

## Odd Harmonious Labeling of Some Graphs

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**Abstract:** The labeling of discrete structures is a potential area of research due to its wide range of applications. The present work is focused on one such labeling called odd harmonious labeling. A graph  $G$  is said to be odd harmonious if there exist an injection  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$  such that the induced function  $f^* : E(G) \rightarrow \{1, 3, \dots, 2q - 1\}$  defined by  $f^*(uv) = f(u) + f(v)$  is a bijection. Here we investigate odd harmonious labeling of some graphs. We prove that the shadow graph and the splitting graph of bistar  $B_{n,n}$  are odd harmonious graphs. Moreover we show that the arbitrary supersubdivision of path  $P_n$  admits odd harmonious labeling. We also prove that the joint sum of two copies of cycle  $C_n$  for  $n \equiv 0 \pmod{4}$  and the graph  $H_{n,n}$  are odd harmonious graphs.

**Key Words:** Harmonious labeling, Smarandachely  $p$ -harmonious labeling, odd harmonious labeling, shadow graph, splitting graph, arbitrary supersubdivision.

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### §1. Introduction

We begin with simple, finite, connected and undirected graph  $G = (V(G), E(G))$  with  $|V(G)| = p$  and  $|E(G)| = q$ . For standard terminology and notation we follow Gross and Yellen [5]. We will provide brief summary of definitions and other information which are necessary and useful for the present investigations.

**Definition 1.1** *If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.*

Any graph labeling will have the following three common characteristics:

- (1) a set of numbers from which the vertex labels are chosen;
- (2) a rule that assigns a value to each edge;
- (3) a condition that these values must satisfy.

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Graph labelings is an active area of research in graph theory which is mainly evolved through its rigorous applications in coding theory, communication networks, optimal circuits layouts and graph decomposition problems. According to Beineke and Hegde [1] graph labeling serves as a frontier between number theory and structure of graphs. For a dynamic survey of various graph labeling problems along with an extensive bibliography we refer to Gallian [2].

**Definition 1.2** *A function  $f$  is called graceful labeling of a graph  $G$  if  $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$  is injective and the induced function  $f^* : E(G) \rightarrow \{1, 2, \dots, q\}$  defined as  $f^*(e = uv) = |f(u) - f(v)|$  is bijective.*

A graph which admits graceful labeling is called a *graceful graph*. Rosa [8] called such a labeling a  $\beta$  - valuation and Golomb [3] subsequently called it *graceful labeling*. Several infinite families of graceful as well as non-graceful graphs have been reported. The famous Ringel-Kotzig tree conjecture [7] and many illustrious works on graceful graphs brought a tide of different ways of labeling the elements of graph such as odd graceful labeling, harmonious labeling etc. Graham and Sloane [4] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes.

**Definition 1.3** *A graph  $G$  is said to be harmonious if there exist an injection  $f : V(G) \rightarrow Z_q$  such that the induced function  $f^* : E(G) \rightarrow Z_q$  defined by  $f^*(uv) = (f(u) + f(v)) \pmod{q}$  is a bijection and  $f$  is said to be harmonious labelling of  $G$ .*

If  $G$  is a tree or it has a component that is a tree, then exactly one label may be used on two vertices and the labeling function is not an injection. After this many researchers have studied harmonious labeling. A labeling is also introduced with minor variation in harmonious theme, which is defined as follows.

**Definition 1.4** *Let  $k, p$  be integers with  $p \geq 1$  and  $k \leq p$ . A graph  $G$  is said to be Smarandachely  $p$ -harmonious labeling if there exist an injection  $f : V(G) \rightarrow \{0, 1, 2, \dots, kp - 1\}$  such that the induced function  $f^* : E(G) \rightarrow \{1, p + 1, \dots, pq - 1\}$  defined by  $f^*(uv) = f(u) + f(v)$  is a bijection. Particularly, if  $p = k = 2$ , such a Smarandachely 2-harmonious labeling is called an odd harmonious labeling of  $G$ ,  $f$  and  $f^*$  are called vertex function and edge function respectively.*

Liang and Bai [6] have obtained a necessary conditions for the existence of odd harmonious labelling of graph. It has been also shown that many graphs admit odd harmonious labeling and odd harmoniousness of graph is useful for the solution of undetermined equations. In the same paper many ways to construct an odd harmonious graph were reported. Vaidya and Shah [9] have also proved that the shadow and the splitting graphs of path  $P_n$  and star  $K_{1,n}$  are odd harmonious graphs.

Generally there are three types of problems that can be considered in this area.

- (1) How odd harmonious labeling is affected under various graph operations;
- (2) To construct new families of odd harmonious graph by investigating suitable function which generates labeling;
- (3) Given a graph theoretic property  $P$ , characterize the class/classes of graphs with prop-

erty  $P$  that are odd harmonious.

The problems of second type are largely discussed while the problems of first and third types are not so often but they are of great importance. The present work is aimed to discuss the problems of first kind.

**Definition 1.5** The shadow graph  $D_2(G)$  of a connected graph  $G$  is constructed by taking two copies of  $G$  say  $G'$  and  $G''$ . Join each vertex  $u'$  in  $G'$  to the neighbours of the corresponding vertex  $v'$  in  $G''$ .

**Definition 1.6** For a graph  $G$  the splitting graph  $S'(G)$  of a graph  $G$  is obtained by adding a new vertex  $v'$  corresponding to each vertex  $v$  of  $G$  such that  $N(v) = N(v')$ .

**Definition 1.7** The arbitrary supersubdivision of a graph  $G$  produces a new graph by replacing each edge of  $G$  by complete bipartite graph  $K_{2,m_i}$  (where  $m_i$  is any positive integer) in such a way that the ends of each  $e_i$  are merged with two vertices of 2-vertices part of  $K_{2,m_i}$  after removing the edge  $e_i$  from the graph  $G$ .

**Definition 1.8** Consider two copies of a graph  $G$  and define a new graph known as joint sum is the graph obtained by connecting a vertex of first copy with a vertex of second copy.

**Definition 1.9**  $H_{n,n}$  is the graph with vertex set  $V(H_{n,n}) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$  and the edge set  $E(H_{n,n}) = \{v_i u_j : 1 \leq i \leq n, n - i + 1 \leq j \leq n\}$ .

## §2. Main Results

**Theorem 2.1**  $D_2(B_{n,n})$  is an odd harmonious graph.

*Proof* Consider two copies of  $B_{n,n}$ . Let  $\{u, v, u_i, v_i, 1 \leq i \leq n\}$  and  $\{u', v', u'_i, v'_i, 1 \leq i \leq n\}$  be the corresponding vertex sets of each copy of  $B_{n,n}$ . Denote  $D_2(B_{n,n})$  as  $G$ . Then  $|V(G)| = 4(n + 1)$  and  $|E(G)| = 4(2n + 1)$ . To define  $f : V(G) \rightarrow \{0, 1, 2, 3, \dots, 16n + 7\}$ , we consider following two cases.

**Case 1.**  $n$  is even

$$\begin{aligned} f(u) &= 2, f(v) = 1, f(u') = 0, f(v') = 5, \\ f(u_i) &= 9 + 4(i - 1), 1 \leq i \leq n, f(u'_i) = f(u_n) + 4i, 1 \leq i \leq n, \\ f(v_1) &= f(u'_n) + 3, f(v_{2i+1}) = f(v_1) + 8i, 1 \leq i \leq \frac{n}{2} - 1, \\ f(v_2) &= f(u'_n) + 5, f(v_{2i}) = f(v_2) + 8(i - 1), 2 \leq i \leq \frac{n}{2}, \\ f(v'_1) &= f(v_n) + 6, f(v'_{2i+1}) = f(v'_1) + 8i, 1 \leq i \leq \frac{n}{2} - 1, \\ f(v'_2) &= f(v_n) + 8, f(v'_{2i}) = f(v'_2) + 8(i - 1), 2 \leq i \leq \frac{n}{2} \end{aligned}$$

**Case 2:**  $n$  is odd

$$\begin{aligned}
 f(u) &= 2, f(v) = 1, f(u') = 0, f(v') = 5, \\
 f(u_i) &= 9 + 4(i - 1), 1 \leq i \leq n, f(u'_i) = f(u_n) + 4i, 1 \leq i \leq n, \\
 f(v_1) &= f(u'_n) + 3, f(v_{2i+1}) = f(v_1) + 8i, 1 \leq i \leq \frac{n-1}{2}, \\
 f(v_2) &= f(u'_n) + 5, f(v_{2i}) = f(v_2) + 8(i - 1), 2 \leq i \leq \frac{n-1}{2}, \\
 f(v'_1) &= f(v_n) + 2, f(v'_{2i+1}) = f(v'_1) + 8i, 1 \leq i \leq \frac{n-1}{2}, \\
 f(v'_2) &= f(v_n) + 8, f(v'_{2i}) = f(v'_2) + 8(i - 1), 2 \leq i \leq \frac{n-1}{2}
 \end{aligned}$$

The vertex function  $f$  defined above induces a bijective edge function  $f^* : E(G) \rightarrow \{1, 3, \dots, 16n + 7\}$ . Thus  $f$  is an odd harmonious labeling for  $G = D_2(B_{n,n})$ . Hence  $G$  is an odd harmonious graph.  $\square$

**Illustration 2.2** Odd harmonious labeling of the graph  $D_2(B_{5,5})$  is shown in Fig. 1.

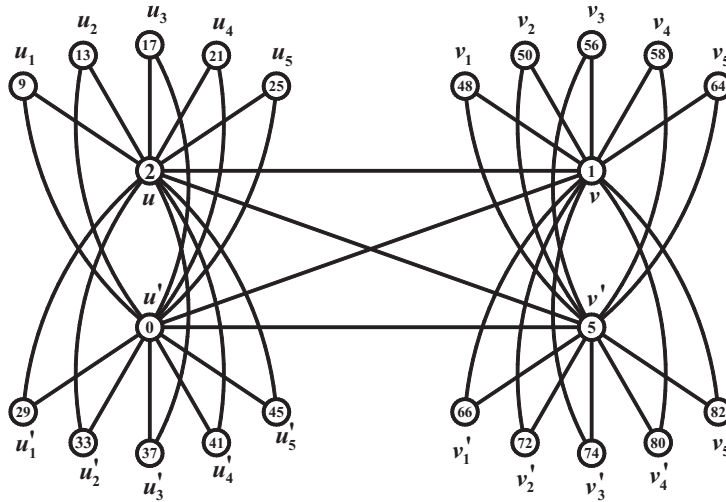


Fig. 1

**Theorem 2.3**  $S'(B_{n,n})$  is an odd harmonious graph.

*Proof* Consider  $B_{n,n}$  with vertex set  $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ , where  $u_i, v_i$  are pendant vertices. In order to obtain  $S'(B_{n,n})$ , add  $u', v', u'_i, v'_i$  vertices corresponding to  $u, v, u_i, v_i$  where,  $1 \leq i \leq n$ . If  $G = S'(B_{n,n})$  then  $|V(G)| = 4(n + 1)$  and  $|E(G)| = 6n + 3$ . We define vertex labeling  $f : V(G) \rightarrow \{0, 1, 2, 3, \dots, 12n + 5\}$  as follows.

$$\begin{aligned}
 f(u) &= 0, f(v) = 3, f(u') = 2, f(v') = 1, \\
 f(u_i) &= 7 + 4(i - 1), 1 \leq i \leq n, f(v_1) = f(u_n) + 3, \\
 f(v_{i+1}) &= f(v_1) + 4i, 1 \leq i \leq n - 1, \\
 f(u'_1) &= f(v_n) + 5, f(u'_{i+1}) = f(u'_1) + 2i, 1 \leq i \leq n - 1, \\
 f(v'_1) &= f(u'_n) - 1, f(v'_{i+1}) = f(v'_1) + 2i, 1 \leq i \leq n - 1.
 \end{aligned}$$

The vertex function  $f$  defined above induces a bijective edge function  $f^* : E(G) \rightarrow \{1, 3, \dots, 12n + 5\}$ . Thus  $f$  is an odd harmonious labeling of  $G = S'(B_{n,n})$  and  $G$  is an odd harmonious graph.  $\square$

**Illustration 2.4** Odd harmonious labeling of the graph  $S'(B_{5,5})$  is shown in Fig. 2.

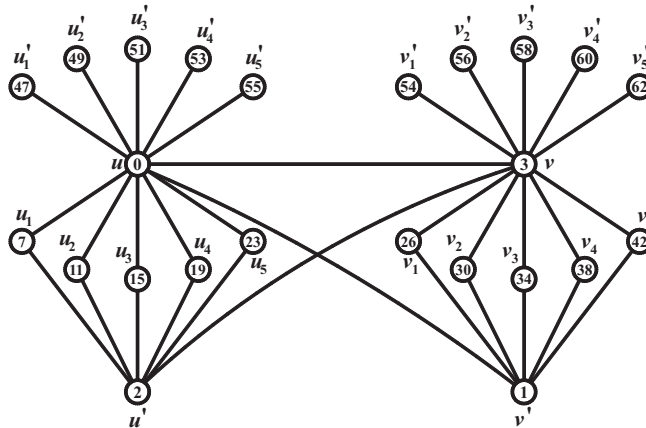


Fig. 2

**Theorem 2.5** Arbitrary supersubdivision of path  $P_n$  is an odd harmonious graph.

*Proof* Let  $P_n$  be the path with  $n$  vertices and  $v_i$  ( $1 \leq i \leq n$ ) be the vertices of  $P_n$ . Arbitrary supersubdivision of  $P_n$  is obtained by replacing every edge  $e_i$  of  $P_n$  with  $K_{2,m_i}$  and we denote this graph by  $G$ . Let  $u_{ij}$  be the vertices of  $m_i$ -vertices part of  $K_{2,m_i}$  where  $1 \leq i \leq n - 1$  and  $1 \leq j \leq \max\{m_i\}$ . Let  $\alpha = \sum_{i=1}^{n-1} m_i$  and  $q = 2\alpha$ . We define vertex labeling  $f : V(G) \rightarrow \{0, 1, 2, 3, \dots, 2q - 1\}$  as follows.

$$\begin{aligned}
 f(v_{i+1}) &= 2i, \quad 0 \leq i \leq n - 1, \\
 f(u_{1j}) &= 1 + 4(j - 1), \quad 1 \leq j \leq m_1, \\
 f(u_{ij}) &= f(u_{(i-1)n}) + 2 + 4(j - 1), \quad 1 \leq j \leq m_i, 2 \leq i \leq n.
 \end{aligned}$$

The vertex function  $f$  defined above induces a bijective edge function  $f^* : E(G) \rightarrow \{1, 3, \dots, 2q - 1\}$ . Thus  $f$  is an odd harmonious labeling of  $G$ . Hence arbitrary supersubdivision of path  $P_n$  is an odd harmonious graph.  $\square$

**Illustration 2.6** Odd harmonious labeling of arbitrary supersubdivision of path  $P_5$  is shown in Fig. 3.

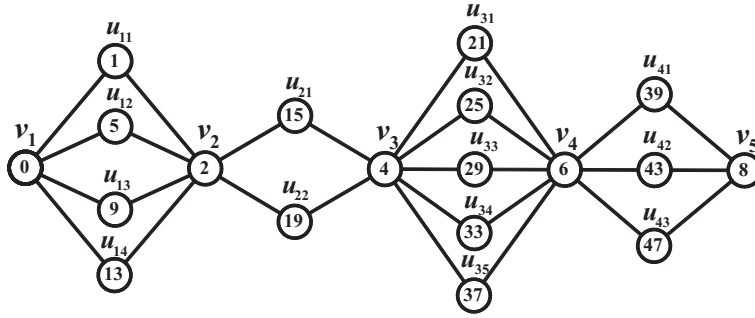


Fig. 3

**Theorem 2.7** Joint sum of two copies of  $C_n$  admits an odd harmonious labeling for  $n \equiv 0(mod 4)$ .

*Proof* We denote the vertices of first copy of  $C_n$  by  $v_1, v_2, \dots, v_n$  and vertices of second copy by  $v_{n+1}, v_{n+2}, \dots, v_{2n}$ . Join the two copies of  $C_n$  with a new edge and denote the resultant graph by  $G$  then  $|V(G)| = 2n$  and  $|E(G)| = 2n + 1$ . Without loss of generality we assume that the new edge by  $v_n v_{n+1}$  and  $v_1, v_2, \dots, v_n, v_{n+1}, v_{n+2}, \dots, v_{2n}$  will form a spanning path in  $G$ . Define  $f : V(G) \rightarrow \{0, 1, 2, 3, \dots, 4n + 1\}$  as follows.

$$\begin{aligned}
 f(v_{2i+1}) &= 2i, \quad 0 \leq i \leq \frac{3n}{4} - 1, \\
 f\left(v_{\frac{3n}{2}+2i-1}\right) &= \frac{3n}{2} + 2i, \quad 1 \leq i \leq \frac{n}{4}, \\
 f(v_{2i}) &= 2i - 1, \quad 1 \leq i \leq \frac{n}{4}, \\
 f\left(v_{\frac{n}{2}+2i+2}\right) &= \frac{n}{2} + 3 + 2i, \quad 0 \leq i \leq \frac{3n}{4} - 1.
 \end{aligned}$$

The vertex function  $f$  defined above induces a bijective edge function  $f^* : E(G) \rightarrow \{1, 3, \dots, 4n + 1\}$ . Thus  $f$  is an odd harmonious labeling of  $G$ . Hence joint sum of two copies of  $C_n$  admits odd harmonious labeling for  $n \equiv 0(mod 4)$ .  $\square$

**Illustration 2.8** Odd harmonious labeling of the joint sum of two copies of  $C_{12}$  is shown in Fig. 4.

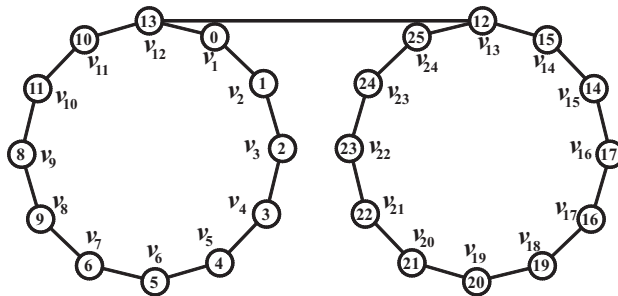


Fig. 4

**Theorem 2.9** The graph  $H_{n,n}$  is on odd harmonious graph.

*Proof* Let  $V = \{v_1, v_2, \dots, v_n\}$ ,  $U = \{u_1, u_2, \dots, u_n\}$  be the partition of  $V(H_{n,n})$ . Let  $G = H_{n,n}$  then  $|V(G)| = 2n$  and  $|E(G)| = \frac{n(n+1)}{2}$ . We define odd harmonious labeling  $f : V(G) \rightarrow \{0, 1, 2, 3, \dots, (n^2 + n - 1)\}$  as follows.

$$f(v_i) = i(i - 1), 1 \leq i \leq n,$$

$$f(u_i) = (2n + 1) - 2i, 1 \leq i \leq n.$$

The vertex function  $f$  defined above induces a bijective edge function  $f^* : E(G) \rightarrow \{1, 3, \dots, n^2 + n - 1\}$ . Thus  $f$  is an odd harmonious labeling of  $G$ . Hence the graph  $H_{n,n}$  is on odd harmonious graph.  $\square$

**Illustration 2.10** Odd harmonious labeling of the graph  $H_{5,5}$  is shown in Fig. 5.

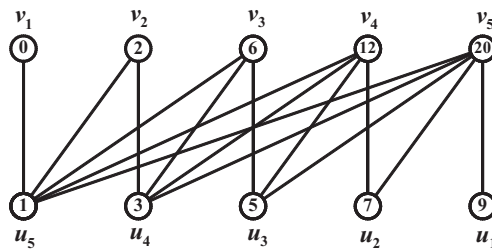


Fig. 5.

### §3. Concluding Remarks

Liang and Bai have proved that  $P_n$ ,  $B_{n,n}$  are odd harmonious graphs for all  $n$  and  $C_n$  is odd harmonious graph for  $n \equiv 0(mod 4)$  while we proved that the shadow and the splitting graphs of  $B_{n,n}$  admit odd harmonious labeling. Thus odd harmoniousness remains invariant for the shadow graph and splitting graph of  $B_{n,n}$ . It is also invariant under arbitrary supersubdivision of  $P_n$ . To investigate similar results for other graph families and in the context of various graph labeling problems is a potential area of research.

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