

Mathematical modeling of hydroelastic walls oscillations of the channel on Winkler foundation under vibrations

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Abstract. The bending oscillations of a narrow slit channel walls with highly viscous liquid inside and put on a vibrating Winkler foundation are investigated. The channel walls bending oscillations laws are discovered on the basis of hydroelasticity problem solution, as well as pressure in the liquid ones. The deflections amplitudes distribution and liquid pressure along the channel functions are constructed. The obtained results allow investigating dynamic processes, conditioned by constructions elastic elements and viscous liquid interaction in lubrication system, damping and various devices and units.

Keywords: hydroelastic oscillations, viscous liquid, plate, vibrations, Winkler foundation.

1. Introduction

The elastic elements of constructions with liquid oscillations are of great importance for modern machines, units and devices analysis [1]. In reference [2] the rectangular plate oscillations plunged into an ideal incompressible liquid with a free surface are investigated. Reference [3] studies the plate chaotic oscillations under interaction with ideal incompressible flow. The problem of round plate oscillations on incompressible liquid free surface is considered in reference [4], whereas reference [5] investigates round plate oscillations in water with free surface. The problem of bending oscillations of channel wall interacting with ideal liquid, which fills the channel, is solved in [6], and vibration cavitations in cooling liquid, surrounding internal-combustion engine shell having been solved on its basis. The investigation of the beam hydroelastic oscillation in viscous flow with application to piezoelectric element which can be used for receiving energy from the flow is made in [7]. Reference [8] studies the cantilevering fixed beam oscillations in an unlimited volume of viscous incompressible flow. Reference [9] investigates interminable plate oscillations on viscous liquid layer. The investigation of terminable length vibrating plates interaction with viscous incompressible liquid layer between them is made in [10]. References [11, 12] consider analogues problems for round disks with one of them being three-layered. The plate oscillations, caused by pressure pulsating in viscous liquid layer are considered in [13]. References [14, 15] deal with three-layered plates interaction with viscous liquid layer. The oscillations of membrane with elastic foundation, situated on the channel bottom, with ideal incompressible liquid with a free surface inside the channel are investigated in [16]. However, for practical purposes, the consideration of thin liquid layer viscosity impact on elastic channel walls oscillations, for the case of the channel having vibrating elastic foundation, is of great interest.

2. Statement of the problem

Let us consider a narrow slit channel, filled with highly viscous incompressible liquid (see Fig. 1). The channel walls make two elastic plates 1 and 2, being of similar geometrical sizes in the plan and simply supported at the butt ends. The channel size $b \gg \ell$. The plates thicknesses are h_1, h_2 , the liquid layer thickness is $\delta_0 \ll \ell$. The plate 1 is put on Winkler elastic foundation. The considered mechanical system performs harmonic oscillations with assigned vibro-acceleration $\ddot{z}_0 = E_z d^2 f_0(\omega t)/dt^2 = -E_z \omega^2 f_0(\omega t)$, $f_0(\omega t) = \sin \omega t$. Here $E_z \omega^2$ is the vibro-acceleration

amplitude, ω is the oscillation frequency, t is the time. The vibro-acceleration amplitude can be assigned in gravity acceleration units g , i.e. $E_z \omega^2 = kg$, here k is the vibro-overloading coefficient. Plates 1, 2 perform bending oscillations with amplitudes $w_{1m} \ll \delta_0$, $w_{2m} \ll \delta_0$. In the considered assignment, the liquid flow in the channel is viewed as a creeping one and liquid leakage at the butt ends being free in the same liquid with constant pressure p_0 .

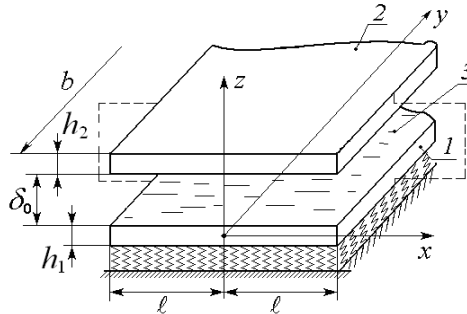


Fig. 1. A schematic diagram of a narrow slit channel on Winkler foundation

Let us put Cartesian coordinate system in the middle plate 1 surface center. Taking into account, that $b \gg l$ we will consider the problem in a flat setting. Strong damping, conditioned by liquid viscosity, takes place in the considered oscillating system. Therefore, transition processes in the system will get out rapidly and the stationary oscillations will emerge. Further, according to reference [17], we will focus on studying stationary forced harmonic oscillations regime.

3. The theory

The channel wall bending oscillations equations take the form of:

$$D_i \frac{\partial^4 w_i}{\partial x^4} + (2 - i)\chi w_i + \rho_i h_i \left(\ddot{z}_0 + \frac{\partial^2 w_i}{\partial t^2} \right) = (-1)^{i-1} q, \quad i = 1, 2, \quad (1)$$

where w_i is the deflection of i th plate, D_i is the bending stiffness of i th plate, ρ_i is the density of i th plate, χ is the elastic coefficient of Winkler foundation, $q = -p + 2\rho\nu\partial u_z/\partial z$ is the normal stress, acting from the liquid layer on the plate [18], p is the pressure, ρ is the density of the liquid, ν is the kinematical coefficient of the liquid viscosity, u_z is the liquid velocity vector project on coordinate axis z .

Boundary conditions for Eq. (1) are the conditions of simply supported edges:

$$w_i = \frac{\partial^2 w_i}{\partial x^2} = 0, \quad x = \pm l, \quad i = 1, 2. \quad (2)$$

Highly viscous incompressible liquid dynamics in a narrow slit channel is described by Navier-Stokes equations and continuity equation, in which local and convective inertia members are omitted, according to hydrodynamic lubrication theory [18]:

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2} \right), \quad \frac{1}{\rho} \frac{\partial p}{\partial z} = -\ddot{z}_0 + \nu \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2} \right), \quad \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0, \quad (3)$$

where u_x is the liquid velocity vector project on coordinate axis x .

Boundary conditions for Eq. (3) are the conditions of liquid adhesion to the walls and the conditions for pressure at the both ends:

$$u_x = \frac{\partial u_2}{\partial t}, \quad u_z = \frac{\partial w_2}{\partial t}, \quad \text{at } z = \frac{h_1}{2} + \delta_0 + w_2, \quad (4)$$

$$u_x = \frac{\partial u_1}{\partial t}, \quad u_z = \frac{\partial w_1}{\partial t}, \quad \text{at } z = \frac{h_1}{2} + w_1, \quad (5)$$

$$p = p_0, \quad \text{at } x = \pm \ell, \quad (5)$$

where u_i is the longitudinal shift of i th plate, $i = 1, 2$.

Let us introduce dimensionless variables:

$$\xi = \frac{x}{\ell}, \quad \zeta = (z - 0,5h_1)\delta_0^{-1}, \quad \tau = \omega t, \quad u_z = w_{2m}\omega U_\zeta, \quad u_x = w_{2m}\omega \frac{U_\xi}{\psi}, \quad (6)$$

$$w_i = w_{im}W_i, \quad u_i = u_{im}U_i, \quad p = p_0 + P \frac{\nu\rho w_{2m}\omega\psi^{-2}}{\delta_0} - \rho\ddot{z}_0 \left(z - \delta_0 - w_2 - \frac{h_1}{2} \right), \quad (6)$$

$$\lambda_i = \frac{w_{im}}{\delta_0} \ll 1, \quad \psi = \frac{\delta_0}{\ell} \ll 1,$$

where ψ, λ_i are the small parameters, characterizing the problem, $i = 1, 2$.

By substituting Eq. (6) in liquid dynamic Eq. (3) and the corresponding boundary Eq. (4) and (5), we get:

$$\frac{\partial P}{\partial \xi} = \psi^2 \frac{\partial^2 U_\xi}{\partial \xi^2} + \frac{\partial^2 U_\xi}{\partial \zeta^2}, \quad \frac{\partial P}{\partial \zeta} = \psi^2 \left[\psi^2 \frac{\partial^2 U_\zeta}{\partial \xi^2} + \frac{\partial^2 U_\zeta}{\partial \zeta^2} \right], \quad \frac{\partial U_\xi}{\partial \xi} + \frac{\partial U_\zeta}{\partial \zeta} = 0, \quad (7)$$

$$U_\xi = \psi \left(\frac{u_{2m}}{w_{2m}} \right) \frac{\partial U_2}{\partial \tau}, \quad U_\zeta = \frac{\partial W_2}{\partial \tau} \quad \text{at } \zeta = 1 + \lambda_2 W_2, \quad (8)$$

$$U_\xi = \psi \left(\frac{u_{1m}}{w_{2m}} \right) \frac{\partial U_1}{\partial \tau}, \quad U_\zeta = \left(\frac{w_{1m}}{w_{2m}} \right) \frac{\partial W_1}{\partial \tau} \quad \text{at } \zeta = \lambda_1 W_1, \quad (9)$$

$$P = 0 \quad \text{at } \xi = \pm 1. \quad (9)$$

The normal stress, acting from the liquid layer on the plate, will be written down, as:

$$q = \rho\delta_0\ddot{z}_0(\zeta - 1 - \lambda_2 W_2) - p_0 - \nu\rho w_{2m}\omega(\psi^2\delta_0)^{-1}(P - 2\psi^2) \frac{\partial U_\zeta}{\partial \zeta}. \quad (10)$$

Further, we note that in the considered setting $\psi = o(1)$, $u_{im}/w_{2m} = O(1)$, $\lambda_1 = \lambda_2(w_{1m}/w_{2m}) = o(1)$, and therefore, the member at $\psi, \psi^2, \lambda_1, \lambda_2$ in Eq. (7), boundary Eq. (8), (9) and normal stress Eq. (10) can be omitted. As a result, we get liquid dynamic problem in the form of the Eq. (11) and boundary Eq. (12):

$$\frac{\partial P}{\partial \xi} = \frac{\partial^2 U_\xi}{\partial \zeta^2}, \quad \frac{\partial P}{\partial \zeta} = 0, \quad \frac{\partial U_\xi}{\partial \xi} + \frac{\partial U_\zeta}{\partial \zeta} = 0, \quad (11)$$

$$U_\xi = 0, \quad U_\zeta = \frac{\partial W_2}{\partial \tau} \quad \text{at } \zeta = 1, \quad U_\xi = 0, \quad U_\zeta = \left(\frac{w_{1m}}{w_{2m}} \right) \frac{\partial W_1}{\partial \tau} \quad \text{at } \zeta = 0, \quad (12)$$

$$P = 0 \quad \text{at } \xi = \pm 1.$$

The solutions of the Eq. (11) with boundary Eq. (12) has the form of:

$$U_\xi = \frac{\zeta^2 - \zeta}{2} \frac{\partial P}{\partial \xi}, \quad U_\zeta = \frac{w_{1m}}{w_{2m}} \frac{\partial W_1}{\partial \tau} + \frac{3\zeta^2 - 2\zeta^3}{12} \frac{\partial^2 P}{\partial \xi^2},$$

$$P = 12 \int_1^\xi \int \frac{\partial \tilde{W}}{\partial \tau} d\xi d\xi - 6(\xi - 1) \int_{-1}^1 \int \frac{\partial \tilde{W}}{\partial \tau} d\xi d\xi, \quad \tilde{W} = W_2 - \frac{w_{1m}}{w_{2m}} W_1. \quad (13)$$

By substituting Eq. (13) in the expression from the normal stress Eq. (10), written down in zero approximation at ψ and λ_2 , and considering the normal stress in channel walls dynamics Eq. (1), we get integro-differential equations of channel walls hydroelastic oscillations:

$$D_i \ell^{-4} \frac{\partial^4 w_i}{\partial \xi^4} + (2 - i)\chi w_1 + \rho_i h_i \omega^2 \left(\partial^2 w_i / \partial \tau^2 + \ddot{z}_0 \omega^{-2} \left(1 + \frac{(2 - i)\rho \delta_0}{\rho_i h_i} \right) \right)$$

$$= (-1)^i \left(p_0 + 12\nu\rho w_{2m} \omega (\psi^2 \delta_0)^{-1} \left(\int_1^\xi \int \frac{\partial \tilde{W}}{\partial \tau} d\xi d\xi - (\xi - 1)/2 \int_{-1}^1 \int \frac{\partial \tilde{W}}{\partial \tau} d\xi d\xi \right) \right). \quad (14)$$

With consideration of boundary Eq. (2), (5) we will look for the assigned equations solution in the form of:

$$w_i = w_{im} W_i = w_{im} \sum_{k=1}^{\infty} \left(R_{ik}^0 + R_{ik}(\tau) \right) \cos \left(\frac{(2k - 1)\pi \xi}{2} \right), \quad i = 1, 2, \quad (15)$$

here the upper index 0 means the solution, corresponding to static pressure p_0 .

By substituting Eq. (15) into Eq. (14) and setting the remaining members of Eq. (14) in series of trigonometric functions of longitudinal coordinate ξ and then, equating the coefficients at similar trigonometric functions, we obtain the expression for R_{ik}^0 :

$$w_{im} R_{ik}^0 = (-1)^{i-1} 4p_0 (-1)^k ((2k - 1)\pi)^{-1} \left(D_i \ell^{-4} \left(\frac{(2k - 1)\pi}{2} \right)^4 + (2 - i)\chi \right)^{-1}, \quad (16)$$

and the system of ordinary differential equations for $R_{ik}(\tau)$:

$$w_{1m} a_{11} R_{1k} + w_{1m} a_{12} \frac{dR_{1k}}{d\tau} + w_{2m} a_{14} \frac{dR_{2k}}{d\tau} = -d_1 E_z \omega^2 \frac{d^2 f_0}{d\tau^2},$$

$$w_{1m} a_{22} \frac{dR_{1k}}{d\tau} + w_{2m} a_{23} R_{2k} + w_{2m} a_{24} \frac{dR_{2k}}{d\tau} = -d_2 E_z \omega^2 \frac{d^2 f_0}{d\tau^2}. \quad (17)$$

Here, it is taken into account that for the stationary harmonic oscillations regime we have $d^2 R_{ik} / d\tau^2 = -R_{ik}$ and the following symbols having been introduced:

$$a_{11} = D_1 \ell^{-4} ((2k - 1)\pi/2)^4 + \chi - \rho_1 h_1 \omega^2, \quad a_{12} = 2K_k \omega, \quad a_{14} = -a_{12},$$

$$a_{22} = a_{14} = -a_{12}, \quad a_{23} = D_2 \ell^{-4} ((2k - 1)\pi/2)^4 - \rho_2 h_2 \omega^2, \quad a_{24} = a_{12} = -a_{14},$$

$$2K_k = 12\nu\rho\psi^{-2} \delta_0^{-1} (2/((2k - 1)\pi))^2, \quad d_i = -4(-1)^k ((2k - 1)\pi)^{-1} (\rho_i h_i + (2 - i)\rho \delta_0).$$

By solving the system Eq. (17) for the stationary harmonic oscillations regime we find:

$$w_{im} R_{ik} = E_z \omega^2 \left(\frac{F_k C_{ik} - C_k F_{ik}}{F_k^2 + C_k^2} \frac{df_0(\tau)}{d\tau} - \frac{F_k F_{ik} + C_k C_{ik}}{F_k^2 + C_k^2} \frac{d^2 f_0(\tau)}{d\tau^2} \right), \quad (18)$$

here $i = 1, 2$ for plates 1 and 2, the following symbols having been introduced:

$$F_k = a_{11}a_{23}, \quad C_k = a_{12}(a_{11} + a_{23}), \quad F_{1k} = d_1a_{23}, \\ F_{2k} = d_2a_{11}, \quad C_{1k} = C_{2k} = a_{12}(d_1 + d_2).$$

Considering Eq. (16) and (18), we can present the final expressions for plate deflections as:

$$w_i = \frac{p_0 \ell^4}{D_i} \sum_{k=1}^{\infty} \left(\frac{4(-1)^k (-1)^{i-1} \cos((2k-1)\pi x / (2\ell))}{(2k-1)\pi \left((2k-1)\pi/2 \right)^4 + (2-i)\chi \ell^4 / D_i} \right) \\ + E_z \omega^2 A_i(x, \omega) \sin(\omega t + \phi_i(x, \omega)), \\ A_i = \sqrt{Q_i^2 + N_i^2}, \quad \phi_i = \arctg\left(\frac{N_i}{Q_i}\right), \\ Q_i = \sum_{k=1}^{\infty} \frac{F_k F_{ik} + C_k C_{ik}}{F_k^2 + C_k^2} \cos \frac{(2k-1)\pi x}{2\ell}, \quad N_i = \sum_{k=1}^{\infty} \frac{F_k C_{ik} - C_k F_{ik}}{F_k^2 + C_k^2} \cos \frac{(2k-1)\pi x}{2\ell}. \tag{19}$$

Taking into account Eqs. (18), (15) in the expression for pressure Eq. (13) we finally obtain the law of dimensional pressure distribution in the channel in the form:

$$p = p_0 - \rho \ddot{z}_0 \left(z - \delta_0 - \frac{h_1}{2} \right) + E_z \omega^2 \Pi(x, \omega) \sin(\omega t + \phi_p(x, \omega)), \\ \Pi(x, \omega) = \sqrt{B_k^2 + S_k^2}, \quad \phi_p(x, \omega) = \arctg\left(\frac{S_k}{B_k}\right), \\ B_k = \sum_{k=1}^{\infty} 2K_k \omega C_k \frac{F_{1k} - F_{2k}}{F_k^2 + C_k^2} \cos \frac{(2k-1)\pi x}{2\ell}, \\ S_k = \sum_{k=1}^{\infty} 2K_k \omega F_k \frac{F_{1k} - F_{2k}}{F_k^2 + C_k^2} \cos \frac{(2k-1)\pi x}{2\ell}. \tag{20}$$

4. Conclusions

The first member of the Eq. (19) is the channel walls static deflection conditioned by static pressure in the liquid p_0 , the second member is the deflection, conditioned by foundation vibration. The first component in pressure distribution law Eq. (20) represents static liquid pressure in the channel. The second component is dynamic liquid pressure, conditioned by foundation vibro-acceleration (an analogue of pressure due to the gravity). The third component is pressure, conditioned by channel walls oscillations under foundation vibro-acceleration.

It must be noted that the first and the second pressure components cannot exceed the assigned values. The third component pressure value is defined by the function $\Pi(x, \omega)$, which represents pressure distribution frequency dependent function along the channel. Under longitudinal coordinate fixed value x , the assigned function is the pressure amplitude characteristic in the assigned channel cross-section. The same can be stated in relation to the deflections amplitudes distribution functions $A_i(x, \omega)$, being frequency dependent ones. The functions $\phi_i(x, \omega)$, $\phi_p(x, \omega)$ are frequency dependent ones of the plates deflections phase shifts distribution and the pressure phase shift along the channel, correspondingly.

The investigation of the above mentioned functions behavior, depending on oscillations frequency, makes it possible to study dynamic processes in the channel under consideration. The obtained expressions for channel walls deflections analysis allows to state that the elastic coefficient of Winkler foundation in a static state causes deflection reduction of the first plate, deflections amplitude dynamics reducing in both plates in a dynamic state. Thus, the paper results

can be used for calculation and analysis hydroelastic oscillations of channel walls on elastic foundation and for calculation of hydraulic dampers, pumps elements, hydrodrives, lubrication, cooling and fuel supply systems.

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