International J.Math. Combin. Vol.1 (2007), No.1, 94-109

Long Dominating Cycles in Graphs

Yongga A

(Department of Mathematics of Inner Mongolia Normal University, Huhhot 010022, P.R.China)

Zhiren Sun

(Department of Mathematics of Nanjing Normal University, Nanjing 210097, P.R. China)

Abstract: Let G be a connected graph of order n, and $NC2(G)$ denote $\min\{|N(u)\cup N(v)| : dist(u,v) = 2\}$, where $dist(u,v)$ is the distance between u and v in G. A cycle C in G is called a *dominating cycle*, if $V(G)\backslash V(C)$ is an independent set in G. In this paper, we prove that if G contains a dominating cycle and $\delta \geq 2$, then G contains a dominating cycle of length at least $\min\{n, 2NC2(G)-1\}$ and give a family of graphs showing our result is sharp, which proves a conjecture of R. Shen and F. Tian, also related with the cyclic structures of algebraically Smarandache multi-spaces.

Key words: Dominating cycle, neighborhood union, distance.

AMS(2000): 05C38, 05C45.

§1. Introduction

All graphs considered in this paper will be finite and simple. We use Bondy & Murty [1] for terminology and notations not defined here.

Let $G = (V, E)$ be a graph of order n and C be a cycle in G. C is called a *dominating cycle*, or briefly a *D-cycle*, if $V(G)\backslash V(C)$ is an independent set in G. For a vertex v in G, the neighborhood of v is denoted by $N(v)$, and the degree of v is denoted by $d(v)$. For two subsets S and T of $V(G)$, we set $N_T(S) = \{v \in T\backslash S : N(v) \cap S \neq \emptyset\}$. We write $N(u,v)$ instead of $N_{V(G)}(\{u,v\})$ for any $u,v \in V(G)$. If F and H are two subgraphs of G, we also write $N_F(H)$ instead of $N_{V(F)}\big(V(H)\big)$. In the case $F = G$, if no ambiguity can arise, we usually omit the subscript G of $N_G(H)$. We denote by $G[S]$ the subgraph of G induced by any subset S of $V(G)$.

For a connected graph G and $u,v \in V(G)$, we define the *distance between u and v* in G, denoted by $dist(u,v)$, as the minimum value of the lengths of all paths joining u and v in G. If G is non-complete, let $NC(G)$ denote $\min\{|N(u,v)| : uv \notin E(G)\}$ and $NC2(G)$ denote $\min\{|N(u,v)| : dist(u,v) = 2\}$; if G is complete, we set $NC(G) = n-1$ and $NC2(G) = n-1$.

In [2], Broersma and Veldman gave the following result.

Theorem 1([2]) *If G is a 2-connected graph of order n and G contains a D-cycle, then G has a D-cycle of length at least* $\min\{n, 2NC(G)\}$ *unless G is the Petersen graph.*

For given positive integers n_1,n_2 and n_3, let $K(n_1,n_2,n_3)$ denote the set of all graphs

[1]Received August 6, 2007. Accepted September 8, 2007

[2]Supported by the National Science Foundation of China(10671095) and the Tian Yuan Foundation on Mathematics.

of order $n_1 + n_2 + n_3$ consisting of three disjoint complete graphs of order n_1, n_2 and n_3, respectively. For any integer $p \geq 3$, let $\mathcal{J}_1^*$ (resp. $\mathcal{J}_2^*$) denote the family of all graphs of order $2p+3$ (resp. $2p+4$) which can be obtained from a graph H in $K(3,p,p)$ (resp. $K(3,p,p+1)$) by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of H. Let $\mathcal{J}_1 = \{G : G$ is a spanning subgraph of some graph in $\mathcal{J}_1^*\}$ and $\mathcal{J}_2 = \{G : G$ is a spanning subgraph of some graph in $\mathcal{J}_2^*\}$. In [5], Tian and Zhang got the following result.

Theorem 2([5]) *If G is a 2-connected graph of order n such that every longest cycle in G is a D-cycle, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G)\}$ *unless G is the Petersen graph or $G \in \mathcal{J}_1 \cup \mathcal{J}_2$.*

In [4], Shen and Tian weakened the conditions of Theorem 2 and obtained the following theorem.

Theorem 3([4]) *If G contains a D-cycle and $\delta \geq 2$, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G) - 3\}$.

Theorem 4([6]) *If G contains a D-cycle and $\delta \geq 2$, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G) - 2\}$.

In [4], Shen and Tian believed the followings are true.

Conjecture 1 *If G satisfies the conditions of Theorem 3, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G) - \epsilon(n)\}$, *where $\epsilon(n) = 1$ if n is even, and $\epsilon(n) = 2$ if n is odd.*

Conjecture 2 *If G contains a D-cycle and $\delta \geq 2$, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G)\}$ *unless G is one of the exceptional graphs listed in Theorem 2. And the complete bipartite graphs $K_{m,m+q}$ $(q \geq 1)$ show that the bound $2NC2(G)$ is sharp.*

In this paper, we prove the following result, which solves Conjecture 1 due to Shen and Tian, also related with the cyclic structures of algebraically Smarandache multi-spaces (see [3] for details).

Theorem 5 *If G contains a D-cycle and $\delta \geq 2$, then G contains a D-cycle of length at least* $\min\{n, 2NC2(G) - 1\}$ *unless $G \in \mathcal{J}_1$.*

Remark The Petersen graph shows that our bound $2NC2(G) - 1$ is sharp.

§2. Proof of Theorem 5

In order to prove Theorem 5, we introduce some additional notations.

Let C be a cycle in G. We denote by $\overrightarrow{C}$ the cycle C with a given orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor. We write $u^{+2} := (u^+)^+$ and $u^{-2} := (u^-)^-$, etc. If $A \subseteq V(C)$, then $A^+ = \{v^+ : v \in A\}$ and $A^- = \{v^- : v \in A\}$. For any subset S of $V(G)$, we write $N^+(S)$ and $N^-(S)$ instead of $(N(S))^+$ and $(N(S))^-$,respectively.

Let G be a graph satisfying the conditions of Theorem 4, i.e. G contains a D-cycle and $\delta \geq 2$. Throughout, we suppose that

——G is non-hamiltonian and C is a longest D-cycle in G,

——$|V(C)| \leq 2NC2(G) - 2$,

——$R = G\backslash V(C)$ and $x \in R$, such that $d(x)$ is as large as possible.

First of all, we prove some claims.

By the maximality of C and the definition of D-cycle, we have

Claim 1 $N(x) \subseteq V(C)$.

Claim 2 $N(x) \cap N^+(x) = N(x) \cap N^-(x) = \emptyset$.

Let $v_1, v_2, \ldots, v_k$ be the vertices of $N(x)$, in cyclic order around $\overrightarrow{C}$. Then $k \geq 2$ since $\delta \geq 2$. For any $i \in \{1, 2, \ldots, k\}$, we have $v_i^+ \neq v_{i+1}$ (indices taken modulo k) by Claim 2. Let $u_i = v_i^+$, $w_i = v_{i+1}^-$ (indices taken modulo k), $T_i = u_i\overrightarrow{C}w_i$, $t_i = |T_i|$.

Claim 3 $N_R(y_1) \cap N_R(y_2) = \emptyset$, if y_1, $y_2 \in N^+(x)$ or y_1, $y_2 \in N^-(x)$.In particular, $N^+(x) \cap N(u_i) = N^-(x) \cap N(w_i) = \emptyset$.

For any $i, j \in \{1, 2, \ldots, k\} (i \neq j)$, we also have the following Claims.

Claim 4 Each of the followings does not hold :

(1) There are two paths $P_1[w_j, z]$ and $P_2[u_i, z^-]$, $(z \in v_{j+1}\overrightarrow{C}v_i)$ of length at most two that are internally disjoint from C and each other ;

(2) There are two paths $P_1[w_j, z]$ and $P_2[u_i, z^+]$ $(z \in v_{j+1}\overrightarrow{C}v_i)$ of length at two that are internally disjoint from C and each other ;

(3) There are two paths $P_1[u_i, z]$ and $P_2[u_j, z^+]$ ($z \in u_j^+\overrightarrow{C}v_i)$ of length at most two that are internally disjoint from C and each other, and similarly for $P_1[u_i, z]$ and $P_2[u_j, z^-]$ $(z \in u_i^+\overrightarrow{C}v_j)$.

Claim 5 For any $v \in V(G)$, we have $d_R(v) \leq 1$.

If not, then by Claim 1, there exists a vertex, say v, in C such that $d_R(v) > 1$. Let $x_1, x_2 \in N_R(v)$, then $|N(x_1, x_2)| \geq NC2(G)$.

First, we prove that $|N(x_1, x_2) \cap N^+(x_1, x_2)| \leq 2$. Otherwise, let y_1, y_2 and y_3 be three distinct vertices in $N(x_1, x_2) \cap N^+(x_1, x_2)$. By Claim 2, we know $y_i \in N(x_1) \cap N^+(x_2)$ or $y_i \in N(x_2) \cap N^+(x_1)$ for any $i \in \{1, 2, 3\}$. Thus, there must exist i and j $(i \neq j, i, j \in \{1, 2, 3\})$ such that $y_i, y_j \in N(x_1) \cap N^+(x_2)$ or $y_i, y_j \in N(x_2) \cap N^+(x_1)$. In either case, it contradicts Claim 3. So we have that $|N(x_1, x_2) \cap N^+(x_1, x_2)| \leq 2$.

Now we have

$$\begin{aligned}|V(C)| &\geq |N(x_1,x_2)\cup N^+(x_1,x_2)|\\ &\geq 2|N(x_1,x_2)|-2\\ &\geq 2NC2(G)-2,\end{aligned}$$

so $V(C)=N(x_1,x_2)\cup N^+(x_1,x_2)$ by assumption on $|V(C)|$, and in particular, $N(x_1,x_2)\cap N^+(x_1,x_2)=\{y_1,y_2\}$.Therefore $y_1\in N(x_1)\cap N^+(x_2)$ and $y_2\in N^+(x_1)\cap N(x_2)$.

Now, we prove that $d_R(v^+)\leq 1, d_R(v^-)\leq 1$. If not,suppose $d_R(v^-)>1$, let $z_1,z_2\in N_R(v^-)$, by Claim 1 and $V(C)=N(x_1,x_2)\cup N^+(x_1,x_2), N(z_1,z_2)\subseteq N^+(x_1,x_2)$, so we have x_1 (or x_2) $\in N(v^{-2})$. Using a similar argument as above, we have z_1 (or z_2) $\in N(v^{-3})$, which contradicts Claim 3. Thus, we have $d_R(v^-)\leq 1$; similarly, $d_R(v^+)\leq 1$.

Now, we consider $N(x_2,v^-)\cup N^-(x_1,v^+)$.Since $dist(x_2,v^-)=dist(x_1,v^+)=2$ and $|N(x_2,v^-)|\geq NC2(G), |N^-(x_1,v^+)|=|N(x_1,v^+)|\geq NC2(G)$. We prove that $|N_C(x_2,v^-)\cap N_C^-(x_1,v^+)|\leq 1$. Let $z\in\{N_C(x_2,v^-)\cap N_C^-(x_1,v^+)\}\backslash\{y_1^-\}$.

We consider following cases.

(i) Let $z\in y_1^+\overrightarrow{C}y_2^{-2}$, if $zx_2\in E(G)$ and $x_1z^+\in E(G)$, or $zx_2\in E(G)$ and $v^+z^+\in E(G)$, or $v^-z\in E(G)$ and $x_1z^+\in E(G)$, each case contradicts Claim 3; if $v^-z\in E(G)$ and $v^+z^+\in E(G)$, then $C'=x_1y_2^-\overleftarrow{C}z^+v^+\overrightarrow{C}zv^-\overleftarrow{C}y_2x_2vx_1$ is a D-cycle longer than C,a contradiction.

(ii) Let $z\in y_2^+\overrightarrow{C}y_1^{-2}$, if $x_2z\in E(G)$ and $x_1z^+\in E(G)$, or $x_2z\in E(G)$ and $v^+z^+\in E(G)$, both contradict Claim 3; if $v^-z\in E(G)$ and $x_1z^+\in E(G)$, it contradicts Claim 3; if $v^-x_1\in E(G)$ and $z^+v^+\in E(G)$, then $C'=x_1y_1\overrightarrow{C}v^-z\overleftarrow{C}v^+z^+\overrightarrow{C}y_1^-x_2vx_1$ is a D-cycle longer than C, for $z\in v\overrightarrow{C}y_1^-$; and $C'=x_1y_2^-\overleftarrow{C}v^+z^+\overrightarrow{C}v^-z\overleftarrow{C}y_2x_2vx_1$ is a D-cycle longer than C for $z\in y_2\overrightarrow{C}v^-$.

So, we have $|N_C(x_2,v^-)\cap N_C^-(x_1,v^+)|\leq 1$. Moreover, $y_1,y_2^-\notin N(x_2,v^-)\cup N^-(x_1,v^+)$. Otherwise, if $y_1\in N(v^-)$, then $C'=x_1y_2^-\overleftarrow{C}y_1v^-\overleftarrow{C}y_2x_2y_1^-\overleftarrow{C}vx_1$ is a D-cycle longer than C. By Claim 2, $y_1\notin N(x_2)\cup N^-(x_1,v^+)$, so we have $y_1\notin N(x_2,v^-)\cup N^-(x_1,v^+)$. By Claims 1 and 3 we have $y_2^-\notin N(x_2,v^-)\cup N^-(x_1,v^+)$. Thus, we have

$$\begin{aligned}|V(C)| &\geq |N_C(x_2,v^-)\cup N_C^-(x_1,v^+)|+2\\ &\geq |N_C(x_2,v^-)|+|N_C^-(x_1,v^+)|-1+2\\ &= |N(x_2,v^-)\backslash N_R(x_2,v^-)|+|N(x_1,v^+)\backslash N_R(x_1,v^+)|+1\\ &\geq 2NC2(G)-2+1\\ &= 2NC2(G)-1,\end{aligned}$$

a contradiction with $|V(C)|\leq 2NC2(G)-2$. So, we have $d_R(v)\leq 1$, for any $v\in V(G)$.

Claim 6 $t_i\geq 2$.

If $t_i=1$ for all of i, then $N_R(u_i)=\emptyset$ for all of i (if not, let $z\in N_R(u_i)$ for some i, by Claim 1 and Claim 5 $N(z)\subseteq V(C)$ and $u_jz\in E(G)$ for some j. then, $z\in N_R(u_i)\cap N_R(u_j)$, a contradiction). Then $N(u_i)\cap N^+(u_i)=\emptyset$(otherwise, $y\in N(u_i)\cap N^+(u_i)$, then $C'=$

$xv_{i+1}\overrightarrow{C}y^-u_iy\overrightarrow{C}v_ix$ is a D-cycle longer than C). Moreover, we have $N(x)\cap N^+(x)=\emptyset$ by Claim 2, $N^+(x)\cap N(u_i)=N^+(u_i)\cap N(x)=\emptyset$ by Claim 3. Hence, $N(x,u_i)\cap N^+(x,u_i)=\emptyset$. So we have

$$|V(C)|\geq |N(x,u_i)\cup N^+(x,u_i)|\geq 2|N(x,u_i)|\geq 2NC2(G),$$

a contradiction. So we may assume $t_i=1$ for some i, without loss of generality, suppose $t_1=1$ and $N_R(w_k)\neq\emptyset$. Let $y\in N_R(w_k)$, choose $y_1\in N(y)$ such that $N(y)\cap(y_1^+\overrightarrow{C}w_k^-)=\emptyset$. Using a similar argument as above and $d_R(u_1)\leq 1$, by Claim 5, we have

$$|V(C)|=|N_C(x,u_1)\cup N_C^+(x,u_1)|\geq 2NC2(G)-2.$$

So $V(C)=N_C(x,u_1)\cup N_C^+(x,u_1)$. Similarly, we know that $V(C)=N_C(x,u_1)\cup N_C^-(x,u_1)$. Moreover, $u_1w_k^-\in E(G)$. If $|y_1^+\overrightarrow{C}w_k^-|=1$, then $C'=xv_2\overrightarrow{C}y_1yw_kw_k^-u_1v_1x$ is a D–cycle longer than C, a contradiction. So we may assume that $|y_1^+\overrightarrow{C}w_k^-|\geq 2$.

Now, we consider $N_C(y,y_1^+)\cup N_C^-(x,u_1)$. Since $dist(y,y_1^+)=dist(x,u_1)=2$, $|N(y,y_1^+|\geq NC2(G), |N^-(x,u_1)|=|N(x,u_1)|\geq NC2(G)$. Moreover, we have $v_1,v_2\notin N_C(y,y_1^+)\cup N_C^-(x,u_1)$ and $N_C(y,y_1^+)\cap N_C^-(x,u_1)\subseteq\{w_k\}$. In fact, $v_1\notin N(y,y_1^+)$ by Claims 3 and 5, if $v_1\in N^-(x,u_1)$, then $v_1^+x\in E(G)$ or $v_1^+u_1\in E(G)$, which contradicts to Claims 2 and 3. So $v_1\notin N_C(y,y_1^+)\cup N_C^-(x,u_1)$;if $v_2\in N_C(y,y_1^+)$, then $v_2y^+\in E(G)$ by Claim 5, which contradicts to Claim 4. If $v_2\in N_C^-(x,u_1)$ then $v_2^+\in N(x,u_1)$, which contradicts to Claims 2 and 3. So $v_2\notin N_C(y,y_1^+)\cup N_C^-(x,u_1)$. Suppose $z\in N_C(y,y_1^+)\cap N_C^-(x,u_1)\backslash\{w_k\}$. Now, we consider the following cases.

(i) $z\in v_2\overrightarrow{C}y_1^-$. If $yz\in E(G)$ and $xz^+\in E(G)$, then, it contradicts to Claim 3. Put

$$C'=\begin{cases} yz\overleftarrow{C}v_2xv_1u_1z^+\overleftarrow{C}w_ky & \text{if } yz\in E(G)\text{and}u_1z^+\in E(G);\\ xz^+\overrightarrow{C}y_1yw_k\overleftarrow{C}y_1^+z\overleftarrow{C}v_1x & \text{if } y_1^+z\in E(G)\text{ and}xz^+\in E(G);\\ xv_2\overrightarrow{C}zy_1^+\overrightarrow{C}w_kyy_1\overleftarrow{C}z^+u_1v_1x & \text{if } y_1^+z\in E(G)\text{ and } u_1z^+\in E(G).\end{cases}$$

(ii) $z\in y_1\overrightarrow{C}w_k^-$, then $z\in N(y_1^+)$ since $N(y)\cap(y_1^+\overrightarrow{C}w_k^-)=\emptyset$. Let $zy_1^+\in E(G)$ and $z^+\in N_C(x,u_1)$. Since $V(C)=N_C(x,u_1)\cup N_C^-(x,u_1)$, So $y_1^+\in N_C(x,u_1)\cup N_C^-(x,u_1)$. If $u_1y_1^+\in E(G)$ then $C'=xv_2\overrightarrow{C}y_1yw_k\overleftarrow{C}y_1^+u_1v_1x$ is a D–cycle longer than C , a contradiction; if $xy_1^+\in E(G)$, then it contradicts with Claim 3. Then, $y_1^+\in N^-(x,u_1)$. If $xz^+\in E(G)$ and $y_1^{+2}x\in E(G)$, then it contradicts to Claim 3; Put

$$C'=\begin{cases} xy^{+2}\overrightarrow{C}zy_1^+\overleftarrow{C}u_1z^+\overleftarrow{C}v_1x & \text{if } y_1^{+2}x\in E(G)\text{ and } u_1z^+\in E(G);\\ xv_2\overrightarrow{C}y_1^+z\overleftarrow{C}y_1^{+2}u_1\overleftarrow{C}z^+x & \text{if } y_1^{+2}u_1\in E(G)\text{ and } xz^+\in E(G);\\ xv_2\overrightarrow{C}y_1^+z\overleftarrow{C}y_1^{+2}u_1z^+\overleftarrow{C}v_1x & \text{if } y_1^{+2}u_1\in E(G)\text{ and } u_1z^+\in E(G).\end{cases}$$

In any cases, C' is a D–cycle longer than C, a contradiction. Therefore, $v_1,v_2\notin N_C(y,y_1^+)\cup N_C^-(x,u_1), N_C(y,y_1^+)\cap N_C^-(x,u_1)\subseteq\{w_k\}$. Hence, we have

$$\begin{aligned} |V(C)| &\geq |N_C(y,y_1^+)\cup N_C^-(x_1,u_1)|+2\\ &\geq |N_C(y,y_1^+)|+|N_C^-(x_1,u_1)|-1+2\\ &= |N(y,y_1^+)\backslash N_R(y,y_1^+)|+|N(x_1,u_1)\backslash N_R(x_1,u_1)|+1\\ &\geq 2NC2(G)-2+1\\ &= 2NC2(G)-1,\end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$.

Claim 7 If $\bigcup_{i=1}^{k} N_R(y_i) \neq \emptyset$, then $N_R(y_i) \neq \emptyset$ for all $i \in \{1, 2, \ldots, k\}$, where $y_i = u_i$ (w_i, respectively).

If not, without loss of generality, we assume that $N_R(u_1) \neq \emptyset$ and $N_R(u_k) = \emptyset$. Suppose $x_1 \in N_R(u_1)$ and $y \in N(x_1)$ ($y \neq u_1$). Then $dist(x_1, y^+) = dist(x_1, y^-) = 2$ and $|N(x_1, y^+)| \geq NC2(G)$, $|N(x_1, y^-)| \geq NC2(G)$.

Case 1 $N(x_1) \cap (u_1^+ \overrightarrow{C} v_k) = \emptyset$.

If not, we may choose $y, y \in N(x_1) \cap (u_1^+ \overrightarrow{C} v_k)$, such that $N(x_1) \cap (u_1^+ \overrightarrow{C} y^-) = \emptyset$. We define a mapping f on $V(C)$ as follows:

$$f(v) = \begin{cases} v^- & \text{if } v \in u_k \overrightarrow{C} y^-; \\ v^+ & \text{if } v \in y \overrightarrow{C} w_{k-1}; \\ y^- & \text{if } v = v_k. \end{cases}$$

Then $|f(N_C(x, u_k))| = |N_C(x, u_k)| = |N(x, u_k)| \geq NC2(G)$ by Claim 1 and the assumption $N_R(u_k) = \emptyset$. Moreover, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1\}$. In fact, suppose that $z \in f(N_C(x, u_k)) \cap N(x_1, y^-) \backslash \{w_k, u_1\}$. Obviously, $z \neq v_1, y^-$ by Claims 2 and 4. Now we consider the following cases.

(i) If $z \in u_k \overrightarrow{C} w_k^-$, then $z \in N_C^-(u_k)$ since $N(x) \cap (u_k \overrightarrow{C} w_k) = \emptyset$. Put

$$C' = \begin{cases} u_k z^+ \overrightarrow{C} v_1 x v_k \overleftarrow{C} u_1 x_1 z \overleftarrow{C} u_k & \text{if } x_1 z \in E(G); \\ u_k z^+ \overrightarrow{C} v_1 x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- z \overleftarrow{C} u_k & \text{if } y^- z \in E(G). \end{cases}$$

(ii) If $z \in u_1^+ \overrightarrow{C} y^{-2}$, then $zy^- \in E(G)$ since $N(x_1) \cap (u_1^+ \overrightarrow{C} y^-) = \emptyset$. Put

$$C' = \begin{cases} u_1 \overrightarrow{C} z y^- \overleftarrow{C} z^+ x v_1 \overleftarrow{C} y x_1 u_1 & \text{if } xz^+ \in E(G); \\ u_1 \overrightarrow{C} z y^- \overleftarrow{C} z^+ u_k \overrightarrow{C} v_1 x v_k \overleftarrow{C} y x_1 u_1 & \text{if } u_k z^+ \in E(G). \end{cases}$$

(iii) If $z \in y^+ \overrightarrow{C} v_k$, we put

$$C' = \begin{cases} u_1 \overrightarrow{C} z^- x v_1 \overleftarrow{C} z x_1 u_1 & \text{if} xz^- \in E(G) \text{ and } x_1 z \in E(G); \\ u_1 \overrightarrow{C} y^- z \overrightarrow{C} v_1 x z^- \overleftarrow{C} y x_1 u_1 & \text{if } xz^- \in E(G) \text{ and } y^- z \in E(G); \\ u_1 \overrightarrow{C} z^- u_k \overrightarrow{C} v_1 x v_k \overleftarrow{C} z x_1 u_1 & \text{if } u_k z^- \in E(G) \text{ and } x_1 z \in E(G); \\ u_1 \overrightarrow{C} y^- z \overrightarrow{C} v_k x v_1 \overleftarrow{C} u_k z^- \overleftarrow{C} y x_1 u_1 & \text{if } u_k z^- \in E(G) \text{ and } y^- z \in E(G). \end{cases}$$

In any cases, C' is a D-cycle longer than C, a contradiction. Therefore, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1\}$. By Claims 2 and 4, we have $u_1 \notin N(x, u_k)$ and $v_1 \notin N(x_1, y^-)$. Then $v_1 \notin f(N_C(x, u_k)) \cup N(x_1, y^-)$. Hence, by Claim 6 we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x, u_k)) \cup N_C(x_1, y^-)| + 1 \\ &\geq |f(N_C(x, u_k))| + |N_C(x_1, y^-)| - 2 + 1 \\ &\geq 2NC2(G) - 2. \end{aligned}$$

So, we have $V(C) = N_C(x_1, y^-) \cup f(N_C(x, u_k)) \cup \{v_1\}, N_C(x_1, y^-) \cap f(N_C(x,\ u_k)) = \{w_k, u_1\}$. Hence, $y^- w_k \in E(G)$ and $u_k u_1^+ \in E(G)$ since $t_i \geq 2$.

Now, we prove that $N_R(y^-) = \emptyset$. If not, there exist $y_1 \in N_R(y^-), z \in N_C(y_1)\,(z \neq y^-)$ by Claim 1 and $\delta \geq 2$.

Subcase 1 $N(y_1) \cap (u_1 \overrightarrow{C} y^{-2}) = \emptyset$.

If not, we choose $z \in N(y_1)$, such that $N(y_1) \cap (z^+ \overrightarrow{C} y^{-2}) = \emptyset$. Therefore we can define a mapping f_1 on $V(C)$ as follows:

$$f_1(v) = \begin{cases} v^- & \text{if } v \in u_k^+ \overrightarrow{C} z^+; \\ v^+ & \text{if } v \in z^{+2} \overrightarrow{C} w_{k-1}; \\ z^{+2} & \text{if } v = v_k; \\ z^+ & \text{if } v = u_k. \end{cases}$$

Using an argument as above , we have $|f_1(N_C(x, u_k)| \geq NC2(G)$. Moreover, we have $z^+, v_1, y \notin N_C(y_1, z^+) \cup f_1(N_C(x, u_k))$ and $N_C(y_1, z^+) \cap f_1(N_C(x, u_k)) \subseteq \{z^{+2}, y^-, w_k\}$. Clearly, $z^+ \notin N_C(y_1, z^+)$. If $z^+ \in f_1(N_C(x, u_k))$, then, $u_k \in N_C(x, u_k)$, a contradiction. $y_1 v_1 \notin E(G)$ by Claim 5. If $v_1 z^+ \in E(G)$, since $y, z^+ \in N^+(y_1)$, the two paths $y x_1 u_1$ and $z^+ v_1$ contradict with Claim 4; By Claims 2 and 4 , we have $y \notin N(y_1, z^+)$, if $y \in f_1(N_C(x, u_k))$ then $y^- \in N_C(x, u_k)$, by Claim 3 $y^- \notin N(x)$, so $y^- \in N(u_k)$, then $C' = x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- u_k \overrightarrow{C} v_1 x$ is a D –cycle longer than C, a contradiction. So we have $z^+, v_1, y \notin N_C(y_1, z^+) \cup f_1(N_C(x, u_k))$. Suppose $s \in N_C(y_1, z^+) \cap f_1(N_C(x, u_k)) \backslash \{z^{+2}, y^-, w_k\}$.

Now, we consider the following cases.

(i) $s \in y^+ \overrightarrow{C} v_k$. If $y_1 s \in E(G)$ and $x s^- \in E(G)$ then it contradicts with Claim 4. We put

$$C' = \begin{cases} x v_k \overleftarrow{C} s y_1 y^- \overleftarrow{C} u_1 x_1 y \overleftarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } y_1 s, u_k s^- \in E(G); \\ x s^- \overleftarrow{C} y x_1 u_1 \overrightarrow{C} z y_1 y^- \overleftarrow{C} z^+ s \overrightarrow{C} v_1 x & \text{if } z^+ s, x s^- \in E(G); \\ x v_k \overleftarrow{C} s z^+ \overrightarrow{C} y^- y_1 z \overleftarrow{C} u_1 x_1 y \overrightarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } z^+ s, u_k s^- \in E(G). \end{cases}$$

(ii) $s \in u_k \overrightarrow{C} w_{k-1}$. We have $s \in N^-(u_k)$ since $N(x) \cap (u_k \overrightarrow{C} w_k) = \emptyset$.Put

$$C' = \begin{cases} x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} y^- y_1 s \overleftarrow{C} u_k s^+ \overrightarrow{C} v_1 x & \text{if } y_1 s, u_k s^+ \in E(G); \\ x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} z y_1 y^- \overleftarrow{C} z^+ s \overleftarrow{C} u_k s^+ \overrightarrow{C} v_1 x & \text{if } z^+ s, u_k s^+ \in E(G); \end{cases}$$

(iii) $s \in u_1 \overrightarrow{C} y^{-2}$. If $y_1 s, x s^+ \in E(G)$ then contradicts to Claim 4. If $y_1 s, u_k s^+ \in E(G)$, then

$$C' = x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} s y_1 y^- \overleftarrow{C} s^+ u_k \overrightarrow{C} v_1 x$$

is a D–cycle longer than C, a contradiction. If $s \in z^+ \overrightarrow{C} y^-$, we put

$$C' = \begin{cases} x s^- \overleftarrow{C} z^+ s \overrightarrow{C} y^- y_1 z \overleftarrow{C} u_1 x_1 y \overrightarrow{C} v_1 x & \text{if } z^+ s, s^- x \in E(G); \\ x v_k \overleftarrow{C} y x_1 u_1 \overrightarrow{C} z y_1 y^- \overleftarrow{C} s z^+ \overrightarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } z^+ s, s^- u_k \in E(G). \end{cases}$$

If $s \in u_1\overrightarrow{C}z$, we put

$$C' = \begin{cases} xs^+\overrightarrow{C}zy_1y^-\overleftarrow{C}z^+s\overleftarrow{C}u_1x_1y\overrightarrow{C}v_1x & \text{if } z^+s, xs^+ \in E(G); \\ xv_k\overleftarrow{C}yx_1u_1\overrightarrow{C}sz^+\overrightarrow{C}y^-y_1z\overleftarrow{C}s^+u_k\overrightarrow{C}v_1x & if\ z^+s, u_ks^+ \in E(G). \end{cases}$$

In any cases, C' is a D–cycle longer than C, a contradiction. Hence, by Claim 5 we have

$$\begin{aligned} |V(C)| &\geq |f_1(N_C(x,u_k)) \cup N_C(y_1,z^+)| + 3 \\ &\geq |f_1(N_C(x,u_k))| + |N_C(y_1,z^+)| - 3 + 3 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. So $N(y_1) \cap (u_1\overrightarrow{C}y^{-2}) = \emptyset$,

Subcase 2 $N(y_1) \cap (y\overrightarrow{C}v_k) = \emptyset$.

If not, we may choose $z \in N(y_1) \cap (y\overrightarrow{C}v_k)$, such that $N(y_1) \cap (y\overrightarrow{C}z^-) = \emptyset$. Therefore, we can define a mapping f_2 on $V(C)$ as follows:

$$f_2(v) = \begin{cases} v^+ & \text{if } v \in u_1\overrightarrow{C}y^{-2} \cup z^-\overrightarrow{C}w_{k-1}; \\ v^- & \text{if } v \in y^+\overrightarrow{C}z^{-2} \cup u_k^+\overrightarrow{C}v_1; \\ z^- & \text{if } v = v_k; \\ v_1 & \text{if } v = u_k; \\ z^{-2} & \text{if } v = y; \\ u_1 & \text{if } v = y^- \end{cases}$$

Using a similar argument as above , we have $|f_2(N_C(x,u_k))| \geq NC2(G)$. We consider $N_C(y_1,z^-) \cup f_2(N_C(x,u_k))$, then $v_1, u_1^+ \notin N_C(y_1,z^-) \cup f_2(N_C(x,u_k))$, and $N_C(y_1,z^-) \cap f_2(N_C(x,u_k)) \subseteq \{y^-, w_k\}$. In fact, $v_1 \notin N(y_1,z^-)$ by Claims 4, 5 ; if $v_1 \in f_2(N(x,u_k))$ then $u_k \in N(x,u_k)$, a contradiction; if $u_1^+ \in N(z^-)$, then the paths yx_1u_1 and $z^-u_1^+$ contradict with Claim 5; if $u_1^+ \in f_2(N_C(x,u_k))$, then $u_1 \in N(x,u_k)$, a contradiction. So we have $v_1, u_1^+, \notin N_C(y_1,z^-) \cup f_2(N_C(x,u_k))$. For $s \in N_C(y_1,z^-) \cap f_2(N_C(x,u_k)) \backslash \{y^-, w_k\}$, we consider the following cases.

(i) If $s \in u_1\overrightarrow{C}y$. We have $s \in N(z^-)$ since $N(y_1) \cap (u_1\overrightarrow{C}y^{-2}) = \emptyset$. Put

$$C' = \begin{cases} xs^-\overleftarrow{C}u_1x_1y\overrightarrow{C}z^-s\overrightarrow{C}y^-y_1z\overrightarrow{C}v_1x & \text{if } s^-x \in E(G); \\ xv_k\overleftarrow{C}zy_1y^-\overleftarrow{C}sz\overleftarrow{C}yx_1u_1\overrightarrow{C}s^-u_k\overrightarrow{C}v_1x & \text{if } s^-u_k \in E(G). \end{cases}$$

(ii) If $s \in u_k\overrightarrow{C}v_1$, then $s^+ \in N(u_k)$ since $N(x) \cap (u_k\overrightarrow{C}w_k) = \emptyset$. Put

$$C' = \begin{cases} xv_k\overleftarrow{C}zy_1y^-\overleftarrow{C}u_1x_1y\overrightarrow{C}z^-s\overleftarrow{C}u_ks^+\overrightarrow{C}v_1x & \text{if } z^-s \in E(G); \\ xv_k\overleftarrow{C}yx_1u_1\overrightarrow{C}y^-y_1s\overleftarrow{C}u_ks^+\overrightarrow{C}v_1x & \text{if } y_1s \in E(G). \end{cases}$$

(iii) If $s \in y\overrightarrow{C}z^{-2}$, then we have $s \in N(z^-)$ since $N(y_1) \cap (y\overrightarrow{C}z^{-2}) = \emptyset$. Put

$$C' = \begin{cases} x_1y\overrightarrow{C}sz^-\overleftarrow{C}s^+xv_1\overleftarrow{C}zy_1y^-\overleftarrow{C}u_1x_1 & \text{if } xs^+ \in E(G); \\ xv_k\overleftarrow{C}zy_1y^-\overleftarrow{C}u_1x_1y\overrightarrow{C}sz^-s^+u_k\overrightarrow{C}v_1x & \text{if } u_ks^+ \in E(G). \end{cases}$$

(iv) If $s \in z^- \overrightarrow{C} v_k$. If $y_1 s, xs^- \in E(G)$ then it contradicts to Claim 4. We put

$$C' = \begin{cases} xv_k \overleftarrow{C} sy_1 y^- \overleftarrow{C} u_1 x_1 y \overrightarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } y_1 s, u_k s^- \in E(G); \\ xs^- \overleftarrow{C} zy_1 y^- \overleftarrow{C} u_1 x_1 y \overrightarrow{C} z^- s \overrightarrow{C} v_1 x & \text{if } z^- s, s^- x \in E(G); \\ xv_k \overleftarrow{C} sz^- \overleftarrow{C} yx_1 u_1 \overrightarrow{C} y^- y_1 z \overrightarrow{C} s^- u_k \overrightarrow{C} v_1 x & \text{if } z^- s, s^- u_k \in E(G). \end{cases}$$

In any cases, C' is a D-cycle longer than C, a contradiction. Therefore, we have $v_1, u_1^+, \notin N_C(y_1, z^-) \cup f_2(N_C(x, u_k))$, and $N_C(y_1, z^-) \cap f_2(N_C(x, u_k)) \subseteq \{y^-, w_k\}$. So

$$\begin{aligned} |V(C)| &\geq |N_C(y_1, z^-) \cup f_2(N_C(x, u_k))| + 2 \\ &\geq |N_C(y_1, z^-)| + |N_C(x, u_k)| - 2 + 2 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$. Hence, $N(y_1) \backslash \{y^-\} \subseteq (u_k \overrightarrow{C} u_1)$.

Subcase 3 $N(y_1) \cap (u_k \overrightarrow{C} u_1) = \emptyset$.

If not, we may choose $z \in N(y_1) \cap (u_k \overrightarrow{C} u_1)$, such that $N(y_1) \cap (z^+ \overrightarrow{C} u_1) = \emptyset$. We define a mapping f_3 on $V(C)$ as follows:

$$f_3(v) = \begin{cases} v^- & \text{if } v \in y^+ \overrightarrow{C} v_k \cup u_k^+ \overrightarrow{C} z^+; \\ v^+ & \text{if } v \in z^{+2} \overrightarrow{C} y^{-2}; \\ z^+ & \text{if } v = u_k; \\ v_k & \text{if } v = y; \\ z^{+2} & \text{if } v = y^-. \end{cases}$$

Using a similar argument as above , we have $|f_3(N_C(x, u_k))| \geq NC2(G)$. Moreover, $z^+, u_1^+ \notin N_C(y_1, z^+) \cup f_3(N_C(x, u_k)), N_C(y_1, z^+) \cap f_3(N_C(x, u_k)) \subseteq \{y^-, w_k\}$. In fact, clearly, $z^+ \notin N_C(y_1, z^+)$, if $z^+ \in f_3(N_C(x, u_k))$, then $u_k \in N_C(x, u_k)$, a contradiction; if $u_1^+ \in N_C(y_1, z^+)$, then $u_1^+ \in N(z^+)$ since $N_C(y_1) \cap (y^{-2} \overrightarrow{C} u_k) = \emptyset$, so $C' = x_1 y \overrightarrow{C} zy_1 y^- \overleftarrow{C} u_1^+ z^+ \overrightarrow{C} u_1 x_1$ is a D –cycle longer than C, a contradiction; if $u_1^+ \in f_3(N_C(x, u_k))$ then $u_1 \in N_C(x, u_k)$,a contradiction; so we have $z^+, u_1^+ \notin N_C(y_1, z^+) \cup f_3(N_C(x, u_k))$. Suppose $s \in N_C(y_1, z^+) \cap f_3(N_C(x, u_k)) \backslash \{y^-, w_k\}$. Now, we consider the following cases.

(i) If $s \in v_k \overrightarrow{C} z^+$, then We have $s^+ u_k \in E(G)$ since $N(x) \cap (u_k \overrightarrow{C} w_k) = \emptyset$. Put

$$C' = \begin{cases} xv_k \overleftarrow{C} yx_1 u_1 \overrightarrow{C} y^- y_1 s \overleftarrow{C} u_k s^+ \overrightarrow{C} v_1 x & \text{if } y_1 s \in E(G); \\ xv_k \overleftarrow{C} yx_1 u_1 \overrightarrow{C} y^- y_1 z \overleftarrow{C} s^+ u_k \overrightarrow{C} sz^+ \overrightarrow{C} v_1 x & \text{if } z^+ s \in E(G). \end{cases}$$

(ii) If $s \in z^{+2} \overrightarrow{C} w_k^-$, then we have $s^- u_k, sz^+ \in E(G)$ since $N(x) \cap (u_k \overrightarrow{C} w_k) = N(y_1) \cap (z^+ \overrightarrow{C} v_1) = \emptyset$. Put

$$C' = xv_k \overrightarrow{C} yx_1 u_1 \overrightarrow{C} y^- y_1 z \overleftarrow{C} u_k s^- \overleftarrow{C} z^+ s \overrightarrow{C} v_1 x$$

.

(iii) If $s \in u_1 \overrightarrow{C} y^{-2}$, then we have $sz^+ \in E(G)$ since $N(y_1) \cap (u_1 \overrightarrow{C} y^{-2}) = \emptyset$. Put

$$C' = \begin{cases} xs^-\overleftarrow{C}u_1x_1y\overrightarrow{C}zy_1y^-\overleftarrow{C}sz^+\overrightarrow{C}v_1x & \text{if } xs^- \in E(G); \\ xv_k\overleftarrow{C}yx_1u_1\overleftarrow{C}s^-u_k\overrightarrow{C}zy_1y^-\overleftarrow{C}sz^+\overrightarrow{C}v_1x & \text{if } u_ks^- \in E(G). \end{cases}$$

(iv) If $s \in y\overrightarrow{C}v_k$, then we have $sz^+ \in E(G)$ since $N(y_1) \cap (y\overrightarrow{C}v_k) = \emptyset$.Put

$$C' = \begin{cases} xs^+\overrightarrow{C}zy_1y^-\overleftarrow{C}u_1x_1y\overrightarrow{C}sz^+\overrightarrow{C}v_1x & \text{if } xs^+ \in E(G); \\ xv_k\overleftarrow{C}s^+u_k\overrightarrow{C}zy_1y^-\overleftarrow{C}u_1x_1y\overrightarrow{C}sz^+\overrightarrow{C}v_1x & \text{if } u_ks^+ \in E(G). \end{cases}$$

In any cases, C' is a D–cycle longer than C, a contradiction. Therefore we have $N_C(y_1, z^+) \cap f_3(N_C(x, u_k)) \subseteq \{y^-, w_k\}$. So we have

$$\begin{aligned} |V(C)| &\geq |N_C(y_1, z^+) \cup f_3(N_C(x, u_k)| + 2 \\ &\geq |N_C(y_1, z^+| + |N_C(x, u_k)| - 2 + 2 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction with $|V(C)| \leq 2NC2(G) - 2$. Hence, $N(y_1) \cap (u_k\overrightarrow{C}v_1) = \emptyset$.

Thus, $N(y_1) = \{y^-\}$, which contradicts to $\delta \geq 2$. Therefore, we know that $N_R(y^-) = \emptyset$.

So we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x, u_k)) \cup N_C(x_1, y^-)| + 1 \\ &\geq |f(N_C(x, u_k))| + |N_C(x_1, y^-)| - 2 + 1 \\ &= |N(x, u_k)\backslash N_R(x, u_k)| + |N(x_1, y^-)\backslash N_R(x_1, y^-)| - 1 \\ &= |N(x, u_k)| + |N(x_1, y^-)| - 1 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. So we have $N(x_1) \cap (u_1^+\overrightarrow{C}v_k) = \emptyset$, hence, $N(x_1) \subseteq u_k\overrightarrow{C}u_1$.

Case 2 $N(x_1) \cap (u_k\overrightarrow{C}v_1) = \emptyset$.

Otherwise, since $v_1x_1 \notin E(G)$, we can choose $y, y \in u_k\overrightarrow{C}w_k$, such that $N(x_1) \cap (y^+\overrightarrow{C}v_1) = \emptyset$. Therefore, we can define a mapping g on $V(C)$ as follows:

$$g(v) = \begin{cases} v^- & \text{if } v \in u_1^+\overrightarrow{C}y; \\ v^+ & \text{if } v \in y^+\overrightarrow{C}w_k; \\ y^+ & \text{if } v = u_1, \\ y & \text{if } v = v_1. \end{cases}$$

Using a similar argument as before, we have $|g(N_C(x, u_k))| \geq NC2(G)$, $y^+ \notin g(N_C(x, u_k)) \cup N(x_1, y^+)$ and $g(N_C(x, u_k)) \cap N(x_1, y^+) \subseteq \{u_1\}$. Hence, by Claim 6 we have

$$\begin{aligned} |V(C)| &\geq |g(N_C(x, u_k)) \cup N(x_1, y^+)| + 1 \\ &\geq |g(N_C(x, u_k))| + |N(x_1, y^+)| - 1 + 1 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. So $N(x_1) \cap (u_k \overrightarrow{C} v_1) = \emptyset$. Then $N(x_1) = \{u_1\}$, which contradicts to $\delta \geq 2$.

Claim 8 If $x_1 \in N_R(u_1)$ and $N(x_1) \cap (u_1^+ \overrightarrow{C} v_k) \neq \emptyset$, then $|\{u_k u_1^+, y^- w_k\} \cap E(G)| = 1$ for $y \in N(x_1) \cap (u_1^+ \overrightarrow{C} v_k)$ with $N(x_1) \cap (u_1^+ \overrightarrow{C} y^-) = \emptyset$.

First we have $d(x_1, y^-) = 2$ and $|N(x_1, y^-)| \geq NC2(G)$.Let $u_k u_1^+ \notin E(G)$. Now we define a mapping f on $V(C)$ as follows:

$$f(v) = \begin{cases} v^- & \text{if } v \in u_k^{+2} \overrightarrow{C} v_1 \cup u_1^{+2} \overrightarrow{C} y^-; \\ v^+ & \text{if } v \in y \overrightarrow{C} w_{k-1}; \\ y^- & \text{if } v = u_k; \\ y & \text{if } v = v_k; \\ u_1 & \text{if } v = u_k^+; \\ v_1 & \text{if } v = u_1^+; \\ u_k & \text{if } v = u_1. \end{cases}$$

Then $|f(N_C(x, u_k))| = |N_C(x, u_k)| \geq NC2(G) - 1$ by Claim 5. Moreover using a similar argument as in Claim 7, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1, y\}$. But we have $y^-, v_1, u_k \notin f(N_C(x, u_k)) \cup N(x_1, y^-)$ by the choice of y Claims 2 and 4, respectively. Therefore, by Claim 5 we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x, u_k)) \cup N_C(x_1, y^-)| + 3 \\ &\geq |f(N_C(x, u_k))| + |N_C(x_1, y^-)| - 3 + 3 \\ &\geq 2NC2(G) - 2. \end{aligned}$$

So $V(C) = f(N_C(x, u_k)) \cup N_C(x_1, y^-) \cup \{v_1, y^-, u_k\}$ by the assumption on $|V(C)|$, and in particular, $f(N_C(x, u_k)) \cap N_C(x_1, y^-) = \{w_k, u_1, y\}$. Therefore, $y^- w_k \in E(G)$. Using a similar argument as above, we have if $y^- w_k \notin E(G)$, then $u_k u_1^+ \in E(G)$.

Claim 9 There exists a vertex x with $x \notin V(C)$ such that $N_R(u_i) = N_R(w_i) = \emptyset$.

We only prove $N_R(u_i) = \emptyset$. If not, we may choose $x \notin V(C)$ such that $\min\{t_i\}$ is as small as possible. By Claim 7, without loss of generality, suppose that $t_k = \min\{t_i\}$ for the vertex x. Let $x_1 \in N_R(u_1), x_2 \in N_R(u_k)$. By Claims 2 and 3, $x \neq x_1, x_2; x_1 \neq x_2$. And by Claim 5 and the choice of x, we have $N(x_i) \cap (u_k \overrightarrow{C} v_1) = \emptyset$, for $i = 1, 2$. Since $\delta \geq 2$, $N(x_1) \cap (u_1^+ \overrightarrow{C} v_k) \neq \emptyset$. Choose $y \in N(x_1) \cap (u_k \overrightarrow{C} v_k)$ such that $N(x_1) \cap (u_1^+ \overrightarrow{C} y^-) = \emptyset$, then $d(x_1, y^-) = 2$ and $|N(x_1, y^-)| \geq NC2(G)$. By Claim 8, we have $u_k u_1^+$ or $y^- w_k \in E(G)$.

First we prove that $N(x_2) \cap (y \overrightarrow{C} v_k) = \emptyset$. If not, we may choose $z \in y^+ \overrightarrow{C} v_k^-$ such that $N(x_2) \cap (z^+ \overrightarrow{C} v_k) = \emptyset$ by Claim 5. Then $d(x_2, z^+) = 2$ and $|N(x_2, z^+)| \geq NC2(G)$. Now we define a mapping f on $V(C)$ as follows:

$$f(v) = \begin{cases} v^- & \text{if } v \in u_1^+\overrightarrow{C}y^- \cup z^{+2}\overrightarrow{C}v_k; \\ v^+ & \text{if } v \in y\overrightarrow{C}z^- \cup u_k\overrightarrow{C}w_k; \\ y & \text{if } v = z; \\ v_k & \text{if } v = z^+; \\ u_k & \text{if } v = v_1; \\ y^- & \text{if } v = u_1. \end{cases}$$

Then $|f(N_C(x_2, z^+))| = |N_C(x_2, z^+)| \geq NC2(G) - 1$ by Claim 5. Moreover using a similar argument as in Claim 7, we have $f(N_C(x_2, z^+)) \cap N(x_1, y^-) \subseteq \{u_1, y\}$. But $y^-, v_k, v_1 \notin f(N_C(x_2, z^+)) \cup N(x_1, y^-)$, otherwise, $u_1z^+ \in E(G)$ or $y^-v_k \in E(G)$ or $z^+w_k \in E(G)$ by Claim 5, and hence the D-cycle

$$C' = \begin{cases} u_1\overrightarrow{C}zx_2u_k\overrightarrow{C}v_1xv_k\overleftarrow{C}z^+u_1 & \text{if } u_1z^+ \in E(G); \\ u_1x_1y\overrightarrow{C}v_ky^-\overleftarrow{C}u_1^+u_k\overrightarrow{C}u_1 & \text{if } y^-v_k \in E(G); \\ xv_k\overleftarrow{C}z^+w_k\overleftarrow{C}u_kx_2z\overleftarrow{C}v_1x & \text{if } z^+w_k \in E(G). \end{cases}$$

is longer than C, a contradiction. Therefore, by Claim 5 we have

$$\begin{aligned} |V(C)| &\geq |f(N_C(x_2, z^+)) \cup N_C(x_1, y^-)| + 3 \\ &\geq |f(N_C(x_2, z^+))| + |N_C(x_1, y^-)| - 2 + 3 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

which contradicts to that $|V(C)| \leq 2NC2(G) - 2$. So we have $N(x_2) \cap (y\overrightarrow{C}v_k) = \emptyset$. Hence $N(x_2)_(u_1^+\overrightarrow{C}y^-) \cup \{u_k\}$.

Now, we prove that $N(x_2) \cap (u_1^+\overrightarrow{C}y^-) = \emptyset$. In fact, we may choose $z \in u_1^+\overrightarrow{C}y^{-2}$ with $z \in N(x_2)$ such that $N(x_2) \cap (u_1^+\overrightarrow{C}z^-) = \emptyset$. (Since $x_2y^- \notin E(G)$, otherwise, $C' = u_1\overrightarrow{C}y^-x_2u_k\overrightarrow{C}v_1xv_k\overleftarrow{C}yx_1u_1$ is a D-cycle longer than C, a contradiction.) Then $d(x_2, z^-) = 2$ and $|N(x_2, z^-)| \geq NC2(G)$. We define a mapping g on $V(C)$ as follows:

$$g(v) = \begin{cases} v^- & \text{if } v \in z^+\overrightarrow{C}v_k; \\ v^+ & \text{if } v \in u_k\overrightarrow{C}z^{-2}; \\ v_k & \text{if } v = z; \\ u_k & \text{if } v = z^-. \end{cases}$$

Then we have $|g(N_C(x_2, z^-))| \geq NC2(G) - 1$ by Claim 5. Moreover using a similar argument as in Claim 7, we have $g(N_C(x_2, z^-)) \cap N(x_1, y^-) \subseteq \{u_1\}$. But $v_1, u_k \notin g(N_C(x_2, z^-)) \cup N(x_1, y^-)$, otherwise since $u_k \notin g(N_C(x_2, z^-)) \cup N(x_1, y^-)$, $w_kz^- \in E(G)$ by Claims 2 and 4, and hence the D-cycle $u_1\overrightarrow{C}z^-w_k\overleftarrow{C}\ u_kx_2z\overrightarrow{C}v_kxv_1u_1$ is longer than C, a contradiction. Therefore, by Claim 5 we have

$$\begin{aligned}|V(C)| &\geq |g(N_C(x_2, z^-)) \cap N(x_1, y^-)| + 2\\ &\geq |g(N_C(x_2, z^-))| + |N(x_1, y^-)| - 1 + 2\\ &\geq 2NC2(G) - 1,\end{aligned}$$

which contradicts to that $|V(C)| \leq 2NC2(G) - 2$. So we have $N(x_2) \cap (u_1^+ \overrightarrow{C} y^-) = \emptyset$.

Therefore, $N(x_2) = \{u_k\}$, which contradicts to $\delta \geq 2$.

Claim 10 For any $x \notin V(C), t_i \geq 3$.

Otherwise, there exists a vertex $x, x \notin V(C)$, such that $\min\{t_i\} = 2$ by Claim 6. Note that the choice of the vertex x in Claim 9, we have $N_R(u_i) = N_R(w_i) = \emptyset$ for the vertex x. Without loss of generality, suppose $t_1 = 2$, then $N_C^-(u_1) \cap N_C(w_1) = \{u_1\}$ by Claim 4, $N(x) \cap N^+(x) = \emptyset$ by Claim 2, and $N_C^-(u_1) \cap N(x) = N^-(x) \cap N_C(w_1) = \emptyset$ by Claim 3. Hence, $N_C^-(x, u_1) \cap N_C(x, w_1) = \{u_1\}$. We also have $|N_C(x, u_1)| \geq NC2(G)$ and $|N_C(x, w_1)| \geq NC2(G)$ since $d(x, u_1) = d(x, w_1) = 2$. Then

$$\begin{aligned}|V(C)| &\geq |N_C^-(x, u_1) \cup N_C(x, w_1)|\\ &\geq |N_C^-(x, u_1)| + |N_C(x, w_1)| - 1\\ &\geq 2NC2(G) - 1,\end{aligned}$$

which contradicts to that $|V(C)| \leq 2NC2(G) - 2$.

By Claim 10, we have $|V(C)| = k + \sum\limits_{i=1}^{k} t_i \geq 4k$. Thus we get the following.

Claim 11 For any $x, x \notin V(C)$,

$$d(x) \leq \frac{|V(C)|}{4} \leq \frac{2NC2(G) - 2}{4} = (NC2(G) - 1)/2.$$

Claim 12 $u_i^+ u_j \notin E(G)$, for the vertex x as in Claim 9.

In fact, if $u_i^+ u_j \in E(G)$, then the cycle $u_i^+ \overrightarrow{C} v_j x v_i \overleftarrow{C} u_j u_i^+$ is a longest D-cycle not containing u_i, by Claim 9. Thus $d(u_i) \leq (NC2(G) - 1)/2$ by Claim 11. So we have

$$NC2(G) \leq |N(x, u_i)| \leq d(x) + d(u_i) \leq NC2(G) - 1,$$

a contradiction. We choose x as in Claim 9, and define a mapping f on $V(C)$ as follows:

$$f(v) = \begin{cases} v^+ & \text{if } v \in u_1 \overrightarrow{C} v_k^-;\\ v^- & \text{if } v \in u_k^+ \overrightarrow{C} v_1;\\ u_1 & \text{if } v = v_k;\\ v_1 & \text{if } v = u_k.\end{cases}$$

Then $|f(N_C(x, u_k))| \geq NC2(G)$ and $|N_C(x, u_1)| \geq NC2(G)$ by Claim 10. Moreover, we have $f(N_C(x, u_k)) \cap N_C(x, u_1)_{\{} v_2, v_3, \ldots, v_k, w_k\}$. By Claims 2, 4, and 12, we also have $u_2^+, u_3^+, \ldots, u_{k-1}^+ \notin f(N_C(x, u_k)) \cup N_C(x, u_1)$. Therefore, we have

$$\begin{aligned}|V(C)| &\geq |f(N_C(x,u_k)) \cup N_C(x,u_1)| + k - 2\\ &\geq |f(N_C(x,u_k))| + |N_C(x,u_1)| - k + k - 2\\ &\geq 2NC2(G) - 2.\end{aligned}$$

So

$$V(C) = f(N_C(x,u_k)) \cup N_C(x,u_1) \cup \{u_2^+, u_3^+, \ldots, u_{k-1}^+\}$$

by the assumption on $|V(C)|$, and in particular,

$$f(N_C(x,u_k)) \cap N_C(x,u_1) = \{v_2, v_3, \ldots, v_k, w_k\}.$$

Then $u_1w_k, u_kw_{k-1} \in E(G)$.

Claim 13 $k = 2$.

If there exists $v \in V(C)\backslash\{v_1, v_k\}$, by partition of $V(C)$, we have $v^{+2} \in f(N_C(x,u_k)) \cup N_C(x,u_1) \cup \{u_2^+, u_3^+, ..., u_{k-1}^+\}$. If $v^{+2} \in N_C(x,u_1)$, then $v^{+2}u_1 \in E(G)$, and the cycle $u_1v^{+2}\overrightarrow{C}v_1x v\overleftarrow{C}u_1$ is a D-cycle not containing v^+ by Claim 9. Thus $d(v^+) \leq (NC2(G)-1)/2$ by Claim 11. So we have

$$NC2(G) \leq |N(x,v^+)| \leq d(x) + d(v^+) \leq NC2(G) - 1,$$

a contradiction.So $v^+ \in N(x,u_k)$, which contradicts to Claims 2,3. Hence we have $k = 2$.

Claim 14 Each of the followings does not hold :

(1) There is $u \in u_1\overrightarrow{C}v_2$, such that $u^+u_1 \in E(G)$ and $u^-u_2 \in E(G)$.
(2) There is $u \in u_2\overrightarrow{C}v_1$, such that $u^-u_1 \in E(G)$ and $u^+u_2 \in E(G)$.
(3) There is $u \in u_2\overrightarrow{C}v_1$, such that $u^+w_1 \in E(G)$ and $u^-w_2 \in E(G)$.
(4) There is $u \in u_1\overrightarrow{C}v_2$, such that $u^+w_2 \in E(G)$ and $u^-w_1 \in E(G)$.

If not, suppose there is $u \in u_1\overrightarrow{C}v_2$, such that $u^+u_1 \in E(G)$ and $u^-u_2 \in E(G)$. We define a mapping h on $V(C)$ as follows :

$$h(v) = \begin{cases} v^+ & \text{if } v \in u_1\overrightarrow{C}u^-u_2 \cup u^+\overrightarrow{C}w_1;\\ v^- & \text{if } v \in u_2^+\overrightarrow{C}v_1;\\ u^+ & \text{if } v = v_2;\\ v_1 & \text{if } v = u_2;\\ u_1 & \text{if } v = u;\\ u & \text{if } v = u_2^+.\end{cases}$$

Then $|h(N_C(x,u_2))| \geq NC2(G)$ and $|N_C(x,u_1)| \geq NC2(G)$. Moreover we have $u_1 \notin N(x,u_1) \cup h(N(x,u_2))$, and $N(x,u_1) \cap h(N(x,u_2)) \subseteq \{v_2, u^+\}$. In fact, clearly $u_1 \notin N(x,u_1)$, if $u_1 \in h(N(x,u_2))$, then $u \in N(x,u_2)$, a contradiction. Let $s \in N(x,u_1) \cap h(N(x,u_2))\backslash\{v_2,u^+\}$, if $s \in u_1^+\overrightarrow{C}v_2 \cap N(x,u_1) \cap h(N(x,u_2))\backslash\{v_2,u^+\}$ then $su_1 \in E(G)$ and $s^-u_2 \in E(G)$; or if

$s \in u_2\overrightarrow{C}w_2 \cap N(x,u_1) \cap h(N(x,u_2))$, then $su_1 \in E(G)$ and $s^+u_2 \in E(G)$, both cases contradict to Claim 3. So $u_1 \notin N(x,u_1) \cup h(N(x,u_2))$, $N(x,u_1) \cap h(N(x,u_2)) \subseteq \{v_2, u^+\}$. Hence

$$\begin{aligned} |V(C)| &\geq |h(N_C(x,u_2)) \cup N_C(x,u_1)| + 1 \\ &\geq |h(N_C(x,u_2))| + |N_C(x,u_1)| - 2 + 1 \\ &\geq 2NC2(G) - 1, \end{aligned}$$

a contradiction. Similarly, (2), (3) and (4) are true.

Claim 15 $N(u_2) \cap (u_1\overrightarrow{C}w_1^-) = N(u_1) \cap (u_2\overrightarrow{C}w_2^-) = \emptyset$.

If not, we may choose $z \in N(u_2) \cap (u_1\overrightarrow{C}w_1^-)$, such that $N(u_2) \cap (u_1\overrightarrow{C}z^-) = \emptyset$. then $u_1z \in E(G)$ (if not, $u_1z \notin E(G)$ then $u_2z^- \in E(G)$ by partition of $V(G)$, which contradicts the choice of z) and $N(u_1) \cap (z^+\overrightarrow{C}w_1) = \emptyset$ (if not, we may choose $s \in N(u_1) \cap (z^+\overrightarrow{C}w_1)$, such that $N(u_1) \cap (z^+\overrightarrow{C}s^-) = \emptyset$ since $z^+u_1 \notin E(G)$. So $s^-u_1 \notin E(G)$,by partition of the $V(C)$, $s^{-2}u_2 \in E(G)$. Which contradicts Claim 14) Moreover $u_1^+\overrightarrow{C}z \subseteq N(u_1)$, and $z\overrightarrow{C}v_2 \subseteq N(u_2)$. Similarly , we have $y \in u_2\overrightarrow{C}w_2$,such that $u_2y, u_1y \in E(G)$ and $N(u_1) \cap (u_2\overrightarrow{C}y^-) = N(u_2) \cap (y^+\overrightarrow{C}w_2) = \emptyset$, $y\overrightarrow{C}v_1 \subseteq N(u_1)$ and $u_2^+\overrightarrow{C}y \subseteq N(u_2)$.

Now we define a mapping g on $V(C)$ as follows:

$$g(v) = \begin{cases} v^+ & \text{if } v \in v_2\overrightarrow{C}w_2^-; \\ v^- & \text{if } v \in u_1\overrightarrow{C}w_1; \\ v_2 & \text{if } v = w_2; \\ w_1 & \text{if } v = v_1. \end{cases}$$

Using similar argument as above , consider $N(x,w_1) \cup g(N(x,w_2))$, there exists $u \in V(C)$, such that $w_1u, w_2u \in E(G)$. Without loss generality, we may assume $u \in u_1\overrightarrow{C}w_1$, Moreover then $N(w_2) \cap (u^+\overrightarrow{C}w_1) = N(w_1) \cap (u_1\overrightarrow{C}u^-) = \emptyset$, and $v_1\overrightarrow{C}u \subseteq N(w_2)$, $u\overrightarrow{C}v_2 \subseteq N(w_1)$. Let $u \neq z$. If $u \in z\overrightarrow{C}w_1^-$, $u^-u_2 \in E(G)$ by partition of $V(C)$ since $uu_1 \notin E(G)$, which contradicts to Claim 4 ; if $u \in u_1\overrightarrow{C}z$, then $C' = xv_2w_1u\overrightarrow{C}w_1^-u_2\overrightarrow{C}w_2u^-\overleftarrow{C}v_1x$ is a D-cycle longer than C, a contradiction. If $u = z$, since $z^{+2}u_1 \notin E(G), z^+u_2 \in E(G)$ by partition of $V(C)$, which contradicts to Claim 4. Hence $N(u_2) \cap (u_1\overrightarrow{C}w_2^-) = \emptyset$. Similarly $N(u_1) \cap (u_2\overrightarrow{C}w_1^-) = \emptyset$.

By Claim 15 we have

Claim 16 If there exists $z \in v_1\overrightarrow{C}v_2$, such that $u_2z \in E(G)$, then $u_1z \in E(G)$ and $u_1^+\overrightarrow{C}z \subseteq N(u_1)$, $z\overrightarrow{C}w_1 \subseteq N(u_2)$. similarly if there exists $z \in v_2\overrightarrow{C}v_1$, such that $u_2z \in E(G)$, then $u_1z \in E(G)$ and $u_2^+\overrightarrow{C}z \subseteq N(u_2)$, $z\overrightarrow{C}w_2 \subseteq N(u_1)$.

Proof of Theorem 5

Now we are going to complete the proof of Theorem 5. We choose x as in Claim 9. By Claim 13, we know that $k = 2$.

First we prove that there exists $u \in V(C)$ such that $u_1, u_2 \in N(u)$. If there is not any $u \in V(C)\backslash\{v_2, w_1, u_2^+\}$ such that $u_2u \notin E(G)$, then $w_1^-u_1 \in E(G)$ (if not, $w_1^{-2}u_2 \in E(G)$ by partition of $V(C)$). If $u_1w_1 \notin E(G)$ then $u_2w_1^- \in E(G)$, so we have $u_1, u_2 \in N(w_1^-)$; if there is $u \in V(C)$, such that $u_2u \in E(G)$ then, by Claim 16, $u_1u \in E(G)$, hence $u_1, u_2 \in N(u)$.

By Claim 16, clearly, there are not $z \in u_1\overrightarrow{C}w_1, y \in u_2\overrightarrow{C}w_2$, such that $yz \in E(G)$.

So we have $G \in \mathcal{J}_1$. The proof of Theorem 5 is finished.

Acknowledgment

We were helped in completing this paper by conversations with Prof.F.Tian and L.Zhang.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York(1976).
[2] H.J. Broersma and H.J. Veldman, *Long dominating cycles and paths in graphs with large neighborhood unions*, Journal of Graph Theory 15(1991), 29-38.
[3] L.F.Mao, *Smarandache multi-space theory*, Hexis, Phoenix, AZ2006.
[4] R.Q.Shen and F.Tian, *Long dominating cycles in graphs*, *Discrete Mathematics*, Vol.177(1997), 287-294.
[5] F. Tian and L.Z.Zhang *Long dominating cycle in a kind of 2-connected graphs*, Systems Science and Mathematical Science 8(1995), 66-74.
[6] Z.R.Sun, *Long dominating cycle in graphs* (Submitted).