A NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE PROPERTY*

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ABSTRACT. Let p be a prime, n be any positive integer, $\alpha(n,p)$ denotes the power of p in the factorization of $n!$. In this paper, we study the asymptotic properties of the mean value $\sum_{p\leq n}\alpha(n,p)$, and give an interesting asymptotic formula for it.

1. INTRODUCTION

Let p be a prime, $e_p(n)$ denotes the largest exponent (of power p) which divides n, $\alpha(n,p)=\sum_{k\leq n}e_p(k)$. In problem 68 of [1], Professor F.Smarandach asked us to study the properties of the sequences $e_p(n)$. This problem is interesting because there are close relations between $e_p(n)$ and the factorization of $n!$. In fact, $\alpha(n,p)$ is the power of p in the factorization of $n!$. In this paper, we use the elementary methods to study the asymptotic properties of the mean value $\sum_{p\leq n}\alpha(n,p)$, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any prime p and any fixed positive integer n, we have the asymptotic formula*

$$\sum_{p\leq n}\alpha(n,p)=n\ln\ln n+cn+c_1\frac{n}{\ln n}+c_2\frac{n}{\ln^2 n}+\cdots+c_k\frac{n}{\ln^k n}+O\left(\frac{n}{\ln^{k+1}n}\right).$$

where k is any fixed positive integer, c_i $(i=1,2,\cdots,)$ are some computable constants.

2. PROOF OF THE THEOREM

In this section, we complete the proof of the Theorem. First for any prime p and any fixed positive integer n, we let $a(n,p)$ denote the sum of the base p digits of n. That is, if $n=a_1p^{\alpha_1}+a_2p^{\alpha_2}+\cdots+a_sp^{\alpha_s}$ with $\alpha_s>\alpha_{s-1}>\cdots>\alpha_1\geq 0$, where $1\leq a_i\leq p-1$, $i=1,2,\cdots,s$, then $a(n,p)=\sum_{i=1}^{s}a_i$, and for this number theoretic function, we have the following two simple Lemmas:

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Lemma 1. *For any integer $n \geq 1$, we have the identity*

$$\alpha_p(n) \equiv \alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \frac{1}{p-1}(n - a(n,p)),$$

where $[x]$ denotes the greatest integer not exceeding x.

Proof. From the properties of $[x]$ we know that

$$\left[\frac{n}{p^i}\right] = \left[\frac{a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}}{p^i}\right]$$
$$= \begin{cases} \sum_{j=k}^{s} a_j p^{\alpha_j - i}, & \text{if } \alpha_{k-1} < i \leq \alpha_k \\ 0, & \text{if } i > \alpha_s. \end{cases}$$

So from this formula we have

$$\alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \sum_{i=1}^{+\infty} \left[\frac{a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}}{p^i}\right]$$
$$= \sum_{j=1}^{s} \sum_{k=1}^{\alpha_j} a_j p^{\alpha_j - k} = \sum_{j=1}^{s} a_j \left(1 + p + p^2 + \cdots + p^{\alpha_j - 1}\right)$$
$$= \sum_{j=1}^{s} a_j \cdot \frac{p^{\alpha_j} - 1}{p-1} = \frac{1}{p-1} \sum_{j=1}^{s} (a_j p^{\alpha_j} - a_j)$$
$$= \frac{1}{p-1}(n - a(n,p)).$$

This completes the proof of Lemma 1.

Lemma 2. *For any positive integer n, we have the estimate*

$$a(n,p) \leq \frac{p-1}{\ln p} \ln n.$$

Proof. Let $n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$, where $1 \leq a_i \leq p-1$, $i = 1, 2, \cdots, s$. Then from the definition of $a(n,p)$ we have

(1) $$a(n,p) = \sum_{i=1}^{s} a_i \leq \sum_{i=1}^{s} (p-1) = (p-1)s.$$

On the other hand, using the mathematical induction we can easily get the inequality

$$n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s} \geq a_s p^s,$$

or

(2) $$s \leq \frac{\ln(n/a_s)}{\ln p} \leq \frac{\ln n}{\ln p}.$$

Combining (1) and (2) we immediately get the estimate

$$a(n,p) \leq \frac{p-1}{\ln p} \ln n.$$

This proves Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of the Theorem. First, we separate the summation in the Theorem into two parts.

$$\sum_{p\leq n} \alpha(n,p) = \sum_{p\leq\sqrt{n}} \alpha(n,p) + \sum_{\sqrt{n}<p\leq n} \alpha(n,p). \tag{3}$$

For the first part, from Lemma 1 we have

$$\begin{aligned}
\sum_{p\leq\sqrt{n}} \alpha(n,p) &= \sum_{p\leq\sqrt{n}} \frac{1}{p-1}(n-a(n,p)) \\
&= n\sum_{p\leq\sqrt{n}}\left(\frac{1}{p}+\frac{1}{p(p-1)}\right) - \sum_{p\leq\sqrt{n}}\frac{a(n,p)}{p-1} \\
&= n\left(\sum_{p\leq\sqrt{n}}\frac{1}{p}+\sum_{p}\frac{1}{p(p-1)}+O\left(\sum_{m>\sqrt{n}}\frac{1}{m^2}\right)\right) - \sum_{p\leq\sqrt{n}}\frac{a(n,p)}{p-1} \\
&= n\left(\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x}d\pi(x)+A+O\left(\frac{1}{\sqrt{n}}\right)\right) - \sum_{p\leq\sqrt{n}}\frac{a(n,p)}{p-1}. \qquad (4)
\end{aligned}$$

where $\pi(x)$ denotes the number of all prime not exceeding x. For $\pi(x)$, we have the asymptotic formula

$$\pi(x) = \frac{x}{\ln x} + a_2\frac{x}{\ln^2 x} + \cdots + a_k\frac{x}{\ln^k x} + O\left(\frac{x}{\ln^{k+1} x}\right) \tag{5}$$

and

$$\begin{aligned}
\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x}d\pi(x) &= \frac{\pi(\sqrt{n})}{\sqrt{n}} + \int_{\frac{3}{2}}^{\sqrt{n}}\frac{\pi(x)}{x^2}dx \\
&= \frac{1}{\ln\sqrt{n}}+\frac{a_2}{\ln^2\sqrt{n}}+\cdots+\frac{a_k}{\ln^k\sqrt{n}}+O\left(\frac{1}{\ln^{k+1}\sqrt{n}}\right)+\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x\ln x}dx \\
&\quad + a_2\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x\ln^2 x}dx+\cdots+a_{k+1}\int_{\frac{3}{2}}^{\sqrt{n}}\frac{1}{x\ln^{k+1} x}dx+O\left(\frac{1}{\ln^{k+1} n}\right) \\
&= \frac{a_{11}}{\ln n}+\frac{a_{12}}{\ln^2 n}+\cdots+\frac{a_{1k}}{\ln^k n}+\ln\ln n+B+\frac{a_{21}}{\ln n}+\frac{a_{22}}{\ln^2 n} \\
&\quad +\cdots+\frac{a_{2k}}{\ln^k n}+O\left(\frac{1}{\ln^{k+1} n}\right) \\
&= \ln\ln n+B+\frac{a_{31}}{\ln n}+\frac{a_{32}}{\ln^2 n}+\cdots+\frac{a_{3k}}{\ln^k n}+O\left(\frac{1}{\ln^{k+1} n}\right). \qquad (6)
\end{aligned}$$

From Lemma 2 we have

$$\sum_{p\le\sqrt{n}}\frac{a(n,p)}{p-1}\le\sum_{p\le\sqrt{n}}\frac{\ln n}{\ln p}=\ln n\sum_{p\le\sqrt{n}}\frac{1}{\ln p}\le\ln n\sum_{p\le\sqrt{n}}1\le\sqrt{n}\ln n. \tag{7}$$

Combining (4), (6) and (7) we obtain

$$\begin{aligned}\sum_{p\le\sqrt{n}}\alpha(n,p)&=n\ln\ln n+c_0n+a_{31}\frac{n}{\ln n}+a_{32}\frac{n}{\ln^2 n}\\&\quad+\cdots+a_{3k}\frac{n}{\ln^k n}+O\left(\frac{n}{\ln^{k+1}n}\right).\end{aligned}\tag{8}$$

For the second part, we have

$$\begin{aligned}\sum_{\sqrt{n}<p\le n}\alpha(n,p)&=\sum_{\sqrt{n}<p\le n}\sum_{i=1}^{+\infty}\left[\frac{n}{p^i}\right]=\sum_{\sqrt{n}<p\le n}\left[\frac{n}{p}\right]=\sum_{\sqrt{n}<p\le n}\sum_{m\le\frac{n}{p}}1\\&=\sum_{m\le\sqrt{n}}\sum_{\sqrt{n}<p\le\frac{n}{m}}1=\sum_{m\le\sqrt{n}}\left(\pi\left(\frac{n}{m}\right)-\pi(\sqrt{n})\right)\\&=\sum_{m\le\sqrt{n}}\pi\left(\frac{n}{m}\right)-[\sqrt{n}]\pi(\sqrt{n}).\end{aligned}\tag{9}$$

Applying Euler's summation formula (see [2] Theorem 3.1) and the expansion into power-series we have

$$\begin{aligned}\sum_{m\le\sqrt{n}}\frac{1}{m(\ln n-\ln m)^r}&=\sum_{m\le\sqrt{n}}\frac{1}{m\ln^r n(1-\frac{\ln m}{\ln n})^r}\\&=\sum_{s=0}^{+\infty}\sum_{m\le\sqrt{n}}\frac{\binom{r-1+s}{r-1}\ln^s m}{m\ln^{s+r}n}\\&=\sum_{s=0}^{+\infty}\binom{r-1+s}{r-1}\left(\sum_{m\le\sqrt{n}}\frac{\ln^s m}{m\ln^{s+r}n}\right)\\&=\sum_{s=0}^{+\infty}\frac{\binom{r-1+s}{r-1}}{\ln^{s+r}n}\left(\frac{\ln^{s+1}n}{(s+1)2^{s+1}}+d_{s+1}+O\left(\frac{\ln^s n}{2^s\sqrt{n}}\right)\right)\\&=\sum_{i=r-1}^{k}\frac{d_{1i}}{\ln^i n}+O\left(\frac{\ln^s n}{\sqrt{n}}\right).\end{aligned}$$

From this and (5) we get

$$\sum_{m\le\sqrt{n}} \pi\left(\frac{n}{m}\right)$$

$$= \sum_{m\le\sqrt{n}} \left(\frac{\frac{n}{m}}{\ln\left(\frac{n}{m}\right)} + a_2\frac{\frac{n}{m}}{\ln^2\left(\frac{n}{m}\right)} + \cdots + a_{k+1}\frac{\frac{n}{m}}{\ln^{k+1}\left(\frac{n}{m}\right)} + O\left(\frac{\frac{n}{m}}{\ln^{k+2}\left(\frac{n}{m}\right)}\right)\right)$$

$$= n\sum_{m\le\sqrt{n}} \left(\frac{1}{m(\ln n - \ln m)} + a_2\frac{1}{m(\ln n-\ln m)^2} + \cdots \right.$$

$$\left. + a_{k+1}\frac{1}{m(\ln n - \ln m)^{k+1}} + O\left(\frac{1}{m(\ln n-\ln m)^{k+2}}\right)\right)$$

$$= n\left(b_0 + \frac{b_1}{\ln n} + \frac{b_2}{\ln^2 n} + \cdots + \frac{b_k}{\ln^k n} + O\left(\frac{1}{\ln^{k+1} n}\right)\right)$$

(10)

$$= b_0 n + b_1\frac{n}{\ln n} + b_2\frac{n}{\ln^2 n} + \cdots + b_k\frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right)$$

and

$$[\sqrt{n}]\pi(\sqrt{n}) = \frac{n}{\ln\sqrt{n}} + a_2\frac{n}{\ln^2\sqrt{n}} + \cdots + a_k\frac{n}{\ln^k\sqrt{n}} + O\left(\frac{n}{\ln^{k+1}\sqrt{n}}\right)$$

(11)

$$= a_{41}\frac{n}{\ln n} + a_{42}\frac{n}{\ln^2 n} + \cdots + a_{4k}\frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right).$$

Combining (9), (10) and (11) we have

(12) $$\sum_{\sqrt{n}<p\le n} \alpha(n,p) = b_0 n + a_{51}\frac{n}{\ln n} + a_{52}\frac{n}{\ln^2 n} + \cdots + a_{5k}\frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right).$$

From (3), (8) and (12) we obtain the asymptotic formula

$$\sum_{p\le n}\alpha(n,p) = n\ln\ln n + cn + c_1\frac{n}{\ln n} + c_2\frac{n}{\ln^2 n} + \cdots + c_k\frac{n}{\ln^k n} + O\left(\frac{n}{\ln^{k+1} n}\right).$$

This completes the proof of the Theorem.

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