Periodic Skeletons of Nonlinear Dynamical Systems in the Problems of Global Bifurcation Analysis

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Abstract. The construction of the periodic skeleton is a search for stable and unstable periodic regimes for a given parameter space of nonlinear dynamical periodicals systems. This stage of the main non-linear bifurcation theory, which is designed for a global analysis of nonlinear dynamical systems and the state of the parameter space that allows for complete bifurcation analysis, build complex bifurcation group and discover new previously unknown solutions.

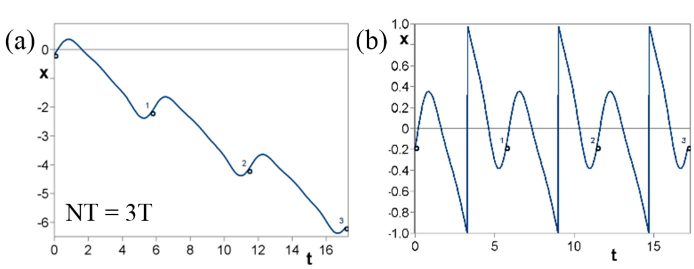
1. Introduction

For the global analysis of nonlinear dynamical periodic systems, before commencing the research it is useful to find – in the given point of parameter space – all stable and unstable orbits (solutions, regimes) and their basic passports for oscillating regimes P1, P2, P3, …, P*n*, and for rotating regimes R1, R2, R3, …, R*n*, where *n* – a maximum order of findable solutions [1-7]. For this purpose, according to a certain rule, on the plane of states a certain number (sometimes a quite large number) of initial points is set, and they are used to find periodic regimes (coordinates of fixed points) on the basis of the Newton-Kantorovich method and to evaluate the stability of these regimes according to multipliers. Searching the fixed points or the rotational orbits the cylindrical phase space is taken into account with period $Lx = 2$ (see Fig. 1).

The found stable and unstable regimes with their passports are used for the global analysis both in the state space and parameter space. The basic passport of each periodic regime includes information about the order of the regime, coordinates of its fixed point and values of multipliers, quantitatively characterizing stability or instability of the found periodic regimes. For example, the line below:

P5 (3/5), Fixed Point (2.815634, -0.293778), Multipliers $\rho_1 = -0.467$, $\rho_2 = -1.813$ (1)

The subharmonic regime with the period $T = 5T_\omega$ has a projection of phase periodic trajectory with three loops; the coordinates of a fixed point on the Poincaré plane are presented in brackets. The unstable regime – inverted saddle – since $\rho_2 = -1.813$. It is worth pointing out that a basic passport of periodic regime was applied by T. Hayashi, J. Ueda in works [8, 9].



(c)

N	t	x	v
0	0.000000	-0.177442	0.850614
1	5.711987	-2.177443	0.850615
2	11.423973	-4.177442	0.850614

(d)

N	t	x	v
0	0.000000	-0.177442	0.850614
1	5.711987	-0.177442	0.850615
2	11.423973	-0.177442	0.850614

Figure 1. The example of calculation of periodic rotating orbit of period-1 without (a), (c) and with (b), (d) taking into account the cylindrical phase space with period $Lx = 2$.

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The most valuable information for the global analysis in the parameter space is presented by the found unstable regimes, the continuation of which along the branches of their solutions in the parameter space enables the discovery of new bifurcation groups and rare attractors. The procedures of constructing the periodic skeleton are included into the software SPRING [2], created in Institute of Mechanics of Riga Technical University.

In this work the constructions of periodic skeletons are considered for four nonlinear dynamical systems: Duffing-Ueda system, system with one degree of freedom and with five equilibrium positions (three-well potential system), pendulum system with several equilibrium positions and six body system with cubic elastic connection.

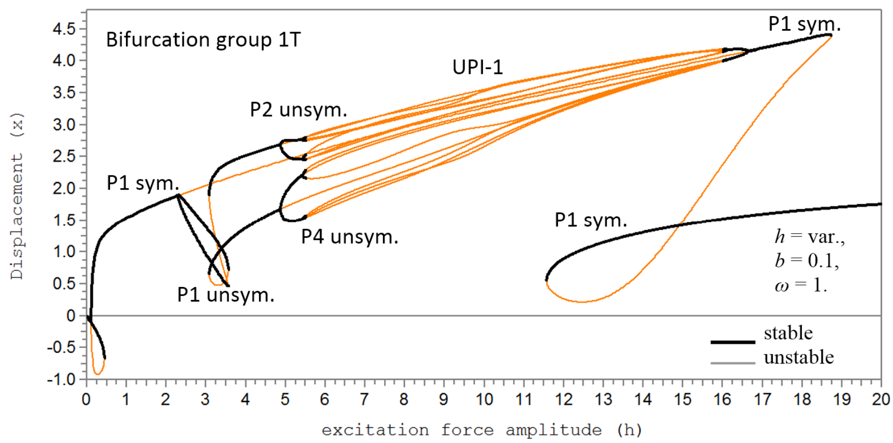


Figure 2. Bifurcation diagram for Duffing-Ueda system with linear friction and harmonic driving force. Shown only bifurcation group 1T, but there are else bifurcation groups 5T and two bifurcation groups 3T with rare attractors (RA). Stable solutions are plotted by black solid lines and unstable – by thin lines (reddish online). Eq.: $\ddot{x}+b\dot{x}+x^3=h\cos(\omega t)$. Parameters: b = 0.1, h = var., ω = 1, k = 7. For h = 10 the system has global chaotic attractor.

Table 1. The example of periodic skeleton for the Duffing-Ueda system with linear friction and harmonic driving force. There are shown some unstable periodic regimes from period P1 to P3, at this parameter system have four different bifurcation groups. Parameters: $b=0.1$, $h=10$, $\omega=1$, $k=7$.

N	Orbits	x	v	ρ_1	ρ_2	α	Bif.group
1.	P1u	2.84812	-0.921283	-0.306	-1.743	180°	1T
2.	P1u	3.550309	-0.789157	-0.305	-1.743	180°	1T
3.	P1u	3.267975	0.634068	14.9	0.035	180°	1T
4.	P2u	2.716156	-1.851278	-0.010	-27.6	180°	1T
5.	P2u	3.313121	1.933647	-0.010	-27.6	180°	1T
6.	P3u	3.196589	3.368722	-0.045	-3.35	180°	3T
7.	P3u	3.314358	-1.627349	-0.045	-3.35	180°	3T
8.	P3u	2.417191	0.376985	3.51	0.043	0°	3T
9.	P3u	2.607611	0.038841	44.9	0.007	0°	3T

2. Duffing-Ueda system [9]

The first example of the construction of periodic skeleton (PSK) in studied Duffing-Ueda system with linear friction and harmonic driving force. The equation for this system in dimensionless form is such

$$\ddot{x}+b\dot{x}+x^3=h\cos(\omega t) \quad (2)$$

The system has one degree of freedom and restoring force has one equilibrium position. For this model has been constructed periodic skeleton for regimes order from P1 up to P5 for $h=10$. For this parameters the system has global chaotic attractor which forms from different bifurcation groups (see Table 1).

3. System with one degree of freedom and five equilibrium positions (3-well potential)
Next example – three-well potential symmetrical damped system described by equation

$$\ddot{x} + b\dot{x} + f(x) = h\cos(\omega t) \tag{3}$$

where

$$f(x) = 1.2x - 1.8x|x| + 0.6x^3 \tag{4}$$

$$\Pi(x) = 0.6x^2 - 0.6x^3 sign(x) + 0.15x^4 \tag{5}$$

Periodic skeleton was build for $h_1 = 1.1$ where the system has three period one attractors. Many different bifurcation groups such as 3T, 5T, 7T have been found.

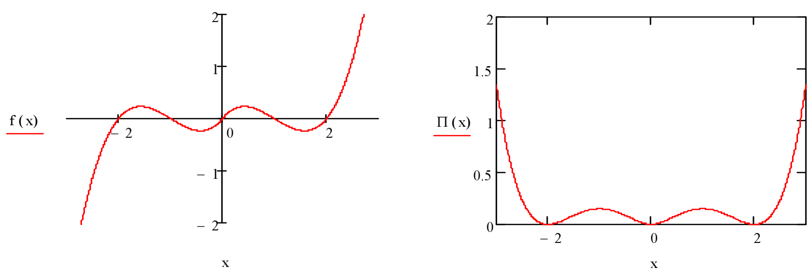


Figure 3. Restoring force and potential well. See (4), (5).

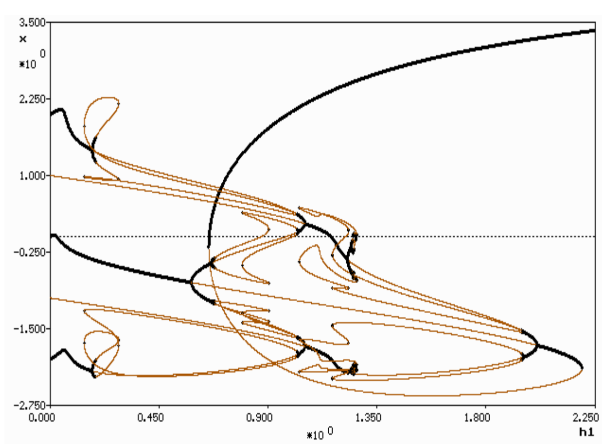


Figure 4. 1T Bifurcation diagram for three – well potential system. Parameters: h_1 = var., ω = 1, b = 0.2.

Table 2. The example of periodic skeleton (a part) for $h_1 = 1.1$. (Shown table is not complete).

N	n	Stab	x	v	ρ_{max}	α	x_0	v_0
1	1	U	-1.094243	0.206176	10.74	0.0	-2.4400	0.0
2	1	S	-1.859098	0.151827	0.533	125.68	-2.4100	0.0
3	1	U	-2.366013	2.063071	5.824	0.0	-2.1100	0.0
4	1	S	2.268269	2.852704	0.533	133.16	-1.4500	0.0
23	3	U	0.492823	0.178453	279.0	0.0	0.2300	0.0060
24	3	U	-0.856195	-0.000677	418.4	0.0	-0.8800	0.0120
25	3	U	-1.180742	0.198907	-562.3	0.0	-0.8200	0.0300
59	5	U	0.439931	0.266829	530.1	0.0	0.5000	0.2220
63	6	U	-0.780318	0.142094	129.6	0.0	-0.7900	0.1200
76	7	U	0.500373	0.215648	-582.8	0.0	0.5000	0.2160
77	7	U	-1.017628	0.240051	-3300	0.0	-1.0900	0.2820

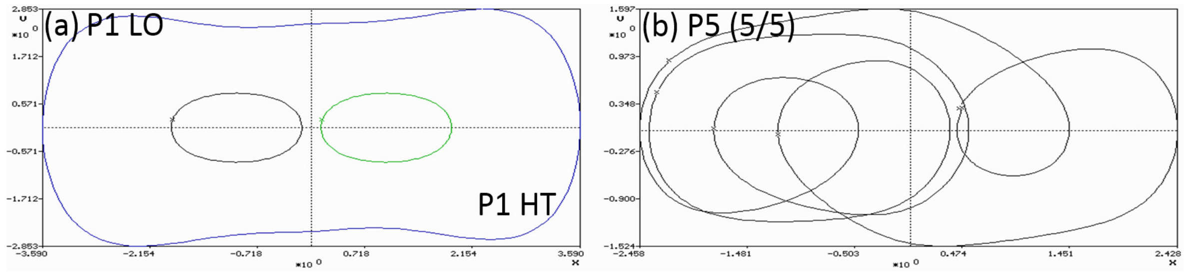


Figure 5. (a) Hilltop stable period-one twin orbits (point 2 from Table 2).
(b) Unstable period-five (point 59 from Table 2). Parameters: $h_1 = 1.1$, $\omega = 1$, $b = 0.2$.

4. Pendulum system with several equilibrium positions [10, 11]

The example of the construction of periodic skeleton (PSK) in studied pendulum system with several equilibrium positions (6) for several oscillating and rotating orbits (see Table 3) by using the Newton-Kantorovich method, is shown. For rotational orbits the cylindrical phase space was taken into account with period $L_\beta = 2\pi$. Full results of complete bifurcation analysis of this system are performed in work [7]. The equation of motion of studied pendulum system in dimensionless form is

$$\ddot{\beta} + c\dot{\beta} - p\sin\beta + \left[1 - \frac{1}{\sqrt{1+\alpha\sin\beta}} - \left(q_0 + q_1\sin\omega t\right)\right]\cos\beta = 0 \qquad (6)$$

where α is the geometrical coefficient which depends from the length of the pendulum H and the horizontal distance L; q_0 is an offset of amplitude of excitation force; ω is the frequency of external excitation force.

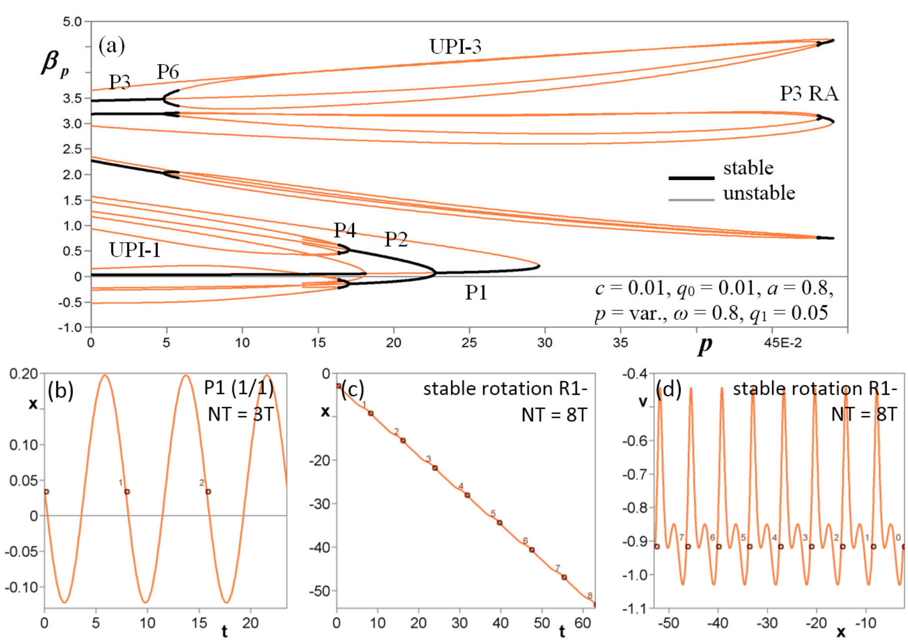


Figure 6. (a) Bifurcation diagram Sn (p) of the fixed periodic points of the coordinate β_p of the oscillating orbits versus gravitation coefficient p. In the diagram the 1T and 3T bifurcation groups are shown. There are other different oscillating and rotating bifurcation groups in this system. (b) Time history for stable periodic oscillating orbit P1 (1/1) (see fixed point in Table 3 by orbit Nr.3). (c), (d) Time history and phase portrait for stable R1 counter-clockwise rotational orbit (see in Table 3 the orbit Nr.4).

Table 3. The example of periodic skeleton of five oscillating and rotating orbits for the pendulum system with linear spring and several equilibrium positions (6). Others regimes of periodic skeleton were shown in Table 7.1 [7]. Parameters: $c = 0.01$, $q_0 = 0.01$, $a = 0.8$, $p = 0.05$, $\omega = 0.8$, $q_1 = 0.05$, $k = 7$.

N	Orbits	β	$\dot{\beta}$	ρ_1	ρ_2	α	Bif.group
1.	R1 u-	-0.585939	-0.86374	3.745	0.247	0°	R1 -
2.	R1 s-	-2.497295	-0.911182	0.961	0.961	160.1°	R1 -
3.	P1 (1/1) s	0.036278	-0.127843	0.961	0.961	211.4°	1T ($\beta = 0$)
4.	R1 s+	-0.595455	0.832939	0.962	0.962	160.3°	R1 +
5.	P1 (1/1) s	3.109178	0.185429	0.961	0.961	142.3°	1T ($\beta = \pi$)

5. Six body system with cubic connections

In this chapter shown example for system with six degree of freedom with linear damping, cubic restoring force and external force. The six body model are shown on Fig. 7 and the equation of motion are (7). In the Table 4 shown periodic skeleton for this model.

$$\begin{cases} m_1\ddot{x}_1 + Fv_1 + Fx_1 - Fv_2 - Fx_2 = Ft \\ m_2\ddot{x}_2 + Fv_2 + Fx_2 - Fv_3 - Fx_3 = 0 \\ m_3\ddot{x}_3 + Fv_3 + Fx_3 - Fv_4 - Fx_4 = 0 \\ m_4\ddot{x}_4 + Fv_4 + Fx_4 - Fv_5 - Fx_5 = 0 \\ m_6\ddot{x}_6 + Fv_6 + Fx_6 = 0 \end{cases} \tag{7}$$

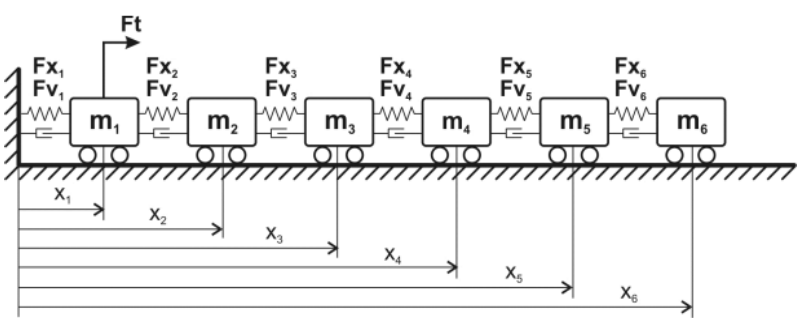
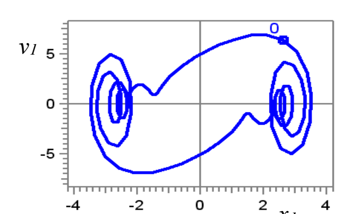


Figure 7. Six body model.

Full periodic skeleton for regime P1 (in Table 4 first position), there are shown all coordinates for six mass, phase diagram for first mass and stability multipliers ρ and α.



Fixed point

$x_1 = 2.597699$	$v_1 = 6.422203$
$x_2 = 2.151884$	$v_2 = 6.466516$
$x_3 = 2.328209$	$v_3 = 6.748152$
$x_4 = 2.377062$	$v_4 = 6.744016$
$x_5 = 2.051667$	$v_5 = 6.394883$
$x_6 = 2.131998$	$v_6 = 6.637183$

Stability

$\rho_1 = 0.853461$	$\alpha_1 = -31°$	$\rho_7 = 0.288625$	$\alpha_7 = 36°$
$\rho_2 = 0.853461$	$\alpha_2 = 31°$	$\rho_8 = 0.288625$	$\alpha_8 = -36°$
$\rho_3 = 0.490673$	$\alpha_3 = -132°$	$\rho_9 = 0.472005$	$\alpha_9 = -72°$
$\rho_4 = 0.490673$	$\alpha_4 = 132°$	$\rho_{10} = 0.472005$	$\alpha_{10} = 72°$
$\rho_5 = 0.458196$	$\alpha_5 = -130°$	$\rho_{11} = 0.1944$	$\alpha_{11} = -141°$
$\rho_6 = 0.458196$	$\alpha_6 = 130°$	$\rho_{12} = 0.1944$	$\alpha_{12} = 141°$

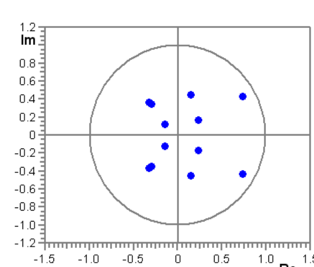


Table 4. The example of periodic skeleton of the six body model with six degree of freedom with linear damping, cubic restoring force and external force.
Parameters: $\omega = 0.66$, $h = 10$, $m = 1$, $b = 0.1$, $c_1 = 1$, $c_3 = 1$.

No	Ord.	x_1	v_1	$x_2, v_2, \dots x_5, v_5$	x_6	v_6	ρ_{max}
1	P1	2.597699	6.422203	…	2.131998	6.637183	0.853
2	P1	1.396726	0.407374	…	-3.2801	-1.38399	1.455
3	P1	1.180483	0.680359	…	-2.52102	-1.11373	1.317
4	P1	1.687686	-0.24746	…	-3.95983	-1.19284	1.317
5	P1	1.811503	-0.28237	…	-3.94447	-1.36865	1.174
6	P1	1.108979	0.656915	…	-2.19438	-0.41891	1.174
7	P1	1.437817	0.403361	…	-3.323	-1.45517	1.447
8	P1	1.357571	0.399155	…	-3.23412	-1.30715	1.447
9	P4	0.885893	0.027445	…	-5.47703	0.423135	2.017
10	P4	2.019645	1.495407	…	-0.80929	-1.58506	2.017
11	P4	1.942505	0.644733	…	-0.22094	0.780237	0.626

6. Conclusions

This work shows usefulness to carry out the preliminary construction of periodic skeleton for the given parameter $p = p_0$ in order to successfully construct a complete bifurcation diagrams in nonlinear dynamical periodic systems. Founded stable and unstable periodic orbits allow conducting the global bifurcation analysis, building complex bifurcation group and discovering new previously unknown solutions in Duffing-Ueda system, system with one degree of freedom and with five equilibrium positions, pendulum system several equilibrium positions and six body system with cubic connections.

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