Scientia Magna
Vol. 3 (2007), No. 3, 58-62

On the mean value of the Smarandache LCM function

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Abstract For any positive integer n, the F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$. The main purpose of this paper is using the elementary methods to study the mean value properties of $P(n)SL(n)$ and $p(n)SL(n)$, and give two sharper asymptotic formulas for them.

Keywords F.Smarandache LCM function, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [2] and [3].

For example, Lv Zhongtian [4] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

Jianbin Chen [5] studied the value distribution properties of $SL(n)$, and proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(SL(n) - P(n))^2 = \frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot\frac{x^{\frac{5}{2}}}{\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

The main purpose of this paper is using the elementary methods to study the mean value properties of $P(n)SL(n)$ and $p(n)SL(n)$, and give two sharper asymptotic formulas for them. That is, we shall prove the following two conclusions:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} P(n)SL(n) = x^3 \cdot \sum_{i=1}^{k} \frac{c_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $P(n)$ denotes the largest prime divisor of n, and c_i $(i = 1, 2, \cdots, k)$ are computable constants.

Theorem 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x} p(n)SL(n) = x^3 \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $p(n)$ denotes the smallest prime divisor of n, b_i $(i = 1, 2, \cdots, k)$ are computable constants and $b_1 = \frac{1}{3}$.

Whether there exist an asymptotic formula for $\sum_{n\le x} \frac{p(n)}{SL(n)}$ and $\sum_{n\le x} \frac{P(n)}{SL(n)}$ is an open problem.

§2. Proof of the theorems

In this section, we shall use the elementary methods to complete the proof the theorems.

First we prove Theorem 1. In fact for any positive integer $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [2] we know that

$$SL(n) = \max\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_s^{\alpha_s}\}. \tag{1}$$

Now we consider the summation

$$\sum_{n\le x} P(n)SL(n). \tag{2}$$

We separate all integer n in the interval $[1, x]$ into four subsets A, B, C and D as follows:

A: $P(n) \ge \sqrt{n}$ and $n = m \cdot P(n)$, $m < P(n)$;

B: $n^{\frac{1}{3}} < P(n) \le \sqrt{n}$ and $n = m \cdot P^2(n)$, $m < n^{\frac{1}{3}}$;

C: $n^{\frac{1}{3}} < p_1 < P(n) \le \sqrt{n}$ and $n = m \cdot p_1 \cdot P(n)$, where p_1 is a prime;

D: $P(n) \le n^{\frac{1}{3}}$.

It is clear that if $n \in A$, then from (1) we know that $SL(n) = P(n)$. Therefore, by the Abel's summation formula (See Theorem 4.2 of [6]) and the Prime Theorem (See Theorem 3.2 of [7]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1, 2, \ldots, k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}\sum_{n\in A} P(n)SL(n) &= \sum_{n\in A} P(n)^2 = \sum_{\substack{m\cdot p\leq x\\ m<p}} p^2 = \sum_{m\leq\sqrt{x}} \sum_{m<p<\frac{x}{m}} p^2 \\ &= \sum_{m\leq\sqrt{x}} \left[\frac{x^2}{m^2}\pi\left(\frac{x}{m}\right) - \int_m^{\frac{x}{m}} 2y\pi(y)dy + O\left(m^3\right)\right] \\ &= \sum_{m\leq\sqrt{x}} \left(\frac{x^3}{m^3}\sum_{i=1}^k \frac{b_i}{\ln^i \frac{x}{m}} + O\left(\frac{x^3}{m^3\cdot\ln^{k+1}\frac{x}{m}}\right)\right) \\ &= \zeta(3)\cdot x^3\cdot\sum_{i=1}^k \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right), \qquad (3)\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function, $b_1 = \frac{1}{3}$, b_i $(i = 2, 3, \cdots, k)$ are computable constants.

Similarly, if $n \in C$, then we also have $SL(n) = P(n)$. So

$$\begin{aligned}\sum_{n\in C} P(n)SL(n) &= \sum_{\substack{mp_1p\leq x\\ m<p_1<p}} p^2 = \sum_{m\leq x^{\frac{1}{3}}} \sum_{m<p_1\leq\sqrt{\frac{x}{m}}} \sum_{p_1<p\leq\frac{x}{p_1m}} p^2 \\ &= \sum_{m\leq x^{\frac{1}{3}}} \sum_{m<p_1\leq\sqrt{\frac{x}{m}}} \left[\frac{x^2}{p_1^2m^2}\pi\left(\frac{x}{p_1m}\right) - p_1^2\pi(p_1) - \int_{p_1}^{\frac{x}{p_1m}} 2y\pi(y)dy\right] \\ &= \sum_{m\leq x^{\frac{1}{3}}} \frac{x^3}{m^3}\sum_{i=1}^k \frac{d_i(m)}{\ln^i\frac{x}{m}} + O\left(\frac{x^3}{m^3\ln^{k+1}\frac{x}{m}}\right) \\ &= x^3\cdot\sum_{i=1}^k \frac{h_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right), \qquad (4)\end{aligned}$$

where h_i $(i = 1, 2, \cdots, k)$ are computable constants.

Now we estimate the error terms in set B. Using the same method of proving (3),we have

$$\begin{aligned}\sum_{n\in B} P(n)SL(n) &= \sum_{m\cdot p^2\leq x} p^3 = \sum_{m\leq x^{\frac{1}{3}}} \sum_{m<p<\sqrt{\frac{x}{m}}} p^3 \\ &= \zeta(2)\cdot x^2\cdot\sum_{i=1}^k \frac{e_i}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right) \\ &= O(x^2). \qquad (5)\end{aligned}$$

Finally, we estimate the error terms in set D. For any integer $n \in D$, let $SL(n) = p^\alpha$. We assume that $\alpha \geq 1$. This time note that $P(n) \leq n^{\frac{1}{3}}$, we have

$$\begin{aligned}\sum_{n\in D} P(n)SL(n) &= \sum_{\substack{mp^\alpha \leq x\\ \alpha\geq 1, p\leq x^{\frac{1}{3}}}} p^{\alpha+1} \ll \sum_{\substack{mp^\alpha \leq x\\ \alpha\geq 1, p\leq x^{\frac{1}{3}}}} p^{2\alpha}\\ &\ll \sum_{\substack{p^\alpha \leq x\\ \alpha\geq 1, p\leq x^{\frac{1}{3}}}} p^{2\alpha} \sum_{m\leq \frac{x}{p^\alpha}} 1\\ &\ll x \sum_{\substack{p^\alpha \leq x\\ \alpha\geq 1, p\leq x^{\frac{1}{3}}}} p^{\alpha} \ll x^{\frac{7}{3}}. \end{aligned} \quad (6)$$

Combining (2), (3), (4), (5) and (6) we may immediately obtain the asymptotic formula

$$\begin{aligned}\sum_{n\leq x} P(n)SL(n) &= \sum_{n\in A} P(n)SL(n) + \sum_{n\in B} P(n)SL(n)\\ &+ \sum_{n\in C} P(n)SL(n) + \sum_{n\in D} P(n)SL(n)\\ &= x^3 \cdot \sum_{i=1}^{k} \frac{c_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right).\end{aligned}$$

where $P(n)$ denotes the largest prime divisor of n, and c_i $(i = 1, 2, \cdots, k)$ are computable constants.

This proves Theorem 1.

Now we prove Theorem 2. We separate all integer n in the interval $[1, x]$ into four subsets $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ as follows: $\bar{A}$: $n = 1$; $\bar{B}$: $n = p^\alpha$, $\alpha \geq 1$; $\bar{C}$: $n = p_1^{\alpha_1} p_2^{\alpha_2}$, $\alpha_i \geq 1$, $(i = 1, 2)$; $\bar{D}$: $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, $\alpha_i \geq 1$, $(i = 1, 2, \cdots, s)$, $s \geq 3$. $p(n)$ denotes the smallest prime divisor of n, $p(1) = 0$ and $SL(1) = 1$. Then we have

$$\sum_{n\leq x} p(n)SL(n) = \sum_{n\in\bar{B}} p(n)SL(n) + \sum_{n\in\bar{C}} p(n)SL(n) + \sum_{n\in\bar{D}} p(n)SL(n). \quad (7)$$

Obviously if $n \in \bar{B}$, then from (1) we know that $SL(n) = p^\alpha$. Therefore,

$$\begin{aligned}\sum_{n\in\bar{B}} p(n)SL(n) &= \sum_{p^\alpha\leq x} pp^\alpha = \sum_{p\leq x} p^2 + \sum_{\substack{p^\alpha\leq x\\ \alpha\geq 2}} p^{\alpha+1}\\ &= x^2\pi(x) - \int_{\frac{3}{2}}^{x} 2y\pi(y)dy + O\left(\sum_{2\leq\alpha\leq\ln x} \sum_{p\leq x^{\frac{1}{\alpha}}} p^{2\alpha}\right)\\ &= x^2\pi(x) - \int_{\frac{3}{2}}^{x} 2y\pi(y)dy + O\left(x^{\frac{5}{2}}\right)\\ &= x^3 \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),\end{aligned} \quad (8)$$

where $b_1 = \dfrac{1}{3}$, b_i $(i = 2, 3, \cdots, k)$ are computable constants.

If $n \in \bar{C}$, then $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where $p_1 < p_2$, and $SL(n) \geq \sqrt{n}$, so we have

$$\begin{aligned}\sum_{n\in\bar{C}} p(n)SL(n) &= \sum_{p_1^{\alpha_1} p_2^{\alpha_2} \leq x} SL(p_1^{\alpha_1} p_2^{\alpha_2}) p_1 \\ &= \sum_{p_1^{\alpha_1} \leq \sqrt{x}} \sum_{p_2^{\alpha_2} \leq \frac{x}{p_1^{\alpha_1}}} p_2^{\alpha_2} p_1 + \sum_{p_2^{\alpha_2} \leq \sqrt{x}} \sum_{p_1^{\alpha_1} \leq \frac{x}{p_2^{\alpha_2}}} p_1^{\alpha_1} p_1 \ll x^{\frac{11}{4}}. \qquad (9)\end{aligned}$$

Finally, we estimate the error terms in set $\bar{D}$, this time, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $s \geq 3$. Therefore, $n^{\frac{1}{3}} \leq SL(n) \leq \sqrt{n}$, and $p(n) \leq n^{\frac{1}{3}}$, so we have

$$\sum_{n\in\bar{D}} p(n)SL(n) \ll \sum_{n\leq x} n^{\frac{1}{3}} n^{\frac{1}{2}} \ll x^{\frac{11}{6}}. \qquad (10)$$

Combining (1), (7), (8), (9) and (10) we may immediately obtain the asymptotic formula

$$\sum_{n\leq x} p(n)SL(n) = x^3 \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $p(n)$ denotes the smallest prime divisor of n, and $b_1 = \dfrac{1}{3}$, b_i $(i = 2, 3, \cdots, k)$ are computable constants.

This completes the proof of Theorem 2.

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