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Total Minimal Dominating Signed Graph

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. In this paper, we define the *total minimal dominating signed graph* $M_t(S) = (M_t(G), \sigma)$ of a given signed digraph $S = (G, \sigma)$ and offer a structural characterization of total minimal dominating signed graphs. Further, we characterize signed graphs S for which $S \sim M_t(S)$ and $L(S) \sim M_t(S)$, where $\sim$ denotes switching equivalence and $M_t(S)$ and $L(S)$ are denotes total minimal dominating signed graph and line signed graph of S respectively.

Key Words: Smarandachely k-signed graphs, Smarandachely k-marked graphs, signed graphs, marked graphs, balance, switching, total minimal dominating signed graph, line signed graphs, negation.

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§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [8]. We consider only finite, simple graphs free from self-loops.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. Cartwright and Harary [5] considered graphs in which vertices represent persons and the edges represent symmetric dyadic relations amongst persons each of which designated as being positive or negative according to whether the nature of the

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relationship is positive (friendly, like, etc.) or negative (hostile, dislike, etc.). Such a network S is called a *signed graph* (Chartrand [6]; Harary et al. [11]).

Signed graphs are much studied in literature because of their extensive use in modeling a variety socio-psychological process (e.g., see Katai and Iwai [13], Roberts [15] and Roberts and Xu [16]) and also because of their interesting connections with many classical mathematical systems (Zaslavsky [22]).

A cycle in a signed graph S is said to be *positive* if the product of signs of its edges is positive. A cycle which is not positive is said to be *negative*. A signed graph is then said to be *balanced* if every cycle in it is positive (Harary [9]). Harary and Kabell [22] developed a simple algorithm to detect balance in signed graphs as also enumerated them.

A *marking* of S is a function $\mu : V(G) \to \{+,-\}$; A signed graph S together with a marking μ is denoted by S_μ. Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S.

The following characterization of balanced signed graphs is well known.

Theorem 1(E. Sampathkumar [17]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking μ of a signed graph S. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [20]) or *cycle isomorphic* (see [21]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [21]):

Theorem(T. Zaslavsky [21]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Total Minimal Dominating Signed Graph

The total minimal dominating graph $M_t(G)$ of a graph G is the intersection graph on the family of all total minimal dominating sets of vertices in G. This concept was introduced by Kulli and Iyer [14].

We now extend the notion of $M_t(G)$ to the realm of signed graphs. The *total minimal dominating signed graph* $M_t(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $M_t(G)$ and sign of any edge uv is $M_t(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called total minimal dominating signed graph, if $S \cong M_t(S')$ for some signed graph S'. The following result restricts the class of total minimal dominating signed graphs.

Theorem 3 *For any signed graph $S = (G, \sigma)$, its total minimal dominating signed graph $M_t(S)$ is balanced.*

Proof Since sign of any edge uv is $M_t(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Theorem 1, $M_t(S)$ is balanced. □

For any positive integer k, the k^{th} iterated total minimal dominating signed graph, $M_t^k(S)$ of S is defined as follows:

$$M_t^0(S) = S,\ M_t^k(S) = M_t(M_t^{k-1}(S))$$

Corollary 4 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $M_t^k(S)$ is balanced.*

The following result characterizes signed graphs which are total minimal dominating signed graphs.

Theorem 5 *A signed graph $S = (G, \sigma)$ is a total minimal dominating signed graph if, and only if, S is balanced signed graph and its underlying digraph G is a total minimal dominating graph.*

Proof Suppose that S is total minimal dominating signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $M_t(S') \cong S$. Hence by definition $M_t(G) \cong G'$ and by Theorem 3, S is balanced.

Conversely, suppose that $S = (G, \sigma)$ is balanced and G is total minimal dominating graph. That is there exists a graph G' such that $M_t(G') \cong G$. Since S is balanced by Theorem 1, there exists a marking μ of vertices of S such that for any edge $uv \in G$, $\sigma(uv) = \mu(u)\mu(v)$. Also since $G \cong M_t(G')$, vertices in G are in one-to-one correspondence with the edges of G'. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e' in G' to be the marking on the corresponding vertex in G. Then clearly $M_t(S') \cong S$ and so S is total minimal dominating graph. □

In [3], the authors proved the following for a graph G its total minimal dominating graph $M_t(G)$ is isomorphic to G then G is either C_3 or C_4. Analogously we have the following.

Theorem 6 *For any signed graph $S = (G, \sigma)$, $S \sim M_t(S)$ if, and only if, G is isomorphic to either C_3 or C_4 and S is balanced.*

Proof Suppose $S \sim M_t(S)$. This implies, $G \cong M_t(G)$ and hence by the above observation we see that the graph G must be isomorphic to either C_3 or C_4. Now, if S is any signed graph on any one of these graphs, Theorem 3 implies that $M_t(S)$ is balanced and hence if S is unbalanced its $M_t(S)$ being balanced cannot be switching equivalent to S in accordance with Theorem 2. Therefore, S must be balanced.

Conversely, suppose that S is balanced signed graph on C_3 or C_4. Then, since $M_t(S)$ is balanced as per Theorem 3, the result follows from Theorem 2 again. $\square$

Behzad and Chartrand [4] introduced the notion of line signed graph $L(S)$ of a given signed graph S as follows: Given a signed graph $S = (G, \sigma)$ its *line signed graph* $L(S) = (L(G), \sigma')$ is the signed graph whose underlying graph is $L(G)$, the line graph of G, where for any edge e_ie_j in $L(S)$, $\sigma'(e_ie_j)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [7], is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [7] (See also E. Sampathkumar et al. [18,19]).

Theorem 7(M. Acharya [2]) *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

We now characterize signed graphs whose total minimal dominating signed graphs and its line signed graphs are switching equivalent. In the case of graphs the following result is due to Kulli and Iyer [14].

Theorem 8(Kulli and Iyer [14]) *If G is a $(p-2)$-regular graph then, $M_t(G) \cong L(G)$.*

Theorem 9 *For any signed graph $S = (G, \sigma)$, $M_t(S) \sim L(S)$, if, and only if, G is $(p-2)$-regular.*

Proof Suppose $M_t(S) \sim L(S)$. This implies, $M_t(G) \cong L(G)$ and hence by Theorem 8, we see that the graph G must be $(p-2)$-regular.

Conversely, suppose that G is $(p-2)$-regular. Then $M_t(G) \cong L(G)$ by Theorem 8. Now if S is signed graph with $(p-2)$-regular, then by Theorem 3 and Theorem 7, $M_t(S)$ and $L(S)$ are balanced and hence, the result follows from Theorem 2. $\square$

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [10] as follows:

$\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

Theorem 6 provides easy solutions to two other signed graph switching equivalence relations, which are given in the following results.

Corollary 10 *For any signed graph $S = (G, \sigma)$, $M_t(\eta(S)) \sim M_t(S)$.*

Corollary 11 *For any signed graph $S = (G, \sigma)$, $\eta(S) \sim M_t(S)$ if, and only if, S is an unbalanced signed graph and $G = C_3$.*

For a signed graph $S = (G, \sigma)$, the $M_t(S)$ is balanced (Theorem 3). We now examine, the conditions under which negation $\eta(S)$ of $M_t(S)$ is balanced.

Corollary 12 *Let $S = (G, \sigma)$ be a signed graph. If $M_t(G)$ is bipartite then $\eta(M_t(S))$ is balanced.*

Proof Since, by Theorem 3 $M_t(S)$ is balanced, if each cycle C in $M_t(S)$ contains even number of negative edges. Also, since $M_t(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $M_t(S)$ is also even. Hence $\eta(M_t(S))$ is balanced.□

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