ON THE PERMUTATION SEQUENCE AND ITS SOME PROPERTIES*

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ABSTRACT. The main purpose of this paper is to prove that there is no any perfect power among the permutation sequence: 12, 1342, 135642, 13578642, 13579108642, ⋯⋯. This answered the question 20 of F.Smarandach in [1].

for $n \leq 9$ partially

1. INTRODUCTION

For any positive integer n, we define the permutation sequence $\{P(n)\}$ as follows: $P(1) = 12$, $P(2) = 1342$, $P(3) = 135642$, $P(4) = 13578642$, $P(5) = 13579108642$, $\cdots\cdots$, $P(n) = 135\cdots(2n-1)(2n)(2n-2)\cdots 42$, $\cdots\cdots$,. In problem 20 of [1], Professor F.Smarandach asked us to answer such a question: Is there any perfect power among these numbers? Conjecture: no! This problem is interesting, because it can help us to find some new properties of permutation sequence. In this paper, we shall study the properties of the permutation sequence $P(n)$, and proved that the F.Smarandach conjecture is true. This solved the problem 20 of [1], and more, we also obtained some new divisible properties of $P(n)$. That is, we shall prove the following conclusion:

Theorem. *There is no any perfect power among permutation sequence, and*

$$P(n) = \frac{1}{81}\left(11 \cdot 10^{2n} - 13 \cdot 10^{n} + 2\right) = \overbrace{11\cdots1}^{n} \times 1\overbrace{22\cdots2}^{n}, \text{ for } n \leq 9.$$

2. PROOF OF THE THEOREM

In this section, we complete the proof of the Theorem. First for any positive integer n, we have

$$\begin{aligned} P(n) &= 10^{2n-1} + 3 \times 10^{2n-2} + \cdots + (2n-1) \times 10^{n} \\ &\quad + 2n \times 10^{n-1} + (2n-2) \times 10^{n-2} + \cdots 4 \times 10 + 2 \\ &= \left[10^{2n-1} + 3 \times 10^{2n-2} + \cdots + (2n-1) \times 10^{n}\right] \\ &\quad + \left[2n \times 10^{n-1} + (2n-2) \times 10^{n-2} + \cdots 4 \times 10 + 2\right] \end{aligned}$$

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(1) $$\equiv S_1 + S_2.$$

Now we compute S_1 and S_2 in (1) respectively. Note that

$$\begin{aligned} 9S_1 = 10S_1 - S_1 &= 10^{2n} + 3 \times 10^{2n-1} + \cdots (2n-1) \times 10^{n+1} \\ &\quad - 10^{2n-1} - 3 \times 10^{2n-2} - \cdots - (2n-1) \times 10^n \\ &= 10^{2n} + 2 \times 10^{2n-1} + 2 \times 10^{2n-2} + \cdots + 2 \times 10^{n+1} - (2n-1) \times 10^n \\ &= 10^{2n} + 2 \times 10^{n+1} \times \frac{10^{n-1}-1}{9} - (2n-1) \times 10^n \end{aligned}$$

and

$$\begin{aligned} 9S_2 = 10S_2 - S_2 &= 2n \times 10^n + (2n-2) \times 10^{n-1} + \cdots 4 \times 10^2 + 2 \times 10 \\ &\quad - 2n \times 10^{n-1} - (2n-2) \times 10^{n-2} - \cdots 4 \times 10 - 2 \\ &= 2n \times 10^n - 2 \times 10^{n-1} - 2 \times 10^{n-2} - \cdots 2 \times 10 - 2 \\ &= 2n \times 10^n - 2 \times \frac{10^n - 1}{9}. \end{aligned}$$

So that

(2) $$S_1 = \frac{1}{81} \times \left[11 \times 10^{2n} - 18n \times 10^n - 11 \times 10^n\right]$$

and

(3) $$S_2 = \frac{1}{81}\left[18n \times 10^n - 2 \times 10^n + 2\right].$$

Thus combining (1), (2) and (3) we have

$$\begin{aligned} P(n) = S_1 + S_2 &= \frac{1}{81} \times \left[11 \times 10^{2n} - 18n \times 10^n - 11 \times 10^n\right] \\ &\quad + \frac{1}{81}\left[18n \times 10^n - 2 \times 10^n + 2\right] \end{aligned}$$

(4) $$= \frac{1}{81}\left(11 \cdot 10^{2n} - 13 \cdot 10^n + 2\right) = \overbrace{11 \cdots 1}^{n} \times 1\overbrace{22 \cdots 2}^{n}.$$

From (4) we can easily find that $2 \mid P(n)$, but $4 \nmid P(n)$, if $n \geq 2$, (and $n \leq 9$.) So that $P(n)$ can not be a perfect power, if $n \geq 2$. In fact, if we assume $P(n)$ be a perfect power, then $P(n) = m^k$, for some positive integer $m \geq 2$ and $k \geq 2$. Since $2 \mid P(n)$, so that m must be an even number. Thus we have $4 \mid P(n)$. This contradiction with $4 \nmid P(n)$, if $n \geq 2$. Note that $P(1)$ is not a perfect power, so that $P(n)$ can be a perfect power for all $n \geq 1$, (and $n \leq 9$.) This completes the proof of the Theorem.

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