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Some properties of the LCM sequence

Hailong Li and Qianli Yang

Department of Mathematics, Weinan Teacher's College
Weinan, Shaanxi, P.R.China

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Abstract The main purpose of this paper is using the elementary method to study the properties of the Smarandache LCM sequence, and give some interesting identities.

Keywords Smarandache LCM sequence, elementary method, identities.

§1. Introduction and results

For any positive integer n, we define $L(n)$ as the Least Common Multiply (LCM) of the natural number from 1 to n. That is,

$$L(n) = [1,\ 2,\ \cdots,\ n].$$

The Smarandache Least Common Multiply Sequence is defined by:

$$\text{SLS} \longrightarrow L(1),\ L(2),\ L(3),\ \cdots,\ L(n),\ L(n+1),\ \cdots.$$

For example, the first few values in the sequence $\{L(n)\}$ are: $L(1) = 1$, $L(2) = 2$, $L(3) = 6$, $L(4) = 12$, $L(5) = 60$, $L(6) = 60$, $L(7) = 420$, $L(8) = 840$, $L(9) = 2520$, $L(10) = 2520$, $\cdots$.

About the elementary arithmetical properties of $L(n)$, there are many results in elementary number theory text books (See references [2] and [3]), such as:

$$[a, b] = \frac{ab}{(a, b)} \quad \text{and} \quad [a, b, c] = \frac{abc \cdot (a, b, c)}{(a, b)(b, c)(c, a)},$$

where $(a_1, a_2, \cdots, a_k)$ denotes the Greatest Common Divisor of a_1, a_2, $\cdots$, a_{k-1} and a_k.

Recently, Pan Xiaowei [4] studied the deeply arithmetical properties of L(n), and proved that for any positive integer $n > 2$, we have the asymptotic formula:

$$\left(\frac{L(n^2)}{\prod\limits_{p \le n^2} p} \right)^{\frac{1}{n}} = e + O\left(\exp\left(-c \frac{(\ln n)^{\frac{3}{5}}}{(\ln \ln n)^{\frac{1}{5}}} \right) \right),$$

where c is a positive constant, and $\prod\limits_{p \le n^2}$ denotes the product over all primes $p \le n^2$.

In this paper, we shall use the elementary method to study the calculating problem of $L(n)$, and give an exact calculating formula for it. That is, we shall prove the following:

Theorem 1. For any positive integer $n > 1$, we have the calculating formula

$$L(n) = \exp\left(\sum_{k=1}^{\infty} \theta\left(n^{\frac{1}{k}}\right)\right) = \exp\left(\sum_{k\le n} \Lambda(k)\right),$$

where $\exp(y) = e^y$, $\theta(x) = \sum_{p\le x} \ln p$, $\sum_{p\le x}$ denotes the summation over all primes $p \le x$, and $\Lambda(n)$ is the Mangoldt function defined as follows:

$$\Lambda(n) = \begin{cases} \ln p, & \text{if } n = p^\alpha, \ p \text{ be a prime, and } \alpha \text{ be a positive integer;} \\ 0, & \text{otherwise.} \end{cases}$$

Now let $d(n)$ denotes the Dirichlet divisor function, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers. We define the function $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. Then we have the following:

Theorem 2. For any positive integer $n > 1$, we have the calculating formula

$$\Omega\left(L(n)\right) = \sum_{k=1}^{\infty} \pi\left(n^{\frac{1}{k}}\right).$$

Theorem 3. For all positive integer $n \ge 2$, we also have

$$d\left(L(n)\right) = \exp\left(\sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k}\right) \pi\left(n^{\frac{1}{k}}\right)\right),$$

where $\exp(y) = e^y$ and $\pi(x) = \sum_{p\le x} 1$.

From these theorems and the famous Prime Theorem we may immediately deduce the following two corollaries:

Corollary 1. Under the notations of the above, we have

$$\lim_{n\to\infty} [L(n)]^{\frac{1}{n}} = e \quad \text{and} \quad \lim_{n\to\infty} \left[d\left(L(n)\right)\right]^{\frac{1}{\Omega(L(n))}} = 2,$$

where $e = 2.718281828459\cdots$ is a constant.

Corollary 2. For any integer $n > 1$, we have the asymptotic formula

$$\Omega\left(L(n)\right) = \frac{n}{\ln n} + O\left(\frac{n}{\ln^2 n}\right).$$

§2. Proof of the theorems

In this section, we shall complete the proof of these theorems. First we prove Theorem 1. Let

$$L(n) = [1,\ 2,\ \cdots,\ n] = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} = \prod_{p\le n} p^{\alpha(p)} \tag{1}$$

be the factorization of $L(n)$ into prime powers. Then for each $1 \le i \le s$, there exists a positive integer $1 < k \le n$ such that $p_i^{\alpha_i} \parallel k$. So from (1) we have

$$\begin{aligned} L(n) &= [1,\ 2,\ \cdots,\ n] = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} = \exp\left(\sum_{t=1}^{s} \alpha_t \ln p_t\right) = \exp\left(\sum_{p\leq n} \alpha(p) \ln p\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \sum_{n^{\frac{1}{k+1}} < p \leq n^{\frac{1}{k}}} \alpha(p) \ln p\right). \end{aligned} \tag{2}$$

Note that if $n^{\frac{1}{k+1}} < p \leq n^{\frac{1}{k}}$, then $p^k \leq n$, $p^{k+1} > n$ and $\alpha(p) = k$. So from (2) we have

$$\begin{aligned} L(n) &= \exp\left(\sum_{k=1}^{\infty} \sum_{n^{\frac{1}{k+1}} < p \leq n^{\frac{1}{k}}} k \cdot \ln p\right) \\ &= \exp\left(\sum_{k=1}^{\infty} k \left(\sum_{n^{\frac{1}{k+1}} < p \leq n^{\frac{1}{k}}} \ln p\right)\right) \\ &= \exp\left(\sum_{k=1}^{\infty} k \left[\theta\left(n^{\frac{1}{k}}\right) - \theta\left(n^{\frac{1}{k+1}}\right)\right]\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \left[k\theta\left(n^{\frac{1}{k}}\right) - (k+1)\theta\left(n^{\frac{1}{k+1}}\right) + \theta\left(n^{\frac{1}{k+1}}\right)\right]\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \theta\left(n^{\frac{1}{k}}\right)\right) = \exp\left(\sum_{k\leq n} \Lambda(k)\right), \end{aligned}$$

where $\theta(x) = \sum_{p\leq x} \ln p$, and $\Lambda(n)$ is the Mangoldt function. This proves Theorem 1.

Now we prove Theorem 2. In fact from the definition of $\Omega(n)$ and the method of proving Theorem 1 we have

$$\begin{aligned} \Omega\left(L(n)\right) &= \sum_{p\leq n} \alpha(p) = \sum_{k=1}^{\infty} \sum_{n^{\frac{1}{k+1}} < p \leq n^{\frac{1}{k}}} \alpha(p) = \sum_{k=1}^{\infty} \sum_{n^{\frac{1}{k+1}} < p \leq n^{\frac{1}{k}}} k \\ &= \sum_{k=1}^{\infty} k \left(\sum_{n^{\frac{1}{k+1}} < p \leq n^{\frac{1}{k}}} 1\right) \\ &= \sum_{k=1}^{\infty} k \left[\pi\left(n^{\frac{1}{k}}\right) - \pi\left(n^{\frac{1}{k+1}}\right)\right] \\ &= \sum_{k=1}^{\infty} \left[k\pi\left(n^{\frac{1}{k}}\right) - (k+1)\pi\left(n^{\frac{1}{k+1}}\right) + \pi\left(n^{\frac{1}{k+1}}\right)\right] \\ &= \sum_{k=1}^{\infty} \pi\left(n^{\frac{1}{k}}\right), \end{aligned}$$

where $\pi(x) = \sum_{p\leq x} 1$. This proves Theorem 2.

Note that the definition of the Dirichlet divisor function $d(n)$ we have

$$
\begin{aligned}
d\left(L(n)\right) &= \prod_{p\leq n}\left(\alpha(p)+1\right) = \exp\left(\sum_{p\leq n}\ln[\alpha(p)+1]\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\sum_{n^{\frac{1}{k+1}}<p\leq n^{\frac{1}{k}}}\ln[\alpha(p)+1]\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\sum_{n^{\frac{1}{k+1}}<p\leq n^{\frac{1}{k}}}\ln(k+1)\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\ln(k+1)\sum_{n^{\frac{1}{k+1}}<p\leq n^{\frac{1}{k}}}1\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\ln(k+1)\left[\pi\left(n^{\frac{1}{k}}\right)-\pi\left(n^{\frac{1}{k+1}}\right)\right]\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\left[\ln(k)\pi\left(n^{\frac{1}{k}}\right)-\ln(k+1)\pi\left(n^{\frac{1}{k+1}}\right)+\ln\left(1+\frac{1}{k}\right)\pi\left(n^{\frac{1}{k}}\right)\right]\right)\\
&= \exp\left(\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k}\right)\pi\left(n^{\frac{1}{k}}\right)\right).
\end{aligned}
$$

This completes the proof of Theorem 3.

Corollary 1 and Corollary 2 follows from our theorems and the asymptotic formulae:

$$\theta(x)=\sum_{p\leq x}\ln p = x + O\left(x\exp\left(-c\frac{(\ln x)^{\frac{3}{5}}}{(\ln\ln x)^{\frac{1}{5}}}\right)\right) \quad \text{and} \quad \pi(x)=\frac{x}{\ln x}+O\left(\frac{x}{\ln^2 x}\right),$$

where $c > 0$ is a constant. These formulae can be found in reference [5].

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On the generalization of the primitive number function

Miaohua Liu

Department of Mthematics, Northwest University
Xi'an, Shaanxi, P.R.China

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Abstract Let k be any fixed positive integer, n be any positive integer, $S_k(n)$ denotes the smallest positive integer m such that $m!$ is divisible by k^n. In this paper, we use the elementary methods to study the asymptotic properties of $S_k(n)$, and give an interesting asymptotic formula for it.

Keywords F.Smarandache problem, primitive numbers, asymptotic formula.

§1. Introduction

For any fixed positive integer $k > 1$ and any positive integer n, we define function $S_k(n)$ as the smallest positive integer m such that $k^n \mid m!$. That is,

$$S_k(n) = \min\{m : m \in N, k^n \mid m!\}.$$

For example, $S_4(1) = 4$, $S_4(2) = 6$, $S_4(3) = 8$, $S_4(4) = 10$, $S_4(5) = 12, \cdots$. In problem 49 of book [1], Professor F.Smarandache asked us to study the properties of the sequence $\{S_p(n)\}$, where p is a prime. The problem is interesting because it can help us to calculate the Smarandache function. About this problem, many scholars have shown their interest on it, see [2], [3], [4] and [5]. For example, professor Zhang Wenpeng and Liu Duansen had studied the asymptotic properties of $S_p(n)$ in reference [2], and give an interesting asymptotic formula:

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p}\ln n\right).$$

Yi Yuan [3] had studied the mean value distribution property of $|S_p(n+1) - S_p(n)|$, and obtained the following asymptotic formula: for any real number $x \geq 2$, let p be a prime and n be any positive integer, then

$$\frac{1}{p}\sum_{n\leq x}|S_p(n+1) - S_p(n)| = x\left(1 - \frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

Xu Zhefeng [4] had studied the relationship between the Riemann zeta-function and an infinite series involving $S_p(n)$, and obtained some interesting identities and asymptotic formula for $S_p(n)$. That is, for any prime p and complex number s with Re $s > 1$, we have the identity:

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

And let p be any fixed prime, then for any real number $x \geq 1$,

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p \ln p}{p-1}\right) + O\left(x^{-\frac{1}{2}+\varepsilon}\right),$$

where γ is the Euler constant, ε denotes any fixed positive number.

Zhao Yuan-e [5] had studied an equation involving the function $S_p(n)$, and obtained some interesting results: let p be a fixed prime, for any positive integer n with $n \leq p$, the equation

$$\sum_{d|n} S_p(d) = 2pn$$

holds if and only if n be a perfect number. If n be an even perfect number, then $n = 2^{r-1}(2^r - 1), r \geq 2$, where $2^r - 1$ is a Mersenne prime.

In this paper, we shall use the elementary methods to study the asymptotic properties of $S_k(n)$, and get a more general asymptotic formula. That is, we shall prove the following conclusion:

Theorem. For any fixed positive integer $k > 1$ and any positive integer n, we have the asymptotic formula

$$S_k(n) = \alpha(p-1)n + O\left(\frac{p}{\ln p} \ln n\right),$$

where $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of k into prime powers, and $\alpha(p-1) = \max_{1 \leq i \leq r} \{\alpha_i(p_i - 1)\}$.

§2. Some lemmas

To complete the proof of Theorem, we need the following several lemmas. First for any fixed prime p and positive integer n, we let $\alpha(n,p)$ denote the sum of the base p digits of n. That is, if $n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$, where $1 \leq a_i \leq p-1, i = 1, 2, \cdots, s$, then $\alpha(n,p) = \sum_{i=1}^{s} \alpha_i$, and for this number theoretic function, we have the following:

Lemma 1. For any integer $n \geq 1$, we have the identity

$$\alpha_p(n) \equiv \alpha(n) \equiv \sum_{i=1}^{+\infty} \left[\frac{n}{p^i}\right] = \frac{1}{p-1}(n - \alpha(n,p)),$$

where $[x]$ denotes the greatest integer not exceeding x.

Proof. (See Lemma 1 of reference [2]).

Lemma 2. For any positive integer n with $p \mid n$, we have the estimate

$$\alpha(n,p) \le \frac{p}{\ln p}\ln n.$$

Proof. (See Lemma 2 of reference [2]).

§3. Proof of the theorem

In this section, we use Lemma 1 and Lemma 2 to complete the proof of Theorem. For any fixed positive integer k and any positive integer n, let $S_k(n) = m$, and $k = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$. Then from the definition of $S_k(n)$, we know that $k^n \mid m!$ and $k^n \dagger (m-1)!$. So we also get $p_1^{\alpha_1 n}p_2^{\alpha_2 n}\cdots p_r^{\alpha_r n} \mid m!$ and $p_1^{\alpha_1 n}p_2^{\alpha_2 n}\cdots p_r^{\alpha_r n} \dagger (m-1)!$. From the definition of F.Smarandache function $S(n)$ we may immediately get $S_k(n) = m = \max\limits_{1\le i\le r}\{S(p_i^{\alpha_i n})\}$.

For convenient, let

$$m_i = S(p_i^{\alpha_i n}),$$

so we have

$$m = \max_{1\le i\le r}\{m_i\}.$$

Let $m_i = a_{i1}p_i^{\beta_{i1}} + a_{i2}p_i^{\beta_{i2}} + \cdots + a_{is}p_i^{\beta_{is}}$ with $\beta_{is} > \beta_{i(s-1)} > \cdots > \beta_{i1} \ge 0$ under the base p_i. From the definition of $S(p_i^{\alpha_i n})$, we know that $p_i^{\alpha_i n} \parallel m_i!$, so that $\beta_{i1} \ge 1$. Note that the factorization of $m_i!$ into prime powers is

$$m_i! = \prod_{q\le m_i} q^{\alpha_q(m_i)},$$

where $\prod\limits_{q\le m_i}$ denotes the product over all prime $\le m_i$, and $\alpha_q(m_i) = \sum\limits_{j=1}^{+\infty}\left[\frac{m_i}{q^j}\right]$. From Lemma 1 we may immediately get the inequality

$$\alpha_{p_i}(m_i) - \beta_{i1} < \alpha_i n \le \alpha_{p_i}(m_i),$$

or

$$\frac{1}{p_i-1}(m_i - \alpha(m_i,p_i)) - \beta_{i1} < \alpha_i n \le \frac{1}{p_i-1}(m_i - \alpha(m_i,p_i)),$$

$$\alpha_i(p_i-1)n + \alpha(m_i,p_i) \le m_i \le \alpha_i(p_i-1)n + \alpha(m_i,p_i) + (p_i-1)(\beta_{i1}-1).$$

Combining this inequality and Lemma 2 we obtain the asymptotic formula

$$m_i = \alpha_i(p_i-1)n + O\left(\frac{p_i}{\ln p_i}\ln m_i\right).$$

From above asymptotic formula we can easily see that m_i can achieve the maxima if $\alpha_i(p_i-1)$ come to the maxima. So taking $\alpha(p-1) = \max\limits_{1\le i\le r}\{\alpha_i(p_i-1)\}$, we can obtain

$$m = \alpha(p-1)n + O\left(\frac{p}{\ln p}\ln m\right) = \alpha(p-1)n + O\left(\frac{p}{\ln p}\ln n\right).$$

This completes the proof of Theorem.

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