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# An inequality of the Smarandache function

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**Abstract** For any positive integer $n$, the famous Smarandache function $S(n)$ is defined as the smallest positive integer $m$ such that $n|m!$. That is, $S(n) = \min\{m : \ m \in N, \ n|m!\}$. In an unpublished paper, Dr. Kenichiro Kashihara asked us to solve the following inequalities

$$S\left(x_1^n\right) + S\left(x_2^n\right) + \cdots\cdots + S\left(x_n^n\right) \geq nS\left(x_1\right) \cdot S\left(x_1\right) \cdots\cdots S\left(x_n\right).$$

In this paper, we using the elementary method to study this problem, and prove that for any integer $n \geq 1$, the inequality has infinite group positive integer solutions $(x_1, \ x_2, \ \cdots, \ x_n)$.

**Keywords** F.Smarandache function, inequalities, solution, necessary condition.

## §1. Introduction and Results

For any positive integer $n$, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer $m$ such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ n \in N\}$. For example, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $\cdots\cdots$. About the elementary properties of $S(n)$, many authors had studied it, and obtained some interesting results, see reference [2], [3], [4] and [5]. For example, Wang Yongxing [3] studied the mean value properties of $S(n)$, and obtained a sharper asymptotic formula about this function:

$$\sum_{n \leq x} S(n) = \frac{\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Lu Yaming [4] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \geq 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1, m_2, \cdots, m_k)$.

Jozsef Sandor [5] proved for any positive integer $k \geq 2$, there exist infinite group positive integers $(m_1, \ m_2, \ \cdots, \ m_k)$ satisfied the following inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite group positive integers $(m_1, m_2, \cdots, m_k)$ such that

$$S(m_1 + m_2 + \cdots + m_k) < S(m_1) + S(m_2) + \cdots + S(m_k).$$

In [6], Fu Jing proved more deeply conclusion, i.e., if the positive integer $k$ and $m$ satisfying one of the following conditions:

(a) $k > 2$ and $m \geq 1$ are all odd numbers.

(b) $k \geq 5$ is odd, $m \geq 2$ is even.

(c) Any even numbers $k \geq 4$ and any positive integer $m$;

then the equation

$$m \cdot S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite group positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

On the other hand, Xu Zhefeng [7] studied the value distribution properties of $S(n)$, and obtained a more interesting result. That is, he proved the following conclusion:

Let $P(n)$ be the largest prime factor of $n$, then for any real numbers $x > 1$, we have the asymptotic formula:

$$\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

In an unpublished paper, Dr. Kenichiro Kashihara asked us to solve the following inequalities

$$S\left(x_1^n\right) + S\left(x_2^n\right) + \cdots\cdots + S\left(x_n^n\right) \geq nS\left(x_1\right) \cdot S\left(x_1\right) \cdots\cdots S\left(x_n\right). \tag{1}$$

About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. The main purpose of this paper is using the elementary methods to study this problem, and prove the following:

**Theorem 1.** For any fixed positive integer $n > 1$, the inequality (1) has infinite group positive integer solutions $(x_1,\ x_2,\ \cdots,\ x_n)$.

**Theorem 2.** For any fixed positive integer $n \geq 3$, if $(x_1,\ x_2,\ \cdots,\ x_n)$ satisfying the inequality (1), then at least $n-1$ of $x_1,\ x_2,\ \cdots,\ x_n$ are 1.

It is clear that the condition $n \geq 3$ in Theorem 2 is necessary. In fact if $n = 2$, we can take $x_1 = x_2 = 2$, then we have the identity

$$S(x_1^2) + S(x_2^2) = S(2^2) + S(2^2) = 4 + 4 = 8 = 2S(2)S(2) = 2S(x_1)S(x_2).$$

So if $n = 2$, then Theorem 2 is not correct.

## §2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. If $n = 1$, then this time, the inequality (1) become $S(x_1) \geq S(x_1)$, and it holds for all positive integers $x_1$. So without lose of generality we can assume that $n \geq 2$. We taking $x_1 = x_2 = \cdots x_{n-1} = 1$, $x_n = p > n$, where $p$ be a prime. Note that $S(1) = 1$, $S(p) = p$ and $S(p^n) = np$, so we have

$$S\left(x_1^n\right) + S\left(x_2^n\right) + \cdots\cdots + S\left(x_n^n\right) = n - 1 + S(p^n) = n - 1 + np \tag{2}$$

and

$$nS\left(x_1\right) \cdot S\left(x_1\right) \cdot \cdots \cdot S\left(x_n\right) = nS(p) = np. \tag{3}$$

From (2) and (3) we may immediately deduce that

$$S\left(x_1^n\right) + S\left(x_2^n\right) + \cdots\cdots + S\left(x_n^n\right) \geq nS\left(x_1\right) \cdot S\left(x_1\right) \cdots\cdots S\left(x_n\right). \tag{4}$$

Since there are infinite primes $p > n$, so all positive integer groups

$$\left(x_1,\ x_2,\ \cdots,\ x_n\right) = (1,\ 1,\ \cdots, p)$$

are the solutions of the inequality (1). Therefore, the inequality (1) has infinite group positive integer solutions $(x_1,\ x_2,\ \cdots,\ x_n)$. This proves Theorem 1.

Now we prove Theorem 2. Let $n \geq 3$, if $(x_1,\ x_2,\ \cdots,\ x_n)$ satisfying the inequality (1), then at least $n-1$ of $x_1$, $x_2$, $\cdots$, $x_n$ are 1. In fact if there exist $x_1 > 1$, $x_2 > 1$, $\cdots$, $x_k > 1$ with $2 \leq k \leq n$ such that the inequality

$$S\left(x_1^n\right) + S\left(x_2^n\right) + \cdots\cdots + S\left(x_n^n\right) \geq nS\left(x_1\right) \cdot S\left(x_1\right) \cdots\cdots S\left(x_n\right). \tag{5}$$

Then from the definition and properties of the function $S(n)$ we have $S(x_i) > 1$ and $S\left(x_i^n\right) \leq nS(x_i)$, $i = 1,\ 2,\ \cdots,\ k$. Note that $a_1 + a_2 + \cdots + a_k < a_1 a_2 \cdots a_k$ if $a_i > 1$ and $k \geq 3$, $i = 1,\ 2,\ \cdots,\ k$; If $k = 2$, then $a_1 + a_2 \leq a_1 a_2$, and the equality holds if and only if $a_1 = a_2 = 2$ $(a_1 > 1,\ a_2 > 1)$. So this time, the inequality (5) become

$$n - k + S\left(x_1^n\right) + S\left(x_2^n\right) + \cdots\cdots + S\left(x_k^n\right) \geq nS(x_1)S(x_2)\cdots S(x_k). \tag{6}$$

If $k \geq 3$, then from (6) and the properties of $S(n)$ we have

$$n - k + n\left[S\left(x_1\right) + S\left(x_2\right) + \cdots\cdots + S\left(x_k\right)\right] \geq nS(x_1)S(x_2)\cdots S(x_k)$$

or

$$\frac{n-k}{n} + S\left(x_1\right) + S\left(x_2\right) + \cdots\cdots + S\left(x_k\right) \geq S(x_1)S(x_2)\cdots S(x_k). \tag{7}$$

Note that $0 \leq \frac{n-k}{n} < 1$, so the inequality (7) is not possible, because

$$S(x_1)S(x_2)\cdots S(x_k) \geq S(x_1) + S(x_2) + \cdots + S(x_k) + 1.$$

If $k = 2$, then the inequality (6) become

$$n - 2 + S\left(x_1^n\right) + S\left(x_2^n\right) \geq nS(x_1)S(x_2). \tag{8}$$

Note that $S(x^n) \leq nS(x)$, $S(x_1) + S(x_2) \leq S(x_1)S(x_2)$ and the equality holds if and only if $x_1 = x_2 = 2$, so if $S(x_1) > 2$ or $S(x_2) > 2$, then (8) is not possible. If $S(x_1) = S(x_2) = 2$, then $x_1 = x_2 = 2$. Therefore, the inequality (8) become

$$S\left(2^n\right) \geq \frac{3n}{2} + 1. \tag{9}$$

Let $S(2^n) = m$, then $m \geq 4$, if $n \geq 3$. From the definition and properties of $S(n)$ we have

$$\sum_{i=1}^{\infty} \left[\frac{m-1}{2^i}\right] < n \leq \sum_{i=1}^{\infty} \left[\frac{m}{2^i}\right].$$

Thus,

$$n \geq 1 + \sum_{i=1}^{\infty} \left[\frac{m-1}{2^i}\right] > \frac{m-1}{2} + \frac{m-1}{4} = \frac{3(m-1)}{4},$$

from (9) we have

$$m = S(2^n) \geq \frac{3n}{2} + 1 \geq \frac{9}{8}(m-1) + 1 = m + \frac{m-1}{8} > m.$$

This inequality is not possible. So if $n \geq 3$ and $(x_1, x_2, \cdots, x_n)$ satisfying the inequality (1), then at least $n-1$ of $x_1, x_2, \cdots, x_n$ are 1. This completes the proof of Theorem 2.

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