Perfect Powers in Smarandache Type Expressions

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In [2] and [3] the authors ask how many primes are of the Smarandache form (see [10]) $x^y + y^x$, where $\gcd(x, y) = 1$ and $x, y \geq 2$. In [6] the author showed that there are only finitely many numbers of the above form which are products of factorials.

In this article we propose the following

Conjecture 1. *Let a, b, and c be three integers with $ab \neq 0$. Then the equation*

$$ax^y + by^x = cz^n \qquad \text{with } x,\ y,\ n \geq 2, \text{ and } \gcd(x,\ y) = 1, \tag{1}$$

has finitely many solutions (x, y, z, n).

We announce the following result:

Theorem 1. *The* "abc Conjecture" *implies Conjecture 1.*

The proof of Theorem 1 is based on an idea of Lang (see [5]).

For any integer k let $P(k)$ be the largest prime number dividing k with the convention that $P(0) = P(\pm 1) = 1$. We have the following result.

Theorem 2. *Let a, b, and c be three integers with $ab \neq 0$. Let $P > 0$ be a fixed positive integer. Then the equation*

$$ax^y + by^x = cz^n \qquad \text{with } x,\ y,\ n \geq 2,\ \gcd(x,\ y) = 1, \text{and } P(y) < P, \tag{2}$$

has finitely many solutions (x, y, z, n). Moreover, there exists a computable positive number C depending only on a, b, c, and P such that all the solutions of equation (2) *satisfy* $\max(x, y) < C$.

The proof of theorem 2 uses lower bounds for linear forms in logarithms of algebraic numbers.

Conjecture 2. *The only solutions of the equation*

$$x^y \pm y^x = z^n \qquad \text{with } x,\ y,\ n \geq 2,\ z > 0,\ \gcd(x,\ y) = 1, \tag{3}$$

are $(x, y, z, n) = (3, 2, 1, n)$.

We have the following results:

Theorem 3. *The equation*

$$x^y \pm y^x = z^2 \qquad \text{with } x,\ y \geq 2, \text{ and } \gcd(x,\ y) = 1, \tag{4}$$

has finitely many solutions (x, y, z) with $2 \mid xy$. Moreover, all such solutions satisfy $\max(x, y) < 3 \cdot 10^{143}$.

The proof of Theorem 3 uses lower bounds for linear forms in logarithms of algebraic numbers.

Theorem 4. *The equation*

$$2^y + y^2 = z^n \tag{5}$$

has no solutions $(y,\ z,\ n)$ *such that* y *is odd and* $n > 1$.

The proof of theorem 4 is elementary and uses the fact that $\mathrm{Z}[i\sqrt{2}]$ is an UFD.

2. Preliminary Results

We begin by stating the *abc Conjecture* as it appears in [5]. Let k be a nonzero integer. Define the *radical* of k to be

$$N_0(k) = \prod_{p \mid k} p \tag{6}$$

i.e. the product of the distinct primes dividing k. Notice that if x and y are integers, then

$$N_0(xy) \leq N_0(x)N_0(y),$$

and if gcd $(x,\ y) = 1$, then

$$N_0(xy) = N_0(x)N_0(y).$$

The abc Conjecture ([5]). *Given* $\epsilon > 0$ *there exists a number* $C(\epsilon)$ *having the following property. For any nonzero relatively prime integers* $a,\ b,\ c$ *such that* $a + b = c$ *we have*

$$\max(|a|,\ |b|,\ |c|) < C(\epsilon)N_0(abc)^{1+\epsilon}.$$

The proofs of theorems 2 and 3 use estimations of linear forms in logarithms of algebraic numbers.

Suppose that $\zeta_1,\ \ldots,\ \zeta_l$ are algebraic numbers, not 0 or 1, of heights not exceeding $A_1,\ \ldots,\ A_l$, respectively. We assume $A_m \geq e^e$ for $m = 1,\ \ldots,\ l$. Put $\Omega = \log A_1 \ldots \log A_l$. Let $\mathrm{F} = \mathrm{Q}[\zeta_1,\ \ldots,\ \zeta_l]$. Let $n_1,\ \ldots,\ n_l$ be integers, not all 0, and let $B \geq \max |n_m|$. We assume $B \geq e^2$. The following result is due to Baker and Wüstholz.

Theorem BW ([1]). *If* $\zeta_1^{n_1} \ldots \zeta_l^{n_l} \neq 1$, *then*

$$|\zeta_1^{n_1} \ldots \zeta_l^{n_l} - 1| > \frac{1}{2}\exp\left(-(16(l+1)d_{\mathrm{F}})^{2(l+3)}\Omega \log B\right). \tag{7}$$

In fact, Baker and Würtholz showed that if $\log \zeta_1,\ \ldots,\ \log \zeta_l$ are any fixed values of the logarithms, and $\Lambda = n_1 \log \zeta_1 + \ldots + n_l \log \zeta_l \neq 0$, then

$$\log |\Lambda| > -(16 l d_{\mathrm{F}})^{2(l+2)}\Omega \log B. \tag{8}$$

Now (7) follows easily from (8) via an argument similar to the one used by Shorey *et al.* in their paper [9].

We also need the following p-adic analogue of theorem BW which is due to Alf van der Poorten.

Theorem vdP ([7]). *Let π be a prime ideal of* F *lying above a prime integer p. Then,*

$$\operatorname{ord}_\pi\left(\zeta_1^{n_1}\ldots\zeta_l^{n_l}-1\right)<\left(16(l+1)d_{\mathrm{F}}\right)^{12(l+1)}\frac{p^{d_F}}{\log p}\Omega\left(\log B\right)^2. \tag{9}$$

We also need the following two results.

Theorem K ([4]). *Let A and B be nonzero rational integers. Let $m\geq 2$ and $n\geq 2$ with $mn\geq 6$ be rational integers. For any two integers x and y let $X=\max\ (|x|,\ |y|)$. Then*

$$P(Ax^m+By^n)>C\left(\log_2 X\log_3 X\right)^{1/2} \tag{10}$$

where $C>0$ is a computable constant depending only on A, B, m and n.

Theorem S ([8]). *Let $n>1$ and A, B be nonzero integers. For integers $m>3$, x and y with $|x|>1$, $\gcd\ (x,\ y)=1$, and $Ax^m+By^n\neq 0$, we have*

$$P(Ax^m+By^n)\geq C\left((\log m)(\log\log m)\right)^{1/2} \tag{11}$$

and

$$|Ax^m+By^n|\geq\exp\left(C\left((\log m)(\log\log m)\right)^{1/2}\right) \tag{12}$$

where $C>0$ is a computable number depending only on A, B and n.

Let K be a finite extension of Q of degree d, and let $\mathcal{O}_{\mathrm{K}}$ be the ring of algebraic integers inside K. For any element $\gamma\in\mathcal{O}_{\mathrm{K}}$, let $[\gamma]$ be the ideal generated by γ in $\mathcal{O}_{\mathrm{K}}$. For any ideal I in $\mathcal{O}_{\mathrm{K}}$, let $N(I)$ be the norm of I. Let $\pi_1,\ \pi_2,\ \ldots,\ \pi_l$ be a set of prime ideals in $\mathcal{O}_{\mathrm{K}}$. Put

$$p=\max\ P\left(N(\pi_i)\right).$$

Write

$$\pi_i^h=[p_i]\qquad\text{for } i=1,\ \ldots,\ l$$

where $p_1,\ p_2,\ \ldots,\ p_l\in\mathcal{O}_{\mathrm{K}}$ and h is the class number of K. Denote by $\mathcal{S}$ the set of all elements α of $\mathcal{O}_{\mathrm{K}}$ such that $[\alpha]$ is exclusively composed of prime ideals $\pi_1,\ \pi_2,\ \ldots,\ \pi_l$. Then we have

Lemma T. ([9]). *Let $\alpha \in S$. Assume that*

$$[\alpha] = \pi_1^{b_1}\pi_2^{b_2}...\pi_l^{b_l}.$$

There exist a $\beta \in O_K$ with $|N(\beta)| \leq p^{dhl}$ and a unit $\epsilon \in O_K$ such that

$$\alpha = \epsilon\beta p_1^{a_1}p_2^{a_2}...p_l^{a_l}.$$

Moreover,

$$b_i = a_i h + c_i \qquad \text{for some } 0 \leq c_i < h.$$

3. The Proofs

The Proof of Theorem 1. We may assume that gcd $(a,\ b,\ c) = 1$. By $C_1,\ C_2,\ ...,$ we shall denote computable positive numbers depending only on $a,\ b,\ c$. Let $(x,\ y,\ z,\ n)$ be a solution of (1). Assume that $x > y$, and that $x > 3$. Let $d = \gcd\ (ax^y,\ by^x)$. Notice that $d \mid ab$. Equation (1) becomes

$$\frac{ax^y}{d} + \frac{by^x}{d} = \frac{cz^n}{d}. \tag{13}$$

By the abc Conjecture for $\epsilon = 2/3$ it follows that

$$\max\ (|ax^y|,\ |by^x|\ |cz^n|) < \frac{C(2/3)N_0(abc)^{5/3}}{d^2}\ N_0(xyz)^{5/3}. \tag{14}$$

Let

$$C_1 = C(2/3)N_0(abc)^{5/3}$$

Since $d \geq 1$, and $|b| \geq 1$, from inequality (14) it follows that

$$y^x \leq |by^x| < C_1(xy|z|)^{5/3} < C_1 x^{10/3}|z|^{5/3}. \tag{15}$$

Since $x > \min\ (y,\ 3)$, it follows easily that $y^x > x^y$. Hence,

$$|z|^n = \left|\frac{a}{c}x^y + \frac{b}{c}y^x\right| < C_2 y^x$$

where $C_2 = \dfrac{|a| + |b|}{|c|}$. We conclude that

$$|z| < C_2^{1/n} y^{x/n} \leq C_2^{1/2} y^{x/n}. \tag{16}$$

Combining inequalities (15) and (16) it follows that

$$y^x < C_1 C_2^{5/6} x^{10/3} y^{(5x/3n)},$$

or

$$y^{x(1-5/3n)} < C_3 x^{10/3}, \tag{17}$$

where $C_3 = C_1 C_2^{5/6}$. Since $2 \le y$ and $2 \le n$, it follows that

$$2^{x/6} \le 2^{x(1-5/3n)} < C_3 x^{10/3}. \tag{18}$$

Inequality (18) clearly shows that $x < C_4$.

The Proof of Theorem 2. We may assume that

$$P \ge \max\big(P(a),\ P(b),\ P(c)\big).$$

By C_1, C_2, ..., we shall denote computable positive numbers depending only on a, b, c, P. We begin by showing that n is bounded. Fix $d \in \{2, 3, ..., P-1\}$. Suppose that x, y, z, n is a solution of (2) with $n > 3$ and $d \mid y$. Since

$$by^x = cz^n - a\big(x^{y/d}\big)^d \tag{19}$$

it follows, by Theorem S, that

$$P = P(by^x) = P\Big(cz^n - a\big(x^{y/d}\big)^d\Big) > C_1\big((\log n)(\log\log n)\big)^{1/2} \tag{20}$$

where C_1 is a computable number depending only on a, c, d. Inequality (20) shows that $n < C_2$.

Suppose now that $ny \ge 6$. Let $X = \max(x, |z|)$. From equation (19) and theorem K, it follows that

$$P = P(by^x) = P(cz^n - ax^y) > C_3\big(\log_2 X \log_3 X\big)^{1/2}, \tag{21}$$

where $C_3 > 0$ is a computable constant depending only on a, c, and C_2. From inequality (21) it follows that $X < C_3$. Let $C_4 = \max(C_2, C_3)$. It follows that, if $ny \ge 6$, then $\max(x, |z|, n) < C_4$. We now show that y is bounded as well. Suppose that $y > \max(C_4, e^2)$. Rewrite equation (2) as

$$\frac{|cz|^n}{|a|x^y} = \left|1 - \left(\frac{-b}{a}\right) y^x x^{-y}\right|. \tag{22}$$

Let $A > e^e$ be an upper bound for the height of $-b/a$ and C_4. Let $\Omega = (\log A)^3$. From theorem BW we conclude that

$$\log|c| + n\log|z| - \log|a| - y\log x > -\log 2 - 64^{12}\Omega\log y. \tag{23}$$

Since $x \ge 2$, and $\max(x, |z|, n) < C_4$, it follows, by inequality (23), that

$$y\log 2 - 64^{12}\Omega\log y \le y\log x - 64^{12}\Omega\log y < C_4\log C_4 - \log|a| + \log|c| + \log 2. \tag{24}$$

From equation (24) it follows that $y < C_5$.

Suppose now that $n = y = 2$. We first bound z in terms of x. Rewrite equation (2) as

$$z^2 = 2^x \left|\frac{b}{c}\right| \cdot \left|1 + \left(\frac{a}{b}\right)\left(\frac{x^2}{2^x}\right)\right|. \tag{25}$$

Let $C_6 > 0$ be a computable positive number depending only on a and b such that

$$\left|\frac{a}{b}\right|\left(\frac{x^2}{2^x}\right) < \frac{1}{2} \qquad \text{for } x > C_6. \tag{26}$$

From equation (25) and inequality (26), it follows that

$$2^x\left|\frac{b}{2c}\right| < 2^x\left|\frac{b}{c}\right|\left(1 - \left|\frac{a}{b}\right|\left(\frac{x^2}{2^x}\right)\right) < z^2 < 2^x\left|\frac{b}{c}\right|\left(1 + \left|\frac{a}{b}\right|\left(\frac{x^2}{2^x}\right)\right) < 2^x\left|\frac{3b}{2c}\right| \tag{27}$$

for $x > C_6$. Taking logarithms in inequality (27) we obtain

$$xC_7 + C_8 < \log z < xC_7 + C_9 \qquad \text{for } x > C_6 \tag{28}$$

where $C_7 = \dfrac{\log 2}{2}$, $C_8 = \dfrac{\log|b| - \log 2|c|}{2}$, and $C_9 = \dfrac{\log|3b| - \log|2c|}{2}$. We now rewrite equation (2) as

$$(cz)^2 - acx^2 = ab2^x. \tag{29}$$

Let $\alpha = \sqrt{ac}$. Then

$$(cz + \alpha x)(cz - \alpha x) = cb2^x. \tag{30}$$

We distinguish 2 cases.

CASE 1. $ac < 0$. Let $K = Q[\alpha]$. Since $ac < 0$, it follows that all the units of $\mathcal{O}_K$ are roots of unity. Since K is a quadratic field, it follows that the ideal [2] has at most two prime divisors. Since

$$\gcd\Big([cz + \alpha x],\ [cz - \alpha x]\Big) \ \Big|\ 2[\alpha bc]$$

it follows, by lemma T, that

$$cz + \alpha x = \epsilon\beta p^u \tag{31}$$

where $\frac{x}{h} - 1 < u \le \frac{x}{h}$, and $\epsilon, \beta,\ p \in \mathcal{O}_K$ are such that $|\epsilon| = 1$, $|p| = 2^{h/2}$, where h is the class number of K, and $|\beta| < C_{10}$ where C_{10} is a computable number depending only on a, b, and c. Conjugating equation (31) we get

$$cz - \alpha x = \overline{\epsilon}\overline{\beta}\overline{p}^u. \tag{32}$$

From equations (31) and (32) it follows that

$$2\alpha x = \epsilon\beta p^u\left(1 - (-\epsilon^{-2})(\beta)^{-1}\overline{\beta}(p)^{-u}(\overline{p})^u\right).$$

Hence,

$$2|\alpha|x = |\beta||p|^u\left|1-(-\epsilon^{-2})(\beta)^{-1}\overline{\beta}(p)^{-u}(\overline{p})^u\right| \tag{33}$$

Taking logarithms in equation (33) we obtain

$$\log(2|\alpha|)+\log x = \log|\beta|+u\log p+\log\left|1-(-\epsilon^{-2})(\beta)^{-1}\overline{\beta}(p)^{-u}(\overline{p})^u\right|. \tag{34}$$

Let A, and P be upper bounds for the heights of $-\epsilon^{-2}(\beta)^{-1}\overline{\beta}$ and p, respectively. Assume that min $(A,\ P) > e^e$. Let $\Omega = \log A(\log P)^2$. Assume also that $\frac{x}{h} > 1+e^2$. From equation (34), theorem BW, the fact that $|p| = 2^{h/2}$, and the fact that $\frac{x}{h}-1 < u \leq \frac{x}{h}$, we obtain that

$$\log(2|\alpha|)+\log x > \log|\beta|+u\log|p|-\log 2-64^{12}\Omega\log u >$$

$$\log|\beta|+\left(\frac{x}{h}-1\right)\cdot\left(\frac{h}{2}\right)\log 2-\log 2-64^{12}\Omega\log(x/h). \tag{35}$$

Inequality (35) clearly shows that $x < C_{11}$.

CASE 2. $ac > 0$. We may assume that both a and c are positive. If $b < 0$, equation (2) can be rewritten as

$$|a|x^2-|b|2^x = |c|z^2 > 0 \tag{36}$$

Equation (36) clearly shows that $x < C_{12}$. Hence, we assume that $b > 0$. We distinguish two subcases.

CASE 2.1. $\sqrt{ac} \in \mathrm{Z}$. In this case, from equation

$$(c|z|+\alpha x)(c|z|-\alpha x) = bc2^x$$

and from the fact that

$$\gcd\left(c|z|+\alpha x,\ c|z|-\alpha x\right) \mid 2\alpha cb \tag{37}$$

it follows easily that

$$\begin{cases} c|z|+\alpha x = \beta 2^u \\ c|z|-\alpha x = \gamma \end{cases} \tag{38}$$

where $\beta,\ \gamma,\ u$ are positive integers with $0 < \beta < bc$, $\gamma < (bc)\cdot(2\alpha cb)$ and $u > x-\mathrm{ord}_2(2\alpha cb)$. From equation (38) it follows that

$$2\alpha x = \beta 2^x - \gamma. \tag{39}$$

From equation (39), and from the fact that $0 < \beta < bc$, $\gamma < (bc)\cdot(2\alpha cb)$, and $u > x-\mathrm{ord}_2(2\alpha cb)$, it follows that $x < C_{13}$.

CASE 2.2. $\sqrt{ac} \notin \mathbb{Z}$. Let $K = Q[\alpha]$. Let ϵ be a generator of the torsion free subgroup of the units group of $\mathcal{O}_K$. From equation (37) and lemma T, it follows that

$$c|z| + \alpha x = \epsilon^m \beta_1 p_1^u \tag{40}$$

where $\frac{x}{h} - 1 < u \leq \frac{x}{h}$, and β, $p_1 \in \mathcal{O}_K$ are such that $1 < \beta_1 < C_{14}$ for some computable constant C_{14}, and $1 < p_1 < 2^h \cdot \epsilon$. From equation (40), it follows that

$$c|z| - \alpha x = \epsilon^{-m} \beta_2 p_2^u \tag{41}$$

where $\beta_2 = |\beta_1|^2/\beta_1$, and $p_2 = 2^h/p_1$. Suppose now that $x > C_6$. Since

$$\epsilon^m = p_1^{-u} \beta_1^{-1} \left(c|z| + \alpha x\right)$$

it follows, from inequality (28), and from the fact that $\frac{x}{h} - 1 < u \leq \frac{x}{h}$ and $1 < p_1 < 2^h \cdot \epsilon$, that

$$|m| < C_{15} x + C_{16} \qquad \text{for } x > C_6, \tag{42}$$

for some computable constants C_{15} and C_{16} depending only on a, b, and c. From equations (40) and (41), it follows that

$$2\alpha x = \epsilon^m \beta_1 p_1^u \cdot \left(1 - \epsilon^{-2m} (\beta_1)^{-1} \beta_2 (p_1)^{-u} p_2^u\right)$$

or

$$2\alpha x = (c|z| + \alpha x) \cdot \left(1 - \epsilon^{-2m} (\beta_1)^{-1} \beta_2 (p_1)^{-u} p_2^u\right). \tag{43}$$

Let A_1, A_2, A_3, A_4 be upper bounds for the heights of ϵ, $(\beta_1)^{-1}\beta_2$, p_1, p_2 respectively. Assume that min $(A_1, A_2, A_3, A_4) > e^e$. Denote $\Omega = \prod_{i=1}^4 \log A_i$. Denote $C_{17} = \max\ (2C_{15},\ 1/h)$. From inequality (42), it follows that

$$\max\ (2|m|,\ u) < C_{17} x + C_{16}. \tag{44}$$

Let $B = C_{17} x + C_{16}$. Taking logarithms in equation (43), and applying theorem BW, we obtain

$$\log(2\alpha) + \log x = \log(c|z| + \alpha x) + \log\left|1 - \epsilon^{-2m} (\beta_1)^{-1} \beta_2 (p_1)^{-u} p_2^u\right| >$$

$$\log(c|z| + \alpha x) - \log 2 - 80^{14} \Omega \log(C_{17} x + C_{16}). \tag{45}$$

Combining inequalities (28) and (45) we obtain

$$\log(4\alpha) + \log x + 80^{14}\Omega \log(C_{17}x + C_{16}) > \log(c|z| + \alpha x) > \log z > C_7 x + C_8$$

This last inequality clearly shows that $x < C_{18}$.

The Proof of Theorem 3. We treat only the equation

$$x^y + y^x = z^2.$$

We may assume that x is even. First notice that, since gcd $(x,\ y) = 1$, it follows that gcd $(x,\ z) = $ gcd $(y,\ z) = 1$. Rewrite equation (4) as

$$x^y = \left(z + y^{x/2}\right)\left(z - y^{x/2}\right).$$

Since gcd $\left(z,\ y^{x/2}\right) = 1$ and both z and y are odd, it follows that

$$\gcd\left(z + y^{x/2},\ z - y^{x/2}\right) = 2.$$

Write $x = 2d_1d_2$ such that either one of the following holds

$$\begin{cases} z + y^{x/2} = 2^{y-1}d_1^y \\ z - y^{x/2} = 2d_2^y \end{cases} \quad \text{or} \quad \begin{cases} z + y^{x/2} = 2d_1^y \\ z - y^{x/2} = 2^{y-1}d_2^y \end{cases} \tag{46}$$

Hence, either

$$y^{x/2} = 2^{y-2}d_1^y - d_2^y \tag{47}$$

or

$$y^{x/2} = d_1^y - 2^{y-2}d_2^y \tag{48}$$

We proceed in several steps.

Step 1. (1) *If $x > y$ then either $y \le 9$ and $x < 27$, or $y > 9$ and $x < 3y$.*

(2) *If $x < y$ and $y > 2.6 \cdot 10^{21}$, then $y < 4x$.*

(1) Assume first that $x > y$. Since

$$y^{x/2} = 2^{y-2}d_1^y - d_2^y \qquad \text{or} \qquad y^{x/2} = d_1^y - 2^{y-2}d_2^y$$

it follows that

$$y^{x/2} < 2^{y-1}d_1^y < (2d_1)^y < x^y \qquad \text{or} \qquad y^{x/2} < d_1^y < x^y. \tag{49}$$

Hence,

$$\frac{x}{2}\log y < y \log x. \tag{50}$$

Inequality (50) is equivalent to

$$\frac{x}{\log x} < 2\,\frac{y}{\log y}. \tag{51}$$

If $y \le 9$, then one can check easily that (51) implies $x < 27$. Suppose now that $y > 9$. We show that inequality (51) implies $x < 3y$. Indeed, assume that $x \ge 3y$. Then

$$\frac{3y}{\log 3 + \log y} = \frac{3y}{\log(3y)} \le \frac{x}{\log x} < \frac{2y}{\log y}. \tag{52}$$

Inequality (52) is equivalent to

$$3\log y < \log 9 + 2\log y$$

or $y < 9$. This contradiction shows that $x < 3y$ for $y > 9$.

(2) Assume now that $x < y$. Suppose first that

$$y^{x/2} = 2^{y-2}d_1^y - d_2^y.$$

In this case

$$(2d_1)^y > 2^{y-2}d_1^y = d_2^y + y^{x/2} > d_2^y$$

therefore $2d_1 > d_2$. Since $x = 2d_1d_2$, it follows that $2d_1 > \sqrt{x}$, or $d_1 > \dfrac{\sqrt{x}}{2}$.

Suppose now that

$$y^{x/2} = d_1^y - 2^{y-2}d_2^y.$$

In this case,

$$d_1^y > 2^{y-2}d_2^y > d_2^y$$

or $d_1 > d_2$. We obtain that $d_1 > \sqrt{d_1d_2} = \sqrt{\dfrac{x}{2}} > \dfrac{\sqrt{x}}{2}$.

If equality (47) holds, it follows that

$$y^{x/2} = 2^{y-2}d_1^y\left|1 - 2^{-(y-2)}\left(\frac{d_2}{d_1}\right)^y\right| \geq d_1^y\left|1 - 2^{-(y-2)}\left(\frac{d_2}{d_1}\right)^y\right|. \tag{53}$$

On the other hand, if equality (48) holds, then

$$y^{x/2} = d_1^y\left|1 - 2^{y-2}\left(\frac{d_2}{d_1}\right)^y\right|. \tag{54}$$

From inequality (53) and equation (54), we conclude that, in either case,

$$y^{x/2} \geq d_1^y\left|1 - 2^{\epsilon(y-2)}\left(\frac{d_2}{d_1}\right)^y\right| \tag{55}$$

for some $\epsilon \in \{\pm 1\}$. Suppose now that $x > e^e$. By theorem BW, and inequality (55), it follows that

$$\frac{x}{2}\log y \geq y\log d_1 - \log 2 - 48^{10}e\log x\log y \geq$$

$$y\log\frac{\sqrt{x}}{2} - \log 2 - 48^{10}e\log x\log y \tag{56}$$

or

$$48^{10}e\log x\log y + \log 2 + \frac{x}{2}\log y > y\log\frac{\sqrt{x}}{2}. \tag{57}$$

CASE 1. Assume that $x < 2^6$. From inequality (57), it follows that

$$48^{10}e \cdot 6\log 2 \cdot \log y + \log 2 + 2^5 \log y > y \log \frac{\sqrt{e^e}}{2} > \frac{y}{2}$$

or

$$\left(48^{10}e \cdot 6\log 2 + 2^5\right)\log y + \log 2 > \frac{y}{2}$$

or

$$2\left(48^{10}e \cdot 6\log 2 + 2^5 + 1\right) > \frac{y}{\log y}. \tag{58}$$

Let $C_1 = 2\left(48^{10}e \cdot 6\log 2 + 2^5 + 1\right)$. From inequality (58) and lemma 2 in [6], it follows that

$$y < C_1 \log^2 C_1 < 2\left(48^{10}e \cdot 6\log 2 + 2^5 + 1\right) \cdot 42^2 < 2.6 \cdot 10^{21}. \tag{59}$$

CASE 2. Assume that $x \geq 2^6$. Then,

$$d_1 > \frac{\sqrt{x}}{2} \geq \sqrt[3]{x}.$$

Inequality (56) becomes

$$48^{10}e \log x \log y + \log 2 + \frac{x}{2}\log y > \frac{1}{3}\, y \log x$$

or

$$3e48^{10}\log x \log y + \log 8 + \frac{3}{2}\, x \log y > y \log x$$

or

$$(3e48^{10} + 1)\log x \log y + \frac{3}{2}\, x \log y > y \log x$$

or

$$3e48^{10} + 1 + \frac{3}{2}\,\frac{x}{\log x} > \frac{y}{\log y}. \tag{60}$$

Assume first that

$$\frac{3}{2}\,\frac{x}{\log x} < 3e48^{10} + 1. \tag{61}$$

In this case,

$$\frac{x}{\log x} < \frac{2}{3}\left(3e48^{10} + 1\right). \tag{62}$$

Let $C_2 = \frac{2}{3}\left(3e48^{10} + 1\right)$. From inequality (62) and lemma 2 in [6], it follows that

$$x < C_2 \log^2 C_2 < \frac{2}{3}\left(3e48^{10} + 1\right) \cdot 41^2 < 6 \cdot 10^{20}. \tag{63}$$

In this case, from inequalities (60) and (61), it follows that

$$\frac{y}{\log y} < 2(3e48^{10} + 1). \tag{64}$$

Let $C_3 = 2(3e48^{10}+1)$. It follows, by inequality (64) and lemma 2 in [6], that

$$y < C_3 \log^2 C_3 < 2(3e48^{10}+1) \cdot 42^2 < 1.8 \cdot 10^{21}. \tag{65}$$

Assume now that $y > 2.6 \cdot 10^{21}$. From inequality (59), it follows that $x \geq 2^6$. Moreover, since inequality (65) is a consequence of inequality (61), it follows that

$$\frac{3}{2} \frac{x}{\log x} \geq 3e48^{10} + 1. \tag{66}$$

From inequalitites (60) and (66) it follows that

$$\frac{3x}{\log x} > \frac{y}{\log y}. \tag{67}$$

We now show that inequality (67) implies $y < 4x$. Indeed, assume that $y \geq 4x$. Then inequality (67) implies

$$\frac{3x}{\log x} > \frac{y}{\log y} \geq \frac{4x}{\log(4x)} = \frac{4x}{\log x + \log 4}$$

or

$$3 \log x + 3 \log 4 > 4 \log x$$

or $3 \log 4 > \log x$ which contradicts the fact that $x \geq 2^6$.

Step 2. *If $y \geq 3 \cdot 10^{143}$, then y is prime.*

Let

$$y^{x/2} = 2^{y-2} d_1^y - d_2^y \qquad \text{or} \qquad y^{x/2} = d_1^y - 2^{y-2} d_2^y. \tag{68}$$

Notice that if $y^{x/2} = 2^{y-2} d_1^y - d_2^y$, then gcd $(2d_1,\ d_2) = 1$. Let $p \mid y$ be a prime number. Since $p \nmid 2d_1 d_2 = x$, it follows, by theorem vdP, that

$$\frac{x}{2} \leq \max\left(\mathrm{ord}_p(2^{y-2}d_1^y - d_2^y),\ \mathrm{ord}_p(d_1^y - 2^{y-2}d_2^y)\right) < 48^{36} e \frac{p}{\log p} \log^2 y \log x. \tag{69}$$

By step 1, it follows that

$$\frac{1}{4} y < x \leq 2 \cdot 48^{36} e \frac{p}{\log p} \log^2 y \log(4y) < 4 \cdot 48^{36} e \frac{p}{\log p} \log^3 y. \tag{70}$$

Hence,

$$\frac{y}{\log^3 y} < 16 \cdot 48^{36} e \frac{p}{\log p} < 16 \cdot 48^{36} ep. \tag{71}$$

Suppose that y is not prime. Let $p \mid y$ be a prime such that $p \leq \sqrt{y}$. From inequality (71) it follows that

$$\frac{\sqrt{y}}{\log^3 y} < 16 \cdot 48^{36} e$$

or

$$\frac{\sqrt{y}}{\log^3(\sqrt{y})} < 128 \cdot 48^{36} e. \tag{72}$$

Let $k = \sqrt{y}$ and $C_4 = 128 \cdot 48^{36} e$. By inequality (72) and lemma 2 in [6], it follows that

$$\sqrt{y} = k < C_4 \log^4 C_4 = 128 \cdot 48^{36} e \cdot 146^4 < 5.3 \cdot 10^{71} \tag{73}$$

or

$$y < \left(5.3 \cdot 10^{71}\right)^2 < 3 \cdot 10^{143} \tag{74}$$

This last inequality contradicts the assumption that $y \geq 3 \cdot 10^{143}$.

Step 3. *If $y \geq 3 \cdot 10^{143}$, then $x > y$.*

Let $y = p$ be a prime. If $y^{x/2} = 2^{y-2} d_1^y - d_2^y$, it follows, by Fermat's little theorem that

$$2^{-1} d_1 - d_2 \equiv 2^{y-2} d_1^y - d_2^y \equiv y^{x/2} \equiv 0 \pmod{p},$$

therefore

$$d_1 \equiv 2 d_2 \pmod{p}. \tag{75}$$

On the other hand, if $y^{x/2} = d_1^y - 2^{y-2} d_2^y$, then

$$d_1 - 2^{-1} d_2 \equiv d_1^y - 2^{y-2} d_2^y \equiv y^{x/2} \equiv 0 \text{ mod } p),$$

therefore

$$d_2 \equiv 2 d_1 \pmod{p}. \tag{76}$$

Suppose that $x < y$. From congruences (75) and (76), we conclude that, in both cases, x is a perfect square. Hence,

$$y^x = z^2 - \left(\sqrt{x}\right)^{2y} = \left(z + \left(\sqrt{x}\right)^y\right) \cdot \left(z - \left(\sqrt{x}\right)^y\right). \tag{77}$$

From equation (77) it follows that

$$\begin{cases} z - \left(\sqrt{x}\right)^y = 1 \\ z + \left(\sqrt{x}\right)^y = y^x \end{cases} \tag{78}$$

Hence,

$$2\left(\sqrt{x}\right)^y = y^x - 1. \tag{79}$$

It follows, by equation (79) and theorem BW, that

$$0 = \log\left|y^x - 2\left(\sqrt{x}\right)^y\right| = \log(y^x) + \log\left|1 - 2y^{-x}\left(\sqrt{x}\right)^y\right| >$$

$$x \log y - \log 2 - 64^{12} e \log^2 y \log x. \tag{80}$$

From inequality (80) and Step 1 it follows that

$$\log 2 + 64^{12} e \log^3 y > x \log y > \frac{y \log y}{4}$$

or

$$4 \log 2 + 4 \cdot 64^{12} e \log^3 y > y \log y$$

or

$$\left(4 \cdot 64^{12} e + 1\right) \log^2 y > y$$

or

$$4 \cdot 64^{12} e + 1 > \frac{y}{\log^2 y}. \tag{81}$$

Let $C_5 = 4 \cdot 64^{12} e + 1$. By inequality (81) and lemma 2 in [6] it follows that

$$y < C_5 \log^3 C_5 < (4 \cdot 64^{12} e + 1) \cdot 53^3 < 8 \cdot 10^{27}. \tag{82}$$

The last inequality contradicts the fact that $y \geq 3 \cdot 10^{143}$.

Step 4. *Suppose that $y \geq 3 \cdot 10^{143}$. Let $y = p$ be a prime. Then, with the notations of Step 1, every solution of equation (4) is of one of the following forms:*

(1) $y^{x/2} = 2^{y-2} d_1^y - d_2^y$ with $y = p,\ d_1 = 2 + p,\ d_2 = 1,\ x = 4 + 2p$

(2) $y^{x/2} = d_1^y - 2^{y-2} d_2^y$ with $y = p,\ d_1 = \dfrac{3p-1}{2},\ d_2 = 1,\ x = 3p - 1$

(3) $y^{x/2} = d_1^y - 2^{y-2} d_2^y$ with $y = p,\ d_1 = \dfrac{p-1}{2},\ d_2 = 3,\ x = 3p - 9$

We assume that $y \geq 3 \cdot 10^{143}$. In this case, $y = p$ is prime, and $x > y$. From Step 1 we conclude that $x < 3y$. Moreover, from the arguments used at Step 1 it follows that $d_1 > \dfrac{\sqrt{x}}{2}$. Since $x = 2 d_1 d_2$, it follows that

$$d_2 < \sqrt{x} < \sqrt{3y} = \sqrt{3p}.$$

By the arguments used at Step 3 we may assume that x is not a perfect square. We distinguish the following cases.

CASE 1. $d_2 = 1$. By congruences (75) and (76) it follows that $d_1 \equiv 2 \pmod{p}$, or $2d_1 \equiv 1 \pmod{p}$.

Assume that $d_1 \equiv 2 \pmod{p}$. Since $x = 2d_1$, and $p = y < x < 3y = 3p$, it follows that $d_1 = 2 + p$ and $x = 2d_1 = 4 + 2p$.

Assume that $2d_1 \equiv 1 \pmod{p}$. Again, since $x = 2d_1$, and $p = y < x < 3y = 3p$, it follows that $d_1 = \dfrac{3p-1}{2}$, and $x = 3p - 1$.

CASE 2. $d_2 = 2$. By congruences (75) and (76) it follows that $d_1 \equiv 4 \pmod{p}$, or $d_1 \equiv 1 \pmod{p}$. One can easily check that there is no solution in this case. Indeed, if $d_1 \equiv 4 \pmod{p}$, it follows that $d_1 \geq p + 4$. Hence, $x = 2 d_1 d_2 \geq 4(p + 4) > 3p = 3y$ which contradicts the fact that $x < 3y$.

Similar arguments can be used to show that there is no solution for which $d_2 = 2$ and $d_1 \equiv 1 \pmod{p}$.

CASE 3. $d_2 = 3$. By congruences (75) and (76) it follows that $d_1 \equiv 6 \pmod{p}$, or $2d_1 \equiv 3 \pmod{p}$. One can easily check that there is no solution for which $d_1 \equiv 6 \pmod{p}$. Suppose that $2d_1 \equiv 3 \pmod{p}$. Since $p = y < x < 3y = 3p$ and $x = 2d_1d_2 = 6d_1$, it follows easily that $d_1 = \frac{p-3}{2}$, and $x = 3p - 9$.

CASE 4. $d_2 = k \geq 4$.

If k is even, then, by congruences (75) and (76), it follows that $d_1 \equiv 2k \pmod{p}$, or $d_1 \equiv k/2 \pmod{p}$. Since x is not a perfect square it follows that $d_1 \geq p+k/2$, therefore $x \geq 2pk+k^2 > pk \geq 4p > 3p = 3y$ contradicting the fact that $x < 3y$.

If k is odd, then, by congruences (75) and (76), it follows that $d_1 \equiv 2k \pmod{p}$, or $2d_1 \equiv k \pmod{p}$. We conclude that $d_1 \geq \frac{p-k}{2}$, therefore $x = 2d_1d_2 \geq k(p-k)$. Since $k(p-k) > 3p$ for $5 \leq k \leq \sqrt{3p}$ and $p \geq 3 \cdot 10^{143}$, we conclude that $x > 3p = 3y$ contradicting again the fact that $x < 3y$.

Step 5. *There are no solutions of equation* (2) *with* $y \geq 3 \cdot 10^{142}$ *and* x *even.*

According to Step 4 we need to treat the following cases.

CASE 1.

$$y^{x/2} = 2^{y-2}d_1^y - d_2^y \quad \text{with } y = p,\ d_1 = 2 + p,\ d_2 = 1,\ x = 4 + 2p. \tag{83}$$

Hence,

$$p^{2+p} = 2^{p-2}(2+p)^p - 1 > 2^{p-3}(2+p)^p. \tag{84}$$

Taking logarithms in inequality (84) we obtain

$$(2+p)\log p > (p-3)\log 2 + p\log(p+2)$$

or

$$2\log p + p(\log p - \log(p+2)) > (p-3)\log 2. \tag{85}$$

It follows, by inequality (85), that

$$2\log p > (p-3)\log 2$$

or

$$p\log 2 < 2\log p + 3\log 2 < 5\log p. \tag{86}$$

Inequality (86) is certainly false for $p = y \geq 3 \cdot 10^{143}$.

CASE 2.

$$y^{x/2} = d_1^y - 2^{y-2}d_2^y \quad \text{with } y = p,\ d_1 = \frac{3p-1}{2},\ d_2 = 1,\ x = 3p - 1.$$

Hence,

$$p^{(3p-1)/2} = \left(\frac{3p-1}{2}\right)^p - 2^{p-2} < \left(\frac{3p-1}{2}\right)^p < \left(\frac{3p}{2}\right)^p$$

or

$$p^{(p-1)/2} < \left(\frac{3}{2}\right)^p. \tag{87}$$

Taking logarithms in inequality (87) it follows that

$$\frac{p-1}{2}\log p < p\log 1.5$$

or

$$\log p < \frac{2p}{p-1}\log 1.5 < 3\log 1.5 < \log 1.5^3.$$

It follows that $p < 1.5^3 < 4$ which contradicts the fact that $p \geq 3 \cdot 10^{143}$.

CASE 3.

$$y^{x/2} = d_1^y - 2^{y-2}d_2^y \quad \text{with } y = p,\ d_1 = \frac{p-1}{2},\ d_2 = 3,\ x = 3p-9.$$

Hence,

$$p^{(3p-9)/2} = \left(\frac{p-3}{2}\right)^p - 2^{p-2}3^p < \left(\frac{p-3}{2}\right)^p < p^p. \tag{88}$$

From inequality (88) it follows that $\frac{3p-9}{2} < p$ or $p < 9$ which contradicts the fact that $p = y \geq 3 \cdot 10^{143}$.

The Proof of Theorem 4. The given equation has no solution $(y,\ z,\ n)$ with $n > 1$ and y odd, $y < 5$. Assume now that $y \geq 5$. We may assume that n is prime. We first show that n is odd. Indeed, assume that $(y,\ z)$ is a positive solution of $y^2 + 2^y = z^2$ with both y and z odd. Then $(z+y)(z-y) = 2^y$. Since gcd $(z+y,\ z-y) = 2$ it follows that $z - y = 2$ and $z + y = 2^{y-1}$. Hence, $y = 2^{y-2} - 1$. However, one can easily check that $2^{y-2} - 1 > y$ for $y \geq 5$.

Assume now that $n = p \geq 3$ is an odd prime. Write

$$\left(y + 2^{(y-1)/2}\cdot i\sqrt{2}\right)\cdot\left(y - 2^{(y-1)/2}\cdot i\sqrt{2}\right) = z^n$$

Since $\mathrm{Z}[i\sqrt{2}]$ is euclidian and

$$\gcd\left(y + 2^{(y-1)/2}\cdot i\sqrt{2},\ y - 2^{(y-1)/2}\cdot i\sqrt{2}\right) = 1$$

it follows that there exists $a,\ b \in \mathrm{Z}$ such that

$$\begin{cases} y + 2^{(y-1)/2}\cdot i\sqrt{2} = \left(a + bi\sqrt{2}\right)^n \\ y - 2^{(y-1)/2}\cdot i\sqrt{2} = \left(a - bi\sqrt{2}\right)^n \end{cases} \tag{89}$$

From equations (89) it follows that

$$y = \frac{\left(a+bi\sqrt{2}\right)^n + \left(a-bi\sqrt{2}\right)^n}{2} \tag{90}$$

and

$$2^{(y-1)/2} = \frac{\left(a+bi\sqrt{2}\right)^n - \left(a-bi\sqrt{2}\right)^n}{2\sqrt{2}i} \tag{91}$$

From equation (90) we conclude that a is odd. From equation (91), it follows that

$$2^{(y-1)/2} = b(na^{n-1} + s),$$

where s is even. Since both n and a are odd, it follows that $na^{n-1} + s$ is odd as well. Hence, $b = 2^{(y-1)/2}$. Equation (5) can now be rewritten as

$$y^2 + 2^y = z^n = \left((a+bi\sqrt{2})\cdot(a-bi\sqrt{2})\right)^n = (a^2+2b^2)^n$$

or

$$y^2 + 2^y = (a^2+2^y)^n > 2^{ny} \geq 2^{3y} \tag{92}$$

Inequality (92) implies that

$$y^2 > 2^{3y} - 2^y = 2^y(2^{2y}-1) > 2^y,$$

which is false for $y \geq 5$.

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