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Some identities involving the k-th power complements

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Abstract The main purpose of this paper is using the elementary method to study the calculating problem of one kind infinite series involving the k-th power complements, and obtain several interesting identities.
Keywords k-th power complements; Identities; Riemann-zeta function.

§1. Introduction and Results

For any given natural number $k \geq 2$ and any positive integer n, we call $a_k(n)$ as a k-th power complements, if $a_k(n)$ is the smallest positive integer such that $n \cdot a_k(n)$ is a perfect k-th power. That is,

$$a_k(n) = \min\{m : mn = u^k, u \in N\}.$$

Especially, we call $a_2(n), a_3(n), a_4(n)$ as the square complement number, cubic complement number, and the quartic complement number, respectively. In reference [1], Professor F.Smarandache asked us to study the properties of the k-th power complement number sequence. About this problem, there are many people have studied it, see references [4], [5], and [6]. For example, Lou Yuanbing [7] gave an asymptotic formula involving the square complement number $a_2(n)$. Let real number $x \geq 3$, he proved that

$$\sum_{n \leq x} d(a_2(n)) = c_1 x \ln x + c_2 x + O(x^{\frac{1}{2}+\varepsilon}),$$

where $d(n)$ is the divisor function, $\varepsilon > 0$ be any fixed real number, c_1 and c_2 are defined as following:

$$c_1 = \frac{6}{\pi^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right),$$

$$c_2 = \frac{6}{\pi^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right) \left(\sum_p \frac{2(2p+1)\ln p}{(p-1)(p+1)(p+2)} + 2\gamma - 1\right),$$

γ is the Euler's constant, $\prod_p$ denotes the product over all primes.

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In reference [8], Yao Weili obtained an asymptotic formula involving k-th power complement number $a_k(n)$. That is, for any real number $x \geq 1$, we have

$$\sum_{n\leq x} d(na_k(n)) = x(A_0 \ln^k x + A_1 \ln^{k-1} x + \cdots + A_k) + O(x^{\frac{1}{2}+\varepsilon}),$$

where $A_0, A_1, \cdots, A_k$ are computable constant, ε is any fixed positive number.

In reference [4], Zhang Wenpeng obtained some identities involving the k-th power complements. Those are, for any complex numbers s with $Re(s) \geq 1$, we have

$$\sum_{n=1}^{+\infty} \frac{1}{(na_2(n))^s} = \frac{\zeta^2(2s)}{\zeta(4s)},$$

$$\sum_{n=1}^{+\infty} \frac{1}{(na_3(n))^s} = \frac{\zeta^2(3s)}{\zeta(6s)} \prod_p \left(1 + \frac{1}{p^{3s}+1}\right),$$

$$\sum_{n=1}^{+\infty} \frac{1}{(na_4(n))^s} = \frac{\zeta^2(4s)}{\zeta(8s)} \prod_p \left(1 + \frac{1}{p^{4s}+1}\right)\left(1 + \frac{1}{p^{4s}+2}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

On the other hand, F.Russo [9] proposed 21 unsolved problems, the problem 19 asked us evaluate the infinite series

$$\sum_{n=1}^{+\infty} (-1)^n \frac{1}{a_2(n)}.$$

But is problem very easy. In fact, $a_2(4n^2) = 1$ for all positive integer n. So we have

$$\lim_{n\longrightarrow\infty} (-1)^{4n^2} \frac{1}{a_2(4n^2)} = 1 \neq 0.$$

That is, the infinite series $\sum\limits_{n=1}^{+\infty} (-1)^n \frac{1}{a_2(n)}$ dos not convergent.

In this paper, we shall use the elementary method to study the calculating problem of the infinite series

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{na_k(n)},$$

and obtain several interesting identities for it. That is, we shall prove the following:

Theorem. For any given positive integer $k \geq 2$, we have the identity

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{na_k(n)} = \frac{2^k - k - 1}{2^k + k - 1} \zeta(k) \prod_p \left(1 + \frac{k-1}{p^k}\right),$$

where $\zeta(s)$ is the Riemann-zeta function, and $\prod\limits_p$ denotes the product over all different primes.

Taking $k = 2$ in our theorem, and note that the fact $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$ and

$$\prod_p \left(1 + \frac{1}{p^2}\right) = \prod_p \left(1 - \frac{1}{p^4}\right) \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} = \frac{\zeta(2)}{\zeta(4)},$$

we may immediately obtain the following:

Corollary 1. For the square complement number $a_2(n)$, we have the identity

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{na_2(n)}=\frac{1}{2}.$$

Corollary 2. For the cubic complement number $a_3(n)$, we have the identity

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{na_3(n)}=\frac{2}{5}\zeta(3)\prod_{p}\left(1+\frac{2}{p^3}\right).$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. For all positive integers n, we separate n into three parts: $2\dagger n$; $2\mid n$ and $2^k\dagger n$; $2^k\mid n$. Then from the definition of $a_k(n)$ we have:

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{na_k(n)} &= \sum_{\substack{n=1\\2\dagger n}}^{+\infty}\frac{1}{na_k(n)}-\sum_{\alpha=0}^{\infty}\sum_{\beta=1}^{k-1}\sum_{\substack{n=1\\2\dagger n}}^{\infty}\frac{1}{(2^{\alpha k+\beta}n)a_k(2^{\alpha k+\beta}n)}\\ &\quad -\sum_{\alpha=1}^{\infty}\sum_{\substack{n=1\\2\dagger n}}^{\infty}\frac{1}{(2^{\alpha k}n)a_k(2^{\alpha k}n)}\\ &= \sum_{\substack{n=1\\2\dagger n}}^{\infty}\frac{1}{na_k(n)}-\sum_{\alpha=0}^{\infty}\sum_{\beta=1}^{k-1}\frac{1}{2^{(\alpha+1)k}}\sum_{\substack{n=1\\2\dagger n}}^{\infty}\frac{1}{na_k(n)}-\sum_{\alpha=1}^{\infty}\frac{1}{2^{\alpha k}}\sum_{\substack{n=1\\2\dagger n}}^{\infty}\frac{1}{na_k(n)}\\ &= \frac{2^k-k-1}{2^k-1}\sum_{\substack{n=1\\2\dagger n}}^{\infty}\frac{1}{na_k(n)}.\end{aligned}$$

It is clear that the infinite series $\sum_{\substack{n=1\\2\dagger n}}^{\infty}\frac{1}{na_k(n)}$ is absolutely convergent, so from the Euler product formula (see Theorem 11.6 of [2]) we know that the infinite series can be expressed as an absolutely convergent infinite product. That is,

$$\begin{aligned}\sum_{\substack{n=1\\2\dagger n}}^{\infty}\frac{1}{na_k(n)} &= \prod_{\substack{p\\p\neq 2}}\left\{1+\frac{1}{pa_k(p)}+\frac{1}{p^2a_k(p^2)}+\dots\right\}\\ &= \prod_{\substack{p\\p\neq 2}}\sum_{l=0}^{\infty}\frac{1}{p^la_k(p^l)}\\ &= \prod_{\substack{p\\p\neq 2}}\left(\sum_{\alpha=0}^{\infty}\sum_{\beta=1}^{k-1}\frac{1}{p^{\alpha k+\beta}a_k(p^{\alpha k+\beta})}+\sum_{\alpha=0}^{\infty}\frac{1}{p^{\alpha k}a_k(p^{\alpha k})}\right)\end{aligned}$$

$$
\begin{aligned}
&= \prod_{\substack{p\\p\neq 2}} \left(\sum_{\alpha=0}^{\infty} \sum_{\beta=1}^{k-1} \frac{1}{p^{(\alpha+1)k}} + \sum_{\alpha=0}^{\infty} \frac{1}{p^{\alpha k}} \right) \\
&= \prod_{\substack{p\\p\neq 2}} \frac{p^k+k-1}{p^k-1} \\
&= \prod_{p} \frac{1+\frac{k-1}{p^k}}{1-\frac{1}{p^k}} \cdot \frac{1-\frac{1}{2^k}}{1+\frac{k-1}{2^k}} \\
&= \frac{2^k-1}{2^k+k-1}\zeta(k)\prod_{p}\left(1+\frac{k-1}{p^k}\right).
\end{aligned}
$$

So we have

$$
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{na_k(n)} = \frac{2^k-k-1}{2^k+k-1}\zeta(k)\prod_{p}\left(1+\frac{k-1}{p^k}\right).
$$

This completes the proof of the theorem.

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