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Uniformization, accessory parameters and modular forms

Candidate:
Gabriele Bogo

Supervisor:
Prof. Don Zagier

Supervisor:
Prof. Fernando Rodriguez Villegas

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SISSA - Via Bonomea 265 - 34136 TRIESTE - ITALY

Abstract

Several topics related to modular forms and to the accessory parameter problem for the uniformization of hyperbolic Riemann surfaces are discussed. In the first part of the thesis we present an algorithm for the computation of the accessory parameters for the Fuchsian uniformization of certain punctured spheres. Then, via modular forms of rational weight, we show that the knowledge of the uniformizing differential equation leads to the complete knowledge of the ring of modular forms $M_*(\Gamma)$ and of its Rankin-Cohen structure. In the second part of the thesis, a new operator ∂_ρ is defined on the space of quasimodular forms $\widetilde{M}_*(\Gamma)$ from an infinitesimal deformation of the uniformizing differential equation. It is shown that ∂_ρ can be described in terms of well-known derivations on $\widetilde{M}_*(\Gamma)$ and certain integrals of weight four-cusp forms; the relation between the operator ∂_ρ and a classical construction in Teichmüller theory is discussed. The functions $\partial_\rho g$, $g \in \widetilde{M}_*(\Gamma)$, motivate the study and the introduction of a new class of functions, called *extended modular forms*. Extended modular forms are defined as certain components of vector-valued modular forms with respect to symmetric tensor representations. Apart from the functions $\partial_\rho g$, examples of extended modular forms are: Eichler integrals, more general iterated integrals of modular forms, and elliptic multiple zeta values.

Contents

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Introduction

The uniformization theorem states that a simply connected Riemann surface is biholomorphic to exactly one of three basic objects: the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$, the upper half-plane $\mathbb{H}$. In particular, the universal covering of a Riemann surface being simply connected, every Riemann surface is a quotient of one of those three.

The possible cases are completely different, as are the uniformizing functions. The Riemann sphere, being compact and of genus zero, only covers itself. The complex plane covers itself, the once punctured plane via the exponential map, and elliptic curves via elliptic functions. Every other Riemann surface S has the upper half-plane as universal covering and can be identified with the quotient $\mathbb{H}/\Gamma$ for some Fuchsian group Γ. The composition of the covering map $\mathbb{H} \to S$ with any rational function $S \to \widehat{\mathbb{C}}$ is then a modular function.

This thesis has its roots in the classical work of Klein and Poincaré on the uniformization of hyperbolic Riemann surfaces. Given a hyperbolic surface S, the classical approach to the study of the uniformization of S was to find a biholomorphism between the universal covering $\widetilde{S}$ and the upper half-plane $\mathbb{H}$ by finding the "correct" differential equation (or projective connection) defined on S.

To clarify what "correct" means, let us discuss the first non-trivial case, that of spheres with four punctures. Let α be a complex number, and consider the punctured sphere $X = \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha\}$. On this surface the following differential equation (Heun's equation) is defined:

$$\frac{d^2Y}{dt^2} + \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-\alpha}\Big)\frac{dY}{dt} + \frac{t-\rho}{t(t-1)(t-\alpha)}Y = 0. \tag{1}$$

The coefficient $\rho \in \mathbb{C}$ is called the *accessory parameter* and plays a big role in our story.

It is known that the analytic continuation of linearly independent solutions of (1) defines a local biholomorphism

$$\widetilde{X} \to \mathbb{C}, \tag{2}$$

and that all local biholomorphism (2) arise this way. It follows that if there exists a global one, it is necessarily induced by a differential equation of the type (1). The problem of finding the unique value of ρ, called the *Fuchsian value*, that induces a global biholomorphism $\widetilde{X} \simeq \mathbb{H}$ is known as the *accessory parameter problem.*

This approach to uniformization, which was the one originally envisaged by Riemann and Klein, was eventually abandoned since there was no direct way to identify the Fuchsian value or even to prove its existence. Indeed, the existence of the Fuchsian value was proved as a consequence of the uniformization theorem, which was proved later and with different methods. More than a hundred years later we know little more about the nature of the accessory parameters than was known then, and their exact computation has been carried out only in a few special cases.

Our interest in the Fuchsian parameters grows from its relation with modular forms. A holomorphic solution of the uniformizing differential equation with the correct choice of ρ is nothing but a weight one modular form (or a root of a higher weight modular form) expressed in terms of the uniformizing function. This point of view on the differential equation (1) and on the accessory parameter problem is the one that we adopt here.

The first two chapters of the thesis present the backgroung material on modular forms and some generalizations, and on the classical theory of uniformization and Teichmüller theory. Specifically, in Chapter 1 we review the definitions of modular and quasimodular forms and of the Rankin-Cohen brackets, the differential equations satisfied by modular forms and the notion, which will be used later, of vector-valued modular forms. This material is not new except for a slightly simpler proof and slightly stronger statement of a theorem of Choie and Lee relating quasimodular forms to vector-valued modular forms. The first part of Chapter 2 consists in an exposition of the classical theory of uniformization, its relation with differential equations and accessory parameters; in the second part we review the definitions of quasiconformal maps, quasi-Fuchsian groups and Teichmüller spaces, and we briefly discuss the Bers embedding and a result of Takhtajan and Zograf that will be useful in the following.

In Chapter 3 we put together some of this material to discuss in more depth the modular point of view on uniformization. It is known that a modular form of weight k satisfies a linear ODE of order $k+1$ if expressed locally in terms of a modular function. We compute the general form of this differential equation for a weight two modular form on a genus zero group using Rankin-Cohen brackets. This introduces a new set of accessory parameters, called *modular accessory parameters*, related to the classical ones by a simple formula.

The modular accessory parameters are complex numbers that make true a certain relation in the Rankin-Cohen structure of $M_*(\Gamma)$. It follows in particular that the knowledge of the Rankin-Cohen brackets in terms of certain elements of $M_*(\Gamma)$ implies the knowledge of the accessory parameter, and the differential equation associated to any element on $M_k(\Gamma)$.

We show that, under a certain hypothesis on Γ, the converse is also true: the knowledge of the classical accessory parameters gives not only the uniformization group Γ (as a monodromy group) and the covering map, but also the full ring of modular forms on Γ and all the Rankin-Cohen brackets in terms of certain elements. This is proved by studying modular forms of *rational weight* associated to powers of a certain automorphy factor J. The ring of all such rational weight forms turns out to be a free ring and a Rankin-Cohen algebra, with the integral weight moduar forms forming a canonical subalgebra.

In Chapter 4 we exploit some of the result of the preceding chapter to determine the Fuchsian value *algorithmically.* The basic idea is the following: the modular accessory parameters are coefficients of certain Q-expansions that are the classical q-expansions of modular forms whrn $\rho = \rho_F$. We can then test the modular transformation properties of the functions defined by these Q-expansion: if we can find a parameter that makes these properties true, then that is the Fuchsian one.

The problem that must be solved on order to implement this idea is to determine the group with respect to which our function should be a modular form. One possibility would be to compute numerically the monodromy of the differential equation, but this is not what we do. Instead we compute some numbers that will eventually be the cusp representatives of the uniformizing group. We can compute these numbers using the automorphisms of the punctured sphere X : they lift to automorphisms on $\mathbb{H}$ and hence determine fixed points on $\mathbb{H}$, which then project, via the covering map, to the fixed points of the automorphism in X. But the latter can be determined exactly, and the real part of their image under the ratio of the solutions of (1) are the cusp representatives we want. The generators of the group can then be computed from the cusp representatives that we heve determined.

The algorithm will be presented for four-punctured spheres, but works more generally for all punctured spheres with sufficiently many automorphisms. As an application, we compute a local expansion of the function giving the Fuchsian value and are led to rediscover numerically an earlier result of Takhtajan and Zograf.

In Chapter 5, our research takes a different road. We saw that modular forms appear from solutions of the uniformizing equation when ρ is the Fuchsian value ρ_F. The question is: what happens after an infinitesimal deformation of ρ_F? More precisely, we consider the derivatives of the solution of (1) with respect to ρ and then compose these functions with the Hauptmodul to define new functiona on $\mathbb{H}$. Maybe surprisingly, these functions, which are not modular any more, can still be described in terms of well-known modular objects: quasimodular forms and Eichler integrals of cusp forms.

These deformation operators, that we introduce and will be denoted by $\partial_{i,Q}$, can also be extended to quasimodular forms. Recall that the space $\widetilde{M}_*(\Gamma)$ of quasimodular forms has an $\mathfrak{sl}_2(\mathbb{C})$-module structure given by three derivations: the differentiation operator $\mathbf{D}$, the weight operator $\mathbf{W}$ and the derivation $\boldsymbol{\delta}$ which defined by the conditions that $\boldsymbol{\delta} f = 0$ if f is modular and $\boldsymbol{\delta}\phi = 1$ for some quasimodular form ϕ of weight two. The deformation on quasimodular forms can then be expressed in terms of the operators $\mathbf{D}, \mathbf{W}, \boldsymbol{\delta}$ and Eichler integrals (and their derivatives) of weight four cusp forms h_i; if $g \in \widetilde{M}_*(\Gamma)$ we have

$$\partial_{i,Q} g = 2\widetilde{h}_i \, \mathbf{D}g + \widetilde{h}_i' \, \mathbf{W}g + \widetilde{h}_i'' \, \boldsymbol{\delta} g. \qquad (3)$$

The fact that the h_i are of weight four is independent of g. From the point of view of the uniformization theory, the appearance of weight four cusp forms is not a surprise since these are related to the deformation theory of Riemann surfaces, which is in a sense a geometric counterpart of our deformation $\partial_{i,Q}$. In the last part of the chapter, we will relate this operator, when g is a modular function, to a standard construction in Teichmüller theory. After this identification we will recover a formula of Ahlfors for quasiconformal mappings.

In Chapter 6 we start from the identity (3) to pursue the study of the new type of modularity. This is only the first step towards a theory that, hopefully soon, will be more rich and complete. The main goal was to find the natural space where the right-hand side of (3) lives. The best way to proceed seemed to be to consider vector-valued modular forms with respect to certain representations of Γ, where Γ is any Fuchsian group of finite covolume. The point of departure is the fact that the monodromy representation of the linear differential equation satisfied by a modular form f is a symmetric tensor representation Sym^* of Γ. In particular, f can be identified as a component of a vector-valued modular form with respect to this representation. Similarly, as we saw in Chapter 1, quasimodular forms can also be identified with components of vector-valued modular forms with respect to symmetric tensor representations. The new idea is to look at components of vector-valued modular forms with respect to a representation that is an extension, or an iterated extension of symmetric tensor representations.

To see how this works, observe that the new function in (3) is also obtained from a differential equation. If we try to emulate the modular case, i.e. to rewrite the solutions (on $\mathbb{H}$) of the new differential equation as vector-valued forms, we find that the matrix we have to consider is of the form

$$A(\gamma) = \left(\begin{array}{c|c} \mathrm{Sym}^1(\gamma) & M(\gamma) \\ \hline 0 & \mathrm{Sym}^2(\gamma) \end{array}\right), \tag{4}$$

where the matrix $M(\gamma)$ is essentially given by the coefficients of the period polynomial of the cusp form h_i in (3). Since A is the monodromy of a linear differential equation, the map $\gamma \mapsto A(\gamma)$ defines a representation of Γ on $V \simeq \mathrm{Sym}^1\mathbb{C} \oplus \mathrm{Sym}^2\mathbb{C}$, i.e., V is an *extension* of Sym^2 by Sym^1.

We can then consider more generally the vector-valued modular forms arising from arbitrary extensions of symmetric tensor representations. This is only an intermediate step, since we want a space of scalar-valued functions. What we do in Chapter 6 is to define a new modular object, called *extended modular form*, as certain components in vector-valued forms with respect to representations defined by successive extensions of tensor representations.

The last part of the chapter contains examples of extended modular forms. Apart from the functions in (3), natural examples are Eichler integrals and their derivatives, but we can also give an example of a quite different form coming from the world of depth-one elliptic multiple zeta values.

In Chapter 7 we first present some considerations on a conjecture of J. Thompson. He conjectured that the Fuchsian values associated to a sphere with algebraic punctures are also algebraic. Based on extensive computations, performed with the algorithm presented in Chapter 4, a new conjecture in proposed, which contradicts with Thompson's one.

Finally, Chapter 7 also contains a brief report of a somewhat puzzling numerical phenomenon that was discovered experimentally while playing with the functions related to the uniformizing differential equation.

Chapter 1

Review of modular forms

1.1 Modular forms

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be a *Fuchsian group of the first kind*, i.e., a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ with finite covolume. It acts on the upper half-plane $\mathbb{H}$ via Möbius transformations

$$\gamma \mapsto \gamma\tau := \frac{a\tau+b}{c\tau+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \tau \in \mathbb{H}.$$

Let $\mathrm{Hol}_0(\mathbb{H})$ denote the space of holomorphic functions on $\mathbb{H}$ with at most polynomial growth at the cusps of Γ. Let $f \in \mathrm{Hol}_0(\mathbb{H})$ and let $k \in \mathbb{Z}$. For every $\gamma \in \Gamma$ define the *slash operator* by

$$f\big|_k\gamma := f(\gamma\tau)(c\tau+d)^{-k}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This defines an action of Γ on Hol_0. The functions $f \in \mathrm{Hol}_0$ such that $f\big|_k\gamma = f$ are called *modular forms* of weight k; the space of such functions is denoted by $M_k(\Gamma)$. An element $f \in M_k(\Gamma)$ is called a *cusp form* if $\Im(\tau)^{k/2}|f(\tau)|$ is bounded in $\mathbb{H}$. The space of cusp forms is denoted by $S_k(\Gamma)$.

For every $k \in \mathbb{Z}$ the space $M_k(\Gamma)$ is a finite dimensional vector space. The correspondence between modular forms on Γ and differentials on the surface $\overline{\mathbb{H}}/\Gamma$, together with the Riemann-Roch theorem, leads to dimension formulas for the spaces of modular and cusp forms.

In the theorems, for any real number s, we denote by $\lfloor s \rfloor$ the largest integer $\leq s$.

Theorem 1. *Let k be an even integer, g the genus of $\overline{\mathbb{H}}/\Gamma$, $e_1, \ldots, e_r$ the orders of* *inequivalent elliptic points of* Γ *and* n *the number of inequivalent cusps. Then*

$$\dim S_k(\Gamma) = \begin{cases} (k-1)(g-1) + \sum_{i=1}^{r} \lfloor \frac{k}{2}(1-\frac{1}{e_i}) \rfloor + \frac{k-2}{2}n & k > 2, \\ g & k = 2, \\ 1 & k = 0, n = 0, \\ 0 & k = 0, n > 0, \\ 0 & k < 0; \end{cases}$$

$$\dim M_k(\Gamma) = \begin{cases} \dim S_k(\Gamma) + n & k \geq 4, \\ \dim S_2(\Gamma) + n - 1 = g + n - 1 & k = 2, n > 0 \\ \dim S_2(\Gamma) = g & k = 2, n = 0, \\ 1 & k = 0 \\ 0 & k < 0. \end{cases}$$

The next theorem discuss odd weight forms.

Theorem 2. *Let* k *be an odd integer, and assume* $-1 \notin \Gamma$. *Let* g *be the genus of* $\overline{\mathbb{H}}/\Gamma$, $e_1, \ldots, e_r$ *the orders of inequivalent elliptic points of* Γ. *Let* u, v *denote the numbers of regular and irregular cusps of* Γ, *respectively. Then*

$$\dim S_k(\Gamma) = \begin{cases} (k-1)(g-1) + \sum_{i=1}^{r} \lfloor \frac{k}{2}(1-1/e_i) \rfloor + \frac{k-2}{2}u + \frac{k-1}{2}v & k \geq 3, \\ 0 & k < 0; \end{cases}$$

$$\dim M_k(\Gamma) = \begin{cases} \dim S_k(\Gamma) + u & k \geq 3, \\ \dim S_1(\Gamma) + u/2 & k = 1, \\ 0 & k < 0. \end{cases}$$

We remark that the dimension of $S_1(\Gamma)$ for a generic Fuchsian group Γ is not known. The Riemann-Roch theorem only permits to find the difference $\dim M_1(\Gamma) - \dim S_1(\Gamma)$. However, if Γ has genus $g = 0$, we can give an exact formula: there are in fact no cusps of weight one, simply because $\dim S_2(\Gamma) = g = 0$. Then the dimension of the space of weight one modular forms only depends on the number u of regular cusps:

$$\dim M_1(\Gamma) = u/2, \quad \text{if } g = 0.$$

In particular, u is always even.

1.2 Modular forms and differential operators

The derivative of a modular form is not modular. It is easy to verify that, if $f \in M_k(\Gamma)$,

$$f'(\gamma\tau) = f'(\tau)(c\tau + d)^{k+2} + \frac{k}{2\pi i}c(c\tau + d)^{k+1}, \quad f' = \mathbf{D}f := \frac{1}{2\pi i}\frac{df}{dt}. \tag{1.1}$$

In [12] four different ways to deal with this "problem" are presented. Here we discuss some of those.

1.2.1 Almost holomorphic and quasimodular forms

Let $f \in M_k(\Gamma)$, and let $\Im(\tau)$ denote the imaginary part of τ. For every $\gamma \in \mathrm{SL}(2,\mathbb{R})$ we have

$$\frac{1}{\Im(\gamma\tau)} = \frac{|c\tau+d|^2}{\Im(\tau)} = \frac{(c\tau+d)^2}{\Im(\tau)} - 2ic(c\tau+d), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From this observation and (1.1) it follows that the non holomorphic function

$$f' - \frac{kf}{4\pi\Im(\tau)}$$

transform like a modular form of weight $k+2$ with respect to Γ.

More generally we consider the non holomorphic functions

$$\widehat{F}(\tau) = \sum_{j=0}^{p} f_j(\tau)\left(\frac{-1}{4\pi\Im(\tau)}\right)^j, \quad f_j \in \mathrm{Hol}_0(\Gamma),$$

that satisfies, for every $\gamma \in \Gamma$,

$$F\big|_k\gamma = F.$$

We call these functions *almost holomorphic modular forms* of weight k and *depth* $\leq p$. The space of such forms is denoted by $\widehat{M_k}(\Gamma)^{(\leq p)}$.

The space of constant terms f_0 of $\widehat{F}$ as $\widehat{F} \in \widehat{M_k}(\Gamma)^{(\leq p)}$ is denoted by $\widetilde{M_k}(\Gamma)$ and called the space of *quasimodular forms* of weight k and *depth* $\leq p$. Since the constant term $f_0(\tau)$ completely determines the associated almost holomorphic form $\widehat{F} = \sum_{j=0}^{p} f_j(1/4\pi\Im(\tau))^j$, the spaces $\widehat{M_k}(\Gamma)^{(\leq p)}$ and $\widehat{M_k}(\Gamma)^{(\leq p)}$ are canonically isomorphic. It is easy to prove moreover that every $f_j, (j = 0,\ldots,p)$ in the expansion of $\widehat{F}$ is a quasimodular form. We will denote by

$$\widehat{M_*} = \bigoplus_k \widehat{M_k}(\Gamma), \quad \widehat{M_k}(\Gamma) := \bigcup_p \widehat{M_k}(\Gamma)^(\leq p),$$
$$\widetilde{M_*} = \bigoplus_k \widetilde{M_k}(\Gamma), \quad \widetilde{M_k}(\Gamma) := \bigcup_p \widetilde{M_k}(\Gamma)^(\leq p),$$

the graded and filtered ring of almost holomorphic and quasimodular forms.

Quasimodular forms can also be defined directly in terms of the slash operator. A quasimodular form of weight k and depth$\leq p$ is a function $f_0 \in \mathrm{Hol}(\Gamma)$ such that

$$\left(f_0\big|_k\gamma\right)(\tau) = \sum_{j=0}^{p} f_j \left(\frac{c}{c\tau+d}\right)^j, \quad \text{for every fixed } \tau \in \mathbb{H}.$$

Note that the holomorphic functions in the above expansion are the same coefficients appearing in the expansion of the almost holomorphic modular form associated to f_0.

If Γ is a non-cocompact group, there always exists a holomorphic quasimodular form ϕ of weight two which is not modular. The properties of quasimodular forms are given in the following proposition.

Proposition 1. *Suppose that Γ is a non co-compact Fuchsian group. Then*

1. *The space of quasimodular forms is closed under differentiations. More precisely, if $Df := \frac{1}{2\pi i}\frac{df}{d\tau}$, we have $D(\widetilde{M}_k^{(\leq p)}) \subset \widetilde{M}_{k+2}(\Gamma)^{(\leq p+1)}$ for every $k, p \geq\ 0$.*
2. *Every quasimodular form on Γ is a polynomial in ϕ with modular coefficients. More precisely, for every $k, p \geq 0$ we have $\widetilde{M}_k(\Gamma)^{(\leq p)} = \bigoplus_{j=0}^{p} M_{k-2j}(\Gamma)\phi^j$.*
3. *Every quasimodular form on Γ can be written uniquely as a linear combinations of derivatives of modular forms and derivatives of ϕ.*

We remark that in the case Γ is cocompact, the space of quasimodular forms coincides with the space of derivatives of modular forms of Γ. In particular, it is closed under differentiations.

we finally mention that there are three natural derivations on the ring $\widetilde{M}_*(\Gamma)$ of quasimodular forms. One is the differentiation operator $\mathbf{D}$ introduced in (1.1); the *weight operator* $\mathbf{W}$ acts by multiplication by the weight, i.e. $\mathbf{W}g = kg$ if $f : 0 \in \widetilde{M}_k(\Gamma)$. Finally let $\widehat{F} = \sum_{j=0}^{p} f_j(1/4\pi\Im(\tau))^j$. Denote by $\boldsymbol{\delta}$ the operator which sends the quasimodular form f_0 to f_1.

These derivations satisfy the following commutator relations

$$[\mathbf{W}, \mathbf{D}] = 2\mathbf{D}, \quad [\mathbf{W}, \boldsymbol{\delta}] = -2\boldsymbol{\delta}, \quad [\mathbf{D}, \boldsymbol{\delta}] = \mathbf{W},$$

which give to $\widetilde{M}_*(\Gamma)$ (and to $\widehat{M}_*(\Gamma)$) the structure of a $\mathfrak{sl}_2-$module.

1.2.2 Rankin-Cohen brackets and Rankin-Cohen structure

Rankin-Cohen brackets are bilinear operators $[\ ,\ ]_n$, $n \geq 0$, defined on the space of C^∞ functions, usually complex valued and defined on the upper half-plane. Their definition and main property are closely related to the slash operator. Rankin-Cohen brackets were introduced in full generality by Cohen in [16], and in special cases by Rankin [37].

Definition 1. *Let f, g be $\mathcal{C}^\infty$ complex valued functions defined on $\mathbb{H}$ of weight k, l respectively. For every $n \geq 0$, the $n-th$* **Rankin-Cohen bracket** *is the bilinear form defined by*

$$[f, g]_n := \sum_{\substack{r,s>0 \\ r+s=n}} (-1)^r \binom{k+n-1}{s}\binom{l+n-1}{r} \mathbf{D}^r f \mathbf{D}^s g.$$

The main property of Rankin-Cohen brackets is the following

$$\big[f,g\big]_n\big|_{k+l+2n}\gamma = \big[f\big|_k\gamma, g\big|_l\gamma\big]_n.$$

This implies that if f, g are modular forms of weight k, l respectively, the holomorphic function $\big[f,g\big]_n$ is a modular form of weight $k+l+2n$.

Beside this, Rankin-Cohen brackets verify a number of algebraic relations. Important examples of these relations are

$$\big[[f,g]_1,h\big]_1 + \big[[g,h]_1,f\big]_1 + \big[[h,f]_1,g\big]_1 = 0 \qquad \text{(Jacobi identity);}$$

if f, g, h have weight k, l, m respectively,

$$m[f,g]_1 h + k[g,h]_1 f + l[h,f]_1 g = 0, \tag{1.2}$$

and, relating first and second brackets,

$$k^2(k+1)f^2[g,g]_2 = l^2(l+1)g^2[f,f]_2 - (k+1)(l+1)^2[f,g]^2 + b(a+1)(b+1)g[[g,f],f]. \tag{1.3}$$

Zagier, in [49], studied the algebraic structure given to $M_*(\Gamma)$ or more abstract spaces by the collection of Rankin-Cohen brackets.

A *Rankin-Cohen algebra* or *RC algebra* R over a field K is a graded K-vector space $M_* = \bigoplus_{i\geq 0} M_i$ with $\dim M_i$ finite for every $i \geq 0$, together with bilinear operations $[,]_n \colon M_k \otimes M_l \to M_{k+l+2n}$ which satisfy all the algebraic identities satisfied by the Rankin-Cohen brackets.

In the following we will consider graded vector spaces where the grading is not given by a non negative integer i but it is given by a real number ri, where $r \in \mathbb{R}$ and $i \geq 0$ is an integer. This is because we want eventually consider spaces of modular forms of non integral weights as graded rings. The theory developed by Zagier on RC algebras extends directly to this more general setting. The proofs of the theorem that we need from the paper [49] are mostly based on the relations (1.2),(1.3), which holds for real weight. we will nevertheless denote the grading of the algebra simply using the integers, even if it is a real number.

The main example of a Rankin-Cohen algebra is the algebra of modular forms $M_*(\Gamma)$, with the brackets defined above. Another example is given by quasimodular forms; this in particular is a *standard* RC algebra, which simply means that it is closed under the derivation D which defined the RC brackets. The algebra of modular forms then is a RC subalgebra of $\widetilde{M}_*(\Gamma)$, i.e. a subspace closed under all the brackets $[,]_n$ defining $\widetilde{M}_*(\Gamma)$.

Inspired by the modular case, Zagier then discuss and proves a structure theorem for general RC algebras. we recall it here since it will be useful in Chapter 3.

Proposition 2. *Let M_* be a commutative and graded $K-$algebra, with $M_0 = K$, together with a derivation $\partial \colon M_* \to M_{*+2}$ of degree two, and let $\Phi \in M_4$. Define brackets*

$[,]_{\partial,\Phi,n}$ *for* $n \geq 0$ *on* M_* *by*

$$[f,g]_{\partial,\Phi,n} := \sum_{r+s=n} (-1)^r \binom{k+n-1}{s}\binom{l+n-1}{r} f_r g_s, \quad f \in M_k,\, g \in M_l$$

where $f_r \in M_{k+2r}$, $g_s \in M_{l+2s}$, $r, s, \geq 0$ *are defined recursively by*

$$f_{r+1} = \partial f_r + r(r+k-1)\Phi f_{r-1}, \quad g_{s+1} = \partial g_s + s(s+l-1)\Phi g_{s-1},$$

with initial conditions $f_0 = f$, $g_0 = g$. *Then* $(M_*, [,]_{\partial,\Phi,*})$ *is a RC algebra.*

An RC algebra is called *canonical* if its brackets are given as in the previous proposition for some derivation $\partial M_* \to M_{*+2}$ and some element $\Phi \in M_4$.

The idea now is the following: given a RC algebra R for which we know the multiplicative structure (i.e. the zero bracket), the first and the second brackets, to construct a derivation ∂ and a weight four element Φ in such a way that the full Rankin-Cohen structure of R can be described by the brackets $[\,,\,]_{\partial,\Phi,n}$ defined in the previous proposition. It will follow that, to describe the whole RC algebra, we only need the first two brackets and the ring structure.

Let F be a homogeneous and non zero divisor element of weight $N > 0$ in the RC algebra R. For $f \in M_*$ define

$$\partial f := \frac{[F,f]}{NF}, \quad \Phi := \frac{[F,F]_2}{N^2(N+1)F^2}.$$

Theorem 3 (Zagier). *Let R be a RC algebra with brackets $[\,,\,]_*$, and assume it contains a non-zero divisor F of weight $N > 0$ such that*

1. $[F, M_*] \subset M_* F$;

2. $[F,F]_2 \in M_* F^2$.

Then $[\,,\,]_* = [\,,\,]_{\partial,\Phi,n}$ *for* ∂, Φ *as in* (1.2.2)*, so* M_* *is a canonical RC algebra.*

In particular, from this theorem and the relations (1.2),(1.3), it follows that to know the whole RC algebra we need only:

1. the multiplicative structure for arbitrary $f, g \in M_*$;

2. the first bracket of arbitrary $f \in M_*$ with the fixed element F, which is a derivation; and

3. the second bracket of F with itself.

1.2.3 Linear differential equations

The following proposition affirms that any modular form satisfies a linear diffferential equation if expressed in terms of a modular function t. As we will see in detail, this is indeed deeply related to the uniformization theorem.

Proposition 3. *Let Γ be a Fuchsian group. Let f be a (possibly meromorphic) modular form of positive weight k on Γ, and let t be a modular function with respect to Γ. Express $f(\tau)$ locally as $\Phi(t(\tau))$. Then, the function $\Phi(t)$ satisfies a linear differential equation of order $k+1$ with algebraic coeffcients, or with polynomial coefficients if Γ is of genus zero and t is a Hauptmodul.*

we only mention two relevant facts here. The first is that the differential equation satisfied by $f \in M_k(\Gamma)$ can be written in terms of Rankin-Cohen brackets. In particular, we will consider in the following second order ODE associated to $f^{1/k}$ given by

$$D_t^2 f + \frac{[f,t']_1}{kft'^2} D_t f - \frac{[f,f]_2}{k^2(k+1)f^2t'^2} f = 0, \quad D_t := \frac{1}{t'}\frac{d}{d\tau}.$$

Note that the coefficients are weight zero modular forms, as thay should, since they are algebraic functions of t.

The second fact is that a full set of solutions of the differential equation satisfied by $f \in M_k(\Gamma)$ is given by $f, \tau f, \ldots, \tau^k f$, and the monodromy group is just the k-th symmetric power $\mathrm{Sym}^k(\Gamma)$. Both these facts will play a relevant role in the next chapters.

1.3 Vector valued modular forms

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be a discrete group of finite covolume. Let V be a vector space of dimension n; the action

$$\rho_V \colon \Gamma \to \mathrm{GL}(V), \quad \gamma \mapsto \varphi(\gamma)$$

turns V into a representation of Γ. A vector-valued function

$$F(\tau) = \begin{pmatrix} f_n(\tau) \\ \vdots \\ f_1(\tau) \end{pmatrix} : \mathbb{H} \to \mathbb{C}^n \tag{1.4}$$

is a *vector-valued modular form attached to* V of weight k if the following three conditions (V1), (V2), (V3) are satisfied:

(V1) Every component $f_i \colon \mathbb{H} \to \mathbb{C}$, $i = 1, \ldots, n$, of F is holomorphic on $\mathbb{H}$;

(V2) For every $\gamma \in \Gamma$ we have

$$F|_k\gamma = \rho_V(\gamma)F.$$

If Γ is cocompact the above conditions are enough; if not, we need to specify the behaviour of F at the cusps.

(V3) Every component f_i has polynomial growth at the cusps of Γ.

1.3.1 Symmetric tensor representations

Let $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathbb{C}^2$, and denote, for every $n \in \mathbb{Z}_{\geq 0}$, by $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)^n$ the vector in $\mathbb{C}^{n+1}$ whose components are $u^n, u^{n-1}v, \dots, uv^{n-1}, v^n$. Let $\sigma \in \mathrm{GL}(2, \mathbb{R})$ and let

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \sigma \begin{pmatrix} u \\ v \end{pmatrix}.$$

For every $n \in \mathbb{Z}_{\geq 0}$ define a matrix $\mathrm{Sym}^n(\sigma) \in \mathrm{GL}(n+1, \mathbb{R})$ by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}^n = \mathrm{Sym}^n(\sigma) \begin{pmatrix} u \\ v \end{pmatrix}^n.$$

we have, for instance

$$\mathrm{Sym}^0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1, \quad \mathrm{Sym}^1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathrm{Sym}^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2bc & d^2 \end{pmatrix}.$$

Consider the vector space $V_n := \mathrm{Sym}^n(\mathbb{C}^2)$ of complex dimension $n+1$. For every $n \in \mathbb{Z}_{\geq 0}$ define a representation of $\mathrm{GL}(2, \mathbb{R})$, still denoted by Sym^n, by

$$\mathrm{Sym}^n \colon \mathrm{GL}(2, \mathbb{R}) \to V_n, \quad \sigma \mapsto \mathrm{Sym}^n(\sigma).$$

we will often consider the following matrix valued function

$$L_n(\tau) := \mathrm{Sym}^n \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n\tau & \frac{n(n-1)}{2}\tau^2 & \cdots & \tau^n \\ 0 & 1 & (n-1)\tau & \cdots & \tau^{n-1} \\ & \cdots & & \cdots & \\ 0 & \cdots & & \cdots & 1 \end{pmatrix}, \quad \tau \in \mathbb{H}. \tag{1.5}$$

For every $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$ we define $J(\sigma, \tau) := (c\tau + d)^{-1}$. we have the useful equality

$$L_n(\sigma(\tau))^{-1} \mathrm{Sym}^n(\sigma) L_n(\tau) = \mathrm{Sym}^n \begin{pmatrix} J(\sigma, \tau) & 0 \\ c & J(\sigma, \tau)^{-1} \end{pmatrix}. \tag{1.6}$$

when $\sigma = \sigma_r = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$, for some real number r, the above equality reduces to

$$L_n(\sigma_r(\tau)) = \mathrm{Sym}^n(\sigma_r) L_n(\tau). \tag{1.7}$$

Let $F(\tau)$ be a vector-valued modular form attached to the representation V_n. In this setting we can give an alternative condition (V3′) about the behaviour at the cusps of a vector-valued modular form.

Let x be a cusp of Γ and let $\sigma \in \mathrm{SL}(2,\mathbb{R})$ be such that $\sigma(\infty) = x$. we know that there exists $h > 0$ such that

$$\{\pm 1\} \cdot \sigma^{-1}\Gamma_x\sigma = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^n \mid n \in \mathbb{Z} \right\}.$$

It is easy to show, using (1.7), that the following vector-valued function is invariant under the map $\tau \mapsto \tau + h$

$$L_n(\tau)^{-1}\mathrm{Sym}^n(\sigma^{-1})\big(F|_k\sigma\big)(\tau).$$

Therefore, if we put $q = e^{2\pi i\tau/h}$, there exists $n+1$ functions $g_0(q), \ldots, g_n(q)$, holomorphic in $0 < |q| < 1$ such that

$$L_n(\tau)^{-1}\mathrm{Sym}^n(\sigma^{-1})\big(F|_k\sigma\big)(\tau) = \begin{pmatrix} g_n(q) \\ \vdots \\ g_n(q) \end{pmatrix}.$$

The alternative condition is:

(V3') For every cusp x of Γ, the functions $g_0(q), \ldots, g_n(q)$, whose definition depends on x, are holomorphic in $q = 0$.

If moreover the functions $g_0(q), \ldots, g_n(q)$ are zero at $q = 0$, $F(\tau)$ is called *vector-valued cusp form.*

The space of vector-valued modular forms of weight k attached to Sym^n is denoted by $M_k(\Gamma, n)$; by $S_k(\Gamma, n)$ we denote the space of cusp forms.

1.3.2 Relation with quasimodular forms

In [14] Choie and Lee, extending previous results of Kuga-Shimura [32], prove an isomorphism between the spaces of certain vector valued forms and quasimodular forms. More precisely, they show that, if k, n are integers such that $k \equiv n \mod (2)$ and $k > n \geq 0$, and if Γ is commensurable with $\mathrm{SL}(2,\mathbb{Z})$, then

$$M_{k+n}(\Gamma, n) \simeq \widetilde{M}_{k+n}(\Gamma)^{(\leq n)}.$$

The proof gives an explicit isomorphism, using some generalization of Rankin-Cohen brackets, between the direct sum of spaces of modular forms and VVMF. The connection with quasimodular forms is made via modular and quasimodular polynomials.

Here we prove a similar statement in more generality, using a more direct approach.

Theorem 4. *Let Γ be a discrete group of $\mathrm{SL}(2,\mathbb{R})$ of finite covolume. For every $k, n \geq 0$ there is an isomorphism*
$$M_{k+n}(\Gamma, n) \simeq \widetilde{M}_{k+n}(\Gamma)^{(\leq n)}.$$

Proof. Let $F^* \in M_k(\Gamma, n)$. Define a new vector valued function $F(\tau)$ by

$$F^*(\tau) = L_n(\tau)F(\tau),$$

where $L_n(\tau)$ is as in (1.7). By the transformation propery of $F^*(\tau)$ we obtain

$$F(\gamma\tau) = L_n(\gamma\tau)^{-1}\mathrm{Sym}^n(\gamma)L_n(\tau)F(\tau)(c\tau+d)^k, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which gives, using (1.6),

$$F|_k\gamma = \mathrm{Sym}^n \begin{pmatrix} (c\tau+d)^{-1} & 0 \\ c & c\tau+d \end{pmatrix} F.$$

A simple computation shows that the $r-$th row of the above matrix is give by

$$\left(c^r(c\tau+d)^{r-n}, \binom{r}{1}c^{r-1}(c\tau+d)^{r+1-n}, \binom{r}{2}c^{r-2}(c\tau+d)^{r+2-n}, \dots, (c\tau+d)^{2r-n}, 0, \dots, 0\right).$$

It follows that, if $F = {}^t(f_0, \dots, f_n)$, for every $r = 0, \dots, n$ we have

$$f_r(\gamma\tau)(c\tau+d)^{-k} = \sum_{i=0}^{r} c^{r-i}\binom{r}{i}(c\tau+d)^{r+i-n}f_i(\tau).$$

Hence
$$f_r(\gamma\tau)(c\tau+d)^{n-k-2r} = \sum_{j=0}^{r}\binom{r}{r-j}f_{r-j}(\tau)\left(\frac{c}{c\tau+d}\right)^j.$$

This means that, for every $r = 0, \dots, n$ the component f_r of F transforms like a quasimodular form of weight $k+2r-n$ and depth $\leq r$. Since $F^* \in M_k(\Gamma, n)$ it satisfies (V1), (V3), so it follows that the functions f_r all are holomorphic in $\mathbb{H}$ and satisfies suitable growth conditions at the cusps, if any. Then they are actual quasimodular forms, i.e. $f_r \in \widetilde{M}_{k+2r-n}(\Gamma)^{(\leq n)}$.

Consider the function

$$\hat{F} := \sum_{j=0}^{n}\binom{n}{n-j}f_{n-j}\left(\frac{-1}{4\pi y}\right)^j, \quad \tau = x+iy.$$

From the above discussion it follows that $\hat{F}$ is an almost holomorphic modular form of weight $k+n$ and depth $\leq n$. we can then construct a well-defined linear map

$$M_{k+n}(\Gamma, n) \to \widehat{M}_{k+n}(\Gamma)^{(\leq n)}, \quad F^* \mapsto \widehat{F}, \tag{1.8}$$

where $\hat{F}$ is obtained from F^* as in the first part of the proof. Equivalently, we can define a map to the space of quasimodular forms

$$M_{k+n}(\Gamma, n) \to \widetilde{M}_{k+n}(\Gamma)^{(\leq n)}, \quad F^* \mapsto f_n.$$

Conversely, given a quasimodular form $g = g_0 \in \widetilde{M}_{k+n}(\Gamma)^{(\leq n)}$ or, equivalently, an almost holomorphic modular form $\widehat{G} = \sum_{i=0}^n \hat{g}_i (1/4\pi y)^i$ we can construct the vector $G := {}^t(g_0, \ldots, g_n)$, where

$$\hat{g}_i = \binom{n}{n-i} g_i, \quad i = 0, \ldots, n. \tag{1.9}$$

Is then easy to check, using the transformation property of $\widehat{G}$, that

$$G|_k\gamma = \mathrm{Sym}^n \begin{pmatrix} (c\tau+d)^{-1} & 0 \\ c & c\tau+d \end{pmatrix} G, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

Define $G^*(\tau) := L_n(\tau)\widetilde{G}(\tau)$. The above equality, together with (1.6), imples then $G^*(\tau) \in M_k(\Gamma, n)$.

we have constructed a map

$$\widehat{M}_k(\Gamma)^{(\leq n)} \to M_k(\Gamma, n), \quad \widehat{G} \mapsto G^*,$$

which is easily seen to be the inverse of the map (1.8). This gives the isomorphism in the statement. $\square$

Chapter 2

Classical uniformization and Teichmüller theory

2.1 Uniformization via differential equations

Let S be a Riemann surface, and let $\widetilde{S}$ denote its universal covering. We will consider only *hyperbolic surfaces*, i.e. Riemann surfaces S such that $\widetilde{S} \simeq \mathbb{H}$.

We can describe hyperbolic surfaces in terms of their topological invariants. Let S be obtained from a compact Riemann surface of genus g by removing n points (called *punctures*); we say that S is of *type* (g, n). Then S is hyperbolic if

$$2g - 2 + n > 0.$$

From this description we can easily see that almost every Riemann surface, except for compact elliptic curves and the sphere with at most two punctures, is hyperbolic.

The uniformization problem, as originally conceived by Poincaré and Klein for hyperbolic surfaces can be stated in the following form.

Problem 1. *Let S be a Riemann surface. Show that its universal covering $\widetilde{S}$ is conformal to the upper half-plane $\mathbb{H}$, i.e. that S is a quotient of $\mathbb{H}$ by a group $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ of real Möbius transformations.*

In practice, they wanted to find, for a given S, a biholomorphic map $\widetilde{S} \to \mathbb{H}$. Note that this map is in particular a local biholomorphism.

The study of local biholomorphisms $\widetilde{S} \to \hat{\mathbb{C}}$ is related to certain linear differential equations defined on S. In particular, every such local biholomophism is obtained by lifting to $\widetilde{S}$ a certain multivalued function constructed from the solutions of a differential equation on S. To discuss this topic in more detail, we need to recall few basic facts from the theory of differential equations.

Let $U \subset S$ be a connected open set, but not necessarily simply connected. We consider second order linear differential equations on U that can be described locally by

$$\frac{d^2v}{dx^2} + f\frac{dv}{dx} + gv = 0, \tag{2.1}$$

where x is a local parameter and $f, g\colon U \to \mathbb{C}$ are holomorphic functions.

Near every $u_0 \in U$ the differential equation (2.1) has two linearly independent holomorphic solutions. Any local solution can be analytically continued along every path in U, defining a global solution $v\colon \widetilde{U} \to \mathbb{C}$ which in general is a multivalued on U (with holomorphic branches).

Let v_1, v_2 be two linearly independent solutions near $u_0 \in U$. In Poincaré's first approach to uniformization great importance was given to the ratio

$$w := \frac{v_2}{v_1}\colon U \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C}).$$

This function, when analytically continued along every path in U, is also multivalued, but is better behaved than v_1, v_2; its lift to $\widetilde{U}$ defines a local biholomorphism

$$\widetilde{w}\colon \widetilde{U} \to \widehat{\mathbb{C}},$$

and every two branches of w are related by a Möbius transformation. Functions with this kind of multivaluedness are called $\mathbb{P}\mathrm{GL}(2,\mathbb{C})$-*multivalued*.

The key fact is that every local biholomorphism $\widetilde{U} \to \widehat{\mathbb{C}}$ arises from a second order linear ODE on U. To discuss this, we need to introduce the Schwarzian derivative.

Definition 2. *Let $U \subset S$ be an open set and x a coordinate on U. Let f be a locally injective meromorphic function on U. The* **Schwarzian derivative** *$\{f, x\}$ of f with respect to x is defined by*

$$\{f, x\} = \frac{3f''^2 - 2f'f'''}{4f'^2}, \quad f' := \frac{df}{dx}. \tag{2.2}$$

The relevance of the Schwarzian derivative in the above discussion can be understood from the following proposition.

Proposition 4. *The Schwarzian derivative has the following properties:*

1. *Let $x \mapsto \gamma(x)$ be a Möbius transformation, with $\gamma \in \mathrm{GL}(2,\mathbb{C})$. Then*

$$\{\gamma(f), x\} = \{f, x\}.$$

2. *Let $f = g \circ h$. Then*

$$\{g \circ h, x\}dx^2 = \{h, x\}dx^2 + \{g, h\}dh^2.$$

Consider again the $\mathbb{P}\mathrm{GL}(2,\mathbb{C})-$multivalued function w on U defined above. The first property of the Schwarzian derivative tells us that the function $\{w,x\}$ is single valued and holomorphic in U. This permits to complete the description of locally biholomorphic functions $\widetilde{U} \to \widehat{\mathbb{C}}$.

Proposition 5. *Any $\mathbb{P}\mathrm{GL}(2,\mathbb{C})-$multivalued local biholomorphism $w\colon U \to \mathbb{C}$ is locally the quotient of two linearly independent solutions of the following ODE*

$$\frac{d^2v}{dx^2} + \frac{1}{2}\{w,x\}v = 0. \tag{2.3}$$

where x is a local coordinate on U.

Poincaré's idea was that among the local biholomorphisms $\widetilde{w}$ there exists a global one, which would identify $\widetilde{U}$ with a simply connected domain in $\widehat{\mathbb{C}}$. The above proposition imply that the global biholomorphism, if exists, is obtained from the solution w of the nonlinear differential equation

$$\{w,x\} = g(x)$$

for some holomorphic function g on U.

It should be clear now that the important object is the ratio of two solutions of a differential equation, and not the differential equation itself. We then introduce an equivalence relation among second order ODE on U.

Definition 3. *Let E, E' be differential equations on U, and let $w, w'\colon \widetilde{U} \to \mathbb{C}$ be the functions obtained as ratio of solutions of E, E' respectively. We say that E, E' are* **projectively equivalent** *if $w' = \gamma(w)$ for some Möbius transformation γ.*

It can be proved that for every projective equivalence class there is a unique differential equation of the form (2.3), usually called the *reduced equation*. Because of this, in the rest of this section we will consider only this kind of differential equations.

Up to now we discussed only the local theory, i.e., the theory for an open set $U \subset S$. Of course our final interest is not in the universal covering of U, but in the universal covering of S. Because of this, we have to to glue the differential equations defined on open sets of S, i.e., we have to consider projective connections.

Let $\{U_i, x_i\}_{i\in I}$ be an atlas on S, with transition functions $x_i = \phi_{i,j}(x_j)$ on $U_i \cap U_j$. Recall that a *holomorphic projective connection* $\mathcal{G}$ on S is a collection of holomorphic functions $\{g_i\colon U_i \to \mathbb{C}\}_{i\in I}$ with the property that, in the intersections $U_i \cap U_j$,

$$g_i = g_j \circ \phi_{j,i} \left(\frac{d\phi_{j,i}}{dx_i}\right)^2 + \{\phi_{j,i}, x_i\}. \tag{2.4}$$

We associate to $\mathcal{G}$ a differential equation $E = E(\mathcal{G}) := \{E_i\}_{i\in I}$ on S, i.e. a family of differential equations E_i on the open sets U_i, via

$$E_i: \quad \frac{d^2v}{dx_i^2} + \frac{1}{2}g_i v = 0.$$

The properties of projective connections imply that the differential equations E_i, E_j are projectively equivalent in the intersections $U_i \cap U_j$, $i, j \in I$. This implies the following

Proposition 6. *Let S be as above, $\mathcal{G}$ a projective connection, and let $E = \{E_i\}$ be the associated differential equation on S. Denote by $w_i \colon U_i \to \mathbb{C}$ the ratio of linearly independent solutions of E_i on U_i. Then the collection $\{w_i\}_{i\in I}$ induces a local biholomorphism $w \colon \widetilde{S} \to \widehat{\mathbb{C}}$.*

Here we introduce one of the fundamental objects of the thesis.

Definition 4. *A projective connection $\mathcal{G}$ on S is called* **uniformizing** *if the induced function $w \colon \widetilde{S} \to \widehat{\mathbb{C}}$ gives a global biholomorphism $\widetilde{S} \simeq D \subset \widehat{\mathbb{C}}$. If $D = \mathbb{H}$, the connection $\mathcal{G}$ is called* **Fuchsian***.*

Poincaré in the 1880's tried to prove directly the existence of a Fuchsian uniformizing connection for a given S, but did not succeed. It existence follows indirectly from the uniformization theorem, which was proven twenty years later; up to now nobody has been able give a direct proof.

However, Poincaré could prove that if such a connection exists it is unique.

Theorem 5. *There is a unique projective connection $\mathcal{G}_F$ which induces a biholomorphism $\widetilde{w} \colon \widetilde{S} \to \mathbb{H}$.*

The Fuchsian uniformization problem is then reduced to the following:

Problem 2. *Given a hyperbolic Riemann surface S, find the Fuchsian projective connection $\mathcal{G}_F$ on S.*

Let $Q(S)$ denote the space of *regular quadratic differentials* on S. A regular quadratic differential is given locally by holomorphic functions $\{q_i \colon \ U_i \to \ \mathbb{C}\}_{i\in I}$ such that, in the intersections $U_i \cap U_j$,

$$g_i = g_j \circ \phi_{j,i} \left(\frac{d\phi_{j,i}}{dx_i} \right)^2 \tag{2.5}$$

Comparing the expressions (2.4) and (2.5) it is clear that the difference between holomorphic projective connections is given by a regular quadratic differential. It follows that the space of projective connections on S is an affine space on the vector space $Q(S)$. It is well-known that the dimension of $Q(S)$ is $3(g-1)+n$. Then a generic projective connection on S can be written as

$$\mathcal{G} = \mathcal{R} + \sum_{j=1}^{3g-3+n} \lambda_j Q_j, \tag{2.6}$$

where $\mathcal{R}$ is some projective connection and $Q_1, \dots, Q_{3g-3+n}$ is a basis of the space $Q(S)$. The elements $\lambda_1, \dots, \lambda_{3g-3+n}$ are called **accessory parameters**; they depend on the choice of the projective connection $\mathcal{R}$ and on the choice of a basis of $Q(S)$.

We remark that the projective connection $\mathcal{R}$ can be constructed directly from the Riemann surface S using, for example, some *symmetric bidifferential* on S. We will not discuss this here; the details can be found in [46]

From the uniqueness of the Fuchsian connection on S it follows that, for fixed $\mathcal{R}$ and $Q_1, \dots, Q_{3g-3+n}$ in (2.6), there is a unique choice of accessory parameters $\lambda_1, \dots, \lambda_{3g-3+n}$ such that $\mathcal{G}$ is the Fuchsian connection. We can then restate the uniformization problem in the following from.

Problem 3. *Let S be a hyperbolic Riemann surface. Given a projective connection $\mathcal{R}$ on S and a basis of the space of quadratic differentials $Q(S)$, find the unique complex numbers $\lambda_1, \dots, \lambda_{3g-3+n}$, such that the connection*

$$\mathcal{R} + \sum_{j=1}^{3g-3+n} \lambda_j Q_j$$

is the Fuchsian connection. The complex numbers $(\lambda_1, \dots, \lambda_{3g-3+n})$ are called Fuchsian parameters*.*

2.1.1 Genus zero

In this section we deal only with surfaces of type $(0, n)$, i.e. punctured spheres. These surfaces are hyperbolic if $n \geq 3$, and we assume this from now on.

Let $X := \mathbb{P}^1 \setminus \{ x_1, \dots, x_n \}$, where $x_i \in \hat{C}$ and $x_i \neq x_j$ if $i \neq j$, an hyperbolic punctured sphere. In the genus zero case we have a global coordinate t on X. This means that, to discuss the uniformization of X, instead of projective connections on X we only need to consider a second order differential equation on X.

Classically, the differential equations related to the uniformization of X were described with no explicit reference to quadratic differentials; these equations were determined using some classical analysis.

It goes roughly as follows. We can start with a generic second order linear ODE on X, and try to make it suitable for the uniformization of X. This means that we impose some conditions on the coefficients to make the ratio of solutions be a local biholomorphism. The first step is the reduction of the generic ODE, by projective equivalence, to the following equation involoving the Schwarzian derivative of some meromorphic function $w(x)$:

$$\frac{d^2v}{dx^2} + \frac{1}{2}\{w, x\}v = 0.$$

Then one studies under which conditions on $\{w, x\}$ the function $w(x)$ is a local biholomorphism.

A detailed treatment of this theory can be found, for instance, in Ford's book [18]. The final result is the following.

Theorem 6. *Let X be as above, and suppose that $x_n = \infty$. Every multivalued local biholomorphism $w\colon X \to \mathbb{C}$ satisfies the following non-linear differential equation*

$$\{w,t\} = \frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{(t-x_i)^2} + \sum_{i=1}^{n-1}\frac{m_i}{t-x_i}, \tag{2.7}$$

together with the following behaviour at ∞ :

$$\{w,t\} = \frac{1}{2t^2} + \frac{m_n}{t^3} + O\left(\frac{1}{|t^4|}\right), \quad t \mapsto \infty, \tag{2.8}$$

for certain complex numbers $m_1, \ldots, m_n$.

The complex numbers m_i, $i = 1, \ldots, n$ are the accessory parameters. The dimension of the space $Q(X)$ of quadratic differentials on X has dimension $n-3$, so this should also be the number of accessory parameters. Using (2.8) we can show in fact that the accessory parameters $m_1, \ldots, m_n$ satisfy the following three linear relations:

$$\sum_{i=1}^{n-1} m_i = 0, \quad \sum_{i=1}^{n-1} x_i m_i = 1 - \frac{n}{2}, \quad \sum_{i=1}^{n-1} x_i(1 + m_i x_i) = m_n. \tag{2.9}$$

We see in particular that only $n-3$ accessory parameters are linearly independent, as expected.

2.1.2 Examples

In this section we recognize certain classical linear ODE as equations associated to the uniformization of punctured surfaces.

Hypergeometric equation

The uniformizing equation for the thre-punctured sphere contains three accessory parameters m_1, m_2, m_3 with three linear relations (2.9). If we fix the three punctures to be $x_1 = 0, x_2 = 1, x_3 = \infty$ (we can always do it via a Möbius transformation), we can easily compute $m_1 = -1/2, m_0 = 1/2, m_3 = 3/2$. The uniformizing differential equation in this case is projectively equivalent to Gauss's *hypergeometric equation*

$$t(1-t)\frac{d^2Y}{dt^2} + (1-2t)\frac{dY}{dt} - \frac{1}{4}Y = 0.$$

As it is well-known, this differential equation is rigid, i.e. it has no accessory parameters. This is related to the fact that every three points on the sphere can be brought to $0, 1, \infty$ via a Möbius transformation, i.e. the deformation space of three-punctured spheres is a point.

Heun equation

Consider now four-punctured spheres. From (2.7) we can see that we have four accessory parameters, and three linear relations. It follows that for the surfaces of type $(0,4)$ we have a one-dimensional space of differential equations. The generic equation can be expressed in the form

$$\frac{d^2Y}{dt^2} + \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-\alpha}\Big)\frac{dY}{dt} + \frac{t-m}{t(t-1)(t-\alpha)}Y = 0,$$

which is a special form of *Heun equation*. Here m is the accessory parameter, and by m_F we denote the Fuchsian value.

For some special choice of α, the Fuchsian value is known; the corresponding four-punctured spheres are precisely the ones whose uniformizing group is a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. There exists only four conjugacy classes of finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ with four cusps and no torsion; it turns out that all these groups are congruence subgroups [39].

The uniformizing differentia equations for these values were found by the Chudnovsky brothers [15] and by Zagier [48]. We give here the full list of these four-punctured spheres and uniformizing group (up to conjugacy):

1. $\mathbb{P}^1 \smallsetminus \{\infty, 1, 0, 9\}$, uniformized by $\Gamma_1(6)$; $m_F = 3$.
2. $\mathbb{P}^1 \smallsetminus \{\infty, 1, 0, -1\}$, uniformzed by $\Gamma_1(4) \cap \Gamma(2)$; $m_F = 0$.
3. $\mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \frac{1+i\sqrt{3}}{2}\}$, uniformized by $\Gamma(3)$; $m_F = \frac{1}{2} - \frac{i\sqrt{3}}{6}$.
4. $\mathbb{P}^1 \smallsetminus \{\infty, 1, 0, -11\}$, uniformized by $\Gamma_1(5)$; $m_F = -3$.

We remark that the uniformizing differential equation associated to the last punctured sphere is the one related to Apéry's proof of the irrationality of $\zeta(2)$.

In [22] there are other examples related to punctured spheres with many automorphisms.

Lamé equation

Consider a punctured torus S with periods ω_1, ω_2. Let $\wp(z) = \wp(z;\omega_1,\omega_2)$ denote the Weierstrass elliptic function associated to the given periods. The uniformizing differential equation of S is projectively equivalent to the *Lamé equation* with index $n = 1/2$

$$\frac{d^2Y}{dz^2} + \frac{1}{4}\big(\wp(z) + \lambda\big)Y = 0.$$

Here λ is the accessory parameter. The uniformization theorem of the four-punctured sphere and of the punctured torus are related; every four-punctured sphere X_α is doubly-covered by a punctured torus S_{1,ω_α}. Using this correspondence it is possible to determine the Fuchsian value for the uniformization of S_{1,ω_α} if the one for X_α is known and vice versa.

2.2 Quasiconformal maps and quasi-Fuchsian groups

Notation. Let $f\colon D \subset \mathbb{C} \to \mathbb{C}$ be differentiable on D. We denote the partial derivatives of f by

$$f_{\bar{z}} := \frac{\partial f}{\partial \bar{z}}, \quad f_z := \frac{\partial f}{\partial z}.$$

Let $\mu\colon \mathbb{C} \to \mathbb{C}$ be a measurable function. Assume that

$$\text{ess-sup}|\mu(z)| < 1. \tag{2.10}$$

The following theorem is of fundamental importance for what follows [3].

Theorem 7 (Mapping theorem)**.** *Let μ be a measurable function as in* (2.10). *The* Beltrami differential equation

$$f_{\bar{z}} = \mu(z) f_z \tag{2.11}$$

has a solution $f\colon \mathbb{C} \to C$ which is a homeomorphisms. We can normalize such a solution by

$$f(0) = 0, \quad f(1) = 1;$$

in this case the homeomorphic solution is unique.

A homeomorphic solution f of (2.11) is called a *quasiconformal map* with *complex dilatation* μ. As follows from the uniqueness statement in the theorem, the complex dilatation characterizes the map f.

Our interest in hyperbolic Riemann surfaces will let us focus on some special complex dilatations μ. These come from measurable functions defined on $\mathbb{H}$ rather than $\mathbb{C}$ (since $\mathbb{H}$ is the universal covering of hyperbolic surfaces), and extended to measurable functions on $\mathbb{C}$.

Let $\nu\colon \mathbb{H} \to \mathbb{C}$ be a measurable function such that ess-sup$|\nu(z)| < 1$ holds almost everywhere in $\mathbb{H}$. We consider two different ways to estend ν to a function $\mu\colon \mathbb{C} \to \mathbb{C}$.

First, we can extend ν by symmetry with respect to the real line:

$$\mu(z) := \begin{cases} \nu(z) & z \in \mathbb{H} \\ \overline{\nu(\bar{z})} & z \in \mathbb{L}, \end{cases} \tag{2.12}$$

where $\mathbb{L}$ is the lower half-plane. We denote by f^ν the homeomorphic solutions of the Beltrami equation associated to this choice of μ, where f^ν is normalized such that it fixes $0, 1, \infty$. By the defintion of μ in (2.12) it follows that the restriction

$$f^\mu|_{\mathbb{H}}\colon \mathbb{H} \to \mathbb{C}$$

is a quasiconformal self-map of $\mathbb{H}$.

The second possibility is as follows

$$\mu(z) := \begin{cases} \nu(z) & z \in \mathbb{H} \\ 0 & z \in \mathbb{L}. \end{cases} \tag{2.13}$$

We denote by f_ν the normalized quasiconformal solution of the Beltrami equation with this choice of μ. The restriction of f_ν to $\mathbb{H}$ does not define a self-map of $\mathbb{H}$; however, the restriction to $\mathbb{L}$

$$f_\mu|_{\mathbb{L}}\colon \mathbb{L} \to D_\mu \subset \mathbb{C}$$

is a conformal map (since we extended ν by zero in $\mathbb{L}$). Its image D_ν is a Jordan domain, whose boundary Jordan curve is $f_\nu(\mathbb{R})$. The way we extended ν in (2.13), makes the restriction $f_\nu\big|_{\mathbb{R}}$ non smooth, and it can be proved that the image $f_\nu(\mathbb{R})$ is a Jordan curve of Hausdorff dimension ≥ 1.

There is a big difference in the behaviour of f^ν and f_ν when ν depends on parameters.

Theorem 8 (Ahlfors, Bers)**.** *Let $\nu\colon \mathbb{H} \to \mathbb{C}$ be measurable and such that* (2.10) *is satisfied on $\mathbb{H}$. Consider the two extensions of ν to measurable functions $\mu\colon \mathbb{C} \to \mathbb{C}$ as above.*

1. *(ν extended by symmetry) If $\mu = \mu(r_1, \ldots, r_n; z)$ depends analytically on real parameters $r_1, \ldots, r_n$,, the assignment*

$$(r_1, \ldots, r_n) \mapsto f^\mu(r_1, \ldots, r_n; z)$$

is real-analytic in $r_1, \ldots, r_n$ for every $z \in \mathbb{C}$.

2. *(ν extended by the zero function) If $\mu = \mu(c_1, \ldots, c_n; z)$ depends holomorphically on complex parameters $c_1, \ldots, c_n$,, the assignment*

$$(c_1, \ldots, c_n) \mapsto f_\mu(_1, \ldots, c_n; z)$$

is holomorphic in $c_1, \ldots, c_n$ for every $z \in \mathbb{C}$.

Let $\nu\colon \mathbb{H} \to \mathbb{C}$ be as above. Here we should think of $\mathbb{H}$ as the universal cover of a hyperbolic surface. Fix a Fuchsian group $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ and consider its action, via Möbius transformations, on $\mathbb{H}$. We say that ν is a *Beltrami differential* with respect to Γ if

$$\nu(z) = \nu(\gamma z)\frac{\overline{\gamma'(z)}}{\gamma' z}, \quad \text{for every } \gamma \in \Gamma. \tag{2.14}$$

The name is justified by the fact that such ν descends to a $(-1, 1)-$differential on $\mathbb{H}/\Gamma$. We denote by $B(\Gamma)$ the space of Beltrami differentials with respect to Γ.

It is natural to expect some special property from the quasiconformal maps associated to a Beltrami differential (2.14). Again, different extensions of ν to $\mathbb{C}$ lead to very different properties of f^μ, f_μ. Before to state the relevant result, we need the following definition.

A *quasi-Fuchsian group* G is a discrete subgroup of $\mathrm{SL}(2, \mathbb{C})$ which fixes a Jordan curve J in the plane. It acts discontinuously on the two Jordan regions Ω_J, Ω_J^* defined by the Jordan curve. In particular, both the quotients

$$\Omega_J/G, \quad \Omega_J^*/G$$

are finite type hyperbolic Riemann surfaces.

Proposition 7. *Let ν be a Beltrami coefficient. Consider the two extensions of ν to measurable functions $\mu\colon \mathbb{C} \to \mathbb{C}$ as above.*

1. *(ν extended by symmetry) The group*

$$f^{\mu}\Gamma(f^{\mu})^{-1} := \{f^{\mu}\gamma(f^{\mu})^{-1} \,|\, \gamma \in \Gamma\}$$

is Fuchsian, i.e. $f^{\mu}\gamma(f^{\mu})^{-1}$ is a conformal self-map of $\mathbb{H}$.

2. *(ν extended by the zero function) The group*

$$f_{\mu}\Gamma(f_{\mu})^{-1} := \{f_{\mu}\gamma(f_{\mu})^{-1} \,|\, \gamma \in \Gamma\}$$

is quasi-Fuchsian, i.e. $f_{\mu}\gamma(f_{\mu})^{-1}$ is a conformal self-map of the Jordan domain $D_{\mu} = f_{\mu}(\mathbb{L})$.

The definition of quasiconformal maps can be naturally extended to Riemann surfaces. This leads to a definition of the Teichmüller space of a hyperbolic surface S.

Definition 5. *Let S be a hyperbolic Riemann surface. Consider all pairs (S_1, f) where S_1 is a Riemann surface and $f\colon S \to S_1$ is an orientation preserving quasiconformal map. Two pairs (S_1, f_1), (S_2, f_2) are equivalent if*

$$f_2 \circ f_1^{-1} \sim h$$

where $h\colon S_1 \to S_2$ is a conformal map, and $\sim$ denotes homotopy equivalence.

The set of equivalence classes, denoted $T(S)$, is called the **Teichmüller space** *of S. If S is of type (g, n) the Teichmüller space of S is also denoted by $\mathcal{T}_{g,n}$.*

A quasiconformal F map between hyperbolic surfaces S_1, S_2 induces a quasiconfomal map f of the universal covering $\mathbb{H}$

$$\begin{tikzcd} \mathbb{H} \arrow[r, "f"] \arrow[d, "t_1"'] & \mathbb{H} \arrow[d, "t_2"] \\ S_1 \arrow[r, "F"'] & S_2 \end{tikzcd} \tag{2.15}$$

The map f has a complex dilatation μ_f which can be easily checked to be a Beltrami differential for the group Γ which uniformizes S_1. The above discussion affirms that μ_f characterizes f, and that all such quasiconformal maps f arise form elements in $B(\Gamma)$ that satisfies the condition (2.10) on their norm.

It follows that we can identify points in the Teichmüller space $T(S)$ with the elements in $B(\Gamma)$ that satisfy the norm condition (2.10).

2.3 Bers coordinates and Takhtajan-Zograf's result

In this section we introduce the Teichmüller space $T(\Gamma)$ of a Fuchsian group Γ. When Γ is of finite type, i.e., $S = \mathbb{H}/\Gamma$ is a hyperbolic surface of genus g with n punctures, we have an identification $T(\Gamma) = T(S)$ with the Teichmüller space of S previously defined.

Let $\nu\colon \mathbb{H} \to \mathbb{C}$ be a Betrami differential on Γ as defined in (2.14), with $||\mu||_\infty < 1$; we denote by $D(\Gamma)$ the space of such Beltrami differentials.

Given $\nu \in D(\Gamma)$ there exists a quasiconformal self-mapping f^ν of $\mathbb{H}$ obtained by extending ν to $\mathbb{C}$ by symmetry (2.12). If f^ν is normalized in such a way that it fixes $0, 1, \infty$ it is uniquely defined.

We saw in Proposition (7) that $\Gamma^\nu := f^\mu \Gamma (f^\mu)^{-1}$ is a Fuchsian group. This means that each $\nu \in D(\Gamma)$ induces a faithful representation

$$p_\nu\colon \Gamma \to \mathbb{PSL}(2,\mathbb{R}), \quad \gamma \mapsto f^\mu \Gamma (f^\mu)^{-1}.$$

We say that two representations p_{ν_1}, p_{ν_2} are *equivalent* if there exists $\sigma \in \mathbb{PSL}(2,\mathbb{R})$ such that $p_{\nu_2} = \sigma p_{\nu_1} \sigma^{-1}$.

The *Teichmüller space* $T(\Gamma)$ is defined to be the set of equivalence classes of representations p_ν. It has a complex structure characterized by the fact that the map $\Phi\colon D(\Gamma) \to T(\Gamma)$, which sends ν to p_ν, is holomorphic. We can be more explicit and describe how Φ gives coordinate charts on $T(\Gamma)$.

Consider the space $Q(\Gamma)$ of *holomorphic quadratic differentials* on Γ. This is the space of holomorphic fuctions $q\colon \mathbb{H} \to \mathbb{C}$ such that, for every $\gamma \in \Gamma$,

$$q(\gamma\tau)\gamma'^2(\tau) = q(\tau), \quad \tau \in \mathbb{H}$$

and such that $\Im(\tau)^2 |q(\tau)|$ is bounded on $\mathbb{H}$. There is a pairing $B(\Gamma) \times Q(\Gamma) \to \mathbb{C}$

$$(\nu, q) = \int_{\mathbb{H}/\Gamma} \mu(z) q(z)\, \frac{dxdy}{y^2}, \quad z = x + iy \in \mathbb{H}$$

which is degenerate at the subspace $N(\Gamma) \subset B(\Gamma)$ that is also the kernel of the differential $d\Phi$ of Φ at $\nu = 0$. We can realize the quotient space $B(\Gamma)/N(\Gamma)$ as a subspace of $B(\Gamma)$ with the help of the map (Bergman integral)

$$\Lambda\colon B(\Gamma) \to Q(\Gamma), \quad \Lambda(\nu)(\tau) = \frac{6}{Pi} \int_{\mathbb{H}} \frac{\overline{\nu}(z)}{(\bar{z} - \tau^4)}\, \frac{dxdy}{y^2}.$$

Its kernel coincides with $N(\Gamma)$. Moreover, if we define

$$\Lambda^*\colon Q(\Gamma) \to B(\Gamma), \quad \Lambda^*(q) = \Im(\tau)^2 \overline{q}(\tau),$$

we have $\Lambda\Lambda^* = \mathrm{Id}$ on $Q(\Gamma)$. Then we can realize $B(\Gamma)/N(\Gamma)$ as the subspace

$$\mathcal{H}(\Gamma) := \Lambda^*(Q(\Gamma)) \subset B(\Gamma);$$

the elements of this space are called *harmonic Beltrami differentials*. The above construction implies that a small neighborhood of the point $\nu = 0 \in \mathcal{H}$ is mapped injectively in $T(\Gamma)$ via Φ, and that this defines a local coordinate near $\Phi(0)$. The same construction can be carried out for the other points $\nu \in \mathcal{H}$. These coordinates on $T(\Gamma)$ are called the *Bers coordinates.*

Notice that from this construction it follows that $\mathcal{H}$ is the tangent space of $T(\Gamma)$ at $\Phi(0)$, and $Q(\Gamma)$ the cotangent space. This is important because permits to define a Kähler metric on $T(\Gamma)$, called the Weil-Petersson metric. It is defined from the Petersson inner product on $Q(\Gamma)$

$$\langle q_1, q_2 \rangle := \int_{\mathbb{H}/\Gamma} q_1(z)\overline{q}_2(z)\, \frac{dxdy}{y^2}$$

and the map Λ. For $\nu_1, \nu_2 \in \mathcal{H}(\Gamma)$ the *Weil-Petersson metric* on $T(\Gamma)$ is defined by

$$\langle \nu_1, \nu_2 \rangle_{WP} := \overline{\langle \Lambda(\nu_1), \Lambda(\nu_2)\rangle}.$$

The reason why we are interested in this metric on is a result of Takhtajan and Zograf [43],[44] that, among other things, relates the Weil-Petersson metric on $T(\Gamma)$ to the accessory parameters associated to the uniformization of $\mathbb{H}/\Gamma$. In particular, in [43] they consider the uniformization of punctured spheres. Consider the following space

$$W_n := \{(w_1, \dots, w_{n-3}) \in \mathbb{C}^{n-3} \,|\, w_i \neq 0, 1 \text{ and } w_i \neq w_j \text{ if } i \neq j\}.$$

Each point in W_n defines a n-punctured sphere $X := \mathbb{P}^1 \setminus \{w_1, \dots, w_{n-3}, 0, 1, \infty\}$. One can then consider the Fuchsian parameters $m_1, \dots m_n = m_\infty$ associated to the uniformization of X as functions on W_n.

The universal covering space of W_n is $T(\Gamma)$, where Γ is such that $\mathbb{H}/\Gamma$ is a n-punctured sphere. It follows that, due to some invariance properties of $\langle\, ,\, \rangle_{WP}$, we can project the Weil-Petersson metric from $T(\Gamma)$ onto W_n. Then Takhtajan-Zograf's result is

Theorem 9. *The accessory parameters $m_1, \dots, m_{n-3}$ are continuously differentiable on W_n and*

$$\frac{\partial m_i}{\partial \overline{w}_j} = \frac{1}{2\pi}\Big\langle \frac{\partial}{\partial w_i}, \frac{\partial}{\partial w_j} \Big\rangle_{WP}, \quad i, j = 1, \dots, n-3.$$

As a corollary one finds

$$\frac{\partial m_i}{\partial w_j} = \frac{\partial m_j}{\partial w_i}, \quad \frac{\partial m_i}{\partial \overline{w}_j} = \overline{\left(\frac{\partial m_j}{\partial \overline{w}_j}\right)}, \quad i, j = 1, \dots, n-3.$$

In Chapter 4 we will recover this corollary studying numerically the local expansion of the accessory parameter as a function on W_4.

Chapter 3

Uniformization and modular forms

3.1 The uniformizing equation from the Rankin-Cohen structure

In this section, we denote by $\Gamma \subset SL_2(\mathbb{R})$ a genus zero Fuchsian group with no torsion and with $n \geq 3$ inequivalent cusps. The quotient

$$X_\Gamma := \mathbb{H}/\Gamma$$

has a Riemann surface structure. By assumption, it is isomorphic to an n-punctured sphere. Let t be the Hauptmodul realizing the isomorphism

$$t\colon \mathbb{H}/\Gamma \xrightarrow{\sim} \mathbb{P}^1 \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \infty\} = \mathbb{C} \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}, \tag{3.1}$$

where $\alpha_i \neq \alpha_j$ if $i \neq j$.

In Chapter 1 we explained that, given a modular form g of weight k and a modular function, we can construct a linear ODE of order $k+1$ satisfied by g. In Chapter 2, we discussed in detail a special family of second order ODEs related to the uniformization of a given Riemann surface S. The uniformizing differential equation for S, i.e. the unique ODE whose solutions induce a biholomorphims between $\widetilde{S}$ and $\mathbb{H}$, has a natural interpretation in terms of modular forms. Some consequences of this fact are explored in this chapter.

The uniformizing differential equation has order two, so it would be natural to consider a modular form of weight one. However, as one can easily see from Theorem 2 in Chapter 1, the space $M_1(\Gamma)$ can have dimension zero. For example, if all the cusps of Γ are irregular, then $\dim M_1(\Gamma) = 0$.

A possible choice is to discuss these cases separately, but we have a better option; we can treat all cases at once if we consider modular forms of weight two. The space $M_2(\Gamma)$ does not see the difference between regular and irregular cusps, and for the group Γ we are considering, it has always non-zero dimension. To get a differential equation of order two from some $g \in M_2(\Gamma)$, we can use the same trick used in Proposition 3 (see [12]) and consider the ODE associated to $\sqrt{g}$. Recall that this ODE can be defined in terms of g and a Hauptmodul t only.

To carry out this construction, we choose a weight two form, denoted by f, whose zeros are concentrated in a certain cusp; in particular, f is a modular unit. As the next lemma shows, this choice is always possible.

Lemma 1. *Let Γ and t as in (3.1), and denote by c_0 the cusp of Γ where t has its unique pole. There exists $f \in M_2(\Gamma)$ with all its zeros in c_0. In particular, f has no zeros in $\mathbb{H}$.*

Proof. Let $g \in M_2(\Gamma)$ and let $\sigma \in \mathrm{SL}(2,\mathbb{R})$ be such that $\sigma c_0 = \infty$. Let

$$\left(g\big|_2\sigma^{-1}\right)(\tau) = \sum_{m\geq 0} g_m q^m$$

denote the Fourier expansion of g at c_0, where $q = e^{2\pi i\tau/h}, \tau \in \mathbb{H}$, is a local parameter. It is known that the degree of the divisor associated to any $g \in M_2(\Gamma)$ is $d = n-2$. Define a map

$$\phi\colon M_2(\Gamma) \to \mathbb{C}^d, \quad g \mapsto (g_0, g_1, \ldots, g_{d-1}).$$

This map sends a modular form of weight 2 to the vector defined by its first d Fourier coefficients at the cusp c_0. This map is of course linear.

The dimension of $M_2(\Gamma)$ is $n-1 = d+1$ by Theorem 1, so the map ϕ has a non-trivial kernel of dimension ≥ 1. Let $f \in \mathrm{Ker}(\phi)$. Such f can have at most d zeros in $\overline{\mathbb{H}}$, and they are all in c_0 by construction. □

In the following we will normalize f in such a way that its Fourier expansion at the cusp where t has a zero starts with 1. We can fix this cusp to be ∞. We also fix Γ in its conjugacy class in $\mathrm{SL}_2(\mathbb{R})$ in such a way that the width of ∞ is one. The normalization implies that the Fourier expansion of f at ∞ starts

$$f(\tau) = \sum_{m\geq 0} a_m q^m = 1 + a_1 q + \cdots . \tag{3.2}$$

Now let $f \in M_2(\Gamma)$ and t be as in Lemma 1, and consider the functions

$$ft^i, \quad i = 0, \ldots, d = n-2.$$

By construction, all these functions are holomophic in $\mathbb{H}$ and at the cusps. Moreover, they transform like a modular form of weight two, since f is of weight two and t is of weight zero. It follows that $ft^i \in M_2(\Gamma)$ for every $i = 0, \ldots n-2$. These functions are all linearly independent: for every $i = 0, \ldots, n-2$ the function ft^i has i zeros in the cusp where t is zero and $n-2-i$ zeros in c_0. Since $\dim M_2(\Gamma) = n-1$, it follows that these functions are a basis of $M_2(\Gamma)$.

This situation holds in more generality for even weights.

Corollary 1. *Let $k > 0$ be an integer, and let f and t be as above. The functions*

$$f^k t^i, \quad i = 0, \dots, k(n-2),$$

give a basis of the space $M_{2k}(\Gamma)$.

Proof. The weight $2k$ modular form f^{2k} has $k(n-2)$ zeros in a fundamental domain, all concentrated in the cusp c_0 where t has a simple pole. It follows that the functions in the statement are all holomorphic modular forms of weight $2k$. By looking at the location of the zeros, we can prove that they are all linearly independent. Using the dimension formula for $M_{2k}(\Gamma)$, we see that they are a basis. □

When the number of regular cusps of Γ is maximal (n if n is even, or $n-1$ if n is odd) a similar statement holds also for odd weights. We will discuss this later in the thesis.

Remark 1. *One may wonder why the existence of the modular form f constructed above does not give, at least when n is even, a modular form of weight one. One in fact could consider a holomorphic branch of the square root of f; this is a wll defined function on $\mathbb{H}$, and has its zeros in the cusp c_0. This function transforms like a modular form of weight 1 but, in general, with respect to a non-trivial multiplier system.*

Now we are ready to compute the linear ODE of order two associated to a square root of the modular form $f \in M_2(\Gamma)$ and to the Hauptmodul t introduced above. Recall from Chapter 1 that such a differential operator can be computed from f, t via:

$$L = L_f := \frac{d^2}{dt^2} + \frac{[f, t']_1}{2ft'^2}\frac{d}{dt} - \frac{[f, f]_2}{12f^2t'^2}. \tag{3.3}$$

The operator L defines a Fuchsian ODE with rational coefficients on the punctured Riemann surface X_Γ.

Theorem 10. *The differential operator L on associated to a square root of f and t is given by*

$$L = \frac{d}{dt}\left(P(t)\frac{d}{dt}\right) + \sum_{i=0}^{n-3} \rho_i t^i, \tag{3.4}$$

where $P(t) = t(t-\alpha_2)\cdots(t-\alpha_{n-1})$, $\rho_{n-3} = (n/2-1)^2$, and $\rho_0, \dots, \rho_{n-3} \in \mathbb{C}$ are uniquely determined by f, t.

Proof. We have to write the coefficients of L as rational functions of t. First we prove that

$$(-1)^{n-2}\left(\prod_{i=2}^{n-1} \alpha_i\right) t' = fP(t).$$

The ratio t'/f is a meromorphic modular function, so it is a rational function of t. We know the order and the location of the zeros and poles of f and t. We denote, as before, by c_0 the cusp where t has its unique pole and f its zeros. Then the modular function t'/f will have a simple zero at every cusp different from c_0, i.e. $n-1$ simple zeros (since these are the zeros of t'). It has also a unique pole of order $n-1$ at c_0, since f has $n-2$ zeros there and t' a simple pole. The rational functions of t with this zeros and poles are given by the polynomials $\kappa^{-1}P(t)$, $\kappa \in \mathbb{C}^*$. Looking at the coefficient of the $q-$expansion at ∞, we find the correct factor $\kappa = (-1)^{n-2}\prod_{i=2}^{n-1}\alpha_i$.

Next, we compute the brackets $[f, t']$, and $[f, f]_2$. The first one is very easy:

$$[f, t'] = 2ft'' - 2f't' = f(fP(t)\kappa)' - 2f't' = 2f't' + 2\kappa f^2P'(t)t' - 2f't' = 2\kappa f^2t'P'(t).$$

Dividing then by $2ft'^2 = 2\kappa f^2P(t)t'$ we finally get the rational function $P'(t)/P(t)$ as in the statement.

The computation of the bracket $[f, f]_2$ needs a little more work. By construction, we have that $[f, f,]_2$ is a cusp form of weight eight. Moreover, it has a zero of order $2n-4$ where f is zero, so it is necessarily divisible by f^2. There exists then an element $h_4 \in M_4(\Gamma)$ such that $[f, f]_2 = f^2h_4$. By Corollary 1 we know that h_4 is of the form

$$h_4 = f^2Q(t),$$

where $Q(t)$ is a polynomial in t of degree $\dim M_4(\Gamma) = 2n-3$. Since $[f, f]_2$ is a cusp, $[f, f]_2/f^2$ has a zero in every cusp different from c_0, and these zeros are simple. This means that the polynomial $Q(t)$ is divisible by $P(t)$. We have then

$$h_4 = f^2P(t)\big(\hat{\rho}_{n-3}t^{n-3} + \hat{\rho}_{n-2}t^{n-2} + \cdots + \hat{\rho}_0\big),$$

for some $\hat{\rho}_0, \ldots, \hat{\rho}_{n-3} \in \mathbb{C}$. To determine $\hat{\rho}_{n-3}$ we consider the expansion at the cusp c_0. Let q_0 denote the local parameter. We have then in this parameter

$$f = cq_0^{n-2} + \cdots, \quad t = sq_0^{-1} + s_0 + \cdots,$$

for some nonzero $c, s \in \mathbb{C}$ and $s_0 \in \mathbb{C}$. The bracket $[f, f]_2$ has the expansion

$$[f, f]_2 = 6ff'' - 9f'^2 = 3c^2(n-2)^2q_0^{2n-4} + \cdots,$$

while the expansion of h_4 is

$$h_4 = \big(cq_0^{n-2} + \cdots\big)^2\big(\hat{\rho}_{n-3}s^{n-3}q_0^{3-n} + \cdots\big)\big(s^{n-1}q_0^{1-n} + \cdots\big) = \hat{\rho}_{n-3}c^2s^{2n-4}q_0^0 + \cdots.$$

Combining all this information we get

$$\hat{\rho}_{n-3}c^2s^{2n-4} = (n-2)^2.$$

From the relation $t' = \kappa^{-1}P(t)f$ we can compute the constant κ in terms of the coefficients appearing in the expansions at c_0 :

$$\kappa = -cs^{n-2}.$$

This shows

$$\hat{\rho}_{n-3} \;=\; 3\kappa^{-2}(n-2)^2.$$

It finally follows that the ratio $[f,f]_2/(12f^2t'^2)$ is equal to

$$\frac{[f,f]_2}{12f^2t'^2} \;=\; \frac{f^4P(t)(\kappa^{-2}(n-2)^2t^{n-3}+\cdots+\hat{\rho}_0)}{12\kappa^{-2}f^4P(t)^2} \tag{3.5}$$

$$\;=\; \frac{(n/2-1)^2t^{n-3}+\rho_{n-4}t^{n-4}+\cdots+\rho_0}{P(t)}, \tag{3.6}$$

where $\rho_i = \hat{\rho}_i\kappa^2/12$. □

Definition 6. *The elements* $\rho_1,\ldots,\rho_{n-4}$ *in* (5.4) *are called the* **modular accessory parameters** *associated to the group* Γ.

The modular accessory parameters and the accessory parameters defined in Chapter 2 are different objects. The modular ones are defined from a Fuchsian group, and depend on the choice of the Hauptmodul t and the modular form f. The accessory parameters defined in Chapter 2 are defined from a Riemann surface S, and depend on the choice of a basis of quadratic differentials on S. Nevertheless, they play the same role from the point of view of differential equations, and in fact these two sets of parameters are very much related. In general we will refer to the elements $\rho_0,\ldots,\rho_{n-4}$ simply by accessory parameters.

A set of algebraic relations between the elements of the two sets of accessory parameters can be determined from the theory of differential equations. The main reason is the following important corollary.

Corollary 2. *Let* Γ *be as above. The differential equation* (5.4)*, for the correct choice of the modular accessory parameters, is a uniformizing equation for* X_Γ *in the sense of Chapter 2.*

Proof. If the modular accessory parameters are the correct ones, i.e., the ones that make true the identity

$$\frac{[f,f]_2}{12f^2t'^2} = \frac{(n/2-1)^2t^{n-3}+\rho_{n-4}t^{n-4}+\cdots+\rho_0}{P(t)} \tag{3.7}$$

in the Rankin-Cohen algebra $M_*(\Gamma)$, then (5.4) is solved by a modular form.

From Proposition 3 we know that the solutions of a differential equation satisfied by a modular form $g \in \mathrm{M}_k(\Gamma)$ are $g, \tau g, \ldots, \tau^k g$. In the case of second order ODEs then, the ratio of two solutions is $\tau g/g = \tau$, the coordinate on $\mathbb{H}$. This is precisely the property of the uniformizing equation defined in Chapter 2. □

We will call *modular Fuchsian values* the values of the modular accessory parameters which make the identity (3.7) true in the Rankin-Cohen algebra $M_*(\Gamma)$.

It follows from the above Corollary that (5.4) is projectively equivalent to the ODE associated to the Schwarzian derivative introduced in Chapter 2:

$$\frac{d^2}{dt^2}Y(t) + \left(\frac{1}{4}\sum_{i=1}^{n-1}\frac{1}{(t-\alpha_i)^2} + \frac{1}{2}\sum_{i=1}^{n-1}\frac{m_i}{(t-\alpha_i)}\right)Y(t) = 0, \tag{3.8}$$

where the accessory parameters m_i, $i = 1, \ldots, n$, $m_n = m_\infty$, satisfy the constraints in (2.9).

Equation (3.8) is in *canonical form* (or *reduced* form): it simply means that the coefficient of dY/dt is zero. It general, given a family of projectively equivalent second order equations, there is a unique one in canonical form. There exists a standard transformation which brings any second order Fuchsian equation to its canonical form. Given such an equation

$$\frac{d^2}{dx^2}Y(x) + p(x)\frac{d}{dx}Y(x) + u(x)Y(x) = 0,$$

where $p(x), u(x)$ are rational functions, the equation in canonical form is

$$\frac{d^2}{dx^2}Y(x) + \left(q - \frac{p'}{2} - \frac{p^2}{4}\right)Y(x) = 0. \tag{3.9}$$

Applying this transformation to equation (5.4), we obtain the relations between the modular accessory parameters ρ_i, $i = 1, \ldots, n-3$ and the accessory parameters m_j, $j = 1, \ldots, n-1$.

Lemma 2. *Let $\mathbb{P}^1 \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n = \infty\}$ be a $n-$punctured sphere, and let $P(t) = \prod_{j=1}^{n-1}(t-\alpha_j)$. Let $\rho_0, \ldots, \rho_{n-4}$ be the modular accessory parameters in* (5.4) *and define*

$$P_1(t) := \frac{(n/2-1)^2 t^{n-3} + \rho_{n-4}t^{n-4} + \cdots + \rho_0}{P(t)}.$$

We have the follwing relation between the accessory parameters $m_1, \ldots, m_{n-1}$ in (3.8) *and the modular ones:*

$$m_j = \mathrm{Res}_{t=\alpha_j}\left(2P_1(t) + \frac{P''(t)}{2P(t)}\right).$$

Proof. We explained above that the differential equations (5.4) and (3.8) are projectively equivalent, hence we can transform one into the other using (3.9). Explicilty, this leads to the following relation

$$\frac{1}{4}\sum_{i=j}^{n-1}\frac{1}{(t-\alpha_j)^2} + \frac{1}{2}\sum_{i=j}^{n-1}\frac{m_j}{(t-\alpha_j)} = P_1(t) - \frac{1}{2}\left(\frac{P'(t)}{P(t)}\right)' - \left(\frac{P'(t)}{2P(t)}\right)^2, \tag{3.10}$$

It is easy to verify that the right-hand side is equal to $P_1(t) - P''(t)2P(t) + P'(t)^2/4P(t)^2$, and that

$$\frac{1}{4}\sum_{j=1}^{n-1}\frac{1}{(t-\alpha_j)^2} + \frac{P''(t)}{2P(t)} - \frac{P'(t)^2}{4P(t)^2} = -\frac{P''(t)}{4P(t)}.$$

Identity (3.10) then becomes

$$\sum_{j=1}^{n-1}\frac{m_j}{(t-\alpha_j)} = 2P_1(t) + \frac{P''(t)}{2P(t)},$$

from which the statement follows. □

As an example, when $n = 4$ we have the following equalities for the punctured sphere $X = \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha^{-1}\}$:

$$m_0 = \alpha + 1 - 2\rho,\ m_1 = \frac{1-2\rho}{\alpha - 1},\ m_\alpha = \frac{\alpha(2\rho - \alpha)}{\alpha - 1},\ m_\infty = \frac{1+\alpha-2\rho}{\alpha}. \tag{3.11}$$

The above characterization of the accessory parameters as residues will be useful in the next chapter, while dealing with Teichmüller theory.

Example 1. *In Chapter 2 we gave a few examples of known accessory parameters. Here we discuss some of those examples from the point of view of Theorem 10. More details can be found in [48].*

1. *The punctured sphere $\mathbb{P}^1 \smallsetminus \{\infty, 1, 0, 9\}$ has modular Fuchsian value $\rho_F = 3$. The uniformizing group is $\Gamma_1(6)$, the Hauptmodul t is given below. The uniformizing differential equation is solved by the weight one modular form f :*

$$f(\tau) = \frac{\eta(2\tau)\eta(3\tau)^6}{\eta(\tau)^2\eta(6\tau)^3}, \quad t(\tau) = \frac{\eta(\tau)^3\eta(6\tau)^9}{\eta(2\tau)^3\eta(3\tau)^9}.$$

2. *The punctured sphere $\mathbb{P}^1 \smallsetminus \{\infty, 1, \frac{11+5\sqrt{5}}{2}, \frac{-11+5\sqrt{5}}{2}\}$ has modular Fuchsian value $\rho_F = -3$. The uniformizing group is $\Gamma_1(5)$. The uniformizing differential equation is solved by*

$$t(\tau) = q\prod_{n=1}^{\infty}(1-q)^{5\left(\frac{n}{5}\right)},$$

$$f(\tau) = 1 + \sum_{n=1}^{\infty}\sum_{d|n}\left(\frac{3-i}{2}\chi(d) + \frac{3+i}{2}\bar{\chi}(d)\right)q^n,$$

where $\chi(d)$ is the Dirichlet character of conductor 5 with $\chi(2) = i$, $\chi^2 = \left(\frac{\cdot}{5}\right)$ and $q = e^{2\pi i\tau}, \tau \in \mathbb{H}$.

The uniformizing equation associated to this surface can be used to prove the irrationality of $\zeta(2)$, following Apéry's method (see [9]).

3.2 The Rankin-Cohen structure from the uniformizing equation

In the last section we computed the uniformizing differential equation using Rankin-Cohen brackets. It is interesting to notice that the accessory parameters are related only to the second bracket of f with itself:

$$[f,f]_2 = 12f^4P(t)\big((n/2-1)^2t^{n-3} + \rho_{n-2}t^{n-2} + \cdots + \rho_0\big).$$

In particular, the ability of expressing the bracket $[f,f]_2$ in terms of f,t leads to the knowledge of the Fuchsian value of the accessory parameters.

In this section we will show that, under certain assumptions, the converse is also true. Starting with a n-punctured sphere X, we show that the knowledge of the Fuchsian values of the accessory parameters gives not only the uniformization of X, but also the complete ring of modular forms $M_*(\Gamma)$ and even its Rankin-Cohen structure. The result we will prove is the following.

Theorem 11. *Let X be a punctured sphere, and suppose that the Fuchsian values for the uniformization of X are known. Let Γ be such that $X = \mathbb{H}/\Gamma$, and assume it has at most one irregular cusp. Then, from a basis of solutions of the uniformizing differential equation, one can determine the full ring of modular forms $M_*(\Gamma)$ and the Rankin-Cohen structure on $M_*(\Gamma)$.*

3.2.1 Modular forms of rational weight

We will prove a more general result, of which the above theorem is a corollary. To this end, we introduce the concept of modular forms of rational weight. My reference for this are Ibukiyama's papers [24],[25].

Apart from the half-integral case, modular forms of rational weight are not very popular objects. In [4] it was noticed that the ring of modular forms of weight $k/5$ on $\Gamma(5)$ is freely generatd by two elements of weight $1/5$. Ibukiyama, inspired by this work, found more examples and elaborated a more general theory for principal congruence groups $\Gamma(N)$ with $N > 3$ odd. We will prove similar results for rings of rational weight modular forms on certain genus zero groups. As a special case we will recover the $\Gamma(5)$ example above.

We start with the definitions.

Definition 7. *Let $\Gamma \subset SL_2(\mathbb{R})$ be a Fuchsian group. Fix a rational number r. An* **automorphy factor of weight** *r is function $J(\gamma,\tau)\colon \Gamma\times\mathbb{H} \to \mathbb{C}$ holomorphic in τ and such that:*

(1) *$J(\gamma_1\gamma_2,\tau) = J(\gamma_1,\gamma_2\tau)J(\gamma_2,\tau)$, for every $\gamma_1,\gamma_2 \in \Gamma$;*

(2) *$|J(\gamma\tau)| = |c\tau + d|^r$, for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$; equivalently*

$$\frac{J'(\gamma,\tau)}{J(\gamma,\tau)} = \frac{rc}{c\tau + d}, \quad J'(\gamma\tau) = \frac{dJ(\gamma,\tau)}{d\tau}.$$

Let $k \geq 1$ be an integer. A holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is a **modular form of rational weight** *kr with respect to $J(\gamma,\tau)$ if*

$$f(\gamma\tau) = J(\gamma,\tau)^k f(\tau)$$

for every $\gamma \in \Gamma$, and if f is holomorphic at each cusp of Γ.

In some books [37],[27] there is an extra assumption on $J_r(\gamma,\tau)$ when $-1 \in \Gamma$, that $J_r(-1,\tau) = 1$. This will be automatic in the cases we study, where J will be constructed as $f(\gamma\tau)/f(\tau)$ for f a modular form of weight r.

For a fixed $J = J(\gamma,\tau)$ denote by $\mathrm{M}_{J^k}(\Gamma)$ the linear space of modular forms of weight kr, for every integer $k \geq 1$. The direct sum of these spaces

$$M_{J^*}(\Gamma) := \bigoplus_{k\geq 0} M_{J^k}(\Gamma)$$

is a graded ring. Our result is the following

Proposition 8. *Let Γ be a genus zero Fuchsian group with n cusps and no torsion.*

1. *If Γ has only regular cusps, there exists an automorphy factor J of weight $r = 1/(n/2-1)$. The ring $M_{J^*}(\Gamma)$ is freely generated by two elements of weight r.*

2. *If Γ has exactly one irregular cusp, there exists an automorphy factor J of weight $r = 1/(n-2)$. The ring $M_{J^*}(\Gamma)$ is freely generated by an element of weight r and an element of weight $2r$.*

Proof. Suppose first that Γ has only regular cusps. As explained in Chapter 1, this implies that n is even, so that the number $n/2-1$ is an integer.

We showed in Lemma 1 the existence of a weight two modular form with all its zeros concentrated at a cusp c_0. When Γ has only regular cusps, the same argument, together with the dimension formula for $M_1(\Gamma)$, proves that we can find $f \in M_1(\Gamma)$ with all its $n/2-1$ zeros at a given cusp c_0.

If we set $r = 1/(n/2-1)$, the above discussion proves the existence of a holomorphic function $g\colon \mathbb{H} \to \mathbb{C}$ such that g is non zero on $\mathbb{H}$, it is holomorphic at all cusps and has a simple pole at c_0.

Then if we define

$$J(\gamma,\tau) := \frac{g(\gamma,\tau)}{g(\tau)},$$

we see that $J(\gamma, \tau)$ is an automorphy factor of weight r. It is holomorphic everywhere since g has a unique zero at c_0, and, using $g^r = f$, it is easy to see that also $(1), (2)$ in (7) are satisfied. In particular, by construction $g \in M_J(\Gamma)$.

We can easily construct another element in $M_J(\Gamma)$. Let t be an Hauptmodul for Γ with its pole in the cusp c_0. Define

$$g_1 := gt.$$

This function is holomorphic on $\mathbb{H}$ and at every cusp by construction, since the pole of t cancels with the zero of g at c_0. Moreover, t being of weight zero, g_1 transforms with respect to Γ like g, so $g_1 \in M_J(\Gamma)$.

It follows by looking at the location of the zeros that g, g_1 are linearly independent. They are also algebraically independent. This is a standard fact, and holds in general for holomorphic modular forms of the same weight which are not linearly independent. We reproduce the proof for the reader, copying from Zagier's exposition [12].

Lemma 3. *Let h_1, h_2 be two linearly independent modular forms on $\mathbb{H}$ of the same weight k. Then they are algebraically independent.*

Proof. Let $P(X, Y)$ be any polynomial in $\mathbb{C}[X, Y]$ such that $P(h_1, h_2) \equiv 0$. Considering the weights we see that for every homogeneous component P_d of degree d $P_d(h_1, h_2)$ has to vanish. We have $P_d(h_1, h_2)/h_2^d = p(h_1/h_2)$ for some polynomial $p(x)$ in one variable. Since p has only finitely many zeros, we can have $P_d(h_1, h_2) \equiv 0$ only if h_1/h_2 is a constant. $\square$

From this discussion it follows that, for every $k \geq 0$, the space of homogeneous polynomials of degree k, $V_{kr} := \mathbb{C}[g, g_1]_k$ is a vector space of modular forms of weight kr. Its dimension is easy to compute:

$$\dim V_{kr}(\Gamma) = \dim_{\mathbb{C}} \mathrm{Sym}^k(\mathbb{C} \oplus \mathbb{C}) = k + 1.$$

We see in particular that, when $k = k'/r \in \mathbb{Z}$, for some $k' \in \mathbb{Z}_{\geq 0}$,

$$\dim V_{kr} = \frac{k'}{r} + 1 = k'\left(\frac{n}{2} - 1\right) + 1 = \dim M_{k'}(\Gamma), \tag{3.12}$$

where $M_{k'}(\Gamma)$ is a space of modular forms of integral weight k'.

Note that from this it follows that $M_{J^*}(\Gamma)$ is freely generated by g, g_1. Let $h \in M_{J^k}(\Gamma)$ for some $k \in \mathbb{Z}_{\geq 0}$, and suppose that h is not an polynomial combination of g, g_1. Let w be the mimimum positive integer such that hg^w has integral weight. From (3.12) it follows that hg^w is an homogeneous polynomial in g, g_1 of degree $d = k + w$:

$$hg^w = a_{d,0}g^d + a_{d-1,1}g^{d-1}g_1 + \cdots + a_{0,d}g_1^d, \quad a_i \in \mathbb{C}.$$

Since h is by definition holomorphic, it follows that $(a_{d,0}g^d + \cdots + a_{0,d}g_1^d)/g^w$ is holomorphic. This, by the location of zeros of g, g_1, is possible only if

$$h = a_{d,0}g^k + \cdots + a_{w,k}g_1^k,$$

which means that h is a polynomial combination of g, g_1 as element of $M_{J^k}(\Gamma)$.

The second statement in the Proposition is proved in a similar way, but there are some differences. First we fix $r = 1/(n-2)$. As in the previous case, we can construct an element $\hat{f} \in M_1(\Gamma)$ with all its zeros concentrated in a cusp, but this cusp $\hat{c}_0$ is necessarily the irregular one. This fact is proved using the same argument of Lemma 1 and the dimension formula for $M_1(\Gamma)$, together with the consideration that a weight one form has always a zero at the irregular cusp.

We can find, as in the regular case, a holomorphic function $\hat{g}$ such that $\hat{g}^r = \hat{f}$ and g has its unique zero in the cusp $\hat{c}_0$

Then, we can define an automorphy factor $J := \hat{g}(\gamma\tau)/\hat{g}$ of weight r as in the regular case; we have by construction $\hat{g} \in M_J(\Gamma)$.

Let $\hat{t}$ be an Hauptmodul of Γ with a pole where $\hat{g}$ has its zero. The product $\hat{g}_2 := \hat{g}^2\hat{t}^2$ is holomorphic and modular of weight $2r$, so $\hat{g}_2 \in M_{J^2}(\Gamma)$. From the fact that $\hat{g}_2$ has a zero of order two at $\hat{c}_0$, it follows that the functions $\hat{g}^2$ and $\hat{g}_2$ are linearly independent, and hence algebraically independent. It follows than that also $\hat{g}$ and $\hat{g}_2$ are algebraically independent.

For every $k \geq 0$, the space of homogeneous polynomials of degree k, $W_{k\hat{r}} := \mathbb{C}[\hat{g}, \hat{g}_2]_k$ is a vector space of modular forms of weight $k\hat{r}$. Its dimension is

$$\dim(W_{k\hat{r}}) = \begin{cases} (k+1)/2 & k \text{ odd} \\ (k+2)/2 & k \text{ even.} \end{cases}$$

In particular, if $k = k'/\hat{r}$, $k' \in \mathbb{Z}_{\geq 0}$, we have

$$\dim(W_{k\hat{r}}) = \begin{cases} ((n-2)k'+1)/2 & k' \text{ odd} \\ ((n-2)k'+2)/2 & k' \text{ even.} \end{cases}$$

These are precisely the dimensions of the spaces $M_{k'}(\Gamma)$ when k' is odd or even. As before, this implies that $M_{J^*}(\Gamma)$ is freely generated by $\hat{g}, \hat{g}_2$. $\square$

It has not been explored yet what happens if there are more than one irregular cusp. In this case, it is not known either if there exist some J for which the ring of rational weight modular forms is free.

Example 2. *Note that we find, as a special case, Bannai's example on* $\Gamma(5)$. *This group has genus zero and* 12 *regular cusps. From point one of the above Proposition, we find that* $r = 1/5$ *and that the ring* $M_{J^*}(\Gamma)$ *is freely generated by two elements of weight* $1/5$.

Up to now we only discussed the multiplicative structure of the ring of modular forms. In particular we showed how the multiplicative structure on integral weight forms can be reconstructed from the polynomial ring of rational weight forms. An analogous statament is true for the Rankin-Cohen structure.

The ring of rational weight modular forms $M_{\frac{*}{r}}(\Gamma)$ can be naturally endowed with a Rankin-Cohen structure, using the usual Rankin-Cohen brackets defined in Chapter 1. The Rankin-Cohen algebra $M_*(\Gamma)$ is canonically a sub-Rankin-Cohen algebra of $M_{J^*}(\Gamma)$. We show that, if the Fuchsian values for the uniformization of the surface $\mathbb{H}/\Gamma$ are known, we can completely describe the Rankin-Cohen structure on $M_{J^*}(\Gamma)$, hence on $M_*(\Gamma)$, in terms of its multiplicative generators.

As explained in Chapter 1, to describe the Rankin-Cohen structure we only need to know the multiplicative structure of $M_{J^*}(\Gamma)$, and, for a fixed element $0 \neq h \in M_{J^*}(\Gamma)$, its first bracket with every element of the ring, and the second bracket with itself.

The multiplicative structure of $M_{J^*}(\Gamma)$ was the subject of the previous pages. We showed that it is completely determined by two generators $g, g_1 = gt$ where

$$t\colon \mathbb{H} \to \mathbb{P}^1 \smallsetminus \{\alpha_1, \alpha_2, \ldots, \alpha_n = \infty\}$$

is an Hauptmodul given by the uniformization process.

Now let $h = g$. The first bracket of g with every other element of $M_{J^*}(\Gamma)$ is determined by the bracket $[g, g_1]$ since g, g_1 generate the whole ring and $[g, g] = 0$. It can be written only in terms of g and t in the follwing way. Let u denote, as before, the number of regular cusps of Γ. If $u = n$,

$$[g, g_1] = \frac{1}{r}(gg_1' - g'g_1) = \frac{1}{r}(g(gt)' - g'gt) = \frac{1}{\kappa r}(g^{2r+2}P(t)), \quad P(t) = \prod_{i=1}^{n-1}(t - \alpha_i),$$

where $\kappa^{-1} = (-1)^n \prod_{i=2}^{n-1} \alpha_i$ (recall that, by construction and by Theorem 10 we have $t' = g^{2r}P(t)$). If $u = n - 1$,

$$[g, g_1] = \frac{1}{r}gg_1' - \frac{2}{k}g'g_1 = \frac{2}{\kappa r}(g^{2r+2}tP(t)).$$

To express the second bracket $[g, g]_2$ in terms of g and t we need to know the Fuchsian values of the accessory parameters. First notice that, since g is such that $g^r = f \in M_1(\Gamma)$,

$$[f, f]_2 = \frac{2r^3}{r+1}g^{2r-2}[g, g]_2.$$

Then, if the uniformizing equation is known, we can read $[f, f]_2$ from it, as proven in Theorem 10:

$$[f, f]_2 = g^{6r}P(t)\big(\hat{\rho}_{n-3}t^{n-3} + \hat{\rho}_{n-2}t^{n-2} + \cdots + \hat{\rho}_0\big),$$

where $\hat{\rho}_0, \ldots, \hat{\rho}_{n-3}$ are the known Fuchsian values. It follows that

$$[g, g]_2 = \frac{r+1}{2r^3}g^{4r+2}P(t)\big(\hat{\rho}_{n-3}t^{n-3} + \hat{\rho}_{n-2}t^{n-2} + \cdots + \hat{\rho}_0\big).$$

We have proven

Theorem 12. *Let X be a punctured sphere, and suppose that the Fuchsian values for the uniformization of X are known. Let Γ be such that $X = \mathbb{H}/\Gamma$, and assume it has at most one irregular cusp; let r be as in Proposition 8. Then, from a basis of solutions of the uniformizing differential equation, one can determine the full ring of modular forms $M_{J^*}(\Gamma)$ and the Rankin-Cohen structure on $M_{J^*}(\Gamma)$.*

Theorem 11 follows from the above result considering the sub-Rankin-Cohen algebra of integral weight modular forms.

Chapter 4

Finding the Fuchsian value

In this section we present a method to compute the accessory parameter for the uniformization of a generic four-punctured sphere. It depends on the geometry of these objects, in particular on the existence of non-trivial automorphisms. However, the basic idea is quite general, and can be applied also to spheres with more then four punctures with sufficiently many automorphisms.

In the literature other ways to compute the Fuchsian value (for punctured torus) not exploiting the modularity were explored; we mention in particular the work of the Chudnovsky brothers [15], Hoffman's PhD thesis [23] and the paper by Keen, Rauch and Vasquez [30].

4.1 Preliminaries

Let $\alpha \neq 0\,,1$ be a complex number and consider the four-punctured sphere

$$X = X_\alpha = \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha^{-1}\}.$$

By the uniformization theorem there exist a Fuchsian group $\Gamma = \Gamma_\alpha \ \subset \ SL_2(\mathbb{R})$ and an unbranched holomorphic covering $t\colon \mathbb{H} \to X$ such that

$$t\colon \mathbb{H}/\Gamma \to X$$

is a biholomorphism.

By construction, the group Γ is torsion free and has four inequivalent cusps; we denote the equivalence classes of cusps by $c_1, c_2, c_3, c_4 \in \overline{\mathbb{R}}$. We will show later that we can fix $c_3 = [\infty]$, $c_4 = [0]$. The Hauptmodul t sends these classes to distinct punctures of X.

A typical fundamental domain for the action on $\mathbb{H}$ of such groups is represented in Figure 4.1.

In the following we will discuss the explicit uniformization of X by determining a specific uniformizing group and a specific Hauptmodul. Neither the group nor the

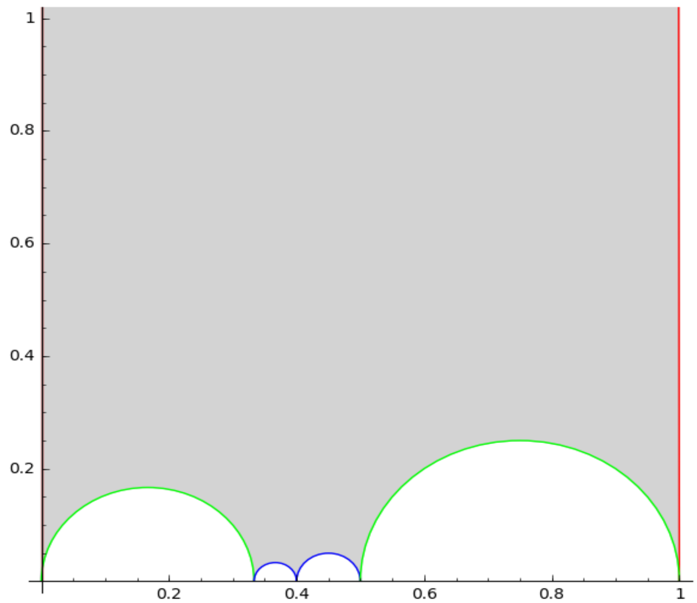

Figure 4.1: Fundamental domain of $\Gamma_1(5)$. The cusp representatives are $0, 1/3, 2/5$ and ∞. The colors indicate how to identify the boundary geodesics.

Hauptmodul are detemined uniquely form the uniformizing differential equation: the uniformizing group is found as the monodromy of the equation, so it is determined only up to conjugacy in $SL_2(\mathbb{C})$, while the Hauptmodul can be composed with any automorphism of X. It follows that we have different choices of Γ and t. To discuss exlpicit uniformizations, we should make this choice unique. For this reason, we fix the following normalization.

Lemma 4. *Let $X = \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha^{-1}\}$ be as above. We can choose the uniformizing group Γ and the Hauptmodul t such that*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$$

generates the stabilizer $\Gamma_\infty \subset \Gamma$, and the value of t at the inequivalent cusps $\infty, 0$ are

$$t(\infty) = 0, \quad t(0) = \alpha^{-1}.$$

These choices uniquely determine Γ and t.

Proof. Let $\hat{\Gamma}, \hat{t}$ be a choice of uniformization of X. We can compose $\hat{t}$ with an automorphism ϕ of X in such a way that $t := \phi \circ \hat{t}$ maps ∞ to the puncture 0 (such automorphism of X always exists, and will be discussed in detail in the next section.)

This operation amounts to determine the coordinate τ on $\mathbb{H}$ given by the uniformizing differential equation up to a linear map $\tau \mapsto a\tau + b$, where $a, b \in \mathbb{R}$ (in principle the variable on $\mathbb{H}$ is determined up to a fractional linear transformation). We see that there are two free real constants a, b. To fix this linear map is equivalent to choosing a matrix

$\sigma \in SL_2(\mathbb{R})$ which sends ∞ to $infty$ and to consider, instead of $\hat{\Gamma}$, the conjugated group

$$\Gamma := \sigma\hat{\Gamma}\sigma^{-1}, \quad \sigma = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}.$$

From this perspective, we can see that to fix a is equivalent to fixing the width of the cusp at ∞ of Γ. Since we are interested in Fourier expansions of modular forms on Γ at ∞, we choose a such that the width at ∞ is one.

Now only b has to be chosen. For different values of b, we would have different coordinates τ on $\mathbb{H}$ and, in turn, different values (among the possible ones) of t at the cusps. Because of this we can fix b by choosing the cusp where t takes a determined value, for example α^{-1}.

A natural idea is to consider a cusp for which we can determine a convenient representative, and we choose 0. That this cusp in not equivalent to ∞ can be proven by contradiction. If the cusps ∞ and 0 were equivalent, there would be an element $\gamma \in \Gamma$ such that $\gamma(\infty) = 0$. Then using γ and T it is possible to construct an elliptic element of Γ; this is a contradiction since Γ is torsion free. Finally, we can fix that t maps 0 to the puncture α^{-1}. □

In the following, while discussing the uniformization of four-punctured spheres, we will always consider Γ and t as in the above proposition.

Notice that, under this normalization, the cusp at ∞ has width one. Together with $t(\infty) = 0$, it implies that the Fourier expansion of t at ∞ reads

$$t = rq + \cdots, \qquad q = e^{2\pi i\tau}, \tau \in \mathbb{H}, \tag{4.1}$$

for some $0 \neq r \in \mathbb{C}$. This number r, which can be though of as a function of α, will play an important role in what follows. In Chapter 7 it will be discussed again as an algebraic function of α and ρ_F.

4.2 Construction of modular forms

The uniformizing differential equation for X can be written in the following form

$$P(t)\frac{d^2}{dt^2}Y(t) + P'(t)\frac{d}{dt}Y(t) + (t - \hat{\rho})Y = 0, \tag{4.2}$$

where $\hat{\rho}$ is the accessory parameter. In the following, we will consider $\hat{\rho}$ and the rescaled parameter $\rho := \alpha\hat{\rho}$ as formal parameters.

In this paragraph, we make concrete the construction of modular forms from the uniformizing equation using power series solutions. In order to do this, we explicitly construct, starting from a Frobenius basis of solutions, some new power series. These will eventually be the Fourier expansions of certain modular forms on Γ if we specialize ρ to the Fuchsian value.

Let $\{y, \hat{y}\}$ be a Frobenius basis of solutions near the regular singular point $t = 0$. As explained in the previous chapter, the nature of equation (4.2) implies that $y(t)$ is holomorphic in a certain neighborhood of $t = 0$. We normalize it by assigning the value $y(0) = 1$. The chosen basis is given by the following power series:

$$y(t) = \sum_{n\geq 0} a_n(\rho)t^n = 1 + \rho t + \frac{1}{4}(\rho^2 - 2\rho(\alpha+1) - \alpha)t^2 + \cdots,$$

$$\hat{y}(t) = \log(t)y(t) + \sum_{n\geq 0} b_n(\rho)t^n = \log(t)y(t) + (-2\rho + \alpha + 1)t + \cdots.$$

The coefficients $a_n = a_n(\rho)$, $b_n = b_n(\rho)$ of the series expansions of $y, \hat{y}$ are computed from the following linear recursions (Frobenius method):

$$\alpha n^2 a_{n-1} - ((\alpha+1)(n^2+n) + \rho)a_n + (n+1)^2 a_{n+1} = 0,$$

$$\alpha n^2 b_{n-1} - ((\alpha+1)(n^2+n) + \rho)b_n + (n+1)^2 b_{n+1} \\ + 2\alpha n a_{n-1} - (2n+1)(\alpha+1)a_n + 2(n-1)a_{n+1} = 0.$$

We explained in Chapter 2 that the relevant function for the uniformization is the ratio of the two solutions $y, \hat{y}$. However, due to the logarithmic term, using power series it is more appropriate to work with the exponential of this ratio

$$Q(t) = \exp(\hat{y}/y) = \sum_{n\geq 0} Q_n(\rho)t^n = t + (-2\rho + \alpha + 1)t^2 + \cdots. \tag{4.3}$$

Inverting this series we find the $Q-$expansion of t around $Q = 0$:

$$t(Q) = \sum_{n\geq 0} \hat{t}_n(\rho)Q^n = Q + (2\rho - \alpha - 1)Q^2 + \cdots. \tag{4.4}$$

Note that this series converges in some open disk, since the function Q is locally biholomorphic in $t = 0$. Finally substitute the above series for t into the holomorphic function $y(t)$ to get a holomorphic function in Q :

$$\hat{f}(Q) := y(t(Q)) = \sum_{n\geq 0} \hat{f}_n(\rho)Q^n = 1 + \rho Q + \frac{1}{4}\big(9\rho^2 - 2\rho(\alpha+1) - \alpha\big)Q^2 + \cdots. \tag{4.5}$$

When the accessory parameter specializes to the Fuchsian value ρ_F, the function $\hat{y}/y$ gives a coordinate on the universal covering $\mathbb{H}$ of X. From our normalization it follows that the $Q-$expansions of f and t are, respectively, expansions at ∞ of the weight one modular form $f = y \circ t$ and of the Hauptmodul t.[1]

[1] when Γ does not admit weight one modular forms, f is a root of some higher weight modular form as explained in Chapter 3.

When $\rho = \rho_F$ is the Fuchsian parameter, a comparison between tha expressions (7.4) and (4.1) gives

$$Q = rq, \quad q = e^{2\pi i\tau}, \tau \in \mathbb{H},$$

for some $0 \neq r \in \mathbb{C}$. It follows that the Q-expansions (7.4),(7.5) of t and f can be converted into q-expansions, which finally make them functions on $\mathbb{H}$:

$$f(\tau) = \sum_{n\geq 0} f_n q^n, \quad f_n := \hat{f}_n(\rho_F) r^n,$$

$$t(\tau) = \sum_{n\geq 0} t_n q^n, \quad t_n := \hat{t}_n(\rho_F) r^n.$$

These constructions have been carried out in the four-punctured sphere case, but clearly they generalize to a general n-punctured sphere (this does not apply to the normalization introduced in Proposition 4, which is tailored to the four-puncured sphere case).

4.3 The uniformizing group

The goal of this section is to write a set of parabolic generators of Γ whose coefficients are functions of cusp representatives. Recall that a set of *parabolic generators* of a torsion free Fuchsian group with n cusps is a collection of n matrices $M_1, \ldots, M_n$, with $\left(\operatorname{trace} M_i\right)^2 = 4$, which generate Γ with the relation

$$\prod_{i=1}^{n} M_i = \mathrm{Id}.$$

To be a generating set, each M_i should be the generator of the stabilizer of a cusp c_i; two cusps c_i, c_j are inequivalent if $i \neq j$.

Let $c \neq \infty$ be a regular cusp of Γ. An element in the stabilizer Γ_c of c in Γ has the form

$$S_c = \begin{pmatrix} 1 + cD_c & -c^2 D_c \\ D_c & 1 - cD_c \end{pmatrix} \tag{4.6}$$

for some positive $D_c \in \mathbb{R}$.

In the case of four-punctured spheres, we have three finite cusps $[0], [c_1], [c_2]$ and the cusp $[\infty]$. Choose $0, \infty$ as representatives of the cusp $[0], [\infty]$ respectively, and let $0 < c_1 < c_2 < 1$ be representatives of the other finite cusps. Then the generators of the stabilizers of $\Gamma_0, \Gamma_{c_1}, \Gamma_{c_2}$, are of the form

$$S_0 = \begin{pmatrix} 1 & 0 \\ D_0 & 1 \end{pmatrix}, \; S_{c_1} = \begin{pmatrix} 1 + c_1 D_1 & -c_1^2 D_1 \\ D_1 & 1 - c_1 D_1 \end{pmatrix}, \; S_{c_2} = \begin{pmatrix} 1 + c_2 D_2 & -c_2^2 D_2 \\ D_2 & 1 - c_2 D_2 \end{pmatrix} \tag{4.7}$$

for some $D_0, D_1, D_2 \in \mathbb{R}_{>0}$.

Lemma 5. *The constants D_0, D_1, D_2 can be determined in terms of the cusps representatives c_1, c_2 as follows*

$$D_0 = \frac{1}{c_1(1-c_2)}, \quad D_1 = \frac{1}{c_1(c_2-c_1)}, \quad D_2 = \frac{1}{(c_2-c_1)(1-c_2)}.$$

Proof. The above choice of cusp representatives fixes a fundamental domain $\mathcal{F}$ for the action of Γ. It is well known that a free generating set for Γ is given by the Möbius transformations which pairs the boundary geodesics of $\mathcal{F}$. Note that among these transformations there is always one which fixes one of the cusp representatives (see for instance Fig 4.1). In our case the cusp representative is c_2 since we fixed $0 < c_1 < c_2$; call S_{c_2} this matrix.

The transformation S_2 also exchange c_1 with its equivalent $c_1' > c2$. There is another transformation which exchanges c_1 with c_1', namely the one that also sends 0 to 1; call it P_{0,c_1}. The product

$$S_{c_1} := S_{c_2}^{-1} P_{0,c_1}$$

gives a transformation that fixes c_1. In the same way, the product $S_0 = P_{0,c_1}^{-1} T$ fixes 0. These matrices by construction satisfy the parabolic relation

$$S_{c_2} S_{c_1} S_0 T^{-1} = \mathrm{Id}. \tag{4.8}$$

Moreover, since we constucted them from a generating set of Γ, they also give a generating set of Γ. The matrices $S_*, * = 0, c_1, c_2$ are of the form (4.7). We can compute the real numbers $D_i\,, i = 0, 1, 2$, solving the system given by the relation (4.8):

$$\begin{pmatrix} 1 + c_2 D_2 & -c_2^2 D_2 \\ D_2 & 1 - c_2 D_2 \end{pmatrix} \begin{pmatrix} 1 + c_1 D_1 & -c_1^2 D_1 \\ D_1 & 1 - c_1 D_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D_0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The statement follows after a straightforward computation. □

We remark that using a similar argument (but not the final computation) one can construct a set of parabolic generators for a genus zero group with any number of cusps. The idea is, as above, to start with a fixed fundamental $\mathcal{F}$ domain and the unique matrix that fixes one of the cusp representatives. Using the transformations that pairs the sides of $\mathcal{F}$ we can compute step by step each matrix that fixes a certain cusp representative; using the description in (4.7) at each step one can express the new matrix in terms of the cusp representatives. Extra care must be considered when the group G contains irregular cusps (for instance if n is odd), where the representation in (4.7) slightly changes.

4.4 Geometry of four-punctured spheres

Consider the surface $X_\alpha = \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha^{-1}\}$. It admits a nontrivial group $\mathrm{Aut}(X_\alpha)$ of holomorphic automorphisms. In general the group $\mathrm{Aut}(X_\alpha)$ is a Klein four-group and is generated by any two of the involutions

$$\phi_0\colon\ t \mapsto \frac{1-\alpha t}{\alpha(1-t)}, \quad \phi_1\colon\ t \mapsto \frac{t-1}{\alpha t-1}, \quad \phi_2\colon\ t \mapsto \frac{1}{\alpha t}, \tag{4.9}$$

where $\phi_0 = \phi_1 \circ \phi_2$. For exceptional choices of α, the automorphism group of X_α is larger: if $\alpha = -1, 1/2, 2$ then $\mathrm{Aut}(X_\alpha)$ has order 8; if $\alpha = 1/2 \pm i\sqrt{3}/2$ then $\mathrm{Aut}(X_\alpha)$ has order 12.

Let $t\colon \mathbb{H}/\Gamma \to X_\alpha$ be a uniformization normalized as in Proposition 4. Every automorphism $\phi \in \mathrm{Aut}(X_\alpha)$ lifts to an involution $\tilde{\phi}$ of the universal covering $\mathbb{H}$

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\tilde{\phi}} & \mathbb{H} \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ X_\alpha & \xrightarrow{\phi} & X_\alpha \end{array}$$

Every such automorphism $\tilde{\phi}$ is given by a Möbius transformation, and, being an involution, is represented by a traceless matrix $W_\phi \in \mathrm{SL}_2(\mathbb{R})$. The following fact is standard.

Lemma 6. *For every $\phi \in \mathrm{Aut}(\Gamma_\alpha)$, the matrix W_ϕ is in the normalizer $N(\Gamma)$ of Γ.*

It is easy to see that every $\phi \in \mathrm{Aut}(X)$ swaps pairs of punctures of X; it follows that the transformation W_ϕ does the same with the corresponding cusps of Γ.

For example, consider the automorphism ϕ_0 in (4.9). The action of ϕ_0 on the punctures $\{\infty, 1, 0, \alpha^{-1}\}$ of X and of W_{ϕ_0} on the cusps $\{[\infty], [0], [c_1], [c_2]\}$ of Γ is displayed in the following table (recall that $t(\infty) = 0$, $t(0) = \alpha^{-1}$).

ϕ_0 on $\overline{X}$	W_{ϕ_0} on $\overline{\mathbb{H}}$
$0 \leftrightarrow \alpha^{-1}$	$[\infty] \leftrightarrow [0]$
$\infty \leftrightarrow 1$	$[c_2] \leftrightarrow [c_1]$

By our normalization, $T \in \Gamma$. Since every $W_\phi \in N(\Gamma)$, for every $\phi \in \mathrm{Aut}(X)$ we have

$$W_\phi T W_\phi^{-1} \in \Gamma.$$

In particular, if W_ϕ sends the cusp c to the cusp ∞, the element $S_c := W_\phi T W_\phi^{-1}$ sends the cusp c into itself, so it belongs to the stabilizer Γ_c. Actually, more is true.

Lemma 7. *The matrix $S_c = W_\phi T W_\phi^{-1}$ is the generator of Γ_c.*

Proof. This follows from the fact that T generates Γ_∞ and W_ϕ normalizes Γ. If S_c is not the generator of Γ_c, there exists $n \in \mathbb{Z}$ such that $S_c = S^n$, where S is the generator of Γ_c. But then, by construction, $W_\phi S^n W_\phi^{-1} = T$ implies

$$T^{1/n} = W_\phi S W_\phi^{-1} \in \Gamma,$$

since $S \in \Gamma$ and W_ϕ is in the normalizer $N(\Gamma)$. By hypothesis the width of ∞ is one, then $T^{1/n} \in \Gamma$ only if $n = \pm 1$. $\square$

Recall that the generator of Γ_c is of the form

$$S_c = \begin{pmatrix} 1 + cD_c & -c^2 D_c \\ D_c & 1 - cD_c \end{pmatrix} \tag{4.10}$$

for some positive $D_c \in \mathbb{R}$. From the lemma and the previous discussion it follows that, if $\phi \in \mathrm{Aut}(X)$ sends $t(c)$ to $t(\infty) = 0$,

$$W_\phi = \sqrt{D_c} \begin{pmatrix} c & \frac{-1-c^2 D_c}{D_c} \\ 1 & -c \end{pmatrix}. \tag{4.11}$$

For example, for the automorphism ϕ_0 (4.9) we have $c = [0]$ and then

$$W_{\phi_0} = \sqrt{D_0} \begin{pmatrix} 0 & -1/D_0 \\ 1 & 0 \end{pmatrix}$$

for some $D_0 \in \mathbb{R}_{>0}$. In other words, we know every lift of $\phi \in \mathrm{Aut}(X)$ up to the positive real constants D_0, D_1, D_2 in (4.7).

These lifts are the key to compute the constants D_0, D_1, D_2 and then the uniformizing group. The reason is that the fixed points of W_ϕ, $\phi \in \mathrm{Aut}(X)$ naturally gives cusp representatives of Γ. Being involutions, the transformations W_ϕ have only one fixed point on $\mathbb{H}$ each. If W_ϕ is related to the cusp c in the sense of (4.11), its fixed point on $\mathbb{H}$ is given by

$$\tau_\phi = \hat{c} + i/\sqrt{D_c},$$

where $\hat{c} \in \mathbb{R}$ is a well-defined cusp representatives of c. It is well defined since the matrix defining W_ϕ is defined by the lifting procedure.

For instance, in the case of the automorphism ϕ_0 (4.9) the cusp is $c = [0]$. The action of W_{ϕ_0} is given by

$$\tau \mapsto \frac{-1}{D_0 \tau},$$

and its fixed point on $\mathbb{H}$ is $\tau_{\phi_0} = i/\sqrt{D_0}$. It follows that the representative $\hat{c}_0$ of the cusp $[0]$ is $\hat{c}_0 = 0$, in accordance with our choice in Lemma 5.

4.5 Computation of the cusps representatives

Let τ_ϕ be the fixed point in $\mathbb{H}$ of the lift W_ϕ. By construction there is an automorphism ϕ of X_α such that

$$t(\tau_\phi) = t(W_\phi(\tau_\phi)) = \phi(t(\tau_\phi)),$$

which means $t(\tau_\phi)$ is a fixed point of the automorphism $\phi \in \mathrm{Aut}(X_\alpha)$.

The fixed points of ϕ are in general determined by a quadratic equation: for instance, the fixed points of ϕ_0 are the solutions of

$$\alpha t^2 - 2\alpha t + 1 = 0. \tag{4.12}$$

The image of $t(\tau_{\phi_0})$ is one of the two roots, and this it is completely determined by the normalization of t. The correct root is chosed by studying the mapping properties of t. We show how it works in the case $|\alpha^{-1}| < 1$ and $\Re(\alpha^{-1}), \Im(\alpha^{-1}) > 0$. Using Möbius transformations and complex conjugation, one can always reduce to this case.

Consider the involution ϕ_0, whose fixed points are determined by the equation above. We know that the fixed point of its involution on $\mathbb{H}$ is $\tau_{\phi_0} = i/\sqrt{D_0}$. This means that it lies on the imaginary axis, so its image on X_α via t should belong to the curve, determined by t, which joines the punctures $0, \alpha^{-1}$. Looking at the two roots of (4.12) and considering the constraints on α^{-1} it follows that

$$t(\tau_{\phi_0}) = z_0 = 1 - \frac{\sqrt{\alpha(\alpha-1)}}{\alpha}.$$

Now look at the involution $\phi_1 \in \mathrm{Aut}(X_\alpha)$ defined in (4.9). The fixed points of this involutions are $\alpha^{-1} \pm \left(\sqrt{1-\alpha}\right)/\alpha$. Since $|\alpha| > 1$, none of these roots is real, and one lies above the real axis and the other below. The fundamental domain for Γ that we considered lies at the left of the boundary geodesic between $\tau = i\infty$ and $\tau = 0$. This implies that the image, via t, of the fundamental domain lies above the curve, determined by t, which joins the cusps $0, \alpha^{-1}$. Then the root we have to choose is the one with positive imaginary part. If $\tau_{\phi_1} = \hat{c}_{\phi_1} + 1/\sqrt{D_1}$ is the fixed point on $\mathbb{H}$ of the lift of ϕ_1, we have

$$t(\tau_{\phi_1}) = z_1 = \frac{1}{\alpha} + \frac{\sqrt{1-\alpha}}{\alpha}.$$

Similar considerations apply to the choice of the fixed points z_2 of the third non-trivial involution of X_α.

Recall that Q denotes the local inverse function to t. We know that $Q = re^{2\pi i\tau}$ in the uniformizing case. Call Q_j the image of z_j via Q. Then, in the uniformizing case, Q_j will have the form

$$Q_j = re^{2\pi i \tau_{\phi_j}}, \quad j = 0, 1, 2,$$

where τ_{ϕ_j} are the fixed points in $\mathbb{H}$ of the lifts of the involutions $\phi_j \in \mathrm{Aut}(X_\alpha)$. Then we see that to compute the cusp representatives $\hat{c}_{\phi_1}, \hat{c}_{\phi_2}$ it is enough to consider the real part of

$$\log(Q_j/Q_0)/(2\pi i) = \tau_j - \tau_0 = \hat{c}_{\phi_j} + i(1/\sqrt{D_j} - 1/\sqrt{D_0}), \quad j = 1, 2.$$

Then, using Lemma 5, from these representative of the cusps we can compute D_0, D_1, D_2 and the generators of the group Γ. Moreover, we can compute the r appearing in the relation $Q = re^{2\pi i\tau}$ simply by

$$r = \frac{Q_0}{\exp(-2\pi/\sqrt{D_0})}. \tag{4.13}$$

4.6 Determination of the Fuchsian value

In the last two sections, to set up the correspondence between cusps representatives $\hat{c}_i$ and fixed points z_i of $\phi_i \in \mathrm{Aut}(X)$ we assumed we already have an Hauptmodul $t\colon \mathbb{H} \to X$. In general, if we start from the differential equation (4.2), we do not have the Hauptmodul t, and everything depends on the accessory parameter ρ.

However, the constructions of the previous sections still make sense even if ρ is a formal parameter, in the sense that we can compute everything (the numbers $\hat{c}_i$ and the matrices defined from these in Lemma 5) as functions of ρ. The matrices $S_{c_i}(\rho)$, $i = 1, 2, 3$, (4.7) that we obtain are functions of ρ, as the group $\Gamma(\rho)$ they generate.

Classical uniformization theory affirms that there exists a unique value ρ_F of ρ which makes the group $\Gamma(\rho)$ a Fuchsian group such that the function $t = t(\rho, Q)|_{\rho=\rho_F}$ in (7.4), is an Hauptmodul $t\colon \mathbb{H}/\Gamma \to X$.

Corollary 2 gives an equivalent statement: the Fuchsian value ρ_F is the unique value that makes the function $f(\rho, Q)$ defined in (7.5) a weight one modular form with respect to the group $\Gamma(\rho)$. This characterization of ρ is the one that we exploit now. If, in analogy with (4.13) we set

$$r(\rho) := Q(\rho, z_0)/\exp(-2\pi/\sqrt{D_0(\rho)}),$$

we can define, for every fixed $\tau^* \in \mathbb{H}$, new functions of ρ by

$$F_j(\rho) := f\big(\rho, r(\rho)e^{2\pi i S_{c_j}(\rho)(\tau^*)}\big) - f\big(\rho, r(\rho)e^{2\pi i\tau^*}\big) J(S_{c_j}(\rho), \tau^*) \quad j = 1, 2, 3,$$

where

$$S_{c_j}(\rho)(\tau^*) := \frac{s_{j,1}(\rho)\tau^* + s_{j,2}(\rho)}{s_{j,3}\tau^* + s_{j,4}}, \quad S_{c_j}(\rho) = \begin{pmatrix} s_{j,1}(\rho) & s_{j,2}(\rho) \\ s_{j,3}(\rho) & s_{j,4}(\rho) \end{pmatrix}$$

and $J, (S_{c_j}(\rho), \tau^*) := (s_{j,3}(\rho)\tau^* + s_{j,4}(\rho))$.

Then Corollary 2 is equivalent to the following statement: the Fuchsian value ρ_F is the unique zero of the system of equations $F_i(\rho)$.

We can then compute numerically the Fuchsian value of the accessory parameter computing the unique number ρ_F such that

$$F_j(\rho_f) = 0, \quad j = 1, 2, 3.$$

For a given $\alpha \in \mathbb{C}$, $\alpha \neq 0, 1$, we can in fact write the corresponding differential equation (5.4), compute the functions introduced above as functions of ρ and use Newton's algorithm to find numerically the common zero of the functions F_j, $j = 1, 2, 3$.

The output gives: the Fuchsian value to desired precision, the generators of the uniformizing group Γ and a basis of the ring of modular forms on Γ.

The Fuchsian value of four-punctured spheres uniformized by congruence groups in Example 1 can be recovered with this method. A different instance of the use of this algorithm is given in the example below.

4.7 Example: local expansion of the Fuchsian parameter function

As in Chapter 2, consider the Fuchsian accessory parameter as a function on the space $W_n = \{(w_1, \dots, w_{n-3}) \mid w_i \neq w_j \text{ if } i \neq j,\ w_i \neq 0, 1\}$:

$$\rho\colon W_n \to \mathbb{C}^{n-3}, \quad w = (w_1, \dots, w_{n-3}) \mapsto \rho(w) = (\rho_1, \dots, \rho_{n-3}),$$

where $\rho_1, \dots, \rho_{n-3}$ are the modular Fuchsian parameters for the surface

$$X = \mathbb{P}^1 \smallsetminus \{w_1, w_2, \dots, w_{n-3}, 0, 1, \infty\}.$$

We have the following result by Kra [31]

Theorem 13. *The map ρ is real analytic, but it is not complex analytic.*

This means that if z is a local parameter on W_n, then the function ρ has a local expansion around every point $z_0 \in W_n$ of the form

$$\rho(z_0 + z) = \sum_{j,k \geq 0} a_{j,k} z^i \bar{z}^j, \quad a_{j,k} \in \mathbb{C}. \tag{4.14}$$

In the following we consider the case $n = 4$ and exploit the algorithm presented in the previous section to compute the local expansion of the (modular) Fuchsian parameter function. In this case it is simply a function

$$\rho\colon \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}, \quad z \mapsto \rho(z).$$

We expand the function ρ around the point $1/2$, corresponding to the four-punctured sphere $\mathbb{P}^1 \setminus \{\infty, 1, 0, 2\}$. We choose this point since we know exactly that $\rho(1/2) = 1$.

First a few words about how we obtained the expansion. We implemented in PARI the algorithm presented before. We then computed many values of the function ρ near the point $z_0 = 1/2$ along the lines $L_n : \Im(z) = \Re(z)/n - 1/(2n)$, for different values of $n \in \mathbb{N}$. The expansion in this case depends only on one real variable x, since $z \in L_n$ if $z = 1/2 + x(1 + i/n)$ and

$$\rho(1/2 + z) = \sum_{j,k \geq 0} a_{j,k} (1 + i/n)^j (1 - i/n)^k x^{k+j},$$

Here the a_{jk} are real numbers, because of the relation $\rho(\bar{z}) = \rho(z)$, (see Chapter 2). Then we had to compute enough expansions along different L_n (the number of expansions depends on the number of a_{ij} one wants to compute) and solve some linear systems to determine numerically the coefficients.

We present the coefficients of the expansion (4.14) we obtained with this method. The constant term a_{00} equals 1 since it is simply the value $\rho(1/2)$ The other coefficients are given in the following table.

$a_{1,0} = -1.231129697228372059$	$a_{0,1} = 0.063875489913862273$	
$a_{2,0} = 2.46225939445674411$	$a_{1,1} = -0.127750979827724546$	$a_{0,2} = 0$
$a_{3,0} = -4.823691858558769882$	$a_{2,1} = 0.189062079397880349$	
$a_{1,2} = 0.011749087780820557$	$a_{0,3} = 0.063020693132626783$	
$a_{4,0} = 9.64738371711753976525587926$	$a_{3,1} = -0.37812415879576069843280002$	
$a_{2,2} = -0.02349817556164111406833394$	$a_{1,3} = -0.12604138626525356614442666$	$a_{0,4} = 0$
$a_{5,0} = -19.09466584514482511$	$a_{4,1} = 0.667376927634543472$	
$a_{3,2} = 0.046637902612272536$	$a_{2,3} = 0.188862509928182591$	
$a_{1,4} = 0.023318951306136268$	$a_{0,5} = 0.133475385526908694$	

We immediatly notice the following relation among the coefficients:

$$a_{2,0} = -2a_{1,0}, \quad a_{1,1} = -2a_{0,1}, \tag{4.15}$$

$$a_{0,2} = 0 = a_{0,4}, \tag{4.16}$$

$$a_{2,1} = 3a_{0,3}, \tag{4.17}$$

$$a_{3,2} = 2a_{1,4}, \quad a_{4,1} = 5a_{0,5}. \tag{4.18}$$

Unfortunately, these do not lead to new discoveries on the accessory parameter function; they can be explained using the symmetry of ρ near 1/2 and Takhtajan-Zograf's result.

The point $z_0 = 1/2$ is the fixed point of the involution $z \mapsto 1 - z$. It is known [30] that the following identity holds

$$\rho(1-z) = \frac{z\rho(z) - 1}{z - 1}.$$

It follows that, near the point $z_0 = 1/2$, one has

$$(z - 1/2)\rho(1/2 - z) = (1/2 + z)\rho(1/2 + z) - 1,$$

which gives

$$\sum_{i,j\geq 0} a_{i,j}[1 + (-1)^{i+j}]z^i\overline{z}^j + 2\sum_{i,j\geq 0} a_{i,j}[1 - (-1)^{i+j}]z^{i+1}\overline{z}^j - 2 = 0. \tag{4.19}$$

The above relation implies that

$$\begin{aligned} a_{0,2n} &= 0 \quad \text{if } n \geq 1, \\ a_{i+1,j} &= -2a_{i,j} \quad \text{if } i + j \text{ is odd.} \end{aligned}$$

This explain why $a_{0,2} = a_{0,4} = 0$.

The Takhtajan-Zograf result in the four-punctured case implies that

$$\frac{\partial m_\alpha}{\partial \bar{z}} = \overline{\left(\frac{\partial m_\alpha}{\partial \bar{z}}\right)}, \tag{4.20}$$

where m_α is the Fuchsian parameter function associated to the puncture $\alpha \neq 0, 1, \infty$. The Fuchsian parameter function we are considering is not m_α, but the modular one. It is related to m_α via the simple formula (3.11)

$$m_\alpha = \frac{2\alpha\rho - 1}{\alpha(1-\alpha)}.$$

In the prevoious computation around $z_0 = 1/2$ we had $\alpha = 1/2 + z$. This, together with (4.20) implies that $\rho(z, \overline{z})$ satisfies the following differential equation

$$(1-2z)\overline{\left(\frac{\partial \rho}{\partial \overline{z}}\right)} = (1-2\overline{z})\frac{\partial \rho}{\partial \overline{z}}.$$

This implies the following relations between the coefficients of the local expansion of ρ :

$$(j+1)a_{i,j+1} - 2ja_{i,j} = (i+1)a_{j,i+1} - 2ia_{j,i}, \quad i, j \geq 0.$$

It is easy to check that the relations (4.15) come from this one and from (4.19). For instance, from the one above for $(i, j) = (0, 1)$ we get

$$2a_{0,2} - 2a_{0,1} = a_{1,1}.$$

This, together with $a_{0,2} = 0$ gives the first equivalence in (4.15).

Chapter 5

Deformation of Fuchsian parameters

Let Γ be a genus zero torsion free group normalized to have width 1 at ∞. Let X be a punctured sphere such that $X \simeq \mathbb{H}/\Gamma$.

In the previous chapter we introduced a function of two variables

$$\hat{f}(\rho, t) = \sum_{m=0}^{\infty} F_n(\rho) Q(\rho, t)^n,$$

where $F_n(\rho)$ is a polynomial in ρ of degree $n-1$. This function was computed starting from the Riemann surface X. We saw that by specializing ρ to a value ρ_F called the Fuchsian value, we obtain a modular form (or a square root of a modular form) for the uniformizing group Γ. More precisely, there exists a non-zero complex number r such that

$$Q(\rho, t)\big|_{\rho=\rho_F} = Q(\rho_F, t) = rq, \quad q = e^{2\pi i \tau}, \tau \in \mathbb{H}.$$

If we set

$$f_m := r^m F_M(\rho_F),$$

the modular form is given by

$$f(\tau) := \hat{F}(\rho, t)\big|_{\rho=\rho_F} = \sum_{m=0}^{\infty} f_m q^m.$$

In this chapter, we are interested in the Taylor expansion of $\hat{f}$ with respect to ρ at the Fuchsian value ρ_F :

$$\hat{f}(\rho, \tau) = \sum_{m \geq 0}^{\infty} \hat{f}_m(\tau)(\rho - \rho_F)^m, \quad \hat{f}_m(\tau) := \frac{1}{m!} \frac{\partial^m \hat{f}(\rho, t)}{\partial \rho^m}|_{\rho=\rho_F}.$$

Notice that the function $\hat{f}_n$ are defined on $\mathbb{H}$ since we specialized to the Fuchsian value, so the coefficients of the Taylor expansion of $\hat{f}(\rho, t)$ at ρ_F are functions of $\tau \in \mathbb{H}$.

From uniformization theory and the above discussion we have $\hat{f}_0(\tau) = f(\tau)$, but we have no information about the higher derivatives.

To get started we compute the first derivative $\hat{f}_1(\tau)$ numerically in a concrete case. Recall that the surface $X = \mathbb{P}^1 \setminus \{\infty, 1, 0, 1/9\}$, in the normalization of Chapter 3, is uniformized as $X \simeq \mathbb{H}/\Gamma_1(6)$ with Hauptmodul

$$t(\tau) = \frac{\eta(\tau)^3\eta(6\tau)^9}{\eta(2\tau)^3\eta(3\tau)^9} = q - 4q^2 + 10q^3 - 20q^4 + \cdots \in \mathbb{Z}[q].$$

The Fuchsian value for X is $\rho_F = 1/3$ and we have $r = 1$ in (**??**). The modular form $f \in M_1(\Gamma_1(6))$ obtained as solution of the uniformizing equation has a description as an eta product:

$$f(\tau) = \frac{\eta(2\tau)\eta(3\tau)^6}{\eta(\tau)^2\eta(6\tau)^3} = 1 + 3q + 3q^2 + 3q^3 + 3q^4 + \cdots \in \mathbb{Z}[q].$$

We define $\hat{f}(\rho, t)$ as in the general construction above (**??**). The first derivative is

$$\begin{aligned}\hat{f}_1(\tau) = \frac{\partial \hat{f}(\rho,t)}{\partial \rho}|_{\rho=\rho_F} = 9q + \frac{45}{/}2q^2 + \frac{21}{2}q^3 + \frac{27}{4}q^4 + \frac{1341}{100}q^5 - \frac{777}{100}q^6 + \\ + \frac{160677}{4900}q^7 + \frac{473229}{9800}q^8 - \frac{90271}{9800}q^9 - \frac{34509}{9800}q^{10} + \\ \frac{16670883}{1185800}q^{11} + \frac{327249}{24200}q^{12} + \cdots .\end{aligned}$$

The Fourier expansion of $\hat{f}_1$ is no longer in $\mathbb{Z}[q]$, and the denominators seem to grow fast. Let us look more closely at the above computation. By definition

$$\frac{\partial \hat{f}(\rho,t)}{\partial \rho} = \sum_{n\geq 0} \frac{\partial}{\partial \rho}\Big(F_n(\rho)Q(\rho,t)^n\Big) \tag{5.1}$$

$$= \sum_{n\geq 0} \frac{\partial F_n(\rho)}{\partial \rho}Q(\rho,t)^n + \frac{1}{Q(\rho,t)}\frac{\partial Q(\rho,t)}{\partial \rho}\sum_{n\geq 0} nF_n(\rho)Q(\rho,t)^n. \tag{5.2}$$

We study the last two summands separately. Define

$$\hat{f}'(\rho,t) = D_Q f := Q(\rho,t)\frac{\partial \hat{f}(\rho,t)}{\partial Q(\rho,t)}.$$

When $\rho = \rho_F$ this becomes the usual derivation in $\mathbb{H}$ given by $\mathrm{d}/\mathrm{d}\tau = 2\pi q(\mathrm{d}/\mathrm{d}q)$. Then

$$\frac{1}{Q(\rho,t)}\frac{\partial Q(\rho,t)}{\partial \rho}\sum_{n\geq 0} nF_n(\rho)Q(\rho,t)^n = \frac{1}{Q(\rho,t)}\frac{\partial Q(\rho,t)}{\partial \rho}f'(\rho,t).$$

For $\rho=\rho_F$ we have

$$\frac{1}{Q(\rho_F,t)}\frac{\partial Q(\rho,t)}{\partial\rho}|_{\rho=\rho_F}=8q+\frac{9}{2}q^2+2q^3-\frac{9}{8}q^4-\frac{108}{125}q^5-\frac{1}{2}q^6+\frac{288}{343}q^7+\frac{9}{32}q^8+\cdots.$$

This expansion has much simpler denominators. In particular, if we multiply the coefficient of q^n by n^3 we get an expansion in $\mathbb{Z}[q]$. This operation correspond to applying the differential operator D_Q three times to the function $\frac{1}{Q(\rho_F,t)}\frac{\partial Q(\rho,t)}{\partial\rho}$ and then substitute $\rho=\rho_F$.

Comparing the $q-$expansions it turns out that

$$D_Q^3\left(\frac{1}{Q(\rho_F,t)}\frac{\partial Q(\rho,t)}{\partial\rho}\right)\Big|_{\rho=\rho_F}=\frac{f(\tau)^2}{2\pi i}\frac{\mathrm{d}t(\tau)}{\mathrm{d}\tau}$$

is a modular form of weight four, specifically a cusp form. In Chapter 3 we denoted this form by h and we do the same here. We denote by $\widetilde{h}$ its Eichler integral. Then

$$\left(\frac{1}{Q(\rho,t)}\frac{\partial Q(\rho,t)}{\partial\rho}\sum_{n\geq 0}nF_n(\rho)Q(\rho,t)^n\right)\Big|_{\rho=\rho_F}=2f'\widetilde{h}.$$

Similarly, we can analyze the summand

$$\left(\sum_{n\geq 0}\frac{\partial F_n(\rho)}{\partial\rho}Q(\rho,t)^n\right)\Big|_{\rho=\rho_F}=9q+\frac{153}{2}q^2+105q^3+\frac{543}{4}q^4+\frac{36057}{200}q^5-\frac{17607}{250}q^6+\cdots.$$

The final reult is

$$\hat{f}_1=\frac{\partial\hat{f}(\rho,t)}{\partial\rho}\Big|_{\rho=\rho_F}=f(\tau)\widetilde{h}'(\tau)+2f'(\tau)\widetilde{h}(\tau).$$

In this chapter we will generalize and prove this kind of observations. A natural connection with the deformation theory of Riemann surfaces will also be explained. The results of this chapter are also the main motivation for the theory developed in Chapter 5, and constitute an important example of the larger class of functions that will be defined there.

5.1 Derivation on differential equations

Let $X=\mathbb{P}^1\smallsetminus\{\alpha_1,\ldots,\alpha_{n-2},\alpha_{n-1}=0,\alpha_n=\infty\}$ be a $n-$punctured sphere. Define

$$P(t):=\prod_{j=1}^{n-1}(t-\alpha_j),\quad P_1(\rho,t):=\sum_{i=0}^{n-3}\rho_i t^i,\tag{5.3}$$

where $\rho_{n-3} = (n/2-1)^2$ and $\rho := (\rho_0, \ldots, \rho_{n-4})$ is a vector of complex parameters, the accessory parameters. The linear differential operator L_n

$$L := \frac{d}{dt}\Big(P(t)\frac{d}{dt}\Big) + P_1(\rho, t) \tag{5.4}$$

is defined on X. As explained in the previous chapters, it is related to the uniformization theory of X.

Consider a Frobenius basis of solutions of $L(Y) = 0$ near the regular singular point $t = 0$:

$$y(\rho, t) = 1 + \sum_{m=1}^{\infty} a_m(\rho)t^m, \quad \hat{y}(\rho, t) = \log(t)y(\rho, t) + \sum_{m=1}^{\infty} b_m(\rho)t^m. \tag{5.5}$$

Here $a_m(\rho), b_m(\rho)$ are polynomials of degree $m-1$ in $\rho_0, \ldots, \rho_{n-4}$ with coefficients in $\mathbb{Q}[\alpha_1, \ldots, \alpha_{n-3}]$. In this section we compute the derivative

$$\partial_{\rho_i} u(\rho, t) := \frac{\partial u(\rho, t)}{\partial \rho_i} \quad i = 1, \ldots, n-4,$$

for every solution $u(\rho, t) = ay(\rho, t) + b\hat{y}(\rho, t)$, $a, b \in \mathbb{C}$ of $L(u) = 0$.

Lemma 8. *For every $i = 0, \ldots, n-4$, the function $\partial_{\rho_i} y(\rho, t)$ satisfies the differential equation*

$$L\Big(\partial_{\rho_i} y(\rho, t)\Big) = t^i y(\rho, t),$$

where L denotes the differential operator in (5.4).

Proof. Let $y(\rho, x)$ be as in (5.7). It is a solution of L if and only if its coefficients $a_m(\rho)$ satisfy a linear recurrence relation

$$a_{m+1}(\rho)p_{-1}(\rho, m) = a_m(\rho)p_0(\rho, m) + \cdots + a_{m-n+3}(\rho)p_{n-3}(\rho, m), \quad m \geq 0,$$

where $a_m = 0$ if $m < 0$ and $p_1(\rho, m), \ldots, p_{n-3}(\rho, m)$ are polynomials of the form

$$p_j(\rho, m) = \rho_j + q_j(m), \quad j = -1, \ldots, n-3,$$

where q_j is a quadratic polynomial which depends only on m (and on the singularities of the differential equation), $\rho_0, \ldots, \rho_{n-3}$ are the accessory parameters and $\rho_{-1} = 0$.

Then if we differentiate $a_{m+1}(\rho)$ with respect to ρ_i, $(i = 0, \ldots, n-4)$ we get

$$p_{-1}(m)\frac{\partial a_{m+1}(\rho)}{\partial \rho_i} = \frac{\partial}{\partial \rho_i}\sum_{j=0}^{n-3} p_j(m, \rho)a_{m-j}(\rho) = \sum_{j=0}^{n-3} p_j(m, \rho)\frac{\partial a_{m-j}(\rho)}{\partial \rho_i} + a_{m-i}(\rho).$$

The statement

$$L\Big(\sum_{m\geq 0}\frac{\partial a_{m+1}}{\partial \rho_i}(\rho)x^{m+1}\Big) = g(x) = \sum_{m\geq 0} g_m x^m$$

is equivalent to

$$\sum_{j=0}^{n-3} p_j(m) \frac{\partial a_{m-j}(\rho)}{\partial \rho_i} = g_m.$$

But we know from the above computation that $g_m = a_{m-i}(\rho)$, so we finally find

$$L\left(\sum_{m\geq 0} \frac{\partial a_{m+1}(\rho)}{\partial \rho_i} x^{m+1}\right) = \sum_{m\geq 0} a_{m-i}(\rho) x^m = x^i y(\rho, x).$$

□

Lemma 9. *Let $P(t), P_1(\rho, t)$ be as in* (5.3)*, and let $u(\rho, t)$ be any solution of $L_n(Y) = 0$ near $t = 0$. For every $i \geq 0$, the function $Y_i(\rho, t) := t^i u(\rho, t)$ satisfies the following second order Fuchsian differential equation*

$$L_i(Y_i) := t^2 P(t) \frac{\mathrm{d}^2 Y_i}{\mathrm{d}t^2} + t\big(tP'(t) - 2iP(t)\big)\frac{\mathrm{d}Y_i}{\mathrm{d}t} + \big(i(i+1)P(t) - itP'(t) + t^2 P_1(\rho, t)\big)Y_i = 0.$$

Proof. Fix a basis $y(\rho, t), \hat{y}(\rho, t)$ of solutions of $L(Y) = 0$. The function t^i satisfies the first order ODE

$$\frac{\mathrm{d}t^i}{\mathrm{d}t} - it^{i-1} = 0. \tag{5.6}$$

Let V be the vector space spanned by $t^i y(\rho, t), t^i \hat{y}(\rho, t)$ over $\mathbb{C}$. Is it well-known that V is the solution space of some linear differential equation. The coefficients of this differential equaton can be easily computed via the following determinant

$$\det \begin{pmatrix} v & v' & v'' \\ t^i y & (t^i y)' & (t^i y)'' \\ t^i \hat{y} & (t^i \hat{y})' & (t^i \hat{y})'' \end{pmatrix} = 0$$

where v is a generic element in V and $' := \mathrm{d}/\mathrm{d}t$.

Using the relations

$$P(t) = y(\rho, t)\hat{y}'(\rho, t) - y'(\rho, t)\hat{y}(\rho, t), \quad P_1(\rho, t) = y'(\rho, t)\hat{y}''(\rho, t) - y''(\rho, t)\hat{y}'(\rho, t),$$

which follows from $L(y(\rho, t)) = 0$, we obtain the differential equation in the statement.

□

Let

$$M_i := L_i \circ L$$

be the differential operator obtained by composing L with L_i.

A corollary of the two lemmas above is that, for every $i = 0, \ldots, n-4$, the space of solutions of M_i is spanned by

$$V_i := \big\langle y(\rho, t),\, \hat{y}(\rho, t),\, \partial_{\rho_i} y(\rho, t),\, \partial_{\rho_i} \hat{y}(\rho, t)\big\rangle_{\mathbb{C}}. \tag{5.7}$$

We can describe the functions in V_i in terms only of $y(\rho, t)$, $\hat{y}(\rho, t)$ and integration.

Proposition 9. *The following equality holds for every $i = 0, \ldots, n-4$:*

$$\partial_{\rho_i} y(\rho, t) = y(\rho, t) \int_0^t \frac{\int_0^t t_2^i \, y^2(\rho, t_2) \, \mathrm{d}t_2}{y^2(\rho, t_1) P(t_1)} \, \mathrm{d}t_1. \tag{5.8}$$

Proof. To avoid heavy notation, we denote by $\varphi_i(\rho, t)$ the right-hand side of (5.8). It can be proved by direct verification that, for every $i = 0, \ldots, n-4$,

$$M_i\big(\varphi_i(\rho, t)\big) = 0$$

and that $\varphi_i(\rho, t)$ is a non-zero holomorphic function of t near $t = 0$.

Every holomorphic function in the solution space V_i of M_i is of the form

$$ay(\rho, t) + b\partial_{\rho_i} y(\rho, t) \tag{5.9}$$

for some $a, b \in \mathbb{C}$, as follows from (5.7).

The expansion of $\varphi_i(\rho, t)$ in $t = 0$ starts

$$\varphi_i(\rho, t) = \frac{(-1)^{n-2}}{(i+1)^2 \prod_{j=1}^{n-2} \alpha_j} t^{i+i} + \cdots , \tag{5.10}$$

while the expansion of the function in (5.9) in $t = 0$ starts

$$ay(\rho, t) + b\partial_{\rho_i} y(\rho, t) = a + \cdots .$$

This implies that $a = 0$ and that $\varphi_i(\rho, t)$ and $\partial_{\rho_i} y(\rho, t)$ are proportional.

Using the linear recursion defining the coefficients of $y(\rho, t)$ one than preves that the expansion of $\partial_{\rho_i} y(\rho, t)$ in $t = 0$ starts as the one in (5.10). This proves $\varphi_i(\rho, t) = \partial_{\rho_i} y(\rho, t)$. □

The same argument gives a similar description for the solution $\partial_{\rho_i} \hat{y}(\rho, t)$. This imples that every element in M_i can be written as linear combination of $y(\rho, t), \hat{y}(\rho, t)$ and integrals of these functions like the one in Proposition 9.

5.2 Deformation of modular forms

Let $y(\rho, t), \hat{y}(\rho, t)$ be solutions of (5.4) as in the previous section. In Chapter 3 we defined

$$Q(\rho, t) := \exp\big(\hat{y}(\rho, t)/y(\rho, t)\big), \tag{5.11}$$

and functions

$$T(\rho, Q) := Q(\rho, T)^{-1} = \sum_{m \geq 0} T_m(\rho) Q^m, \tag{5.12}$$

$$F(\rho, Q) := y(\rho, T(\rho, Q)) = \sum_{m \geq 0} F_m(\rho) Q^m, \tag{5.13}$$

from the solutions of (5.4). Moreover, for $i = 0, \dots, n-4$, define

$$H_i(\rho, Q) := F^4(\rho, Q) P(T(\rho, Q)) T(\rho, Q)^i = \sum_{m=1}^{\infty} C_{i,m}(\rho) Q^m \tag{5.14}$$

where $P = \prod_{j=1}^{n-1}(t - \alpha_j)$ is the polynomial in (5.3).

Recall that when we specialize ρ to the Fuchsian value ρ_F, the functions

$$t(\tau) := T(\rho_F, Q), \quad f^*(\tau) := F(\rho_F, Q),$$

are a Hauptmodul for the uniformizing group Γ, and a root of a modular form $f(\tau)$ of weight two respectively. Here τ denotes a coordinate on $\mathbb{H}$. Moreover, each of the functions

$$h_i(\tau) := H_i(\rho_F, Q)$$

is a cusp form of weight four.

In the following, for a power series

$$K = \sum_{m \geq 0} K_m Q^m,$$

we will adopt the following notation

$$K'(Q) := Q \frac{dK(Q)}{dQ} = \sum_{m=0}^{\infty} m K_m Q^m.$$

Similarly, if K has no constant term we will write $\int K$ for

$$\int_0^Q K(Q) \, \frac{\mathrm{d}Q}{Q} = \sum_{m \geq 1} \frac{K_m}{m} Q^m.$$

The functions $H_i(\rho, Q)$ ($i = 1, \dots, n-4$) and all its integrals have no constant term. We can then define $\widetilde{H}_i(\rho, Q)$ to be the three times iterated integral of H, i.e.:

$$\widetilde{H_i}(\rho, Q) := \sum_{m \geq 1} \frac{C_{i,m}(\rho)}{m^3} Q^m.$$

When $\rho = \rho_F$, the function $\widetilde{h}_i(\tau) := \widetilde{H}_i(\rho_F, Q)$ will be the Eichler integral of h_i.

Our goal is to define and study a differential operator on modular forms induced by the accessory parameters. To begin with, we will define a differential operator for $F(\rho, Q)$; we can do it in two different ways. For some purposes, it is better to give first a general definition.

Let $\mathcal{L}$ be the smallest differential field extension of $(\mathbb{C}(t), d/dt)$ which contains the solutions $y(\rho, t), \hat{y}(\rho, t)$ of the linear differential equation $L = 0$ (5.4). This means that $\mathcal{L}$ contains $\mathbb{C}(t), y(\rho, t), \hat{y}(\rho, t)$, it is a field and it is closed under the differential operator d/dt.

Definition 8. *Let* $w(\rho,t) \in \mathcal{L}_n$*, and define* $W(\rho,Q) := w(\rho, T(\rho,Q))$*. For every* $i = 0,\dots,n-4$*, define*

$$\partial_{i,T}W(\rho,Q) := \left(\frac{\partial w(\rho,t)}{\partial \rho_i}\right)\circ T(\rho,Q), \tag{5.15}$$

$$\partial_{i,Q}W(\rho,Q) := \frac{\partial w(\rho,T(\rho,Q))}{\partial \rho_i}. \tag{5.16}$$

The above series are obviously related:

$$\partial_{i,Q}W(\rho,Q) = \partial_{i,T}W(\rho,Q) + \frac{\partial T(\rho,Q)}{\partial \rho_i}\left(\frac{dw(\rho,t)}{dt}\circ T\right), \tag{5.17}$$

Sometimes, when the discussion permits it, we will write $\partial_{i,*}$ instead of writing both $\partial_{i,Q}, \partial_{i,T}$.

We study the action of the operators $\partial_{i,T}, \partial_{i,Q}$ on $F(\rho,Q)$; this case corresponds to the choice $w(\rho,t) = y(\rho,t)$ in the definition. It follows from (5.17) that to describe completely $\partial_{i,T}F(\rho,Q)$ and $\partial_{i,Q}F(\rho,Q)$ in terms of their $Q-$expansion, we only need to study $\partial_{i,T}F(\rho,Q)$ and $\partial T(\rho,Q)/\partial\rho_i$. This is the content of the next two lemmas.

Lemma 10. *Let* $F(\rho,Q)$ *and* $\widetilde{H}(\rho,Q)$ *be as above. For every* $i = 0,\dots,n-4$,

$$\partial_{i,T}\Big(F(\rho,Q)\Big) = F(\rho,Q)\widetilde{H}_i'(\rho,Q).$$

Proof. By definition of $\partial_{i,T}F(\rho,Q)$ and Proposition 9 we have

$$\partial_{i,T}F(\rho,Q) = \frac{\partial y(\rho,t)}{\partial\rho_i}\circ T(\rho,Q) = y(\rho,T)\int^T \frac{\int^{t_1} y(\rho,t_2)^2\,\mathrm{d}t_2}{y^2(\rho,t_1)P(t_1)}\,\mathrm{d}t_1.$$

We can easily compute the above integral as a function of Q using the formula

$$\frac{1}{Q}dQ = \frac{\kappa}{P(t)y(\rho,t)^2}dt, \quad \kappa := (-1)^{n-2}\prod_{j=1}^{n-2}\alpha_j \tag{5.18}$$

The relation (5.18) can be proved using the Wronskian determinant $W(t)$ of $y(\rho,t), \hat{y}(\rho,t)$

$$W(t) := \hat{y}'(\rho,t)y(\rho,t) - \hat{y}(\rho,t)y'(\rho,t)$$

which is known to be equal to

$$W(t) = c\exp\left(-\int^t \frac{P'(t_1)}{P(t_1)}\,\mathrm{d}t_1\right)$$

for some constant c. Using the two descriptions of the Wronskian above it is easy to prove that $c = (-1)^{n-2} \prod_{i=1}^{n-2} \alpha_i = \kappa$ and that

$$W(t) = \frac{\kappa}{P(t)}.$$

We recover then (5.18) from

$$\frac{\mathrm{d}Q(\rho,t)}{\mathrm{d}t} = Q(\rho,t)\frac{\mathrm{d}\big(\hat{y}(\rho,t)/y(\rho,t)\big)}{\mathrm{d}t} = \frac{W(t)}{y^2(\rho,t)}.$$

It follows form (5.18) and the definition (5.14) that

$$\kappa \int_0^T t^i y(\rho,t)^2 \mathrm{d}t = \int_0^Q P(T(\rho,Q_1))T(\rho,Q_1)^i y^4(\rho,T(\rho,Q_1))\ \frac{\mathrm{d}Q_1}{Q_1} = \int_0^Q H_i(\rho,Q_1)\ \frac{\mathrm{d}Q_1}{Q_1}.$$

Using this formula and (5.18) we find

$$\kappa \int_0^T \frac{\int_0^{t_1} y(\rho,t_2)^2\, \mathrm{d}t_2}{y^2(\rho,t_1)P(t_1)}\, \mathrm{d}t_1 = \int_0^Q \left(\int_0^{Q_1} H_i(\rho,Q_2)\frac{\mathrm{d}Q_2}{Q_2}\right)\frac{\mathrm{d}Q_1}{Q_1},$$

and finally

$$\int_0^Q \left(\int_0^{Q_1} H_i(\rho,Q_2)\frac{\mathrm{d}Q_2}{Q_2}\right)\frac{\mathrm{d}Q_1}{Q_1} = \sum_{n\geq 1} \frac{C_{i,n}}{n^2} Q^n = \widetilde{H}'(\rho,Q).$$

□

Lemma 11. *Let $T(\rho,Q)$ and $\widetilde{H}(\rho,Q)$ be as above. For every $i = 0,\dots,n-4$,*

$$\frac{\partial T(\rho,Q)}{\partial \rho_i} = 2T'(\rho,Q)\widetilde{H}(\rho,Q).$$

Proof. Since $T(\rho,Q)$ is the compositional inverse of $Q(\rho,t)$ we compute from

$$\frac{\partial}{\partial \rho_i}\big(T(\rho,Q(\rho,t))\big) = 0$$

that

$$\frac{\partial}{\partial \rho_i} T(\rho,Q) = -Q\frac{\mathrm{d}T(\rho,Q)}{\mathrm{d}Q}\frac{\partial\big(\hat{y}(\rho,t)/y(\rho,t)\big)}{\partial \rho_i}.$$

We can compute the derivative $\partial\big(\hat{y}(\rho,t)/y(\rho,t)\big)/\partial\rho_i$ using a Wronskian argument like in the previous lemma: from

$$\frac{\mathrm{d}\big(\hat{y}(\rho,t)/y(\rho,t)\big)}{\mathrm{d}t} = \frac{W(t)}{y^2(\rho,t)} = \frac{\kappa}{y^2(\rho,t)P(t)}$$

we get the integral representation

$$\frac{\hat{y}(\rho,t)}{y(\rho,t)} = \int_0^t \frac{\kappa}{y^2(\rho,t_1)P(t_1)}\,\mathrm{d}t_1.$$

From this identity and Proposition 9 it follows that

$$\kappa\frac{\partial}{\partial\rho_i}\left(\frac{\hat{y}(\rho,t)}{y(\rho,t)}\right) = -2\int_0^t \frac{\int_0^{t_1}\frac{\int_0^{t_2} y^2(\rho,t_3)\mathrm{d}t_3}{y^2(\rho,t_2)P(t_2)}\mathrm{d}t_2}{y^2(\rho,t_1)P(t_1)}\mathrm{d}t_1.$$

Substituting $t = T(\rho, Q)$ and using the previous lemma and (5.18), we finally get

$$\frac{\partial}{\partial\rho_i}\left(\frac{\hat{y}(\rho,t)}{y(\rho,t)}\right) = -2\widetilde{H}(\rho,Q),$$

which, together with the previous calculation, proves the lemma. □

Formula (5.18) permits us to write $dy(\rho,t)/dt \circ T$ explicitly as a function of Q. We have

$$\frac{dy(\rho,t)}{dt}\circ T = \frac{\kappa f'(\rho,Q)}{f^2(\rho,Q)P(T)} = \frac{F'(\rho,Q)}{T'(\rho,Q)}.$$

Substituting this identity in (5.17) we find

$$\partial_{i,Q}F(\rho,Q) = \partial_{i,T}F(\rho,Q) + 2F'(\rho,Q)\widetilde{H}_i(\rho,Q). \tag{5.19}$$

This and the two previous lemmas prove the following

Proposition 10. *For every $i = 0,\dots,n-4$,*

$$\partial_{i,T}F(\rho,Q) = F(\rho,Q)\widetilde{H}_i'(\rho,Q),$$

$$\partial_{i,Q}F(\rho,Q) = F(\rho,Q)\widetilde{H}_i'(\rho,Q) + 2F'(\rho,Q)\widetilde{H}_i(\rho,Q)$$

5.3 Specialization to the Fuchsian value

We specialize the results obtained in the previous section to the case of modular forms, i.e. $\rho = \rho_F$. Recall that $f^*(\tau) = F(\rho_F, Q)$ is a square-root of a weight two modular form f, and $t(\tau) = T(\rho_F, Q)$ is an Hauptmodul for the uniformizing group Γ.

The operators $\partial_{i,T}, \partial_{i,Q}$ lead to the definition on some new operators on the whole space of modular forms as follows.

Definition 9. *Let $g \in M_k(\Gamma)$, and let $f \in M_2(\Gamma)$ be as above. Let r, s be coprime positive integers such that $r/s = 2/k$, and let $R(t)$ be the rational function of t such that $g^r = R(t)f^s$. For every $i = 0, \ldots, n-4$, define*

$$\partial_{i,Q} g := \frac{1}{rg^{r-1}} \left[\partial_{i,Q} \left(F(\rho, Q)^{2s} R(T(\rho, Q)) \right) \right] \Big|_{\rho=\rho_F},$$

$$\partial_{i,T} g := \frac{1}{rg^{r-1}} \left[\partial_{i,T} \left(F(\rho, Q)^{2s} R(T(\rho, Q)) \right) \right] \Big|_{\rho=\rho_F}.$$

A simple computation shows that the operators $\partial_{i,Q}$and $\partial_{i,T}$ defined on $M_*(\Gamma)$ satisfy the Leibniz rule.

The next proposition describes the action of $\partial_{i,Q}, \partial_{i,T}$ on the space of modular forms. In particular, if $g \in M_k(\Gamma)$, the proposition shows that $\partial_{i,Q} g$ and $\partial_{i,T} g$ are not modular forms, but combinations of quasimodular forms and Eichler intergrals.

Proposition 11. *Let $g \in M_k(\Gamma)$ as in Definition 9. Then, for every $i = 0, \ldots, n-4$,*

$$\partial_{i,T} g(\tau) = k g(\tau) \widetilde{h}_i'(\tau),$$

$$\partial_{i,Q} g(\tau) = \left[g(\tau), \widetilde{h}_i(\tau) \right]_1,$$

where $[\ ,\ ]_1$ denotes the first Rankin-Cohen bracket defined in Chapter 1.

Proof. These identities follow from the results of the previous section; one has only to apply the definition and compute the action of $\partial_{i,Q}$ and $\partial_{i,T}$ on the products. The fact that $2s = rk$, where s, r are as in Definition 9, permits to write every coefficient in term of k and to avoid the use of r, s. □

Notice that, since $\partial_{i,Q}$ and $\partial_{i,T}$ satisfy the Leibniz rule, it is possible to extend the above operators to the ring of quasimodular forms. Recall from Chapter 1 that, for every non cocompact Fuchsian group Γ there exists a (non-unique) holomorphic function ϕ such that

$$\phi(\gamma\tau) = \phi(\tau)(c\tau + d)^2 + \frac{c}{2\pi i}(c\tau + d), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.20}$$

As explained in Chapter 1, every quasimodular form for Γ is a polynomial in ϕ with modular coefficients (since our Γ is not cocompact). To compute the action of $\partial_{i,T}$ and $\partial_{i,Q}$ on quasimodular forms we construct such a quasimodular form ϕ Recall that we have a distinguished element $f \in M_2(\gamma)$ with all its zeros in a cusp. The function

$$\phi := \frac{f'}{2f} \tag{5.21}$$

is holomophic on $\mathbb{H}$ and at the cusp by the location of zeros of f. Moreover, it is easy to see that it transforms like in (5.20).

We can define, for $* = T, Q$,

$$\partial_{i,*}\phi(\tau) := \partial_{i,*}\left(\frac{f'(\tau)}{2f(\tau)}\right). \tag{5.22}$$

Note that this definition makes sense, since $f' = \big(F(\rho_F, Q)^2\big)' = 2F(\rho, Q)F'(\rho, Q)$ and $\partial_{i,*}$ is defined for $F'(\rho, Q)$ choosing as $w(\rho, t)$ in Definition (**??**) the following function:

$$w(\rho, t) = y(\rho, t)^2 P(t)\frac{dy(\rho, t)}{dt}.$$

where $y(\rho, t)$ is as in (5.7). We have

Lemma 12. *For every $i = 0, \ldots, n-4$,*

$$\partial_{i,*}\varphi = \widetilde{h}_i'' + 2\varphi\widetilde{h}_i', \quad \partial_{i,Q}\varphi = \widetilde{h}_i'' + [\varphi, \widetilde{h}_i]_1.$$

Proof. We prove only the second equality, the proof of the first one is similar. We have by definition

$$\partial_{i,Q}\varphi = \partial_{i,Q}\left(\frac{f'}{2f}\right).$$

We know from the previous Proposition how to compute every part of the above derivative except for $\partial_{i,Q}f'$. We have

$$\partial_{i,Q}F'(\rho, Q) = \frac{\partial}{\partial\rho}\left(y(\rho, T(\rho, Q))^2 P(T(\rho, Q))\left(\frac{dy(\rho, t)}{dt} \circ T(\rho, Q)\right)\right).$$

We can compute everithing using the results in the first part of this chapter. We find

$$\partial_{i,Q}F'(\rho, Q) = 2F''(\rho, Q)\widetilde{H}_i(\rho, Q) + 3F'(\rho, Q)\widetilde{H}_i'(\rho, Q) + F(\rho, Q)\widetilde{H}_i''(\rho, Q),$$

which leads to

$$\partial_{i,Q}f'(\tau) = 2\widetilde{h}_i(\tau)''f(\tau) + [f'(\tau), \widetilde{h}_i(\tau)]_1.$$

Substituting this in the definition of $\partial_{i,Q}\varphi$ we get the equality in the statement. $\square$

Before stating the main result of this section, which generalizes Proposition 11, recall from Chapter 1 that the space $\widetilde{M_*}(\Gamma)$ of quasimodular forms has a natural $\mathfrak{sl}_2(\mathbb{C})$ structure. It is given by three derivations: the differentiation operator $\mathbf{D}$, the weight operator $\mathbf{W}$ and the operator $\boldsymbol{\delta}$. The next theorem shows that the action of the operators $\partial_{i,*}$ on the space of quasimodular forms can be described using the three derivations above and Eichler integrals of cusp forms of weight four.

Theorem 14. *Let $g \in \widetilde{M}_*(\Gamma)$. Then, for every $i = 0, \ldots, n-4$,*

$$\partial_{i,T} g = \widetilde{h}_i' \, \mathbf{W} g + \widetilde{h}_i'' \, \boldsymbol{\delta} g, \tag{5.23}$$

$$\partial_{i,Q} g = 2\widetilde{h}_i \, \mathbf{D} g + \widetilde{h}_i' \, \mathbf{W} g + \widetilde{h}_i'' \, \boldsymbol{\delta} g. \tag{5.24}$$

Proof. We prove the second equality; the first one can be done similarly. Let ϕ be as in (5.21). From Lemma 12 we compute

$$\partial_{i,Q} \phi^n = n \phi^{n-1} \widetilde{h}_i'' + [\phi^n, \widetilde{h}_i]_1.$$

Now write $g = \sum_{j=0}^p g_j \phi^j$, for some $g_j \in M_{k-2j}(\Gamma)$. Then

$$\begin{aligned}
\partial_{i,Q} g = \partial_{i,Q}\left(\sum_{j=0}^p g_j \phi^j\right) &= \sum_{j=0}^p \Big(\partial_{i,Q} g_{k-2j}\Big)\phi^j + \sum_{j=0}^p g_{k-2j}\Big(\partial_{i,Q}\phi^j\Big) \\
&= \sum_{j=0}^p \Big([g_{k-2j}, \widetilde{h}_i]_1 \phi^j + g_{k-2j}[\phi^j, \widetilde{h}_i]_1\Big) + \widetilde{h}_i'' \sum_{j=0}^p j g_{k-2j} \phi^{j-1}.
\end{aligned}$$

Using the following identity, which holds in general for elements A, B, C of an abstract Rankin-Cohen algebra

$$[AB, C]_1 = [A, C]_1 B + [B, C]_1 A,$$

we can simplify the above summation and obtain

$$\partial_{i,Q} g = \sum_{j=0}^p [g_{k-2j}\phi^j, \widetilde{h}_i]_1 + \widetilde{h}_i'' \sum_{j=0}^p j g_{k-2j} \phi^{j-1}.$$

By the bilinearity of $[\,,]_1$ and the discussion and the definition of $\boldsymbol{\delta}$ before the theorem, we finally get

$$\begin{aligned}
\partial_{i,Q} g &= \left[\sum_{j=0}^p g_{k-2j}\phi^j, \widetilde{h}_i\right]_1 + \widetilde{h}_i'' \sum_{j=0}^p j g_{k-2j}\phi^{j-1} \\
&= [g, \widetilde{h}_i]_1 + \widetilde{h}_i'' \boldsymbol{\delta} g \;=\; 2\widetilde{h}_i \mathbf{D} g + \widetilde{h}_i' \mathbf{W} g + \widetilde{h}_i'' \boldsymbol{\delta} g.
\end{aligned}$$

□

5.4 Infinitesimal deformations

Intuitively, the operators $\partial_{j,*}$, $j = 0, \ldots, n-4$, we studied in this chapter can be related to small perturbations of the complex structure of the $n-$punctured sphere X. This final section makes this observation precise. We show that, if we restrict the action of the operator $\partial_{j,*}$ to the Hauptmoduln we recover a result in the theory of quasiconformal maps sometimes called infinitesimal deformation [1].

Moreover, the connection with Teichmüller theory makes natural the appareance of Eichler integrals of weight four cusp forms, i.e. quadratic differentials. These are deeply related to the deformation theory of Riemann surfaces.

Let Γ be a Fuchsian group such that $\mathbb{H}/\Gamma$ is isomorphic to a punctured sphere X. Recall from Chapter 2 that, if $T(\Gamma)$ is the Teichmüller space of the Fuchsian group Γ, we have an holomorphic map

$$\Phi\colon B_1(\Gamma) \to T(\Gamma),$$

where $B_1(\Gamma)$ is the space of Beltrami differentials on Γ of bounded norm $||\mu||_\infty < 1$. The holomorphic tangent space to $T(\Gamma)$ at $\Phi(0) \in T(\Gamma)$ is the space $\mathcal{H}(\Gamma)$ of harmonic Beltrami differentials, and the holomorphic cotangent space is the space of cusp forms of weight four $S_4(\Gamma) = Q(\Gamma)$. These two spaces are related by the linear map

$$\Lambda^*\colon Q(\Gamma) \to \mathcal{H}(\Gamma), \quad q \mapsto \Im(\tau)^2 \bar{q}(\tau). \tag{5.25}$$

We are going to define a special basis of the space $Q(\Gamma)$; this, together with (7.6), will give a basis on $\mathcal{H}(\Gamma)$. The basis we choose for $Q(\Gamma)$ is the one in Chapter 3. Let $t\colon \mathbb{H}/\Gamma \to X$ be a Hauptmodul, and let $g \in M_2(\Gamma)$ be a modular form with all its zeros concentrated in the cusp c_0 where t has its unique pole, normalized by

$$g(\tau) = 1 + \sum_{s\geq 1} g_s q^s, \quad q = e^{2\pi i \tau}. \tag{5.26}$$

We know from Corollary 1 in Chapter 3 that

$$h_j := g^2 P(t) t^j, \quad j = 0, \ldots, n-4, \tag{5.27}$$

give a basis for the space $Q(\Gamma)$. Define

$$\nu_j := \Lambda^*(h_j) = \Im(\tau)^2 g(\tau)^2 P(t) t(\tau)^j, \quad j = 0, \ldots, n-4.$$

Then $\nu_0, \ldots, \nu_{n-4}$ is a basis of $\mathcal{H}(\Gamma)$.

For each $j = 0, \ldots, n-4$, denote by f^{ν_j} the solution of the Beltrami equation

$$f_{\bar{z}} = \mu_j(z) f_z, \quad z \in \mathbb{C},$$

where $\mu_j(z)$ is defined by extending ν_j to $\mathbb{C}$ by symmetry, and f^{ν_j} is normalized such that it fixes $0, 1, \infty$. Recall in particular that f^{ν_j} is a quasiconformal homeomorphism of $\mathbb{H}$ into itself.

Now let $\varepsilon > 0$ be small. For every $j = 0, \ldots, n-4$ we can consider the following diagram

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{f^{\varepsilon\nu_j}} & \mathbb{H} \\ {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t^{\varepsilon\nu_j}} \\ X & \xrightarrow[F^{\varepsilon\nu_j}]{} & X_j \end{array} \tag{5.28}$$

where

$$X_j := \mathbb{H}/\Gamma_j, \quad \Gamma_j := f^{\varepsilon\nu_j}\Gamma\left(f^{\varepsilon\nu_j}\right)^{-1},$$

and $t^{\varepsilon\nu_j}\colon \mathbb{H} \to X_j$ is a holomorphic Hauptmodul normalized by

$$t^{\varepsilon\nu_j}(\infty) = t(\infty),\ t^{\varepsilon\nu_j}(0) = t(0).$$

(It is possible to fix this normalization since f^{ν_j} fixes $\infty, 0, 1$ and so these are still cusps of Γ_j.)

Finally, recall that $F^{\varepsilon\nu_j}$ is a quasiconformal map of Riemann sufaces and is holomorphic in ε, while both $f^{\varepsilon\nu_j}$ and $t^{\varepsilon\nu_j}$ are only real-analytic functions in ε, for every $j = 0, \ldots, n-4$. In particular, it makes sense to consider the derivatives of the above functions with respect to ε and $\bar{\varepsilon}$.

Theorem 15. *Let X be an n-punctured sphere, and let $t\colon \mathbb{H}/\Gamma \to X$ be a Hauptmodul. Let $\nu_j \in \mathcal{H}(\Gamma)$, $j = 0, \ldots, n-4$ be defined as above, and let $\partial_{j,Q}, j = 0, \ldots, n-4$ be the differential operator on $\widetilde{M}_*(\Gamma)$ defined above. Then*

$$\partial_{j,Q} t = \frac{\partial t^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}}\Big|_{\varepsilon=0}.$$

Proof. Let $X = \mathbb{P}^1 \smallsetminus \{\alpha_1, \ldots, \alpha_n = \infty\}$. Let $\rho_0, \ldots, \rho_{n-4}$ be the modular accessory parameters, and let $m_1, \ldots, m_{n-1}, m_n = m_\infty$ be the accessory parameters defined from the Schwarzian derivative in Chapter 2. Recall that in Chapter 3 we proved that

$$m_i = \mathrm{Res}_{t=\alpha_i}\left(2P_1(t) + \frac{P''(t)}{P(t)}\right), \quad i = 1, \ldots, n-1,$$

where

$$P(t) = \prod_{i=1}^{n-1}(t-\alpha_i), \quad P_1(t) = \frac{(1-n/2)^2 t^{n-3} + \sum_{j=0}^{n-4}\rho_j t^j}{P(t)}.$$

In particular, we can see each $m_i = m_i(\rho)$ as a function of $\rho_0, \ldots, \rho_{n-4}$ for a fixed X.

Now consider the harmonic Beltrami differentials $\nu_j, j = 0, \ldots, n-4$, and let $f^{\varepsilon\nu_j}, f^{\varepsilon\nu_j}$ and X_j be defined as above. We have

$$X_j = \mathbb{P}^1 \smallsetminus \{\alpha_1^{\varepsilon\nu_j}, \ldots, \alpha_n^{\varepsilon\nu_j} = \infty\},$$

where $\alpha_i^{\varepsilon\nu_j} := F^{\varepsilon\nu_j}(\alpha_i)$, To the Fuchsian uniformization of X_j are associated new accessory parameters $m_i^{\varepsilon\nu_j}, \ldots, m_n^{\varepsilon\nu_j}$. These accessory parameters are continuously differentialble in ε since they are coefficients of the q-expansion of $t^{\varepsilon\nu_j}$ and this function is real-analytic in ε (see [43]).

The theorem will be proved if we show that

$$\frac{\partial m_i^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}}\Big|_{\varepsilon=0} = \frac{\partial m_i(\rho)}{\partial\rho_j}\Big|_{\rho_F}, \quad i = 1, \ldots, n-1,\ j = 0, \ldots, n-4, \tag{5.29}$$

where ρ_F denotes the Fuchsian value of the modular accessory parameters. This claim is justified by the following observation. Let $t^{\varepsilon\nu_j}(\tau) = \sum_{s=1}^{\infty} t_s^{\varepsilon\nu_j} q^m$ be the $q-$expansion at ∞ of $t^{\varepsilon\nu_j}$, where the dependence of the Fourier coefficients on ε is determined by $\alpha_i^{\varepsilon\nu_j}$ and $m_i^{\varepsilon\nu_j}$, i.e, if we denote $m = (m_1, \ldots, m_{n-1})$, $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$,

$$t_s^{\varepsilon\nu_j} = t_s(m^{\varepsilon\nu_j}, \alpha^{\varepsilon\nu_j}),$$

where $t(\tau) = \sum_{m=1}^{\infty} t_s(m, \alpha)$ is the $q-$expansion of the Hauptmodul t. Note that $\alpha_i^{\varepsilon\nu_j}$ is holomorphic in ε; this follows from the definition $\alpha_i^{\varepsilon\nu_j} := F^{\varepsilon\nu_j}(\alpha_i)$, and the fact that $F^{\varepsilon\nu_j}$ is holomorphic in ε. This implies that the derivative of $\alpha_i^{\varepsilon\nu_j}$ with respect to $\bar{\varepsilon}$ is zero, and then

$$\frac{\partial t_s^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}} = \sum_{i=1}^{n-1} \frac{\partial t_s}{\partial m_i} \frac{\partial m_i^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}}.$$

It follows that

$$\frac{\partial t^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}}\Big|_{\varepsilon=0} = \sum_{s=1}^{\infty} \left(\sum_{i=1}^{n-1} \frac{\partial t_s}{\partial m_i}\Big|_{\varepsilon=0} \frac{\partial m_i^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}}\Big|_{\varepsilon=0} \right) q^s.$$

If we consider $m_i = m_i(\rho)$, the action of $\partial_{j,Q}$ on $t(\tau) = \sum_{s=1}^{\infty} t_s(m(\rho))q^s$ is given, by definition, by

$$\partial_{j,Q} t = \sum_{s=1}^{\infty} \frac{\partial t_s(m(\rho))}{\partial\rho_j} = \sum_{s=1}^{\infty} \left(\sum_{i=1}^{m-1} \frac{\partial t_s}{\partial m_i}\Big|_{\rho=\rho_F} \frac{\partial m_i}{\partial\rho_j}\Big|_{\rho=\rho_F} \right) q^s.$$

The claim preceding (5.29) follows from the above expansions with the observation that

$$\frac{\partial t_s}{\partial m_i}\Big|_{\varepsilon=0} = \frac{\partial t_s}{\partial m_i}\Big|_{\rho=\rho_F}, \quad \text{for every } s, i.$$

We now turn to the proof of (5.29). To this end, we recall that there is a linear isomorphism between the space $Q(\Gamma)$ of cusp forms of weight four and the space $D_2(X)$ of rational functions on $\hat{C}$ having at worst simple poles at the points $\alpha_1, \ldots, \alpha_{n-1}$, and order $O(|x|^{-3})$ as $x \to \infty$. If t is a Hauptmodul as above, an isomorphism is given by

$$J\colon D_2(X) \to Q(\Gamma), \quad R(x) \mapsto \big(R(x) \circ t\big)t'^2.$$

For details see [33]. In particular, we saw in Chapter 3 that if $g \in M_2(\Gamma)$ is as in (5.26), then

$$t' = gP(t), \quad P(t) = \prod_{i=1}^{n-1}(t - \alpha_i).$$

This implies that the rational function $R_j(x) \in D_2(X)$ associated to the weight four cusp form $h_j = g^2P(t)t^j$, $j = 0, \ldots, n-4$, is

$$R_j(x) = \frac{x^j}{P(x)}, \quad j = 0, \ldots, n-4.$$

In [43] it is proved that

$$\left.\frac{\partial m_i^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}}\right|_{\varepsilon=0} = \mathrm{Res}_{x=\alpha_i} R_j(x),$$

where ν_j and h_j are related by $\nu_j(\tau) = \Lambda^*(h_j) = \Im(\tau)^2 h_j(\tau)$ as before. On the other hand, we have form the results of Chapter 3 that

$$\frac{\partial m_i(\rho)}{\partial\rho_j} = \frac{\partial}{\partial\rho_j}\mathrm{Res}_{t=\alpha_i}\left(2P_1(t) + \frac{P''(t)}{P(t)}\right) = \mathrm{Res}_{t=\alpha_i}\left[\frac{\partial}{\partial\rho_j}\left(2P_1(t) + \frac{P''(t)}{P(t)}\right)\right],$$

where $P(t), P_1(t)$, are as above. But $P(t)$ is independent of ρ and $P_1(t) = \sum_{j=0}^{n-4}\rho_j t^j/P(t)$. Then,

$$\frac{\partial m_i(\rho)}{\partial\rho_j} = \mathrm{Res}_{t=\alpha_i}\left[\frac{\partial}{\partial\rho_j}\left(2P_1(t) + \frac{P''(t)}{P(t)}\right)\right] = Res_{t=\alpha_i}\left(\frac{t^j}{P(t)}\right).$$

□

As a corollary, we find a well-known result (which holds also in higher genus) called Ahlfor's formula [1],[2].

Corollary 3. *Let ν_j, h_j $j = 0, \dots, n-4$ be as above. Then, if $f_\tau = (2\pi i)^{-1}df/d\tau$, we have*

$$\left.\frac{\partial f_{\tau\tau\tau}^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}}\right|_{\varepsilon=0} = -\frac{1}{2}h_j, \quad j = 0, \dots, n-4$$

where h_j and ν_j are related by $\nu_j(\tau) = \Im(\tau)^2 h_j(\tau)$.

Proof. From diagram (5.28) we have

$$F^{\varepsilon\nu_j} \circ t = t^{\varepsilon\nu_j} \circ f^{\varepsilon\nu_j}, \quad j = 0, \dots, n-4.$$

If we differentiate both sides by $\partial/\partial\bar{\varepsilon}$, using the fact that $F^{\varepsilon\nu_j}$ is holomorphic in ε and t does not depend on ε, we get

$$\left.\frac{\partial f^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}}\right|_{\varepsilon=0} = -\frac{1}{t'}\left.\frac{\partial t^{\varepsilon\nu_j}}{\bar{\varepsilon}}\right|_{\varepsilon=0}.$$

By the previous theorem and Lemma 11, we have

$$\frac{1}{t'}\left.\frac{\partial t^{\varepsilon\nu_j}}{\bar{\varepsilon}}\right|_{\varepsilon=0} = \partial_{j,Q} t = 2t'\widetilde{h}_j,$$

where $\widetilde{q}_j$ is the Eichler integral of q_j. The two formulas together give

$$\left.\frac{\partial f^{\varepsilon\nu_j}}{\partial\bar{\varepsilon}}\right|_{\varepsilon=0} = -2\widetilde{h}_j,$$

and the statement follows by differentiating each side three times. □

Chapter 6

Extended modular forms: definition and examples

6.1 Motivation from Chapter 5

As an example, consider the case of a four-punctured sphere $X = \mathbb{P}^1 \smallsetminus \{\infty, 1, 0, \alpha\}$. Let $P(t) = t(t-1)(t-\alpha)$, $Q(t) = t - \rho$, where ρ is a free parameter. The differential equation associated to the uniformization of X is

$$L: \; P\frac{d^2}{dt^2}Y(t) + P'\frac{d}{dt}Y(t) + QY(t) = 0.$$

Denote by $y(\rho, t), \hat{y}(\rho, t)$ a basis of solutions near $t = 0$, where $y(\rho, t)$ is holomorphic. We know from Lemma 9 that the linear differential equation satisfied by $\partial y(\rho, t)/\partial\rho$ is $L^2 = L \circ L$, which explicitly looks as follows

$$\begin{aligned} L^2: \; P^2\frac{d^4}{dt^4}Y(t) + 4PP'\frac{d^3}{dt^3}Y(t) + (3PP'' + 2P'^2 + PQ)\frac{d^2}{dt^2}Y(t) \\ + (6P + P'P'' + P'Q)\frac{d}{dt}Y(t) + (P' + Q^2)Y(t) = 0. \end{aligned} \tag{6.1}$$

When $\rho = \rho_F$ is the Fuchsian parameter, we know that the solutions of $L(Y) = 0$ lift to functions $f, \tau f$ on $\mathbb{H}$ such that $f^2 \in M_2(\Gamma)$, where Γ is the uniformizing group. In the following we will assume that f is itself a modular form, i.e. $f \in M_1(\Gamma)$. This assumption makes the exposition smoother, and almost nothing is lost in terms of generality.

From the results of the previous chapter, we know how to describe in modular terms also the holomorphic solution $\partial y(\rho, t)/\partial\rho$ of L^2 (when $\rho = \rho_F$). It is given by $f\widetilde{h}'$, where $\widetilde{h}$ is the Eichler integral of the unique cusp form $h \in S_4(\Gamma)$ whose expansion at infinity starts $h = q + \cdots$.

The function $f\widetilde{h}'$ transforms as follows under the action of Γ :

$$\big(f\widetilde{h}'\big)(\gamma\tau) = c\Big[\tau f(\tau)\widetilde{h}'(\tau) - 2f(\tau)\widetilde{h}(\tau)\Big] + d\big[f(\tau)\widetilde{h}'(\tau)\big] \\ + f(\tau)[\tau p'_\gamma(\tau) - 2cp_\gamma(\tau)], \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{6.2}$$

where

$$p(\gamma;\tau) := r_2(\gamma)\tau^2 + r_1(\gamma)\tau + r_0(\gamma)$$

is the period polynomial of the cusp form $h \in S_4(\Gamma)$.

The above transformation formula looks complicated and not very useful. We rewrite it using vector valued functions, in the spirit of Proposition 3 of Chapter 1.

Consider the following 2×2 matrix $B(\gamma)$, defined from the coefficients of $p(\gamma;\tau)$

$$B(\gamma) := \begin{pmatrix} r_1(\gamma) & -2r_0(\gamma) \\ 2r_2(\gamma) & -r_1(\gamma) \end{pmatrix}, \tag{6.3}$$

Then one can prove that the map

$$\Gamma \to \mathrm{GL}(4,\mathbb{C}) : \quad \gamma \mapsto A(\gamma) := \left(\begin{array}{c|c} \mathrm{Sym}^1(\gamma) & \mathrm{Sym}^1(\gamma)\cdot B(\gamma) \\ \hline 0 & \mathrm{Sym}^1(\gamma) \end{array}\right)$$

is a homomorphism. All we need to prove is that $A(\gamma_1\gamma_2) = A(\gamma_1)A(\gamma_2)$ for $\gamma_1, \gamma_2 \in \Gamma$. This translates into the following statement for the matrix $B(\gamma)$ defined in (6.3):

$$B(\gamma_1\gamma_2) = \gamma_2^{-1}B(\gamma_1)\gamma_2 + B(\gamma_2)$$

which is true thanks to the cocycle property of the period polynomial $p(\gamma;\tau)$.

The reason why we introduced the above homomorphism is the following. Let v_h be the vector-valued function

$$v_h(\tau) := \begin{pmatrix} \tau f(\tau)\widetilde{h}'(\tau) - f(\tau)\widetilde{h}(\tau) \\ f(\tau)\widetilde{h}'(\tau) \\ \tau f(\tau) \\ f(\tau) \end{pmatrix} : \mathbb{H} \to \mathbb{C}^4 \tag{6.4}$$

Then for every $\gamma \in \Gamma$, we have

$$v_h(\gamma\tau) = A(\gamma)v_h(\tau).$$

This means that v_h transforms like a vector-valued modular form for the matrices $A(\gamma)$. Notice that the if we consider the map $\gamma \mapsto \mathrm{Sym}^1(\Gamma)$ as associated to a one dimensional representation V_1 of Γ, then we can think of $A(\gamma)$ as associated to an *extension* W of the representations V_1, V_1. The study of certain functions arising from vector-valued forms associated to extensions of symmetric tensor representation is the subject of this chapter.

Finally notice that (6.4) gives another proof of the fact that $A(\gamma)$ is a homomorphism: that identity implicitly tells that the image of Γ in $\mathrm{GL}(4,\mathbb{R})$ is the monodromy group of the equation (6.1), of which $\widetilde{h}f$ is a solution (in the usual sense involving an Hauptmodul). That the monodromy has to have that precise form is also a general fact, and we will discuss it in the example section below.

6.2 Extended modular forms

Let G be a group, and let V and V' be (left) G-modules. An *extension* over V with kernel V' is an exact sequence

$$0 \longrightarrow V' \xrightarrow{\phi} W \xrightarrow{\psi} V \longrightarrow 0, \tag{6.5}$$

where W is a G-module and ϕ and ψ are G-morphisms. Given another extension

$$0 \longrightarrow V' \xrightarrow{\phi'} W' \xrightarrow{\psi'} V \longrightarrow 0.$$

we say that W and W' are *equivalent* if there is a G-equivariant morphism $w\colon W \to W'$ such that the following diagram commutes:

$$\begin{array}{ccccc}
V' & \xrightarrow{\phi} & W & \xrightarrow{\psi} & V \\
\Big\downarrow{\scriptstyle \mathrm{Id}} & & \Big\downarrow{\scriptstyle w} & & \Big\downarrow{\scriptstyle \mathrm{Id}} \\
V' & \xrightarrow{\phi'} & W' & \xrightarrow{\psi'} & V.
\end{array}$$

We denote by $\mathrm{Ext}^1_G(V,V')$ the set of equivalence classes of extensions (6.5); it is actually an abelian group, as we will now see. For more details on extensions and the functor Ext^1_G see [13],[11].

Let $[W]$ be an equivalence class in $\mathrm{Ext}^1_G(V,V')$ and let W be a representative. As a vector space, it is (non uniquely) isomorphic to the direct sum $V' \oplus V$. It follows that the action of G on W can be described by

$$\gamma \mapsto A^W(g) := \left(\begin{array}{c|c} \rho_{V'}(g) & M(g) \\ \hline 0 & \rho_V(g) \end{array}\right), \tag{6.6}$$

where $M\colon G \to \mathrm{Hom}_{\mathbb{C}}(V,V')$ is such that

$$M(g_1g_2) = \rho_{V'}(g_1)M(g_2) + M(g_1)\rho_V(g_2), \quad \text{for every } g_1,g_2 \in G. \tag{6.7}$$

The above condition on M is essential for A^W to be a representation. In this setting, a function $M\colon G \to \mathrm{Hom}(V,V')$ which satisfies (6.7) is called a *cocycle*. The space of cocycles is denoted by $Z_G(V,V')$. We also have coboundaries: given a linear map $\Phi\colon V \to V'$, the *coboundary* of Φ is the map

$$d\Phi\colon G \to \mathrm{Hom}(V,V'), \quad g \mapsto \quad d\Phi(g) := g\cdot\Phi - \Phi g.$$

Denote by $B_G(V,V')$ the space of coboundaries. The space $\mathrm{Ext}^1_G(V,V')$ can be realized as the quotient $Z_G(V,V')/B_G(V,V')$. It follows that in (6.8) we can replace $M(g)$ by $M(g)+B(g)$, where $B(g)$ is any coboundary, and still get a G-action on the extension W. This is due to the fact that we consider extensions as equivalence classes, and a description of the G-action (6.8) corresponds to the choice of a specific representative.

In the following we fix $G=\Gamma$ a Fuchsian group with finite covolume, and consider extensions of symmetric tensor representations $V=V_s$ and $V'=V_r$, of Γ.

Consider as before a class $[W]\in\mathrm{Ext}^1_\Gamma(V_s,V_r)$, and a representative W. If we fix an identification $W\simeq V_r\oplus V_s$ we can represent the action of Γ on W as in (6.6) by

$$\gamma \mapsto A^W(\gamma) := \left(\begin{array}{c|c} \mathrm{Sym}^r(\gamma) & M(\gamma) \\ \hline 0 & \mathrm{Sym}^s(\gamma)\end{array}\right), \tag{6.8}$$

where $M(\gamma)$ has to satisfy

$$M(\gamma_1\gamma_2) = \mathrm{Sym}^r(\gamma_1)M(\gamma_2) + M(\gamma_1)\mathrm{Sym}^s(\gamma_2), \quad \text{for every } \gamma_1,\gamma_2\in\Gamma.$$

We can in particular consider vector-valued modular forms F attached to W with the Γ-action (6.8). We will denote these vector-valued forms by

$$F(\tau) = {}^{\mathrm{t}}(g_r(\tau),\dots,g_0(\tau),f_s(\tau),\dots,f_0(\tau)),$$

We are interested in these vector-valued forms and in the functions appearing as their components. To get a feeling about these, we compute some examples related to Eichler integrals.

6.2.1 Eichler integrals of cusp forms

Let $k\geq 2$ and let $h\in S_k(\Gamma)$. Consider the period map $P_h\colon\Gamma\to\mathbb{C}^{k-2}$

$$\gamma\mapsto P_h(\gamma) := \left(\int_{\gamma^{-1}(\infty)}^{\infty}\frac{h(z)}{(k-2)!}\,\mathrm{d}z, -\int_{\gamma^{-1}(\infty)}^{\infty}\frac{(k-2)zh(z)}{(k-2)!}\,\mathrm{d}z,\dots,\int_{\gamma^{-1}(\infty)}^{\infty}\frac{(-z)^{k-2}h(z)}{(k-2)!}\,\mathrm{d}z\right).$$

Denote by W_h the unique extension of V_{k-2} by V_0 which is determined by the following Γ-action:

$$\gamma\mapsto A^{W_h}\left(\begin{array}{c|c} 1 & P_h(\gamma) \\ \hline 0 & \mathrm{Sym}^{k-2}(\gamma)\end{array}\right). \tag{6.9}$$

That this is an action of Γ can be seen by using the cocycle property of the period polynomial $p_h(\gamma;\tau)$ of h :

$$p_h(\gamma;\tau) = \tau^{k-2}\int_{\gamma^{-1}(\infty)}^{\infty} \frac{h(z)}{(k-2)!}\,\mathrm{d}z + \cdots + \int_{\gamma^{-1}(\infty)}^{\infty} \frac{(-z)^{k-2}h(z)}{(k-2)!}\,\mathrm{d}z.$$

Since we will discuss for a while vector-valued modular forms attached to W_g, we fix the notation for these vectors

$$F(\tau) = \big(g, f_{k-2}, f_{k-1}, \dots, f_0\big) \tag{6.10}$$

Let $\widetilde{h}$ denote the Eichler integral of h. We know that

$$\widetilde{h}\big|_{2-k}\gamma = \widetilde{h} + p_h(\gamma;\tau). \tag{6.11}$$

This is equivalent to say that the vector-valued function

$$\mathcal{E}_h := \begin{pmatrix} \widetilde{h} \\ \tau^{k-2} \\ \vdots \\ \tau \\ 1 \end{pmatrix} \tag{6.12}$$

satisfies, for every $\gamma \in \Gamma$,

$$\mathcal{E}_h\big|_{2-k}\gamma = A^{W_h}\mathcal{E}_h. \tag{6.13}$$

If the representation (6.9) is given, the component g of (6.10) is uniquely determined by the elements $f_{k-2}, \dots, f_0$ of F. These in turn are determined by f_0 and Theorem 4 affirms that f_0 is a quasimodular form. The vector in (6.12) corresponds to the case $f_0 = 1$; we now look at the other cases.

1. Suppose that f_0 is a modular form. Then, for every $i = 0, \dots, k-2$, the function f_i in (6.10) is given by $\tau^i f$. Using as before the transformation formula for $f_0\widetilde{h}$ it is easy to see that the function g in (6.10) is given by $g = f_0\widetilde{h}$.

2. The situation is more interesting if f_0 has depth one. Supose that $f_0 \in \widetilde{M_l}(\Gamma)^{(\leq 1)}$. Using (1.9), we see that the elements $f_0, f_1, \dots, f_{k-2}$ in (6.10) are given by the product

$$L_{k-2}(\tau)\cdot\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \boldsymbol{\delta} f_0/(k-2) \\ f_0 \end{pmatrix} = \begin{pmatrix} 1 & (k-2)\tau & \cdots & \tau^{k-2} \\ 0 & 1 & \cdots & \tau^{k-3} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \boldsymbol{\delta} f_0/(k-2) \\ f_0 \end{pmatrix} = \begin{pmatrix} f_{k-2} \\ f_{k-1} \\ \vdots \\ f_1 \\ f_0 \end{pmatrix}. \tag{6.14}$$

Now suppose that the vector-valued function F in (6.10), where $f_0, \dots, f_{k-2}$, are given by (6.14), is such that

$$F\big|_w \gamma = A^{W_h} F$$

where necessarily $w = l - (k-2)$. This is true because f_0 is of weight l and it defines a vector-valued form with respect to V_{k-2}; its total weight l is then the sum of $k-2$, coming from V_{k-2}, and an automorphy factor of weight $l-(k-2)$. This automorphy factor then has to be considered while studying the vector F.

From the explicit description (6.14) and (6.9) we find that

$$g(\gamma\tau)\big(c\tau+d\big)^{-w} = g(\tau) + \begin{pmatrix} \int h & (k-2)\tau \int h & \cdots & \tau^{k-2}\int h \\ 0 & \int zh & \cdots & \tau^{k-3}\int zh \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \int z^{k-2}h \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \boldsymbol{\delta} f_0/(k-2) \\ f_0 \end{pmatrix},$$

where in the matrix $\int z^i h$ stands for $(-1)^i \binom{k-2}{i} \int_{\gamma^{-1}\infty}^{\infty} \frac{z^i h(z)}{(k-2)!}\, dz$.

This gives ($w = l-k+2$)

$$g\big|_{l-k+2} = g + f_0 p_h(\gamma) + \frac{\boldsymbol{\delta} f_0}{k-2} p'_h(\gamma).$$

It is not difficult to prove the identity

$$\left(\widetilde{h} f_0 + \frac{\widetilde{h}' \boldsymbol{\delta} f_0}{k-2}\right)\Big|_{l-k+2}\gamma = \widetilde{h} f_0 + \frac{\widetilde{h}' \boldsymbol{\delta} f_0}{k-2} + f_0 p_h(\gamma) + \frac{\boldsymbol{\delta} f_0}{k-2} p'_h(\gamma),$$

so we find that

$$g = \frac{(k-2)\widetilde{h} f_0 + \widetilde{h}'\, \boldsymbol{\delta} f_0}{k-2}.$$

In particular, if $f_0 = f'$ where $f \in M_{l-2}(\Gamma)$ we have

$$g = \frac{[f, \widetilde{h}]}{k-2}.$$

3. If f_0 has depth 2, a the same argument leads to

$$g = \frac{(k-2)\widetilde{h}\, f_0 + \widetilde{h}'\, \boldsymbol{\delta} f_0 + \widetilde{h}''\, \boldsymbol{\delta}^2 f_0}{k-2}.$$

In particular, if $f_0 = f''$ where $f \in M_{l-4}(\Gamma)$, we have

$$g = \frac{(k-2)\widetilde{h}\, \mathbf{D} f' + \widetilde{h}'\, \mathbf{W} f' + \widetilde{h}''\, \boldsymbol{\delta} f'}{k-2} = \frac{[f', \widetilde{h}] + \widetilde{h}''\, \boldsymbol{\delta} f'}{k-2}.$$

Notice that if Γ is a genus zero torsion-free group and $f_0 \in \widetilde{M}_*(\Gamma)$, the functions in 2,3 reduce to $\partial_{i,Q} f_0$ if $h = h_i$, where $h_i, \partial_{i,Q}$ are as in the previous chapter.

Motivated by the above examples, we define *extended modular forms* as certain components of vector-valued modulars form with respect to representations obtained by iterated extensions of symmetric tensor representations. An extended modular form has a weight, which depends on its position in the associated vector-valued modular form. Let W be an extension of V_s by V_r, the Γ-action on W being described by (6.8). Let

$$F(\tau) = {}^{\mathrm{t}}(g_r(\tau), \dots, g_0(\tau), f_s(\tau), \dots, f_0(\tau)),$$

be such that $F\big|_w \gamma = A^W(\gamma)F$ for every $\gamma \in \Gamma$. For every $i = 0, \dots, s$, $j = 0, \dots, r$, the *weight* of the component f_i of F is $s+w-2i$; the weight of g_i is $r+w-2i$. This definition extends in an obious way to the case of successive extensions of tensor representations.

Quasimodular forms are extended modular forms, associated to trivial extensions; the usual notion of weight corresponds to the one defined above.

Eichler integrals and the other functions that appeared in the examples above are extended modular forms, obtained from the extension of two tensor representations. Given a weight k cusp form, the weight of its Eichler integral as extended modular form is $2 - k$, as shown by the identity is (6.13).

Examples of extended modular forms with respect to extensions of more than two tensor representations are given by composition of differential equations solved by modular forms.

6.2.2 Description of $\mathrm{Ext}^1_\Gamma(V_s, V_r)$ in terms of Eichler integrals

To study extended modular forms we have to understand the elements of the space $\mathrm{Ext}^1_\Gamma(V_s, V_r)$. The next theorem relates them to known modular objects.

Theorem 16. *Let $k \geq l$. There exists a short exact sequence*

$$0 \to M_{r+2}(\Gamma, V_s) \to \mathrm{Ext}^1_\Gamma(V_s, V_r) \to S_{r+2}(\Gamma, V_s) \to 0.$$

Proof. Here we study the group $\mathrm{Ext}^1(V_s, V_r)$ by using the following identification with cohomolgy:

$$\mathrm{Ext}^1_\Gamma(V_s, V_r) = \mathrm{Ext}^1_\Gamma(\mathbb{C}, \mathrm{Hom}_\mathbb{C}(V_s, V_r)) = H^1(\Gamma, \mathrm{Hom}_\mathbb{C}(V_s, V_r)).$$

This is possible since the $\Gamma-$modules V_r, V_s are $\mathbb{C}-$vector spaces (see [11]). We have the well known equalities:

$$\mathrm{Ext}^1_\Gamma(V_s, V_r) = H^1(\Gamma, \mathrm{Hom}_\mathbb{C}(V_s, V_r)) = H^1(\Gamma, V_s^* \otimes V_r).$$

By the self-duality of the representation V_s, and using the Clebsch-Gordan decomposition we get

$$H^1(\Gamma, V_s^* \otimes V_r) = H^1(\Gamma, V_s \otimes V_r) = H^1(\Gamma, \oplus_{i=0}^s V_{s+r-2i}).$$

We remark that the Clebsch-Gordan decomposition here works thanks to a general Zariski density argument for cofinite subgroups of $\mathrm{SL}(2,\mathbb{R})$.

We have an isomorphism

$$\bigoplus_{i=0}^{s} H^1(\Gamma, V_{s+r-2i}) \simeq H^1(\Gamma, \oplus_{i=0}^{s} V_{s+r-2i}).$$

Gunning's group cohomological verson of Eichler-Shimura theory affirms that there exists a short exact sequence

$$0 \to M_{r+2}(\Gamma) \to H^1(\Gamma, V_r) \to S_{r+2}(\Gamma) \to 0.$$

This, togheter with the above isomorphism between cohomology groups, gives another short exact sequence

$$0 \to \bigoplus_{i=0}^{s} M_{s+r+2-2i}(\Gamma) \to \mathrm{Ext}^1_\Gamma(V_s, V_r) \to \bigoplus_{i=0}^{s} S_{s+r+2-2i}(\Gamma) \to 0.$$

It is not difficult to prove that

$$\bigoplus_{i=0}^{s} M_{s+r+2-2i}(\Gamma) \simeq \widetilde{M}_{s+r+2}(\Gamma)^{(\leq s)}. \tag{6.15}$$

This, together with Theorem 4, implies

$$\bigoplus_{i=0}^{s} M_{s+r+2-2i}(\Gamma) \simeq M_{r+2}(\Gamma, V_s).$$

Now we show that $\bigoplus_{i=0}^{s} S_{s+r+2-2i}(\Gamma)$ is identified to $S_{r+2}(\Gamma, V_s)$ via the above map.

Let g_0 be a quasimodular form and let $\widehat{G} = \sum_{i=0}^{s} \hat{g}_i (1/4\pi y)^i$ be its associated almost holomorphic modular form. The coefficient $\hat{g}_p$ of the highest power of $1/(4\pi y)$ of $\widehat{G}$ is a modular form. If g_0 is the image under (6.15) of a cusp form, then $\hat{g}_p$ is a cusp form. Consider the isomorphism of Theorem 4

$$\widehat{G} \mapsto G = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g_p \\ \vdots \\ g_0 \end{pmatrix}.$$

The function G is a vector valued cusp form since g_p is. $\square$

6.3 More examples

We proved in the examples above that $\partial_{i,Q} f$ is an extended modular form. The operator $\partial_{i,T}$ also defines extended modular forms, but is a slighlty different way; we present our results below.

In the second example we introduce elliptic multiple zeta values. We discuss one particular example in terms of extended modular forms.

6.3.1 Deformation of Fuchsian parameters

Let Γ be a genus zero torsion-free Fuchsian group with n inequivalent cusps. In the following, the index i always belongs to $\{0, 1, \ldots, n-4\}$. We also keep the notation and the definition of the weight four cusp forms h_i as defined in Chapter 5.

The differential operator $\partial_{i,T}$

Let $f \in \widetilde{M}_k(\Gamma)$. Recall from Chapter 4 that

$$\partial_{i,T} f = \widetilde{h}_i' \, kf + \widetilde{h}_i \, \boldsymbol{\delta} f, \tag{6.16}$$

where $\widetilde{h}_i$ is the Eichler integral of $h_i \in S_4(\Gamma)$, and $\boldsymbol{\delta}$ is the derivation on $\widetilde{M}_*(\Gamma)$ we defined in Chapter 1.

For every $\gamma \in \Gamma$, let

$$p_i(\gamma; \tau) = r_{i,2}(\gamma)\tau^2 + r_{i,1}(\gamma)\tau + r_{i,0}(\gamma).$$

be the period polynomial associated to h_i. We already showed in this chapter that, if $f \in M_1(\Gamma)$, for every $\gamma \in \Gamma$ and

$$B(\gamma) := \begin{pmatrix} r_{i,1} & -2r_{i,0} \\ 2r_{i,2} & -r_{i,1} \end{pmatrix} \tag{6.17}$$

then

$$\begin{pmatrix} \tau f \widetilde{h}_i' - f \widetilde{h}_i \\ f \widetilde{h}_i' \\ \tau f \\ f \end{pmatrix} (\gamma\tau) = \left(\begin{array}{c|c} \mathrm{Sym}^1(\gamma) & \mathrm{Sym}^1(\gamma) \cdot B(\gamma) \\ \hline 0 & \mathrm{Sym}^1(\gamma) \end{array} \right) \cdot \begin{pmatrix} \tau f \widetilde{h}' - f \widetilde{h} \\ f \widetilde{h}' \\ \tau f \\ f \end{pmatrix} (\tau). \tag{6.18}$$

A similar description can be computed also for quasimodular forms of small weight and depth. In general, we can prove

Proposition 12. *For every $f \in \widetilde{M}_k(\Gamma)$, the function $\partial_{i,T} f$ is an extended modular form of weight k.*

This can be proved by looking at the monodromy matrices of the differential equation associated to $\partial_{i,T}f$; they are in fact of the form

$$\left(\begin{array}{c|c} \mathrm{Sym}^k(\gamma) & M(\gamma) \\ \hline 0 & \mathrm{Sym}^k(\gamma) \end{array}\right) \tag{6.19}$$

simply beacuse the ODE satisfied by $\partial_{i,T}f$ is obtained by composition of linear ODEs of the same order (this is why the blocks with $\mathrm{Sym}^k(\gamma)$ in the above matrix have the same exponent k), and both these ODE are satisfied by, possibly non holomorphic, modular forms. This justifies the appareance of the blocks with $\mathrm{Sym}^k(\gamma)$, which are basically related to modularity. The fact that the monodromy is a representation concludes the proof.

6.3.2 Depth one elliptic multiple zeta values

In this section the discussion is naturally restricted to $\Gamma = \Gamma_1 = \mathrm{SL}(2, \mathbb{Z})$. We first recall the definition of (depth one) elliptic multizeta values. Roughly speaking, the elliptic multiple zeta values (also called elliptic MZV) are iterated integrals of certain functions $f_n(u, \tau)$ defined by the Jacobi form

$$F(u, \alpha, \tau) := \frac{\theta'_\tau(0)\theta_\tau(u+\alpha)}{\theta_\tau(u)\theta_\tau(\alpha)} = \sum_{n\geq 0} f_n(u,\tau)(2\pi i\alpha)^{n-1}, \tag{6.20}$$

where α is a formal parameter and $\theta(u)_\tau$ is the classical Jacobi theta function

$$\theta_\tau(u) := \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{(n+1/2)^2}{2}} e^{(n+1/2)u}, \quad q = e^{2\pi i\tau}$$

and $\theta'_\tau(u) = \mathrm{d}\theta_\tau(u)/\mathrm{d}u$. The function in (6.20) is called the *Kronecker function.* Its main properties are studied in [51], where also a relation with the period polynomials of Hecke eigenforms is given. This connection to period polynomials is probably very related to our discussion above on extended modular forms, but we do not develop this here.

To define elliptic MZV, one considers the modified function

$$\Omega(u,\alpha,\tau) := e^{\frac{\Im(u)}{\Im(\tau)}\alpha} F(u,\alpha,\tau) = \sum_{n\geq 0} \omega_n(u,\tau)(2\pi i\alpha)^n,$$

form which follow the equalities for all $n \geq 0$:

$$\omega_n(u,\tau) = \sum_{k=0}^{n} \left(\frac{\Im(u)}{\Im(\tau)}\right)^k f_{n-k}(u,\tau).$$

It can be proved that the function $f_1(u,\tau)$, and also, by definition, $\omega_1(u,\tau)$, has a pole at every lattice point $u \in \mathbb{Z}+\tau\mathbb{Z}$. Because of this, we will distinguish two cases in the definition of elliptic MZV.

Let $r \geq 1$ be an integer, and let $n_1, \dots n_r \in \mathbb{Z}_{\geq 0}$ with $n_1 \neq 1, n_r \neq 1$. The *elliptic multiple zeta value* are given by the iterated integrals

$$A(n_1,\dots,n_r;\tau) := \int_{1\geq u_1\geq\cdots\geq u_r\geq 0} \omega_{n_1}(u_1,\tau)\cdots\omega_{n_r}(u_r,\tau)\mathrm{d}u_1\cdots\mathrm{d}u_r.$$

The above iterated integral is usually denoted by the more compact formula

$$\int_{[0,1]} \omega_{n_1}(u_1,\tau)\mathrm{d}u_1\cdots\omega_{n_r}(u_r,\tau)\mathrm{d}u_r.$$

The number r is the *length* of the elliptic MZV, and the number $n_1+\cdots+n_r+r$ is called the *weight*. The number of nonzero $n_i, i=1,\dots r$, is called the *depth*.

If $n_1 = 1$ or $n_r = 1$ the above defintion does not work because of the poles of $\omega_1(u,\tau)$. However, it turns out that for a small $\varepsilon > 0$

$$\int_{[\varepsilon,1-\varepsilon]} \omega_{n_1}(u_1,\tau)\mathrm{d}u_1\cdots\omega_{n_r}(u_r,\tau)\mathrm{d}u_r = I_\varepsilon(n_1,\dots,n_r;\tau)+\sum_{k=0}^{r} A_k(n_1,\dots,n_r;\tau)\log(-2\pi i\varepsilon)^k,$$

where the branch of log is choosen such that $2\log(\pm i) = \pm\pi i$. The function $I_\varepsilon(n_1,\dots,n_r)$ is $O(\varepsilon^v)$ for some $v>0$ as $\varepsilon\to 0$, and $A_i(n_1,\dots,a_n;\tau)$ are holomorphic functions of τ. We define, if $n_1 = 1$ or $n_r = 1$, the *regularized elliptic MZV* by

$$A(n_1,\dots,n_r) := A_0(n_1,\dots,n_r).$$

We will discuss only depth one MZV. From the definition above it follows that they are linear combinations of the following integrals

$$A_{n,r}(\tau) = A(n,\underbrace{0,\dots,0}_{r-1};\tau) = \int_0^1 \frac{(2\pi i t)^{r-1}}{(r-1)!} f_n(t,\tau)\,\mathrm{d}t,$$

with the above regularization if $n=1$. In [52] the following result is proved.

Theorem 17. *For all $n \geq 2$ the vector spaces*

$$\langle A_{n,r}(\tau)|_{1-r}\gamma : \gamma\in\Gamma_1\rangle_{\mathbb{C}}$$

are finite dimensional, as well as the vector spaces

$$\langle \hat{A}_{1,r}(\tau)|_{1-r}\gamma : \gamma\in\Gamma_1\rangle_{\mathbb{C}}$$

where $\hat{A}_{1,r} := A_{1,r} - \frac{(2\pi i)^{r-2}}{(r-1)!}A_{1,2}(\tau)$.

In particular, depth one elliptic MZV can be seen as components of a weight $1-r$ vector valued modular form with respect to some representation of Γ_1.

In [52] one can find some examples of the above statement. Let $K := (2\pi i)^4/720$ and consider the vector $V_{1,4}$ associated to the elliptic MZV $\hat{A}_{1,4}$:

$$V_{1,4}(\tau) = \begin{pmatrix} \tau\hat{A}_{1,4} - K\tau^4 \\ \hat{A}_{1,4}(\tau) \\ K\tau^3 \\ K\tau^2 \\ K\tau \\ K \end{pmatrix}.$$

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the generators of Γ_1. Then

$$V_{1,4}(\tau)|_{-3}T = \begin{pmatrix} 1 & 1 & -4 & -6 & -4 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} V_{1,4}(\tau), \tag{6.21}$$

$$V_{1,4}(\tau)|_{-3}S = \begin{pmatrix} 0 & -1 & 1 & 0 & -5 & 0 \\ 1 & 0 & 0 & 5 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} V_{1,4}(\tau), \tag{6.22}$$

We see immediately that the matrices in (6.21),(6.22) are elements of $\mathrm{Ext}^1_{\Gamma_1}(\mathrm{Sym}^1, \mathrm{Sym}^3)$. This, together with the holomorphicity of $\hat{A}_{1,4}$ proves the following result.

Proposition 13. *The elliptic MZV $\hat{A}_{1,4}$ is an extended modular form of degree* 1 *and weight* -2.

The degree is 1 since $\hat{A}_{1,4}$ is not a quasimodular form, as one can see from the transformation with respect to $\gamma \in \Gamma_1$. Using Theorem 16 and the fact that $S_k(\Gamma_1) = \{0\}$ if $k < 12$, we can prove that this extension is induced by a quasimodular form of weight six, i.e. by a linear combination

$$a\mathrm{E}_4' + b\mathrm{E}_6,$$

where E_{2m} denotes the following Eisenstein series on Γ_1 :

$$E_{2m}(\tau) := \frac{1}{2}\zeta(1-2m) + \sum_{j\geq 1} \sigma_{2m-1} q^j.$$

A similar result holds for the other examples of depth one elliptic MZV in [52]. This is partially justified by the following result of Zerbini.

Proposition 14. *For $m \geq 2$ consider the following Eichler integral $\widetilde{\mathrm{E}}_{2m}$ of the Eisenstein series* E_{2m}

$$\widetilde{\mathrm{E}}_{2m}(\tau) := \frac{1}{2}\zeta(1-2m)\frac{(2\pi i\tau)^{2m-1}}{(2m-1)!} + \frac{\zeta(2m-1)}{2} + \sum_{j\geq 1}\sigma_{1-2m}q^j.$$

Then

$$\hat{A}_{1,r} = -(2\pi i)^r \sum_{j=1}^{r-2} \frac{B_{2+j}\tau^{j+1}}{(j+2)!(r-j-1)!} - (2\pi i)^r \sum_{j=1}^{r-2} \frac{2(2\pi i)^{-1-j}}{(r-j-1)!}\widetilde{\mathrm{E}}_{2+j}(\tau),$$

where B_n denote the n-th Bernoulli number and $\widetilde{\mathrm{E}}_{2m+1}(\tau) := 0$.

For instance, we have

$$\hat{A}_{1,4} = \frac{(2\pi i)^4\tau^3}{720} - 2\pi i\widetilde{\mathrm{E}}_4(\tau) = \pi i\zeta(3) + 2\pi i\sum_{j\geq 1}\sigma_{-3}q^j.$$

Notice that the weight of $\widetilde{\mathrm{E}}_4$ is -2 in the theory of modular forms, and this agrees with the weight we assigned to $\hat{A}_{1,4}$ using the definition of extended modular forms.

This result, together with Theorem 17, suggests that every depth one MVZ can be a (degree one) extended modular form on Γ_1.

Chapter 7

Open problems and Conjectures

7.1 Thompson's conjecture

We call a punctured sphere *arithmetic* if its uniformizing group Γ is conjugated to a finite index subgroup of $SL(2,\mathbb{Z})$.

In the following we use the description of accessory parameters introduced in Chapter 2. Recall in particular that, given $X := \mathbb{P}^1 \smallsetminus \{\alpha_1, \dots, \alpha_{n-1}, \alpha_n = \infty\}$ and a Hauptmodul

$$t\colon \mathbb{H}/\Gamma \to X,$$

there is a unique choice of numbers $m_1, \dots, m_n = m_\infty \in \mathbb{C}$, called Fuchsian values, such that

$$\{\eta; t\} = \frac{1}{2}\sum_{i=1}^{n-1} \frac{1}{(t-\alpha_i)^2} + \sum_{i=1}^{n-1} \frac{m_i}{t-\alpha_i},$$

where η is a multivalued inverse of t. The follwing relations hold between the accessory parameters m_i

$$\sum_{i=1}^{n-1} m_i = 0, \quad \sum_{i=1}^{n-1} \alpha_i m_i = 1 - \frac{n}{2}, \quad \sum_{i=1}^{n-1} \alpha_i(1+\alpha_i m_i) = m_\infty.$$

In [45] J. Thompson considered the following problem. Let $\alpha_1, \dots, \alpha_{n-1}$ be the finite punctures of X, and suppose that $\alpha_i \in \overline{\mathbb{Q}}$ for every $i = 1, \dots, n-1$; is it true that the Fuchsian values m_i are algebraic numbers for every $i = 1, \dots n$?

He first proves the following interesting result.

Theorem 18 (Thompson)**.** *Let $S := \{\alpha_1, \dots, \alpha_{n-1}\} \subset \overline{\mathbb{Q}}$ be the set of finite punctures of X. Then there exists a finite set S_1 containing $S \cup \{\infty\}$ such that the Fuchsian values $\widetilde{m}_1, \dots, \widetilde{m}_N$ associated to the uniformization of $\mathbb{P}^1 \smallsetminus S_1$, are algebraic numbers.*

We give in detail a proof of this theorem, following the one in [45]. The main ingredient in the proof is the following theorem of Belyi.

Theorem 19 (Belyi)**.** *A complete nonsingular algebraic curve C defined over a field of characteristic zero is defined over $\overline{\mathbb{Q}}$ if and only if there exists a covering $X \to \mathbb{P}^1$ ramified at three points.*

Here is the proof of Thompson's result.

Proof. Consider a polynomial $p(x) = p_S(x) \in \mathbb{Q}[x]$ with the properties:

1. $p(s) \in \{0, 1\}$ for every $s \in S$;
2. if $\xi \in \bar{\mathbb{Q}}$ is such that $p'(\xi) = 0$, then $p(\xi) \in \{0, 1\}$.

The existence of such a polynomial is guaranteed by Belyi's theorem. Let d be the degree of p. Consider the Riemann surface R associated to the equation $p(x) - w = 0$. If we view $\mathbb{P}^1(\mathbb{C})$ as the w-line, this gives a $d-$sheeted holomorphic covering $R \to \mathbb{P}^1(\mathbb{C})$. From condition 2. it follows that it is branched only over $\{0, 1, \infty\}$. Using the Riemann-Hurwitz formula, we can see that the genus of R is zero and that its universal covering is biholomorphic to $\mathbb{H}$. In particular, R is biholomorphic to a punctured sphere $\mathbb{P}^1 \setminus S_1$ (still denoted R), where S_1 is a finite set with $|S_1| \geq 3$. It follows from Belyi's theorem that R is uniformized by a subgroup Γ of $\Gamma(2)$ of index n.

If we set $B = \mathbb{P}^1 \smallsetminus \{\infty, 1, 0\}$, the situation is summarized in the following commutative diagram

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\text{id}} & \mathbb{H} \\ {\scriptstyle\mu}\big\downarrow & & \big\downarrow{\scriptstyle\lambda} \\ R & \xrightarrow[p]{} & B \end{array} \tag{7.1}$$

where $\lambda(\tau)$ is the Legendre modular function, and $\mu \colon \mathbb{H} \to R$ is a holomorphic uniformizing covering map for R.

Recall that $\lambda(\tau)$ has rational coefficients in its expansion at the cusps. This, together with the relation

$$p(\mu(\tau)) = \lambda(\tau), \quad \tau \in \mathbb{H},$$

and the fact that $p \in \mathbb{Q}[x]$, implies that the Fourier expansion of $\mu(\tau)$ at the cusps has algebraic coefficients (with respect to a suitable parameter at the cusp). This implies that the Fuchsian values for the uniformization of $\mathbb{P}^1 \smallsetminus S_1$ are algebraic numbers, since they appears af Fourier coefficients of μ.

To conclude, we have to show that $S_1 \subset \overline{\mathbb{Q}} \cup \{\infty\}$ and that $S \subset S_1$. Both properties follow from the explicit description of S_1 as $S_0 = \mu(\mathbb{Q} \cup \{\infty\})$.

Using the fact that the relation (7.1) extends to the set of cusps and that p is a polynomial one indeed shows that

$$S_1 = p^{-1}\big(\{0, 1\}\big) \cup \{\infty\}. \tag{7.2}$$

From the properties 1. and 2. of p it follows that $S_1 \setminus \{\infty\} \subset \bar{\mathbb{Q}}$ and that $S_0 \subset S_1$. $\square$

We can compute the polynomial p in the theorem for some punctured spheres. For instance:

- $\mathbb{P}^1 \setminus \{\infty, 1, 0, 2\}$ is uniformized by $\Gamma_0(8) \cap \Gamma(2)$; we have $p(x) = 2x - x^2$.
- $\mathbb{P}^1 \setminus \{\infty, 1, 0, e^{\pi i/3}\}$ is uniformized by $\Gamma(3)$; we have $p(x) = 1 - x^6$.

Moreover, the proof of the Thompson's theorem gives an algorithm to find arithmetic punctured spheres. We can start with a finite set S (with more than two elements) and try to compute a polynomial p as in the theorem. If this is possible, then we can find the set S_1 as described in (7.2): the punctured sphere $\mathbb{P}^1 \smallsetminus S_1$ is then arithmetic.

For instance we can start with $S = \{0, 1, 3\}$. It is straightforward to see that the polynomial

$$\widetilde{p}(x) = \frac{1}{4} x(x-3)^2$$

satisfies the properties 1,2, in Theorem 18. The set S_1 associated to p is

$$S_1 = \{0, 1, 3, 4, \infty\}.$$

It follows that the surface $\mathbb{P}^1 \smallsetminus \{\infty, 0, 1, 3, 4\}$ is uniformized by a subgroup of $SL(2, \mathbb{Z})$ of finite index.

After having proved the above theorem, Thompson went further and stated, with some reserve, the following conjecture.

Conjecture 1 (Thompson)**.** *Let X be an n-punctured sphere with algebraic punctures. Then the Fuchsian values for the uniformization of X are algebraic numbers.*

We first notice that, under the assumption that the punctures are algebraic numbers, it is equivalent to discuss the algebraicity of the Fuchsian values $m_i, i = 1, \ldots, n$ or of the modular Fuchsian values $\rho_0, \ldots, \rho_{n-4}$. This follows from the relation between the two sets of parameters we give in Chapter 3, where we showed that

$$m_i = \mathrm{Res}_{t=\alpha_i}\left(2P_1(t) + \frac{P''(t)}{2P(t)}\right), \quad i = 1, \ldots, n-1$$

where $P(t) = \prod_{i=1}^{n-1}(t - \alpha_i)$ and $P_1 = \sum_{j=0}^{n-3} \rho_j t^j$ with $\rho_{n-3} = (1 - n/2)^2$.

It follows from this observation that we can study Thompson's conjecture numerically by using the algorithm we developed and explained in Chapter 4.

The results of the experiments are clear: not only does it seem that Thompson's conjecture is wrong, but even that the Fuchsian values are algebraic in a few cases only. These corresponds to the four cases discovered by Zagier [12] and the Chudnovsky brothers [15].

Based on these computations we can propose a new conjecture; if true, this will in particular disprove Thompsons's conjecture.

To state the new conjecture we need to recall the follwing fact from Chapter 4. Assume that the puncture $\alpha_{n-1} = 0$. From a Frobenius basis $y(\rho, t), \hat{y}(\rho, t)$ of solutions near $t = 0$ of the differential equation associated to X we defined

$$Q(\rho, t) := \exp\bigl(\hat{y}(\rho, t)/y(\rho, t)\bigr).$$

Then we showed that, under a certain normalization, when $\rho = \rho_F$ is the Fuchsian value we have

$$Q(\rho_F, t) = rq, \quad q = e^{2\pi i \tau}, \tau \in \mathbb{H},$$

for some $r \in \mathbb{C}^*$. This number r plays a role in our conjecture.

Conjecture 2. *Let X be an n-punctured sphere with algebraic punctures. The modular Fuchsian value ρ_F is algebraic if and only if r is algebraic.*

Now recall some results from Chapter 3. There we proved, at least in the case where Γ has at most one irregular cusp, that the modular form f obtained from the uniformization gives, together with the Hauptmodul t, generators for the full ring of modular forms $M_*(\Gamma)$.

Recall that the Fourier coefficients of the q-expansion of f and t only depends on the punctures α_i, the Fuchsian value ρ_F and the number r. If these are all algebraic numbers, then the ring $M_*(\Gamma)$ is generated by modular forms with algebraic Fourier coefficients. This would put some serious constraints on the uniformizing group Γ.

We conclude this section with an open problem; this makes sense only if Thompson's conjecture is wrong, independently of the correctness of Conjecture 2.

If Thompson's conjecture is wrong, to an n-punctured sphere with algebraic punctures there may not necessarily correspond an arithmetic group. Nevertheless, Theorem 18 is still true, and we can add a finite number of algebraic punctures to get a punctured sphere uniformized by some $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ of finite index. It is then interesting to study which is the minimum number of punctures one has to add to obtain an arithmetic surface.

Problem 4. *Let $\mathbb{P}^1 \smallsetminus S$ be an n-punctured sphere with algebraic punctures. If it is not uniformized by a finite index subgroup of $\mathrm{SL}(2, Z)$, find the smallest set S_1 such that $S \subset S_1$ and $\mathbb{P}^1 \smallsetminus S_1$ is arithmetic.*

We remark that in general the set S_1 defined by (7.2) is not the smallest possible (see [41]).

7.2 A strange convergence phenomenon

In this final section we report on some surprising experimental results arising from certain computations with functions related to Heun's equation. Recall the standard form of the Heun equation

$$\frac{d^2Y}{dt^2} + \Big(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-\alpha}\Big)\frac{dY}{dt} + \frac{t-\rho}{t(t-1)(t-\alpha)}Y = 0,$$

where $\alpha \in \mathbb{C} \smallsetminus \{0,1\}$ and ρ is the accessory parameter. In the following we will consider α as fixed and ρ as a free parameter.

Let $y(\rho,t), \hat{y}(\rho,t)$ denote a Frobenius basis of solutions near $t=0$. Construct, as we did many times in the thesis, the following functions:

$$Q(\rho,t) = \exp(\hat{y}(\rho,t)/y(\rho,t)) = \sum_{n\geq 0} Q_n(\rho)t^n = t + (-2\rho+\alpha+1)t^2 + \cdots . \tag{7.3}$$

$$T(\rho,Q) = Q(\rho,t)^{-1} = \sum_{n\geq 0} T_n(\rho)Q^n = Q + (2\rho-\alpha-1)Q^2 + \cdots . \tag{7.4}$$

Substitute the above series for t into the holomorphic function $y(t)$ to get a holomorphic function in Q :

$$F(\rho,Q) := y(t(Q)) = \sum_{n\geq 0} F_n(\rho)Q^n = 1 + \rho Q + \frac{1}{4}(9\rho^2 - 2\rho(\alpha+1) - \alpha)Q^2 + \cdots . \tag{7.5}$$

Finally define

$$H(\rho,Q) := F(\rho,Q)^4 T(\rho,Q)\big(T(\rho,Q)-1\big)\big(T(\rho,Q)-\alpha\big) = \sum_{n\geq 1} H_n(\rho)Q^n. \tag{7.6}$$

For every $n \geq 1$ the coefficient $H_n(\rho)$ of $H(\rho,Q)$ is a polynomial in ρ of degree $n-1$. From the coefficients of $H(\rho,Q)$, we define a family of new polynomials in ρ. For every $m > n \in \mathbb{Z}_{\geq 0}$, define

$$\Delta_{m,n}(\rho) := H_m(\rho) - H_n(\rho).$$

This is a polynomial of degree $m-1$. Denote by $U_{m,n}$ the set of roots of $\Delta_{m,n}$. In the following we will fix $n = N \in \mathbb{Z}_{\geq 0}$ and consider the polynomials $\Delta_{m,N}$ for $m > N$.

The experimental discovery is the following: there exist $N-1$ complex numbers

$$l_{N,1}, l_{N,2}, \ldots, l_{N,N-1} \tag{7.7}$$

and, for every $m > N$, a set

$$\big\{u_{N,i}(m)\big\}_{i=1,\ldots,N} \subset U_{m,N}$$

such that

$$u_{N,i}(m) = l_{N,i} + \varepsilon_{N,i}(m)$$

where $\varepsilon(m) = \varepsilon_{N,i}(m)$, is such that $\varepsilon(m) \to 0$ as $m \to \infty$. The striking fact is that $\varepsilon(m)$ goes to zero extremely quickly (see the examples below).

We call the numbers in (7.7) *limits*; they depend on the singular point α of Heun equation. These limits can be identified conjecturally (we did not prove their existence) with the roots of the polynomial $H_N(\rho) = -\Delta_{0,N}$. Before stating a more precise conjecture, we shall give some examples for different values of α. In the examples we only consider the cases $N = 2, 3$; we find the roots of $H_2(\rho)$ and $H_3(\rho)$ in closed form and relate the limits $l_{N,i}$ to these roots.

Example 1

Fix $\alpha = 1/3$. The limits found experimentally are

$$l_{2,1} = 1.3333333333, \quad l_{3,1} = 1.1883470545, \quad l_{3,2} = 1.4783196120$$

Note that the unique root of $H_2(\rho)$ is

$$\rho_2 = 4/3 \sim 1.3333333333$$

while the two roots of $H_3(\rho)$ are

$$\rho_{3,1} = \frac{4}{3} - \frac{1}{3}\sqrt{\frac{7}{37}} \sim 1.1883470545, \quad \rho_{3,2} = \frac{4}{3} + \frac{1}{3}\sqrt{\frac{7}{37}} \sim 1.4783196120.$$

The conjecture is then that $\rho_{i,N} = l_{i,N}$.

In the table below we reproduce the roots values

$$u_{N,i}(m) - \rho_{n,i} = \varepsilon_{N,i}(m)$$

to emphatize the rapid decay of $\varepsilon(m)$.

m	$u_{2,1}(m) - \rho_{2,1}$	$u_{3,1}(m) - \rho_{3,1}$	$u_{3,2}(m) - \rho_{3,2}$
5	-5.01×10^{-3}	-2.12×10^{-1}	-2.60×10^{-1}
10	-1.49×10^{-3}	6.00×10^{-3}	6.03×10^{-2}
25	-2.53×10^{-8}	-1.35×10^{-5}	4.46×10^{-3}
50	8.73×10^{-17}	2.55×10^{-11}	6.23×10^{-7}
75	1.20×10^{-25}	-8.66×10^{-17}	6.05×10^{-10}

We make two remarks:

1. The elements $u^m_{N,i}$ in the table are very easy to identify. This is to say that the property for some $u^m \in U_{N,m}$ of being close to $l_{N,i}$ really defines the numbers $u^m_{N,i}$. For instance, we see from the table that

$$u_2^{25} - l_2 = 1.14 \times 10^{-9}.$$

The elements $u^m \in U_{25,2}$ which are the closest to l_2 other than u_2^{25} are $\tilde{u}_2^{25} = 0.44382 + 0.06573i$ and its complex conjugate; we have

$$\tilde{u}_2^{25} - l_2 = 6.57 \times 10^{-2}.$$

2. One may think that this limits phenomenon exists because some zeros of the polynomials $H_m(\rho)$ are getting closer and closer to the limits $l_{N,i}$. That the zeros are getting closer to the limits $l_{N,i}$ as m grows is true, but this seems not enough to explain the rapid decay of the error function $\varepsilon(m)$. Consider for insance the case of l_2 above: we see from the table that

$$u_2^{50} - l_2 = 2.04 \times 10^{-20}. \tag{7.8}$$

If we compute the roots of $H_{50}(\rho)$, we find that the closest to l_2 is $\rho_{50} = 0.4423818911$ and

$$\rho_{50} - l_2 = 2.06 \times 10^{-3},$$

which is much bigger than the value in (7.8).

We give other two examples, for which the same remarks as above apply.

Example 2

Fix $\alpha = 9$. The limits are

$$l_{2,1} = 0.370370370, \quad l_{3,1} = 0.318347201, \quad l_{3,2} = 0.4223935393.$$

Thay again agree with the roots ρ_2 and $\rho_{3,1}, \rho_{3,2}$ of $H_2(\rho)$ and $H_3(\rho)$:

$$\rho_{2,1} = \frac{10}{27}, \quad \rho_{3,1} = \frac{10}{27} - \frac{1}{27}\sqrt{\frac{73}{37}}, \quad \rho_{3,2} = \frac{10}{27} + \frac{1}{27}\sqrt{\frac{73}{37}}.$$

In the table we give, as before, the values of $u_{N,i}(m) - \rho_{n,i}$.

m	$u_{2,1}(m) - \rho_{2,1}$	$u_{3,1}(m) - \rho_{3,1}$	$u_{3,2}(m) - \rho_{3,2}$
5	$-1,13 \times 10^{-4}$	1.17×10^{-4}	-7.26×10^{-3}
10	-2.67×10^{-7}	-2.24×10^{-8}	3.57×10^{-5}
25	-4.12×10^{-18}	5.38×10^{-20}	5.53×10^{-13}
50	8.05×10^{-37}	3.37×10^{-41}	1.80×10^{-27}
75	3.80×10^{-56}	8.96×10^{-61}	7.22×10^{-41}

Example 3

Fix $\alpha = 50$.The limits are

$$l_2 = 0.34000000, \quad l_{3,1} = 0.28574003, \quad l_{3,2} = 0.39425996.$$

Thay again agree with the roots ρ_2 and $\rho_{3,1}, \rho_{3,2}$ of $H_2(\rho)$ and $H_3(\rho)$:

$$\rho_2 = \frac{17}{50}, \quad \rho_{3,1} = \frac{17}{50} - \frac{1}{50}\sqrt{\frac{817}{111}}, \quad \rho_{3,2} = \frac{17}{50} + \frac{1}{50}\sqrt{\frac{817}{111}}.$$

In the table we give, as before, the values of $u_{N,i}(m) - \rho_{n,i}$.

m	$u_{2,1}(m) - \rho_{2,1}$	$u_{3,1}(m) - \rho_{3,1}$	$u_{3,2}(m) - \rho_{3,2}$
5	-1.33×10^{-4}	-9.21×10^{-6}	-8.33×10^{-3}
10	-4.08×10^{-7}	2.22×10^{-10}	5.24×10^{-5}
25	-1.37×10^{-17}	8.06×10^{-24}	1.61×10^{-12}
50	9.21×10^{-36}	1.04×10^{-48}	1.63×10^{-26}
75	1.48×10^{-54}	-2.00×10^{-74}	2.25×10^{-39}

We finally state clearly our conjecture.

Conjecture 3. *For every $N \geq 2$, the set of limits $\{l_{N,i}\}_{i=1,\dots,N}$ coincides with the set of roots of H_N.*

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