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On Pathos Semitotal and Total Block Graph of a Tree

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Abstract: In this communications, the concept of pathos semitotal and total block graph of a graph is introduced. Its study is concentrated only on trees. We present a characterization of those graphs whose pathos semitotal block graphs are planar, maximal outer planar, non-minimally non-outer planar, non-Eulerian and hamiltonian. Also, we present a characterization of graphs whose pathos total block graphs are planar, maximal outer planar, minimally non-outer planar, non-Eulerian, hamiltonian and graphs with crossing number one.

Key Words: Pathos, path number, Smarandachely block graph, semitotal block graph, Total block graph, pathos semitotal graph, pathos total block graph, pathos length, pathos point, inner point number.

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§1. Introduction

The concept of pathos of a graph G was introduced by Harary [2], as a collection of minimum number of line disjoint open paths whose union is G. The path number of a graph G is the number of paths in pathos. A new concept of a graph valued functions called the semitotal and total block graph of a graph was introduced by Kulli [6]. For a graph $G(p, q)$ if $B = \{u_1, u_2, u_3, \ldots, u_r; r \geq 2\}$ is a block of G, then we say that point u_1 and block B are incident with each other, as are u_2 and B and so on. If two distinct blocks B_1 and B_2 are incident with a common cut point, then they are adjacent blocks. The points and blocks of a graph are called its members. A *Smarandachely block graph* $T_S^V(G)$ for a subset $V \subset V(G)$ is such a graph with vertices $V \cup \mathcal{B}$ in which two points are adjacent if and only if the corresponding members of G are adjacent in $\langle V \rangle_G$ or incident in G, where $\mathcal{B}$ is the set of blocks of G. The semitotal block graph of a graph G denoted by $T_b(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of G in which two points are adjacent if and only if members of G are incident, thus a Smarandachely block graph with $V = \emptyset$. The total block graph of a graph

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G denoted by $T_B(G)$ is defined as the graph whose point set is the union of set of points, set of blocks of G in which two points are adjacent if and only if the corresponding members of G are adjacent or incident, i.e., a Smarandachely block graph with $V = V(G)$. Stanton [11] and Harary [3] have calculated the path number for certain classes of graphs like trees and complete graphs.

All undefined terminology will conform with that in Harary [1]. All graphs considered here are finite, undirected and without loops or multiple lines.

The pathos semitotal block graph of a tree T denoted by $P_{T_B}(T)$ is defined as the graph whose point set is the union of set of points, set of blocks and the set of path of pathos of T in which two points are adjacent if and only if the corresponding members of G are incident and the lines lie on the corresponding path P_i of pathos. Since the system of pathos for a tree is not unique, the corresponding pathos semitotal and pathos total block graph of a tree T is also not unique.

In Fig.1, a tree T, its semitotal block graph $T_b(T)$ and their pathos semi total block $P_{T_b}(T)$ graph are shown. In Fig. 2, a tree T, its semitotal block graph $T_b(T)$ and their pathos total block $P_{T_B}(T)$ graph are shown.

The line degree of a line uv in a tree T, pathos length, pathos point in T was defined by Muddebihal [10]. If G is planar, the inner point number $i(G)$ of a graph G is the minimum number of points not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be minimally nonouterplanar if $i(G) = 1$, as was given by Kulli [4].

We need the following results to prove further results.

Theorem [A][Ref.6] *If G is connected graph with p points and q lines and if b_i is the number of blocks to which v_i belongs in G, then the semitotal block graph $T_b(G)$ has $\left(\sum_{i=1}^{p} b_i\right) + 1$, points and $q + \left(\sum_{i=1}^{p} b_i\right)$ lines.*

Theorem [B][Ref.6] *If G is connected graph with p points and q lines and if b_i is the number of blocks to which v_i belongs in G, then the total block graph $T_B(G)$ has $\left(\sum_{i=1}^{p} b_i\right) + 1$, points and $q + \sum_{i=1}^{p} \binom{b_i + 1}{2}$ lines.*

Theorem [C][Ref.8] *The total block graph $T_B(G)$ of a graph G is planar if and only if G is outerplanar and every cutpoint of G lies on atmost three blocks.*

Theorem [D] [Ref.7] *The total block graph $T_B(G)$ of a connected graph G is minimally nonouter planar if and only if,*

(1) *G is a cycle, or*

(2) *G is a path P of length $n \geq 2$, together with a point which is adjacent to any two adjacent points of P.*

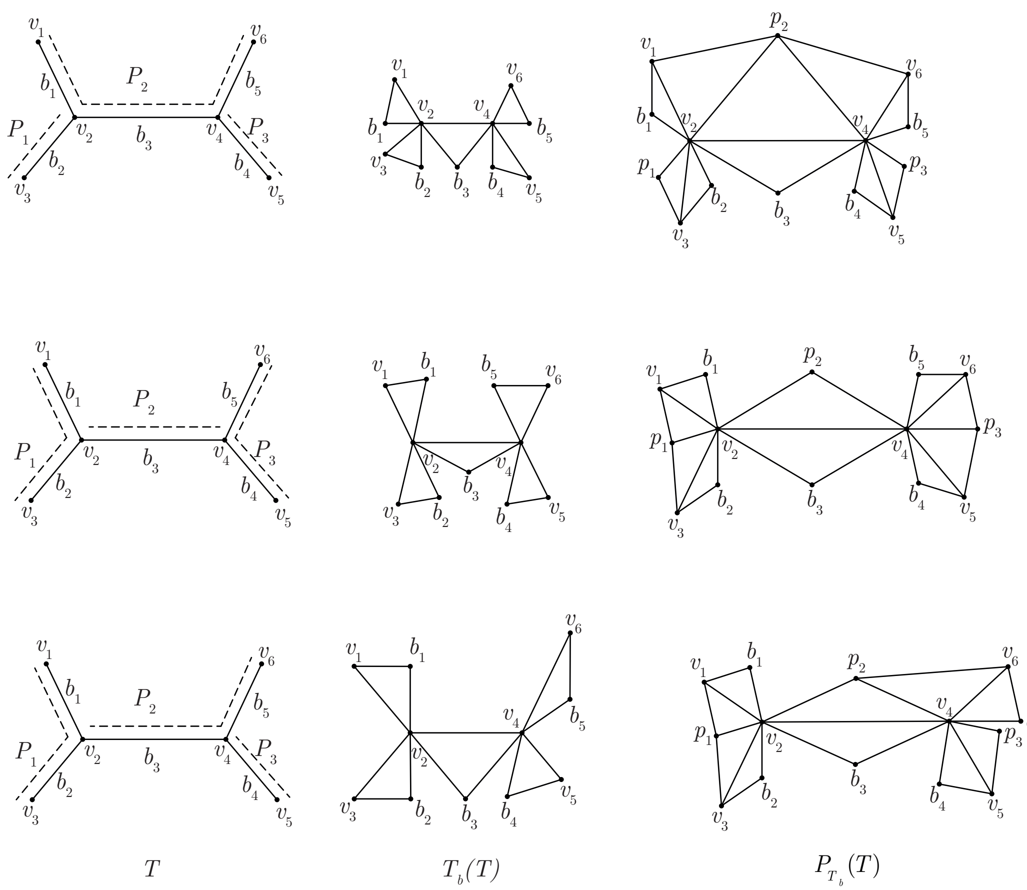

Figure 1:

Theorem [E][Ref.9] *The total block graph $T_B(G)$ of a graph G crossing number 1 if and only if*

(1) *G is outer planar and every cut point in G lies on at most 4 blocks and G has a unique cut point which lies on 4 blocks, or*

(2) *G is minimally non-outer planar, every cut point of G lies on at most 3 blocks and exactly one block of G is theta-minimally non-outer planar.*

Corollary [A][Ref.1] *Every nontrivial tree contains at least two end points.*

Theorem [F][Ref.1] *Every maximal outerplanar graph G with p points has $(2p-3)$ lines.*

Theorem [G][Ref.5] *A graph G is a non empty path if and only if it is connected graph with $p \geq 2$ points and $\sum_{i=1}^{p} {d_i}^2 - 4p + 6 = 0$.*

§2. Pathos Semitotal Block Graph of a Tree

We start with a few preliminary results.

Remark 1 *The number of blocks in pathos semitotal block graph of $P_{T_b}(T)$ of a tree T is equal to the number of pathos in T.*

Remark 2 *If the degree of a pathos point in pathos semi total block graph $P_{T_b}(T)$ of a tree T is n, then the pathos length of the corresponding path P_i of pathos in T is $n-1$.*

Kulli [6] developed the new concept in graph valued functions i.e., semi total and total block graph of a graph. In this article the number of points and lines of a semi total block graph of a graph has been expressed in terms of blocks of G. Now using this we have a modified theorem as shown below in which we have expressed the number of points and lines in terms of lines and degrees of the points of G which is a tree.

Theorem 1 *For any (p,q) tree T, the semitotal block graph $T_b(T)$ has $(2q+1)$ points and $3q$ lines.*

Proof By Theorem [A], the number of points in $T_b(G)$ is $\left(\sum\limits_{i=1}^{p} b_i\right)+1$, where b_i are the number of blocks in T to which the points v_i belongs in G. Since $\sum b_i = 2q$, for G is a tree. Thus the number of points in $T_b(G) = 2q+1$. Also, by Theorem [A] the number of lines in $T_b(G)$ are $q+\left(\sum\limits_{i=1}^{b} b_i\right)$, since $\sum b_i = 2q$ for G is a tree. Thus the number of lines in $T_b(G)$ is $q+2q=3q$. □

In the following theorem we obtain the number of points and lines in $P_{T_b}(T)$.

Theorem 2 *For any non trivial tree T, the pathos semitotal block graph of a tree T, whose points have degree d_i, then the number of points in $P_{T_b}(T)$ are $(2q+k+1)$ and the number of lines are $\left(2q+2+\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2\right)$, where k is the path number.*

Proof By Theorem 1, the number of points in $T_b(T)$ are $2q+1$, and by definition of $P_{T_b}(T)$, the number of points in (2q+k+1), where k is the path number. Also by Theorem 1, the number of lines in $T_b(T)$ are $3q$. The number of lines in $P_{T_b}(T)$ is the sum of lines in $T_b(T)$ and the number of lines which lie on the points of pathos of T which are to $\left(-q+2+\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2\right)$. Thus the number of lines in is equal to $\left(3q+(-q+2+\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2)\right) = \left(2q+2+\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2\right)$.

§2. Planar Pathos Semitotal Block Graphs

A criterion for pathos semi total block graph to be planar is presented in our next theorem.

Theorem 3 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is planar.*

Proof Let T be a non trivial tree, then in $T_b(T)$ each block is a triangle. We have the following cases.

Case 1 Suppose G is a path, $G = P_n : u_1, u_2, u_3, \ldots, u_n, n > 1$. Further, $V\left[T_b\left(T\right)\right] =$

$\{u_1, u_2, u_3, \ldots, u_n, b_1, b_2, b_3, \ldots, b_{n-1}\}$, where $b_1, b_2, b_3, \ldots, b_{n-1}$ are the corresponding block points. In pathos semi total block graph $P_{T_b}(T)$ of a tree T, $\{u_1b_1u_2w, u_2b_2u_3w, u_3b_3u_4w, \ldots, u_{n-1}b_{n-1}u_nw\} \in V[P_{T_b}(T)]$, each set $\{u_{n-1}b_{n-1}u_nw\}$ forms an induced subgraph as $K_4 - x$. Hence one can easily verify that $P_{T_b}(T)$ is planar.

Case 2 Suppose G is not a path. Then $V[T_b(G)] = \{u_1, u_2, u_3, \ldots, u_n, b_1, b_2, b_3, \ldots, b_{n-1}\}$ and $w_1, w_2, w_3, \ldots, w_k$ be the pathos points. Since $u_{n-1}u_n$ is a line and $u_{n-1}u_n = b_{n-1} \in V[T_b(G)]$. Then in $P_{T_b}(G)$ the set $\{u_{n-1}b_{n-1}u_nw\}$ $\forall\, n > 1$, forms $K_4 - x$ as an induced subgraphs. Hence $P_{T_b}(G)$ is planar. □

Further we develop the maximal outer planarity of $P_{T_b}(G)$ in the following theorem.

Theorem 4 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is maximal outer planar if and only if T is a path.*

Proof Suppose $P_{T_b}(T)$ is maximal outer planar. Then $P_{T_b}(T)$ is connected. Hence T is connected. If $P_{T_b}(T)$, is $K_4 - x$, then obviously T is k_2.

Let T be any connected tree with $p \geq 2$, q lines b_i blocks and path number k, then clearly $P_{T_b}(T)$ has $(2q + k + 1)$ points and $\left(2q + 2 + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$ lines. Since $P_{T_b}(T)$ is maximal outer planar, by Theorem [F], it has $[2(2q + k + 1) - 3]$ lines. Hence,

$$2 + 2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2 = 2\,(2\mathrm{q} + \mathrm{k} + 1) - 3 = \ 4\mathrm{q} + 2\mathrm{k} + 2 - 3 = \ 4\mathrm{q} + 2\mathrm{q} - 1$$

$$\frac{1}{2}\sum_{i=1}^{p} d_i^2 = 2\mathrm{q} + 2\mathrm{k} - 3$$

$$\sum_{i=1}^{p} d_i^2 = 4\mathrm{q} + 4\mathrm{k} - 6$$

$$\sum_{i=1}^{p} d_i^2 = \ 4\,(\mathrm{p} - 1)\ \ + 4\mathrm{k} - 6$$

$$\sum_{i=1}^{p} d_i^2 = 4\mathrm{p} + 4\mathrm{k} - 10.$$

But for a path, $k = 1$.

$$\sum_{i=1}^{p} d_i^2 = 4\mathrm{p} + 4\,(1) - 10 = \ 4\mathrm{p} - 6$$

$$\sum_{i=1}^{p} d_i^2 - 4\mathrm{p} + 6 = 0.$$

By Theorem [G], it follows that T is a non empty path. Thus necessity is proved.

For sufficiency, suppose T is a non empty path. We prove that $P_{T_b}(T)$ is maximal outer planar. By induction on the number of points $p_i \geq 2$ of T. It is easy to observe that $P_{T_b}(T)$ of a path P with 2 points is $K_4 - x$, which is maximal outer planar. As the inductive hypothesis, let the pathos semitotal block graph of a non empty path P with n points be maximal outer planar. We now show that the pathos semitotal block graph of a path P' with $(n+1)$ points is maximal outer planar. First we prove that it is outer planar. Let the point and line sequence of the path

P' be $v_1, e_1, v_2, e_2, v_3, \ldots, v_n, e_n, v_{n+1}$, Where $v_1v_2 = e_1 = b_1, v_2v_3 = e_2 = b_2, \ldots, v_{n-1}v_n = e_{n-1} = b_{n1}, v_nv_{n+1} = e_n = b_n$.

The graphs P, P', $T_b(P)$, $T_b(P')$, $P_{T_b}(P)$ and $P_{T_b}(P')$ are shown in the figure 2. Without loss of generality $P' - v_{n+1} = P$.

By inductive hypothesis, $P_{T_b}(P)$ is maximal outer planar. Now the point v_{n+1} is one more point more in $P_{T_b}(P')$ than $P_{T_b}(P)$. Also there are only four lines $(v_{n+1}, v_n)(v_n, b_n)(b_n, v_{n+1})$ and (v_{n+1}, K_1) more in $P_{T_b}(P')$. Clearly the induced subgraph on the points v_{n+1}, v_n, b_n, K_1 is not K_4. Hence $P_{T_b}(P')$ is outer planar.

We now prove that $P_{T_b}(P')$ is maximal outer planar. Since $P_{T_b}(P)$ is maximal outer planar, it has $2(2q + k + 1) - 3$ lines. The outer planar graph $P_{T_b}(P')$ has $2(2q + k + 1) - 3 + 4 = 2(2q + k + 1 + 2) - 3$

$$= 2\left[(2q + 1) + (k + 1) + 1\right] - 3 \text{ lines.}$$

By Theorem [F], $P_{T_b}(P')$ is maximal outer planar. □

The next theorem gives a non-minimally non-outer planar $P_{T_b}(T)$.

Theorem 5 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is non-minimally non-outer planar.*

Proof We have the following cases.

Case 1 Suppose T is a path, then $\Delta(T) \leq 2$, then by Theorem 4, $P_{T_B}(T)$ is maximal outer planar.

Case 2 Suppose T is not a path, then $\Delta(T) \geq 3$, then by theorem 3, $P_{T_b}(T)$ is planar. On embedding $P_{T_b}(T)$ in any plane, the points with degree greater than two of T forms the cut points. In $P_{T_b}(T)$ which lie on at least two blocks. Since each block of $P_{T_b}(T)$ is a maximal outer planar than one can easily verify that $P_{T_b}(T)$ is outer planar. Hence for any non trivial tree with $\Delta(T) \geq 3$, $P_{T_b}(T)$ is non minimally non-outer planar. □

In the next theorem, we characterize the non-Eulerian $P_{T_b}(T)$.

Theorem 6 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is non-Eulerian.*

Proof We have the following cases.

Case 1 Suppose T is a path with 2 points, then $P_{T_b}(T) = K_4 - x$, which is non-Eulerian. If T is a path with $p > 2$ points. Then in $T_b(T)$ each block is a triangle such that they are in sequence with the vertices of $T_b(T)$ as $\{v_1, b_1, v_2, v_1\}$ an induced subgraph as a triangle $T_b(T)$. Further $\{v_2, b_2, v_3, v_2\}$, $\{v_3, b_3, v_4, v_3\}, \ldots, \{v_{n-1}, b_n, v_n, v_{n-1}\}$, in which each set form a triangle as an induced subgraph of $T_b(T)$. Clearly one can easily verify that $T_b(T)$ is Eulerian. Now this path has exactly one pathos point say k_1, which is adjacent to $v_1, v_2, v_3, \ldots, v_n$ in $P_{T_b}(T)$ in which all the points $v_1, v_2, v_3, \ldots, v_n \in P_{T_b}(T)$ are of odd degree. Hence $P_{T_b}(T)$ is non-Eulerian.

Case 2 Suppose $\Delta(T) \geq 3$. Assume T has a unique point of degree ≥ 3 and also assume that $T = K_{1.n}$. Then in $T_b(T)$ each block is a triangle, such that the number of blocks which are K_3

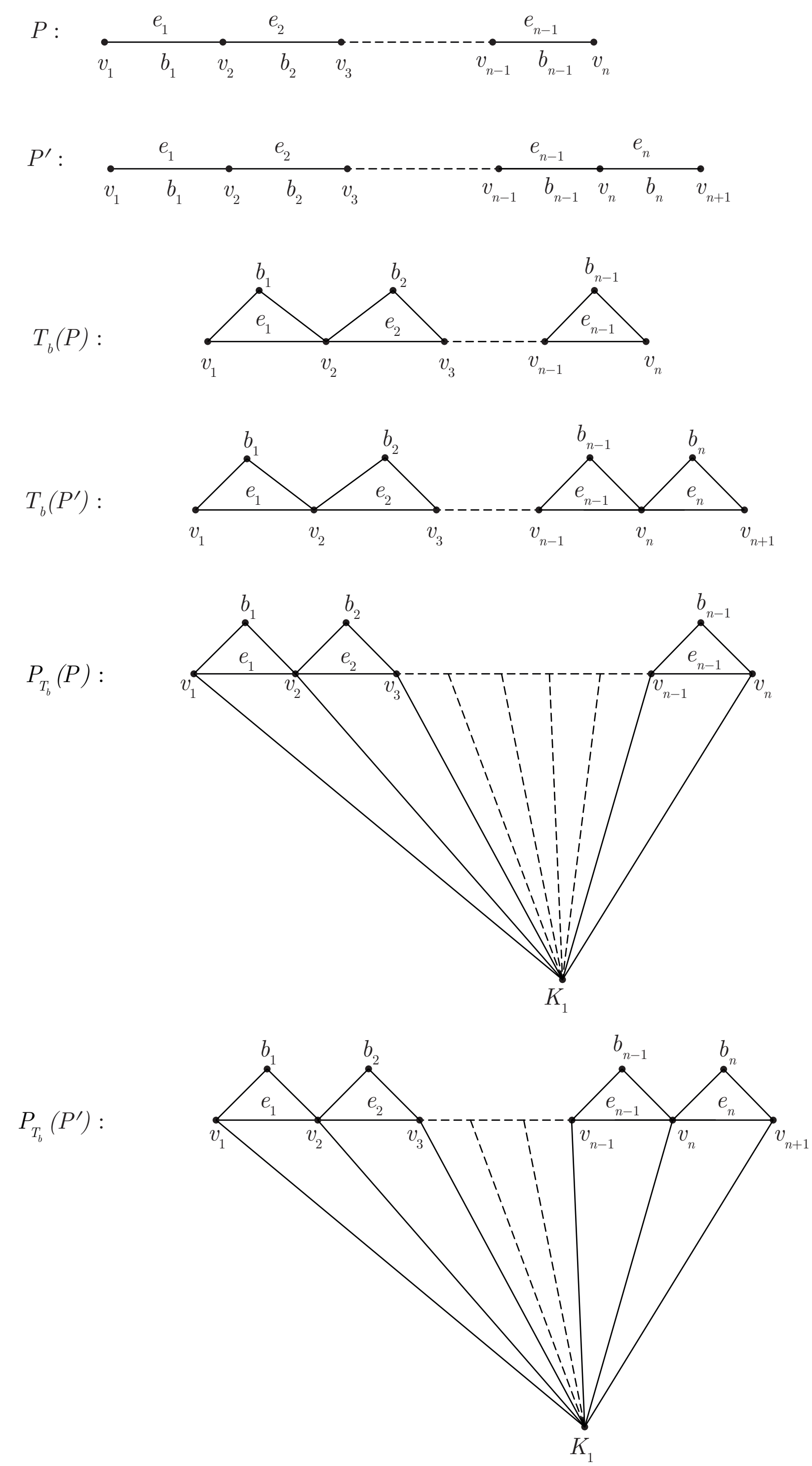
P :
e1 e2 en−1
v1 b1 v2 b2 v3 vn−1 bn−1 vn
P′ :
e1 e2 en−1 en
v1 b1 v2 b2 v3 vn−1 bn−1 vn bn vn+1
Tb(P) :
b1 b2 bn−1
e1 e2 en−1
v1 v2 v3 vn−1 vn
Tb(P′) :
b1 b2 bn−1 bn
e1 e2 en−1 en
v1 v2 v3 vn−1 vn vn+1
PTb(P) :
b1 b2 bn−1
e1 e2 en−1
v1 v2 v3 vn−1 vn
K1
PTb(P′) :
b1 b2 bn−1 bn
e1 e2 en−1 en
v1 v2 v3 vn−1 vn vn+1
K1

Figure 2:

are n with a common cut point k. Since the degree of a vertex $k = 2n$. One can easily verify that $T_b(K_{1,3})$ is Eulerian. To form $P_{T_b}(T)$, $T = K_{1,n}$, the points of degree 2 and the point k are joined by the corresponding pathos point which give $(n+1)$ points of odd degree in $P_{T_b}(T)$. Hence $P_{T_b}(T)$ is non-Eulerian. □

In the next theorem we characterize the hamiltonian $P_{T_b}(T)$.

Theorem 7 *For any non trivial tree T, the pathos semitotal block graph $P_{T_b}(T)$ of a tree T is hamiltonian if and only if T is a path.*

Proof For the necessity, suppose T is a path and has exactly one path of pathos. Let $V[T_b(T)] = \{u_1, u_2, u_3, \ldots, u_n\} \cup \{b_1, b_2, b_3, \ldots, b_{n-1}\}$, where $b_1, b_2, b_3, \ldots, b_{n-1}$ are block points of T. Since each block is a triangle and each block consists of points as $B_1 = \{u_1, b_1, u_2\}, B_2 = \{u_2, b_2, u_3\}, \ldots, B_m = \{u_m, b_m, u_{m+1}\}$. In $P_{T_b}(T)$ the pathos point w is adjacent to $\{u_1, u_2, u_3, \ldots, u_n\}$. Hence $V[P_{T_b}(T)] = \{u_1, u_2, u_3, \ldots, u_n\} \cup \{b_1, b_2, b_3, \ldots, b_{n-1}\} \cup w$ form a cycle as $w, u_1, b_1, u_2, b_2, u_2, \ldots$ u_{n-1}, b_{n-1}, u_n, w. Containing all the points of $P_{T_b}(T)$. Clearly $P_{T_b}(T)$ is hamiltonian. Thus necessity is proved.

For the sufficiency, suppose $P_{T_b}(T)$ is hamiltonian, now we consider the following cases.

Case 1 Assume T is a path. Then T has at least one point with deg $v \geq 3$, $\forall v \in V(T)$, assume that T has exactly one point u such that degree $u > 2$, then $G = T = K_{1.n}$. Now we consider the following subcases of Case 1.

Subcase 1.1 For $K_{1.n}$, $n > 2$ and n is even, then in $T_b(T)$ each block is k_3. The number of path of pathos are $\frac{n}{2}$. Since n is even we get $\frac{n}{2}$ blocks. Each block contains two lines of $\langle K_4 - x \rangle$, which is a non line disjoint subgraph of $P_{T_b}(T)$. Since $P_{T_b}(T)$ has a cut point, one can easily verify that there does not exist any hamiltonian cycle, a contradiction.

Subcase 1.2 For $K_{1.n}$, $n > 2$ and n is odd, then the number of path of pathos are $\dfrac{n+1}{2}$, since n is odd we get $\dfrac{n-1}{2} + 1$ blocks in which $\dfrac{n-1}{2}$ blocks contains two times of $\langle K_4 - x \rangle$ which is nonline disjoint subgraph of $P_{T_b}(T)$ and remaining block is $\langle K_4 - x \rangle$. Since $P_{T_b}(T)$ contain a cut point, clearly $P_{T_b}(T)$ does not contain a hamiltonian cycle, a contradiction. Hence the sufficient condition.

§3. Pathos Total Block Graph of a Tree

A tree T, its total block graph $T_B(T)$, and their pathos total block graphs $P_{T_B}(T)$ are shown in the Fig.3. We start with a few preliminary results.

Remark 3 For any non trivial path, the inner point number of the pathos total block graph $P_{T_B}(T)$ of a tree T is equal to the number of cut points in T.

Remark 4 The degree of a pathos point in $P_{T_B}(T)$ is n, then the pathos length of the corresponding path P_i of pathos in T is $n-1$.

Remark 5 For any non trivial tree T, $P_{T_B}(T)$ is a block.

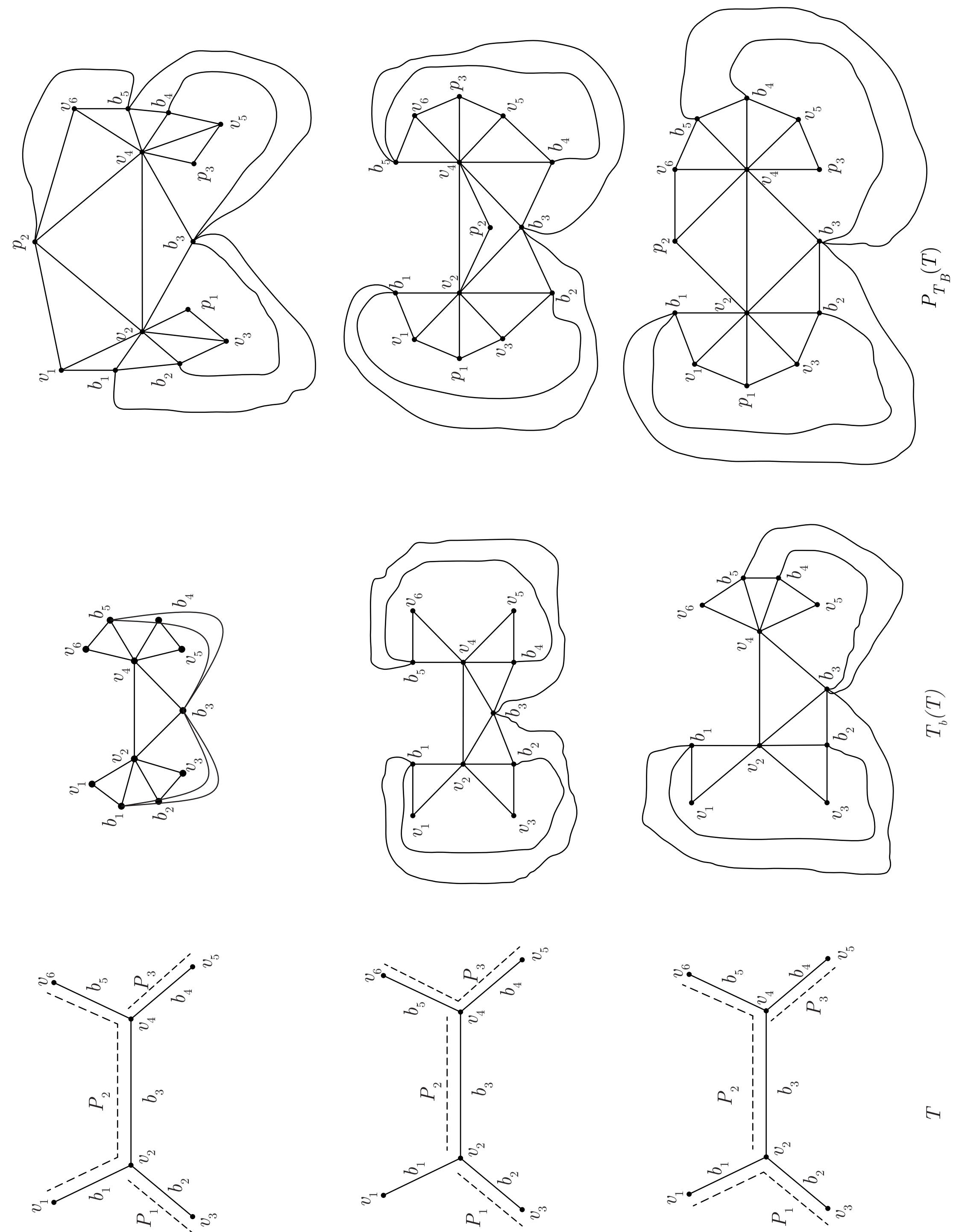

Figure 3:

Also in Kulli [4], developed the number of points and lines of a total block graph of a graph has been expressed in terms of blocks of G. Now using this we have a modified theorem as shown below in which we have expressed the number of points and lines in terms of lines and degrees of the points of G which is a tree.

Theorem 8 *For any non trivial (p, q) tree whose points have degree d_i, the number of points and lines in total block graph of a tree T are $(2q+1)$ and $\left(2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$.*

Proof By Theorem [B], the number of points in $T_b(T)$ is $\left(\sum_{i=1}^{b} b_i\right) + 1$, where b_i are the number of blocks in T to which the points v_i belongs in G. Since $\sum b_i = 2q$, for G is a tree. Thus the number of points in $T_B(G) = 2q+1$. Also, by Theorem [B], the number of lines in $T_B(G)$ are $q + \sum_{i=1}^{b} \binom{b_i+1}{2} = \left(\sum_{i=1}^{b} b_i\right) + \left(\frac{1}{2}\sum_{i=1}^{p} d_i^2\right) = \left(2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$, for G is a tree. □

In the following theorem we obtain the number of points and lines in $P_{T_B}(T)$.

Theorem 9 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T, whose points have degree d_i, then the number of points in $P_{T_B}(T)$ are $(2q+k+1)$ and the number of lines are $\left(q + 2 + \sum_{i=1}^{p} d_i^2\right)$, where k is the path number.*

Proof By Theorem 7, the number of points in $T_B(T)$ are $2q+1$, and by definition of $P_{T_B}(T)$, the number of points in $P_{T_B}(T)$ are $(2q+k+1)$, where k is the path number in T. Also by Theorem 7, the number of lines in $T_B(T)$ are $\left(2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$. The number of lines in $P_{T_B}(T)$ is the sum of lines in $T_B(T)$ and the number of lines which lie on the points of pathos of T which are to $\left(-q + 2 + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right)$. Thus the number of lines in $P_{T_B}(T)$ is equal to $\left(2q + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right) + \left(-q + 2 + \frac{1}{2}\sum_{i=1}^{p} d_i^2\right) = \left(q + 2 + \sum_{i=1}^{p} d_i^2\right)$. □

§4. Planar Pathos Total Block Graphs

A criterion for pathos total block graph to be planar is presented in our next theorem.

Theorem 10 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is planar if and only if $\Delta(T) \leq 3$.*

Proof Suppose $P_{T_B}(T)$ is planar. Then by Theorem [C], each cut point of T lie on at most 3 blocks. Since each block is a line in a tree, now we can consider the degree of cutpoints instead of number of blocks incident with the cut points. Now suppose if $\Delta(T) \leq 3$, then by Theorem [C], $T_B(T)$ is planar. Let $\{b_1, b_2, b_3, \ldots, b_{p-1}\}$ be the blocks of T with p points such that $b_1 = e_1, b_2 = e_2, \ldots, b_{p-1} = e_{p-1}$ and P_i be the number of pathos of T. Now $V[P_{T_B}(T)] = V(G) \cup \{b_1, b_2, \ldots b_{p-1}\} \cup \{P_i\}$. By Theorem [C], and by the definition of pathos, the embedding of $P_{T_B}(T)$ in any plane gives a planar $P_{T_B}(T)$.

Suppose $\Delta(T) \geq 4$ and assume that $P_{T_B}(T)$ is planar. Then there exists at least one point of degree 4, assume that there exists a vertex v such that $\deg v = 4$. Then in $T_B(T)$, this point together with the block points form k_5 as an induced subgraph. Further the corresponding pathos point are adjacent to the V(T) in $T_B(T)$ which gives $P_{T_B}(T)$ in which again k_5 as an induced subgraph, a contradiction to the planarity of $P_{T_B}(T)$. This completes the proof. □

The following theorem results the maximal outer planar $P_{T_B}(T)$.

Theorem 11 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is maximal outer planar if and only if $T = k_2$.*

Proof Suppose $T = k_3$ and $P_{T_B}(T)$ is maximal outer planar. Then $T_B(T) = k_4$ and one can easily verify that, $i[P_{T_B}(T)] > 1$, a contradiction. Further we assume that $T = K_{1,2}$ and $P_{T_B}(T)$ is maximal outer planar, then $T_B(T)$ is $W_p - x$, where x is outer line of W_p. Since $K_{1,2}$ has exactly one pathos, this point together with $W_p - x$ gives W_{p+1}. Clearly, $P_{T_B}(T)$ is non maximal outer planar, a contradiction. For the converse, if $T = k_2$, $T_B(T) = k_3$ and $P_{T_B}(T)$ $= K_4 - x$ which is a maximal outer planar. This completes the proof of the theorem. □

Now we have a pathos total block graph of a path $p \geq 2$ point as shown in the below remarks, and also a cycle with $p \geq 3$ points.

Remark 6 For any non trivial path with p points, $i[P_{T_B}(T)] = p - 2$.

Remark 7 For any cycle C_p, $p \geq 3$, $i[P_{T_B}(C_p)] = p - 1$.

The next theorem gives a minimally non-outer planar $P_{T_B}(T)$.

Theorem 12 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is minimally non-outer planar if and only if T is a path with 3 points.*

Proof Suppose $P_{T_B}(T)$ is minimally non-outer planar. Assume T is not a path. We consider the following cases.

Case 1 Suppose T is a tree with $\Delta(T) \geq 3$. Then there exists at least one point of degree at least 3. Assume v be a point of degree 3. Clearly, $T = K_{1,3}$. Then by the Theorem [D], $i[T_B(T)] > 1$ since $T_B(T)$ is a subgraph of $P_{T_B}(T)$. Clearly $i[P_{T_B}(T)] \geqslant 2$ a contradiction.

Case 2 Suppose T is a closed path with p points, then it is a cycle with p points. By Theorem [D], $P_{T_B}(T)$ is minimally non-outer planar. By Remark 7, $i[P_{T_B}(T)] > 1$, a contradiction.

Case 3 Suppose T is a closed path with $p \geq 4$ points, clearly by Remark 6, $i[P_{T_B}(T)] > 2$, a contradiction.

Conversely, suppose T is a path with 3 points, clearly by Remark 6, $i[P_{T_B}(T)] = 1$. This gives the required result. □

In the following theorem we characterize the non-Eulerian $P_{T_B}(T)$.

Theorem 13 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is non-Eulerian.*

Proof We consider the following cases.

Case 1 Suppose T is a path. For $p = 2$ points, then $P_{T_B}(T) = K_4 - x$, which is non-Eulerian. For $p = 3$ points, then $P_{T_B}(T)$ is a wheel, which is non-Eulerian.

For $p \geq 4$ we have a path $P : v_1, v_2, v_3, \ldots, v_p$. Now in path each line is a block. Then $v_1v_2 = e_1 = b_1, v_2v_3 = e_2 = b_2, \ldots, v_{p-1}v_p = e_{p-1} = b_{p-1}$, $\forall e_{p-1} \in E(G)$, and $\forall b_{p-1} \in V[T_B(P)]$. In $T_B(P)$, the degree of each point is even except b_1 and b_{p-1}. Since the path P has exactly one pathos which is a point of $P_{T_B}(P)$ and is adjacent to the points $v_1, v_2, v_3, \ldots, v_p$, of $T_B(P)$ which are of even degree, gives as an odd degree points in $P_{T_B}(P)$ including odd degree points b_1 and b_2. Clearly $P_{T_B}(P)$ is non-Eulerian.

Case 2 Suppose T is not a path. We consider the following subcases,

Subcase 2.1 Assume T has a unique point degree ≥ 3 and $T = K_{1.n}$, where n is odd. Then in $T_B(T)$ each block is a triangle such that there are n number of triangles with a common cut point k which has a degree $2n$. Since the degree of each point in $T_B(K_{1,n})$ is Eulerian. To form $P_{T_B}(T)$ where $T = K_{1,n}$, the points of degree 2 and the point k are joined by the corresponding pathos point which gives $(n+1)$ points of odd degree in $P_{T_B}(K_{1.n})$. $P_{T_B}(K_{1.n})$ has n points of odd degree. Hence $P_{T_B}(T)$ non-Eulerian.

Assume that $T = K_{1.n}$, where n is even, then in $T_B(T)$ each block is a triangle, which are $2n$ in number with a common cut point k. Since the degree of each point other than k is either 2 or $(n+1)$ and the degree of the point k is $2n$. One can easily verify that $T_B(K_{1,n})$ is non-Eulerian. To form $P_{T_B}(T)$ where $T = K_{1,n}$, the points of degree 2 and the point k are joined by the corresponding pathos point which gives $(n+2)$ points of odd degree in $P_{T_B}(T)$. Hence $P_{T_B}(T)$ non-Eulerian.

Subcase 2.2 Assume T has at least two points of degree ≥ 3. Then $V[T_B(T)] = V(G) \cup \{b_1, b_2, b_3, \ldots, b_p\}$, $\forall e_p \in E(G)$. In $T_B(T)$, each endpoint has degree 2 and these points are adjacent to the corresponding pathos points in $P_{T_B}(T)$ gives degree 3, From Case 1, Tree T has at least 4 points and by Corollary [A], $P_{T_B}(T)$ has at least two points of degree 3. Hence $P_{T_B}(T)$ is non-Eulerian. □

In the next theorem we characterize the hamiltonian $P_{T_B}(T)$.

Theorem 14 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T is hamiltonian.*

Proof We consider the following cases.

Case 1 Suppose T is a path with $\{u_1, u_2, u_3, \ldots, u_n\} \in V(T)$ and $b_1, b_2, b_3, \ldots, b_m$ be the number of blocks of T such that $m = n-1$. Then it has exactly one path of pathos. Now point set of $T_B(T)$ is $V[T_B(T)] = \{u_1, u_2, \ldots, u_n\} \cup \{b_1, b_2, \ldots, b_m\}$. Since given graph is a path then in $T_B(T)$, $b_1 = e_1, b_2 = e_2, \ldots, b_m = e_m$, such that $b_1, b_2, b_3, \ldots, b_m \subset V[T_B(T)]$. Then by the definition of total block graph $\{u_1, u_2, \ldots, u_m\} \cup \{b_1, b_2, \ldots, b_{m-1}, b_m\} \cup \{b_1, u_1, b_2u_2, \ldots, b_mu_{n-1}, b_mu_n\}$ form line set of $T_B(T)$[see Fig. 4].

Now this path has exactly one pathos say w. In forming pathos total block graph of a path, the pathos w becomes a point, then $V[P_{T_B}(T)] = \{u_1, u_2, \ldots, u_n\} \cup \{b_1, b_2, \ldots, b_m\} \cup \{w\}$ and

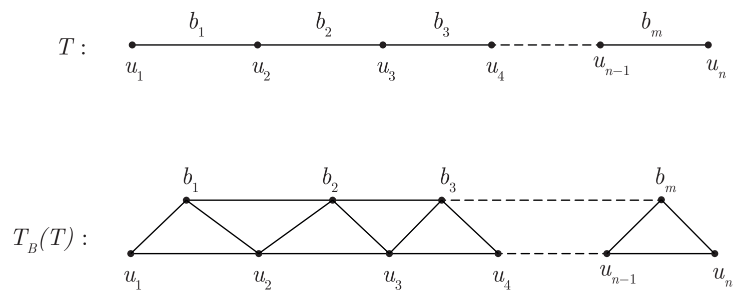

Figure 4:

w is adjacent to all the points $\{u_1, u_2, \ldots, u_m\}$ shown in the Fig.5.

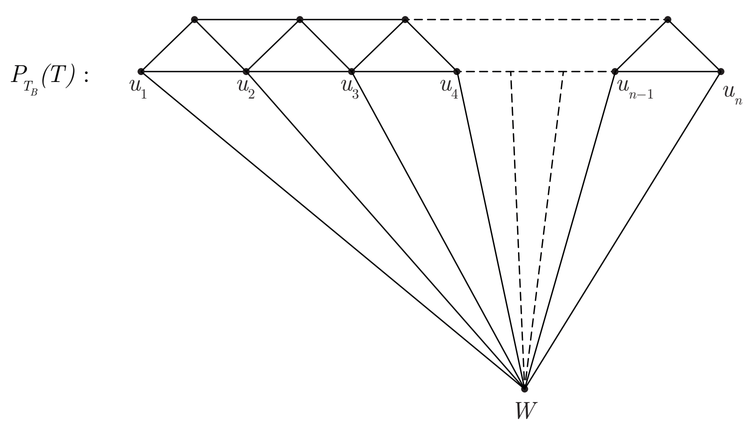

Figure 5:

In $P_{T_B}(T)$, the hamiltonian cycle $w, u_1, b_1, u_2, b_2, u_2, u_3, b_3, \ldots, u_{n-1}, b_m, u_n, w$ exist. Clearly the pathos total block graph of a path is hamiltonian graph.

Case 2 Suppose T is not a path. Then T has at least one point with degree at least 3. Assume that T has exactly one point u such that degree> 2. Now we consider the following subcases of case 2.

Subcase 2.1 Assume $T = K_{1.n}$, $n > 2$ and is odd. Then the number of paths of pathos are $\frac{n+1}{2}$. Let $V[T_B(T)] = \{u_1, u_2, \ldots, u_n, b_1, b_2, \ldots, b_{m-1}\}$. By the definition of $P_{T_B}(T)$, $V[P_{T_B}(T)] = \{u_1, u_2, \ldots, u_n, b_1 b_2, \ldots, b_{n-1}\} \cup \{p_1, p_2, \ldots, p_{n+1/2}\}$. Then there exists a cycle containing the points of $P_{T_B}(T)$ as $p_1, u_1, b_1, b_2, u_3, p_2, u_2, b_3, u_4, \ldots p_1$ and is a hamiltonian cycle. Hence $P_{T_B}(T)$ is a hamiltonian.

Subcase 2.2 Assume $T = K_{1.n}$, $n > 2$ and is even. Then the number of path of pathos are $\frac{n}{2}$, then $V[T_B(T)] = \{u_1, u_2, \ldots, u_n, b_1, b_2, \ldots b_{n-1}\}$. By the definition of $P_{T_B}(T)$. $V[P_{T_B}(T)] = \{u_1, u_2, \ldots, u_n, b_1, b_2, \ldots, b_{n-1}\} \cup \{p_1, p_2, \ldots, p_{n/2}\}$. Then there exist a cycle containing the points of $P_{T_B}(T)$ as $p_1, u_1, b_1, b_2, u_3, p_2, u_4, b_3, b_4, \ldots, p_1$ and is a hamiltonian cycle. Hence $P_{T_B}(T)$ is a hamiltonian.

Suppose T is neither a path or a star. Then T contains at least two points of degree> 2. Let $u_1, u_2, u_3, \ldots, u_n$ be the points of degree ≥ 2 and $v_1, v_2, v_3, \ldots, v_m$ be the end points of T. Since end block is a line in T, and denoted as $b_1, b_2, \ldots, b_k$, then tree T has p_i pathos points, $i > 1$ and each pathos point is adjacent to the point of T where the corresponding pathos lie on the points of T. Let $\{p_1, p_2, \ldots\ldots, p_i\}$ be the pathos points of T. Then there exists a cycle C containing all the points of $P_{T_B}(T)$ as $p_1, v_1, b_1, b_2, v_2, p_2, u_1, b_3, u_2, p_3, v_3, b_4, \ldots, v_{n-1}, b_{n-1}, b_n, v_n, \ldots, p_1$. Hence $P_{T_B}(T)$ is a hamiltonian cycle. Hence $P_{T_B}(T)$ is a hamiltonian graph. □

In the next theorem we characterize $P_{T_B}(T)$ in terms of crossing number one.

Theorem 15 *For any non trivial tree T, the pathos total block graph $P_{T_B}(T)$ of a tree T has crossing number one if and only if $\Delta(T) \leq 4$, and there exist a unique point in T of degree 4.*

Proof Suppose $P_{T_B}(T)$ has crossing number one. Then it is non-planar. Then by Theorem 10, we have $\Delta(T) \geq 4$. We now consider the following cases.

Case 1 Assume $\Delta(T) = 5$. Then by Theorem [E], $T_B\,(T)$ is non-planar with crossing number more than one. Since $T_B\,(T)$ is a subgraph of $P_{T_B}(T)$. Clearly $\mathrm{cr}(P_{T_B}\,(T)) > 1$, a contradiction.

Case 2 Assume $\Delta(T) = 4$. Suppose T has two points of degree 4. Then by Theorem [E], $T_B\,(T)$ has crossing number at least two. But $T_B\,(T)$ is a subgraph of $P_{T_B}(T)$. Hence $\mathrm{cr}(P_{T_B}\,(T)) > 1$, a contradiction.

Conversely, suppose T satisfies the given condition and assume T has a unique point v of degree 4. The lines which are blocks in T such that they are the points in $T_B\,(T)$. In $T_B\,(T)$, these block points and a point v together forms an induced subgraph as k_5. In forming $P_{T_B}(T)$, the pathos points are adjacent to at most two points of this induced subgraph. Hence in all these cases the $\mathrm{cr}(P_{T_B}\,(T)) = 1$. This completes the proof. □

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