SOME NEW RESULTS CONCERNING THE SMARANDACHE CEIL FUNCTION

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Abstract: In this article we present two new results concerning the Smarandache Ceil function. The first result proposes an equation for the number of fixed-point number of the Smarandache ceil function. Based on this result we prove that the average of the Smarandache ceil function is $\Theta(n)$.

1. INTRODUCTION

In this section we review briefly the main results that are used in this article. These concern the Smarandache ceil and functions. The Smarandache ceil function of order k [see www.gallup.unm.edu/~smarandache] is denoted by $S_k : N^* \to N$ and has the following definition

$$S_k(n) = \min\{x \in N \mid x^k \vdots n\} (\forall n \in N^*). \tag{1}$$

This was introduced by Smarandache [1993] who proposed many open problems concerning it. Ibstedt [1997, 1999] studied this function both theoretically and computationally. The main properties proposed in [Ibstedt, 1997] are presented in the following

$$(\forall a, b \in N^*)\,(a,b) = 1 \Rightarrow S_k(a \cdot b) = S_k(a) \cdot S_k(b), \tag{2.a}$$

$$S_k(p_1^{a_1} \cdot \ldots \cdot p_s^{a_s}) = S(p_1^{a_1}) \cdot \ldots \cdot S(p_s^{a_s})\} \text{ and} \tag{2.b}$$

$$S_k(p^a) = p^{\left\lceil \frac{a}{k} \right\rceil}. \tag{2.b}$$

Therefore, if $n = p_1^{a_1} \cdot \ldots \cdot p_s^{a_s}$ is the prime number decomposition of n, then the equation of this function is given by

$$S_k(p_1^{a_1} \cdot \ldots \cdot p_s^{a_s}) = p_1^{\left\lceil \frac{a_1}{k} \right\rceil} \cdot \ldots \cdot p_s^{\left\lceil \frac{a_s}{k} \right\rceil}. \tag{3}$$

Based on these properties, Ibstedt proposed the following results

$$S_{k+1}(n) \vdots S_k(n)\ \forall n>1 \qquad (4)$$

$$n=p_1\cdot\ldots\cdot p_s \Rightarrow S_2(n)=n. \qquad (5)$$

Table 1 shows the values of the Smarandache ceil function of order 2 for $n<25$.

n	$S_2(n)$	N	$S_2(n)$	N	$S_2(n)$	N	$S_2(n)$	n	$S_2(n)$
1	1	6	6	11	11	16	4	21	21
2	2	7	7	12	6	17	17	22	22
3	3	8	4	13	13	18	6	23	23
4	2	9	3	14	14	19	19	24	12
5	5	10	10	15	15	20	10	25	5

Table 1. The Smarandache ceil function.

The Mobius function $\mu: N \to Z$ is defined as follows

$$\mu(1)=1 \qquad (6.a)$$

$$\mu(n)=(-1)^s \text{ if } n=p_1\cdot\ldots\cdot p_s \qquad (6.b)$$

$$\mu(n)=0 \text{ otherwise}. \qquad (6.c)$$

This is an important function both in Number Theory and Combinatorics because gives two inversion equations. The first Mobius inversion formula [Chandrasekharan, 1970] is

$$g(n)=\sum_{d|n} f(d) \Leftrightarrow f(n)=\sum_{d|n}\mu(d)\cdot g(\frac{n}{d}) \qquad (7.a)$$

while the second Mobius formula is

$$g(x)=\sum_{n\le x} f(\frac{x}{n}) \Leftrightarrow f(x)=\sum_{n\le x}\mu(n)\cdot g(\frac{x}{n}). \qquad (7.b)$$

There are several equations concerning series involving the Mobius function [Apostol, 1976]. Among them an important series is

$$\sum_{n>0}\frac{\mu(n)}{n^2}=\frac{6}{\pi^2} \qquad (8.a)$$

that has the following asymptotic form

$$\sum_{0<n\le x}\frac{\mu(n)}{n^2}=\frac{6}{\pi^2}+O(\frac{1}{x}). \qquad (8.b)$$

2. THE ASYMPTOTIC DENSITY OF FIXED POINTS

In this section we present an equation for the asymptotic density of the function S_k's fixed points. The main result presented can also be found in [Keng, 1981] but we give it a detailed proof. We start by remarking that the function S_2 has quit many points. For example, there are 16 fixed points for the first 25 numbers.

Let $q(x)$ be the number of the fixed points less than x: $q(x) = \#\{n \le x : S_k(n) = n\}$. We say that the fixed points have the asymptotic density equal to a if $\lim_{x \to \infty} \frac{q(x)}{x} = a$.

Ibstedt [1997] found that if n is a square free number then it is a fixed point for S_2. Actually, the result holds for any Smarandache ceil function.

Proposition 1. $n = p_1 \cdot \ldots \cdot p_s \Leftrightarrow S_k(n) = n$.

Proof Let $n = p_1^{a_1} \cdot \ldots \cdot p_s^{a_s}$ be the prime number decomposition of n. The following equivalence gives the proof:

$$S_k(n) = n \Leftrightarrow p_1^{a_1} \cdot \ldots \cdot p_s^{a_s} = p_1^{\left\lceil \frac{a_1}{k} \right\rceil} \cdot \ldots \cdot p_s^{\left\lceil \frac{a_s}{k} \right\rceil} \Leftrightarrow$$

$$\left\lceil \frac{a_i}{k} \right\rceil = a_i, i = 1,2,\ldots,s \Leftrightarrow a_i = 1, i = 1,2,\ldots,s \Leftrightarrow n = p_1 \cdot \ldots \cdot p_s.$$

Therefore, n is a square free number. ♦

Proposition 2. $(\forall\, n \in N)(\exists!\, d \mid n) \frac{n}{d^2}$ is square free. (9)

Proof. Firstly, we prove that there is such as divisor. If $n = p_1^{a_1} \cdot \ldots \cdot p_s^{a_s}$ the prime number decomposition, then $d = p_1^{\left\lfloor \frac{a_1}{2} \right\rfloor} \cdot \ldots \cdot p_s^{\left\lfloor \frac{a_s}{2} \right\rfloor}$ satisfies $\frac{n}{d^2}$ is square free. Actually, $\frac{n}{d^2}$ is the product of all prime numbers that have odd power in the prime number decomposition of n. Now, we prove that d is unique. Assume that there are distinct divisors such that $\frac{n}{d_1^2}, \frac{n}{d_2^2}$ are square free. We can write this as follows $n = d_1^2 \cdot p_1 \cdot \ldots \cdot p_s = d_2^2 \cdot q_1 \cdot \ldots \cdot q_r$. Let p be a prime number that does not appear in the both sites $p_1, \ldots, p_s$ and $q_1, \ldots, q_r$ (choose that it is in the first). p

should also appear in the prime number decomposition of d_2^2. Therefore, we find that the power of p is even for the right hand side and odd for the left hand side. ♦

Proposition 3. $\{0<n\le x\}=\bigcup_{d=1}^{\lfloor\sqrt{x}\rfloor} d^2\cdot\{i\le\frac{x}{d^2}:\text{i is square free}\}$. (10)

Proof. It is enough to prove this equation just for natural number. Consider n>1 a natural number. Equation (10) becomes

$$\{1,2,...,n\}=\bigcup_{d=1}^{\lfloor\sqrt{n}\rfloor} d^2\cdot\{i\le\frac{n}{d^2}:\text{i is square free}\}. \quad (11)$$

The inclusion $\{1,2,...,n\}\supseteq\bigcup_{d=1}^{\lfloor\sqrt{n}\rfloor} d^2\cdot\{i\le\frac{n}{d^2}:\text{i is square free}\}$ is obviously true. A number $i\le n$ can be written uniquely as $i=d^2\cdot d_1$ where $d\le\lfloor\sqrt{i}\rfloor\le\lfloor\sqrt{n}\rfloor$ and d_1 is square free. We find that it belongs to $d^2\cdot\{i\le\frac{n}{d^2}:\text{i is square free}\}$, thus Equation (10) holds. ♦

Consequence: Taking the number of elements in Equation (10) we find

$$\lfloor x\rfloor=\sum_{i=1}^{\lfloor\sqrt{x}\rfloor} q\left(\frac{x}{i^2}\right)\forall x>0. \quad (12)$$

Based on this result and on Equations (7-8) the following theorem is found.

Theorem 4. [Keng] $q(x)=\frac{6}{\pi^2}\cdot x+O(\sqrt{x})$ (13)

Proof. For $x=y^2$, Equation (12) gives $\lfloor y^2\rfloor=\sum_{i=1}^{\lfloor y\rfloor} q\left(\left(\frac{y}{i}\right)^2\right)$. The second Mobius inversion formula gives

$$q(y^2)=\sum_{i=1}^{\lfloor y\rfloor}\mu(i)\cdot\left\lfloor\frac{y^2}{i^2}\right\rfloor. \quad (14)$$

Equation (14) is transformed based on Equation (8.b) as follows

$$q(y^2)=\sum_{i=1}^{\lfloor y\rfloor}\mu(i)\cdot\left\lfloor\frac{y^2}{i^2}\right\rfloor=\sum_{i=1}^{\lfloor y\rfloor}\mu(i)\cdot\left(\frac{y^2}{i^2}-\left\{\frac{y^2}{i^2}\right\}\right)=\sum_{i=1}^{\lfloor y\rfloor}\mu(i)\cdot\frac{y^2}{i^2}-\sum_{i=1}^{\lfloor y\rfloor}\mu(i)\cdot\left\{\frac{y^2}{i^2}\right\}$$

$$=y^2\cdot\sum_{i=1}^{\lfloor y\rfloor}\frac{\mu(i)}{i^2}+O(y)=\frac{6}{\pi^2}\cdot y^2+y^2\cdot O(\frac{1}{y})+O(y)=\frac{6}{\pi^2}\cdot y^2+O(y).$$

Equation (13) is obtained from the last one by substituting $x=y^2$. ♦

Consequence: $\lim_{x\to\infty}\frac{q(x)}{x}=\frac{6}{\pi^2}$. (15)

Equation (15) gives that the asymptotic density for the fixed points of the Smarandache ceil function is $\frac{6}{\pi^2}$. Because $\frac{6}{\pi^2}=0.607927...$, we find that more than 60% of points are fixed points. Equation (15) also produces an algorithm for approximating π that is described in the following.

Step 1. Find the number of fixed points for the Smarandache ceil function S_2.

Step 2. Find the approximation of π by using $\pi \approx \sqrt{\frac{6\cdot x}{q(x)}}$.

3. THE AVERAGE OF THE SMARANDACHE CEIL FUNCTION

In this section we study the Θ complexity of the average of the Smarandache ceil function. Let $\overline{S_k}(n)=\frac{\sum_{i=1}^{n}S_k(i)}{n}$ be the average of the Smarandache ceil function. Recall that $f(n)=\Theta(g(n))$ if $(\exists C_1,C_2>0)(\forall n>n_0)C_1\cdot g(n)\le f(n)\le C_2\cdot g(n)$ [Bach, 1996].

Theorem 5. The Θ-complexity of the average $\overline{S_k}(n)$ is given by

$$\overline{S_k}(n)=\Theta(n). \qquad (16)$$

Proof. This result is obtained from Equation (15). One inequality is obviously obtained as follows $\overline{S_k}(n)=\frac{\sum_{i=1}^{n}S_k(i)}{n}\le\frac{\sum_{i=1}^{n}i}{n}=\frac{n+1}{2}$.

Because $\lim_{x\to\infty}\frac{q(x)}{x}=\frac{6}{\pi^2}>\frac{1}{2}$, we find that $q(x)>\frac{x}{2},\forall x>x_0$. Therefore, there are at least 50% fixed points. Consider that $i_1=1,i_2=2,...,i_{q(n)}$ are the fixed points less than n for the Smarandache ceil function. These obviously satisfy $i_j\ge j, j=1,2,...,q(n)$.

Now, we keep in the average only the fixed points

$$\overline{S}_k(n)=\frac{\sum_{i=1}^{n} S_k(i)}{n}\geq\frac{\sum_{j=1}^{q(n)} S_k(i_j)}{n}=\frac{\sum_{j=1}^{q(n)} i_j}{n}\geq\frac{\sum_{j=1}^{q(n)} j}{n}=\frac{q(n)\cdot(q(n)+1)}{2\cdot n}.$$

Because $q(n)>\frac{n}{2}$, we find that $\overline{S}_k(n)\geq\frac{\frac{n}{2}\cdot(\frac{n}{2}+1)}{2\cdot n}=\frac{n}{8}+\frac{1}{4}$ for each $n>x_0$.

Therefore, the average function satisfies

$$\frac{n}{8}+\frac{1}{4}\leq\overline{S}_k(n)\leq\frac{n}{2}+\frac{1}{2}\forall n>x_0 \tag{16}$$

that gives the Θ-complexity is $\overline{S}_k(n)=\Theta(n)$. ♦

This Θ-complexity complexity gives that the average of the Smarandache ceil function is linear. Unfortunately, we have not been able to find more details about the average function behavior. What is ideally to find is $C\in\left(\frac{1}{8},\frac{1}{2}\right)$ such that

$$\overline{S}_k(n)=C\cdot n+O(n^{1-\varepsilon}). \tag{17}$$

From Equation (17) we find the constant C is $C=\lim_{n\to\infty}\frac{\overline{S}_k(n)}{n}=\lim_{n\to\infty}\frac{\sum_{i=1}^{n} S_k(i)}{n^2}$.

Example. For the Smarandache ceil function S_2 we have found by using a simple computation that $\frac{\sum_{i=1}^{n} S_2(i)}{n^2}\approx 0.3654...$ and $\sqrt{n}\cdot\left[\frac{\sum_{i=1}^{n} S_2(i)}{n^2}-0.3654\right]\approx 0.038...$, which give the $\overline{S}_2(n)\approx 0.3654\cdot n+0.038\cdot\sqrt{n}$

This example makes us to believe that the following conjecture holds.

Conjecture: There is a constant $C\in\left(\frac{1}{8},\frac{1}{2}\right)$ such that $\overline{S}_k(n)=C\cdot n+O(n^{1-\frac{1}{k}})$. (18)

4. CONCLUSSIONS

This article has presented two important results concerning the Smarandache ceil function. We firstly have established that the asymptotic density of fixed points is $\frac{6}{\pi^2}$. Based on this we have found the average function of the Smarandache ceil function behaves linearly. Based on a simple computation the following Equation (18) has been conjectured.

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