

Comparison of natural frequencies of isotropic plate using DSM with Wittrick-Williams algorithm

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Abstract. In this paper, comparison of natural undamped frequencies of isotropic plates are investigated by using the dynamics stiffness element for isotropic plates. The DS Matrix for isotropic has been formulated by the application of classical plate theory. The generalized DS matrix has to solve by using Wittrick-Williams algorithm. Results of different aspect ratio have been validated from the existing literature.

Keywords: dynamic stiffness matrix, isotropic plate, classical plate theory, Wittrick-Williams algorithm.

1. Introduction

Pure metals have limited applicability in engineering applications because of the requirement of the inconsistent property of materials. In engineering, materials may be required to be hard as well as ductile. To solve this requirement, there is no such material that exists in nature. The combination of one metal (in molten state) with another parent metal or non-metals is required to solve the engineering problem [1]. Another advanced material which is formed by the combination of one or more than one material with different physical and chemical properties in solid states are called composite material. The properties of the composite material are better than the individual parent materials [2]. In this article, we study the comparison of the natural frequency of isotropic plate using DSM and FEM. DSM provides very accurate results without depending on the number of elements in the nodal analysis and it is another effective method to the FEM [3]. Boscolo and Banerjee [4] studied about the free vibration analysis of homogeneous isotropic plates. In this paper, we present the comparison of nondimensional natural frequency of plate using DSM from the literature survey. We have used CPT on DSM formulation to derive the dynamic stiffness matrix. Wittrick-Williams algorithm [5] has to solve the transcendental nature of element with find out the nondimensional frequencies of the structure. We show that the natural frequencies are more accurate than available literature and set a standard value for comparison purpose.

2. Theory

2.1. Model development for functionally graded plates

The isotropic plate system has taken in this paper consist of the flexible square plate has different boundary condition at all edges, which are not movable in the neutral surface. CPT and Wittrick-Williams algorithm is described as the displacement field of plate surfaces are brief summaries below.

The Cartesian coordinate system (x, y, z) , of the isotropic plate, is fixed as represented in Fig. 1 plate's thickness is h . We assumed that the harmonic distributed force per unit area, $f_1 \cos(\omega t)$ in the z direction. $u_o(x, y, z)$, $v_o(x, y, z)$, and $w_o(x, y, z)$ are the displacement component of an arbitrary point of the plane of the plate in x , y , and z direction, respectively.

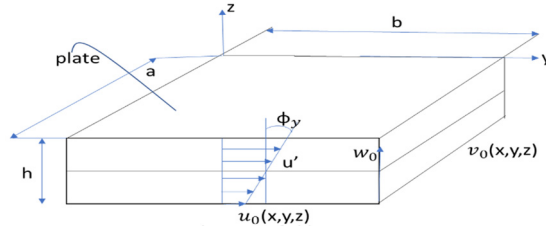


Fig. 1. Cartesian coordinate of displacement field of a isotropic plate

2.2. Classical plate theory (CPT)

The displacement component of an isotropic plate is $u_o(x, y, z)$, $v_o(x, y, z)$, and $w_o(x, y, z)$ by using classical plate theory, shown in Fig. 1, are given by [4]:

$$u_o(x, y, z) = u'(x, y) - z \frac{\partial w'}{\partial x}, \quad v_o(x, y, z) = v'(x, y) - z \frac{\partial w'}{\partial y}, \quad w_o(x, y, z) = w'(x, y). \quad (1)$$

In the above expression, the index (') consider the displacement component of the plate geometric. The mid-surface displacement, $u_o(x, y)$ and $v_o(x, y)$ of a thin homogenous isotropic place can be neglected. The only unknown in the transverse displacement w_o is considered in Eq. (1).

Hamilton’s principle has to investigate the nondimensional natural frequency of isotropic plate using the fourth order differential equation of a plate with different boundary conditions:

$$D_{eff} \left(\frac{\partial^4 w'}{\partial x^4} + 2 \frac{\partial^4 w'}{\partial x^2 \partial y^2} + \frac{\partial^4 w'}{\partial y^4} \right) + \rho h \frac{\partial^4 w'}{\partial t^4} = 0. \quad (2)$$

The BCs are¹:

$$V_x: -D_{eff} \left(\frac{\partial^3 w'}{\partial x^3} + (2 - \nu) \frac{\partial^3 w'}{\partial x \partial y^2} \right) \delta w', \quad M_{xx}: -D_{eff} \left(\frac{\partial^2 w'}{\partial x^2} + \nu \frac{\partial^2 w'}{\partial y^2} \right) \delta \phi_y, \quad (3)$$

where $D_{eff} = \epsilon h^3 / \{12(1 - \nu^2)\}$ is the flexural stiffness, ϵ is the property of the material called Young’s modulus, h the plate thickness, and ρ is the density of the material.

2.3. DSM formulation

For DSM formulation the first basic concept is to solve the fourth order differential Eq. (2) of the Isotropic plate. The levy-type [6] boundary condition for exact solution is sought. An isotropic plate has given as simply supported (SS) on two sides and remaining sides can be fixed (F) or clamped (C) The basic formulation of the Eq. (2) which satisfied the boundary condition in following form:

$$w'(x, y, t) = \sum_{m=1}^{\infty} W_m(x) e^{i\omega t} \sin(\alpha_m y), \quad \alpha_m = \frac{m\pi}{L}, \quad m = 1, 2, \quad (4)$$

where ω is the unknow frequency. The fourth order ordinary differential equation is to be obtained by substituting Eq. (4) into Eq. (2) as follows:

$$\frac{d^4 W_m}{dx^4} - 2 \alpha_m^2 \frac{d^2 W_m}{dx^2} + \left(\alpha_m^4 - \frac{\rho h \omega^2}{D_{eff}} \right) W_m = 0, \quad m = 1, 2, \dots, \infty. \quad (5)$$

The solution of the above quadratic equation determined by applying a trial solution [4]. For solving the differential equation, the two solutions are possible which is to rely depend on the roots:

- Condition 1: $\alpha_m^2 \geq \omega \sqrt{\frac{I_0}{D_{eff}}} \Rightarrow$ all roots are real ($\alpha_{1m}, -\alpha_{1m}, \alpha_{2m}, -\alpha_{2m}$):

$$\alpha_{1m} = \sqrt{\alpha_m^2 + \omega \sqrt{\frac{I_0}{D_{eff}}}}, \quad \alpha_{2m} = \sqrt{\alpha_m^2 - \omega \sqrt{\frac{I_0}{D_{eff}}}}. \quad (6)$$

The solution is:

$$W_m(x) = A_m \cosh(\alpha_{1m} x) + B_m \sinh(\alpha_{1m} x) + C_m \cosh(\alpha_{2m} x) + D_m \sinh(\alpha_{2m} x). \quad (7)$$

- Condition 2: $\alpha_m^2 < \omega \sqrt{\frac{I_0}{D_{eff}}} \Rightarrow$ different roots are ($\alpha_{1m}, -\alpha_{1m}, i \alpha_{2m}, -i \alpha_{2m}$):

$$\alpha_{1m} = \sqrt{\alpha_m^2 + \omega \sqrt{\frac{I_0}{D_{eff}}}}, \quad \alpha_{2m} = \sqrt{-\alpha_m^2 + \omega \sqrt{\frac{I_0}{D_{eff}}}}.$$

The solution is:

$$W_m(x) = A_m \cosh(\alpha_{1m} x) + B_m \sinh(\alpha_{1m} x) - C_m \cosh(\alpha_{2m} x) + D_m \sinh(\alpha_{2m} x). \quad (8)$$

Method to solve DS matrix for case 1 is explain below and similarly to solve in second case but is not show for brevity.

In (Eqs. (6) and (4)), the known displacement, the rotation ϕ_y , moment M_{xx} and shear force V_x explain in the following form Eq. (3):

$$\phi_{ym}(x, y) = \phi_{ym}(x) \sin(\alpha_m y) = - (A_m \alpha_{1m} \sinh(\alpha_{1m} x) + B_m \alpha_{1m} \cosh(\alpha_{1m} x) - C_m \alpha_{2m} \sinh(\alpha_{2m} x) + D_m \alpha_{2m} \cosh(\alpha_{2m} x)) \sin(\alpha_m y), \quad (9)$$

$$V_{xm}(x, y) = V_{xm}(x) \sin(\alpha_m y) = D_{eff} ((A_m (\alpha_{1m}^3 - (2 - \nu) \alpha_m^2 \alpha_{1m} \sinh(\alpha_{1m} x) + B_m (\alpha_{1m}^3 - (2 - \nu) \alpha_m^2 \alpha_{1m} \cosh(\alpha_{1m} x) + C_m (\alpha_{2m}^3 - (2 - \nu) \alpha_m^2 \alpha_{2m} \sinh(\alpha_{2m} x) + D_m (\alpha_{2m}^3 - (2 - \nu) \alpha_m^2 \alpha_{2m} \cosh(\alpha_{2m} x))) \sin(\alpha_m y), \quad (10)$$

$$M_{xxm}(x, y) = M_{xxm}(x) \sin(\alpha_m y) = -D_{eff} (A_m (\alpha_{1m}^2 - \nu \alpha_m^2) \cosh(\alpha_{1m} x) + B_m (\alpha_{1m}^2 - \nu \alpha_m^2) \sinh(\alpha_{1m} x) + C_m (\alpha_{2m}^2 + \nu \alpha_m^2) \cos(\alpha_{2m} x) - D_m (\alpha_{2m}^2 + \nu \alpha_m^2) \sin(\alpha_{2m} x)) \sin(\alpha_m y). \quad (11)$$

The displacements boundary condition for the plate are:

$$\begin{aligned} x = 0, \quad W_m &= W_1, \quad \phi_{ym} = \phi_{y1}, \\ x = b, \quad W_m &= W_2, \quad \phi_{ym} = \phi_{y2}. \end{aligned} \quad (12)$$

Similarly, the boundary condition for the forces are:

$$\begin{aligned} x = 0, \quad V_{xm} &= -V_1, \quad M_{xxm} = -M_1, \\ x = b, \quad V_{xm} &= -V_2, \quad M_{xxm} = M_2. \end{aligned} \quad (13)$$

From Fig. 2, we are implementing the different BCs, i.e. putting Eq. (12) in to Eqs. (6) and (9), the following expression is forming:

$$\begin{bmatrix} W_1 \\ \phi_{y1} \\ W_2 \\ \phi_{y1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -\alpha_{1m} & 0 & -\alpha_{2m} \\ C_{h1} & S_{h1} & C_{h2} & S_{h2} \\ -\alpha_{1m} S_{h1} & -\alpha_{1m} C_{h1} & -\alpha_{1m} S_{h2} & -\alpha_{1m} C_{h1} \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}, \quad (14)$$

i.e.:

$$\delta = AC, \quad (15)$$

where:

$$\begin{aligned} C_{h1} &= \cosh(\alpha_{im} b), \quad S_{h1} = \sinh(\alpha_{im} b), \\ C_i &= \cos(\alpha_{im} b), \quad S_i = \sin(\alpha_{im} b), \quad i = 1, 2. \end{aligned} \quad (16)$$

The BCs for forces, i.e. putting in Eq. (13) into Eqs. (10) and (11), the following matrix has formed:

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 0 & R_1 & 0 & R_2 \\ L_1 & 0 & L_1 & 0 \\ -R_1 S_{h1} & -R_1 C_{h1} & -R_1 S_{h2} & -R_1 C_{h2} \\ -L_1 C_{h1} & -L_1 S_{h1} & -L_2 C_{h1} & -L_2 S_{h2} \end{bmatrix} \begin{bmatrix} A_m \\ B_m \\ C_m \\ D_m \end{bmatrix}, \quad (17)$$

i.e.:

$$F = RC, \quad (18)$$

where:

$$R_i = D_{eff}(\alpha_{im}^3 - \alpha^2 \alpha_{im} (2 - \nu)), \quad L_i = D_{eff}(\alpha_{im}^2 - \alpha^2 \nu), \quad (19)$$

where $i = 1, 2$.

From Eqs. (15) and (18) the DS matrix K for the isotropic plate can be obtained and BE leave out the constant factor C to get:

$$F = K\delta, \quad (20)$$

where:

$$K = RA^{-1}. \quad (21)$$

From the general expression Eq. (21) to form DS matrix which is similar to six variable terms $S_{vv}, S_{vm}, S_{mm}, f_{vv}, f_{vm}, f_{mm}$ this variable terms describe the effect on shear and moment reason of displacement put on the “same” (s) nodal line, and the “far” nodal line. Thus, the dynamic stiffness matrix (K) expressed in following way:

$$K = \begin{bmatrix} S_{vv} & S_{vm} & f_{vv} & f_{vm} \\ S_{mm} & -f_{vm} & f_{mm} & \\ \text{Sym} & S_{vv} & -S_{vm} & \\ & & S_{mm} & \end{bmatrix}. \quad (22)$$

2.4. Algorithm used for the DSM element

The global DS matrix for natural frequency of plate is to obtain by using Wittrick-Williams algorithm [5]. Due to nonlinear behavior of dynamic stiffness element, to drive frequency determinant is excessive difficult. Wittrick and Williams algorithm [5] must be used to solving this problem and ensures that there are no frequencies missed out of the structure:

$$\bar{\omega} = \omega a^2 \sqrt{\frac{\rho_c h}{D_c}} \tag{23}$$

3. Numerical results

The numerical results of the DSM formulation have been inserted in a MATLAB and to find the frequencies and mode shape of the structure. This part is to analysis the natural frequencies of the isotropic plate by using DSM with CPT based on Levi-type BCs to the available literature Leissa [7] values.

The natural frequencies of simple supported (SS) isotropic plates are shown in Table 1, and Table 2. The four different edge boundary condition are applying as notification SS-F-SS-F C.

Table 1. Non-dimensional natural frequencies
 ($\bar{\omega} = \omega a^2 \sqrt{\rho_c h / D_c}$ for SS-F-SS-F plate)

S-F-S-F	a/b	$\nu = 0$		$\nu = 0.3$		$\nu = 0.5$	
mn		Present	Ref [10]	Present	Ref [10]	Present	Ref [10]
1 1	0.4	9.8696	9.8696	9.7600	9.7600	9.4506	9.4506
2 1		39.4784	39.4784	39.2386	39.2387	38.3771	38.3771
1 1		9.8696	9.8696	9.6313	9.6314	9.0792	9.0793
2 1	1	39.4784	39.4787	38.9449	38.9450	37.5191	37.5192
3 1		88.8264	88.8264	87.9866	87.9867	85.4898	85.4899
1 1	2.5	9.8696	9.8696	9.4841	9.4841	8.7042	8.7042
2 1		39.4784	39.4787	38.3629	38.3629	35.8798	35.8799
3 1		88.8264	88.8264	86.9684	86.9684	82.5093	82.5093
4 1		157.9136	157.9137	155.3211	155.3211	148.725	148.7256

Table 2. Non-dimensional natural frequencies
 ($\bar{\omega} = \omega a^2 \sqrt{\rho_c h / D_c}$ for SS-SS-SS-SS isotropic square plate)

S-S-S-S				
mn	Present	Ref [8]	Ref [9]	Ref [10]
1 1	19.7392	19.7392	19.739	19.739
1 2	49.3480	49.3490	49.348	49.348
2 1	49.3480	49.3490	49.348	49.348
2 2	78.9568	79.4007	78.957	78.957
1 3	98.6960	100.1729	99.304	100.17
3 1	98.6960	100.1868	99.304	-
3 2	128.3048	130.3895	-	-

4. Conclusions

Present work is used to understand the vibration analysis of isotropic plate was studied using classical plate theory under different levy type boundary conditions. This present study to solve the natural frequencies of plate by formulating the dynamic stiffness matrix together with CPT. Wittrick-William algorithm to take into account the nonlinear behaviour of the DSM element. Formulation is used to solve the dynamics stiffness matrix with applying boundary condition has

been import in MATLAB and extract the nondimensional frequencies of plate beside a high degree of desired accuracy. Thus, the result has been investigated to the acceptable comparison of natural frequencies of isotropic plate.

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