ON CERTAIN INEQUALITIES INVOLVING THE SMARANDACHE FUNCTION

by

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1. The Smarandache function satisfies certain elementary inequalities which have importance in the deduction of properties of this (or related) functions. We quote here the following relations which have appeared in the Smarandache Function Journal:

Let p be a prime number. Then

$$S(p^x) \le S(p^y) \qquad \text{for } x \le y \tag{1}$$

$$\frac{S(p^a)}{p^a} \ge \frac{S(p^{a+1})}{p^{a+1}} \qquad \text{for } a \ge 0 \tag{2}$$

where x, y, a are nonnegative integers;

$$S(p^a) \le S(q^a) \qquad \text{for } p \le q \text{ primes;} \tag{3}$$

$$(p-1)a + 1 \le S(p^a) \le pa; \tag{4}$$

If $p > \frac{a}{2}$ and $p \le a-1$ $(a \ge 2)$, then

$$S(p^a) \le p(a-1) \tag{5}$$

For inequalities (3), (4), (5), see [2], and for (1), (2), see [1].

We have also the result ([4]):

$$\text{For composite } n \ne 4, \quad \frac{S(n)}{n} \le \frac{2}{3} \tag{6}$$

Clearly, $1 \le S(n)$ for $n \ge 1$ and $1 < S(n)$ for $n \ge 2$ (7)

and $S(n) \le n$ (8)

which follow easily from tfe definition $S(n) = \min\{k \in N^* : n \text{ divides } k!\}$

2. Inequality (2), written in the form $S(p^{a+1}) \le pS(p^a)$, gives by successive application $S(p^{a+2}) \le pS(p^{a+1}) \le p^2 S(p^a)$, ..., that is

$$S(p^{a+c}) \le p^c \cdot S(p^a) \tag{9}$$

where a and c are natural numbers (For c = 0 there is equality, and for a = 0 this follows by (8)).

Relation (9) suggest the following result:

Theorem 1.

For all positive integers m and n holds true the inequality

$$S(mn) \le m \cdot S(n) \tag{10}$$

Proof.

For a general proof, suppose that m and n have a canonical factorization

$$m = p_1^{a_1} \ldots p_r^{a_r} \cdot q_1^{b_1} \ldots q_s^{b_s}, \quad n = p_1^{c_1} \ldots p_r^{c_r} \cdot t_1^{d_1} \ldots t_k^{d_k},$$

where $p_i\,(i = \overline{1,r})$, $q_j\,(j = \overline{1,s})$, $t_p\,(p = \overline{1,k})$ are distinct primes and $a_i \ge 0$, $c_j \ge 0$, $b_j \ge 1$, $d_p \ge 1$ are integers.

By a well known result of Smarandache (see [3]) we can write

$$\begin{aligned} S(m \cdot n) &= \max\{S(p_1^{a_1+c_1}), \ldots, S(p_r^{a_r+c_r}), S(q_1^{b_1}), \ldots, S(q_s^{b_s}), S(t_1^{d_1}), \ldots, S(t_k^{d_k})\} \\ &\le \max\{p_1^{a_1} S(p_1^{c_1}), \ldots, p_r^{a_r} S(p_r^{c_r}), S(q_1^{b_1}), \ldots, S(q_s^{b_s}), \ldots, S(t_k^{d_k})\} \end{aligned}$$

by (9). Now, a simple remark and inequality (8) easily give

$$S(m \cdot n) \le p_1^{a_1} \ldots p_r^{a_r} q_1^{b_1} \ldots q_s^{b_s} \cdot \max\{S(p_1^{c_1}), \ldots, S(p_r^{c_r}), S(t_1^{d_1}), \ldots, S(t_k^{d_k})\} = mS(n)$$

proving relation (10).

Remark.

For (m,n)=1, inequality (10) appears as

$$\max\{S(m), S(n)\} \le mS(n)$$

This can be proved more generally, for all m and n

Theorem 2.

For all m, n we have:

$$\max\{S(m), S(n)\} \le mS(n) \tag{11}$$

Proof.

The proof is very simple. Indeed, if $S(m) \ge S(n)$, then $S(m) \le mS(n)$ holds, since $S(n) \ge 1$ and $S(m) \le m$, see (7), (8). For $S(m) \le S(n)$ we have $S(n) \le mS(n)$ which is t me by $m \ge 1$. In all cases, relation (11) follows.

This proof has an independent interest. As we shall see, Theorem 2 will follow also from Theorem 1 and the following result:

Theorem 3.

For all m, n we have

$$S(mn) \geq \max \{S(m), S(n)\} \tag{12}$$

Proof.

Inequality (1) implies that $S(p^a) \leq S(p^{a+c})$, $S(p^c) \leq S(p^{a+c})$, so by using the representations of m and n, as in the proof of Theorem 1, by Smarandache's theorem and the above inequalities we get relation (12).

We note that, equality holds in (12) only when all $a_i = 0$ or all $c_i = 0$ $(i = \overline{1, r})$, i.e. when m and n are coprime.

3. As an application of (10), we get:

Corollary 1.

a) $\frac{S(a)}{a} \leq \frac{S(b)}{b}$, if $b \mid a$ (13)

b) If a has a composite divisor $b \neq 4$, then

$$\frac{S(a)}{a} \leq \frac{S(b)}{b} \leq \frac{2}{3} \tag{14}$$

Proof.

Let $a = b \cdot k$. Then $\frac{S(bk)}{bk} \leq \frac{S(b)}{b}$ is equivalent with $S(kb) \leq kS(b)$, which is relation (10) for m=k, n=b.

Relation (14) is a consequence of (13) and (6). We note that (14) offers an improvement of inequality (6).

We now prove:

Corollary 2.

Let m, n, r, s be positive integers. Then:

$$S(mn) + S(rs) \geq \max \{ S(m) + S(r), S(n) + S(s) \} \tag{15}$$

Proof.

We apply the known relation:

$$\max \{ a + c, b + d \} \leq \max \{ a, b \} + \max \{ c, d \} \tag{16}$$

By Theorem 3 we can write $S(mn) \geq \max \{S(m), S(n)\}$ and $S(rs) \geq \max \{S(r), S(s)\}$, so by consideration of (16) with

$$a \equiv S(m),\ b \equiv S(r),\ c \equiv S(n),\ d \equiv S(s)$$

we get the desired result.

Remark.

Since (16) can be generalized to n numbers $(n \geq 2)$, and also Theorem 1-3 do hard for the general case (which follow by induction; however these results are based essentially on (10) - (15), we can obtain extensions of these theorems to n numbers.

Corollary 3.

Let a, b composite numbers, $a \neq 4$, $b \neq 4$. Then

$$\frac{S(ab)}{ab} \leq \frac{S(a)+S(b)}{a+b} \leq \frac{2}{3};$$

and

$$S^2(ab) \leq ab[S^2(a) + S^2(b)]$$

where $S^2(a) = (S(a))^2$, etc.

Proof.

By (10) we have $S(a) \geq \frac{S(ab)}{b}$, $S(b) \geq \frac{S(ab)}{a}$, so by addition

$S(a) + S(b) \geq S(ab)\left(\frac{1}{a} + \frac{1}{b}\right)$, giving the first part of (16).

For the second, we have by (6):

$S(a) \leq \frac{2}{3}a$, $S(b) \leq \frac{2}{3}b$, so $S(a)+S(b) \leq \frac{2}{3}(a+b)$, yielding the second part of (16).

For the proof of (17), remark that by $2(n^2 + r^2) \geq (n + r)^2$, the first past of (16), as well as the inequality $2ab \leq (a + b)^2$ we can write successively:

$$S^2(ab) \leq \frac{a^2b^2}{(a+b)^2} \cdot [S(a)+S(b)]^2 \leq \frac{2a^2b^2}{(a+b)^2} \cdot [S^2(a)+S^2(b)] \leq ab[S^2(a)+S^2(b)]$$

References

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