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Smarandache sums of products

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Introduction This paper deals with the sums of products of first n natural numbers, taken r at a time. Many interesting results about the summations are obtained. Mr. Ramasubramanian [1] has already made some work in this direction. This paper is an extension of his work.

In next part, the sums of odd and even natural numbers are discussed, and also of natural numbers, not necessarily beginning with one. After that, properties of sequences, arising out of these sums are obtained. Interestingly, the numbers thus obtained are Stirlings numbers.

1.1 Definition. The Smarandache sum of products is denoted by $S(n, r)$, and is defined as sum of products of first n natural numbers, taken r at a time, without repeatition, $r \leq n$.

For example:

$S(4, 1) = 1 + 2 + 3 + 4 = 10$,

$S(4, 2) = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35$,

$S(4, 3) = 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 = 50$,

$S(4, 4) = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

We assume that $S(n, 0) = 1$.

1.2 Following are some elementary properties of $S(n, r)$:

1. $S(n, n) = \llcorner n =$factorial $n = nS(n-1, n-1)$,

2. $S(n, 1) = n(n+1)/2$; these are triangular numbers,

3. $(p+1)(p+2)(p+3)\dots(p+n) = S(n, 0)p^n + S(n, 1)p^{n-1} + S(n, 2)p^{n-2} + S(n, 3)p^{n-3} + \cdots + S(n, n-1)p + S(n, n)$,

4. $S(n, 0) + S(n, 1) + S(n, 2) + \cdots + S(n, n) = S(n+1, n+1) = \llcorner(n+1)$,

5. Number of terms in $S(n, r) =^n Cr$.

The 4th property can be obtained by putting $p = 1$ in the 3-rd property.

Verification of 4th property for $n = 5$.

$S(5, 0) = 1$,

$S(5, 1) = 1 + 2 + 3 + 4 + 5 = 15$,

$S(5, 2) = 1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 1 \cdot 5 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 + 3 \cdot 4 + 3 \cdot 5 + 4 \cdot 5 = 85$,

$S(5, 3) = 1{\cdot}2{\cdot}3+1{\cdot}2{\cdot}4+1{\cdot}2{\cdot}5+1{\cdot}3{\cdot}4+1{\cdot}3{\cdot}5+2{\cdot}3{\cdot}4+2{\cdot}3{\cdot}5+3{\cdot}4{\cdot}5+2{\cdot}4{\cdot}5+2{\cdot}4{\cdot}5 = 225$,

$S(5, 4) = 1 \cdot 2 \cdot 3 \cdot 4 + 1 \cdot 2 \cdot 3 \cdot 5 + 1 \cdot 3 \cdot 4 \cdot 5 + 2 \cdot 3 \cdot 4 \cdot 5 + 1 \cdot 2 \cdot 4 \cdot 5 = 274$,

$S(5, 5) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

Hence, left side is

$$1 + 15 + 85 + 225 + 274 + 120 = 720,$$

right side is

$$S(6, 6) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720.$$

1.3 We have,

$$\begin{aligned}(p-1)(p-2)(p-3)\dots(p-n) &= S(n,0) - S(n,1)p^n + S(n,2)p^{n-1} - S(n,3)p^{n-3} \\ &\quad + \dots + S(n, n-1)p + (-1)^n S(n,n).\end{aligned}$$

Put $p = 1$,

$$S(n, 0) - S(n, 1) + S(n, 2) - S(n, 3) + \cdots + S(n, n-1) + (-1)^n S(n, n) = 0.$$

If n is even, then

$S(n, 0) + S(n, 2) + S(n, 4) + \dots + S(n, n) = S(n, 1) + S(n, 3) + S(n, 5) + \dots + S(n, n-1)$,

for odd n,

$S(n, 0) + S(n, 2) + S(n, 4) + \dots + S(n, n-1) = S(n, 1) + S(n, 3) + S(n, 5) + \dots + S(n, n)$.

1.4 To verify $\mathbf{S(n, r) = S(n-1, r) + nS(n-1, r-1), r < n}$.

Put $r = 1$, then, the left side is

$$S(n, 1) = n(n+1)/2.$$

Right side is

$$S(n-1, 1) + nS(n-1, 0) = (n-1)n/2 + n \cdot 1 = n \cdot (n+1)/2.$$

2.1 Reduction formula (I) for $\mathbf{S(n, r)}$.

Here we use the result of 1.4 repeatedly.

We have,

$$S(n, r) = S(n-1, r) + nS(n-1, r-1),$$

$$S(n-1, r) = S(n-2, r) + (n-1)S(n-2, r-1),$$

$$S(n-2, r) = S(n-3, r) + (n-2)S(n-3, r-1),$$

$$S(n-3, r) = S(n-4, r) + (n-1)S(n-4, r-1),$$

$$\dots$$

$$S(r+1, r) = S(r, r) + (r+1)S(r, r-1).$$

Adding, we get,

$$\begin{aligned}S(n, r) &= S(r, r) + (n)S(n-1, r-1) + (n-1)S(n-2, r-1) + (n-2)S(n-3, r-1) \\ &\quad +\dots + (r+1)S(r, r-1)\end{aligned}$$

Verification:

Put $n = 5,\ r = 2$, the left side is

$$S(5, 2) = 85,$$

right side is

$$S(2, 2) + 5S(4, 1) + 4S(3, 1) + 3S(2, 1) = 2 + 50 + 24 + 9 = 85,$$

2.2 Reduction formula (II) for $\mathbf{S(n, r)}$.

We have,

$$\begin{aligned} & (p + 1)(p + 2)(p + 3) \ldots (p + n)(p + n + 1) \\ = \ & S(n + 1, 0)p^{n+1} + S(n + 1, 1)p^{n} + S(n + 1, 2)p^{n-1} + S(n + 1, 3)p^{n-2} \\ & + \ldots + S(n + 1, r + 1)p^{n-r} + \ldots + S(n + 1, n + 1). \end{aligned} \tag{1}$$

Left side of (1) is

$$\begin{aligned} & (p + 1)\{(p + 1 + 1)(p + 1 + 2)...(p + 1 + n - 1)(p + 1 + n)\} \\ = \ & (p + 1)\{S(n, 0)(p + 1)^{n} + S(n, 1)(p + 1)^{n-1} + \ldots + S(n, r)(p + 1)^{n-r} \ldots + S(n, n)\} \\ = \ & S(n, 0)(p + 1)^{n+1} + S(n, 1)(p + 1)^{n} + \ldots + S(n, r)(p + 1)^{n-r+1} + \ldots + S(n, n). \end{aligned}$$

Expanding each of

$(p + 1)^{n+1}, (p + 1)^{n}, (p + 1)^{n-1}, \ldots, (p + 1)^{n-r+1}$,

by binomial theorem, we get the left side of (1) is

$$\begin{aligned} & S(n, 0)[C(n + 1, 0)p^{n+1} + C(n + 1, 1)p^{n} + \ldots + C(n + 1, r + 1)p^{n-r} \ldots + C(n + 1, n + 1)] \\ & + S(n, 1)\{C(n, 0)p^{n} + C(n, 1)p^{n-1} + \ldots + C(n, r)p^{n-r} \ldots + C(n, n)\} \\ & + S(n, 2)\{C(n - 1, 0)p^{n-1} + \ldots + C(n - 1, r - 1)p^{n-r} \ldots + C(n + 1, n + 1)\} + \ldots \\ & + S(n, r)\{C(n - r + 1, 1)p^{n-r} + \ldots\} \\ & + S(n, r + 1)\{C(n - r, 0)p^{n-r}\} \\ & + \ldots + S(n, n)(p + 1), \end{aligned} \tag{2}$$

where $C(n, r)$ =combinations of n things, taken r at a time $(=^{n} C_{r})$.

Now, comparing the coefficients of p^{n-r} from right side of (1), and that from (2), we get,

$$\begin{aligned} S(n + 1, r + 1) \ = \ & C(n + 1, r + 1)S(n, 0) + C(n, r)S(n, 1) + C(n - 1, r - 1)S(n, 2) \\ & + \ldots + C(n - r + 1, 1)S(n, r) + \ldots + S(n + 1, r). \end{aligned}$$

because $C(n - r, 0) = 1$.

Now,

$$S(n + 1, r + 1) = S(n, r + 1) + (n + 1)S(n, r),$$

from (1.4) above. Hence,

$$\begin{aligned} S(n, r + 1) + (n + 1)S(n, r) \ = \ & C(n + 1, r + 1)S(n, 0) + C(n, r)S(n, 1) + C(n - 1, r - 1) \\ & S(n, 2) + \ldots + C(n - r + 1, 1)S(n, r) + \ldots + S(n + 1, r), \end{aligned}$$

or

$$\begin{aligned}(n+1)S(n,r) - C(n-r+1,1)S(n,r) &= C(n+1,r+1)S(n,0) + C(n,r)S(n,1) + \ldots + \\ &\quad C(n-1,r-1)S(n,2) + \ldots + C(n-r+2,2) \\ &\quad S(n,r-1),\end{aligned}$$

or

$$\begin{aligned}S(n,r)[(n+1) - (n-r+1)] &= C(n+1,r+1)S(n,0) + C(n,r)S(n,1) + \ldots + \\ &\quad C(n-1,r-1)S(n,2) + \ldots + C(n-r+2,2)S(n,r-1),\end{aligned}$$

that is

$$\begin{aligned}r \cdot S(n,r) &= C(n+1,r+1)S(n,0) + C(n,r)S(n,1) + C(n-1,r-1)S(n,2) + \\ &\quad \ldots + C(n-r+2,2)S(n,r-1).\end{aligned}$$

This is the required result.

Verification:

Put $r = 2$,

$$\begin{aligned}S(n,2) &= C(n+1,3) \cdot S(n,0) + C(n,2) \cdot S(n,1) \\ &= (n+1)n(n-1)/6 + n(n-1) \cdot n \cdot (n+1)/2,\end{aligned}$$

or

$$S(n,2) = (n-1) \cdot n \cdot (n+1) \cdot (3n+2)/24.$$

For $n = 5$,

$$S(5,2) = 4 \cdot 5 \cdot 6 \cdot 17/24 = 85,$$

which is true.

For $n = 4$,

$$S(4,2) = 3 \cdot 4 \cdot 5 \cdot 14/24 = 35,$$

which is also true.

2.3 Reduction formula for $\mathbf{S(n,2)}$.

We have,

$$S(n,r) = S(n-1,r) + nS(n-1,r-1), \ r < n.$$

Put $r = 2$,

$$S(n,2) = S(n-1,2) + nS(n-1,1),$$

$$S(n-1,2) = S(n-2,2) + (n-1)S(n-2,1),$$

$$S(n-2,2) = S(n-3,2) + (n-2)S(n-3,1),$$

$$\ldots$$

$$S(3,2) = S(2,2) + 3S(2,1),$$

$$S(2,2) = 0 + 2S(1,1).$$

Now, $S(1,1) = 1$. Hence adding these results,

$$\begin{aligned} S(n,2) &= 2 + \sum_{3}^{n} pS(p-1,1)/2 \\ &= 2 + \sum_{3}^{n} p(p-1)p/2 \\ &= 2 + \sum_{1}^{n} (p^3 - p^2)/2 - (1/2)[(1^3 + 2^3) - (1^2 + 2^2)] \\ &= 2 + n^2(n+1)^2/8 - n(n+1)(2n+1)/12 - (9-5)/2 \\ &= (n-1)n(n+1)(3n+2)/24. \end{aligned}$$

This results is already obtain.

2.4 Similarly,

$$S(n,3) = C(n+1,4)3n(n+1)/64,$$

$$S(n,4) = C(n+1,5)(15n^3 + 15n^2 - 10n + 8)/48,$$

$$S(n,5) = C(n+1,6)(3n^4 + 2n^3 - 7n^2 - 6n)/16.$$

3.1 Definition.

$E(n,r)$ is sum of products of first n even natural numbers, taken r at a time.

$O(n,r)$ is sum of products of first n odd natural numbers, taken r at a time.

We have,

$$\begin{aligned} (p+2)\cdot(p+4)\cdot(p+6) &= p^3 + 12p^2 + 44p + 48 + \dots \\ &= E(3,0)p^3 + E(3,1)p^2 + E(3,2)p + E(3,3), \end{aligned} \tag{3}$$

where,

$E(3,0) = 1$,

$E(3,1) = 2 + 4 + 6 = 12$,

$E(3,2) = 2\cdot 4 + 2\cdot 6 + 4\cdot 6 = 44$,

$E(3,3) = 2\cdot 4\cdot 6 = 48$.

Now, put $p = 1$ in (3),

Left side is

$$3\cdot 5\cdot 7 = O(3,3).$$

Hence,

$$O(3,3) = E(3,0) + E(3,1) + E(3,2) + E(3,3) = 105,$$

$$O(5,5) = E(4,0) + E(4,1) + E(4,2) + E(4,3) + E(4,4) = 945.$$

Similarly, we have,

$$E(3,3) = O(3,0) + O(3,1) + O(3,2) + O(3,3) = 48.$$

Therefore, in general, it can be conjectured that ,

$$O(n+1,n+1) = \sum_{r=0}^{n} E(n,r),$$

and

$$E(n,n) = \sum_{r=0}^{n} O(n,r).$$

3.2 Now we extend the definition of summation for a set of natural numbers, not necessarily beginning from 1.

$S[(p+1,p+n),r]$ is sum of products of n natural numbers, beginning from a natural number $p+1$, taken r at a time.

We state,

$$\begin{aligned} & S[(p+1,p+n),0] + S[(p+1,p+n),1] + S[(p+1,p+n),2] \\ & + \ldots + S[(p+1,p+n),r] + \ldots + S[(p+1,p+n),n] \\ = \ & S[(p+1,p+n+1),r] \ldots . \end{aligned} \tag{4}$$

Verification:

Put $p = n = 3$.

To verify:

$$S[(3,6),0] + S[(3,6),1] + S[(3,6),2] + S[(3,6),3] + S[(3,6),4] = S[(4,7),4].$$

Left side is

$$\begin{aligned} & 1 + (3+4+5+6) + (3\cdot 4 + 3\cdot 5 + 3\cdot 6 + 4\cdot 5 + 4\cdot 6 + 5\cdot 6) \\ & +(3\cdot 4\cdot 5 + 3\cdot 4\cdot 6 + 4\cdot 5\cdot 6 + 4\cdot 5\cdot 6) + (3\cdot 4\cdot 5\cdot 6) \\ = \ & 1 + 18 + 119 + 342 + 360 \\ = \ & 840. \end{aligned}$$

Right side is

$$4\cdot 5\cdot 6\cdot 7 = 840.$$

Hence verified. Hence the result (4) is true.

Similar results are true for odd and even integers,as

$$O[(2,7),0] + O[(2,7),1] + O[(2,7),2] + O[(2,7),3] + O[(2,7),4] = E[(3,8),4],$$

and

$$E[(2,7),0] + E[(2,7),1] + E[(2,7),2] + E[(2,7),3] + E[(2,7),4] = O[(3,8),4].$$

The verification of these results are simple.

3.3 To prove:

1. $S(n-1,1) = nS(n,1) - n$,
2. $S(n-1,2) = S(n,2) - nS(n,1) + n^2$,
3. $S(n-1,3) = S(n,3) - nS(n,2) + n^2S(n,1) - n^3$,

...

4. $S(n-1,r) = S(n,r) - nS(n,r-1) + n^2S(n,r-2) - n^3S(n,r-3) + \dots$

Proof. We have

$$\begin{aligned} & (p+1)(p+2)(p+3)\dots(p+n-1)(p+n) \\ = \; & S(n,0)p^n + S(n,1)p^{n-1} + S(n,2)p^{n-2} + S(n,3)p^{n-3} + \dots, \end{aligned} \tag{5}$$

divide both sides by $(p+n)$,

$$\begin{aligned} & (p+1)(p+2)(p+3)\dots(p+n-1) \\ = \; & [S(n,0)p^n + S(n,1)p^{n-1} + S(n,2)p^{n-2} + S(n,3)p^{n-3} + \dots]/(p+n). \end{aligned} \tag{6}$$

Now using result (5), the left side of (6) is

$$S(n-1,0)p^{n-1} + S(n-1,1)p^{n-2} + S(n-1,2)p^{n-3} + S(n-1,3)p^{n-4} + \dots. \tag{7}$$

By actual division, the right side of (6) is

$$p^{n-1} + p^{n-2}\cdot[S(n,1)-n] + p^{n-3}[S(n,2) - n\cdot S(n,1) + n] + \dots. \tag{8}$$

Equating the coefficients of like powers of p from (7) and (8),
we have,

$S(n-1,1) = n\cdot S(n,1) - n,$

$S(n-1,2) = S(n,2) - n\cdot S(n,1) + n$

$S(n-1,3) = S(n,3) - n\cdot S(n,2) + n^2\cdot S(n,1) - n^3\cdots.$

$S(n-1,r) = S(n,r) - n\cdot S(n,r-1) + n^2S(n,r-2) - n^3\cdot S(n,r-3) + \cdots.$

Verification :

For $n = 5$, and $r = 3$,
right side is

$$S(4,3) = 50,$$

also, left side is

$$S(5,3) - 5\cdot S(5,2) + 52\cdot S(5,1) - 53 = 225 - 5(85) + 25(15) = 50.$$

We have,

$$S(n,r) = S(n-1,r) + n\cdot S(n-1,r-1), r < n.$$

Also

$$S(n-1,r) = S(n-2,r) + (n-1)\cdot S(n-2,r-1).$$

Adding,

$$S(n,r) = S(n-2,r) + (n-1)\cdot S(n-2,r-1) + n\cdot S(n-1,r-1).$$

Again,

$$S(n-1,r-1) = S(n-2,r-1) + n-1\cdot S(n-2,r-2),$$

hence,

$$S(n,r) = S(n-2,r) + n-1\cdot S(n-2,r-1) + n\cdot S(n-2,r-1) + n\cdot(n-1)S(n-2,r-2),$$

$$S(n,r) = S(n-2,r) + (2n-1) \cdot S(n-2,r-1) + n \cdot (n-1) \cdot S(n-2,r-2).$$

Verification:

Put $n = 5$ and $r = 3$, the left side is

$$S(5,3) = 225,$$

right side is

$$S(3,3) + (9) \cdot S(3,2) + 5 \cdot 4 \cdot S(3,1) = 6 + 9.11 + 5.4.6 = 225,$$

hence verified.

4.1 Interestingly, the set of numbers $S(n,r)$, forms a sequence when r is fixed.

For,

$S(2,2) = 2, S(3,2) = 11, S(4,2) = 35, S(5,2) = 85, S(6,2) = 175\ldots$

The numbers

$$\{2, 11, 35, 85, 175, 322, 546, 870, 1320, 1925, 2717, 3731, \ldots, \}$$

are the **Stirling numbers** of first kind.

These numbers are the numbers of edges of a complete k-partite graph of order $S(k,1)$, that is of order $k(k+1)/2$.

The n-th term of this sequence is given by

$$\mathbf{a_n = a_{n-1} + [n(n+1)2]/2, n \geq 2},$$

and $a_1 = 2$.

4.2 A similar sequence for $S(n,3)$ is

$$\{6, 50, 225, 1960, 4536, 9450, 18150, 32670, \ldots\}$$

with similar properties , thus the sequences generated by $S(n,r)$ create additional good results. These are also **Stirling numbers**.

References

[1] S.H.Ramsubramanian, Patterns in product summations, AMTI, Chennai, India, 1991.