^i_{ml}\phi^m_j - q_kq_l\phi^m_j\phi^i_m\end{aligned} \tag{4.19}$$

After some simplifications we obtain

$$\begin{aligned}\widetilde{W}^i_{jkl} &= W^i_{jkl} + \phi^i_j\left(\partial_l q_k - \partial_k q_l\right) + q_k\left(\partial_l \phi^i_j - \Gamma^m_{jl}\phi^i_m + \Gamma^i_{ml}\phi^m_j\right) \\ &- q_l\left(\partial_k \phi^i_j - \Gamma^m_{jk}\phi^i_m + \Gamma^i_{mk}\phi^m_j\right). \end{aligned} \tag{4.20}$$

If we use the definition of prolonged covariant derivative for ϕ^i_j and η_k, then we get

$$\widetilde{W}^i_{jkl} = W^i_{jkl} + \phi^i_j\left(\dot{D}_l q_k - \dot{D}_k q_l\right) + q_k \dot{D}_l \phi^i_j - q_l \dot{D}_k \phi^i_j. \tag{4.21}$$

Since $\dot{D}_l \phi^i_j = 0$, we find that

$$\widetilde{W}^i_{jkl} = W^i_{jkl} + \phi^i_j\left(\dot{D}_l q_k - \dot{D}_k q_l\right). \tag{4.22}$$

From (4.22), the covariant curvature tensor $\widetilde{W}_{ijkl}$ is given by

$$\widetilde{W}_{ijkl} = W_{ijkl} + 2\phi_{ji}\dot{D}_{[l}q_{k]}, \tag{4.23}$$

where $\dot{D}_{[l}q_{k]}$ is anti symmetric part of $\dot{D}_l q_k$.

Contracting the tensor $\widetilde{W}^i_{jkl}$ with respect to i and l and using the fact that $\eta_i\phi^i_j = 0$, gives us

$$\widetilde{W}_{jk} = W_{jk} + \phi^i_j \dot{D}_i q_k. \tag{4.24}$$

Theorem 4.2. *For an almost contact Weyl manifold with a quarter symmetric linear connection, we have*

$$\widetilde{s} = s, \tag{4.25}$$

where $\widetilde{s}$ and s are scalar curvature of the manifold with respect to the connections $\widetilde{D}$ and D, respectively.

Proof. Multiplying (4.24) by g^{jk} and using the identity $\xi^j\phi^i_j = 0$ gives

$$\begin{aligned}\widetilde{s} &= s + \dot{D}_i\left(q_k g^{jk}\phi^i_j\right) \\ &= s + \dot{D}_i\left(f\eta_k g^{jk}\phi^i_j\right) \\ &= s + \dot{D}_i\left(f\xi^j\phi^i_j\right) \\ &= s.\end{aligned}$$

□

Theorem 4.3. *On an almost contact Weyl manifold with a quarter symmetric linear connection, if the 1-form q is locally a gradient, then*

$$\begin{aligned}\widetilde{W}^i_{jkl} &= W^i_{jkl}, \\ \widetilde{W}_{jk} &= W_{jk}.\end{aligned}$$

Proof. The proof is immediate from (4.22). □

Proposition 4.4. *On an almost contact Weyl manifold with a quarter symmetric connection, the following relations hold:*

(i) $\widetilde{W}_{ijkl} + \widetilde{W}_{ijlk} = 0$

(ii) $\widetilde{W}_{ijkl} + \widetilde{W}_{jikl} = \frac{4}{n} g_{ij} W_{[lk]}$

(iii) $\widetilde{W}^i_{ikl} = 2nD_{[l}\omega_{k]}$

(iv) $\widetilde{W}_{klij} + \widetilde{W}_{kijl} + \widetilde{W}_{kjli} = 2\left(\phi_{kl}\dot{D}_{[i}q_{j]} + \phi_{ki}\dot{D}_{[j}q_{l]} + \phi_{kj}\dot{D}_{[l}q_{i]}\right)$

Proof. (i). Changing the indices k and l in (4.23) yields

$$\widetilde{W}_{ijlk} = W_{ijlk} - 2\phi_{ji}\dot{\nabla}_{[l}q_{k]}. \tag{4.26}$$

By adding (4.23) to (4.26) we obtain the result.

(ii). Similar to proof (i).

(iii). Contracting (4.22) with respect to i and j yields

$$\widetilde{W}^i_{ikl} = W^i_{ikl} + \phi^i_i\left(\dot{D}_l q_k - \dot{D}_k q_l\right). \tag{4.27}$$

Since $\phi^i_i = 0$, we find that

$$\widetilde{W}^i_{ikl} = W^i_{ikl} = 2nD_{[l}\omega_{k]}. \tag{4.28}$$

(iv). Using the 1^{st} Bianchi Identity

$$W_{klij} + W_{kijl} + W_{kjli} = 0, \tag{4.29}$$

and after the straightforward calculations, we get the result. □

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Hyper-Generalized Quasi Einstein Manifolds Satisfying Certain Ricci Conditions

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Abstract. In the present paper, we deal with hyper-generalized quasi Einstein manifold. First, we investigate geometric properties of this manifold with respect to its generators. Then, we study some classes of hyper-generalized quasi Einstein manifold satisfying certain Ricci conditions. Precisely, we obtain some necessary conditions for hyper-generalized quasi Einstein manifold to be a generalized quasi Einstein or a quasi Einstein manifold.
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1 Introduction

The notion of hyper-generalized quasi Einstein manifold has been first introduced by A. A. Shaikh, C. Özgür and A. Patra, in 2011 [18]. An n-dimensional Riemannian manifold (M^n, g), $(n > 2)$ is called a hyper-generalized quasi Einstein manifold if its Ricci tensor of type $(0,2)$ is non-zero and satisfies the following condition [2]

$$\begin{aligned} S(X,Y) =& ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \\ &+ d[A(X)D(Y) + A(Y)D(X)] \end{aligned} \tag{1.1}$$

for all $X, Y \in \chi(M)$, where a, b, c and d are real valued, non-zero scalar functions on (M^n, g), A, B and D are non-zero 1-forms such that

$$g(X, \rho_1) = A(X), \quad g(X, \rho_2) = B(X), \quad g(X, \rho_3) = D(X), \tag{1.2}$$

where ρ_1, ρ_2 and ρ_3 are three unit vector fields mutually orthogonal to each other at every point on M. The scalars a, b, c and d are called associated scalars of the manifold, A, B and D are called associated 1-forms and ρ_1, ρ_2 and ρ_3 are called the generators of the manifold. Throughout this work, such an n-dimensional manifold will be denoted by $(HGQE)_n$.

The name "hyper" is used as in the case of hyper-real numbers. Especially, if ρ_2 and ρ_3 are linearly dependent or if $d = 0$, then the notion of hyper-generalized quasi Einstein manifold turns into the notion of generalized quasi Einstein manifold introduced by M.C. Chaki in 2001, [2]. In [2, 6, 10], many authors studied this kind of manifolds. Recently, in [11], the authors obtained some properties of the generalized quasi Einstein manifolds satisfying some curvature conditions on the conformal, concircular, projective and the quasi-conformal curvature tensors. Furthermore, in [12], the authors investigated some geometric and physical properties of the generalized quasi Einstein manifolds with applications in general relativity. In addition to these studies, in [17] a non-trivial example of $(HGQE)_{2n+1}$ was given by Shaikh and Matsuyama which can be briefly summarized as follows:

Example 1.1. Let M^{2n+1} be an almost contact metric manifold [1] admitting an $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \phi^2 = -I + \eta \otimes \xi, \tag{1.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad \eta(X) = g(X, \xi), \qquad \eta(\xi) = 1, \tag{1.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.5}$$

for all vector fields $X, Y \in M^{2n+1}$ and such kind of manifold is denoted by $M^{2n+1}(\phi, \xi, \eta, g)$.

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a trans-Sasakian manifold [13] if the following condition holds:

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{1.6}$$

where α, β are smooth functions on M and such a structure is called *trans-Sasakian structure of type* (α, β). In [17], it was shown that in a conformally flat trans-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ of type (α, β), the Ricci tensor is of the form:

$$\begin{aligned} S(X, Y) =& ag(X, Y) + b\eta(X)\eta(Y) + c[\eta(X)\omega(Y) + \eta(Y)\omega(X)] \\ &+ d[\eta(X)\pi(Y) + \eta(Y)\pi(X)], \end{aligned} \tag{1.7}$$

where a, b, c and d are non-zero scalars given by

$$a = \frac{r}{2n} - (\alpha^2 - \beta^2), \quad b = -\frac{r}{2n} + (2n+1)(\alpha^2 - \beta^2), \quad c = 1, \quad d = -(2n-1) \tag{1.8}$$

and η, ω and π are non-zero 1-forms such that $\eta(X) = g(X, \xi)$ so,

$$\omega(X) = -((\phi X)\alpha) = g(X, \phi(grad\alpha)), \quad \pi(X) = (X\beta) = g(X, grad\beta) \tag{1.9}$$

for all X. Here, ξ is always orthogonal to $grad\beta$ and $\phi(grad\alpha)$ so we may only consider $grad\beta$ and $\phi(grad\alpha)$ are orthogonal to each other. Then, such a trans-Sasakian manifold is a $(HGQE)_{2n+1}$, which is neither $(QE)_{2n+1}$ nor $G(QE)_{2n+1}$, [17].

Let $\{e_i: i = 1, 2, \cdots, n\}$ be an orthonormal frame field at any point of the manifold. Then, setting $X = Y = e_i$ in (1.1) and taking summation over i; $(1 \leq i \leq n)$, we obtain

$$r = an + b, \tag{1.10}$$

where r is the curvature of the manifold. In view of the equations (1.1) and (1.2), in a hyper-generalized quasi Einstein manifold, we have

$$S(X, \rho_1) = (a + b)A(X) + cB(X) + dD(X), \tag{1.11}$$

$$S(X, \rho_2) = aB(X) + cA(X), \quad S(X, \rho_3) = aD(X) + dA(X), \tag{1.12}$$

$$S(\rho_1, \rho_1) = (a + b), \quad S(\rho_2, \rho_2) = S(\rho_3, \rho_3) = a, \quad S(\rho_1, \rho_2) = c, \quad S(\rho_1, \rho_3) = d. \tag{1.13}$$

If $d = c = 0$ in the fundamental equation (1.1) of $(HGQE)_n$, then the manifold reduces to a quasi Einstein manifold. Quasi Einstein manifolds have been studied by several authors, such as M.C. Chaki [3], U.C. De and G. C. Ghosh [5] and S. Guha [10]. Also, in [14]; A.A. Shaikh, Y.H. Kim and S.K. Hui and in [4]; A. De, C. Özgür and U.C. De studied on Lorentzian quasi Einstein spacetimes.

Similarly, if $d = c = b = 0$ in (1.1), then the manifold reduces to an Einstein manifold which is characterized by the proportionality of the Ricci tensor to the metric tensor.

Let R denote the Riemannian curvature tensor of M. The k-nullity distribution $N(k)$ [19] of a Riemannian manifold M is defined by the set of all vector fields $Z \in T_p(M)$ satisfying the condition $R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]$, for all $X, Y \in T_p(M)$ where k is some smooth function on M. In a quasi Einstein manifold M, if the generator U belongs to some k-nullity distribution, then M said to be an $N(k)$-quasi Einstein manifold [20]. C. Özgür and M. M. Tripathi [15] proved that in an n-dimensional $N(k)$-quasi Einstein manifold, $k = \frac{a+b}{n-1}$.

The importance of these manifolds is in fact due to the existence of certain spacetimes endowed with semi-Riemannian metrics. In general relativity and cosmology, the purpose of studying various types of semi-Riemannian manifolds is to represent the different phases in the evolution of the universe. Quasi Einstein spacetimes arose during the study of exact solutions of Einstein's field equations. For instance, the Robertson-Walker spacetimes are quasi Einstein spacetimes. While $(QE)_4$ can be taken as a model of perfect fluid spacetime, the importance of $G(QE)_4$ lies in the fact that such a 4-dimensional semi-Riemannian manifold is related to the study of general relativistic fluid spacetime admitting heat flux [16, 5]. Thus, the investigations on these manifolds with Riemannian or semi-Riemannian metric are very important in differential geometry as well as in general relativity and cosmology.

In this direction, this paper is organized as follows: First, in Section 2 we investigate geometric properties of $(HGQE)_n$ with respect to its generators. Then, some pseudo-symmetry types of such manifolds are considered and some necessary conditions for hyper-generalized quasi Einstein manifold to be a generalized quasi Einstein or a quasi Einstein manifold are obtained.

2 Some Geometric Properties of $(HGQE)_n$

In this section, we investigate some geometric properties of $(HGQE)_n$. First, we consider a $(HGQE)_n$ whose generator ρ_1 is a parallel vector field, that is

$$\nabla_X \rho_1 = 0, \tag{2.1}$$

where ∇ is the Levi-Civita connection. Then for all X and Y clearly we have, $(\nabla_X A)(Y) = g(\nabla_X \rho_1, Y) = 0$ and so $R(X,Y)\rho_1 = 0$. Contracting the last equation we get, $S(X, \rho_1) = 0$. Combining the last equation with (1.11), we obtain

$$(a+b)A(X) + cB(X) + dD(X) = 0. \tag{2.2}$$

Putting $X = \rho_1$ in (2.2), we get $a + b = 0$, putting $X = \rho_2$ in (2.2), we get $c = 0$ and putting $X = \rho_3$ in (2.2), we get $d = 0$ which means that such a manifold reduces to a quasi Einstein manifold and the sum of its associated scalar functions is zero.

On the other hand, if we assume that the generator ρ_2 is parallel vector field, then similarly we have $\nabla_X \rho_2 = 0$ and so $S(X, \rho_2) = 0$. Thus, in view of (1.11), we get

$$aB(X) + cA(X) = 0. \tag{2.3}$$

Putting $X = \rho_2$ in (2.3), we have $a = 0$ which is a contradiction. In a similar manner, if we assume that the generator ρ_3 is parallel vector field, then we have $\nabla_X \rho_3 = 0$ and so $S(X, \rho_3) = 0$ so, in view of (1.12), we get

$$aD(X) + dA(X) = 0. \tag{2.4}$$

Putting $X = \rho_3$ in (2.3), again we have $a = 0$. Hence we can state that:

Theorem 2.1. *In a $(HGQE)_n$, if the generator ρ_1 is a parallel vector field, then this manifold reduces to a quasi Einstein manifold in which the sum of its associated scalar functions is zero. But, the generators ρ_2 and ρ_3 can not be parallel vector fields.*

From Theorem (2.1), the Ricci tensor of the $(HGQE)_n$, whose generator ρ_1 is a parallel vector field, can be expressed as follows:

$$S(X,Y) = a[g(X,Y) - A(X)A(Y)]. \tag{2.5}$$

Taking the covariant derivative of the Ricci tensor and using the fact that ρ_1 is a parallel vector field, we obtain

$$(\nabla_Z S)(X,Y) = Z(a)[g(X,Y) - A(X)A(Y)]. \tag{2.6}$$

Contracting (2.6) over X and Z and using contracted second Bianchi Identity, we get

$$\frac{1}{2}Y(r) = Y(a) - \rho_1(a)A(Y). \tag{2.7}$$

Since $r = (n-1)a$, from (2.7) we obtain

$$(\frac{n-3}{2})Y(a) = -\rho_1(a)A(Y). \tag{2.8}$$

Putting $Y = \rho_1$ in (2.8), we obtain $\rho_1(a) = 0$. In this case, when $n > 3$, we get $Y(a) = 0$, for all Y. That is, a is a constant. Then, from (2.6), we get $\nabla S = 0$, which leads us the following result:

Theorem 2.2. *If the generator ρ_1 of $(HGQE)_n$, $(n > 3)$ is a parallel vector field, then the associated scalars of the manifold are constants and this manifold is Ricci symmetric.*

Moreover, since $S(X,Y) = g(QX,Y)$, for all X, Y where Q is a Ricci operator, from (1.11)-(1.13) and as $a \neq 0$, we can easily state that:

Corollary 2.3. *In a $(HGQE)_n$, the following statements hold:*

(1) $Q\rho_1$ is orthogonal to ρ_1 if and only if $a + b = 0$.

(2) $Q\rho_2$ is orthogonal to ρ_1, then $c = 0$. That is, the manifold becomes a generalized quasi Einstein manifold.

(3) $Q\rho_3$ is orthogonal to ρ_1, then $d = 0$. That is, the manifold becomes a generalized quasi Einstein manifold.

(4) $Q\rho_2$ can not be orthogonal to ρ_2 and ρ_3.

(5) $Q\rho_2$ is always orthogonal to ρ_3.

3 Some Pseudo-symmetry Types of $(HGQE)_n$

Definition 3.1. [8] An n-dimensional Riemannian manifold (M^n, g) is called Ricci-pseudosymmetric, if the tensor $R \cdot S$ and the Tachibana tensor $Q(g,S)$ are linearly dependent, where for all $X, Y, Z, W \in \chi(M)$;

$$(R(X,Y) \cdot S)(Z,W) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \tag{3.1}$$

$$Q(g,S)(Z,W;X,Y) = -S((X \wedge_g Y)Z,W) - S(Z,(X \wedge_g Y)W) \tag{3.2}$$

and

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{3.3}$$

That is, the necessary and sufficient condition for (M^n, g) to be a Ricci-pseudo symmetric manifold is that the following equation is satisfied

$$(R(X,Y) \cdot S)(Z,W) = L_S Q(g,S)(Z,W;X,Y) \tag{3.4}$$

on the set $U_S = \{x \in M : S \neq \frac{r}{n} g \ at \ x\}$ and L_S is a certain function on U_S. Then, by using (3.1)-(3.4), we can write

$$\begin{aligned} S(R(X,Y)Z,W)+S(Z,R(X,Y)W) = L_S[&g(Y,Z)S(X,W) \\ &-g(X,Z)S(Y,W) + g(Y,W)S(Z,X) - g(X,W)S(Y,Z)]. \end{aligned} \tag{3.5}$$

Now, we consider Ricci-pseudosymmetric $(HGQE)_n$. Using the fundamental equation (1.1) of $(HGQE)_n$ in (3.5) and in the resulting equation, putting $Z = \rho_1$, $W = \rho_2$ and $Z = \rho_1$, $W = \rho_3$ respectively, we obtain the following expressions:

$$\begin{aligned} bR(X,Y,\rho_1,\rho_2)+&dR(X,Y,\rho_3,\rho_2) \\ &= L_S\Big[b[A(Y)B(X) - A(X)B(Y)] - dB(Y)D(X)\Big], \end{aligned} \tag{3.6}$$

$$\begin{aligned} bR(X,Y,\rho_1,\rho_3)+cR(X,Y,\rho_2,\rho_3) = L_S\Big[&b[A(Y)B(X) - A(X)B(Y)] \\ &+c[B(Y)D(X) - B(X)D(Y)] - D(X)D(Y)\Big]. \end{aligned} \tag{3.7}$$

Contracting (3.7) over X and Y and remembering that the generators ρ_1, ρ_2 and ρ_3 are orthonormal, we obtain $d = 0$. Thus, the manifold under consideration reduces to a $G(QE)_n$. In [11], the authors proved that every Ricci-pseudosymmetric $G(QE)_n$ is an $N(k)$-quasi Einstein manifold. Hence, we can summarize above results by the following theorem:

Theorem 3.2. *Every Ricci-pseudosymmetric $(HGQE)_n$ is an $N(k)$-quasi Einstein manifold with $L_S = \frac{a+b}{n-1}$.*

If the function L_S in (3.4) vanishes, the Ricci-pseudosymmetric manifold turns into a Ricci semi-symmetric manifold. Thus, the next result is obtained directly:

Corollary 3.3. *Every Ricci semi-symmetric $(HGQE)_n$ is a quasi Einstein manifold whose Ricci tensor is of the form $S(X,Y) = a[g(X,Y) - A(X)A(Y)]$.*

As a generalization of Ricci-pseudosymmetric manifolds, R.Deszcz introduced the following notion:

Definition 3.4. [8] A semi-Riemannian manifold (M^n, g), $(n \geq 3)$ is said to be *Ricci-generalized pseudosymmetric* if at every point of (M^n, g), $R \cdot R$ and $Q(S,R)$ are linearly dependent.

That is, the necessary and sufficient condition for (M^n, g) to be a Ricci-generalized pseudosymmetric manifold is that the following equation is satisfied

$$R \cdot R = L_R Q(S,R) \tag{3.8}$$

at every point of the manifold, where L_R is some function on M.

Very important subclasses of Ricci-generalized pseudosymmetric manifolds form manifolds fulfilling the following condition

$$R \cdot R = Q(S,R). \tag{3.9}$$

Such manifolds are said to be *special Ricci-generalized pseudosymmetric manifolds*, [7].

The condition (3.9) arises during the study of the Riemannian manifolds satisfying the condition

$$\omega(X)R(Y,Z)+\omega(Y)R(Z,X)+\omega(Z)R(X,Y)=0, \tag{3.10}$$

where ω is a non-zero 1-form and $X,Y,Z\in\chi(M)$. R. Deszcz and W. Grycak obtained the following characterization theorem for this kind of manifold:

Theorem 3.5. *[9] (see Theorem 1) If at a point $x\in M$, the non-zero 1-form ω satisfies the condition* (3.10)*, then the relation* (3.9) *holds at $x\in M$.*

Motivated by the previous theorem, we will investigate the $(HGQE)_n$ satisfying the condition

$$\sum_{X,Y,Z}\omega(X)R(Y,Z)=0, \tag{3.11}$$

where $\sum$ denotes the cyclic sum over X,Y,Z. In this case, the following assumptions can be examined:

Case 1: First, we choose the 1-form ω as the associated 1-form A of $(HGQE)_n$. Then, we have

$$A(X)R(Y,Z)W+A(Y)R(Z,X)W+A(Z)R(X,Y)W=0. \tag{3.12}$$

Contracting (3.12) over Z and W, we get

$$A(X)S(Y,Z)+R(\rho_1,Y,X,Z)-A(Z)S(Y,X)=0. \tag{3.13}$$

Again, contracting (3.13) over X and Y, we obtain

$$2S(\rho_1,Z)-rA(Z)=0. \tag{3.14}$$

By virtue of (1.11) and (1.10), (3.14) yields

$$[(2-n)a+b]A(Z)+2cB(Z)+2dB(Z)=0. \tag{3.15}$$

Putting $Z=\rho_1$ in (3.15), we get $b=(n-2)a$, putting $Z=\rho_2$ in (3.15), we get $c=0$ and putting $Z=\rho_3$ in (3.15), we get $d=0$. Hence the Ricci tensor can be expressed in the following form:

$$S(X,Y)=a[g(X,Y)+(n-2)A(X)A(Y)]. \tag{3.16}$$

Since $n>2$, the manifold reduces to a $(QE)_n$. Also, in view of (3.16), (3.13) yields

$$R(X,Y)\rho_1=a[A(Y)X-A(X)Y] \tag{3.17}$$

which implies that the generator ρ_1 belongs to a-nullity distribution. Hence, such a manifold becomes an $N(k)$-quasi Einstein manifold with $k=\frac{a+b}{n-1}=a$.

Case 2-3: If we choose the 1-form ω as the associated 1-form B or D of $(HGQE)_n$, then from similar calculations with the Case 1, we get $c = 0$ and $b = a(2-n)$. Thus, the Ricci tensor can be expressed as

$$S(X,Y) = ag(X,Y) + a(2-n)A(X)A(Y) + d[A(X)\omega(Y) + A(Y)\omega(X)] \tag{3.18}$$

which means that the manifold reduces to a $G(QE)_n$.

As a result of these examinations, we can state the following theorem:

Theorem 3.6. *Let M be a $(HGQE)_n$ $(n > 2)$ satisfying the condition*

$$\sum_{X,Y,Z} \omega(X)R(Y,Z) = 0,$$

where ω is a certain 1-form and $\sum$ denotes the cyclic sum over $X, Y, Z \in \chi(M)$. Then, the following conditions hold:

(1) If $\omega = A$, then M reduces to an $N(a)$-quasi Einstein manifold, where a is an associated scalar function of M.

(2) If $\omega = B$ or D, then M reduces to a $G(QE)_n$.

Also, as a result of the last two theorems, the following corollary is obtained:

Corollary 3.7. *Every $(HGQE)_n$ $(n > 2)$ satisfying the condition*

$$\sum_{X,Y,Z} \omega(X)R(Y,Z) = 0,$$

where ω is one of the associated 1-forms of the manifold, is a special Ricci-generalized pseudosymmetric manifold.

Analogously, we can examine the $(HGQE)_n$ satisfying the condition

$$\sum_{X,Y,Z} \omega(X)C(Y,Z) = 0, \tag{3.19}$$

where ω is a non-zero 1-form, $\sum$ denotes the cyclic sum over X, Y, Z and C denotes the conformal curvature tensor defined by

$$\begin{aligned} C(X,Y)Z =& R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.20}$$

where Q is Ricci operator, r is scalar curvature tensor. Similarly, the following cases can be investigated:

Case 1: We can choose the 1-form ω as the associated 1-form A of the $(HGQE)_n$. Then, we have

$$A(X)C(Y,Z)W + A(Y)C(Z,X)W + A(Z)C(X,Y)W = 0. \tag{3.21}$$

Then, contracting (3.21), first we obtain $C(X,Y)Z = 0$ for all X, Y, Z. That is, the manifold is conformally flat. Thus from (3.20), we get

$$\begin{aligned} R(X,Y)Z =& \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.22}$$

Putting $Z = \rho_1$ in (3.22), we get

$$R(X,Y)\rho_1 = \frac{a+b}{n-1}[A(Y)X - A(X)Y] - \frac{d}{n-2}[D(Y)X - D(X)Y] \tag{3.23}$$

and contracting (3.23) over X, we get

$$S(X,\rho_1) = (a+b)A(X) + \frac{d(n-1)}{n-2}D(X). \tag{3.24}$$

Comparing the equations (1.11) and (3.24) and putting $X = \rho_2$ and $X = \rho_3$ in the resulting equation, we get $c = 0$ and $d = 0$, respectively. Similarly, putting $Z = \rho_2$ in (3.22), we get

$$\begin{aligned} R(X,Y)\rho_2 =& \Big(\frac{2a}{n-2} - \frac{r}{(n-1)(n-2)}\Big)[B(Y)X - B(X)Y] \\ &+ \frac{c-d}{n-2}[A(Y)X - A(X)Y] \end{aligned} \tag{3.25}$$

and contracting (3.25) over X, we get

$$S(X,\rho_2) = \frac{2a(n-1) - r}{(n-2)}B(X) + \frac{(c-d)(n-1)}{n-2}A(X). \tag{3.26}$$

Comparing the equations (1.12) and (3.26) and putting $X = \rho_2$ in the resulting equation, we also have $b = 0$. In summary, we have $b = c = d = 0$. This implies that the manifold under consideration reduces to an Einstein manifold, which is a contradiction.

Case 2-3: If we choose the 1-form ω as the associated 1-form B or D of the $(HGQE)_n$, then by direct calculations, we obtain a contradiction similar with the Case 1. Hence, we obtain the following result:

Theorem 3.8. *There does not exist any $(HGQE)_n$ satisfying the condition*

$$\sum_{X,Y,Z} \omega(X)C(Y,Z) = 0,$$

where ω is one of the associated 1-forms of the manifold, $\sum$ denotes the cyclic sum over X, Y, Z and C denotes the conformal curvature tensor.

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