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The function equation $S(n) = Z(n)$ [1]

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Abstract For any positive integer n, let $S(n)$ and $Z(n)$ denote the Smarandache function and the pseudo Smarandache function respectively. In this paper we prove that the equation $S(n) = Z(n)$ has infinitely many positive integer solutions n.

Keywords Smarandache function; Pseudo Smarandache function; Diophantine equation.

For any positive integers n, let $S(n)$ and $Z(n)$ denote the Smarandache function and pseudo Smarandache function respectively. In [1], Ashbacher proposed two problems concerning the equation

$$S(n) = Z(n) \tag{1}$$

as follows.

Problem 1. Prove that if n is an even perfect number, then n satisfies (1).

Problem 2. Prove that (1) has infinitely many positive integer solutions n.

In this paper we completely solve these problems as follows.

Theorem 1. If n is an even perfect number, then (1) holds.

Theorem 2. (1) has infinitely many positive integer solutions n.

Proof of Theorem 1. By [2, Theorem 277], if n is an even perfect number, then

$$n = 2^{p-1}(2^p - 1), \tag{2}$$

where p is a prime. By [3] and [4], we have

$$S(n) = 2^p - 1. \tag{3}$$

On the other hand, since

$$\frac{1}{2}\left(2^p - 1\right)\left(\left(2^p - 1\right) + 1\right) = n, \tag{4}$$

by (2), we get

$$Z(n) = 2^p - 1 \tag{5}$$

immediately. The combination of (3) and (5) yields (1). Thus, the theorem is proved.

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Proof of Theorem 2. Let p be an odd prime with $p \equiv 3(\mod 4)$. Since $S(2) = 2$ and $S(p) = p$, we have

$$S(2p) = \max(S(2), S(p)) = \max(2, p) = p. \tag{6}$$

Let $t = Z(2p)$, By the define of $Z(n)$, we have

$$\frac{1}{2}t(t+1) \equiv 0(\mathrm{mod} 2p). \tag{7}$$

It implies that either $t \equiv 0(\mathrm{mod} p)$ or $t + 1 \equiv 0(\mathrm{mod} p)$. Hence, we get $t \geq p - 1$. If $t = p - 1$, then from (7) we obtain

$$\frac{1}{2}(p-1)p \equiv 0(\mathrm{mod} 2p). \tag{8}$$

whence we get

$$\frac{1}{2}(p-1)p \equiv 0(\mathrm{mod} 2). \tag{9}$$

But, since $p \equiv 3(\mathrm{mod} 4)$, (9) is impossible. So we have

$$t \geq p. \tag{10}$$

Since $p + 1 \equiv 0(\mathrm{mod} 4)$, we get

$$\frac{1}{2}p(p+1) \equiv 0(\mathrm{mod} 2p) \tag{11}$$

and $t = p$ by (10). Therefore, by (6), $n = 2p$ is a solution of (1). Notice that there exist infinitely many primes p with $p \equiv 3(\mathrm{mod} 4)$. It implies that (1) has infinitely many positive integer solutions n. The theorem is proved.

References

[1] C.Ashbacher, Problems, Smrandache Notions J. **9**(1998), 141-151.

[2] G.H.Hardy and E.M.Wright, An introduction to the theory of numbers, Oxford University Press, Oxford, 1938.