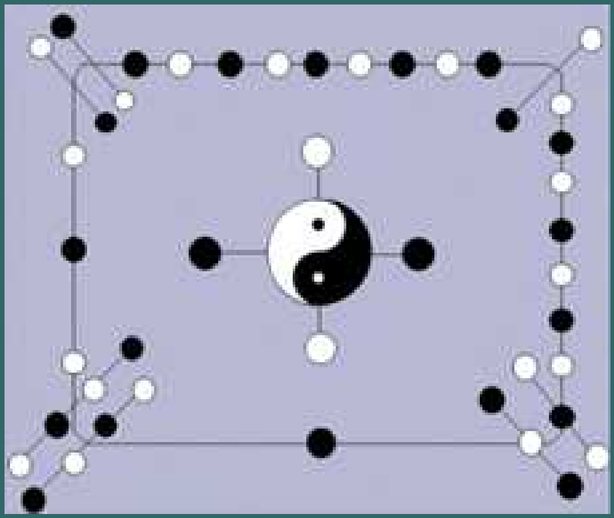
ISSN 1937 - 1055

VOLUME 2, 2013

INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

BEIJING UNIVERSITY OF CIVIL ENGINEERING AND ARCHITECTURE

June, 2013

Vol.2, 2013 ISSN 1937-1055

International Journal of

Mathematical Combinatorics

Edited By

The Madis of Chinese Academy of Sciences and

Beijing University of Civil Engineering and Architecture

June, 2013

Aims and Scope: The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

Smarandache multi-spaces with applications to other sciences, such as those of algebraic multi-systems, multi-metric spaces,$\cdots$, etc.. Smarandache geometries;

Differential Geometry; Geometry on manifolds;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Applications of Smarandache multi-spaces to theoretical physics; Applications of Combinatorics to mathematics and theoretical physics;

Mathematical theory on gravitational fields; Mathematical theory on parallel universes;

Other applications of Smarandache multi-space and combinatorics.

Generally, papers on mathematics with its applications not including in above topics are also welcome.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews(USA), Zentralblatt fur Mathematik(Germany), Referativnyi Zhurnal (Russia), Mathematika (Russia), Computing Review (USA), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA).

Subscription A subscription can be ordered by an email to *j.mathematicalcombinatorics@gmail.com* or directly to

Linfan Mao
The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science
Beijing, 100190, P.R.China
Email: maolinfan@163.com

Price: US$48.00

Editorial Board (3nd)

Famous Words:

I want to bring out the secrets of nature and apply them for the happiness of man. I don't know of any better service to offer for the short time we are in the world.

By Thomas Edison, an American inventor.

International J.Math. Combin. Vol.2(2013), 01-07

S-Denying a Theory

Florentin Smarandache

(Department of Mathematics, University of New Mexico - Gallup, USA)

E-mail: fsmarandache@gmail.com

Abstract: In this paper we introduce the operators of validation and invalidation of a proposition, and we extend the operator of S-denying a proposition, or an axiomatic system, from the geometric space to respectively any theory in any domain of knowledge, and show six examples in geometry, in mathematical analysis, and in topology.

Key Words: operator of S-denying, axiomatic system

AMS(2010): 51M15, 53B15, 53B40, 57N16

§1. Introduction

Let T be a theory in any domain of knowledge, endowed with an ensemble of sentences E, on a given space M.

E can be for example an axiomatic system of this theory, or a set of primary propositions of this theory, or all valid logical formulas of this theory, etc. E should be closed under the logical implications, i.e. given any subset of propositions $P_1, P_2, \cdots$ in this theory, if Q is a logical consequence of them then Q must also belong to this theory.

A sentence is a logic formula whose each variable is quantified i.e. inside the scope of a quantifier such as: $\exists$ (exist), $\forall$ (for all), modal logic quantifiers, and other various modern logics' quantifiers. With respect to this theory, let P be a proposition, or a sentence, or an axiom, or a theorem, or a lemma, or a logical formula, or a statement, etc. of E. It is said that P is S-denied on the space M if P is valid for some elements of M and invalid for other elements of M, or P is only invalid on M but in at least two different ways.

An ensemble of sentences E is considered S-denied if at least one of its propositions is S-denied. And a theory T is S-denied if its ensemble of sentences is S-denied, which is equivalent to at least one of its propositions being S-denied.

The proposition P is partially or totally denied/negated on M. The proposition P can be simultaneously validated in one way and invalidated in (finitely or infinitely) many different ways on the same space M, or only invalidated in (finitely or infinitely) many different ways.

[1]Reported at the First International Conference on Smarandache Multispaces and Multistructures, June 28-30,2013, Beijing, P.R.China.

[2]Received March 27,2013, Accepted June 5, 2013.

[3]The multispace operator S-denied (Smarandachely-denied) has been inherited from the previously published scientific literature (see for example Ref. [1] and [2]).

The invalidation can be done in many different ways. For example the statement $A =:$ $x \neq 5$ can be invalidated as $x = 5$ (total negation), but $x \in \{5, 6\}$ (partial negation). (Use a notation for S-denying, for invalidating in a way, for invalidating in another way a different notation; consider it as an operator: neutrosophic operator? A notation for invalidation as well.)

But the statement $B =:$ $x > 3$ can be invalidated in many ways, such as $x \leq 3$, or $x = 3$, or $x < 3$, or $x = -7$, or $x = 2$, etc. A negation is an invalidation, but not reciprocally - since an invalidation signifies a (partial or total) degree of negation, so invalidation may not necessarily be a complete negation. The negation of B is $B =:$ $x \leq 3$, while $x = -7$ is a partial negation (therefore an invalidation) of B.

Also, the statement $C =:$ *John's car is blue and Steve's car is red* can be invalidated in many ways, as: *John's car is yellow and Steve's car is red*, or *John's car is blue and Steve's car is black*, or *John's car is white and Steve's car is orange*, or *John's car is not blue and Steve's car is not red*, or *John's car is not blue and Steve's car is red*, etc.

Therefore, we can S-deny a theory in finitely or infinitely many ways, giving birth to many partially or totally denied versions/deviations/alternatives theories: $T_1, T_2, \cdots$. These new theories represent degrees of negations of the original theory T.

Some of them could be useful in future development of sciences.

Why do we study such S-denying operator? Because our reality is heterogeneous, composed of a multitude of spaces, each space with different structures. Therefore, in one space a statement may be valid, in another space it may be invalid, and invalidation can be done in various ways. Or a proposition may be false in one space and true in another space or we may have a degree of truth and a degree of falsehood and a degree of indeterminacy. Yet, we live in this mosaic of distinct (even opposite structured) spaces put together.

S-denying involved the creation of the *multi-space* in geometry and of the *S-geometries* (1969). It was spelt *multi-space*, or *multispace*, of *S-multispace*, or *mu-space*, and similarly for its: *multi-structure*, or *multistructure*, or *S-multistructure*, or *mu-structure*.

§2. Notations

Let $< A >$ be a statement (or proposition, axiom, theorem, etc.).

a) For the classical Boolean *logic negation* we use the same notation. The negation of $< A >$ is noted by $\neg A$ and $\neg A =< nonA >$. An invalidation of $< A >$ is noted by $i(A)$, while a validation of $< A >$ is noted by $v(A)$:

$$i(A) \subset 2^{<nonA>} \backslash \{\emptyset\} \text{ and } v(A) \subset 2^{<A>} \backslash \{\emptyset\},$$

where 2^X means the power-set of X, or all subsets of X.

All possible invalidations of $< A >$ form a set of invalidations, notated by $I(A)$. Similarly for all possible validations of $< A >$ that form a set of validations, and noted by $V(A)$.

b) S-denying of $< A >$ is noted by $S_\neg(A)$. S-denying of $< A >$ means some validations of $< A >$ together with some invalidations of $< A >$ in the same space, or only invalidations of

$< A >$ in the same space but in many ways. Therefore, $S_\neg(A) \subset V(A) \bigcup I(A)$ or $S_\neg(A) \subset I(A)^k$ for $k \geq 2$.

§3. Examples

Let's see some models of S-denying, three in a geometrical space, and other three in mathematical analysis (calculus) and topology.

3.1 The first S-denying model was constructed in 1969. This section is a compilation of ideas from paper [1]:

An axiom is said Smarandachely denied if the axiom behaves in at least two different ways within the same space (i.e., validated and invalided, or only invalidated but in multiple distinct ways). A Smarandache Geometry [SG] is a geometry which has at least one Smarandachely denied axiom.

Let's note any point, line, plane, space, triangle, etc. in such geometry by s-point, s-line, s-plane, s-space, s-triangle respectively in order to distinguish them from other geometries. *Why these hybrid geometries*? Because in reality there does not exist isolated homogeneous spaces, but a mixture of them, interconnected, and each having a different structure. These geometries are becoming very important now since they combine many spaces into one, because our world is not formed by perfect homogeneous spaces as in pure mathematics, but by non-homogeneous spaces. Also, SG introduce the degree of negation in geometry for the first time (for example an axiom is denied 40% and accepted 60% of the space) that's why they can become revolutionary in science and it thanks to the idea of partial denying/accepting of axioms/propositions in a space (making multi-spaces, i.e. a space formed by combination of many different other spaces), as in fuzzy logic the degree of truth (40% false and 60% true). They are starting to have applications in physics and engineering because of dealing with non-homogeneous spaces.

The first model of S-denying and of SG was the following:

The axiom that through a point exterior to a given line there is only one parallel passing through it (Euclid's Fifth Postulate), was S-denied by having in the same space: no parallel, one parallel only, and many parallels.

In the Euclidean geometry, also called parabolic geometry, the fifth Euclidean postulate that there is only one parallel to a given line passing through an exterior point, is kept or validated. In the Lobachevsky-Bolyai-Gauss geometry, called hyperbolic geometry, this fifth Euclidean postulate is invalidated in the following way: there are infinitely many lines parallels to a given line passing through an exterior point.

While in the Riemannian geometry, called elliptic geometry, the fifth Euclidean postulate is also invalidated as follows: there is no parallel to a given line passing through an exterior point. Thus, as a particular case, Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries may be united altogether, in the same space, by some SG's. These last geometries can be partially Euclidean and partially Non-Euclidean simultaneously.

3.2 Geometric Model

Suppose we have a rectangle ABCD. See Fig.1 below.

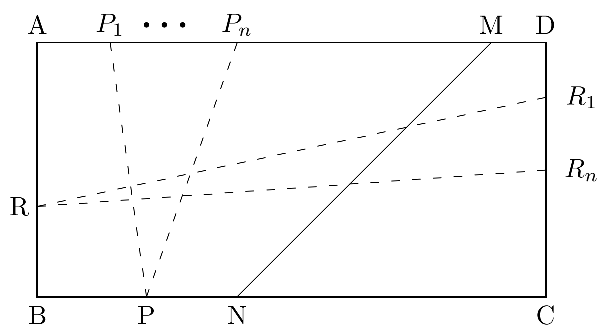


Fig.1

In this model we define as:

Point = any point inside or on the sides of this rectangle;

Line = a segment of line that connects two points of opposite sides of the rectangle;

Parallel lines = lines that do not have any common point (do not intersect);

Concurrent lines = lines that have a common point.

Let's take the line MN, where M lies on side AD and N on side BC as in the above Fig. 1. Let P be a point on side BC, and R a point on side AB.

Through P there are passing infinitely many parallels ($PP_1, \cdots, PP_n, \cdots$) to the line MN, but through R there is no parallel to the line MN (the lines $RR_1, \cdots, RR_n$ cut line MN). Therefore, the Fifth Postulate of Euclid (that though a point exterior to a line, in a given plane, there is only one parallel to that line) in S-denied on the space of the rectangle ABCD since it is invalidated in two distinct ways.

3.3 Another Geometric Model

We change a little the Geometric Model 1 such that:

The rectangle ABCD is such that side AB is smaller than side BC. And we define as line the arc of circle inside (and on the borders) of ABCD, centered in the rectangle's vertices A, B, C, or D.

The axiom that: through two distinct points there exist only one line that passes through is S-denied (in three different ways):

a) Through the points A and B there is no passing line in this model, since there is no arc of circle centered in A, B, C, or D that passes through both points. See Fig.2.

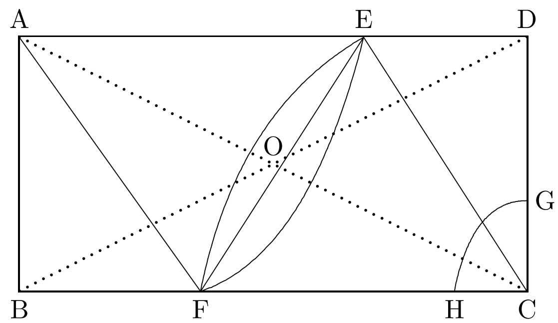


Fig.2

b) We construct the perpendicular EF⊥AC that passes through the point of intersection of the diagonals AC and BD. Through the points E and F there are two distinct lines the dark green (left side) arc of circle centered in C since CE≡FC, and the light green (right side) arc of circle centered in A since AE≡AF. And because the right triangles $\bigsqcup$COE, $\bigsqcup$COF, $\bigsqcup$AOE, and $\bigsqcup$AOF are all four congruent, we get CE≡FC≡AE≡AF.

c) Through the points G and H such that CG≡CH (their lengths are equal) there is only one passing line (the dark green arc of circle GH, centered in C) since AG≠AH (their lengths are different), and similarly BG≠BH and DG≠DH.

3.4 Example for the Axiom of Separation

The Axiom of Separation of Hausdorff is the following:

$$\forall x, y \in M, \exists N(x), N(y) \Rightarrow N(x) \bigcap N(y) = \emptyset,$$

where $N(x)$ is a neighborhood of x, and respectively $N(y)$ is a neighborhood of y.

We can S-deny this axiom on a space M in the following way:

a) $\exists x_1, y_1 \in M$ and $\exists N_1(x_1), N_1(y_1) \Rightarrow N_1(x_1) \bigcap N_1(y_1) = \emptyset$, where $N_1(x_1)$ is a neighborhood of x_1, and respectively $N_1(y_1)$ is a neighborhood of y_1. [validated]

b) $\exists x_2, y_2 \in M \Rightarrow \forall N_2(x_2), N_2(y_2), N_2(x_2) \bigcap N_2(y_2) = \emptyset$, where $N_2(x_2)$ is a neighborhood of x_2, and respectively $N_2(y_2)$ is a neighborhood of y_2. [invalidated]

Therefore we have two categories of points in M: some points that verify The Axiom of Separation of Hausdorff and other points that do not verify it. So M becomes a partially separable and partially inseparable space, or we can see that M has some degrees of separation.

3.5 Example for the Norm

If we remove one or more axioms (or properties) from the definition of a notion $< A >$ we get a pseudo-notion $< pseudoA >$. For example, if we remove the third axiom (inequality of the triangle) from the definition of the $< norm >$ we get a $< pseudonorm >$. The axioms of a norm on a real or complex vectorial space V over a field F, $x \rightarrow \| \cdot \|$, are the following:

a) $\|x\| = 0 \Leftrightarrow x = 0$;

b) $\forall x \in V, \forall \alpha \in F$, $\|\alpha x\| = |\alpha|\|x\|$;

c) $\forall x, y \in V$, $\|x + y\| \leq \|x\| \cdot \|y\|$ (inequality of the triangle).

For example, a pseudo-norm on a real or complex vectorial space V over a field F, $x \rightarrow {}_p\|\cdot\|$, may verify only the first two above axioms of the norm.

A pseudo-norm is a particular case of an S-denied norm since we may have vectorial spaces over some given scalar fields where there are some vectors and scalars that satisfy the third axiom [validation], but others that do not satisfy [invalidation]; or for all vectors and scalars we may have either $\|x + y\| = 5\|x\| \cdot \|y\|$ or $\|x + y| = 6\|x\| \cdot \|y\|$, so invalidation (since we get $\|x + y\| > \|x\| \cdot \|y\|$) in two different ways.

Let's consider the complex vectorial space $\mathscr{C} = \{a + bi, \text{ where } a, b \in R, i = \sqrt{-1}\}$ over the field of real numbers R. If $z = a + bi \in \mathscr{C}$ then its pseudo-norm is $\|z\| = \sqrt{a^2 + b^2}$. This verifies the first two axioms of the norm, but do not satisfy the third axiom of the norm since:

For $x = 0 + bi$ and $y = a + 0i$ we get $\|x + y\| = \|a + bi\| = \sqrt{a^2 + b^2} \leq \|x\| \cdot \|y\| = \|0 + bi\| \cdot \|a + 0i\| = |ab|$, or $a^2 + b^2 \leq a^2b^2$. But this is true for example when $a = b \geq \sqrt{2}$ (validation), and false if one of a or b is zero and the other is strictly positive (invalidation).

Pseudo-norms are already in use in today's scientific research, because for some applications the norms are considered too restrictive. Similarly one can define a pseudo-manifold (relaxing some properties of the manifold), etc.

3.6 Example in Topology

A topology $\mathscr{O}$ on a given set E is the ensemble of all parts of E verifying the following properties:

a) E and the empty set $\emptyset$ belong to $\mathscr{O}$;

b) Intersection of any two elements of $\mathscr{O}$ belongs to $\mathscr{O}$ too;

c) Union of any family of elements of $\mathscr{O}$ belongs to $\mathscr{O}$ too.

Let's go backwards. Suppose we have a topology $\mathscr{O}_1$ on a given set E_1, and the second or third (or both) previous axioms have been S-denied, resulting an S-denied topology $S\neg(\mathscr{O}_1)$ on the given set E_1.

In general, we can go back and *recover* (*reconstruct*) the original topology $\mathscr{O}_1$ from $S\neg(\mathscr{O}_1)$ by recurrence: if two elements belong to $S\neg(\mathscr{O}_1)$ then we set these elements and their intersection to belong to $\mathscr{O}_1$, and if a family of elements belong to $S\neg(\mathscr{O}_1)$ then we set these family elements and their union to belong to $\mathscr{O}_1$; and so on: we continue this recurrent process until it does not bring any new element to $\mathscr{O}_1$.

§4. **Conclusion**

Decidability changes in an S-denied theory, i.e. a defined sentence in an S-denied theory can be partially deducible and partially undeducible (we talk about degrees of deducibility of a sentence in an S-denied theory).

Since in classical deducible research, a theory T of language L is said complete if any sentence of L is decidable in T, we can say that an S-denied theory is partially complete (or has some degrees of completeness and degrees of incompleteness).

References

[1] Kuciuk L. and Antholy M., An Introduction to Smarandache Geometries, *Mathematics Magazine*, Aurora, Canada, Vol.12, 2003.

[2] Mao Linfan, An introduction to Smarandache geometries on maps, *2005 International Conference on Graph Theory and Combinatorics*, Zhejiang Normal University, Jinhua, Zhejiang, P. R. China, June 25-30, 2005. also appear in *Smarandache geometries & map theory with applications*(I), Chinese Branch Xiquan House, 2007.

[3] Dmitri Rabounski, Smarandache spaces as a new extension of the basic space-time of general relativity, *Progress in Physics*, Vol.2, L1-L2, 2010.

[4] F.Smarandache, *Paradoxist Geometry*, State Archives from Valcea, Rm. Valcea, Romania, 1969, and in *Paradoxist Mathematics*, Collected Papers (Vol. II), Kishinev University Press, Kishinev, 5-28, 1997.

[5] F.Smarandache, Multi-space and multi-structure, in *Neutrosophy. Neutrosophic Logic, Set, Probability and Statistics*, American Research Press, 1998.

[6] Matthew Szudzik, *Sentence*, From MathWorld – A Wolfram Web Resource, created by Eric W. Weisstein, http://mathworld.wolfram.com/Sentence.html.

International J.Math. Combin. Vol.2(2013), 08-23

Non-Solvable Equation Systems with Graphs Embedded in $\mathbb{R}^n$

Linfan Mao

(Chinese Academy of Mathematics and System Science, Beijing 100190, P.R.China)

E-mail: maolinfan@163.com

Abstract: Different from the homogenous systems, a Smarandache system is a contradictory system in which an axiom behaves in at least two different ways within the same system, i.e., validated and invalided, or only invalided but in multiple distinct ways. Such systems widely exist in the world. In this report, we discuss such a kind of Smarandache system, i.e., non-solvable equation systems, such as those of non-solvable algebraic equations, non-solvable ordinary differential equations and non-solvable partial differential equations by topological graphs, classify these systems and characterize their global behaviors, particularly, the sum-stability and prod-stability of such equations. Applications of such systems to other sciences, such as those of controlling of infectious diseases, interaction fields and flows in network are also included in this report.

Key Words: Non-solvable equation, Smarandache system, topological graphs, vertex-edge labeled graph, G-solution, sum-stability, prod-stability.

AMS(2010): 05C15, 34A30, 34A34, 37C75, 70F10, 92B05

§1. Introduction

Consider two systems of linear equations following:

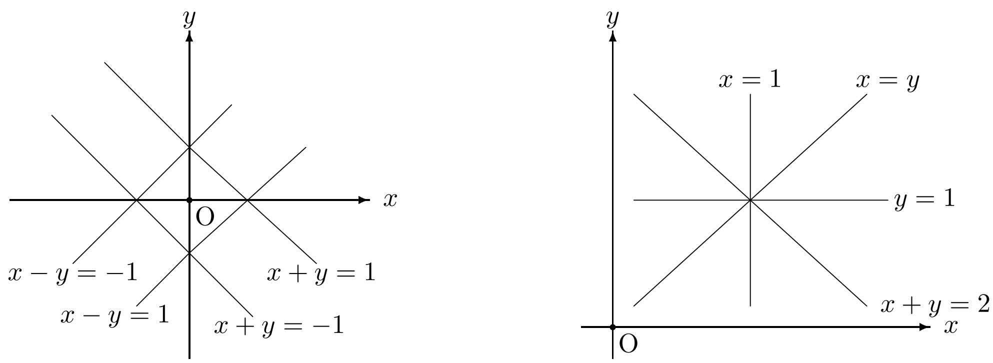
$$(LES_4^N)\quad \begin{cases} x+y = 1 \\ x+y = -1 \\ x-y = -1 \\ x-y = 1 \end{cases} \qquad\qquad (LES_4^S)\quad \begin{cases} x=y \\ x+y=2 \\ x=1 \\ y=1 \end{cases}$$

Clearly, (LES_4^N) is non-solvable because $x+y=-1$ is contradictious to $x+y=1$, and so that for equations $x-y=-1$ and $x-y=1$. Thus there are no solutions x_0, y_0 hold with all equations in this system. But (LES_4^S) is solvable clearly with a solution $x=1$ and $y=1$.

It should be noted that each equation in systems (LES_4^N) and (LES_4^S) is a straight line in Euclidean space $\mathbb{R}^2$, such as those shown in Fig.1.

[1]Reported at the First International Conference on Smarandache Multispaces and Multistructures, June 28-30,2013, Beijing, P.R.China.

[2]Received April 6, 2013, Accepted June 6,2013.



(LES_4^N) (LES_4^S)

Fig.1

What is the geometrical essence of a non-solvable or solvable system of linear equations? It is clear that each linear equation $ax+by=c$ with $ab \neq 0$ is in fact a point set $L_{ax+by=c} = \{(x,y)|ax+by=c\}$ in $\mathbb{R}^2$. Then, the system (LES_4^n) is non-solvable but (LES_4^S) solvable in sense because of

$$L_{x+y=1} \bigcap L_{x+y=-1} \bigcap L_{x-y=1} \bigcap L_{x-y=-1} = \emptyset$$

and

$$L_{x=y} \bigcap L_{x=1} \bigcap L_{y=1} \bigcap L_{x+y=2} = \{(1,1)\}$$

in Euclidean plane $\mathbb{R}^2$. Generally, let

$$(ES_m) \quad \begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_m(x_1, x_2, \cdots, x_n) = 0 \end{cases}$$

be a system of algebraic equations in Euclidean space $\mathbb{R}^n$ for an integer $n \geq 1$ with point set S_{f_i} such that $f_i(x_1, x_2, \cdots, x_n) = 0$ for any point $(x_1, x_2, \cdots, x_n) \in S_{f_i}$, $1 \leq i \leq m$. Then, it is clear that the system (ES_m) is solvable or not dependent on $\bigcap\limits_{i=1}^{m} S_{f_i} = \emptyset$ or $\neq \emptyset$. This fact implies the following interesting result.

Proposition 1.1 *Any system (ES_m) of algebraic equations with each equation solvable posses a geometrical figure in $\mathbb{R}^n$, no matter it is solvable or not.*

Conversely, for a geometrical figure $\mathscr{G}$ in $\mathbb{R}^n, n \geq 2$, *how can we get an algebraic representation for geometrical figure $\mathscr{G}$*? As a special case, let G be a graph embedded in Euclidean space $\mathbb{R}^n$ and

$$(ES_e)\quad \begin{cases} f_1^e(x_1,x_2,\cdots,x_n)=0\\ f_2^e(x_1,x_2,\cdots,x_n)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ f_{n-1}^e(x_1,x_2,\cdots,x_n)=0 \end{cases}$$

be a system of equations for determining an edge $e\in E(G)$ in $\mathbb{R}^n$. Then the system

$$\left.\begin{array}{r} f_1^e(x_1,x_2,\cdots,x_n)=0\\ f_2^e(x_1,x_2,\cdots,x_n)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ f_{n-1}^e(x_1,x_2,\cdots,x_n)=0 \end{array}\right\}\ \forall e\in E(G)$$

is a non-solvable system of equations. Generally, let $\mathscr{G}$ be a geometrical figure consisting of m parts $\mathscr{G}_1,\mathscr{G}_2,\cdots,\mathscr{G}_m$, where $\mathscr{G}_i$ is determined by a system of algebraic equations

$$\begin{cases} f_1^{[i]}(x_1,x_2,\cdots,x_n)=0\\ f_2^{[i]}(x_1,x_2,\cdots,x_n)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ f_{n-1}^{[i]}(x_1,x_2,\cdots,x_n)=0 \end{cases}$$

Similarly, we get a non-solvable system

$$\left.\begin{array}{l} f_1^{[i]}(x_1,x_2,\cdots,x_n)=0\\ f_2^{[i]}(x_1,x_2,\cdots,x_n)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ f_{n-1}^{[i]}(x_1,x_2,\cdots,x_n)=0 \end{array}\right\}\ 1\leq i\leq m.$$

Thus we obtain the following result.

Proposition 1.2 *Any geometrical figure $\mathscr{G}$ consisting of m parts, each of which is determined by a system of algebraic equations in $\mathbb{R}^n, n\geq 2$ posses an algebraic representation by system of equations, solvable or not in $\mathbb{R}^n$.*

For example, let G be a planar graph with vertices v_1,v_2,v_3,v_4 and edges v_1v_2,v_1v_3,v_2v_3, v_3v_4, v_4v_1, shown in Fig.2.

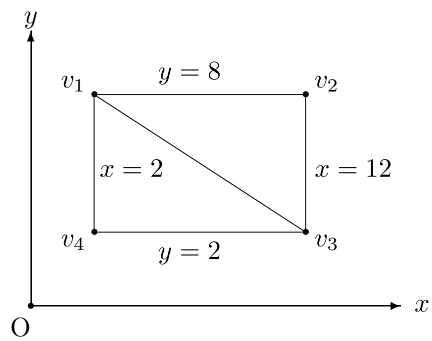


Fig.2

Then we get a non-solvable system of linear equations

$$\begin{cases} x = 2 \\ y = 8 \\ x = 12 \\ y = 2 \\ 3x + 5y = 46. \end{cases}$$

More results on non-solvable linear systems of equations can be found in [9]. Terminologies and notations in this paper are standard. For those not mentioned in this paper, we follow [12] and [15] for partial or ordinary differential equations. [5-7], [13-14] for algebra, topology and Smarandache systems, and [1] for mechanics.

§2. Smarandache Systems with Labeled Topological Graphs

A non-solvable system of algebraic equations is in fact a contradictory system in classical meaning of mathematics. As we have shown, such systems extensively exist in mathematics and possess real meaning even if in classical mathematics. This fact enables one to introduce the conception of Smarandache system following.

Definition 2.1([5-7]) *A rule $\mathcal{R}$ in a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule $\mathcal{R}$.

Without loss of generality, let $(\Sigma_1; \mathcal{R}_1)$ $(\Sigma_2; \mathcal{R}_2), \cdots, (\Sigma_m; \mathcal{R}_m)$ be mathematical systems, where $\mathcal{R}_i$ is a rule on Σ_i for integers $1 \le i \le m$. If for two integers $i, j,\ 1 \le i, j \le m$, $\Sigma_i \ne \Sigma_j$ or $\Sigma_i = \Sigma_j$ but $\mathcal{R}_i \ne \mathcal{R}_j$, then they are said to be *different*, otherwise, *identical*. If we can list all systems of a Smarandache system $(\Sigma; \mathcal{R})$, then we get a *Smarandache multi-space* defined following.

Definition 2.2([5-7],[11]) *Let* $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ *be* $m \geq 2$ *mathematical spaces, different two by two. A Smarandache multi-space* $\widetilde{\Sigma}$ *is a union* $\bigcup\limits_{i=1}^{m}\Sigma_i$ *with rules* $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m}\mathcal{R}_i$ *on* $\widetilde{\Sigma}$, *denoted by* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.

The conception of Smarandache multi-space reflects the notion of the whole $\widetilde{\Sigma}$ is consisting of its parts $(\Sigma_i;\mathcal{R}_i), i \geq 1$ for a thing in philosophy. The laterality of human beings implies that one can only determines lateral feature of a thing in general. Such a typical example is the proverb of blind men with an elephant.

Fig. 3

In this proverb, there are 6 blind men were be asked to determine what an elephant looked like by feeling different parts of the elephant's body. The man touched the elephant's leg, tail, trunk, ear, belly or tusk claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe, respectively. They then entered into an endless argument and each of them insisted his view right. *All of you are right*! A wise man explains to them: *Why are you telling it differently is because each one of you touched the different part of the elephant. So, actually the elephant has all those features what you all said*, i.e., a Smarandache multi-space consisting of these 6 parts.

Usually, a man is blind for an unknowing thing and takes himself side as the dominant factor. That makes him knowing only the lateral features of a thing, not the whole. That is also the reason why one used to harmonious, not contradictory systems in classical mathematics. But the world is filled with contradictions. Being a wise man knowing the world, we need to find the whole, not just the parts. Thus the Smarandache multi-space is important for sciences.

Notice that a Smarandache multi-space $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ naturally inherits a combinatorial structure, i.e., a vertex-edge labeled topological graph defined following.

Definition 2.3(([5-7])) *Let* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *be a Smarandache multi-space with* $\widetilde{\Sigma} = \bigcup\limits_{i=1}^{m}\Sigma_i$ *and* $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m}\mathcal{R}_i$. *Then a inherited graph* $G\left[\widetilde{\Sigma},\widetilde{R}\right]$ *of* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *is a labeled topological graph defined by*

$$V\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right) = \{\Sigma_1,\Sigma_2,\cdots,\Sigma_m\},$$

$$E\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right) = \{\ (\Sigma_i,\Sigma_j)\mid \Sigma_i\bigcap\Sigma_j \neq \emptyset, 1 \leq i,j \leq m\}$$

with an edge labeling

$$l^E: \ (\Sigma_i, \Sigma_j) \in E\left(G\left[\widetilde{S}, \widetilde{R}\right]\right) \to l^E(\Sigma_i, \Sigma_j) = \varpi\left(\Sigma_i \bigcap \Sigma_j\right),$$

where ϖ *is a characteristic on* $\Sigma_i \bigcap \Sigma_j$ *such that* $\Sigma_i \bigcap \Sigma_j$ *is isomorphic to* $\Sigma_k \bigcap \Sigma_l$ *if and only if* $\varpi(\Sigma_i \bigcap \Sigma_j) = \varpi(\Sigma_k \bigcap \Sigma_l)$ *for integers* $1 \leq i, j, k, l \leq m$.

For example, let $S_1 = \{a, b, c\}$, $S_2 = \{c, d, e\}$, $S_3 = \{a, c, e\}$ and $S_4 = \{d, e, f\}$. Then the multi-space $\widetilde{S} = \bigcup\limits_{i=1}^{4} S_i = \{a, b, c, d, e, f\}$ with its labeled topological graph $G[\widetilde{S}]$ is shown in Fig.4.

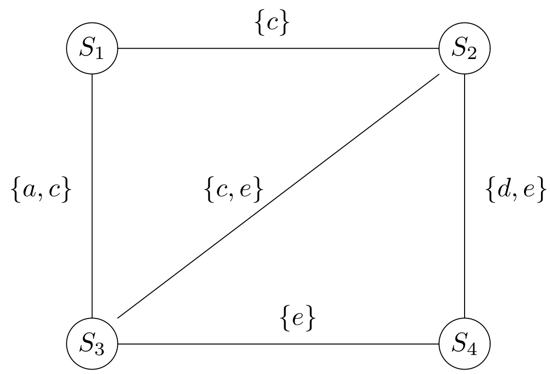


Fig.4

The labeled topological graph $G\left[\widetilde{\Sigma}, \widetilde{R}\right]$ reflects the notion that there exist linkages between things in philosophy. In fact, each edge $(\Sigma_i, \Sigma_j) \in E\left(G\left[\widetilde{\Sigma}, \widetilde{R}\right]\right)$ is such a linkage with coupling $\varpi(\Sigma_i \bigcap \Sigma_j)$. For example, let $a = \{\text{tusk}\}$, $b = \{\text{nose}\}$, $c_1, c_2 = \{\text{ear}\}$, $d = \{\text{head}\}$, $e = \{\text{neck}\}$, $f = \{\text{belly}\}$, $g_1, g_2, g_3, g_4 = \{\text{leg}\}$, $h = \{\text{tail}\}$ for an elephant $\mathscr{C}$. Then its labeled topological graph is shown in Fig.5,

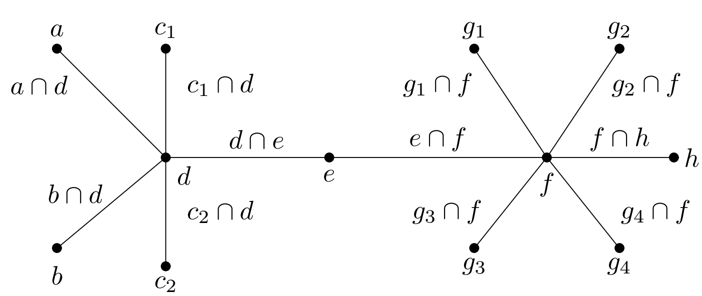


Fig.5

which implies that one can characterizes the geometrical behavior of an elephant combinatorially.

§3. Non-Solvable Systems of Ordinary Differential Equations

3.1 Linear Ordinary Differential Equations

For integers $m,\ n \geq 1$, let

$$\dot{X} = F_1(X),\ \dot{X} = F_2(X), \cdots, \dot{X} = F_m(X) \qquad (DES_m^1)$$

be a differential equation system with continuous $F_i : \mathbf{R}^n \to \mathbf{R}^n$ such that $F_i(\overline{0}) = \overline{0}$, particularly, let

$$\dot{X} = A_1 X, \cdots, \dot{X} = A_k X, \cdots, \dot{X} = A_m X \qquad (LDES_m^1)$$

be a linear ordinary differential equation system of first order with

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0 \leq k \leq m,\ 1 \leq i, j \leq n$.

Definition 3.1 *An ordinary differential equation system (DES_m^1) or $(LDES_m^1)$ are called non-solvable if there are no function $X(t)$ hold with (DES_m^1) or $(LDES_m^1)$ unless the constants.*

As we known, the general solution of the ith differential equation in $(LDES_m^1)$ is a linear space spanned by the elements in the solution basis

$$\mathscr{B}_i = \{\ \overline{\beta}_k(t)e^{\alpha_k t} \mid 1 \leq k \leq n\ \}$$

for integers $1 \leq i \leq m$, where

$$\alpha_i = \begin{cases} \lambda_1, & if\ 1 \leq i \leq k_1; \\ \lambda_2, & if\ k_1 + 1 \leq i \leq k_2; \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots ; \\ \lambda_s, & if\ k_1 + k_2 + \cdots + k_{s-1} + 1 \leq i \leq n, \end{cases}$$

λ_i is the k_i-fold zero of the characteristic equation

$$\det(A - \lambda I_{n\times n}) = |A - \lambda I_{n\times n}| = 0$$

with $k_1 + k_2 + \cdots + k_s = n$ and $\overline{\beta}_i(t)$ is an n-dimensional vector consisting of polynomials in t with degree$\leq k_i - 1$.

In this case, we can simplify the labeled topological graph $G\left[\widetilde{\sum}, \widetilde{R}\right]$ replaced each $\sum_i$ by the solution basis $\mathscr{B}_i$ and $\sum_i \bigcap \sum_j$ by $\mathscr{B}_i \bigcap \mathscr{B}_j$ if $\mathscr{B}_i \bigcap \mathscr{B}_j \neq \emptyset$ for integers $1 \leq i, j \leq m$, called the *basis graph* of $(LDES_m^1)$, denoted by $G[LDES_m^1]$. For example, let $m = 4$ and $\mathscr{B}_1^0 = \{e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}\}$, $\mathscr{B}_2^0 = \{e^{\lambda_3 t}, e^{\lambda_4 t}, e^{\lambda_5 t}\}$, $\mathscr{B}_3^0 = \{e^{\lambda_1 t}, e^{\lambda_3 t}, e^{\lambda_5 t}\}$ and $\mathscr{B}_4^0 = \{e^{\lambda_4 t}, e^{\lambda_5 t}, e^{\lambda_6 t}\}$, where $\lambda_i,\ 1 \leq i \leq 6$ are real numbers different two by two. Then $G[LDES_m^1]$ is shown in Fig.6.

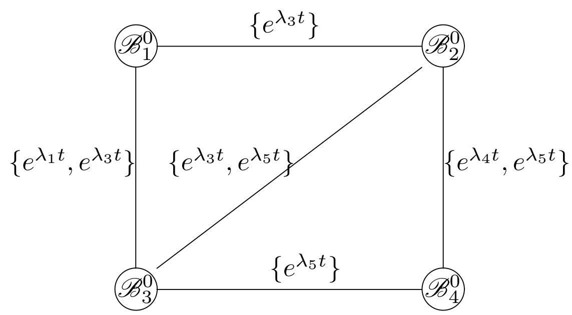


Fig.6

We get the following results.

Theorem 3.2([10]) *Every linear homogeneous differential equation system* $(LDES_m^1)$ *uniquely determines a basis graph* $G[LDES_m^1]$ *inherited in* $(LDES_m^1)$*. Conversely, every basis graph* G *uniquely determines a homogeneous differential equation system* $(LDES_m^1)$ *such that* $G[LDES_m^1] \simeq G$.

Such a basis graph $G[LDES_m^1]$ is called the G-solution of $(LDES_m^1)$. Theorem 3.2 implies that

Theorem 3.3([10]) *Every linear homogeneous differential equation system* $(LDES_m^1)$ *has a unique* G*-solution, and for every basis graph* H*, there is a unique linear homogeneous differential equation system* $(LDES_m^1)$ *with* G*-solution* H.

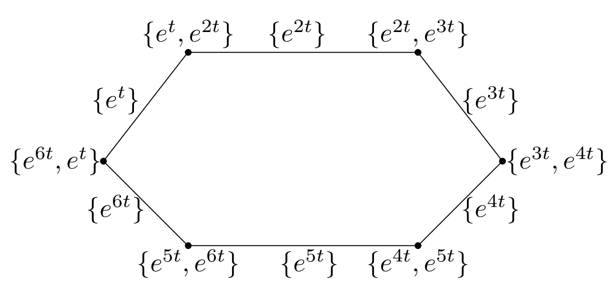


Fig.7 **A basis graph**

Example 3.4 Let (LDE_m^n) be the following linear homogeneous differential equation system

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where $\ddot{x} = \frac{d^2x}{dt^2}$ and $\dot{x} = \frac{dx}{dt}$. Then the solution basis of equations (1) − (6) are respectively $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ and its basis graph is shown in Fig.7.

3.2 Combinatorial Characteristics of Linear Differential Equations

Definition 3.5 *Let $(LDES_m^1)$, $(LDES_m^1)'$ be two linear homogeneous differential equation systems with G-solutions H, H'. They are called combinatorially equivalent if there is an isomorphism $\varphi : H \to H'$, thus there is an isomorphism $\varphi : H \to H'$ of graph and labelings θ, τ on H and H' respectively such that $\varphi\theta(x) = \tau\varphi(x)$ for $\forall x \in V(H) \bigcup E(H)$, denoted by $(LDES_m^1) \stackrel{\varphi}{\simeq} (LDES_m^1)'$.*

We introduce the conception of *integral graph* for $(LDES_m^1)$ following.

Definition 3.6 *Let G be a simple graph. A vertex-edge labeled graph $\theta : G \to \mathbb{Z}^+$ is called integral if $\theta(uv) \le \min\{\theta(u), \theta(v)\}$ for $\forall uv \in E(G)$, denoted by G^{I_θ}.*

Let $G_1^{I_\theta}$ and $G_2^{I_\tau}$ be two integral labeled graphs. They are called identical if $G_1 \stackrel{\varphi}{\simeq} G_2$ and $\theta(x) = \tau(\varphi(x))$ for any graph isomorphism φ and $\forall x \in V(G_1) \bigcup E(G_1)$, denoted by $G_1^{I_\theta} = G_2^{I_\tau}$.

For example, these labeled graphs shown in Fig.8 are all integral on $K_4 - e$, but $G_1^{I_\theta} = G_2^{I_\tau}$, $G_1^{I_\theta} \ne G_3^{I_\sigma}$.

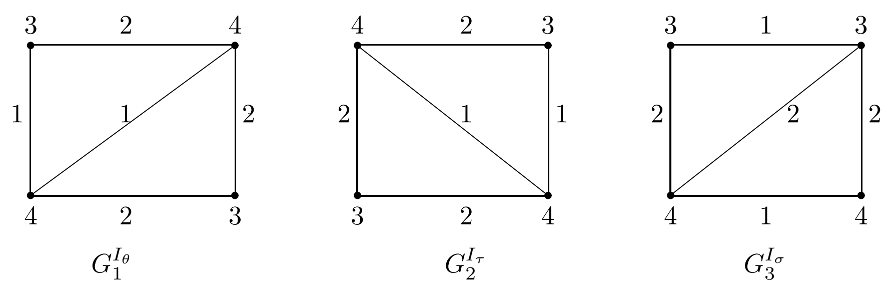


Fig.8

Then we get a combinatorial characteristic for combinatorially equivalent $(LDES_m^1)$ following.

Theorem 3.5([10]) *Let $(LDES_m^1)$, $(LDES_m^1)'$ be two linear homogeneous differential equation systems with integral labeled graphs H, H'. Then $(LDES_m^1) \stackrel{\varphi}{\simeq} (LDES_m^1)'$ if and only if $H = H'$.*

3.3 Non-Linear Ordinary Differential Equations

If some functions $F_i(X)$, $1 \le i \le m$ are non-linear in (DES_m^1), we can linearize these non-linear equations $\dot{X} = F_i(X)$ at the point $\overline{0}$, i.e., if

$$F_i(X) = F_i'(\overline{0})X + R_i(X),$$

where $F_i'(\overline{0})$ is an $n \times n$ matrix, we replace the ith equation $\dot{X} = F_i(X)$ by a linear differential equation

$$\dot{X} = F_i'(\overline{0})X$$

in (DES_m^1). Whence, we get a uniquely linear differential equation system $(LDES_m^1)$ from (DES_m^1) and its basis graph $G[LDES_m^1]$. Such a basis graph $G[LDES_m^1]$ of linearized differential equation system (DES_m^1) is defined to be the *linearized basis graph* of (DES_m^1) and denoted by $G[DES_m^1]$. We can also apply G-solutions $G[DES_m^1]$ for characterizing the behavior of (DES_m^1).

§4. Cauchy Problem on Non-Solvable Partial Differential Equations

Let $(PDES_m)$ be a system of partial differential equations with

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \end{cases}$$

on a function $u(x_1, \cdots, x_n, t)$. Then its *symbol* is determined by

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0, \end{cases}$$

i.e., substitute $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_n^{\alpha_n}$ into $(PDES_m)$ for the term $u_{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}$, where $\alpha_i \geq 0$ for integers $1 \leq i \leq n$.

Definition 4.1 *A non-solvable $(PDES_m)$ is algebraically contradictory if its symbol is non-solvable. Otherwise, differentially contradictory.*

The following result characterizes the non-solvable partial differential equations of first order by applying the method of characteristic curves.

Theorem 4.2([11]) *A Cauchy problem on systems*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} x_i|_{x_n=x_n^0} = x_i^0(s_1, s_2, \cdots, s_{n-1}) \\ u|_{x_n=x_n^0} = u_0(s_1, s_2, \cdots, s_{n-1}) \\ p_i|_{x_n=x_n^0} = p_i^0(s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases}$$

if and only if the system

$$F_k(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0, \ 1 \leq k \leq m$$

is algebraically contradictory, in this case, there must be an integer $k_0, \ 1 \leq k_0 \leq m$ *such that*

$$F_{k_0}(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) \neq 0$$

or it is differentially contradictory itself, i.e., there is an integer $j_0, \ 1 \leq j_0 \leq n-1$ *such that*

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0.$$

Particularly, we get conclusions following by Theorem 4.2.

Corollary 4.3 *Let*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

be an algebraically contradictory system of partial differential equations of first order. Then there are no values $x_i^0, u_0, p_i^0, \ 1 \leq i \leq n$ *such that*

$$\begin{cases} F_1(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) = 0, \\ F_2(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) = 0. \end{cases}$$

Corollary 4.4 *A Cauchy problem* ($LPDES_m^C$) *of quasilinear partial differential equations with initial values* $u|_{x_n=x_n^0} = u_0$ *is non-solvable if and only if the system* ($LPDES_m$) *of partial differential equations is algebraically contradictory.*

Denoted by $\widehat{G}[PDES_m^C]$ such a graph $G[PDES_m^C]$ eradicated all labels. Particularly, replacing each label $S^{[i]}$ by $S_0^{[i]} = \{u_0^{[i]}\}$ and $S^{[i]} \bigcap S^{[j]}$ by $S_0^{[i]} \bigcap S_0^{[j]}$ for integers $1 \leq i, j \leq m$, we get a new labeled topological graph, denoted by $G_0[PDES_m^C]$. Clearly, $\widehat{G}[PDES_m^C] \simeq \widehat{G}_0[PDES_m^C]$.

Theorem 4.5([11]) *For any system* ($PDES_m^C$) *of partial differential equations of first order,* $\widehat{G}[PDES_m^C]$ *is simple. Conversely, for any simple graph* G, *there is a system* ($PDES_m^C$) *of partial differential equations of first order such that* $\widehat{G}[PDES_m^C] \simeq G$.

Particularly, if ($PDES_m^C$) is linear, we can immediately find its underlying graph following.

Corollary 4.6 *Let* ($LPDES_m$) *be a system of linear partial differential equations of first order with maximal contradictory classes* $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s$ *on equations in (LPDES). Then* $\widehat{G}[LPDES_m^C] \simeq K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s)$, *i.e., an s-partite complete graph.*

Definition 4.7 *Let $(PDES_m^C)$ be the Cauchy problem of a partial differential equation system of first order. Then the labeled topological graph $G[PDES_m^C]$ is called its topological graph solution, abbreviated to G-solution.*

Combining this definition with that of Theorems 4.5, the following conclusion is holden immediately.

Theorem 4.8([11]) *A Cauchy problem on system $(PDES_m)$ of partial differential equations of first order with initial values $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}$, $1 \leq i \leq n$ for the kth equation in $(PDES_m)$, $1 \leq k \leq m$ such that*

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^{n} p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0$$

is uniquely G-solvable, i.e., $G[PDES_m^C]$ is uniquely determined.

§5. Global Stability of Non-Solvable Differential Equations

Definition 5.1 *Let H be a spanning subgraph of $G[LDES_m^1]$ of systems $(LDES_m^1)$ with initial value $X_v(0)$. Then $G[LDES_m^1]$ is called sum-stable or asymptotically sum-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$ of the linear differential equations of $(LDES_m^1)$ with $|Y_v(0) - X_v(0)| < \delta_v$ exists for all $t \geq 0$,*

$$\left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| < \varepsilon,$$

or furthermore,

$$\lim_{t \to 0} \left| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right| = 0.$$

Similarly, a system $(PDES_m^C)$ is sum-stable if for any number $\varepsilon > 0$ there exists $\delta_v > 0$, $v \in V(\widehat{G}[0])$ such that each $G(t)$-solution with $\left| {u'}_0^{[v]} - u_0^{[v]} \right| < \delta_v, \forall v \in V(\widehat{G}[0])$ exists for all $t \geq 0$ and with the inequality

$$\left| \sum_{v \in V(\widehat{G}[t])} {u'}^{[v]} - \sum_{v \in V(\widehat{G}[t])} u^{[v]} \right| < \varepsilon$$

holds, denoted by $G[t] \overset{\Sigma}{\sim} G[0]$. Furthermore, if there exists a number $\beta_v > 0$, $v \in V(\widehat{G}[0])$ such that every $G'[t]$-solution with $\left| {u'}_0^{[v]} - u_0^{[v]} \right| < \beta_v, \forall v \in V(\widehat{G}[0])$ satisfies

$$\lim_{t \to \infty} \left| \sum_{v \in V(\widehat{G}[t])} {u'}^{[v]} - \sum_{v \in V(\widehat{G}[t])} u^{[v]} \right| = 0,$$

then the $G[t]$-solution is called asymptotically stable, denoted by $G[t] \overset{\Sigma}{\to} G[0]$.

We get results on the global stability for G-solutions of $(LDES_m^1)$ and $(PDES_m^C)$.

Theorem 5.2([10]) *A zero G-solution of linear homogenous differential equation systems $(LDES_m^1)$ is asymptotically sum-stable on a spanning subgraph H of $G[LDES_m^1]$ if and only if* $\mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *in* $(LDES^1)$ *hold for* $\forall v \in V(H)$.

Example 5.3 Let a G-solution of $(LDES_m^1)$ or (LDE_m^n) be the basis graph shown in Fig.4.1, where $v_1 = \{e^{-2t}, e^{-3t}, e^{3t}\}$, $v_2 = \{e^{-3t}, e^{-4t}\}$, $v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}$, $v_4 = \{e^{-5t}, e^{-6t}, e^{-8t}\}$, $v_5 = \{e^{-t}, e^{-6t}\}$, $v_6 = \{e^{-t}, e^{-2t}, e^{-8t}\}$. Then the zero G-solution is sum-stable on the triangle $v_4v_5v_6$, but it is not on the triangle $v_1v_2v_3$.

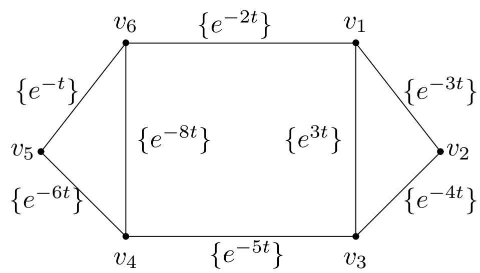


Fig.9

For partial differential equations, let the system $(PDES_m^C)$ be

$$\left.\begin{array}{l}\dfrac{\partial u}{\partial t} = H_i(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1})\\ u|_{t=t_0} = u_0^{[i]}(x_1, x_2, \cdots, x_{n-1})\end{array}\right\} \quad 1 \leq i \leq m \qquad (APDES_m^C)$$

A point $X_0^{[i]} = (t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]})$ with $H_i(t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]}) = 0$ for $1 \leq i \leq m$ is called an *equilibrium point* of the ith equation in $(APDES_m)$. Then we know that

Theorem 5.4([11]) *Let $X_0^{[i]}$ be an equilibrium point of the ith equation in $(APDES_m)$ for each integer $1 \leq i \leq m$. If $\sum\limits_{i=1}^{m} H_i(X) > 0$ and $\sum\limits_{i=1}^{m} \dfrac{\partial H_i}{\partial t} \leq 0$ for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, then the system $(APDES_m)$ is sum-stability, i.e., $G[t] \overset{\Sigma}{\sim} G[0]$. Furthermore, if $\sum\limits_{i=1}^{m} \dfrac{\partial H_i}{\partial t} < 0$ for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, then $G[t] \overset{\Sigma}{\to} G[0]$.*

§6. Applications

6.1 Applications to Geometry

First, it is easily to shown that the G-solution of $(PDES_m^C)$ is nothing but a differentiable manifold.

Theorem 6.1([11]) *Let the Cauchy problem be $(PDES_m^C)$. Then every connected component of $\Gamma[PDES_m^C]$ is a differentiable n-manifold with atlas $\mathscr{A} = \{(U_v, \phi_v) | v \in V(\widehat{G}[0])\}$ underlying graph $\widehat{G}[0]$, where U_v is the n-dimensional graph $G[u^{[v]}] \simeq \mathbb{R}^n$ and ϕ_v the projection $\phi_v : ((x_1, x_2, \cdots, x_n), u(x_1, x_2, \cdots, x_n))) \to (x_1, x_2, \cdots, x_n)$ for $\forall (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$.*

Theorems 4.8 and 6.1 enables one to find the following result for vector fields on differentiable manifolds by non-solvable system $(PDES_m^C)$.

Theorem 6.2([11]) *For any integer $m \geq 1$, let $U_i, 1 \leq i \leq m$ be open sets in $\mathbb{R}^n$ underlying a connected graph defined by*

$$V(G) = \{U_i | 1 \leq i \leq m\}, \quad E(G) = \{(U_i, U_j) | U_i \bigcap U_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

If X_i is a vector field on U_i for integers $1 \leq i \leq m$, then there always exists a differentiable manifold $M \subset \mathbb{R}^n$ with atlas $\mathscr{A} = \{(U_i, \phi_i) | 1 \leq i \leq m\}$ underlying graph G and a function $u_G \in \Omega^0(M)$ such that

$$X_i(u_G) = 0, \quad 1 \leq i \leq m.$$

More results on geometrical structure of manifold can be found in references [2-3] and [8].

6.2 Global Control of Infectious Diseases

Consider two cases of virus for infectious diseases:

Case 1 *There are m known virus $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$ and an person infected a virus $\mathscr{V}_i$ will never infects other viruses $\mathscr{V}_j$ for $j \neq i$.*

Case 2 *There are m varying $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ from a virus $\mathscr{V}$ with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$.*

We are easily to establish a non-solvable differential model for the spread of infectious viruses by applying the SIR model of one infectious disease following:

$$\left\{\begin{array}{l} \dot{S} = -k_1 SI \\ \dot{I} = k_1 SI - h_1 I \\ \dot{R} = h_1 I \end{array}\right. \quad \left\{\begin{array}{l} \dot{S} = -k_2 SI \\ \dot{I} = k_2 SI - h_2 I \\ \dot{R} = h_2 I \end{array}\right. \quad \cdots \quad \left\{\begin{array}{l} \dot{S} = -k_m SI \\ \dot{I} = k_m SI - h_m I \\ \dot{R} = h_m I \end{array}\right. \qquad (DES_m^1)$$

and know the following result by Theorem 5.2 that

Conclusion 6.3([10]) *For m infectious viruses $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ in an area with infected rate k_i, heal rate h_i for integers $1 \leq i \leq m$, then they decline to 0 finally if $0 < S < \sum\limits_{i=1}^m h_i \Big/ \sum\limits_{i=1}^m k_i$, i.e., these infectious viruses are globally controlled. Particularly, they are globally controlled if each of them is controlled in this area.*

6.3 Flows in Network

Let O be a node in N incident with m in-flows and 1 out-flow shown in Fig.10.

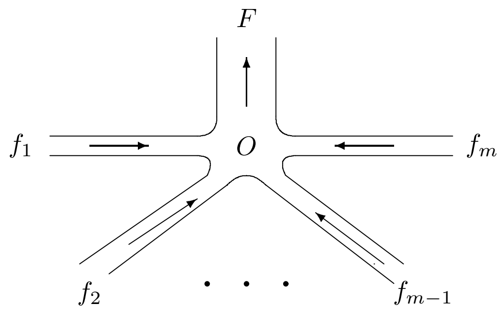


Fig.10

How can we characterize the behavior of flow F? Denote the rate, density of flow f_i by $\rho^{[i]}$ for integers $1 \leq i \leq m$ and that of F by $\rho^{[F]}$, respectively. Then we know that

$$\frac{\partial \rho^{[i]}}{\partial t} + \phi_i(\rho^{[i]})\frac{\partial \rho^{[i]}}{\partial x} = 0, \ 1 \leq i \leq m.$$

We prescribe the initial value of $\rho^{[i]}$ by $\rho^{[i]}(x,t_0)$ at time t_0. Replacing each $\rho^{[i]}$ by ρ in these flow equations of $f_i, \ 1 \leq i \leq m$ enables one getting a non-solvable system $(PDES_m^C)$ of partial differential equations following.

$$\left.\begin{array}{l} \frac{\partial \rho}{\partial t} + \phi_i(\rho)\frac{\partial \rho}{\partial x} = 0 \\ \rho\,|_{t=t_0} \ = \rho^{[i]}(x,t_0) \end{array}\right\} \ 1 \leq i \leq m.$$

Let $\rho_0^{[i]}$ be an equilibrium point of the ith equation, i.e., $\phi_i(\rho_0^{[i]})\frac{\partial \rho_0^{[i]}}{\partial x} = 0$. Applying Theorem 5.4, if

$$\sum_{i=1}^{m} \phi_i(\rho) < 0 \ \text{ and } \ \sum_{i=1}^{m} \phi_(\rho)\left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] \geq 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, then we know that the flow F is stable and furthermore, if

$$\sum_{i=1}^{m} \phi_(\rho)\left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] < 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, then it is also asymptotically stable.

References

[1] R.Abraham and Marsden, *Foundation of Mechanics*(2nd edition), Addison-Wesley, Reading, Mass, 1978.

[2] Linfan Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[3] Linfan Mao, Combinatorial fields - an introduction, *International J. Math. Combin.*, Vol.3, 2009, 01-22.

[4] Linfan Mao, Relativity in combinatorial gravitational fields, *Progress in Physics*, Vol.3(2010), 39-50.

[5] Linfan Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries* (Second edition), Graduate Textbook in Mathematics, The Education Publisher Inc. 2011.

[6] Linfan Mao, *Smarandache Multi-Space Theory* (Second edition), Graduate Textbook in Mathematics, The Education Publisher Inc. 2011.

[7] Linfan Mao, *Combinatorial Geometry with Applications to Field Theory* (Second edition), Graduate Textbook in Mathematics, The Education Publisher Inc. 2011.

[8] Linfan Mao, Graph structure of manifolds with listing, *International J.Contemp. Math. Science*, Vol.5, 2011, No.2, 71-85.

[9] Linfan Mao, Non-solvable spaces of linear equation systems, *International J. Math. Combin.*, Vol.2 (2012), 9-23.

[10] Linfan Mao, Global stability of non-solvable ordinary differential equations with applications, *International J.Math. Combin.*, Vol.1 (2013), 1-37.

[11] Linfan Mao, Non-solvable partial differential equations of first order with applications, Submitted, 2013.

[12] M.Renardy, R.C.Rogers, *An Introduction to Partial Differential Equations*(2th Edition), Springer-Verlag, Berlin Heidelberg, 2004.

[13] F.Smarandache, Mixed noneuclidean geometries, *Eprint arXiv: math/0010119*, 10/2000.

[14] F.Smarandache, *A Unifying Field in Logics–Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[15] Wolfgang Walter, *Ordinary Differential Equations*, Springer-Verlag New York, Inc., 1998.

International J.Math. Combin. Vol.2(2013), 24-33

Some Properties of Birings

A.A.A.Agboola

(Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria)

B.Davvaz

(Department of Mathematics, Yazd University, Yazd, Iran)

E-mail: aaaola2003@yahoo.com, davvaz@yazd.ac.ir

Abstract: Let R be any ring and let $S = R_1 \cup R_2$ be the union of any two subrings of R. Since in general S is not a subring of R but R_1 and R_2 are algebraic structures on their own under the binary operations inherited from the parent ring R, S is recognized as a bialgebraic structure and it is called a biring. The purpose of this paper is to present some properties of such bialgebraic structures.

Key Words: Biring, bi-subring, bi-ideal, bi-field and bidomain

AMS(2010): 08A05, 13A05, 13A15

§1. Introduction

Generally speaking, the unions of any two subgroups of a group, subgroupoids of a groupoid, subsemigroups of a semigroup, submonoids of a monoid, subloops of a loop, subsemirings of a semiring, subfields of a field and subspaces of a vector space do not form any nice algebraic structures other than ordinary sets. Similarly, if S_1 and S_2 are any two subrings of a ring R, I_1 and I_2 any two ideals of R, the unions $S = S_1 \cup S_2$ and $I = I_1 \cup I_2$ generally are not subrings and ideals of R, respectively [2]. However, the concept of bialgebraic structures recently introduced by Vasantha Kandasamy [9] recognises the union $S = S_1 \cup S_2$ as a biring and $I = I_1 \cup I_2$ as a bi-ideal. One of the major advantages of bialgebraic structures is the exhibition of distinct algebraic properties totally different from those inherited from the parent structures. The concept of birings was introduced and studied in [9]. Other related bialgebraic structures introduced in [9] included binear-rings, bisemi-rings, biseminear-rings and group birings. Agboola and Akinola in [1] studied bicoset of a bivector space. Also, we refer the readers to [3-7]. In this paper, we will present and study some properties of birings.

[1]Reported at the First International Conference on Smarandache Multispaces and Multistructures, June 28-30,2013, Beijing, P.R.China.

[2]Received April 13, 2013, Accepted June 8, 2013

§2. Definitions and Elementary Properties of Birings

Definition 2.1 *Let R_1 and R_2 be any two proper subsets of a non-empty set R. Then, $R = R_1 \cup R_2$ is said to be a biring if the following conditions hold:*

(1) *R_1 is a ring;*

(2) *R_2 is a ring.*

Definition 2.2 *A biring $R = R_1 \cup R_2$ is said to be commutative if R_1 and R_2 are commutative rings. $R = R_1 \cup R_2$ is said to be a non-commutative biring if R_1 is non-commutative or R_2 is non-commutative.*

Definition 2.3 *A biring $R = R_1 \cup R_2$ is said to have a zero element if R_1 and R_2 have different zero elements. The zero element 0 is written $0_1 \cup 0_2$ (notation is not set theoretic union) where $0_i, i = 1, 2$ are the zero elements of R_i. If R_1 and R_2 have the same zero element, we say that the biring $R = R_1 \cup R_2$ has a mono-zero element.*

Definition 2.4 *A biring $R = R_1 \cup R_2$ is said to have a unit if R_1 and R_2 have different units. The unit element u is written $u_1 \cup u_2$, where $u_i, i = 1, 2$ are the units of R_i. If R_1 and R_2 have the same unit, we say that the biring $R = R_1 \cup R_2$ has a mono-unit.*

Definition 2.5 *A biring $R = R_1 \cup R_2$ is said to be finite if it has a finite number of elements. Otherwise, R is said to be an infinite biring. If R is finite, the order of R is denoted by $o(R)$.*

Example 1 Let $R = \{0, 2, 4, 6, 7, 8, 10, 12\}$ be a subset of $\mathcal{Z}_{14}$. It is clear that $(R, +, \cdot)$ is not a ring but then, $R_1 = \{0, 7\}$ and $R_2 = \{0, 2, 4, 6, 8, 10, 12\}$ are rings so that $R = R_1 \cup R_2$ is a finite commutative biring.

Definition 2.6 *Let $R = R_1 \cup R_2$ be a biring. A non-empty subset S of R is said to be a sub-biring of R if $S = S_1 \cup S_2$ and S itself is a biring and $S_1 = S \cap R_1$ and $S_2 = S \cap R_2$.*

Theorem 2.7 *Let $R = R_1 \cup R_2$ be a biring. A non-empty subset $S = S_1 \cup S_2$ of R is a sub-biring of R if and only if $S_1 = S \cap R_1$ and $S_2 = S \cap R_2$ are subrings of R_1 and R_2, respectively.*

Definition 2.8 *Let $R = R_1 \cup R_2$ be a biring and let x be a non-zero element of R. Then,*

(1) *x is a zero-divisor in R if there exists a non-zero element y in R such that $xy = 0$;*

(2) *x is an idempotent in R if $x^2 = x$;*

(3) *x is nilpotent in R if $x^n = 0$ for some $n > 0$.*

Example 2 Consider the biring $R = R_1 \cup R_2$, where $R_1 = \mathcal{Z}$ and $R_2 = \{0, 2, 4, 6\}$ a subset of $\mathcal{Z}_8$.

(1) If $S_1 = 4\mathcal{Z}$ and $S_2 = \{0, 4\}$, then S_1 is a subring of R_1 and S_2 is a subring of R_2. Thus, $S = S_1 \cup S_2$ is a bi-subring of R since $S_1 = S \cap R_1$ and $S_2 = S \cap R_2$.

(2) If $S_1 = 3\mathcal{Z}$ and $S_2 = \{0, 4\}$, then $S = S_1 \cup S_2$ is a biring but not a bi-subring of R because $S_1 \neq S \cap R_1$ and $S_2 \neq S \cap R_2$. This can only happen in a biring structure.

Theorem 2.9 *Let $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ be any two birings and let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be sub-birings of R and S, respectively. Then,*

(1) $R \times S = (R_1 \times S_1) \cup (R_2 \times S_2)$ *is a biring;*

(2) $I \times J = (I_1 \times J_1) \cup (I_2 \times J_2)$ *is a sub-biring of* $R \times S$.

Definition 2.10 *Let* $R = R_1 \cup R_2$ *be a biring and let* I *be a non-empty subset of* R.

(1) I *is a right bi-ideal of* R *if* $I = I_1 \cup I_2$, *where* I_1 *is a right ideal of* R_1 *and* I_2 *is a right ideal of* R_2;

(2) I *is a left bi-ideal of* R *if* $I = I_1 \cup I_2$, *where* I_1 *is a left ideal of* R_1 *and* I_2 *is a left ideal of* R_2;

(3) $I = I_1 \cup I_2$ *is a bi-ideal of* R *if* I_1 *is an ideal of* R_1 *and* I_2 *is an ideal of* R_2.

Definition 2.11 *Let* $R = R_1 \cup R_2$ *be a biring and let* I *be a non-empty subset of* R. *Then,* $I = I_1 \cup I_2$ *is a mixed bi-ideal of* R *if* I_1 *is a right (left) ideal of* R_1 *and* I_2 *is a left (right) ideal of* R_2.

Theorem 2.12 *Let* $I = I_1 \cup I_2$, $J = J_1 \cup J_2$ *and* $K = K_1 \cup K_2$ *be left (right) bi-ideals of a biring* $R = R_1 \cup R_2$. *Then,*

(1) $IJ = (I_1J_1) \cup (I_2J_2)$ *is a left(right) bi-ideal of* R;

(2) $I \cap J = (I_1 \cap J_1) \cup (I_2 \cap J_2)$ *is a left(right) bi-ideal of* R;

(3) $I + J = (I_1 + J_1) \cup (I_2 + J_2)$ *is a left(right) bi-ideal of* R;

(4) $I \times J = (I_1 \times J_1) \cup (I_2 \times J_2)$ *is a left(right) bi-ideal of* R;

(5) $(IJ)K = \Big((I_1J_1)K_1\Big) \cup \Big((I_2J_2)K_2\Big) = I(JK) = \Big(I_1(J_1K_1)\Big) \cup \Big(I_2(J_2K_2)\Big)$;

(6) $I(J+K) = \Big(I_1(J_1+K_1)\Big) \cup \Big(I_2(J_2+K_2)\Big) = IJ + IK = (I_1J_1 + I_1K_1) \cup (I_2J_2 + I_2K_2)$;

(7) $(J+K)I = \Big((J_1+K_1)I_1\Big) \cup \Big((J_2+K_2)I_2\Big) = JI + KI = (J_1I_1 + K_1I_1) \cup (J_2I_2 + K_2I_2)$.

Example 3 Let R be the collection of all 2×2 upper triangular and lower triangular matrices over a field F and let

$$R_1 = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in F \right\},$$

$$R_2 = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in F \right\},$$

$$I_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in F \right\},$$

$$I_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} : a \in F \right\}.$$

Then, $R = R_1 \cup R_2$ is a non-commutative biring with a mono-unit $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I = I_1 \cup I_2$ is a right bi-ideal of $R = R_1 \cup R_2$.

Definition 2.13 *Let* $R = R_1 \cup R_2$ *and* $S = S_1 \cup S_2$ *be any two birings. The mapping* $\phi : R \to S$

is called a biring homomorphism if $\phi = \phi_1 \cup \phi_2$ *and* $\phi_1 : R_1 \to S_1$ *and* $\phi_2 : R_2 \to S_2$ *are ring homomorphisms. If* $\phi_1 : R_1 \to S_1$ *and* $\phi_2 : R_2 \to S_2$ *are ring isomorphisms, then* $\phi = \phi_1 \cup \phi_2$ *is a biring isomorphism and we write* $R = R_1 \cup R_2 \cong S = S_1 \cup S_2$. *The image of* ϕ *denoted by* $Im\phi = Im\phi_1 \cup Im\phi_2 = \{y_1 \in S_1, y_2 \in S_2 : y_1 = \phi_1(x_1), y_2 = \phi_2(x_2) \text{ for some } x_1 \in R_1, x_2 \in R_2\}$. *The kernel of* ϕ *denoted by*

$$Ker\phi = Ker\phi_1 \cup Ker\phi_2 = \{a_1 \in R_1, a_2 \in R_2 : \phi_1(a_1) = 0 \text{ and } \phi_2(a_2) = 0\}.$$

Theorem 2.14 *Let* $R = R_1 \cup R_2$ *and* $S = S_1 \cup S_2$ *be any two birings and let* $\phi = \phi_1 \cup \phi_2 : R \to S$ *be a biring homomorphism. Then,*

(1) $Im\phi$ *is a sub-biring of the biring* S;

(2) $Ker\phi$ *is a bi-ideal of the biring* R;

(3) $Ker\phi = \{0\}$ *if and only if* $\phi_i, i = 1, 2$ *are injective.*

Proof (1) It is clear that $Im\phi = Im\phi_1 \cup Im\phi_2$, where $\phi_1 : R_1 \to S_1$ and $\phi_2 : R_2 \to S_2$ are ring homomorphisms, is not an empty set. Since $Im\phi_1$ is a subring of S_1 and $Im\phi_2$ is a subring of S_2, it follows that $Im\phi = Im\phi_1 \cup Im\phi_2$ is a biring. Lastly, it can easily be shown that $Im\phi \cap S_1 = Im\phi_1$,$Im\phi \cap S_2 = Im\phi_2$ and consequently, $Im\phi = Im\phi_1 \cup Im\phi_2$ is a sub-biring of the biring $S = S_1 \cup S_2$.

(2) The proof is similar to (1).

(3) It is clear. □

Let $I = I_1 \cup I_2$ be a left bi-ideal of a biring $R = R_1 \cup R_2$. We know that R_1/I_1 and R_2/I_2 are factor rings and therefore $(R_1/I_1) \cup (R_2/I_2)$ is a biring called *factor-biring*. Since $\phi_1 : R_1 \to R_1/I_1$ and $\phi_2 : R_2 \to R_2/I_2$ are natural homomorphisms with kernels I_1 and I_2, respectively, it follows that $\phi_1 \cup \phi_2 = \phi : R \to R/I$ is a natural biring homomorphism whose kernel is $Ker\phi = I_1 \cup I_2$.

Theorem 2.15(First Isomorphism Theorem) *Let* $R = R_1 \cup R_2$ *and* $S = S_1 \cup S_2$ *be any two birings and let* $\phi_1 \cup \phi_2 = \phi : R \to S$ *be a biring homomorphism with kernel* $K = Ker\phi = Ker\phi_1 \cup Ker\phi_2$. *Then,* $R/K \cong Im\phi$.

Proof Suppose that $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ are birings and suppose that $\phi_1 \cup \phi_2 = \phi : R \to S$ is a biring homomorphism with kernel $K = Ker\phi = Ker\phi_1 \cup Ker\phi_2$. Then, K is a bi-ideal of R, $Im\phi = Im\phi_1 \cup Im\phi_2$ is a bi-subring of S and $R/K = (R_1/Ker\phi_1) \cup (R_2/Ker\phi_2)$ is a biring. From the classical rings (first isomorphism theorem), we have $R_i/Ker\phi_i \cong Im\phi_i, i = 1, 2$ and therefore, $R/K = (R_1/Ker\phi_1) \cup (R_2/Ker\phi_2) \cong Im\phi = Im\phi_1 \cup Im\phi_2$. □

Theorem 2.16(Second Isomorphism Theorem) *Let* $R = R_1 \cup R_2$ *be a biring. If* $S = S_1 \cup S_2$ *is a sub-biring of* R *and* $I = I_1 \cup I_2$ *is a bi-ideal of* R, *then*

(1) $S + I$ *is a sub-biring of* R;

(2) I *is a bi-ideal of* $S + I$;

(3) $S \cap I$ *is a bi-ideal of* S;

(4) $(S + I)/I \cong S/(S \cap I)$.

Proof Suppose that $R = R_1 \cup R_2$ is a biring, $S = S_1 \cup S_2$ a sub-biring and $I = I_1 \cup I_2$ a bi-ideal of R.

(1) $S + I = (S_1 + I_1) \cup (S_2 + I_2)$ is a biring since $S_i + I_i$ are subrings of R_i, where $i = 1, 2$. Now, $R_1 \cap (S + I) = \Big(R_1 \cap (S_1 + I_1)\Big) \cup \Big(R_1 \cap (S_2 + I_2)\Big) = S_1 + I_1$. Similarly, we have $R_2 \cap (S + I) = S_2 + I_2$. Thus, $S + I$ is a sub-biring of R.

(2) and (3) are clear.

(4) It is clear that $(S+I)/I = \Big((S_1+I_1)/I_1\Big) \cup \Big((S_2+I_2)/I_2\Big)$ is a biring since $(S_1+I_1)/I_1$ and $(S_2 + I_2)/I_2$ are rings. Similarly, $S/(S \cap I) = \Big(S_1/(S_1 \cap I_1)\Big) \cup \Big(S_2/(S_2 \cap I_2)\Big)$ is a biring. Consider the mapping $\phi = \phi_1 \cup \phi_2 : S_1 \cup S_2 \to \Big((S_1 + I_1)/I_1\Big) \cup \Big((S_2 + I_2)/I_2\Big)$. It is clear that ϕ is a biring homomorphism since $\phi_i : S_i \to (S_i + I_i)/I_i, i = 1, 2$ are ring homomorphisms. Also, since $Ker\phi_i = S_i \cap I_i, i = 1, 2$, it follows that $Ker\phi = (S_1 \cap I_1) \cup (S_2 \cap I_2)$. The required result follows from the first isomorphism theorem. □

Theorem 2.17(Third Isomorphism Theorem) *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be two bi-ideals of R such that $J_i \subseteq I_i, i = 1, 2$. Then,*

(1) *I/J is a bi-ideal of R/J;*

(2) *$R/I \cong (R/J)/(I/J)$.*

Proof Suppose that $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ are two bi-ideals of the biring $R = R_1 \cup R_2$ such that $J_i \subseteq I_i, i = 1, 2$.

(1) It is clear that $R/J = (R_1/J_1) \cup (R_2/J_2)$ and $I/J = (I_1/J_1) \cup (I_2/J_2)$ are birings. Now, $(R_1/J_1) \cap \Big((I_1/J_1) \cup (I_2/J_2)\Big) = \Big((R_1/J_1) \cap (I_1/J_1)\Big) \cup \Big((R_1/J_1) \cap (I_2/J_2)\Big) = I_1/J_1$ (since $J_i \subseteq I_i \subseteq R_i, i = 1, 2$). Similarly, $(R_2/J_2) \cap \Big((I_1/J_1) \cup (I_2/J_2)\Big) = I_2/J_2$. Consequently, I/J is a sub-biring of R/J and in fact a bi-ideal.

(2) Let us consider the mapping $\phi = \phi_1 \cup \phi_2 : (R_1/J_1) \cup (R_2/J_2) \to (R_1/I_1) \cup (R_2/I_2)$. Since $\phi_i : R_i/J_i \to R_i/I_i, i = 1, 2$ are ring homomorphisms with $Ker\phi_i = I_i/J_i$, it follows that $\phi = \phi_1 \cup \phi_2$ is a biring homomorphism and $Ker\phi = Ker\phi_1 \cup Ker\phi_2 = (I_1/J_1) \cup (I_2/J_2)$. Applying the first isomorphism theorem, we have $\Big((R_1/J_1)/(I_1/J_1)\Big) \cup \Big((R_2/J_2)/(I_2/J_2)\Big) \cong (R_1/I_1) \cup /(R_2/I_2)$. □

Definition 2.18 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ be a bi-ideal of R. Then,*

(1) *I is said to be a principal bi-ideal of R if I_1 is a principal ideal of R_1 and I_2 is a principal ideal of R_2;*

(2) *I is said to be a maximal (minimal) bi-ideal of R if I_1 is a maximal (minimal) ideal of R_1 and I_2 is a maximal (minimal) ideal of R_2;*

(3) *I is said to be a primary bi-ideal of R if I_1 is a primary ideal of R_1 and I_2 is a primary ideal of R_2;*

(4) *I is said to be a prime bi-ideal of R if I_1 is a prime ideal of R_1 and I_2 is a prime ideal of R_2.*

Example 4 Let $R = R_1 \cup R_2$ be a biring, where $R_1 = \mathcal{Z}$, the ring of integers and $R_2 = \mathcal{R}[x]$, the ring of polynomials over $\mathcal{R}$. Let $I_1 = (2)$ and $I_2 = (x^2 + 1)$. Then, $I = I_1 \cup I_2$ is a principal bi-ideal of R.

Definition 2.19 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ be a bi-ideal of R. Then, I is said to be a quasi maximal (minimal) bi-ideal of R if I_1 or I_2 is a maximal (minimal) ideal.*

Definition 2.20 *Let $R = R_1 \cup R_2$ be a biring. Then, R is said to be a simple biring if R has no non-trivial bi-ideals.*

Theorem 2.21 *Let $\phi = \phi_1 \cup \phi_2 : R \to S$ be a biring homomorphism. If $J = J_1 \cup J_2$ is a prime bi-ideal of S, then $\phi^{-1}(J)$ is a prime bi-ideal of R.*

Proof Suppose that $J = J_1 \cup J_2$ is a prime bi-ideal of S. Then, $J_i, i = 1, 2$ are prime ideals of S_i. Since $\phi^{-1}(J_i), i = 1, 2$ are prime ideals of R_i, we have $I = \phi^{-1}(J_1) \cup \phi^{-1}(J_2)$ to be a prime bi-ideal of R. □

Definition 2.22 *Let $R = R_1 \cup R_2$ be a commutative biring. Then,*

(1) *R is said to be a bidomain if R_1 and R_2 are integral domains;*

(2) *R is said to be a pseudo bidomain if R_1 and R_2 are integral domains but R has zero divisors;*

(3) *R is said to be a bifield if R_1 and R_2 are fields. If R is finite, we call R a finite bifield. R is said to be a bifield of finite characteristic if the characteristic of both R_1 and R_2 are finite. We call R a bifield of characteristic zero if the characteristic of both R_1 and R_2 is zero. No characteristic is associated with R if R_1 or R_2 is a field of zero characteristic and one of R_1 or R_2 is of some finite characteristic.*

Definition 2.23 *Let $R = R_1 \cup R_2$ be a biring. Then, R is said to be a bidivision ring if R is non-commutative and has no zero-divisors that is R_1 and R_2 are division rings.*

Example 5 (1) Let $R = R_1 \cup R_2$, where $R_1 = \mathcal{Z}$ and $R_2 = \mathcal{R}[x]$ the ring of integers and the ring of polynomials over $\mathcal{R}$, respectively. Since R_1 and R_2 are integral domains, it follows that R is a bidomain.

(2) The biring $R = R_1 \cup R_2$ of Example 1 is a pseudo bidomain.

(3) Let $F = F_1 \cup F_2$ where $F_1 = \mathcal{Q}(\sqrt{p_1})$, $F_2 = \mathcal{Q}(\sqrt{p_2})$ where $p_i, i = 1, 2$ are different primes. Since F_1 and F_2 are fields of zero characteristics, it follows that F is a bi-field of zero characteristic.

Theorem 2.24 *Let $R = R_1 \cup R_2$ be a biring. Then, R is a bidomain if and only if the zero bi-ideal $(0) = (0_1) \cup (0_2)$ is a prime bi-ideal.*

Proof Suppose that R is a bidomain. Then, $R_i, i = 1, 2$ are integral domains. Since the zero ideals (0_i) in R_i are prime, it follows that $(0) = (0_1) \cup (0_2)$ is a prime bi-ideal.

Conversely, suppose that $(0) = (0_1) \cup (0_2)$ is a prime bi-ideal. Then, $(0_i), i = 1, 2$ are prime ideals in R_i and hence $R_i, i = 1, 2$ are integral domains. Thus $R = R_1 \cup R_2$ is a bidomain. □

Theorem 2.25 *Let $F = F_1 \cup F_2$ be a bi-field. Then, $F[x] = F_1[x] \cup F_2[x]$ is a bidomain.*

Proof Since F_1 and F_2 are fields which are integeral domains, it follows that $F_1[x]$ and $F_2[x]$ are integral domains and consequently, $F[x] = F_1[x] \cup F_2[x]$ is a bidomain. □

§3. Further Properties of Birings

Except otherwise stated in this section, all birings are assumed to be commutative with zero and unit elements.

Theorem 3.1 *Let R be any ring and let S_1 and S_2 be any two distinct subrings of R. Then, $S = S_1 \cup S_2$ is a biring.*

Proof Suppose that S_1 and S_2 are two distinct subrings of R. Then, $S_1 \nsubseteq S_2$ or $S_2 \nsubseteq S_1$ but $S_1 \cap S_2 \neq \emptyset$. Since S_1 and S_2 are rings under the same operations inherited from R, it follows that $S = S_1 \cup S_2$ is a biring. □

Corollary 3.2 *Let R_1 and R_2 be any two unrelated rings that is $R_1 \nsubseteq R_2$ or $R_2 \nsubseteq R_1$ but $R_1 \cap R_2 \neq \emptyset$. Then, $R = R_1 \cup R_2$ is a biring.*

Example 6 (1) Let $R = \mathcal{Z}$ and let $S_1 = 2\mathcal{Z}$, $S_2 = 3\mathcal{Z}$. Then, $S = S_1 \cup S_2$ is a biring.

(2) Let $R_1 = \mathcal{Z}_2$ and $R_2 = \mathcal{Z}_3$ be rings of integers modulo 2 and 3, respectively. Then, $R = R_1 \cup R_2$ is a biring.

Example 7 Let $R = R_1 \cup R_2$ be a biring, where $R_1 = \mathcal{Z}$, the ring of integers and $R_2 = C[0, 1]$, the ring of all real-valued continuous functions on $[0, 1]$. Let $I_1 = (p)$, where p is a prime number and let $I_2 = \{f(x) \in R_2 : f(x) = 0\}$. It is clear that I_1 and I_2 are maximal ideals of R_1 and R_2, respectively. Hence, $I = I_1 \cup I_2$ is a maximal bi-ideal of R.

Theorem 3.3 *Let $R = \{0, a, b\}$ be a set under addition and multiplication modulo 2. Then, R is a biring if and only if a and b $(a \neq b)$ are idempotent (nilpotent) in R.*

Proof Suppose that $R = \{0, a, b\}$ is a set under addition and multiplication modulo 2 and suppose that a and b are idempotent (nilpotent) in R. Let $R_1 = \{0, a\}$ and $R_2 = \{0, b\}$, where $a \neq b$. Then, R_1 and R_2 are rings and hence $R = R_1 \cup R_2$ is a biring. The proof of the converse is clear. □

Corollary 3.4 *There exists a biring of order 3.*

Theorem 3.5 *Let $R = R_1 \cup R_2$ be a finite bidomain. Then, R is a bi-field.*

Proof Suppose that $R = R_1 \cup R_2$ is a finite bidomain. Then, each $R_i, i = 1, 2$ is a finite integral domain which is a field. Therefore, R is a bifield. □

Theorem 3.6 *Let $R = R_1 \cup R_2$ be a bi-field. Then, R is a bidomain.*

Proof Suppose that $R = R_1 \cup R_2$ is a bi-field. Then, each $R_i, i = 1, 2$ is a field which is an integral domain. The required result follows from the definition of a bidomain. □

Remark 1 Every finite bidivision ring is a bi-field.

Indeed, suppose that $R = R_1 \cup R_2$ is a finite bidivision ring. Then, each $R_i, i = 1, 2$ is a finite division ring which is a field. Consequently, R is a bi-field.

Theorem 3.7 *Every biring in general need not have bi-ideals.*

Proof Suppose that $R = R_1 \cup R_2$ is a biring and suppose that $I_i, i = 1, 2$ are ideals of R_i. If $I = I_1 \cup I_2$ is such that $I_i \neq I \cap R_i$, where $i = 1, 2$, then I cannot be a bi-ideal of R. □

Corollary 3.8 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$, where $I_i, i = 1, 2$ are ideals of R_i. Then, I is a bi-ideal of R if and only if $I_i = I \cap R_i$, where $i = 1, 2$.*

Corollary 3.9 *A biring $R = R_1 \cup R_2$ may not have a maximal bi-ideal.*

Theorem 3.10 *Let $R = R_1 \cup R_2$ be a biring and let $M = M_1 \cup M_2$ be a bi-ideal of R. Then, R/M is a bi-field if and only if M is a maximal bi-ideal.*

Proof Suppose that M is a maximal bi-ideal of R. Then, each M_i, $i = 1, 2$ is a maximal ideal in R_i, $i = 1, 2$ and consequently, each R_i/I_i is a field and therefore R/M is a bi-field.

Conversely, suppose that R/M is a bi-field. Then, each R_i/M_i, $i = 1, 2$ is a field so that each M_i, $i = 1, 2$ is a maximal ideal in R_i. Hence, $M = I_1 \cup I_2$ is a maximal bi-ideal. □

Theorem 3.11 *Let $R = R_1 \cup R_2$ be a biring and let $P = P_1 \cup P_2$ be a bi-ideal of R. Then, R/P is a bidomain if and only if P is a prime bi-ideal.*

Proof Suppose that P is a prime bi-ideal of R. Then, each P_i, $i = 1, 2$ is a prime ideal in R_i, $i = 1, 2$ and so, each R_i/P_i is an integral domain and therefore R/P is a bidomain.

Conversely, suppose that R/P is a bidomain. Then, each R_i/P_i, $i = 1, 2$ is an integral domain and therefore each P_i, $i = 1, 2$ is a prime ideal in R_i. Hence, $P = P_1 \cup P_2$ is a prime bi-ideal. □

Theorem 3.12 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ be a bi-ideal of R. If I is maximal, then I is prime.*

Proof Suppose that I is maximal. Then, $I_i, i = 1, 2$ are maximal ideals of R_i so that R_i/I_i are fields which are integral domains. Thus, $R/I = (R_1/I_1) \cup (R_2/I_2)$ is a bidomain and by Theorem 3.11, $I = I_1 \cup I_2$ is a prime bi-ideal. □

Theorem 3.13 *Let $\phi : R \to S$ be a biring homomorphism from a biring $R = R_1 \cup R_2$ onto a biring $S = S_1 \cup S_2$ and let $K = Ker\phi_1 \cup Ker\phi_2$ be the kernel of ϕ.*

(1) *If S is a bi-field, then K is a maximal bi-ideal of R;*

(2) *If S is a bidomain, then K is a prime bi-ideal of R.*

Proof By Theorem 2.7, we have $R/K = (R_1/Ker\phi_1) \cup (R_2/Ker\phi_2) \cong Im\phi = Im\phi_1 \cup Im\phi_2 = S_1 \cup S_2 = S$. The required results follow by applying Theorems 3.10 and 3.11. □

Definition 3.14 *Let $R = R_1 \cup R_2$ be a biring and let $N(R)$ be the set of nilpotent elements of R. Then, $N(R)$ is called the bi-nilradical of R if $N(R) = N(R_1) \cup N(R_2)$, where $N(R_i)$,*

$i = 1, 2$ are the nilradicals of R_i.

Theorem 3.15 *Let $R = R_1 \cup R_2$ be a biring. Then, $N(R)$ is a bi-ideal of R.*

Proof $N(R)$ is non-empty since $0_1 \in N(R_1)$ and $0_2 \in N(R_2)$. Now, if $x = x_1 \cup x_2, y_1 \cup y_2 \in N(R)$ and $r = r_1 \cup r_2 \in R$ where $x_i, y_i \in N(R_i), r_i \in R_i, i = 1, 2$, then it follows that $x - y, xr \in N(R)$. Lastly, $R_1 \cap \Big(N(R_1) \cup N(R_2)\Big) = \Big(R_1 \cap N(R_1)\Big) \cup \Big(R_1 \cap N(R_2)\Big) = N(R_1)$. Similarly, we have $R_2 \cap \Big(N(R_1) \cup N(R_2)\Big) = N(R_2)$. Hence, $N(R)$ is a bi-ideal. □

Definition 3.16 *Let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be any two bi-ideals of a biring $R = R_1 \cup R_2$. The set $(I : J)$ is called a bi-ideal quotient of I and J if $(I : J) = (I_1 : J_1) \cup (I_2 : J_2)$, where $(I_i : J_i), i = 1, 2$ are ideal quotients of I_i and J_i. If $I = (0) = (0_1) \cup (0_2)$, a zero bi-ideal, then $\Big((0) : J\Big) = \Big((0_1) : J_1\Big) \cup \Big((0_2) : J_2\Big)$ which is called a bi-annihilator of J denoted by $Ann(J)$. If $0 \neq x \in R_1$ and $0 \neq y \in R_2$, then $Z(R_1) = \bigcup_x Ann(x)$ and $Z(R_2) = \bigcup_y Ann(y)$, where $Z(R_i), i = 1, 2$ are the sets of zero-divisors of R_i.*

Theorem 3.17 *Let $R = R_1 \cup R_2$ be a biring and let $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ be any two bi-ideals of R. Then, $(I : J)$ is a bi-ideal of R.*

Proof For $0 = 0_1 \cup 0_2 \in R$, we have $0_1 \in (I_1 : J_1)$ and $0_2 \in (I_2 : J_2)$ so that $(I : J) \neq \emptyset$. If $x = x_1 \cup x_2, y = y_1 \cup y_2 \in (I : J)$ and $r = r_1 \cup r_2 \in R$, then $x - y, xr \in (I : J)$ since $(I_i : J_i), i = 1, 2$ are ideals of R_i. It can be shown that $R_1 \cap \Big((I_1 : J_1) \cup (I_2 : J_2)\Big) = (I_1 : J_1)$ and $R_2 \cap \Big((I_1 : J_1) \cup (I_2 : J_2)\Big) = (I_2 : J_2)$. Accordingly, $(I : J)$ is a bi-ideal of R. □

Example 8 Under addition and multplication modulo 6, consider the biring $R = \{0, 2, 3, 4\}$, where $R_1 = \{0, 3\}$ and $R_2 = \{0, 2, 4\}$. It is clear that $Z(R) \neq Z(R_1) \cup Z(R_2)$. Hence, for $0 \neq z = x \cup y \in R$, $0 \neq x \in R_1$ and $0 \neq y \in R_2$, we have

$$\bigcup_{z = x \cup y} Ann(z) \neq \Big(\bigcup_x Ann(x)\Big) \cup \Big(\bigcup_y Ann(y)\Big).$$

Definition 3.18 *Let $I = I_1 \cup I_2$ be any bi-ideal of a biring $R = R_1 \cup R_2$. The set r(I) is called a bi-radical of I if $r(I) = r(I_1) \cup r(I_2)$, where $r(I_i), i = 1, 2$ are radicals of I_i. If $I = (0) = (0_1) \cup (0_2)$, then $r(I) = N(R)$.*

Theorem 3.19 *If $R = R_1 \cup R_2$ is a biring and $I = I_1 \cup I_2$ is a bi-ideal of R, then $r(I)$ is a bi-ideal.*

References

[1] Agboola A.A.A. and Akinola L.S., On the Bicoset of a Bivector Space, Int. J. Math. Comb., 4(2009), 1-8.

[2] Davvaz B., *Fundamentals of Ring Theory*, Yazd University, Yazd , 2006.

[3] Jun Young Bae, Biideals in BCK/BCI-bialgebras, *Kyungpook Math. J.*, 48(2008), 577-584.

[4] Jun Young Bae and Lee Kyoung Ja., Ideal theory in BCK/BCI-bialgebras, *Adv. Stud. Contemp. Math.* (Kyungshang) 20(2010), 481-488.

[5] Jun Young Bae, Öztürk, Mehmet Ali and Roh Eun Hwan, BCK/BCI-bialgebras, *Sci. Math. Jpn.*, 64(2006), 595-600.

[6] Maggu P.L., On Introduction of Bigroup Concept with its Applications in Industry, *Pure and App. Math Sci.*, 39(1994), 171-173.

[7] Maggu P.L. and Rajeev K., On Sub-bigroup and its Applications in Industry, *Pure and App. Math Sci.*, 43(1996), 85-88.

[8] Smarandache F., Special Algebraic Structures, in Collected Papers, Abaddaba, Oradea, 3(2000), 78-81.

[9] Vasantha Kandasamy W.B., *Bialgebraic Structures and Smarandache Bialgebraic Structures*, American Research Press, Rehoboth, 2003. http://www.gallup.unm.edu/ smarandache /eBooks-otherformats.htm.

[10] Vasantha Kandasamy W.B., Bivector Spaces, *U. Sci. Phy. Sci.*, 11(1999), 186-190.

International J.Math. Combin. Vol.2(2013), 34-43

Smarandache Directionally n-Signed Graphs — A Survey

P.Siva Kota Reddy

(Department of Mathematics, Acharya Institute of Technology, Soladevanahalli, Bangalore-560 107, India)

E-mail: reddy_math@yahoo.com; pskreddy@acharya.ac.in

Abstract: Let $G = (V, E)$ be a graph. By *directional labeling (or d-labeling)* of an edge $x = uv$ of G by an ordered n-tuple $(a_1, a_2, \cdots, a_n)$, we mean a labeling of the edge x such that we consider the label on uv as $(a_1, a_2, \cdots, a_n)$ in the direction from u to v, and the label on x as $(a_n, a_{n-1}, \cdots, a_1)$ in the direction from v to u. In this survey, we study graphs, called *(n, d)-sigraphs*, in which every edge is d-labeled by an n-tuple $(a_1, a_2, \cdots, a_n)$, where $a_k \in \{+, -\}$, for $1 \leq k \leq n$. Several variations and characterizations of directionally n-signed graphs have been proposed and studied. These include the various notions of balance and others.

Key Words: Signed graphs, directional labeling, complementation, balance.

AMS(2010): 05C 22

§1. Introduction

For graph theory terminology and notation in this paper we follow the book [3]. All graphs considered here are finite and simple. There are two ways of labeling the edges of a graph by an ordered n-tuple $(a_1, a_2, \cdots, a_n)$ (See [10]).

1. *Undirected labeling* or *labeling*. This is a labeling of each edge uv of G by an ordered n-tuple $(a_1, a_2, \cdots, a_n)$ such that we consider the label on uv as $(a_1, a_2, \cdots, a_n)$ irrespective of the direction from u to v or v to u.

2. *Directional labeling* or *d-labeling*. This is a labeling of each edge uv of G by an ordered n-tuple $(a_1, a_2, \cdots, a_n)$ such that we consider the label on uv as $(a_1, a_2, \cdots, a_n)$ in the direction from u to v, and $(a_n, a_{n-1}, \cdots, a_1)$ in the direction from v to u.

Note that the d-labeling of edges of G by ordered n-tuples is equivalent to labeling the symmetric digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, where uv is a symmetric arc in $\overrightarrow{G}$ if, and only if, uv is an edge in G, so that if $(a_1, a_2, \cdots, a_n)$ is the d-label on uv in G, then the labels on the arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are $(a_1, a_2, \cdots, a_n)$ and $(a_n, a_{n-1}, \cdots, a_1)$ respectively.

Let H_n be the *n-fold sign group*, $H_n = \{+, -\}^n = \{(a_1, a_2, \cdots, a_n) : a_1, a_2, \cdots, a_n \in \{+, -\}\}$ with co-ordinate-wise multiplication. Thus, writing $a = (a_1, a_2, \cdots, a_n)$ and $t =$

[1]Reported at the First International Conference on Smarandache Multispaces and Multistructures, June 28-30,2013, Beijing, P.R.China.

[2]Received May 16, 2013, Accepted June 10, 2013.

$(t_1, t_2, \cdots, t_n)$ then $at := (a_1t_1, a_2t_2, \cdots, a_nt_n)$. For any $t \in H_n$, the *action* of t on H_n is $a^t = at$, the co-ordinate-wise product.

Let $n \geq 1$ be a positive integer. An *n-signed graph (n-signed digraph)* is a graph $G = (V, E)$ in which each edge (arc) is labeled by an ordered n-tuple of signs, i.e., an element of H_n. A *signed graph* $G = (V, E)$ is a graph in which each edge is labeled by + or −. Thus a 1-signed graph is a signed graph. Signed graphs are well studied in literature (See for example [1, 4-7, 13-21, 23, 24].

In this survey, we study graphs in which each edge is labeled by an ordered n-tuple $a = (a_1, a_2, \cdots, a_n)$ of signs (i.e, an element of H_n) in one direction but in the other direction its label is the reverse: $a^r = (a_n, a_{n-1}, \cdots, a_1)$, called *directionally labeled n-signed graphs* (or *(n, d)-signed graphs*).

Note that an n-signed graph $G = (V, E)$ can be considered as a symmetric digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, where both $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are arcs if, and only if, uv is an edge in G. Further, if an edge uv in G is labeled by the n-tuple $(a_1, a_2, \cdots, a_n)$, then in $\overrightarrow{G}$ both the arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are labeled by the n-tuple $(a_1, a_2, \cdots, a_n)$.

In [1], the authors study voltage graph defined as follows: A *voltage graph* is an ordered triple $\overrightarrow{G} = (V, \overrightarrow{E}, M)$, where V and $\overrightarrow{E}$ are the vertex set and arc set respectively and M is a group. Further, each arc is labeled by an element of the group M so that if an arc $\overrightarrow{uv}$ is labeled by an element $a \in M$, then the arc $\overrightarrow{vu}$ is labeled by its inverse, a^{-1}.

Since each n-tuple $(a_1, a_2, \cdots, a_n)$ is its own inverse in the group H_n, we can regard an n-signed graph $G = (V, E)$ as a voltage graph $\overrightarrow{G} = (V, \overrightarrow{E}, H_n)$ as defined above. Note that the d-labeling of edges in an (n, d)-signed graph considering the edges as symmetric directed arcs is different from the above labeling. For example, consider a $(4, d)$-signed graph in **F**igure 1. As mentioned above, this can also be represented by a symmetric 4-signed digraph. Note that this is not a voltage graph as defined in [1], since for example; the label on $\overrightarrow{v_2v_1}$ is not the (group) inverse of the label on $\overrightarrow{v_1v_2}$.

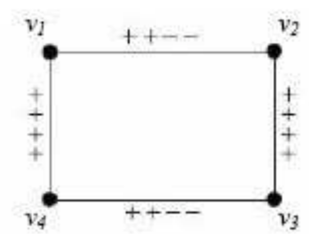


Fig.1

In [8-9], the authors initiated a study of $(3, d)$ and $(4, d)$-Signed graphs. Also, discussed some applications of $(3, d)$ and $(4, d)$-Signed graphs in real life situations.

In [10], the authors introduced the notion of complementation and generalize the notion of balance in signed graphs to the directionally n-signed graphs. In this context, the authors look upon two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge. Also given some motivation to study (n, d)-signed graphs in connection with relations among human beings in society.

In [10], the authors defined complementation and isomorphism for (n, d)-signed graphs as follows: For any $t \in H_n$, the *t-complement* of $a = (a_1, a_2, \cdots, a_n)$ is: $a^t = at$. The *reversal* of $a = (a_1, a_2, \cdots, a_n)$ is: $a^r = (a_n, a_{n-1}, \cdots, a_1)$. For any $T \subseteq H_n$, and $t \in H_n$, the *t-complement* of T is $T^t = \{a^t : a \in T\}$.

For any $t \in H_n$, the *t-complement* of an (n, d)-signed graph $G = (V, E)$, written G^t, is the same graph but with each edge label $a = (a_1, a_2, \cdots, a_n)$ replaced by a^t. The *reversal* G^r is the same graph but with each edge label $a = (a_1, a_2, \cdots, a_n)$ replaced by a^r.

Let $G = (V, E)$ and $G' = (V', E')$ be two (n, d)-signed graphs. Then G is said to be *isomorphic* to G' and we write $G \cong G'$, if there exists a bijection $\phi : V \to V'$ such that if uv is an edge in G which is d-labeled by $a = (a_1, a_2, \cdots, a_n)$, then $\phi(u)\phi(v)$ is an edge in G' which is d-labeled by a, and conversely.

For each $t \in H_n$, an (n, d)-signed graph $G = (V, E)$ is *t-self complementary*, if $G \cong G^t$. Further, G is *self reverse*, if $G \cong G^r$.

Proposition 1.1(E. Sampathkumar et al. [10]) *For all $t \in H_n$, an (n, d)-signed graph $G = (V, E)$ is t-self complementary if, and only if, G^a is t-self complementary, for any $a \in H_n$.*

For any cycle C in G, let $\mathcal{P}(\overrightarrow{C})$ [10] denotes the product of the n-tuples on C given by $(a_{11}, a_{12}, \cdots, a_{1n})(a_{21}, a_{22}, \cdots, a_{2n}) \cdots (a_{m1}, a_{m2}, \cdots, a_{mn})$ and

$\mathcal{P}(\overleftarrow{C}) = (a_{mn}, a_{m(n-1)}, \cdots, a_{m1})(a_{(m-1)n}, a_{(m-1)(n-1)}, \cdots, a_{(m-1)1}) \cdots (a_{1n}, a_{1(n-1)}, \cdots, a_{11})$.

Similarly, for any path P in G, $\mathcal{P}(\overrightarrow{P})$ denotes the product of the n-tuples on P given by $(a_{11}, a_{12}, \cdots, a_{1n})(a_{21}, a_{22}, \cdots, a_{2n}) \cdots (a_{m-1,1}, a_{m-1,2}, \cdots, a_{m-1,n})$ and

$$\mathcal{P}(\overleftarrow{P}) = (a_{(m-1)n}, a_{(m-1)(n-1)}, \cdots, a_{(m-1)1}) \cdots (a_{1n}, a_{1(n-1)}, \cdots, a_{11}).$$

An n-tuple $(a_1, a_2, \cdots, a_n)$ is *identity n-tuple*, if each $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. Further an n-tuple $a = (a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a^r = a$, otherwise it is a *non-symmetric n-tuple*. In (n, d)-signed graph $G = (V, E)$ an edge labeled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Note that the above products $\mathcal{P}(\overrightarrow{C})$ $(\mathcal{P}(\overrightarrow{P}))$ as well as $\mathcal{P}(\overleftarrow{C})$ $(\mathcal{P}(\overleftarrow{P}))$ are n-tuples. In general, these two products need not be equal.

§2. Balance in an (n, d)-Signed Graph

In [10], the authors defined two notions of balance in an (n, d)-signed graph $G = (V, E)$ as follows:

Definition 2.1 *Let $G = (V, E)$ be an (n, d)-sigraph. Then,*

(i) G is identity balanced *(or* i-balanced*), if $P(\overrightarrow{C})$ on each cycle of G is the identity n-tuple, and*

(ii) G is balanced*, if every cycle contains an even number of non-identity edges.*

Note: An i-balanced (n, d)-sigraph need not be balanced and conversely. For example, consider the $(4, d)$-sigraphs in Figure.2. In Figure.2(a) G is an i-balanced but not balanced, and in Figure.2(b) G is balanced but not i-balanced.

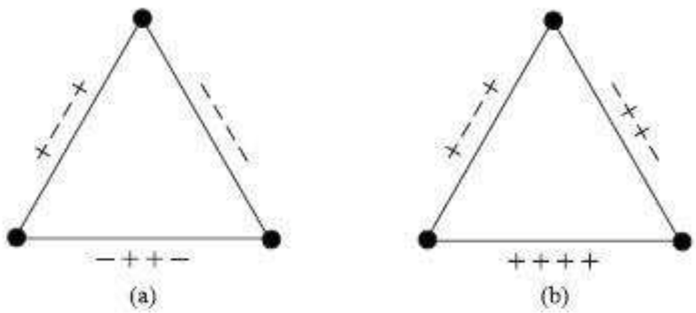


Fig.2

2.1 Criteria for balance

An (n, d)-signed graph $G = (V, E)$ is i-balanced if each non-identity n-tuple appears an even number of times in $P(\overrightarrow{C})$ on any cycle of G.

However, the converse is not true. For example see Figure.3(a). In Figure.3(b), the number of non-identity 4-tuples is even and hence it is balanced. But it is not i-balanced, since the 4-tuple $(++--)$ (as well as $(--++)$) does not appear an even number of times in $P(\overrightarrow{C})$ of 4-tuples.

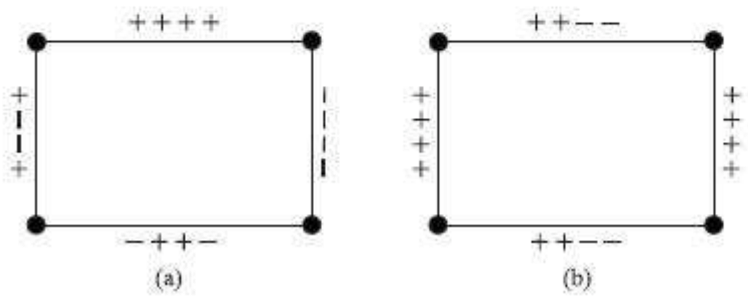


Fig.3

In [10], the authors obtained following characterizations of balanced and i-balanced (n, d)-sigraphs:

Proposition 2.2(E.Sampathkumar et al. [10]) *An (n, d)-signed graph $G = (V, E)$ is balanced if, and only if, there exists a partition $V_1 \cup V_2$ of V such that each identity edge joins two vertices in V_1 or V_2, and each non-identity edge joins a vertex of V_1 and a vertex of V_2.*

As earlier discussed, let $P(C)$ denote the product of the n-tuples in $P(\overrightarrow{C})$ on any cycle C in an (n, d)-sigraph $G = (V, E)$.

Theorem 2.3(E.Sampathkumar et al. [10]) *An (n, d)-signed graph $G = (V, E)$ is i-balanced if, and only if, for each k, $1 \leq k \leq n$, the number of n-tuples in $P(C)$ whose k^{th} co-ordinate is $-$ is even.*

In H_n, let S_1 denote the set of non-identity symmetric n-tuples and S_2 denote the set of non-symmetric n-tuples. The product of all n-tuples in each $S_k, 1 \leq k \leq 2$ is the identity n-tuple.

Theorem 2.4(E.Sampathkumar et al. [10]) *An (n, d)-signed graph $G = (V, E)$ is i-balanced, if both of the following hold:*

(i) In $P(C)$, each n-tuple in S_1 occurs an even number of times, or each n-tuple in S_1 occurs odd number of times (the same parity, or equal mod 2).

(ii) In $P(C)$, each n-tuple in S_2 occurs an even number of times, or each n-tuple in S_2 occurs an odd number of times.

In [11], the authors obtained another characterization of i-balanced (n, d)-signed graphs as follows:

Theorem 2.5(E.Sampathkumar et al. [11]) *An (n, d)-signed graph $G = (V, E)$ is i-balanced if, and only if, any two vertices u and v have the property that for any two edge distinct $u - v$ paths $\overrightarrow{P_1} = (u = u_0, u_1, \cdots, u_m = v$ and $\overrightarrow{P_2} = (u = v_0, v_1, \cdots, v_n = v)$ in G, $\mathcal{P}(\overrightarrow{P_1}) = (\mathcal{P}(\overrightarrow{P_2}))^r$ and $\mathcal{P}(\overrightarrow{P_2}) = (\mathcal{P}(\overrightarrow{P_1}))^r$.*

From the above result, the following are the easy consequences:

Corollary 2.6 *In an i-balanced (n, d)-signed graph G if two vertices are joined by at least 3 paths then the product of n tuples on any paths joining them must be symmetric.*

A graph $G = (V, E)$ is said to be k-connected for some positive integer k, if between any two vertices there exists at least k disjoint paths joining them.

Corollary 2.7 *If the underlying graph of an i-balanced (n, d)-signed graph is 3-connected, then all the edges in G must be labeled by a symmetric n-tuple.*

Corollary 2.8 *A complete (n, d)-signed graph on $p \geq 4$ is i-balanced then all the edges must be labeled by symmetric n-tuple.*

2.2 Complete (n, d)-Signed Graphs

In [11], the authors defined: an (n, d)-sigraph is *complete*, if its underlying graph is complete. Based on the complete (n, d)-signed graphs, the authors proved the following results: An (n, d)-signed graph is *complete*, if its underlying graph is complete.

Proposition 2.9(E.Sampathkumar et al. [11]) *The four triangles constructed on four vertices $\{a, b, c, d\}$ can be directed so that given any pair of vertices say (a, b) the product of the edges of these 4 directed triangles is the product of the n-tuples on the arcs $\overrightarrow{ab}$ and $\overrightarrow{ba}$.*

Corollary 2.10 *The product of the n-tuples of the four triangles constructed on four vertices $\{a, b, c, d\}$ is identity if at least one edge is labeled by a symmetric n-tuple.*

The *i-balance base* with axis a of a complete (n, d)-signed graph $G = (V, E)$ consists list of the product of the n-tuples on the triangles containing a [11].

Theorem 2.11(E.Sampathkumar et al. [11]) *If the i-balance base with axis a and n-tuple of an* *edge adjacent to a is known, the product of the n-tuples on all the triangles of G can be deduced from it.*

In the statement of above result, it is not necessary to know the n-tuple of an edge incident at a. But it is sufficient that an edge incident at a is a symmetric n-tuple.

Theorem 2.12(E.Sampathkumar et al. [11]) *A complete (n, d)-sigraph $G = (V, E)$ is i-balanced if, and only if, all the triangles of a base are identity.*

Theorem 2.13(E.Sampathkumar et al. [11]) *The number of i-balanced complete (n, d)-sigraphs of m vertices is p^{m-1}, where $p = 2^{\lceil n/2 \rceil}$.*

§3. Path Balance in (n, d)-Signed Graphs

In [11], E.Sampathkumar et al. defined the path balance in an (n, d)-signed graphs as follows:

Let $G = (V, E)$ be an (n, d)-sigraph. Then G is

1. *Path i-balanced*, if any two vertices u and v satisfy the property that for any $u - v$ paths P_1 and P_2 from u to v, $\mathcal{P}(\overrightarrow{P}_1) = \mathcal{P}(\overrightarrow{P}_2)$.

2. *Path balanced* if any two vertices u and v satisfy the property that for any $u - v$ paths P_1 and P_2 from u to v have same number of non identity n-tuples.

Clearly, the notion of path balance and balance coincides. That is an (n, d)-signed graph is balanced if, and only if, G is path balanced.

If an (n, d) signed graph G is i-balanced then G need not be path i-balanced and conversely.

In [11], the authors obtained the characterization path i-balanced (n, d)-signed graphs as follows:

Theorem 3.1(Characterization of path i-balanced $(n; d)$ signed graphs) *An (n, d)-signed graph is path i-balanced if, and only if, any two vertices u and v satisfy the property that for any two vertex disjoint $u - v$ paths P_1 and P_2 from u to v, $\mathcal{P}(\overrightarrow{P}_1) = \mathcal{P}(\overrightarrow{P}_2)$.*

§4. Local Balance in (n, d)-Signed Graphs

The notion of local balance in signed graph was introduced by F. Harary [5]. A signed graph $S = (G, \sigma)$ is locally at a vertex v, or S is *balanced at* v, if all cycles containing v are balanced. A cut point in a connected graph G is a vertex whose removal results in a disconnected graph. The following result due to Harary [5] gives interdependence of local balance and cut vertex of a signed graph.

Theorem 4.1(F.Harary [5]) *If a connected signed graph $S = (G, \sigma)$ is balanced at a vertex u. Let v be a vertex on a cycle C passing through u which is not a cut point, then S is balanced at v.*

In [11], the authors extend the notion of local balance in signed graph to (n,d)-signed graphs as follows: Let $G=(V,E)$ be a (n,d)-signed graph. Then for any vertices $v \in V(G)$, G is *locally i-balanced at v* (*locally balanced at v*) if all cycles in G containing v is i-balanced (balanced).

Analogous to the above result, in [11] we have the following for an (n,d) signed graphs:

Theorem 4.2 *If a connected (n,d)-signed graph $G=(V,E)$ is locally i-balanced (locally balanced) at a vertex u and v be a vertex on a cycle C passing through u which is not a cut point, then S is locally i-balanced(locally balanced) at v.*

§5. Symmetric Balance in (n,d)-Signed Graphs

In [22], P.S.K.Reddy and U.K.Misra defined a new notion of balance called *symmetric balance* or *s-balanced* in (n,d)-signed graphs as follows:

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+,-\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil n/2 \rceil$. Let $G=(V,E)$ be an (n,d)-signed graph. Then G is *symmetric balanced* or *s-balanced* if $P(\overrightarrow{C})$ on each cycle C of G is symmetric n-tuple.

Note: If an (n,d)-signed graph $G=(V,E)$ is i-balanced then clearly G is s-balanced. But a s-balanced (n,d)-signed graph need not be i-balanced. For example, the $(4,d)$-signed graphs in Figure 4. G is an s-balanced but not i-balanced.

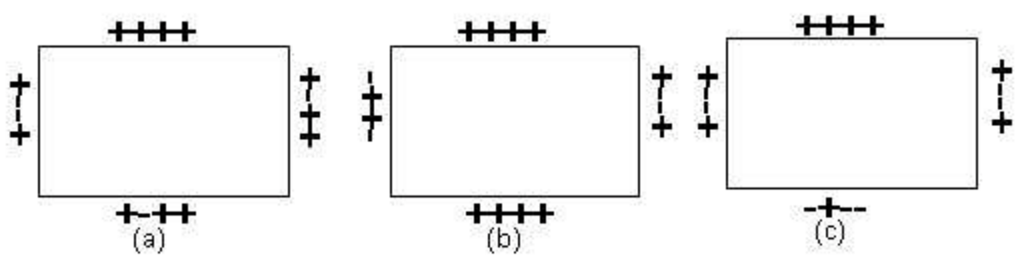


Fig.4

In [22], the authors obtained the following results based on symmetric balance or s-balanced in (n,d)-signed graphs.

Theorem 5.1(P.S.K.Reddy and U.K.Mishra [22]) *A (n,d)-signed graph is s-balanced if and only if every cycle of G contains an even number of non-symmetric n-tuples.*

The following result gives a necessary and sufficient condition for a balanced (n,d)-signed graph to be s-balanced.

Theorem 5.2(P.S.K.Reddy and U.K.Mishra [22]) *A balanced (n,d) signed graph $G=(V,E)$ is s-balanced if and only if every cycle of G contains even number of non identity symmetric n*

tuples.

In [22], the authors obtained another characterization of s-balanced (n,d)-signed graphs, which is analogous to the partition criteria for balance in signed graphs due to Harary [4].

Theorem 5.3(Characterization of s-balanced (n,d)-sigraph) *An (n,d)-signed graph $G=(V,E)$ is s balanced if and only if the vertex set $V(G)$ of G can be partitioned into two sets V_1 and V_2 such that each symmetric edge joins the vertices in the same set and each non-symmetric edge joins a vertex of V_1 and a vertex of V_2.*

An n-marking $\mu : V(G) \to H_n$ of an (n,d)-signed graph $G=(V,E)$ is an assignment n-tuples to the vertices of G. In [22], the authors given another characterization of s-balanced (n,d)-signed graphs which gives a relationship between the n-marking and s-balanced (n,d)-signed graphs.

Theorem 5.4(P.S.K.Reddy and U.K.Mishra [22]) *An (n,d)-signed graph $G=(V,E)$ is s-balanced if and only if there exists an n-marking μ of vertices of G such that if the n-tuple on any arc $\overrightarrow{uv}$ is symmetric or nonsymmetric according as the n-tuple $\mu(u)\mu(v)$ is.*

§6. Directionally 2-Signed Graphs

In [12], E.Sampathkumar et al. proved that the directionally 2-signed graphs are equivalent to bidirected graphs, where each end of an edge has a sign. A bidirected graph implies a signed graph, where each edge has a sign. Signed graphs are the special case $n=1$, where directionality is trivial. Directionally 2-signed graphs (or $(2,d)$-signed graphs) are also special, in a less obvious way. A bidirected graph $\mathrm{B}=(G,\beta)$ is a graph $G=(V,E)$ in which each end (e,u) of an edge $e=uv$ has a sign $\beta(e,u) \in \{+,-\}$. G is the underlying graph and β is the bidirection. (The $+$ sign denotes an arrow on the u-end of e pointed into the vertex u; a $-$ sign denotes an arrow directed out of u. Thus, in a bidirected graph each end of an edge has an independent direction. Bidirected graphs were defined by Edmonds [2].) In view of this, E.Sampathkumar et al. [12] proved the following result:

Theorem 6.1(E.Sampathkumar et al. [12]) *Directionally 2-signed graphs are equivalent to bidirected graphs.*

§7. Conclusion

In this brief survey, we have described directionally n-signed graphs (or (n,d)-signed graphs) and their characterizations. Many of the characterizations are more recent. This in an active area of research. We have included a set of references which have been cited in our description. These references are just a small part of the literature, but they should provide a good start for readers interested in this area.

References

[1] B.D.Acharya and M. Acharya, New algebraic models of a Social System, *Indian J. of Pure and Appl. Math.*, 17(2) 1986, 152-168.

[2] J.Edmonds and E.L.Johnson, Matching: a well-solved class of integral linear programs, in: Richard Guy et al., eds., *Combinatorial Structures and Their Applications* (Proc. Calgary Int. Conf., Calgary, 1969), pp. 89-92. Gordon and Breach, New York, 1970.

[3] F.Harary, *Graph Theory*, Addison-Wesley Publishing Co., 1969.

[4] F.Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, 2(1953), 143-146.

[5] F.Harary, On local balance and N-balance in signed graphs, *Michigan Math. J.*, 3(1955), 37-41.

[6] F.Harary, R. Norman and D. Cartwright, *Structural models: An introduction to the theory of directed graphs*, J. Wiley, New York, 1965.

[7] R.Rangarajan, M. S.Subramanya and P.Siva Kota Reddy, The H-Line Signed Graph of a Signed Graph, *International J. Math. Combin.*, 2 (2010), 37-43.

[8] E.Sampathkumar, P.Siva Kota Reddy and M. S.Subramanya, $(3, d)$-sigraph and its applications, *Advn. Stud. Contemp. Math.*, 17(1) (2008), 57-67.

[9] E.Sampathkumar, P.Siva Kota Reddy and M. S.Subramanya, $(4, d)$-sigraph and its applications, *Advn. Stud. Contemp. Math.*, 20(1) (2010), 115-124.

[10] E.Sampathkumar, P.Siva Kota Reddy, and M.S.Subramanya, Directionally n-signed graphs, in: B.D. Acharya et al., eds., *Advances in Discrete Mathematics and Applications: Mysore, 2008* (Proc. Int. Conf. Discrete Math., ICDM-2008), pp. 153-160, Ramanujan Math. Soc. Lect. Notes Ser., 13, Ramanujan Mathematical Society, Mysore, India, 2010.

[11] E.Sampathkumar, P.Siva Kota Reddy, and M.S.Subramanya, Directionally n-signed graphs-II, *Int. J. Math. Combin.*, 4 (2009), 89-98.

[12] E.Sampathkumar, M.A.Sriraj and T.Zaslavsky, Directionally 2-signed and bidirected graphs.

[13] P.Siva Kota Reddy and M.S.Subramanya, Signed graph equation $L^k(S) \sim \overline{S}$, *International J. Math. Combin.*, 4 (2009), 84-88.

[14] P.Siva Kota Reddy, S.Vijay and V.Lokesha, n^{th} Power signed graphs, *Proceedings of the Jangjeon Math. Soc.*, 12(3) (2009), 307-313.

[15] P.Siva Kota Reddy, S.Vijay and H.C.Savithri, A Note on Path Sidigraphs, *International J. Math. Combin.*, 1 (2010), 42-46.

[16] P.Siva Kota Reddy, S.Vijay and V.Lokesha, n^{th} Power signed graphs-II, *International J. Math. Combin.*, 1 (2010), 74-79.

[17] P.Siva Kota Reddy and S.Vijay, Total minimal dominating signed graph, *International J. Math. Combin.*, 3 (2010), 11-16.

[18] P.Siva Kota Reddy and K.V.Madhusudhan, Negation switching equivalence in signed graphs, *International J. Math. Combin.*, 3 (2010), 85-90.

[19] P.Siva Kota Reddy, t-Path Sigraphs, *Tamsui Oxford J. of Math. Sciences*, 26(4) (2010), 433-441.

[20] P.Siva Kota Reddy, B.Prashanth and Kavita S. Permi, A Note on Antipodal Signed Graphs, *International J. Math. Combin.*, 1 (2011), 107-112.

[21] P.Siva Kota Reddy, B.Prashanth, and T.R.Vasanth Kumar, Antipodal Signed Directed Graphs, *Advn. Stud. Contemp. Math.*, 21(4) (2011), 355-360.

[22] P.Siva Kota Reddy and U.K.Misra, Directionally n-signed graphs-III: The notion of symmetric balance, Submitted.

[23] T.Zaslavsky, Signed Graphs, *Discrete Appl. Math.*, 4(1)(1982), 47-74.

[24] T.Zaslavsky, A mathematical bibliography of Signed and Gain graphs and its allied areas, *Electron. J. Combin.*, Dynamic Surveys in Combinatorics (1998), No. DS8. Eighth ed. (2012).

International J.Math. Combin. Vol.2(2013), 44-53

Characterizations of the Quaternionic Mannheim Curves In Euclidean space $\mathbb{E}^4$

O.Zeki Okuyucu

(Department of Mathematics, University of Bilecik Şeyh Edebali, Turkey)

E-mail: osman.okuyucu@bilecik.edu.tr

Abstract: In [5], Matsuda and Yorozu obtained that Mannheim curves in 4-dimensional Euclidean space. In this study, we define *quaternionic Mannheim curves* and we give some characterizations of them in Euclidean 3−space and 4−space.

Key Words: Quaternion algebra, Mannheim curve, Euclidean space.

AMS(2010): 11R52, 53A04

§1. Introduction

The geometry of curves in a Euclidean space have been developed a long time ago and we have a deep knowledge about it. In the theory of curves in Euclidean space, one of the important and interesting problem is characterizations of a regular curve. We can characterize some curves via their relations between the Frenet vectors of them. For instance Mannheim curve is a special curve and it is characterized using the Frenet vectors of its Mannheim curve couple.

In 2007, the definition of Mannheim curves in Euclidean 3-space is given by H.Liu and F.Wang [4] with the following:

Definition 1.1 *Let α and β be two curves in Euclidean 3-space. If there exists a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of β, then α is called a Mannheim curve, and β is called a Mannheim partner curve of α.*

In their paper, they proved that a given curve is a Mannheim curve if and only if then for $\lambda \in \mathbb{R}$, it has $\lambda\left(\varkappa^2+\tau^2\right)=\varkappa$, where $\varkappa$ and τ are curvature functions of curve. Also in 2009, Matsuda and Yorozu, in [5], defined generalized Mannheim curves in Euclidean 4-space. If the first normal line at each point of α is included in the plane generated by the second normal line and the third normal line of β at corresponding point under a bijection, which is from α to β. Then the curve α is called generalized Mannheim curve and the curve β is called generalized Mannheim mate curve of α. And they gave a theorem such that if the curve α is a generalized Mannheim curve in Euclidean 4-space, then the first curvature function k_1 and

[1]Received January 24, 2013, Accepted June 12, 2013.

second curvature functions k_2 of the curve α satisfy the equality:

$$k_1(s) = \mu\left\{(k_1(s))^2 + (k_2(s))^2\right\},$$

where μ is a positive constant number. The quaternion was introduced by Hamilton. His initial attempt to generalize the complex numbers by introducing a three-dimensional object failed in the sense that the algebra he constructed for these three-dimensional object did not have the desired properties. On the 16th October 1843 Hamilton discovered that the appropriate generalization is one in which the scalar(real) axis is left unchanged whereas the vector(imaginary) axis is supplemented by adding two further vectors axes.

In 1987, The Serret-Frenet formulas for a quaternionic curve in $\mathbb{E}^3$ and $\mathbb{E}^4$ was defined by Bharathi and Nagaraj, in [7], and then in 2004, Serret-Frenet formulas for quaternionic curves and quaternionic inclined curves have been defined in Semi-Euclidean space by Çöken and Tuna in 2004, [1].

In 2011 Güngör and Tosun studied quaternionic rectifying curves, [8]. Also, Gök et.al and Kahraman et.al, in [6,3], defined a new kind of slant helix in Euclidean space $\mathbb{E}^4$ and semi-Euclidean space $\mathbb{E}^4_2$. It called quaternionic B_2-slant helix in Euclidean space $\mathbb{E}^4$ and semi-real quaternionic B_2-slant helix in semi-Euclidean space $\mathbb{E}^4_2$, respectively. Recently, Sağlam, in [2], has studied on the osculating spheres of a real quaternionic curve in Euclidean 4−space. In this paper, we define quaternionic Mannheim curves and we give some characterizations of them in Euclidean 3 and 4 space.

§2. Preliminaries

Let Q_H denotes a four dimensional vector space over the field H of characteristic grater than 2. Let e_i $(1 \le i \le 4)$denote a basis for the vector space. Let the rule of multiplication on Q_H be defined on e_i $(1 \le i \le 4)$ and extended to the whole of the vector space by distributivity as follows:

A real quaternion is defined with $q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + de_4$ where a, b, c, d are ordinary numbers. Such that

$$\begin{aligned} &e_4 = 1, && e_1^2 = e_2^2 = e_3^2 = -1, \\ &e_1e_2 = e_3, && e_2e_3 = e_1,\ e_3e_1 = e_2, \\ &e_2e_1 = -e_3, && e_3e_2 = -e_1,\ e_1e_3 = -e_2. \end{aligned} \tag{2.1}$$

If we denote $S_q = d$ and $\overrightarrow{V_q} = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$, we can rewrite real quaternions the basic algebraic form $q = S_q + \overrightarrow{V_q}$ where S_q is scalar part of q and $\overrightarrow{V_q}$ is vectorial part. Using these basic products we can now expand the product of two quaternions to give

$$p \times q = S_pS_q - \langle\overrightarrow{V_p},\ \overrightarrow{V_q}\rangle + S_p\overrightarrow{V_q} + S_q\overrightarrow{V_p} + \overrightarrow{V_p} \wedge \overrightarrow{V_q} \text{ for every } p, q \in Q_H, \tag{2.2}$$

where we have use the inner and cross products in Euclidean space $\mathbb{E}^3$, [7]. There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol γ and defined as follows:

$$\gamma q = -a\overrightarrow{e_1} - b\overrightarrow{e_2} - c\overrightarrow{e_3} + de_4 \text{ for every } q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + de_4 \in Q_H,$$

which is called the "Hamiltonian conjugation". This defines the symmetric, real valued, non-degenerate, bilinear form h are follows:

$$h(p,\ q) = \frac{1}{2}\,[\ p \times \gamma q + q \times \gamma p]\ \text{ for } p, q \in Q_H.$$

And then, the norm of any q real quaternion denotes

$$\|q\|^2 = h(q,q) = q \times \gamma q. \tag{2.3}$$

The concept of a spatial quaternion will be used of throughout our work. q is called a spatial quaternion whenever $q + \gamma q = 0$, [2].

The Serret-Frenet formulae for quaternionic curves in $\mathbb{E}^3$ and $\mathbb{E}^4$ are follows:

Theorem 2.1([7]) *The three-dimensional Euclidean space $\mathbb{E}^3$ is identified with the space of spatial quaternions $\{p \in Q_H \mid p + \gamma p = 0\}$ in an obvious manner. Let $I = [0,1]$ denotes the unit interval of the real line $\mathbb{R}$. Let*

$$\begin{array}{llll} \alpha : & I \subset \mathbb{R} & \longrightarrow & Q_H \\ & s & \longrightarrow & \alpha(s) = \sum\limits_{i=1}^{3} \alpha_i(s)\overrightarrow{e_i},\ 1 \le i \le 3, \end{array}$$

be an arc-lenghted curve with nonzero curvatures $\{k,\ r\}$ and $\{t(s), n(s), b(s)\}$ denotes the Frenet frame of the curve α. Then Frenet formulas are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{2.4}$$

where k is the principal curvature, r is torsion of α.

Theorem 2.2([7]) *The four-dimensional Euclidean spaces $\mathbb{E}^4$ is identified with the space of quaternions. Let $I = [0,1]$ denotes the unit interval of the real line $\mathbb{R}$. Let*

$$\begin{array}{llll} \alpha^{(4)} : & I \subset \mathbb{R} & \longrightarrow & Q_H \\ & s & \longrightarrow & \alpha^{(4)}(s) = \sum\limits_{i=1}^{4} \alpha_i(s)\overrightarrow{e_i},\ 1 \le i \le 4,\ \overrightarrow{e_4} = 1, \end{array}$$

be a smooth curve in $\mathbb{E}^4$ with nonzero curvatures $\{K,\ k,\ r-K\}$ and $\{T(s), N(s), B_1(s), B_2(s)\}$ denotes the Frenet frame of the curve α. Then Frenet formulas are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r-K) \\ 0 & 0 & -(r-K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2.5}$$

where K is the principal curvature, k is the torsion and $(r-K)$ is bitorsion of $\alpha^{(4)}$.

§3. Characterizations of the Quaternionic Mannheim Curve

In this section, we define quaternionic Mannheim curves and we give some characterizations of them in Euclidean 3 and 4 space.

Definition 3.1 *Let $\alpha(s)$ and $\beta(s^*)$ be two spatial quaternionic curves in $\mathbb{E}^3$. Let $\{t(s),\ n(s),\ b(s)\}$ and $\{t^*(s^*),\ n^*(s^*),\ b^*(s^*)\}$ be Frenet frames,respectively, on these curves. $\alpha(s)$ and $\beta(s^*)$ are called spatial quaternionic Mannheim curve couple if $n(s)$ and $b^*(s^*)$ are linearly dependent.*

Theorem 3.2 *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a spatial quaternionic Mannheim curve with the arc lenght parameter s and $\beta : \overline{I} \subset \mathbb{R} \to \mathbb{E}^3$ be spatial quaternionic Mannheim partner curve of α with the arc lenght parameter s^*. Then*

$$d(\alpha(s), \beta(s^*)) = \text{constant}, \quad \text{for all } s \in I.$$

Proof From Definition 3.1, we can write

$$\alpha(s) = \beta(s^*) + \lambda^*(s^*)\, b^*(s^*) \tag{3.1}$$

Differentiating the Eq.(3.1) with respect to s^* and by using the Frenet equations, we get

$$\frac{d\alpha(s)}{ds}\frac{ds}{ds^*} = t^*(s^*) + \lambda^{*'}(s^*) b^*(s^*) - \lambda^*(s^*) r^*(s^*) n^*(s^*)$$

If we denote $\dfrac{d\alpha(s)}{ds} = t(s)$

$$t(s) = \frac{ds^*}{ds}\left[t^*(s^*) + \lambda^{*'}(s^*) b^*(s^*) - \lambda^*(s^*) r^*(s^*) n^*(s^*)\right]$$

and

$$h(t(s), n(s)) = \frac{ds^*}{ds}\begin{bmatrix} h(t^*(s^*), n(s)) + \lambda^{*'}(s^*)\, h(b^*(s^*), n(s)) \\ -\lambda^*(s^*) r^*(s^*) h(n^*(s^*), n(s)) \end{bmatrix}.$$

Since $\{n(s), b^*(s^*)\}$ is a linearly dependent set, we get

$$\lambda^{*'}(s^*) = 0$$

that is, λ^* is constant function on $\overline{I}$. This completes the proof. □

Theorem 3.3 *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be spatial quaternionic curves with the arc-length parameter s. Then α is spatial quaternionic Mannheim curve if and only if*

$$k(s) = \lambda\left(k^2(s) + r^2(s)\right), \tag{3.2}$$

where λ is constant.

Proof If α is spatial quaternionic Mannheim curve, we can write

$$\beta(s) = \alpha(s) + \lambda(s) n(s)$$

Differentiating the above equality and by using the Frenet equations, we get

$$\frac{d\beta(s)}{ds} = \left[(1-\lambda(s)k\,(s))\,t(s) + \lambda'(s)n(s) + \lambda(s)r\,(s)\,b\,(s)\right].$$

As $\{n\,(s)\,, b^*(s^*)\}$ is a linearly dependent set, we get

$$\lambda'(s) = 0.$$

This means that λ is constant. Thus we have

$$\frac{d\beta(s)}{ds} = (1-\lambda k\,(s))\,t(s) + \lambda r\,(s)\,b\,(s)\,.$$

On the other hand, we have

$$t^* = \frac{d\beta}{ds}\frac{ds}{ds^*} = \left[(1-\lambda k\,(s))\,t(s) + \lambda r\,(s)\,b\,(s)\right]\frac{ds}{ds^*}.$$

By taking the derivative of this equation with respect to s^* and applying the Frenet formulas we obtain

$$\frac{dt^*}{ds}\frac{ds}{ds^*} = \left[-\lambda k'(s)t(s) + \left(k(s) - \lambda k^2(s) - \lambda r^2(s)\right)n(s) + \lambda r'(s)b(s)\right]\left(\frac{ds}{ds^*}\right)^2$$
$$+ \left[(1-\lambda k\,(s))\,t(s) + \lambda r\,(s)\,b\,(s)\right]\frac{d^2s}{ds^{*2}}.$$

From this equation, we get

$$k(s) = \lambda\left(k^2(s) + r^2(s)\right).$$

Conversely, if $k(s) = \lambda\left(k^2(s) + r^2(s)\right)$, then we can easily see that α is a Mannheim curve. □

Theorem 3.4 *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be spatial quaternionic Mannheim curve with arc-length parameter s. Then β is the spatial quaternionic Mannheim partner curve of α if and only if the curvature functions $k^*\,(s^*)$ and $r^*(s^*)$ of β satisfy the following equation*

$$\frac{dr^*}{ds^*} = \frac{k^*}{\mu}\left(1 + \mu^2 {r^*}^2\right),$$

where μ is constant.

Proof Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be spatial quaternionic Mannheim curve. Then, we can write

$$\alpha\,(s^*) = \beta\,(s^*) + \mu\,(s^*)\,b^*\,(s^*) \tag{3.3}$$

for some function $\mu\,(s^*)$. By taking the derivative of Eq.(3.3) with respect to s^* and using the Frenet equations, we obtain

$$t\frac{ds}{ds^*} = t^*\,(s^*) + \mu'\,(s^*)\,b^*\,(s^*) - \mu\,(s^*)\,r^*\,(s^*)\,n^*\,(s^*)\,.$$

And then, we know that $\{n\,(s)\,, b^*\,(s^*)\}$ is a linearly dependent set, so we have

$$\mu'\,(s^*) = 0.$$

This means that $\mu(s^*)$ is a constant function. Thus we have

$$t\frac{ds}{ds^*} = t^*(s^*) - \mu r^*(s^*) n^*(s^*). \tag{3.4}$$

On the other hand, we have

$$t = t^* \cos\theta + n^* \sin\theta, \tag{3.5}$$

where θ is the angle between t and t^* at the corresponding points of the curves α and β. By taking the derivative of Eq.(3.5) with respect to s^* and using the Frenet equations, we obtain

$$kn\frac{ds}{ds^*} = -\left(k^* + \frac{d\theta}{ds^*}\right)\sin\theta t^* + \left(k^* + \frac{d\theta}{ds^*}\right)\cos\theta n^* + r^* \sin\theta b^*.$$

From this equation and the fact that the $\{n(s), b^*(s^*)\}$ is a linearly dependent set, we get

$$\begin{cases} \left(k^* + \frac{d\theta}{ds^*}\right)\sin\theta = 0 \\ \left(k^* + \frac{d\theta}{ds^*}\right)\cos\theta = 0. \end{cases}$$

For this reason, we have

$$\frac{d\theta}{ds^*} = -k^*. \tag{3.6}$$

From the Eq.(3.4), Eq.(3.5) and notice that t^* is orthogonal to b^*, we find that

$$\frac{ds}{ds^*} = \frac{1}{\cos\theta} = -\frac{\mu r^*}{\sin\theta}.$$

Then we have

$$\mu r^* = -\tan\theta.$$

By taking the derivative of this equation and applying Eq.(3-6), we get

$$\mu\frac{dr^*}{ds^*} = k^*\left(1 + \mu^2 {r^*}^2\right)$$

that is

$$\frac{dr^*}{ds^*} = \frac{k^*}{\mu}\left(1 + \mu^2 {r^*}^2\right).$$

Conversely, if the curvature k^* and torsion r^* of the curve β satisfy the equality:

$$\frac{dr^*}{ds^*} = \frac{k^*}{\mu}\left(1 + \mu^2 {r^*}^2\right)$$

for constant μ, then we define a curve by

$$\alpha(s^*) = \beta(s^*) + \mu b^*(s^*) \tag{3.7}$$

and we will show that α is a spatial quaternionic Mannheim curve and β is the spatial quaternionic Mannheim partner curve of α. By taking the derivative of Eq.(3.7) with respect to s^* twice, we get

$$t\frac{ds}{ds^*} = t^* - \mu r^* n^*, \tag{3.8}$$

$$kn\left(\frac{ds}{ds^*}\right)^2 + t\frac{d^2s}{d{s^*}^2} = \mu k^* r^* t^* + \left(k^* - \mu\frac{dr^*}{ds^*}\right)n^* - \mu {r^*}^2 b^*, \tag{3.9}$$

respectively. Taking the cross product of Eq.(3.8) with Eq.(3.9) and noticing that

$$k^* - \mu \frac{dr^*}{ds^*} + \mu^2 k^* {r^*}^2 = 0,$$

we have

$$kb \left(\frac{ds}{ds^*} \right)^3 = \mu^2 {r^*}^3 t^* + \mu {r^*}^2 n^*. \tag{3.10}$$

By taking the cross product of Eq.(3.8) with Eq.(3.10), we get

$$kn \left(\frac{ds}{ds^*} \right)^4 = -\mu {r^*}^2 \left(1 + \mu^2 {r^*}^2 \right) b^*.$$

This means that the principal normal vector field of the spatial quaternionic curve α and binormal vector field of the spatial quaternionic curve β are linearly dependent set. And so α is a spatial quaternionic Mannheim curve and β is spatial quaternionic Mannheim partner curve of α. □

Theorem 3.5 *Let $\{\alpha, \beta\}$ be a spatial quaternionic Mannheim curve couple in $\mathbb{E}^3$. Then measure of the angle θ between the tangent vector fields of spatial quaternionic curves $\alpha(s)$ and $\beta(s^*)$ is constant if and only if the spatial quaternionic curve $\beta(s^*)$ is a geodesic.*

Proof From Eq.(3-6), we know that $\frac{d\theta}{ds^*} = -k^*$, where θ is the angle between t and t^* at the corresponding points of the curves α and β.

If θ is a constant angle, the curvature of the curve β,

$$k^* = 0,$$

that is the curve β is a geodesic.

Conversely, if the curve β is a geodesic, the angel θ between t and t^* at the corresponding points of the curves α and β satisfy the following equality:

$$\frac{d\theta}{ds^*} = 0,$$

that is θ is a constant angle. □

Definition 3.6 *A quaternionic curve $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ is a quaternionic Mannheim curve if there exists a quaternionic curve $\beta^{(4)} : \overline{I} \subset \mathbb{R} \to \mathbb{E}^4$ such that the second Frenet vector at each point of $\alpha^{(4)}$ is included the plane generated by the third Frenet vector and the fourth Frenet vector of $\beta^{(4)}$ at corresponding point under φ, where φ is a bijection from $\alpha^{(4)}$ to $\beta^{(4)}$. The curve $\beta^{(4)}$ is called the quaternionic Mannheim partner curve of $\alpha^{(4)}$.*

Theorem 3.7 *Let $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ and $\beta^{(4)} : \overline{I} \subset \mathbb{R} \to \mathbb{E}^4$ be quaternionic Mannheim curve couple with arc-length s and $\overline{s}$, respectively. Then*

$$d\left(\alpha^{(4)}(s), \beta^{(4)}(\overline{s}) \right) = \lambda(s) = constant, \;\; for\ all\ s \in I \tag{3.11}$$

Proof From the Definition 3.6, quaternionic Mannheim partner curve $\beta^{(4)}$ of $\alpha^{(4)}$ is given by the following equation

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda(s) N(s),$$

where $\lambda(s)$ is a smooth function. A smooth function $\psi : s \in I \to \psi(s) = \overline{s} \in \overline{I}$ is defined by

$$\psi(s) = \int_0^s \left\| \frac{d\alpha^{(4)}(s)}{ds} \right\| ds = \overline{s}.$$

The bijection φ:$\alpha^{(4)} \to \beta^{(4)}$ is defined by $\varphi\left(\alpha^{(4)}(s)\right) = \beta^{(4)}(\psi(s))$. Since the second Frenet vector at each point of $\alpha^{(4)}$ is included the plane generated by the third Frenet vector and the fourth Frenet vector of $\beta^{(4)}$ at corresponding point under φ, for each $s \in I$, the Frenet vector $N(s)$ is given by the linear combination of Frenet vectors $\overline{B}_1(\psi(s))$ and $\overline{B}_2(\psi(s))$ of $\beta^{(4)}$, that is, we can write

$$N(s) = g(s)\overline{B}_1(\psi(s)) + h(s)\overline{B}_2(\psi(s)),$$

where $g(s)$ and $h(s)$ are smooth functions on I. So we can write

$$\beta^{(4)}(\psi(s)) = \alpha^{(4)}(s) + \lambda(s)N(s). \tag{3.12}$$

Differentiating the Eq.(3.12) with respect to s and by using the Frenet equations, we get

$$\overline{T}(\psi(s))\,\psi'(s) = [(1 - \lambda K(s))\,T(s) + \lambda'(s)\,N(s) + \lambda(s)\,k(s)\,B_1(s)].$$

By the fact that:

$$h\left(\overline{T}(\varphi(s)), g(s)\overline{B}_1(\psi(s)) + h(s)\overline{B}_2(\psi(s))\right) = 0,$$

we have

$$\lambda'(s) = 0$$

that is, $\lambda(s)$ is constant function on I. This completes the proof. □

Theorem 3.78 *If the quaternionic curve $\alpha^{(4)} : I \subset \mathbb{R} \to \mathbb{E}^4$ is a quaternionic Mannheim curve, then the first and second curvature functions of $\alpha^{(4)}$ satisfy the equality:*

$$K(s) = \lambda\left\{K^2(s) + k^2(s)\right\},$$

where λ is constant.

Proof Let $\beta^{(4)}$ be a quaternionic Mannheim partner curve of $\alpha^{(4)}$. Then we can write

$$\beta^{(4)}(\psi(s)) = \alpha^{(4)}(s) + \lambda N(s).$$

Differentiating the above equation, we get

$$\overline{T}(\psi(s))\,\psi'(s) = [(1 - \lambda K(s))\,T(s) + \lambda k(s)\,B_1(s)],$$

that is,

$$\overline{T}(\psi(s)) = \frac{1 - \lambda K(s)}{\psi'(s)}T(s) + \frac{\lambda k(s)}{\psi'(s)}B_1(s),$$

where $\psi^{\shortmid}(s)=\sqrt{(1-\lambda K(s))^2+(\lambda k(s))^2}$ for $s\in I$. By differentiation of both sides of the above equality with respect to s, we have

$$\begin{aligned}\psi^{\shortmid}(s)\overline{K}(\overline{s})\overline{N}(\overline{s}) &= \left(\frac{1-\lambda K(s)}{\psi^{\shortmid}(s)}\right)^{\shortmid} T(s) \\ &+ \left(\frac{(1-\lambda K(s))K(s)-\lambda k(s)^2}{\psi^{\shortmid}(s)}\right) N(s) \\ &+ \left(\frac{\lambda k(s)}{\psi^{\shortmid}(s)}\right)^{\shortmid} B_1(s) - \frac{\lambda k(s)(r(s)-K(s))}{\psi^{\shortmid}(s)} B_2(s).\end{aligned}$$

By the fact:

$$h\left(\overline{N}(\varphi(s)), g(s)\overline{B}_1(\psi(s))+h(s)\overline{B}_2(\psi(s))\right)=0,$$

we have that coefficient of N in the above equation is zero, that is,

$$(1-\lambda K(s))K(s)-\lambda k(s)^2=0.$$

Thus, we have

$$K(s)=\lambda\left\{K^2(s)+k^2(s)\right\}$$

for $s\in I$. This completes the proof. □

Theorem 3.9 *Let $\alpha^{(4)}: I\subset\mathbb{R}\to\mathbb{E}^4$ be a quaternionic curve with arc-length s, whose curvature functions $K(s)$ and $k(s)$ are non-constant functions and satisfy the equality: $K(s)=\lambda\left\{K^2(s)+k^2(s)\right\}$, where λ is positive constant number. If the quaternionic curve $\beta^{(4)}$ is given by $\beta^{(4)}(\overline{s})=\alpha^{(4)}(s)+\lambda N(s)$, then $\alpha^{(4)}$ is a quaternionic Mannheim curve and $\beta^{(4)}$ is the quaternionic Mannheim partner curve of $\alpha^{(4)}$.*

Proof Let $\overline{s}$ be the arc-length of the quaternionic curve $\beta^{(4)}$. That is, $\overline{s}$ is defined by

$$\overline{s}=\int_0^s\left\|\frac{d\alpha^{(4)}(s)}{ds}\right\|ds$$

for $s\in I$. We can write a smooth function $\psi: s\in I\to\psi(s)=\overline{s}\in\overline{I}$. By the assumption of the curvature functions $K(s)$ and $k(s)$, we have

$$\begin{aligned}\psi^{\shortmid}(s)&=\sqrt{(1-\lambda K(s))^2+(\lambda k(s))^2},\\ \psi^{\shortmid}(s)&=\sqrt{1-\lambda K(s)}\end{aligned}$$

for $s\in I$. Then we can easily write

$$\begin{aligned}\beta^{(4)}(\overline{s})&=\beta^{(4)}(\psi(s))\\ &=\alpha^{(4)}(s)+\lambda N(s)\end{aligned}$$

for the quaternionic curve $\beta^{(4)}$. If we differentiate both sides of the above equality with respect to s, we get

$$\psi^{\shortmid}(s)\overline{T}(\psi(s))=T(s)+\lambda\left\{-K(s)T(s)+k(s)B_1(s)\right\}.$$

And so we have,

$$\overline{T}\left(\psi(s)\right)=\sqrt{1-\lambda K(s)}T(s)+\frac{\lambda k(s)}{\sqrt{1-\lambda K(s)}}B_1(s). \tag{3.13}$$

Differentiating the above equality with respect to s and by using the Frenet equations, we get

$$\psi'(s)\overline{K}\left(\psi(s)\right)\overline{N}\left(\psi(s)\right)=\left(\sqrt{1-\lambda K(s)}\right)'T(s)+\left(\frac{K(s)\left(1-\lambda K(s)\right)-\lambda k^2(s)}{\sqrt{1-\lambda K(s)}}\right)N(s)$$
$$+\left(\frac{\lambda k(s)}{\sqrt{1-\lambda K(s)}}\right)'B_1(s)+\frac{\lambda k(s)\left(r(s)-K(s)\right)}{\sqrt{1-\lambda K(s)}}B_2(s)$$

From our assumption, it holds

$$\frac{K(s)\left(1-\lambda K(s)\right)-\lambda k^2(s)}{\sqrt{1-\lambda K(s)}}=0.$$

We find the coefficient of $N(s)$ in the above equality vanishes. Thus the vector $\overline{N}\left(\psi(s)\right)$ is given by linear combination of $T(s)$, $B_1(s)$ and $B_2(s)$ for each $s\in I$. And the vector $\overline{T}\left(\psi(s)\right)$ is given by linear combination of $T(s)$ and $B_1(s)$ for each $s\in I$ in the Eq.(3.13). As the curve $\beta^{(4)}$ is quaternionic curve in $\mathbb{E}^4$, the vector $N(s)$ is given by linear combination of $\overline{B}_1\left(\overline{s}\right)$ and $\overline{B}_2\left(\overline{s}\right)$. For this reason, the second Frenet curve at each point of $\alpha^{(4)}$ is included in the plane generated the third Frenet vector and the fourth Frenet vector of $\beta^{(4)}$ at corresponding point under φ. Here the bijection $\varphi:\alpha^{(4)}\to\beta^{(4)}$ is defined by $\varphi\left(\alpha^{(4)}(s)\right)=\beta^{(4)}(\psi\left(s\right))$. This completes the proof. □

References

[1] A.C.Çöken, A. Tuna, On the quaternionic inclined curves in the semi-Euclidean space $\mathbb{E}^4_2$, *Appl. Math. Comput.*, 155 (2004), 373-389.

[2] D.Sağlam, On the Osculating Spheres of a Real Quaternionic Curve in the Euclidean Space $\mathbb{E}^4$, *International J.Math. Combin.*, Vol.3(2012), 46-53.

[3] F.Kahraman, İ. Gök, H. H. Hacısalihoğlu, On the Quaternionic B_2 Slant helices In the semi-Euclidean Space $\mathbb{E}^4_2$, *Appl. Math. Comput.*, 218 (2012) 6391–6400.

[4] H.Liu and F.Wang, Mannheim partner curves in 3-space, *J. Geom.*, 88 (2008) 120 – 126.

[5] H.Matsuda and S.Yorozu, On Generalized Mannheim Curves in Euclidean 4-Space, *Nihonkai Math. J.*, Vol.20 (2009), 33–56.

[6] İ. Gök, O.Z. Okuyucu, F. Kahraman, H. H. Hacısalihoğlu, On the Quaternionic B_2- Slant Helices in the Euclidean Space $\mathbb{E}^4$. *Adv. Appl. Clifford Algebras*, DOI 10.1007/s00006-011-0284-6.

[7] K.Bharathi, M.Nagaraj, Quaternion valued function of a real Serret-Frenet formulae, *Indian J. Pure Appl. Math.*, 16 (1985) 741-756.

[8] M.A.Güngör, M. Tosun, Some characterizations of quaternionic rectifying curves, *Diff. Geom. Dyn. Syst.*, Vol.13, (2011), 89-100.

International J.Math. Combin. Vol.2(2013), 54-61

Introduction to Bihypergroups

B.Davvaz

(Department of Mathematics, Yazd University, Yazd, Iran)

A.A.A.Agboola

(Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria)

E-mail: davvaz@yazd.ac.ir, aaaola2003@yahoo.com

Abstract: Similar to groups, the union of two sub-hypergroups do not form a hypergroup but they find a nice hyperstructure called bihypergroup. This short note is devoted to the introduction of bihypergroups and illustrate them with examples. A characterization theorem about sub-bihypergroups is given and some of their properties are presented.

Key Words: bigroup, hypergroup, sub-hypergroup, bihypergroup, sub-bihypergroup.

AMS(2010): 20N20

§1. Introduction

Bigroups are a very nice tool as the answers to a major problem faced by all groups, that is the union of two subgroups do not form any algebraic structure but they find a nice bialgebraic structure as bigroups. The study of bigroups was carried out in 1994-1996. Maggu [7,8] was the first one to introduce the notion of bigroups. However, the concept of bialgebraic structures was recently studied by Vasantha Kandasamy [11]. Agboola and Akinola in [1]studied bicoset of a bivector space.

The theory of hyperstructures was introduced in 1934 by Marty [9] at the 8th Congress of Scandinavian Mathematicians. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Several books have been written on this topic, see [2-4,6,13]. Hyperstructure theory both extends some well-known group results and introduce new topics leading us to a wide variety of applications, as well as to a broadening of the investigation fields.

§2. Basic Facts and Definitions

This section has two parts. In the first part we recall the definition of bigroups. In the second part we recall the notion of hypergroups.

[1]Received April 28, 2013, Accepted June 15, 2013.

2.1 Bigroups

Definition 2.1 *Let $*_1$ and $*_2$ be any two binary operations defined on a non-empty set G. Then, G is said to be a bigroup if there exists two proper subsets G_1 and G_2 such that*

(1) $G = G_1 \cup G_2$;
(2) $(G_1, *_1)$ *is a group;*
(3) $(G_2, *_2)$ *is a group.*

Definition 2.2 *Let $G = G_1 \cup G_2$ be a bigroup. A non-empty subset A of G is said to be a sub-bigroup of G if $A = A_1 \cup A_2$, A is a bigroup under the binary operations inherited from G, $A_1 = A \cap G_1$ and $A_2 = A \cap G_2$.*

Example 1([11]) Suppose that $G = \mathbb{Z} \cup \{i, -i\}$ under the operations "+" and "·". We consider $G = G_1 \cup G_2$, where $G_1 = \{-1, 1, i, -i\}$ under the operation "·" and $G_2 = \mathbb{Z}$ under the operation "+" are groups. Take $H = \{-1, 0, 1\} = H_1 \cup H_2$, where $H_1 = \{0\}$ is a group under "+" and $H_2 = \{-1, 1\}$ is a group under "·". Thus, H is a sub-bigroup of G. Note that H is not a group under "+" or "· ".

Definition 2.3 *Let $G = G_1 \cup G_2$ be a bigroup. Then, G is said to be commutative if both G_1 and G_2 are commutative.*

Definition 2.4 *Let $A = A_1 \cup A_2$ be a sub-bigroup of a bigroup $G = G_1 \cup G_2$. Then, A is said to be a normal bi-subgroup of G if A_1 is a normal subgroup of G_1 and A_2 is a normal subgroup of G_2.*

2.2 Hypergroups

In this part, we present the notion of hypergroup and some well-known related concepts. These concepts will be used in the building of bihypergroups, for more details we refer the readers to see [2-4, 6, 13].

Let H be a non-empty set and $\circ : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, \circ)$ is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \ A \circ x = A \circ \{x\} \ \text{ and } \ x \circ B = \{x\} \circ B.$$

Definition 2.5 *A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that*

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

A hypergroupoid $(H, \circ)$ is called a quasihypergroup if for all a of H we have $a \circ H = H \circ a = H$. This condition is also called the reproduction axiom.

Definition 2.6 *A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a hypergroup. For any $x, y \in H$, we define the right and the left extensions as follows: $x/y = \{a \in H \mid x \in a \circ y\}$ and $x \backslash y = \{b \in H \mid y \in x \circ b\}$.*

Example 2 Let $(S,\cdot)$ be a semigroup and let P be a non-empty subset of S. For all x,y of S, we define $x\circ y = xPy$. Then, $(S,\circ)$ is a semihypergroup. If $(S,\cdot)$ is a group. then $(S,\circ)$ is a hypergroup.

Example 3 If G is a group and for all x,y of G, $<x,y>$ denotes the subgroup generated by x and y, then we define $x\circ y = <x,y>$. We obtain that $(G,\circ)$ is a hypergroup.

Definition 2.7 *Let $(H,\circ)$ and $(H',\circ')$ be two hypergroupoids. A map $\phi : H \to H'$, is called*

(1) *an inclusion homomorphism if for all x,y of H, we have $\phi(x\circ y) \subseteq \phi(x)\circ'\phi(y)$;*

(2) *a good homomorphism if for all x,y of H, we have $\phi(x\circ y) = \phi(x)\circ'\phi(y)$.*

A good homomorphism ϕ is called a very good homomorphism if for all $x,y\in H$, $\phi(x/y) = \phi(x)/\phi(y)$ and $\phi(x\backslash y) = \phi(x)\backslash\phi(y)$.

Let $(H,\circ)$ be a semihypergroup and R be an equivalence relation on H. If A and B are non-empty subsets of H, then

$$\begin{aligned} A\overline{R}B &\text{ means that } \forall a\in A, \exists b\in B \text{ such that } aRb \text{ and}\\ &\qquad\qquad \forall b'\in B, \exists a'\in A \text{ such that } a'Rb';\\ A\overline{\overline{R}}B &\text{ means that } \forall a\in A, \forall b\in B, \text{ we have } aRb. \end{aligned}$$

Definition 2.8 *The equivalence relation ρ is called*

(1) *regular on the right (on the left) if for all x of H, from $a\rho b$, it follows that $(a\circ x)\overline{\rho}(b\circ x)$ ($(x\circ a)\overline{\rho}(x\circ b)$ respectively);*

(2) *strongly regular on the right (on the left) if for all x of H, from $a\rho b$, it follows that $(a\circ x)\overline{\overline{\rho}}(b\circ x)$ ($(x\circ a)\overline{\overline{\rho}}(x\circ b)$ respectively);*

(3) *ρ is called regular (strongly regular) if it is regular (strongly regular) on the right and on the left.*

Theorem 2.9 *Let $(H,\circ)$ be a semihypergroup and ρ be an equivalence relation on H.*

(1) *If ρ is regular, then H/ρ is a semihypergroup, with respect to the following hyperoperation: $\overline{x}\otimes\overline{y} = \{\overline{z} \mid z\in x\circ y\}$;*

(2) *If the above hyperoperation is well defined on H/ρ, then ρ is regular.*

Corollary 2.10 *If $(H,\circ)$ is a hypergroup and ρ is an equivalence relation on H, then R is regular if and only if $(H/\rho,\otimes)$ is a hypergroup.*

Theorem 2.11 *Let $(H,\circ)$ be a semihypergroup and ρ be an equivalence relation on H.*

(1) *If ρ is strongly regular, then H/ρ is a semigroup, with respect to the following operation: $\overline{x}\otimes\overline{y} = \{\overline{z} \mid z\in x\circ y\}$;*

(2) *If the above operation is well defined on H/ρ, then ρ is strongly regular.*

Corollary 2.12 *If $(H,\circ)$ is a hypergroup and ρ is an equivalence relation on H, then ρ is strongly regular if and only if $(H/\rho,\otimes)$ is a group.*

Definition 2.13 *Let* $(H, \circ)$ *is a semihypergroup and* A *be a non-empty subset of* H. *We say that* A *is a complete part of* H *if for any nonzero natural number* n *and for all* $a_1, \ldots, a_n$ *of* H, *the following implication holds:*

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \implies \prod_{i=1}^{n} a_i \subseteq A.$$

Theorem 2.14 *If* $(H, \circ)$ *is a semihypergroup and* R *is a strongly regular relation on* H, *then for all* z *of* H, *the equivalence class of* z *is a complete part of* H.

§3. Bihypergroup Structures

In this section, we introduce the concept of bihypergroup and illustrate it with examples.

Definition 3.1 *A set* $(H, \circ, \star)$ *with two hyperoperations* $\circ$ *and* $\star$ *is called a bihypergroup if there exist two proper subsets* H_1 *and* H_2 *such that*

(1) $H = H_1 \cup H_2$;

(2) $(H_1, \circ)$ *is a hypergroup;*

(3) $(H_2, \star)$ *is a hypergroup.*

Theorem 3.2 *Every hypergroup is a bihypergroup.*

Proof Suppose that $(H, \circ)$ is a hypergroup. If we consider $H = H_1 = H_2$ and $\circ = \star$, then $(H, \circ, \star)$ is a bihypergroup. □

Example 4 Let $H = \{a, b, c, d, e\}$ and let $\circ$ and $\star$ be two hyperoperations on H defined by the following tables:

$\circ$	a	b	c	d	e
a	a	b	c, d	d	b
b	b	a, b	c, d	c, d	e
c	c	c, d	a, b	a, b	e
d	c, d	c, d	a, b	a, b	e
e	b	e	e	e	e

and

$\star$	a	b	c	d	e
a	a	b	c, d	d	e
b	b	a, e	c, d	c, d	b
c	c	c, d	a, b	a, b	e
d	c, d	c, d	a, b	a, b	e
e	e	b	e	e	a

It is not difficult to see that $H_1 = \{a, b, c, d\}$ is a hypergroup together with the hyperoperation $\circ$ and $H_2 = \{a, b, e\}$ is a hypergroup together with the hyperoperation $\star$. Hence, $H = H_1 \cup H_2$ is a bihypergroup.

Example 5 ([5]) Blood groups are inherited from both parents. The ABO blood type is controlled by a single gene (the ABO gene) with three alleles: I^A, I^B and i. The gene encodes glycosyltransferase that is an enzyme that modifies the carbohydrate content of the red blood cell antigens. The gene is located on the long arm of the ninth chromosome (9q34).

People with blood type A have antigen A on the surfaces of their blood cells, and may be of genotype I^AI^A or I^Ai. People with blood type B have antigen B on their red blood cell surfaces, and may be of genotype I^BI^B or I^Bi. People with the rare blood type AB have both antigens A and B on their cell surfaces, and are genotype I^AI^B. People with blood type O have neither antigen, and are genotype ii. A type A and a type B couple can also have a type O child if they are both heterozygous (I^Ai and I^Bi, respectively).

$\otimes$	O	A	B	AB
O	O	O A	O B	A B
A	O A	O A	AB A B O	AB A B
B	O B	AB A B O	O B	AB A B
AB	A B	AB A B	AB A B	AB A B

Now, we consider $H = \{O, A, B\}$. If $H_1 = \{O, A\}$ and $H_2 = \{O, B\}$, then $H = H_1 \cup H_2$ is a bihypergroup.

Definition 3.3 *Let $H = H_1 \cup H_2$ be a bihypergroup. A non-empty subset A of H is said to be a sub- bihypergroup of G if $A = A_1 \cup A_2$, A is a bihypergroup under the binary operations inherited from H, $A_1 = A \cap H_1$ and $A_2 = A \cap H_2$.*

Remark 1 If $(H, \circ, \star)$ is a bihypergroup and K is a sub-bihypergroup of H, then $(K, \circ)$ and $(K, \star)$ in general are not hypergroups.

Theorem 3.4 *Let $H = H_1 \cup H_2$ be a bihypergroup. A non-empty subset $A = A_1 \cup A_2$ of H is a sub-bihypergroup of H if and only if $A_1 = A \cap H_1$ and $A_2 = A \cap H_2$ are sub-hypergroups of H_1 and H_2, respectively.*

Proof Suppose that $A = A_1 \cup A_2$ is a sub-bihypergroup of H. Then, A_i, $i = 1, 2$ are sub-hypergroups of H_i and therefore $A_i = A \cap H_i$ are sub-hypergroups of H_i.

Conversely, suppose that $A_1 = A \cap G_1$ is a sub-hypergroup of H_1 and $A_2 = A \cap H_2$ is a sub-hypergroup of H_2. It can be shown that $A_1 \cup A_2 = (A \cap H_1) \cup (A \cap H_2) = A$. Hence, A is sub-bihypergroup of H. □

Theorem 3.5 *Let H be any hypergroup and let A_1 and A_2 be any two sub-hypergroups of H such that $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$ but $A_1 \cap A_2 \neq \emptyset$. Then, $A = A_1 \cup A_2$ is a bihypergroup.*

Proof The required result follows from the definition of bihypergroup. □

Theorem 3.6 *Let $(H, \circ, \star)$ and $(H', \circ', \star')$ be any two bihypergroups, where $H = H_1 \cup H_2$ and $H' = H'_1 \cup H'_2$. Then, $(H \times H', \odot, \otimes)$ is a bihypergroup, where*

(1) $H \times H' = (H_1 \times H'_1) \cup (H_2 \times H'_2)$;

(2) $(x_1, x'_1) \odot (y_1, y'_1) = \{(z_1, z'_1) \mid z_1 \in x_1 \circ y_1,\ z'_1 \in x'_1 \star y'_1\}$, *for all* $(x_1, x'_1), (y_1, y'_1) \in H_1 \times H'_1$;

(3) $(x_2, x'_2) \odot (y_2, y'_2) = \{(z_2, z'_2) \mid z_2 \in x_2 \circ' y_2,\ z'_2 \in x'_2 \star' y'_2\}$, *for all* $(x_2, x'_2), (y_2, y'_2) \in H_2 \times H'_2$.

Definition 3.7 *Let $(H, \circ, \star)$ be a bihypergroup, where $H = H_1 \cup H_2$. Then, H is said to be commutative if both $(H_1, \circ)$ and $(H_2, \circ)$ are commutative.*

Let $H = H_1 \cup H_2$ and ρ be an equivalence relation on H. The *restriction* of ρ to H_1 and H_2 are the relations on H_1 and H_2 defined as

$$\rho|_{H_1} := \rho \cap (H_1 \times H_1) \text{ and } \rho|_{H_2} := \rho \cap (H_2 \times H_2).$$

Lemma 3.8 *Let $H = H_1 \cup H_2$ and ρ be an equivalence relation on H. Then, $\rho|_{H_1}$ and $\rho|_{H_2}$ are equivalence relations on H_1 and H_2, respectively.*

Definition 3.9 *Let $(H, \circ, \star)$ be a bihypergroup, where $H = H_1 \cup H_2$ and let ρ be an equivalence relation on H. We say that ρ is a (strongly) regular relation on H, if $\rho|_{H_1}$ is a (strongly) regular relation on H_1 and $\rho|_{H_2}$ is a (strongly) regular relation on H_2.*

Theorem 3.10 *Let $(H, \circ, \star)$ be a bihypergroup, where $H = H_1 \cup H_2$, and let ρ be an equivalence relation on H.*

(1) *If ρ is regular, then $H_1/\rho|_{H_1} \cup H_2/\rho|_{H_2}$ is a bihypergroup;*

(2) *If ρ is strongly regular, then $H_1/\rho|_{H_1} \cup H_2/\rho|_{H_2}$ is a bigroup.*

Proof The proof follows from Lemma 3.8 and Theorems 2.9 and 2.11. □

Definition 3.11 *Let $(H, \circ, \star)$ and $(H', \circ', \star')$ be any two bihypergroups, where $H = H_1 \cup H_2$ and*

$H' = H'_1 \cup H'_2$. The map $\phi : H \to H'$ is said to be a bihypergroup (inclusion, good, very good, respectively) homomorphism if ϕ restricted to H_1 is a hypergroup (inclusion, good, very good, respectively) homomorphism from H_1 to H'_1 and ϕ restricted to H_2 is a hypergroup (inclusion, good, very good, respectively) homomorphism from H_2 to H'_2.

Definition 3.12 *Let $\phi = \phi_1 \cup \phi_2 : (H = H_1 \cup H_2, \circ, \star) \to (H' = H'_1 \cup H'_2, \circ', \star')$ be a good homomorphism and $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ be non-empty subsets of H and H', respectively.*

(1) *The image of A under ϕ denoted by $\phi(A) = \phi_1(A_1) \cup \phi_2(A_2)$, is the set $\{\phi_1(a_1), \phi_2(a_2) \mid a_1 \in A_1, a_2 \in A_2\}$;*

(2) *The inverse image of B under ϕ denoted by $\phi^{-1}(B) = \phi^{-1}(B_1) \cup \phi^{-1}(B_2)$, is the set $\{h_1 \in H_1, h_2 \in H_2 \mid \phi_1(h_1) \in B_1, \phi_2(h_2) \in B_2\}$.*

Lemma 3.13 *Let H and H' be two hypergroups and $\phi : H \to H'$ be a good homomorphism.*

(1) *If A is a sub-hypergroup of H, then $\phi(A)$ is a sub-hypergroup of H';*

(2) *If ϕ is a very good homomorphism and B is a subhypergroup of H', then $\phi^{-1}(B)$ is a sub-hypergroup of H.*

Proof The proof of (1) is clear. We prove (2). Suppose that $x, y \in \phi^{-1}(B)$ are arbitrary elements. Then, $\phi(x), \phi(y) \in B$. For every $z \in x \circ y$, $\phi(z) \in \phi(x \circ y) = \phi(x) \star \phi(y) \subseteq B$. So, $z \in \phi^{-1}(B)$. Hence, $\phi^{-1}(B) \circ \phi^{-1}(B) \subseteq \phi^{-1}(B)$.

Now, suppose that $x, a \in \phi^{-1}(B)$ are arbitrary elements. Then, $\phi(x), \phi(a) \in B$. Since B is a sub-hypergroup of H', by reproduction axiom, there exists $u \in B$ such that $\phi(a) \in u \star \phi(x)$. Thus, $u \in \phi(a)/\phi(x)$. Since ϕ is very good homomorphism, $u \in \phi(a/x)$. Hence, there exists $y \in a/x$ such that $u = \phi(y)$. Thus, $y \in \phi^{-1}(B)$ and $a \in y \circ x$. Hence, for every $x, a \in \phi^{-1}(B)$, there exists $y \in \phi^{-1}(B)$ such that $a \in \phi^{-1}(B) \circ x$. This implies that $\phi^{-1}(B) \subseteq \phi^{-1}(B) \circ x$ for all $x \in \phi^{-1}(B)$. Similarly, we can prove that $\phi^{-1}(B) \subseteq x \circ \phi^{-1}(B)$ for all $x \in \phi^{-1}(B)$. □

Proposition 3.14 *Let $\phi = \phi_1 \cup \phi_2 : (H = H_1 \cup H_2, \circ, \star) \to (H' = H'_1 \cup H'_2, \circ', \star')$ be a good homomorphism and $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ be non-empty subsets of H and H', respectively.*

(1) *$\phi(A)$ is a sub-bihypergroup of H';*

(2) *If ϕ is a very good homomorphism, then $\phi^{-1}(B)$ is a sub-bihypergroup of H.*

Proof (1) Suppose that $A = A_1 \cup A_2$ is a sub-bihypergroup of H. By Lemma 3.13(1), $\phi_1(A_1)$ is a sub-hypergroup of H'_1 and $\phi_2(A_2)$ is a sub-hypergroup of H'_2. Thus, $\phi(A) = \phi_1(A_1) \cup \phi_2(A_2)$ is a bihypergroup. Now,

$$\begin{aligned} \phi(A) \cap H'_1 &= \Big(\phi_1(A_1) \cup \phi_2(A_2)\Big) \cap H'_1 \\ &= \Big(\phi_1(A_1) \cap H'_1\Big) \cup \Big(\phi_2 \cap H'_1\Big) \\ &= \phi(A_1). \end{aligned}$$

Similarly, it can be shown that $\phi(A) \cap H_2' = \phi_2(A_2)$. Accordingly, $\phi(A)$ is a sub-bihypergroup of H'.

The proof of (2) follows from Lemma 3.13(2) and is similar to the proof of (1). □

Definition 3.15 *Let $(H, \circ, \star)$ be a bihypergroup where $H = H_1 \cup H_2$ and let $A = A_1 \cup A_2$ be a non-empty subset of H. Then, A is said to be a complete part of H if A_1 is a complete part of H_1 and A_2 is a complete part of H_2.*

Theorem 3.16 *Let $(H, \circ, \star)$ be a bihypergroup where $H = H_1 \cup H_2$ and let ρ be an equivalence relation on H. If $\rho|_{H_1}$ and $\rho|_{H_2}$ are strongly regular relations on H_1 and H_2 respectively, then for all $x = x_1 \cup x_2 \in H$, $\overline{x_1} \cup \overline{x_2}$ is a complete part of H.*

Proof It is clear. □

References

[1] Agboola A.A.A., and Akinola L.S., On the Bicoset of a Bivector Space, *Int. J. Math. Comb.*, Vol.4,2009, 1-8.

[2] Corsini P., Prolegomena of Hypergroup Theory, Second edition, Aviain editore, 1993.

[3] Corsini P. and Leoreanu V., *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[4] Davvaz B., *Polygroup Theory and Related Systems*, World Sci. Publ., 2013.

[5] Davvaz, Dehghan Nezad A. and Heidari M. M., Inheritance examples of algebraic hyperstructures, Information Sciences, **224**(2013), 180-187.

[6] Davvaz B. and Leoreanu-Fotea V., *Hyperring Theory and Applications*, International Academic Press, USA, 2007.

[7] Maggu P.L., On Introduction of Bigroup Concept with its Applications in Industry, *Pure and App. Math Sci.*, **(39)**(1994), 171-173.

[8] Maggu P.L., and Rajeev K., On Sub-bigroup and its Applications in Industry, *Pure and App. Math Sci.*, **(43)**(1996), 85-88.

[9] Marty F., Sur une Generalization de la Notion de Groupe. 8th Congress Math. Scandinaves, Stockholm, Sweden, 45-49 (1934).

[10] Smarandache F., Special Algebraic Structures, in Collected Papers, Abaddaba, Oradea, **(3)**(2000), 78-81.

[11] Vasantha Kandasamy W.B., *Bialgebraic Structures and Smarandache Bialgebraic Structures*, American Research Press, Rehoboth, 2003.

[12] Vasantha Kandasamy W.B., Bivector Spaces, *U. Sci. Phy. Sci.*, **(11)**(1999), 186-190.

[13] Vougiouklis T., *Hyperstructures and Their Rrepresentations*, Hadronic Press, Inc, 115, Palm Harber, USA, 1994.

International J.Math. Combin. Vol.2(2013), 62-73

Smarandache Seminormal Subgroupoids

H.J.Siamwalla

(Abeda Inamdar Senior College, Azam Campus, Pune, India)

A.S.Muktibodh

(Shri M. Mohota College of Science, Umred Road, Nagpur, India)

E-mail: siamwalla.maths@gmail.com, amukti2000@yahoo.com

Abstract: In this paper, we define Smarandache seminormal subgroupoids. We have proved some results for finding the Smarandache seminormal subgroupoids in $Z(n)$ when n is even and n is odd.

Key Words: Groupoids, Smarandache groupoids, Smarandache seminormal subgroupoids

AMS(2010): 08A05

§1. Introduction

In [5] and [6], W.B.Kandasamy defined new classes of Smarandache groupoids using Z_n. In this paper we define and prove some theorems for construction of Smarandache seminormal subgroupoids according as n is even or odd.

Definition 1.1 *A non-empty set of elements G is said to form a groupoid if in G is defined a binary operation called the product, denoted by $*$ such that $a * b \in G\ \forall a, b \in G$. We denote groupoids by $(G, *)$.*

Definition 1.2 *Let $(G, *)$ be a groupoid. A proper subset $H \subset G$ is a subgroupoid if $(H, *)$ is itself a groupoid.*

Definition 1.3 *Let S be a non-empty set. S is said to be a semigroup if on S is defined a binary operation $*$ such that*

(1) *for all a, $b \in S$ we have $a * b \in S$;*
(2) *for all a, b, $c \in S$ we have $a * (b * c) = (a * b) * c$.*

$(S, *)$ is a semi-group.

Definition 1.4 *A Smarandache groupoid G is a groupoid which has a proper subset S such that S under the operation of G is a semigroup.*

Definition 1.5 *Let $(G, *)$ be a Smarandache groupoid. A non-empty subgroupoid H of G is said*

[1]Received April 3, 2013, Accepted June 18, 2013.

to be a Smarandache subgroupoid if H contains a proper subset K such that K is a semigroup under the operation $*$.

Definition 1.6 *Let G be a Smarandache groupoid. V be a Smarandache subgroupoid of G. We say V is a Smarandache seminormal subgroupoid if $aV = V$ for all $a \in G$ or $Va = V$ for all $a \in G$.*

For example, let $(G, *)$ be groupoid given by the following table:

$*$	a_0	a_1	a_2	a_3	a_4	a_5
a_0	a_0	a_3	a_0	a_3	a_0	a_3
a_1	a_2	a_5	a_2	a_5	a_2	a_5
a_2	a_4	a_1	a_4	a_1	a_4	a_1
a_3	a_0	a_3	a_0	a_3	a_0	a_3
a_4	a_2	a_5	a_2	a_5	a_2	a_5
a_5	a_4	a_1	a_4	a_1	a_4	a_1

It is a Smarandache groupoid as $\{a_3\}$ is a semigroup. $V = \{a_1, a_3, a_5\}$ is a Smarandache subgroupoid,also $aV = V$. Therefore V is Smarandache seminormal subgroupoid in G.

Definition 1.7 *Let $Z_n = \{0, 1, \cdots, n-1\}$, $n \geqslant 3$ and $a, b \in Z_n \backslash \{0\}$. Define a binary operation $*$ on Z_n as follows:*

$a * b = ta + ub \pmod n$, where t, u are two distinct elements in $Z_n \backslash \{0\}$ and $(t, u) = 1$. Here $'+'$ is the usual addition of two integers and $'ta'$ means the product of the two integers t and a.

Elements of Z_n form a groupoid with respect to the binary operation $*$. We denote these groupoid by $\{Z_n(t, u), *\}$ or $Z_n(t, u)$ for fixed integer n and varying $t, u \in Z_n \backslash \{0\}$ such that $(t, u) = 1$. Thus we define a collection of groupoids $Z(n)$ as follows
$Z(n) = \{\{Z_n(t, u), *\}|$ for integers $t, u \in Z_n \backslash \{0\}$ such that $(t, u) = 1\}$.

§2. Smarandache Seminormal Subgroupoids When $n \equiv 0 (\mathrm{mod} 2)$

When n is even we are interested in finding Smarandache seminormal subgroupoid in $Z_n(t, t+1)$.

Theorem 2.1 *Let $Z_n(t, t+1) \in Z(n)$, n is even, $n > 3$ and $t = 1, \cdots, n-2$. Then $Z_n(t, t+1)$ is Smarandache groupoid.*

Proof Let $x = \dfrac{n}{2}$. Then

$$
\begin{aligned}
x * x &= xt + x(t+1) = 2xt + x \\
&= (2t+1)x \equiv x \bmod n
\end{aligned}
$$

Consequently, $\{x\}$ is a semigroup in $Z_n(t, t+1)$. Thus $Z_n(t, t+1)$ is a Smarandache groupoid when n is even. □

Remark In the above theorem we can also show that beside $\{n/2\}$ the other semigroup is $\{0, n/2\}$ in $Z_n(t, t+1) \in Z(n)$.

Proof If t is even, $0*t+\frac{n}{2}*(t+1) \equiv \frac{n}{2} \bmod n$, $\frac{n}{2}*t+0*(t+1) \equiv 0 \bmod n$, $\frac{n}{2}*t+\frac{n}{2}*(t+1) \equiv \frac{n}{2} \bmod n$ and $0*t+0*(t+1) \equiv 0 \bmod n$. So $\{0, \frac{n}{2}\}$ is semigroup in $Z_n(t, t+1)$. If t is odd, $0*t+\frac{n}{2}*(t+1) \equiv 0 \bmod n$, $\frac{n}{2}*t+0*(t+1) \equiv \frac{n}{2} \bmod n$, $\frac{n}{2}*t+\frac{n}{2}*(t+1) \equiv \frac{n}{2} \bmod n$ and $0*t+0*(t+1) \equiv 0 \bmod n$. So $\{0, \frac{n}{2}\}$ is a semigroup in $Z_n(t, t+1)$. □

Theorem 2.2 *Let $n > 3$ be even and $t = 1, \cdots, n-2$,*

(1) *If $\frac{n}{2}$ is even then $A_0 = \{0, 2, \cdots, n-2\} \subseteq Z_n$ is Smarandache subgroupoid in $Z_n(t, t+1) \in Z(n)$.*

(2) *If $\frac{n}{2}$ is odd then $A_1 = \{1, 3, \cdots, n-1\} \subseteq Z_n$ is Smarandache subgroupoid in $Z_n(t, t+1) \in Z(n)$.*

Proof (1) Let $\frac{n}{2}$ is even.$\Rightarrow \frac{n}{2} \in A_0$. We will show that A_0 is subgroupoid.

Let $x_i, x_j \in A_0$ and $x_i \neq x_j$. Then

$$\begin{aligned} x_i * x_j &= x_i t + x_j(t+1) \\ &= (x_i + x_j)t + x_j \equiv x_k \bmod n \end{aligned}$$

for some $x_k \in A_0$ as $(x_i + x_j)t + x_j$ is even. So $x_i * x_j \in A_0$. Thus A_0 is subgroupoid in $Z_n(t, t+1)$.

Let $x = \frac{n}{2}$. Then

$$\begin{aligned} x * x &= xt + x(t+1) \\ &= (2t+1)x \equiv x \bmod n. \end{aligned}$$

Therefore, $\{x\}$ is a semigroup in A_0. Thus A_0 is a subgroupoid in $Z_n(t, t+1)$.

(2) Let $\frac{n}{2}$ is odd. $\Rightarrow \frac{n}{2} \in A_1$. We show that A_1 is subgroupoid.

Let $x_i, x_j \in A_1$ and $x_i \neq x_j$. Then

$$\begin{aligned} x_i * x_j &= x_i t + x_j(t+1) \\ &= (x_i + x_j)t + x_j \equiv x_k \bmod n \end{aligned}$$

for some $x_k \in A_1$ as $(x_i + x_j)t + x_j$ is odd. Therefore, $x_i * x_j \in A_1$. Thus A_1 is subgroupoid in $Z_n(t, t+1)$.

Let $x = \frac{n}{2}$. Then

$$\begin{aligned} x * x &= xt + x(t+1) \\ &= (2t+1)x \equiv x \bmod n. \end{aligned}$$

So $\{x\}$ is a semigroup in A_1. Thus A_1 is a Smarandache subgroupoid in $Z_n(t, t+1)$. □

Theorem 2.3 *Let $n > 3$ be even and $t = 1, \cdots, n-2$,*

(1) *If $\frac{n}{2}$ is even then $A_0 = \{0, 2, \cdots, n-2\} \subseteq Z_n$ is Smarandache seminormal subgroupoid of $Z_n(t, t+1) \in Z(n)$.*

(2) *If $\frac{n}{2}$ is odd then $A_1 = \{1, 3, \cdots, n-1\} \subseteq Z_n$ is Smarandache seminormal subgroupoid of $Z_n(t, t+1) \in Z(n)$.*

Proof By Theorem 2.1, $Z_n(t, t+1)$ is a Smarandache groupoid.

(1) Let $\frac{n}{2}$ is even. Then by Theorem 2.2, $A_0 = \{0, 2, \cdots, n-2\}$ is Smarandache subgroupoid of $Z_n(t, t+1)$. Now we show that either $aA_0 = A_0$ or $A_0 a = A_0$ $\forall$ $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$.

Case 1 t is even.

Let $a_i \in A_0$ and $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$. Then

$$\begin{aligned} a * a_i &= at + a_i(t+1) \\ &\equiv a_j \bmod n \end{aligned}$$

for some $a_j \in A_0$ as $at + a_i(t+1)$ is even. Therefore, $a * a_i \in A_0$ $\forall$ $a_i \in A_0$, $aA_0 = A_0$. Thus, A_0 is a Smarandache seminormal subgroupoid in $Z_n(t, t+1)$.

Case 2 t is odd.

Let $a_i \in A_0$ and $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$. Then

$$\begin{aligned} a_i * a &= a_i t + a(t+1) \\ &\equiv a_j \bmod \text{n} \end{aligned}$$

for some $a_j \in A_0$ as $a_i t + a(t+1)$ is even. Therefore, $a_i * a \in A_0$ $\forall$ $a_i \in A_0$, $A_0 a = A_0$. Thus A_0 is a Smarandache seminormal subgroupoid in $Z_n(t, t+1)$.

(2) Let $\frac{n}{2}$ is odd. Then by Theorem 2.2, $A_1 = \{1, 3, 5, \cdots, n-1\}$ is Smarandache subgroupoid of $Z_n(t, t+1)$. Now we show that either $aA_1 = A_1$ or $A_1 a = A_1$ $\forall$ $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$.

Case 1 t is even.

Let $a_i \in A_1$ and $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$. Then

$$\begin{aligned} a * a_i &= at + a_i(t+1) \\ &= (a + a_i)t + a_i \\ &\equiv a_j \bmod n \end{aligned}$$

for some $a_j \in A_1$ as $(a + a_i)t + a_i$ is odd. Therefore, $a * a_i \in A_1$ $\forall$ $a_i \in A_1$, $\therefore aA_1 = A_1$. Thus A_1 is Smarandache seminormal subgroupoid in $Z_n(t, t+1)$.

Case 2 t is odd.

Let $a_i \in A_1$ and $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$. Then

$$\begin{aligned} a_i * a &= a_i t + a(t+1) \\ &\equiv a_j \bmod \text{n} \end{aligned}$$

for some $a_j \in A_1$ as $a_i t + a(t+1)$ is odd. Therefore, $a_i * a \in A_1 \ \forall \ \ a_i \in A_1,\ A_1 a = A_1$. Thus A_1 is Smarandache seminormal subgroupoid in $Z_n(t, t+1)$. □

By the above theorem we can determine the Smarandache seminormal subgroupoid in $Z_n(t, t+1)$ of $Z(n)$ when n is even and $n > 3$.

n	$n/2$	t	$Z_n(t,t+1)$	Smarandache seminormal subgroupoid in $Z_n(t,t+1)$
4	2	1	$Z_4(1,2)$	$\{0,2\}$
		2	$Z_4(2,3)$	
6	3	1	$Z_6(1,2)$	$\{1,3,5\}$
		2	$Z_6(2,3)$	
		3	$Z_6(3,4)$	
		4	$Z_6(4,5)$	
8	4	1	$Z_8(1,2)$	$\{0,2,4,6\}$
		2	$Z_8(2,3)$	
		3	$Z_8(3,4)$	
		4	$Z_8(4,5)$	
		5	$Z_8(5,6)$	
		6	$Z_8(6,7)$	
10	5	1	$Z_{10}(1,2)$	$\{1,3,5,7,9\}$
		2	$Z_{10}(2,3)$	
		3	$Z_{10}(3,4)$	
		4	$Z_{10}(4,5)$	
		5	$Z_{10}(5,6)$	
		6	$Z_{10}(6,7)$	
		7	$Z_{10}(7,8)$	
		8	$Z_{10}(8,9)$	
12	6	1	$Z_{12}(1,2)$	$\{0,2,4,6,8\}$
		2	$Z_{12}(2,3)$	
		3	$Z_{12}(3,4)$	
		4	$Z_{12}(4,5)$	
		5	$Z_{12}(5,6)$	
		6	$Z_{12}(6,7)$	
		7	$Z_{12}(7,8)$	
		8	$Z_{12}(8,9)$	
		9	$Z_{12}(9,10)$	
		10	$Z_{12}(10,11)$	

§3. Smarandache Seminormal Subgroupoids Depend on t,u when $n \equiv 0(\text{mod}2)$

When n is even we are interested in finding Smarandache seminormal subgroupoid in $Z_n(t,u) \in Z(n)$ when t is even and u is odd or when t is odd and u is even.

Theorem 3.1 *Let $Z_n(t,u) \in Z(n)$,if n is even,$n > 3$ and for each $t,u \in Z_n$,if one is even and other is odd then $Z_n(t,u)$ is Smarandache groupoid.*

Proof Let $x = \frac{n}{2}$. Then

$$\begin{aligned} x * x &= xt + xu \\ &= (t+u)x \equiv x \bmod n. \end{aligned}$$

So $\{x\}$ is a semigroup in $Z_n(t,u)$. Thus $Z_n(t,u)$ is a Smarandache groupoid when n is even. □

Remark In the above theorem we can also show that beside $\{n/2\}$ the other semigroup is $\{0, n/2\}$ in $Z_n(t,u) \in Z(n)$.

Proof If t is even and u is odd, $0 * t + \frac{n}{2} * u \equiv \frac{n}{2} \bmod n$, $\frac{n}{2} * t + 0 * u \equiv 0 \bmod n$, $\frac{n}{2} * t + \frac{n}{2} * u \equiv \frac{n}{2} \bmod n$ and $0 * t + 0 * u \equiv 0 \bmod n$. So $\{0, \frac{n}{2}\}$ is semigroup in $Z_n(t,u)$. If t is odd and u is even, $0 * t + \frac{n}{2} * u \equiv 0 \bmod n$, $\frac{n}{2} * t + 0 * u \equiv \frac{n}{2} \bmod n$, $\frac{n}{2} * t + \frac{n}{2} * u \equiv \frac{n}{2} \bmod n$ and $0 * t + 0 * u \equiv 0 \bmod n$. So $\{0, \frac{n}{2}\}$ is semigroup in $Z_n(t,u)$. □

Theorem 3.2 *Let $n > 3$ be even and $t,u \in Z_n$.*

(1) *If $\frac{n}{2}$ is even then $A_0 = \{0, 2, \cdots, n-2\} \subseteq Z_n$ is Smarandache subgroupoid of $Z_n(t,u) \in Z(n)$ when one of t and u is odd and other is even.*

(2) *If $\frac{n}{2}$ is odd then $A_1 = \{1, 3, \cdots, n-1\} \subseteq Z_n$ is Smarandache subgroupoid of $Z_n(t,u) \in Z(n)$ when one of t and u is odd and other is even.*

Proof (1) Let $\frac{n}{2}$ be even. $\Rightarrow \frac{n}{2} \in A_0$. We show that A_0 is subgroupoid.
Let $x_i, x_j \in A_0$ and $x_i \neq x_j$. Then

$$x_i * x_j = x_i t + x_j u \equiv x_k \bmod n$$

for some $x_k \in A_0$ as $x_i t + x_j u$ is even. So $x_i * x_j \in A_0$. Thus A_0 is a subgroupoid in $Z_n(t,u)$.
Let $x = \frac{n}{2}$. Then

$$\begin{aligned} x * x &= xt + xu \\ &= x(t+u) \equiv x \bmod n. \end{aligned}$$

Whence, $\{x\}$ is a semigroup in A_0. Thus, A_0 is a Smarandache subgroupoid in $Z_n(t,u)$.

(2) Let $\frac{n}{2}$ be odd. $\Rightarrow \frac{n}{2} \in A_1$. We show that A_1 is subgroupoid.
Let $x_i, x_j \in A_1$ and $x_i \neq x_j$. Then

$$x_i * x_j = x_i t + x_j u \equiv x_k \bmod n$$

for some $x_k \in A_1$ as $x_i + x_j u$ is odd. So $x_i * x_j \in A_1$. Consequently, A_1 is subgroupoid in $Z_n(t,u)$.

Let $x = \frac{n}{2}$. Then

$$x * x = xt + xu = x(t+u) \equiv x \bmod n.$$

So $\{x\}$ is a semigroup in A_1. Thus A_1 is a Smarandache subgroupoid in $Z_n(t,u)$. □

Theorem 3.3 *Let $n > 3$ be even and $t = 1, \cdots, n-2$.*

(1) If $\frac{n}{2}$ is even then $A_0 = \{0, 2, \cdots, n-2\} \subseteq Z_n$ is Smarandache seminormal subgroupoid of $Z_n(t,u) \in Z(n)$ when one of t and u is odd and other is even;

(2) If $\frac{n}{2}$ is odd then $A_1 = \{1, 3, \cdots, n-1\} \subseteq Z_n$ is Smarandache seminormal subgroupoid of $Z_n(t,u) \in Z(n)$ when one of t and u is odd and other is even.

Proof By Theorem 3.1, $Z_n(t,u)$ is a Smarandache groupoid.

(1) Let $\frac{n}{2}$ is even. Then by Theorem 3.2, $A_0 = \{0, 2, \cdots, n-2\}$ is Smarandache subgroupoid of $Z_n(t,u)$. Now we show that either $aA_0 = A_0$ or $A_0 a = A_0$ $\forall$ $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$.

Case 1 t is even and u is odd.

Let $a_i \in A_0$ and $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$. Then

$$a * a_i = at + a_i u \equiv a_j \bmod n$$

for some $a_j \in A_0$ as $at + a_i u$ is even. Whence, $a * a_i \in A_0$ $\forall$ $a_i \in A_0$, $aA_0 = A_0$. Thus, A_0 is a Smarandache seminormal subgroupoid in $Z_n(t,u)$.

Case 2 t is odd and u is even.

Let $a_i \in A_0$ and $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$. Then

$$a_i * a = a_i t + au \equiv a_j \bmod \mathrm{n}$$

for some $a_j \in A_0$ as $a_i t + au$ is even. Therefore, $a_i * a \in A_0$ $\forall$ $a_i \in A_0$, $A_0 a = A_0$. Thus, A_0 is Smarandache seminormal subgroupoid in $Z_n(t,u)$.

(2) Let $\frac{n}{2}$ is odd then by Theorem 3.2 is $A_1 = \{1, 3, 5, \cdots, n-1\}$ is Smarandache subgroupoid of $Z_n(t,u)$. We show that either $aA_1 = A_1$ or $A_1 a = A_1$ $\forall$ $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$.

Case 1 t is even and u is odd.

Let $a_i \in A_1$ and $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$. Then

$$a * a_i = at + a_i u \equiv a_j \bmod n$$

for some $a_j \in A_1$ as $at + a_iu$ is odd. So, $a * a_i \in A_1 \ \forall \ a_i \in A_1$, $\therefore \ aA_1 = A_1$. Thus, A_1 is a Smarandache seminormal subgroupoid in $Z_n(t,u)$.

Case 2 t is odd and u is even.

Let $a_i \in A_1$ and $a \in Z_n = \{0, 1, 2, \cdots, n-1\}$.

$$\begin{aligned} a_i * a &= a_it + au \\ &\equiv a_j \bmod n \end{aligned}$$

for some $a_j \in A_1$ as $a_it + au$ is odd. Therefore, $a_i * a \in A_1 \ \forall \ a_i \in A_1$, $A_1a = A_1$. Thus, A_1 is a Smarandache seminormal subgroupoid in $Z_n(t,u)$. □

By the above theorem we can determine Smarandache seminormal subgroupoid in $Z_n(t,u) \in Z(n)$ for $n > 3$, when n is even and when one of t and u is odd and other is even.

n	$n/2$	t	$Z_n(t,u)$	Smarandache seminormal subgroupoid
4	2	1	$Z_4(1,2)$	$\{0,2\}$
		2	$Z_4(2,3)$	
6	3	1	$Z_6(1,2), Z_6(1,4)$	$\{1,3,5\}$
		2	$Z_6(2,1), Z_6(2,3), Z_6(2,5)$	
		3	$Z_6(3,2), Z_6(3,4)$	
		4	$Z_6(4,1), Z_6(4,3), Z_6(4,5)$	
		5	$Z_6(5,2), Z_6(5,4)$	
8	4	1	$Z_8(1,2), Z_8(1,4), Z_8(1,6)$	$\{0,2,4,6\}$
		2	$Z_8(2,1), Z_8(2,3), Z_8(2,5)$, $Z_8(2,7)$	
		3	$Z_8(3,2), Z_8(3,4)$	
		4	$Z_8(4,1), Z_8(4,3), Z_8(4,5)$, $Z_8(4,7)$	
		5	$Z_8(5,2), Z_8(5,4), Z_8(5,6)$	
		6	$Z_8(6,1), Z_8(6,5), Z_8(6,7)$,	
		7	$Z_8(7,2), Z_8(7,4), Z_8(7,6)$,	

n	$n/2$	t	$Z_n(t,u)$	Smarandache seminormal subgroupoid
10	5	1	$Z_{10}(1,2), Z_{10}(1,4), Z_{10}(1,6), Z_{10}(1,8)$	$\{1,3,5,7,9\}$
		2	$Z_{10}(2,1), Z_{10}(2,3), Z_{10}(2,5), Z_{10}(2,7), Z_{10}(2,9)$	
		3	$Z_{10}(3,2), Z_{10}(3,4), Z_{10}(3,8),$	
		4	$Z_{10}(4,1), Z_{10}(4,3), Z_{10}(4,5), Z_{10}(4,7), Z_{10}(4,9)$	
		5	$Z_{10}(5,2), Z_{10}(5,4), Z_{10}(5,6), Z_{10}(5,8)$	
		6	$Z_{10}(6,1), Z_{10}(6,5), Z_{10}(6,7),$	
		7	$Z_{10}(7,2), Z_{10}(7,4), Z_{10}(7,6), Z_{10}(7,8)$	
		8	$Z_{10}(8,1), Z_{10}(8,3), Z_{10}(8,5), Z_{10}(8,7), Z_{10}(8,9)$	
		9	$Z_{10}(9,2), Z_{10}(9,4), Z_{10}(9,8)$	
12	6	1	$Z_{12}(1,2), Z_{12}(1,4), Z_{12}(1,6), Z_{12}(1,8), Z_{12}(1,10)$	$\{0,2,4,6,8,10\}$
		2	$Z_{12}(2,1), Z_{12}(2,3), Z_{12}(2,5), Z_{12}(2,7), Z_{12}(2,9), Z_{12}(2,11)$	
		3	$Z_{12}(3,2), Z_{12}(3,4), Z_{12}(3,8), Z_{12}(3,10)$	
		4	$Z_{12}(4,1), Z_{12}(4,3), Z_{12}(4,5), Z_{12}(4,7), Z_{12}(4,9), Z_{12}(4,11)$	
		5	$Z_{12}(5,2), Z_{12}(5,4), Z_{12}(5,6), Z_{12}(5,8)$	
		6	$Z_{12}(6,1), Z_{12}(6,3), Z_{12}(6,5), Z_{12}(6,7), Z_{12}(6,11)$	
		7	$Z_{12}(7,2), Z_{12}(7,4), Z_{12}(7,6), Z_{12}(7,8), Z_{12}(7,10)$	
		8	$Z_{12}(8,1), Z_{12}(8,3), Z_{12}(8,5), Z_{12}(8,7), Z_{12}(8,9), Z_{12}(8,11)$	
		9	$Z_{12}(9,2), Z_{12}(9,4), Z_{12}(9,8), Z_{12}(9,10)$	
		10	$Z_{12}(10,1), Z_{12}(10,3), Z_{12}(10,7), Z_{12}(10,9), Z_{12}(10,11)$	
		11	$Z_{12}(11,2), Z_{12}(11,4), Z_{12}(11,6), Z_{12}(11,8), Z_{12}(11,10)$	

§4. Smarandache Seminormal Subgroupoids When $n \equiv 1(\mathrm{mod} 2)$

When n is odd we are interested in finding Smarandache seminormal subgroupoid in $Z_n(t,u) \in Z(n)$.We have proved the similiar result in [4].

Theorem 4.1 *Let $Z_n(t,u) \in Z(n)$. If n is odd, $n > 4$ and for each $t = 2, \cdots, \frac{n-1}{2}$ and $u = n-(t-1)(t,u) = 1$, then $Z_n(t,u)$ is a Smarandache groupoid.*

Proof Let $x \in \{0, \cdots, n-1\}$. Then

$$x * x = xt + xu = (n+1)x \equiv x \bmod n.$$

So $\{x\}$ is semigroup in Z_n. Thus $Z_n(t,u)$ is a Smarandache groupoid in $Z(n)$. □

Remark We note that all $\{x\}$ where $x \in \{1, \cdots, n-1\}$ are proper subsets which are semigroups in $Z_n(t,u)$.

Theorem 4.2 *Let $n > 4$ be odd and $t = 2, \cdots, \frac{n-1}{2}$ and $u = n-(t-1)$ such that $(t,u) = 1$ if $s = (n,t)$ or $s = (n,u)$ then $A_k = \{k, k+s, \cdots, k+(r-1)s\}$ for $k = 0, 1, \cdots, s-1$ where $r = \frac{n}{s}$ is a Smarandache subgroupoid in $Z_n(t,u) \in Z(n)$.*

Proof Let $x_p, x_q \in A_k$. Then

$$x_p \neq x_q \Rightarrow \left.\begin{array}{l} x_p = k+ps \\ x_q = k+qs \end{array}\right\} p,\ q \in \{0, 1, \cdots, r-1\}.$$

Also,

$$\begin{aligned} x_p * x_q &= x_p t + x_q u \\ &= (k+ps)t + (k+qs)(n-(t-1)) \\ &= k(n+1) + ((p-q)t + q(n+1))s \\ &\equiv (k+ls) \bmod n \\ &\equiv x_l \bmod n \end{aligned}$$

$x_l \in A_k$ as $x_l = k+ls$ for some $l \in \{0, 1, \cdots, r-1\}$. Whence, $x_p * x_q \in A_k$. Consequently, A_k is a subgroupoid in $Z_n(t,u)$. By the above remark all singleton sets are semigroup. Thus,A_k is a Smarandache subgroupoid. □

Theorem 4.3 *Let $n > 4$ be odd and $t = 2, \cdots, \frac{n-1}{2}$ and $u = n-(t-1)$ such that $(t,u) = 1$ if $s = (n,t)$ or $s = (n,u)$ then $A_k = \{k, k+s, \cdots, k+(r-1)s\}$ for $k = 0, 1, \cdots, s-1$ where $r = \frac{n}{s}$ is a Smarandache seminormal subgroupoid in $Z_n(t,u) \in Z(n)$.*

Proof By Theorem 4.1, $Z_n(t,u)$ is a Smarandache groupoid. Also by Theorem 4.2, $A_k = \{k, k+s, \cdots, k+(r-1)s\}$ for $k = 0, 1, \cdots, s-1$ is Smarandache subgroupoid of $Z_n(t,u)$.

If $s=(n,t)$, let $x_p \in A_k$ and $a \in Z_n = \{0,1,2,\cdots,n-1\}$. Then

$$\begin{aligned} a * x_p &= at + x_p u \\ &= at + (k+ps)(n-t+1) \\ &= k(n+1) + [(a-k)v_1 + (pn - pt + p)]s \text{ where } t = v_1 s \\ &\equiv k + ls \bmod n \end{aligned}$$

$x_l \in A_k$ as $x_l = k + ls$ for some $l \in \{0,1,\cdots,r-1\}$. So, $a * x_p \in A_k$, $a * A_k = A_k$. Thus, A_k is a Smarandache seminormal subgroupoid in $Z_n(t,u)$.

If $s=(n,u)$, let $x_p \in A_k$ and $a \in Z_n = \{0,1,2,\cdots,n-1\}$. Then

$$\begin{aligned} x_p * a &= x_p t + au \\ &= (k+ps)(n-u+1) + au \\ &= k(n+1) + [(a-k)v_2 + (pn - pu + p)]s \text{ where } t = v_2 s \\ &\equiv (k + ls) \bmod n \end{aligned}$$

$x_l \in A_k$ as $x_l = k + ls$ for some $l \in \{0,1,\cdots,r-1\}$. Therefore, $a * x_p \in A_k$, $a * A_k = A_k$. Thus A_k is a Smarandache seminormal subgroupoid in $Z_n(t,u)$. □

By the above theorem we can determine Smarandache seminormal subgroupoid in $Z_n(t,u)$ when n is odd and $n > 4$.

n	t	u	$Z_n(t,u)$	$s=(n,u)$ or $s=(n,t)$	$r=n/s$	Smarandache seminormal subgroupoid in $Z_n(t,u)$
9	3	7	$Z_9(3,7)$	3 = (9,3)	3	$A_0=\{0,3,6\}$
						$A_1=\{1,4,7\}$
						$A_2=\{2,5,8\}$
15	3	13	$Z_{15}(3,13)$	3 = (15,3)	5	$A_0=\{0,3,6,9,12\}$
						$A_1=\{1,4,7,10,13\}$
						$A_2=\{2,5,8,11,14\}$
	5	11	$Z_{15}(5,11)$	5 = (15, 5)	3	$A_0=\{0,5,10\}$
						$A_1=\{1,6,11\}$
						$A_2=\{2,7,12\}$
						$A_3=\{3,8,13\}$
						$A_4=\{4,9,14\}$
	7	9	$Z_{15}(7,9)$	3 = (15,9)	5	$A_0=\{0,3,6,9,12\}$
						$A_1=\{1,4,7,10,13\}$
						$A_2=\{2,5,8,11,14\}$

n	t	u	$Z_n(t,u)$	$s=(n,u)$ or $s=(n,t)$	$r=n/s$	Smarandache seminormal subgroupoid in $Z_n(t,u)$
21	3	19	$Z_{21}(3,19)$	3 = (21,3)	7	$A_0=\{0,3,6,9,12,15,18\}$
						$A_1=\{1,4,7,10,13,16,19\}$
						$A_2=\{2,5,8,11,14,17,20\}$
	7	15	$Z_{21}(7,15)$	7 = (21,7)	3	$A_0=\{0,7,14\}$
						$A_1=\{1,8,15\}$
						$A_2=\{2,9,16\}$
						$A_3=\{3,10,17\}$
						$A_4=\{4,11,18\}$
						$A_5=\{5,12,19\}$
						$A_6=\{6,13,14\}$
				3 = (21,15)	7	$A_0=\{0,3,6,9,12,15,18\}$
						$A_1=\{1,4,7,10,13,16,19\}$
						$A_2=\{2,5,8,11,14,17,20\}$
	9	13	$Z_{21}(9,13)$	3 = (21,9)	7	$A_0=\{0,3,6,9,12,15,18\}$
						$A_1=\{1,4,7,10,13,16,19\}$
						$A_2=\{2,5,8,11,14,17,20\}$

References

[1] G.Birkhoff and S.S.Maclane, *A Brief Survey of Modern Algebra*, New York, U.S.A. The Macmillan and Co., 1965.

[2] R.H.Bruck, *A Survey of Binary Systems*, Springer Verlag, 1958.

[3] Ivan Nivan and H.S.Zukerman, *Introduction to Number theory*, Wiley Eastern Limited, 1989.

[4] H.J.Siamwalla and A.S.Muktibodh, Some results on Smarandache groupoids, *Scientia Magna* ,Vol.8, 2(2012), 111-117.

[5] W.B.Vasantha Kandasamy, New Classes of Finite Groupoids using Z_n, *Varamihir Journal of Mathematical Science*, Vol.1(2001), 135-143.

[6] W.B.Vasantha Kandasamy, *Smarandache Groupoids*, http: //www/gallup.unm.edu∼/smar -andache/Groupoids.pdf.

International J.Math. Combin. Vol.2(2013), 74-83

The Kropina-Randers Change of Finsler Metric and Relation Between Imbedding Class Numbers of Their Tangent Riemannian Spaces

H.S.Shukla, O.P.Pandey and Honey Dutt Joshi

(Department of Mathematics & Statistics, DDU Gorakhpur University, Gorakhpur, India)

E-mail: profhsshuklagkp@rediffmail.com, oppandey1988@gmail.com, honeythms@gmail.com

Abstract: In the present paper the relation between imbedding class numbers of tangent Riemannian spaces of (M^n, L) and (M^n, L^*) have been obtained, where the Finsler metric L^* is obtained from L by $L^* = \frac{L^2}{\beta} + \beta$ and M^n is the differentiable manifold.

Key Words: Finsler metric, Randers change, Kropina change, Kropina-Randers change, imbedding class.

AMS(2010): 53B40, 53C60

§1. Introduction

Let (M^n, L) be an n-dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function $L(x, y)$. In 1971, Matsumoto [2] introduced the transformation of Finsler metric:

$$L^*(x, y) = L(x, y) + \beta(x, y) \tag{1.1}$$

where $\beta(x, y) = b_i(x)y^i$ is a differentiable one-form on M^n. In 1984 Shibata [6] has studied the properties of Finsler space (M^n, L^*) whose metric function $L^*(x, y)$ is obtained from $L(x, y)$ by the relation $L^*(x, y) = f(L, \beta)$ where f is positively homogeneous of degree one in L and β. This change of metric function is called a $\beta-$change. The change (1.1) is a particular case of $\beta-$change called Randers change. The following theorem has importance under Randers change.

Theorem (1.1)([2]) *Let (M^n, L^*) be a locally Minkowskian n-space obtained from a locally Minkowskian n-space (M^n, L) by the change (1.1). If the tangent Riemannian n-space (M_x^n, g_x) to (M^n, L) is of imbedding class r, then tangent Riemannian n-space (M_x^n, g_x^*) to (M^n, L^*) is of imbedding class at most $r + 2$.*

In [5] it has been proved that Theorem (1.1) is valid for Kropina change of Finsler metric

[1]Received April 8, 2013, Accepted June 20, 2013.

function given by

$$L^*(x,y) = \frac{L^2(x,y)}{\beta(x,y)}. \tag{1.2}$$

In 1990, Prasad, Shukla and Singh [4] proved that Theorem (1.1) is valid for the transformation given by (1.1) in which $b_i(x)$ in β is replaced by h-vector $b_i(x,y)$ such that $\dfrac{\partial b_i}{\partial y^j}$ is proportional to angular metric tensor.

Recently Prasad, Shukla and Pandey [3] have proved that Theorem (1.1) is also valid for exponential change of Finsler metric given by

$$L^*(x,y) = Le^{\beta/L}.$$

In the present paper we consider Kropina-Randers change of Finsler metric given by

$$L^* = \frac{L^2}{\beta} + \beta$$

and prove that Theorem (1.1) is valid for this transformation also.

§2. **The Finsler Space** (M^n, L^*)

Let (M^n, L) be a given Finsler space and let $b_i(x)dx^i$ be a one-form on M^n. We shall define on M^n a function $L^*(x,y)\,(>0)$ by the equation

$$L^* = \frac{L^2}{\beta} + \beta, \tag{2.1}$$

where we put $\beta(x,y) = b_i(x)y^i$. To find the metric tensor g^*_{ij}, the angular metric tensor h^*_{ij}, the Cartan tensor C^*_{ijk} and the v-curvature tensor of (M^n, L^*) we use the following results:

$$\dot\partial_i\beta = b_i \qquad \dot\partial_i L = l_i, \qquad \dot\partial_j l_i = L^{-1}h_{ij}, \tag{2.2}$$

where $\dot\partial_i$ stands for $\dfrac{\partial}{\partial y^i}$ and h_{ij} are components of angular metric tensor of (M^n, L) given by $h_{ij} = g_{ij} - l_il_j = L\dot\partial_i\dot\partial_j L$.

The successive differentiation of (2.1) with respect to y^i and y^j gives

$$l^*_i = \frac{2L}{\beta}l_i + \left(1 - \frac{L^2}{\beta^2}\right) b_i, \tag{2.3}$$

$$h^*_{ij} = 2\left(\frac{L^2}{\beta^2} + 1\right)\left\{h_{ij} + l_il_j - \frac{L}{\beta}(l_ib_j + l_jb_i) + \frac{L^2}{\beta^2}\, b_ib_j\right\}. \tag{2.4}$$

From (2.3) and (2.4) we get the following relation between metric tensors of (M^n, L) and (M^n, L^*):

$$g^*_{ij} = 2\left(\frac{L^2}{\beta^2} + 1\right) g_{ij} + \frac{4L^2}{\beta^2}l_il_j + \left(\frac{3L^4}{\beta^4} + 1\right) b_ib_j - \frac{4L^3}{\beta^3}(l_ib_j + l_jb_i). \tag{2.5}$$

The contravariant components of the metric tensor of (M^n, L^*) is derived from (2.5) and are given by

$$g^{*ij} = \frac{\beta^2}{2(\beta^2+L^2)}g^{ij} - \frac{\beta^2\{\beta^2(\beta^2+L^2)+\triangle L^2(L^2-\beta^2)\}}{b^2(\beta^2+L^2)^3}l^il^j + \frac{\beta^3 L}{b^2(\beta^2+L^2)^2}(l^ib^j+l^jb^i) - \frac{\beta^2}{2b^2(\beta^2+L^2)}b^ib^j \tag{2.6}$$

where we put $l^i = g^{ij}l_j$, $b^i = g^{ij}b_j$, $b^2 = g^{ij}b_ib_j$ and $\triangle = b^2 - \dfrac{\beta^2}{L^2}$.

Differentiating (2.5) with respect to y^k and using (2.2) we get the following relation between the Cartan tensors of (M^n, L) and (M^n, L^*):

$$\begin{aligned} C^*_{ijk} &= \frac{1}{2}\dot{\partial}_k g^*_{ij} \\ &= 2\left(\frac{L^2}{\beta^2}+1\right)C_{ijk} - \frac{2L^2}{\beta^2}(h_{ij}m_k+h_{jk}m_i+h_{ik}m_j) - \frac{6L^4}{\beta^5}m_im_jm_k, \end{aligned} \tag{2.7}$$

where $m_i = b_i - \dfrac{\beta}{L}l_i$. It is to be noted that

$$m_il^i = 0, \quad m_ib^i = \triangle = m_im^i, \quad h_{ij}l^j = 0, \quad h_{ij}m^j = h_{ij}b^j = m_i, \tag{2.8}$$

where $m^i = g^{ij}m_j = b^i - \dfrac{\beta}{L}l^i$.

The quantities corresponding to (M^n, L^*) will be denoted by putting $*$ on those quantities. To find $C^{*i}_{jk} = g^{*ih}C^*_{jhk}$ we use (2.6), (2.7) and (2.8). We get

$$\begin{aligned} C^{*i}_{jk} &= C^i_{jk} - \frac{L^2}{\beta(\beta^2+L^2)}(h_{jk}m^i + h^i_jm_k + h^i_km_j) \\ &\quad - \frac{\beta^4}{2b^2L^2(\beta^2+L^2)}C_{.jk}n^i - \frac{3L^4}{b^2\beta^3(\beta^2+L^2)}m_jm_km^i \\ &\quad + \frac{\triangle\beta^3}{2b^2(\beta^2+L^2)^2}h_{jk}n^i + \frac{\beta(2\beta^2+3\triangle L^2)}{2b^2(\beta^2+L^2)^2}m_jm_kn^i, \end{aligned} \tag{2.9}$$

where $n^i = \dfrac{2L^2}{\beta^4}\{(\beta^2+L^2)b^i - 2\beta Ll^i\}$ and $C_{.jk} = C_{hjk}b^h$.

Throughout this paper we use the symbol . to denote the contraction with b^i. To find the v-curvature tensor of (M^n, L^*) we use the following:

$$\begin{aligned} &C_{ijk}m^i = C_{.ij}, \quad C_{ijk}n^i = \frac{2L^2}{\beta^4}(\beta^2+L^2)C_{.jk}, \\ &m_in^i = \frac{2\triangle L^2}{\beta^4}(\beta^2+L^2), \quad m^im_i = \triangle, \\ &h_{ij}n^i = \frac{2L^2}{\beta^4}(\beta^2+L^2)m_j, \quad C^h_{ij}h_{hk} = C_{ijk}, \quad h^r_jh^i_r = h^i_j. \end{aligned} \tag{2.10}$$

The v-curvature tensor S^*_{hijk} of (M^n, L^*) is defined as

$$S^*_{hijk} = C^{*r}_{hk}C^*_{rij} - C^{*r}_{hj}C^*_{rik} \tag{2.11}$$

From (2.7), (2.8), (2.9), (2.10) and (2.11) we get the following relation between v-curvature tensors of (M^n, L) and (M^n, L^*):

$$S^*_{hijk} = 2\left(\frac{L^2}{\beta^2}+1\right)S_{hijk} + d_{hj}d_{ik} - d_{hk}d_{ij} + E_{hk}E_{ij} - E_{hj}E_{ik}, \tag{2.12}$$

where

$$d_{ij} = \frac{\sqrt{2(\beta^2+L^2)}}{b\beta}\left[C_{.ij} + \frac{\beta}{\beta^2+L^2}h_{ij} + \frac{2L^2}{\beta(\beta^2+L^2)}m_i m_j\right], \tag{2.13}$$

$$E_{ij} = -\frac{\sqrt{2}L}{\beta\sqrt{\beta^2+L^2}}\left[h_{ij} + \frac{2L^2}{\beta^2}m_i m_j\right]. \tag{2.14}$$

By direct calculation we get the following results which will be used in the latter section of the paper:

$$\dot\partial_i b^2 = -2C_{..i}, \qquad \dot\partial_i \triangle = -2C_{..i} - \frac{2\beta}{L^2}m_i. \tag{2.15}$$

§3. Imbedding Class Numbers

The tangent vector space M^n_x to M^n at every point x is considered as the Riemannian n-space (M^n_x, g_x) with the Riemannian metric $g_x = g_{ij}(x,y)dy^i dy^j$. Then the components of the Cartan tensor are the Christoffel symbols associated with g_x:

$$C^i_{jk} = \frac{1}{2}g^{ih}(\dot\partial_k g_{jh} + \dot\partial_j g_{hk} - \dot\partial_h g_{jk}).$$

Thus C^i_{jk} defines the components of the Riemannian connection on M^n_x and v-covariant derivative, say

$$X_i|_j = \dot\partial_j X_i - X_h C^h_{ij} \tag{3.0}$$

is the covariant derivative of covariant vector X_i with respect to Riemannian connection C^i_{jk} on M^n_x. It is observed that the v-curvature tensor S_{hijk} of (M^n, L) is the Riemannian Christoffel curvature tensor of the Riemannian space (M^n, g_x) at a point x. The space (M^n, g_x) equipped with such a Riemannian connection is called the tangent Riemannian n-space [2].

It is well known [1] that any Riemannian n-space V^n can be imbedded isometrically in a Euclidean space of dimension $\frac{n(n+1)}{2}$. If $n+r$ is the lowest dimension of the Euclidean space in which V^n is imbedded isometrically, then the integer r is called the imbedding class number of V^n. The fundamental theorem of isometric imbedding ([1] page 190) is that the tangent Riemannian n-space (M^n_x, g_x) is locally imbedded isometrically in a Euclidean $(n+r)-$space if and only if there exist $r-$number $\epsilon_P = \pm 1$, $r-$symmetric tensors $H_{(P)ij}$ and $\frac{r(r-1)}{2}$ covariant vector fields $H_{(P,Q)i} = -H_{(Q,P)i}$; $P, Q = 1, 2, \cdots, r$, satisfying the Gauss equations

$$S_{hijk} = \sum_P \epsilon_P\{H_{(P)hj}H_{(P)ik} - H_{(P)ij}H_{(P)hk}\}, \tag{3.1}$$

The Codazzi equations

$$H_{(P)ij}|_k - H_{(P)ik}|_j = \sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q,P)k} - H_{(Q)ik}H_{(Q,P)j}\}, \tag{3.2}$$

and the Ricci-Kühne equations

$$\begin{aligned} H_{(P,Q)i}|_j - H_{(P,Q)j}|_i \quad &+ \quad \sum_R \epsilon_R \{H_{(R,P)i}H_{(R,Q)j} - H_{(R,P)j}H_{(R,Q)i}\} \\ &+ \quad g^{hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\} = 0. \end{aligned} \tag{3.3}$$

The numbers $\epsilon_P = \pm 1$ are the indicators of unit normal vector N_P to M^n and $H_{(P)ij}$ are the second fundamental tensors of M^n with respect to the normals N_P.

Proof of Theorem (1.1) In order to prove Theorem (1.1), we put

$$\begin{aligned} &(a) \quad H^*_{(P)ij} = \left[2\left(\frac{L^2}{\beta^2}+1\right)\right]^{1/2} H_{(P)ij}, \qquad \epsilon^*_P = \epsilon_P, \quad P = 1, 2, \cdots, r \\ &(b) \quad H^*_{(r+1)ij} = d_{ij}, \qquad \epsilon^*_{r+1} = 1 \\ &(c) \quad H^*_{(r+2)ij} = E_{ij}, \qquad \epsilon^*_{r+2} = -1. \end{aligned} \tag{3.4}$$

Then it follows from (2.12) and (3.1) that

$$S^*_{hijk} = \sum_{\lambda=1}^{r+2} \epsilon^*_\lambda \{H^*_{(\lambda)hj}H^*_{(\lambda)ik} - H^*_{(\lambda)hk}H^*_{(\lambda)ij}\},$$

which is the Gauss equation of (M^n_x, g^*_x).

Moreover, to verify Codazzi and Ricci Kühne equation of (M^n_x, g^*_x), we put

$$\begin{aligned} &(a) \quad H^*_{(P,Q)i} = -H^*_{(Q,P)i} = H_{(P,Q)i}, \quad P, Q = 1, 2, , \cdots, r \\ &(b) \quad H^*_{(P,r+1)i} = -H^*_{(r+1,P)i} = \frac{1}{b}H_{(P).i}, \quad P = 1, 2, \cdots, r \\ &(c) \quad H^*_{(P,r+2)i} = -H^*_{(r+2,P)i} = 0, \quad P = 1, 2, \cdots, r. \\ &(d) \quad H^*_{(r+1,r+2)i} = -H^*_{(r+2,r+1)i} = -\frac{L^2\sqrt{2}}{b(\beta^2+L^2)\sqrt{\beta L}} m_i. \end{aligned} \tag{3.5}$$

The Codazzi equations of (M^n_x, g^*_x) consists of the following three equations:

$$\begin{aligned} (a) \quad H^*_{(P)ij}\overset{*}{|}_k - H^*_{(P)ik}\overset{*}{|}_j = &\sum_Q \epsilon^*_Q \{H^*_{(Q)ij}H^*_{(Q,P)k} - H^*_{(Q)ik}H^*_{(Q,P)j}\} \\ &+ \epsilon^*_{r+1}\{H^*_{(r+1)ij}H^*_{(r+1,P)k} - H^*_{(r+1)ik}H^*_{(r+1,P)j}\} \\ &+ \epsilon^*_{r+2}\{H^*_{(r+2)ij}H^*_{(r+2,P)k} - H^*_{(r+2)ik}H^*_{(r+2,P)j}\} \end{aligned} \tag{3.6}$$

$$\begin{aligned} (b) \quad H^*_{(r+1)ij}\overset{*}{|}_k - H^*_{(r+1)ik}\overset{*}{|}_j = &\sum_Q \epsilon^*_Q \{H^*_{(Q)ij}H^*_{(Q,r+1)k} - H^*_{(Q)ik}H^*_{(Q,r+1)j}\} \\ &+ \epsilon^*_{r+2}\{H^*_{(r+2)ij}H^*_{(r+2,r+1)k} - H^*_{(r+2)ik}H^*_{(r+2,r+1)j}\} \\ (c) \quad H^*_{(r+2)ij}\overset{*}{|}_k - H^*_{(r+2)ik}\overset{*}{|}_j = &\sum_Q \epsilon^*_Q \{H^*_{(Q)ij}H^*_{(Q,r+2)k} - H^*_{(Q)ik}H^*_{(Q,r+2)j}\} \\ &+ \epsilon^*_{r+1}\{H^*_{(r+1)ij}H^*_{(r+1,r+2)k} - H^*_{(r+1)ik}H^*_{(r+1,r+2)j}\}. \end{aligned}$$

To prove these equations we note that for any symmetric tensor X_{ij} satisfying $X_{ij}l^i = X_{ij}l^j = 0$, we have from (2.9) and (3.0),

$$\begin{aligned} X_{ij}\overset{*}{|}_k - X_{ik}\overset{*}{|}_j &= X_{ij}|_k - X_{ik}|_j + \frac{1}{b^2}\{C_{.ik}X_{.j} - C_{.ij}X_{.k}\} \\ &\quad + \frac{L^2}{\beta(L^2+\beta^2)}(X_{ij}m_k - X_{ik}m_j) + \frac{L^2}{b^2\beta(L^2+\beta^2)} \\ &\quad \times (X_{.j}m_k - X_{.k}m_j)m_i + \frac{\beta}{b^2(L^2+\beta^2)}(h_{ik}X_{.j} - h_{ij}X_{.k}). \end{aligned} \tag{3.7}$$

In view of (3.4) and (3.5), equation (3.6)a is equivalent to

$$\begin{aligned} &\left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.H_{(P)ij}\right)\overset{*}{\Big|}_k - \left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.H_{(P)ik}\right)\overset{*}{\Big|}_j \\ &= \sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.\sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q,P)k} - H_{(Q)ik}H_{(Q,P)j}\} \\ &\quad - \frac{1}{b}\{H_{(P).k}d_{ij} - H_{(P).j}d_{ik}\}. \end{aligned} \tag{3.8}$$

Since $\left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}\right)\overset{*}{\Big|}_k = \dot{\partial}_k\left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}\right) = -\frac{\sqrt{2}\,L^2}{\beta^2\sqrt{L^2+\beta^2}}\,m_k$, applying formula (3.7) for $H_{(P)ij}$ and using equation (2.13), we get

$$\begin{aligned} &\left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.H_{(P)ij}\right)\overset{*}{\Big|}_k - \left(\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}.H_{(P)ik}\right)\overset{*}{\Big|}_j = \sqrt{2\left(\frac{L^2}{\beta^2}+1\right)} \\ &\quad \times\{H_{(P)ij}|_k - H_{(P)ik}|_j\} - \frac{1}{b}\{H_{(P).k}d_{ij} - H_{(P).j}d_{ik}\}, \end{aligned}$$

which after using equation (3.2), gives equation (3.8).

In view of (3.4) and (3.5), equation (3.6)b is equivalent to

$$\begin{aligned} d_{ij}\overset{*}{|}_k - d_{ik}\overset{*}{|}_j &= \frac{\sqrt{2(\beta^2+L^2)}}{b\beta}\sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q).k} - H_{(Q)ik}H_{(Q).j}\} \\ &\quad - \frac{L^2\sqrt{2}}{b(\beta^2+L^2)\sqrt{\beta L}}\{E_{ij}m_k - E_{ik}m_j\}. \end{aligned} \tag{3.9}$$

To verify (3.9), we note that

$$C_{.ij}|_k - C_{.ik}|_j = b_h S^h_{ijk} \tag{3.10}$$

$$b|_k = -\frac{1}{b}C_{..k}, \qquad h_{ij}|_k - h_{ik}|_j = L^{-1}(h_{ij}l_k - h_{ik}l_j) \tag{3.11}$$

$$m_i|_k = -C_{.ik} - \frac{\beta}{L^2}h_{ik} - L^{-1}l_i m_k. \tag{3.12}$$

The v-covariant differentiation of (2.13) will give the value of $d_{ij}|_k$. Then taking skew-symmetric part of $d_{ij}|_k$ in j and k, we get

$$\begin{aligned} d_{ij}|_k - d_{ik}|_j &= A(C_{.ij}|_k - C_{.ik}|_j) + B(h_{ij}|_k - h_{ik}|_j) + D(m_i|_k m_j \\ &\quad + m_j|_k m_i - m_i|_j m_k - m_k|_j m_i) + (\dot{\partial}_k A)C_{.ij} - (\dot{\partial}_j A)C_{.ik} \\ &\quad + (\dot{\partial}_k B)h_{ij} - (\dot{\partial}_j B)h_{ik} + (\dot{\partial}_k D)m_i m_j - (\dot{\partial}_j D)m_i m_k, \end{aligned} \tag{3.13}$$

where $A = \dfrac{\sqrt{2(\beta^2+L^2)}}{b\beta}$, $B = \dfrac{\sqrt{2}}{b\sqrt{(\beta^2+L^2)}}$, $D = \dfrac{4L^2}{b\beta^2\sqrt{2(\beta^2+L^2)}}$.

Applying formula (3.7) for d_{ij} and using (3.13), (3.10), (3.11), (3.12), (2.15), we get

$$d_{ij}\overset{*}{|}_k - d_{ik}\overset{*}{|}_j = \frac{\sqrt{2(\beta^2+L^2)}}{b\beta} b_h S^h_{ijk} + \frac{2L^3}{b\beta(L^2+\beta^2)^{3/2}\sqrt{\beta L}}(h_{ij}m_k - h_{ik}m_j). \tag{3.14}$$

Substituting (3.1) and (2.14) in the right hand side of (3.14), we get equation (3.9).

In view of (3.4) and (3.5), equation (3.6)c is equivalent to

$$E_{ij}\overset{*}{|}_k - E_{ik}\overset{*}{|}_j = \frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}(d_{ij}m_k - d_{ik}m_j). \tag{3.15}$$

The v-covariant differentiation of (2.14) and use of (2.15) will give the value of $E_{ij}|_k$. Then taking skew-symmetric part of $E_{ij}|_k$ in j and k and using (3.11), (3.12), we get

$$E_{ij}|_k - E_{ik}|_j = -\frac{2\sqrt{2}L^3}{\beta^3\sqrt{\beta^2+L^2}}(C_{.ij}m_k - C_{.ik}m_j) - \frac{2\sqrt{2}L^3}{\beta^2(L^2+\beta^2)^{3/2}}(h_{ij}m_k - h_{ik}m_j). \tag{3.16}$$

Applying formula (3.7) for E_{ij} and using (3.16), we get (3.15). This completes the proof of Codazzi equations of (M^n_x, g^*_x).

The Ricci Kühne equations of (M^n_x, g^*_x) consist of the following four equations:

$$\text{(a)}\quad H^*_{(P,Q)i}\overset{*}{|}_j - H^*_{(P,Q)j}\overset{*}{|}_i + \sum_Q \epsilon^*_Q\{H^*_{(R,P)i}H^*_{(R,Q)j} - H^*_{(R,P)j}H^*_{(R,Q)i}\} + \epsilon^*_{r+1}\{H^*_{(r+1,P)i}H^*_{(r+1,Q)j} - H^*_{(r+1,P)j}H^*_{(r+1,Q)i}\} + \epsilon^*_{r+2}\{H^*_{(r+2,P)i}H^*_{(r+2,Q)j} - H^*_{(r+2,P)j}H^*_{(r+2,Q)i}\} + g^{*hk}\{H^*_{(P)hi}H^*_{(Q)kj} - H^*_{(P)hj}H^*_{(Q)ki}\} = 0, \qquad P,Q = 1,2,\cdots,r \tag{3.17}$$

$$\text{(b)}\quad H^*_{(P,r+1)i}\overset{*}{|}_j - H^*_{(P,r+1)j}\overset{*}{|}_i + \sum_R \epsilon^*_R\{H^*_{(R,P)i}H^*_{(R,r+1)j} - H^*_{(R,P)j}H^*_{(R,r+1)i}\} + \epsilon^*_{r+2}\{H^*_{(r+2,P)i}H^*_{(r+2,r+1)j} - H^*_{(r+2,P)j}H^*_{(r+2,r+1)i}\} + g^{*hk}\{H^*_{(P)hi}H^*_{(r+1)kj} - H^*_{(P)hj}H^*_{(r+1)ki}\} = 0, \quad P = 1,2,\cdots,r$$

$$\text{(c)}\quad H^*_{(P,r+2)i}\overset{*}{|}_j - H^*_{(P,r+2)j}\overset{*}{|}_i + \sum_R \epsilon^*_R\{H^*_{(R,P)i}H^*_{(R,r+2)j} - H^*_{(R,P)j}H^*_{(R,r+2)i}\} + \epsilon^*_{r+1}\{H^*_{(r+1,P)i}H^*_{(r+1,r+2)j} - H^*_{(r+1,P)j}H^*_{(r+1,r+2)i}\} + g^{*hk}\{H^*_{(P)hi}H^*_{(r+2)kj} - H^*_{(P)hj}H^*_{(r+2)ki}\} = 0, \quad P = 1,2,\cdots,r$$

$$\text{(d)}\quad H^*_{(r+1,r+2)i}\overset{*}{|}_j - H^*_{(r+1,r+2)j}\overset{*}{|}_i + \sum_R \epsilon^*_R\{H^*_{(R,r+1)i}H^*_{(R,r+2)j} - H^*_{(R,r+1)j} \times H^*_{(R,r+2)i}\} + g^{*hk}\{H^*_{(r+1)hi}H^*_{(r+2)kj} - H^*_{(r+1)hj}H^*_{(r+2)ki}\} = 0.$$

In view of (3.4) and (3.5), equation (3.17)a is equivalent to

$$H_{(P,Q)i}\overset{*}{|}_j - H_{(P,Q)j}\overset{*}{|}_i + \sum_R \epsilon_R\{H_{(R,P)i}H_{(R,Q)j} - H_{(R,P)j}H_{(R,Q)i}\} \tag{3.18}$$
$$+\frac{1}{b^2}\{H_{(P).i}H_{(Q).j} - H_{(P).j}H_{(Q).i}\} + g^{*hk}\{H_{(P)hi}H_{(Q)kj}$$
$$-H_{(P)hj}H_{(Q)ki}\}.2\left(\frac{L^2}{\beta^2}+1\right) = 0. \qquad P, Q = 1, 2, \cdots, r.$$

Since $H_{(P)ij}l^i = 0 = H_{(P,Q)i}l^i$, from (2.6), we get

$$g^{*hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\}\left(\frac{L^2}{\beta^2}+1\right) = g^{hk}\left(\frac{L^2}{\beta^2}+1\right)\{H_{(P)hi}\times$$
$$H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\} - \frac{1}{2b^2}\{H_{(P).i}H_{(P).j} - H_{(P).j}H_{(P).i}\}.$$

Also, we have $H_{(P,Q)i}\overset{*}{|}_j - H_{(P,Q)j}\overset{*}{|}_i = H_{(P,Q)i}|_j - H_{(P,Q)j}|_i$. Hence equation (3.18) is satisfied identically by virtue of (3.3).

In view of (3.4) and (3.5), equation (3.17)b is equivalent to

$$\left(\frac{1}{b}H_{(P).i}\right)\overset{*}{\Big|}_j - \left(\frac{1}{b}H_{(P).j}\right)\overset{*}{\Big|}_i + \frac{1}{b}\sum_R \epsilon_R\{H_{(R,P)i}H_{(R).j} - H_{(R,P)j}H_{(R).i}\} \tag{3.19}$$
$$+ g^{*hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)} = 0. \quad P, Q = 1, 2, \cdots, r.$$

Since $b^h|_j = -g^{hk}C_{.jk}$, $H_{(P)hi}l^i = 0$, we have

$$H_{(P).i}\overset{*}{|}_j - H_{(P).j}\overset{*}{|}_i = H_{(P).i}|_j - H_{(P).j}|_i = [H_{(P)hi}|_j - H_{(P)hj}|_i]b^h \tag{3.20}$$
$$-g^{hk}\{H_{(P)hi}C_{.kj} - H_{(P)hj}C_{.ki}\}$$

$$\frac{1}{b}\overset{*}{\Big|}_j = \dot{\partial}_j\left(\frac{1}{b}\right) = \frac{1}{b^3}C_{..j} \tag{3.21}$$

and

$$g^{*hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)} = \frac{\beta}{\sqrt{2(L^2+\beta^2)}}g^{hk}\times \tag{3.22}$$
$$\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\} - \frac{\beta}{b^2\sqrt{2(L^2+\beta^2)}}\{H_{(P).i}d_{.j} - H_{(P).j}d_{.i}\}.$$

After using (2.13) the equation (3.22) may be written as

$$g^{*hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)} = \frac{1}{b}g^{hk}\times \tag{3.23}$$
$$\{H_{(P)hi}C_{.kj} - H_{(P)hj}C_{.ki}\} - \frac{1}{b^3}\{H_{(P).i}C_{..j} - H_{(P).j}C_{..i}\}.$$

From (3.2), (3.20), (3.21) and (3.23) it follows that equation (3.19) holds identically.

In view of (3.4) and (3.5), equation (3.17)c is equivalent to

$$\frac{\sqrt{2}L^2}{b^2(L^2+\beta^2)\sqrt{\beta L}}\{H_{(P).i}m_j - H_{(P).j}m_i) \\ +\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}g^{*hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\} = 0, \tag{3.24}$$

Since $E_{ij}l^i = E_{ij}l^j = 0$, from (2.5), we have

$$\sqrt{2\left(\frac{L^2}{\beta^2}+1\right)}g^{*hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\} = \frac{\beta}{\sqrt{2(L^2+\beta^2)}}g^{hk} \times \\ \{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\} - \frac{\beta}{b^2\sqrt{2(L^2+\beta^2)}}\{H_{(P).i}E_{.j} - H_{(P).j}E_{.i}\}.$$

In view of (2.14) the right hand side of the last equation is equal to

$$-\frac{\sqrt{2}L^2}{b^2(L^2+\beta^2)\sqrt{\beta L}}\{H_{(P).i}m_j - H_{(P).j}m_i\}.$$

Hence equation (3.24) is satisfied identically.

In view of (3.4) and (e3.5), equation (3.17)d is equivalent to

$$\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_i\right)\Big|^{*}_{j} - \left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_j\right)\Big|^{*}_{i} \\ + g^{*hk}(d_{hi}E_{kj} - d_{hj}E_{ki}) = 0. \tag{3.25}$$

Since $E_{ij}l^i = 0$, $d_{ij}l^i = 0$, from (2.6), it follows that

$$g^{*hk}\{d_{hi}E_{kj} - d_{hj}E_{ki}\} = \frac{\beta^2}{2(L^2+\beta^2)}g^{hk}\{d_{hi}E_{kj} - d_{hj}E_{ki}\} \\ -\frac{\beta^2}{2b^2(L^2+\beta^2)}\{d_{.i}E_{.j} - d_{.j}E_{.i}\}.$$

In view of (2.13) the right hand side of the last equation is equal to

$$-\frac{2L}{b^3(L^2+\beta^2)}\{C_{..i}m_j - C_{..j}m_i\}.$$

Also,

$$\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_i\right)\Big|^{*}_{j} - \left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_j\right)\Big|^{*}_{i} \\ = -\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}(m_i\overset{*}{|}_j - m_j\overset{*}{|}_i) + \dot{\partial}_j\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}\right)m_i \\ -\dot{\partial}_i\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}\right)m_j.$$

Since $m_i\overset{*}{|}_j - m_j\overset{*}{|}_i = L^{-1}(l_j m_i - l_i m_j)$ and

$$\dot{\partial}_j\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}\right) = -\frac{\sqrt{2}L}{b(L^2+\beta^2)\sqrt{\beta L}}l_j - \frac{2L}{b^3(L^2+\beta^2)}C_{..j},$$

we have

$$\begin{aligned}\left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_i\right)\overset{*}{\Big|}_j \quad &- \quad \left(-\frac{\sqrt{2}L^2}{b(L^2+\beta^2)\sqrt{\beta L}}m_j\right)\overset{*}{\Big|}_i \\ &= \quad -\frac{2L}{b^3(L^2+\beta^2)}\{C_{..j}m_i - C_{..i}m_j\}.\end{aligned} \tag{3.26}$$

Hence equation (3.25) is satisfied identically. Therefore Ricci-Kühne equations are satisfied for (M_x^n, g_x^*) given in (3.17) are satisfied.

Hence Theorem (1.1) given in introduction is satisfied for Kropina-Randers change of Finsler metric.

References

[1] Eisenhart L.P., *Riemannian Geometry*, Princeton, 1926.

[2] Matsumoto M., On transformations of locally Minkowskian space, *Tensor N. S.*, 22 (1971), 103-111.

[3] Prasad B.N., Shukla H. S. and Pandey O. P., The exponential change of Finsler metric and relation between imbedding class numbers of their tangent Riemannian spaces, (under publication).

[4] Prasad B. N., Shukla H. S. and Singh D.D., On a transformation of the Finsler metric, *Math. Vesnik*, 42 (1990), 45-53.

[5] Singh U.P., Prasad B.N. and Kumari Bindu, On a Kropina change of Finsler metric, *Tensor N. S.*, 64 (2003), 181-188.

[6] Shibata C., On invariant tensors of $\beta-$changes of Finsler metrics, *J. Math. Kyoto University*, 24 (1984), 163-188.

International J.Math. Combin. Vol.2(2013), 84-90

The Bisector Surface of Rational Space Curves in Minkowski 3-Space

Mustafa Dede

(Department of Mathematics, Faculty of Arts and Sciences, University of 7 Aralık , 79000 Kilis, Turkey)

E-mail: mustafadede@kilis.edu.tr

Abstract: The aim of this paper is to compare bisector surfaces of rational space curves in Euclidean and Minkowski 3-spaces.

Key Words: Minkowski 3-space, bisector surface, medial surface, Voronoi surface.

AMS(2010): 53B40, 53C60

§1. Introduction

Bisector construction plays an important role in many geometric computations, such as Voronoi diagrams construction, medial axis transformation, shape decomposition, mesh generation, collision-avoidance motion planning, and NC tool path generation (Dutta 1993, Elber 1998, Pottmann 1995, Peternell 2000, Farouki 1994b).

Let $\mathbb{R}^3_1$ be a Minkowski 3-space with Lorentzian metric

$$ds^2 = dx^2 + dy^2 - dz^2 \tag{1}$$

If $\langle X, Y\rangle = 0$ for all X and Y, the vectors X and Y are called perpendicular in the sense of Lorentz, where $\langle , \rangle$ is the induced inner product in $\mathbb{R}^3_1$. The norm of $X \in \mathbb{R}^3_1$ is denoted by $\|X\|$ and defined as

$$\|X\| = \sqrt{|\langle X, X\rangle|} \tag{2}$$

We say that a Lorentzian vector X is spacelike, lightlike or timelike if $\langle X, X\rangle > 0$ and $X = 0$, $\langle X, X\rangle = 0$, $\langle X, X\rangle < 0$, respectively. A smooth regular curve is said to be a timelike, spacelike or lightlike curve if the tangent vector is a timelike, spacelike, or lightlike vector, respectively (Turgut 1998, Turgut 1997, O'Neill 1983) .

For any $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, the Lorentz vector product of X and Y is defined as follows:

$$X \wedge Y = (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2).$$

This yields

[1]Received November 16, 2012, Accepted June 22, 2013.

$$e_1 \wedge e_2 = -e_3, \qquad e_3 \wedge e_1 = -e_2, \qquad e_2 \wedge e_3 = e_1$$

where $\{e_1, e_2, e_3\}$ are the base of the space $\mathbb{R}_1^3$.

§2. Bisector Surface of Two Space Curves in Minkowski 3-Space

To introduce the subject of bisector in Minkowski 3-space, we deal with an elementary example. Let $A = (a, b, c)$ and $N = (m, n, l)$ be two points in $\mathbb{R}_1^3$. Since the bisector $B = (x, y, z)$ is the set of points equidistant from the two points A and N, we have

$$\left|(x-a)^2 + (y-b)^2 - (z-c)^2\right| = \left|(x-m)^2 + (y-n)^2 - (z-l)^2\right| \tag{3}$$

There are two cases in Equation (3). Now, let us discuss the following two cases.

Case 1 If $(x-a)^2 + (y-b)^2 - (z-c)^2 = (x-m)^2 + (y-n)^2 - (z-l)^2$ then, we have a plane equation in $\mathbb{R}_1^3$ given by

$$x(m-a) + y(n-b) + z(c-l) + \frac{1}{2}(a^2 + b^2 - c^2 + m^2 + n^2 - l^2) = 0 \tag{4}$$

Case 2 If $(x-a)^2 + (y-b)^2 - (z-c)^2 = (z-l)^2 - (x-m)^2 - (y-n)^2$ then, we have a hyperboloid equation in $\mathbb{R}_1^3$ given by

$$x^2 + y^2 - z^2 - x(a+m) - y(b+n) + z(c+l) + \frac{1}{2}(a^2 + b^2 - c^2 + m^2 + n^2 - l^2) = 0 \tag{5}$$

We now investigate the bisector surface of two rational space curves. Let

$$\begin{aligned} C_1(s) &= (x_1(s), y_1(s), z_1(s)) \\ C_2(t) &= (x_2(t), y_2(t), z_2(t)) \end{aligned} \tag{6}$$

be two regular parametric C^1-continuous space curves in Minkowski 3-space. The tangent vectors of $C_1(s)$ and $C_2(t)$ are determined by, respectively

$$\begin{aligned} T_1(s) &= (x_1'(s), y_1'(s), z_1'(s)) \\ T_2(t) &= (x_2'(t), y_2'(t), z_2'(t)) \end{aligned} \tag{7}$$

When a point P is on the bisector of two curves, there exist (at least) two points $C_1(s)$ and $C_2(t)$ such that point P is simultaneously contained in the normal planes $L_1(s)$ and $L_2(t)$. As a result, the point P satisfies the following two linear equations:

$$L_1(s) :< P - C_1(s), T_1(s) >= 0 \tag{8}$$

$$L_2(t) :< P - C_2(t), T_2(t) >= 0 \tag{9}$$

Moreover, point P is also contained in the bisector plane $L_{12}(s,t)$ between the two points $C_1(s)$ and $C_2(t)$. The plane $L_{12}(s,t)$ is orthogonal to the vector $C_1(s) - C_2(t)$ and passes through the mid point $[C_1(s) + C_2(t)]/2$ of $C_1(s)$ and $C_2(t)$. Therefore, the bisector plane $L_{12}(s,t)$ is defined by the following linear equation:

$$L_{12}(s,t) :< P - \frac{C_1(s) + C_2(t)}{2}, C_1(s) - C_2(t) >= 0 \tag{10}$$

Any bisector point P must be a common intersection point of the three planes of $L_1(s)$, $L_2(t)$, and $L_{12}(s,t)$, for some s and t. Therefore, the point P can be computed by solving the following simultaneous linear equations in P:

$$\left.\begin{aligned} L_1(s): &\quad < P, T_1(s) >=< C_1(s), T_1(s) > \\ L_2(t): &\quad < P, T_2(t) >=< C_2(t), T_2(t) > \\ L_{12}(s,t): &\quad < P, C_1(s) - C_2(t) >= \frac{C_1(s)^2 - C_2(t)^2}{2} \end{aligned}\right\} \tag{11}$$

Using Equations (6), we have

$$C_1(s) - C_2(t) = (x_{12}(s,t), y_{12}(s,t), z_{12}(s,t)) \tag{12}$$

where $x_{12}(s,t) = x_1(s) - x_2(t)$, $y_{12}(s,t) = y_1(s) - y_2(t)$ and $z_{12}(s,t) = z_1(t) - z_2(s)$.

Substituting Equations (12), (1) and (7) into Equation (11) then, we obtain the implicit equations of the planes $L_1(s)$, $L_2(t)$, and $L_{12}(s,t)$ as

$$\left.\begin{aligned} L_1(s): &\quad = \quad x_1'(s)P_x + y_1'(s)P_y - z_1'(s)P_z = d_1(s) \\ L_2(t): &\quad = \quad x_2'(t)P_x + y_2'(t)P_y - z_2'(t)P_z = d_2(t) \\ L_{12}(s,t): &\quad = \quad x_{12}(s,t)P_x + y_{12}(s,t)P_y - z_{12}(s,t)P_z = m(s,t) \end{aligned}\right\} \tag{13}$$

where $P = (P_x, P_y, P_z)$ is the bisector point, and $d_1(s)$, $d_2(t)$ and $m(s,t)$ are given by

$$d_1(s) =< C_1(s), T_1(s) >, \quad d_2(t) =< C_2(t), T_2(t) > \tag{14}$$

$$m(s,t) = \frac{C_1(s)^2 - C_2(t)^2}{2} \tag{15}$$

We may express results in the matrix form as

$$\begin{bmatrix} x_1'(s) & y_1'(s) & -z_1'(s) \\ x_2'(t) & y_2'(t) & -z_2'(t) \\ x_{12}(s,t) & y_{12}(s,t) & -z_{12}(s,t) \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} d_1(s) \\ d_2(t) \\ m(s,t) \end{bmatrix} \tag{16}$$

By Cramer's rule, Equation (13) can be solved as follows:

$$P_x(s,t)=\frac{\begin{vmatrix} d_1(s) & y_1'(s) & -z_1'(s)\\ d_2(t) & y_2'(t) & -z_2'(t)\\ m(s,t) & y_{12}(s,t) & -z_{12}(s,t)\end{vmatrix}}{\begin{vmatrix} x_1'(s) & y_1'(s) & -z_1'(s)\\ x_2'(t) & y_2'(t) & -z_2'(t)\\ x_{12}(s,t) & y_{12}(s,t) & -z_{12}(s,t)\end{vmatrix}}, \tag{17}$$

$$P_y(s,t)=\frac{\begin{vmatrix} x_1'(s) & d_1(s) & -z_1'(s)\\ x_2'(t) & d_2(t) & -z_2'(t)\\ x_{12}(s,t) & m(s,t) & -z_{12}(s,t)\end{vmatrix}}{\begin{vmatrix} x_1'(s) & y_1'(s) & -z_1'(s)\\ x_2'(t) & y_2'(t) & -z_2'(t)\\ x_{12}(s,t) & y_{12}(s,t) & -z_{12}(s,t)\end{vmatrix}}, \tag{18}$$

and

$$P_z(s,t)=\frac{\begin{vmatrix} x_1'(s) & y_1'(s) & d_1(s)\\ x_2'(t) & y_2'(t) & d_2(t)\\ x_{12}(s,t) & y_{12}(s,t) & m(s,t)\end{vmatrix}}{\begin{vmatrix} x_1'(s) & y_1'(s) & -z_1'(s)\\ x_2'(t) & y_2'(t) & -z_2'(t)\\ x_{12}(s,t) & y_{12}(s,t) & -z_{12}(s,t)\end{vmatrix}}. \tag{19}$$

The bisector surface $P(s,t) = (P_x(s,t), P_y(s,t), P_z(s,t))$ has a simple rational representation as long as the common denominator of P_x, P_y and P_z in equation (13) does not vanish.

Example 2.1 Let $C_1(s)$ and $C_2(t)$ be two non-intersecting orthogonal straight lines in Minkowski space given by parametrization

$$C_1(s) = (1, s, 0), \quad C_2(t) = (0, 0, t) \tag{20}$$

By using Equations (20), (14) and (15), we have

$$d_1(s) = s, \quad d_2(t) = -t, \quad m(s,t) = \frac{1+s^2+t^2}{2} \tag{21}$$

Substituting Equation (21) into Equations (17), (18) and (19). Finally, we have the bisector surface $P(s,t)$ given by parametrization

$$P(s,t) = (\frac{1-s^2-t^2}{2}, s, t)$$

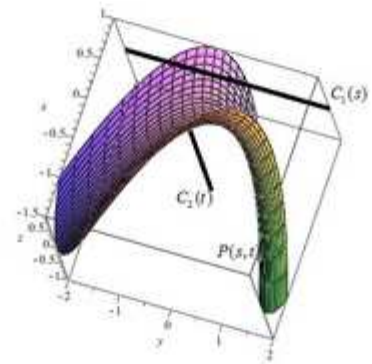


Figure 2.1a

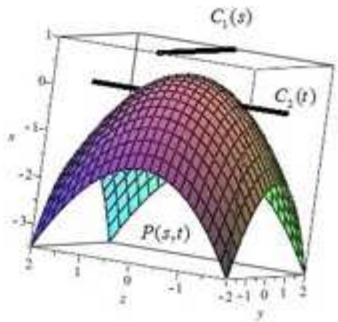


Figure 2.1b

where, Figure 2.1a shows the bisector surface of two lines in Euclidean space, Figure 2.1b shows that the bisector surface of two lines in Minkowski space.

We observe that the bisector of $C_1(s)$ and $C_2(t)$ lines, shown in Fig. 1(a), is a hyperbolic paraboloid of one sheet in Euclidean 3-space (Elber 1998). On the other hand, the bisector surface of $C_1(s)$ and $C_2(t)$ lines, shown in Fig. 1(b), is elliptic paraboloid in Minkowski 3-space.

Example 2.2 Figure 2(a) and Figure 2(b) illustrates the bisector surfaces of a Euclidean circle and a line, given by parametrization

$$C_1(s) = (\cos(s), \sin(s), 0), \qquad C_2(t) = (0, 0, t) \tag{22}$$

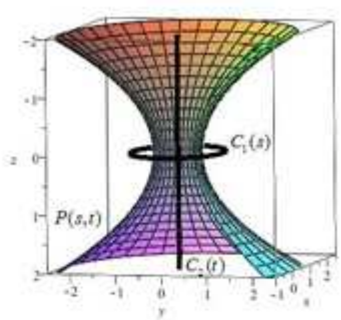


Figure 2.2a

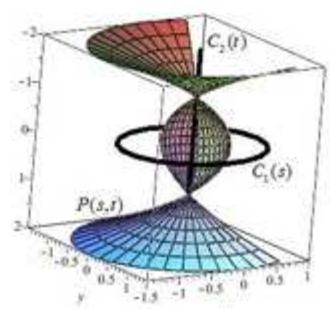


Figure 2.2b

where, Figure 2.2a shows the bisector surface of a circle and a line in Euclidean space, Figure 2.2b shows that the bisector surface of a circle and a line in Minkowski space.

From (22), (12) and (13), we get

$$d_1(s) = 0, \qquad d_2(t) = -t, \quad m(s,t) = \frac{1+t^2}{2} \tag{23}$$

$$(x_{12}(s,t), y_{12}(s,t), z_{12}(s,t)) = (\cos(s), \sin(s), -t) \tag{24}$$

Substituting above equations into Equations (17), (18) and (19), we have the bisector surface given by parametrization

$$P(s,t) = (\frac{1-t^2}{2}\cos(s), \frac{1-t^2}{2}\sin(s), t) \tag{25}$$

Consequently, Fig.3(a) and Fig.3(b) shows an example of the bisector surface of a non-planar curve (Euclidean helix) and a line.

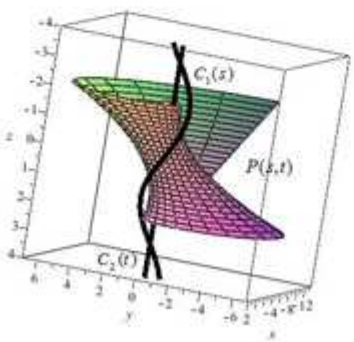


Figure 2.3a

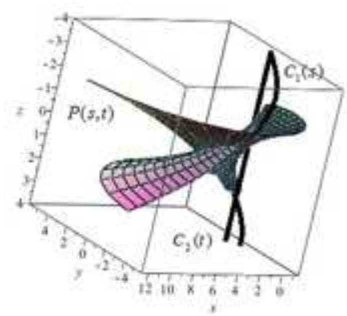


Figure 2.3b

where, Figure 2.3a shows the bisector surface of a helix and a line in Euclidean space, Figure 2.3 shows that the bisector surface of a helix and a line in Minkowski space.

§3. Conclusions

In this paper, we have shown that the bisector surface of curve/curve in Minkowski 3-space. Bisector surface of point/curve and surface/surface are not included in this paper. The different studies on bisector surface in Minkowski 3-space may be presented in a future publication.

References

[1] Dutta D.and Hoffman C., On the skeleton of simple CSG, *ASME J. Mech. Des. Objects*, 115(1993), 87-94.

[2] Elber G. and Myung-Soo K., The bisector surface of rational space curves, *ACM Transactions on Graphics*, 17(1998), 32-49.

[3] Farouki R. and Johnstone J., Computing point/curve and curve/curve bisectors, *Design and Application of Curves and Surfaces: Mathematics of Surfaces V*, R. B. Fisher, Ed., Oxford University Press, 1994b, 327-354.

[4] O'Neill, B., *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[5] Peternell M., Geometric properties of bisector surfaces, *Graphical Models*, 62(2000), 202-236.

[6] Pottmann H. Rational curves and surfaces with rational offsets, *Computer-Aided Geometric Design*, 12(1995), 175-192.

[7] Turgut A., Hacısalihoglu H.H., Timelike ruled surfaces in the Minkowski 3-space-II, *Turkish Journal of Mathematics* 22(1998), 33-46.

[8] Turgut A., Hacısalihoğlu H.H., Timelike ruled surfaces in the Minkowski 3-space, *Far East J. Math. Sci.*, 5(1997), 83-90.

International J.Math. Combin. Vol.2(2013), 91-96

A Note on Odd Graceful Labeling of a Class of Trees

Mathew Varkey T.K.

(Department of Mathematics, T.K.M. College of Engineering, Kollam-5, India)

Shajahan A.

(Department of Mathematics, M.S.M. College Kayamkulam, India)

E-mail: mathewvarkeytk@gmail.com, shajahan_safa@yahoo.com

Abstract: A connected graph with n vertices and q edges is called odd graceful if it is possible to label the vertices x with pairwise distinct integers $f(x)$ in $\{0, 1, 2, 3, \cdots, 2q-1\}$ so that when each edge, xy is labeled $|f(x) - f(y)|$, the resulting edge labels are pairwise distinct and thus form the entire set $\{1, 3, 5, \cdots, 2q-1\}$. In this paper we study the odd graceful labeling of class of T_n trees.

Key Words: Labeling, Odd graceful graph, Tree.

AMS(2010): 05C78

§1. Introduction

Unless mentioned otherwise, a graph in this paper shall mean a simple finite graph without isolated vertices.

For all terminology and notations in graph theory, we follow Harary [1] and for all terminology regarding odd graceful labeling, we follow [2]. A connected graph with n vertices and q edges is called odd graceful if it is possible to label the vertices x with pairwise distinct integers $f(x)$ in $\{0, 1, 2, 3, \cdots, 2q-1\}$ so that each edge, xy, is labeled $|f(x) - f(y)|$, the resulting edge labels are pairwise distinct. (and thus form the entire set $\{1, 3, 5, \cdots, 2q-1\}$). In this article we study the odd graceful labeling of typical class of T_n trees.

§2. On T_n-Class of Trees

Definition 2.1([3]) *Let T be a tree and x and y be two adjacent vertices in T. Let there be two end vertices (non-adjacent vertices of degree one) $x_1, y_1 \in T$ such that the length of the path $x - x_1$ is equal to the length of the path $y - y_1$. If the edge xy is deleted from T and x_1, y_1 are joined by an edge x_1y_1; then such a transformations of the edge from xy to x_1y_1 is called an elementary parallel transformation (or an EPT of T) and the edge xy is called a transformable edge.*

[1]Received November 16, 2012, Accepted June 22, 2013.

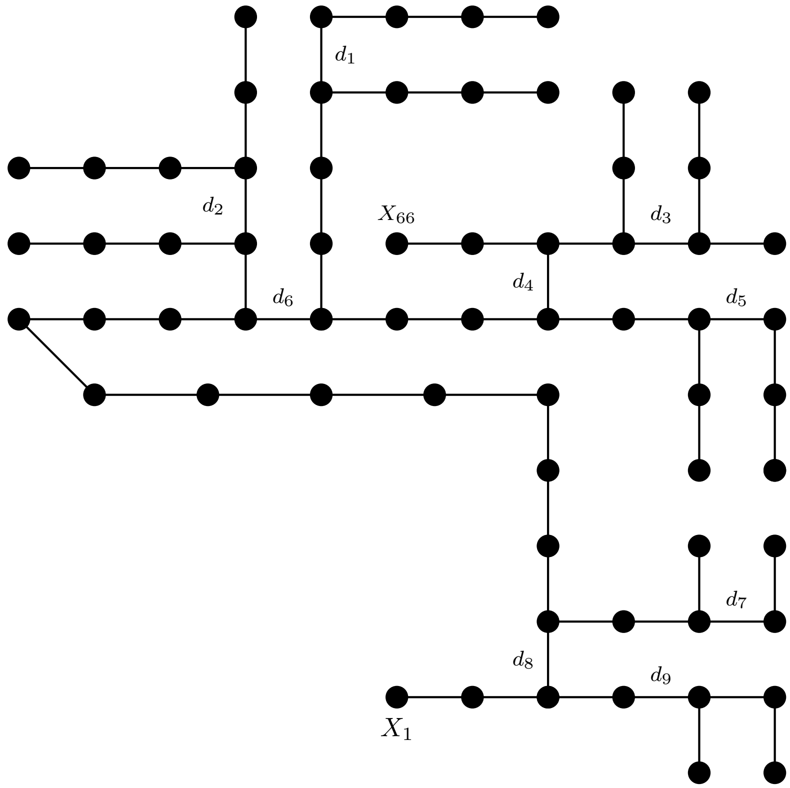


Fig.1 A T_{66}-tree T

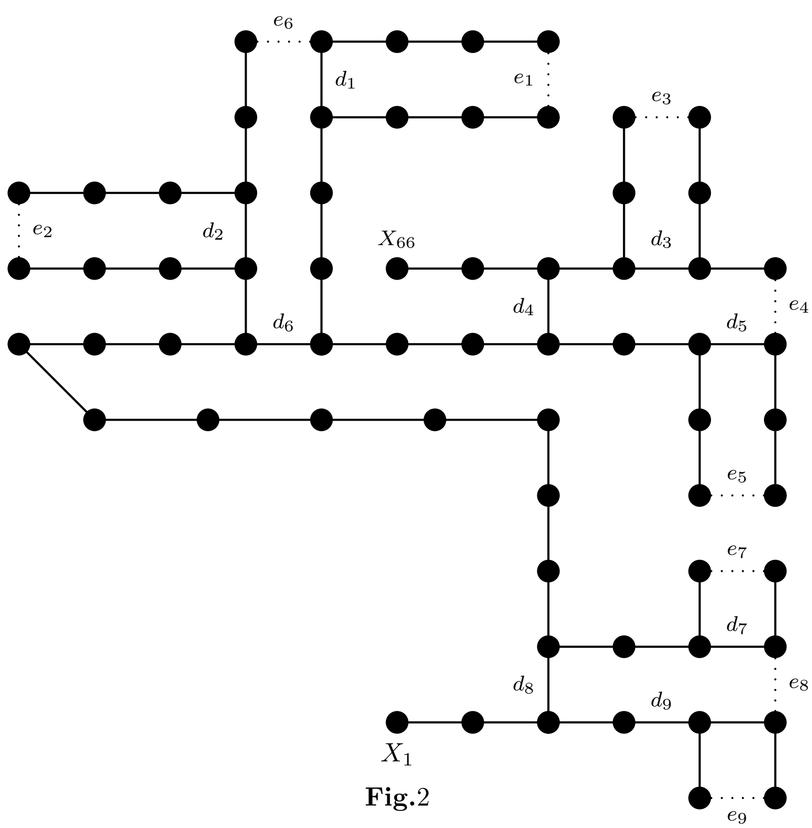


Fig.2

Definition 2.2 *If by a sequence of EPT's, the tree, T can be reduced to a Hamiltonian path, then T is called a T_n-tree (transformed tree) and such a Hamiltonian path is denoted as $P^H(T)$. Any such sequence regarded as a composition mapping (EPT's) denoted by P is called parallel transformation of T*[3].

A T_n-tree and a sequence of nine EPT's reducing it to a hamiltonian path are illustrated in Fig.1 to Fig.3.

In Fig.2, let $d_1, d_2, \cdots d_9$ are the deleted edges and $e_1, e_2, \cdots, e_9$ are the corresponding added edges (Given in broken lines).

An EPT $P_i^H(T)$; for $i = 1, 2, \cdots, 9$. The hamiltonian path $P^H(T)$ for the tree in Fig. 1 is given in Fig.3.

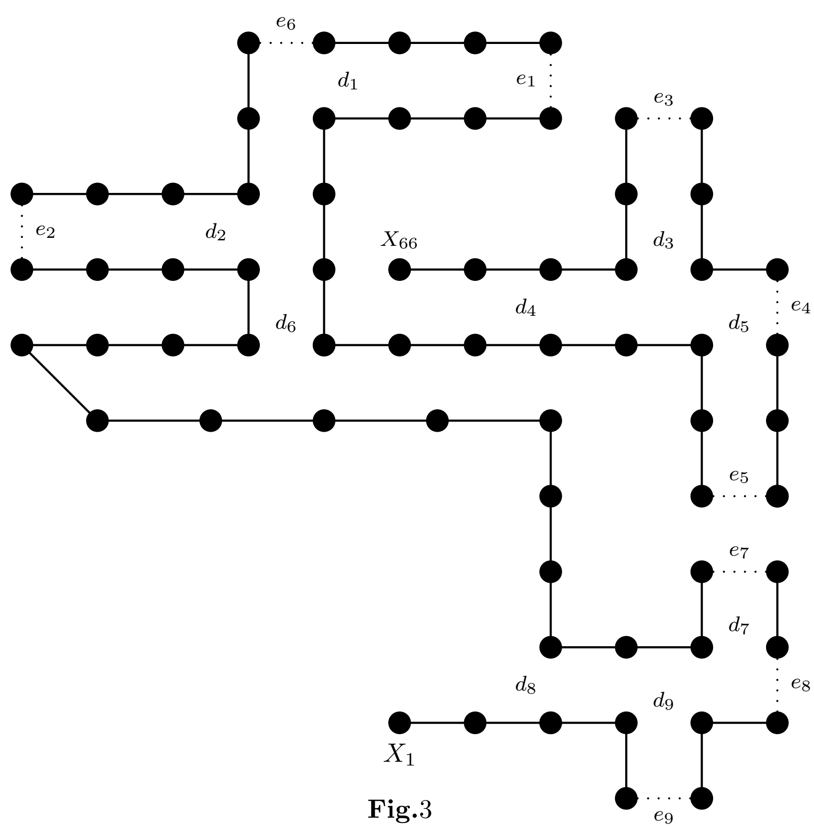


Fig.3

Theorem 2.3 *Every T_n tree is odd graceful.*

Proof Let T be a T_n tree with $(n+1)$ vertices. By definition there exist a path $P^H(T)$ corresponding to T_n. Let $E_d = \{d_1, d_2, \cdots, d_r\}$ be the set of edges deleted from tree T and E_p is the set of edges newly added through the sequence $\{e_1, e_2, \cdots, e_r\}$ of the EPT's used to arrive at the path (Hamiltonian path) $P^H(T)$. Clearly E_d and E_p have the same number of edges. Now we have $V(P^H(T)) = V(T)$ and $E(P^H(T)) = \{E\{T\} - E_d\} \cup E_p$: Now denote the vertices of $P^H(T)$ successively as $v_1, v_2, \cdots, v_{n+1}$ starting from one pendant vertex of $P^H(T)$ right up to other. Define the vertex numbering of f from $V(P^H(T)) \to \{0, 1, 2, \cdots, 2q-1\}$ as follows.

$$\begin{aligned} f(v_i) &= 2\Big[\frac{i-1}{2}\Big] \text{ if } i \text{ is odd, } 1 \leqslant i \leqslant n+1 \\ &= (2q-1) - 2\Big[\frac{i-2}{2}\Big] \text{ if } i \text{ is even, } 2 \leqslant i \leqslant n+1 \end{aligned}$$

where, q is the number of edges of T and [.] denote the integer part.

Now it can be easily seen that f is injective. Let g_f^* be the induced mapping defined from the edge set of $P^H(T)$ in to the set $\{1, 3, 5, \cdots, 2q-1\}$ as follows:
$g_f^*(uv) = |f(u) - f(v)|$ whenever $uv \in E(PH(T))$.
Since $P^H(T)$ is a path, every edge in $P^H(T)$ is of the form v_iv_{i+1} for $i = 1, 2, \cdots, n$.

Case 1 When i is even, then

$$\begin{aligned} g_f^*(v_iv_{i+1}) &= |f(v_i) - f(v_{i+1})| \\ &= \left|(2q-1) - 2\left[\frac{i-2}{2}\right] - 2\left[\frac{i+1-1}{2}\right]\right| \\ &= \left|(2q-1) - 2\left\{\left[\frac{i-2}{2}\right] + \left[\frac{i}{2}\right]\right\}\right| \\ &= \left|(2q-1) - 2\left[\frac{i-2+i}{2}\right]\right| \\ &= \left|(2q-1) - 2\left[\frac{2i-2}{2}\right]\right| \\ &= \left|(2q-1) - 4\left[\frac{i-1}{2}\right]\right| \qquad (1) \end{aligned}$$

Case 2 When i is odd, then

$$\begin{aligned} g_f^*(v_iv_{i+1}) &= |f(v_i) - f(v_{i+1})| \\ &= \left[2\left[\frac{i-1}{2}\right] - \left((2q-1) - 2\left[\frac{i+1-2}{2}\right]\right)\right] \\ &= \left|2\left[\frac{i-1}{2}\right] - (2q-1) + 2\left[\frac{i-1}{2}\right]\right| \\ &= \left|(2q-1) - 4\left[\frac{i-1}{2}\right]\right| \qquad (2) \end{aligned}$$

From (1) and (2), we get for all i,

$$g_f^*(v_iv_{i+1}) = \left|(2q-1) - 4\left[\frac{i-1}{2}\right]\right| \qquad (3)$$

From (3), it is clear that g_f^* is injective and its range is $\{1, 3, 5, \cdots, 2q-1\}$. Then f is odd graceful on $P^H(T)$.

In order to prove that f is also odd graceful on T_n, it is enough to show that $g_f^*(d_s) = g_f^*(e_s)$. Let $d_s = v_iv_j$ be an edge of T for same indices i and j, $1 \leqslant i \leqslant n+1; 1 \leqslant j \leqslant n+1$ and d_s be deleted and e_s be the corresponding edge joined to obtain $P^H(T)$ at a distance k from u_i and u_j. Then $e_s = v_{i+k}v_{j-k}$. Since e_s is an edge in $P^H(T)$, it must be of the form $e_s = v_{i+k}v_{i+k+1}$.

We have $(v_{i+k}, v_{j-k}) = (v_{i+k}, v_{i+k+1}) \Longrightarrow j-k = i+k+1 \Longrightarrow j = i+2k+1$. Therefore i and j are of opposite parity $\Longrightarrow$ one of i, j is odd and other is even.

Case a When i is odd, $1 \leqslant i \leqslant n$. The value of the edge $e_s = v_iv_j$ is given by

$$\begin{aligned}
g_f^*(d_s) &= g_f^*(v_iv_j) \\
&= g_f^*(v_iv_{i+2k+1}) \\
&= |f(v_i) - f(v_{i+2k+1}| \qquad (4) \\
&= \left|(2q-1) - 2\left[\frac{i-2}{2}\right] - 2\left[\frac{i+2k+1-1}{2}\right]\right| \\
&= \left|(2q-1) - 2\left\{\left[\frac{i-2}{2}\right] + 2\left[\frac{i+2k}{2}\right]\right\}\right| \\
&= |(2q-1) - (2i+2k-2)| \\
&= |(2q-1) - 2(i+k-1)| \qquad (5)
\end{aligned}$$

Case b When i is even, $2 \leqslant i \leqslant n$.

$$\begin{aligned}
g_f^*(d_s) &= |f(v_i) - f(v_{i+2k+1})| \\
&= \left|2\left[\frac{i-2}{2}\right] - \left((2q-1) - 2\left[\frac{i+2k+1-2}{2}\right]\right)\right| \\
&= \left|2\left[\frac{i-2}{2}\right] + 2\left[\frac{i+2k-1}{2}\right] - (2q-1)\right| \\
&= |(2i+2k-2) - 2 - (2q-1)| \\
&= |(2q-1) - 2(i+k-1)| \qquad (6)
\end{aligned}$$

From (4), (5) and (6) it follows that

$$g_f^*(d_s) = g_f^*(v_iv_j) = |(2q-1) - 2(i+k-1)|, 1 \leqslant i \leqslant n \qquad (7)$$

Now again,

$$\begin{aligned}
g_f^*(e_s) &= g_f^*(v_{i+k}v_{j-k}) = g_f^*(v_kv_{i+k+1}) \\
&= |f(v_{i+k}) - f(v_{i+k+1})| \\
&= \left|(2q-1) - 2\left[\frac{i+k-2}{2}\right] - 2\left[\frac{i+k+1-1}{2}\right]\right. \\
&= |(2q-1) - (2i+2k-2)| \\
&= |(2q-1) - 2(i+k-1)|, 1 \leqslant i \leqslant n \qquad (8)
\end{aligned}$$

From (7) and (8), it follows that

$$g_f^*(e_s) = g_f^*(d_s).$$

Then f is odd graceful on T_n also. Hence the graph T_n-tree is odd graceful. The proof is complete. □

For example, an odd graceful labelling of a T_n-tree using 2.3, is shown in Fig.4.

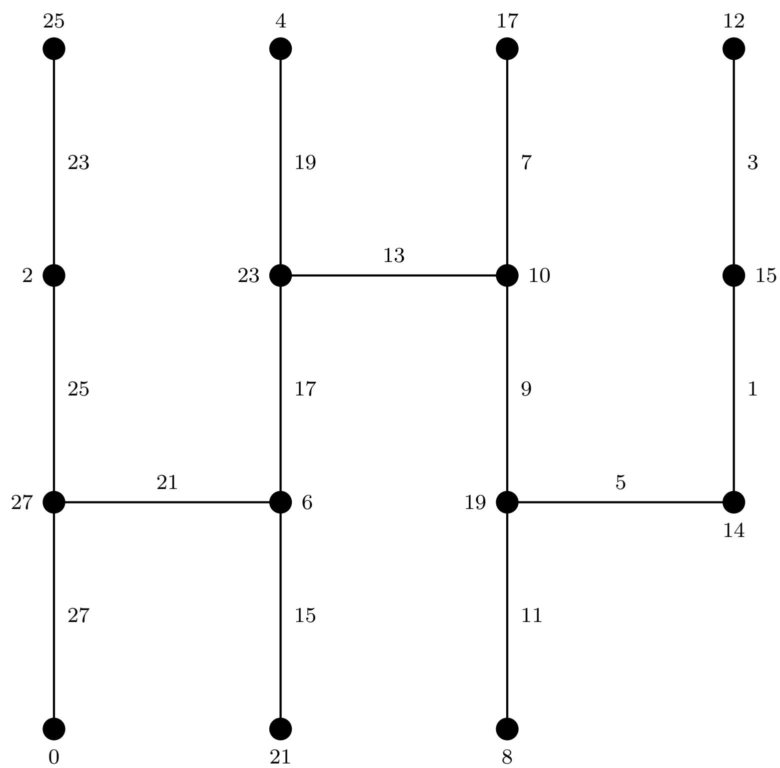


Fig.4

An odd graceful labeling of a T_n-tree using Theorem 2.3.

References

[1] F.Harary, *Graph Theory*, Addision Wesley, Reading, M.A., 1969.

[2] Joseph A. Gallian, A dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics*, 2012, pp 1-178.

[3] T.K.Mathew Varkey, Graceful labeling of a class of trees, *Proceedings of the National Seminar on Algebra and Discrete Mathematics*, Dept. of Mathematics, University of Kerala, Trivandrum, Kerala, November 2003, pp 156-159.

International J.Math. Combin. Vol.2(2013), 97-115

Graph Folding and Incidence Matrices

E.M.El-Kholy

(Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt)

El-Said R.Lashin and Salama N.Daoud

(Department of Mathematics, Faculty of Science, El-Minufiya University, Shebeen El-Kom, Egypt)

E-mail: salama_nagy2005@yahoo.com

Abstract: In this paper we described the graph folding of two given graphs under some known operations such as, intersection, union, join product, Cartesian product, tensor product, normal product and composition product by using incidence matrices.

Key Words: Incidence matrices, Cartesian product, tensor product, normal product, composition product, join product

AMS(2010): 57M10, 57M20

§1. Introduction

By a simple graph G, we mean that a graph with no loops or multiple edges. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple graphs. Then

(1) The simple graph $G = (V, E)$, where $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2$ is called the union of G_1 and G_2, and is denoted by $G_1 \bigcup G_2$ ([2]). When G_1 and G_2 are vertex disjoint, $G_1 \bigcup G_2$ is denoted by $G_1 + G_2$ and is called the sum of the graphs G_1 and G_2.

(2) If $V_1 \bigcap V_2 \neq \emptyset$, the graph $G = (V, E)$, where $V = V_1 \bigcap V_2$ and $E = E_1 \bigcap E_2$ is called the intersection of G_1, G_2 and is written as $G_1 \bigcap G_2$ ([2]).

(3) If G_1 and G_2 are vertex-disjoint graphs. Then the join, $G_1 \vee G_2$ is the supergraph of $G_1 + G_2$, in which each vertex of G_1 is adjacent to every vertex of G_2.

(4) The cartesian product $G_1 \times G_2$ is the simple graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and edge set $E(G_1 \times G_2) = (E_1 \times V_2) \bigcup (V_1 \times E_2)$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ iff either

(i) $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 , or

(ii) u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$ ([1]).

(5) The composition, or lexicographic product $G_1[G_2]$ is the simple graph with $V_1 \times V_2$ as the vertex set in which the vertices (u_1, u_2) , (v_1, v_2) are adjacent if either u_1 is adjacent to v_1

[1]Received September 25, 2011, Accepted June 24, 2013.

or $u_1 = v_1$ and u_2 is adjacent to v_2.

The graph $G_1[G_2]$ need not to be isomorphic to $G_2[G_1]$ ([2]).

(6) The normal product, or the strong product $G_1 \circ G_2$ is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \circ G_2$ iff either

(i) $u_1 = v_1$ and u_2 is adjacent to v_2 , or

(ii) u_1 is adjacent to v_1 and $u_2 = v_2$, or

(iii) u_1 is adjacent to v_1 and u_2 is adjacent to v_2 ([2]).

(7) The tensor product or Kronecher product $G_1 \otimes G_2$ is the simple graph with $V(G_1 \otimes G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ iff u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2.

Notice that $G_1 \circ G_2 = (G_1 \otimes G_2) \bigcup (G_1 \otimes G_2)$ ([2]).

§2. Graph Folding

Let G_1 and G_2 be graphs and $f : G_1 \to G_2$ be a continuous function. Then f is called a graph map, if

(i) for each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$;

(ii) for each edge $e \in E(G_1)$, $\dim(f(e)) \leq \dim(e)$.

A graph map $f : G_1 \to G_2$ is called a graph folding iff f maps vertices to vertices and edges to edges, i.e., for each $v \in V(G_1)$, $f(v) \in V(G_2)$ and for $e \in E(G_1)$, $f(e) \in E(G_2)$ ([3]).

The set of graph foldings between graphs G_1 and G_2 is denoted by $\mu(G_1, G_2)$ and the set of graph foldings of G_1 into itself by $\mu(G_1)$.

§3. Incidence Matrices

Let G be a finite graph with the set of vertices $V(G) = \{v_1, \cdots, v_{r_1}, v_{r_1+1}, \cdots, v_r\}$ and the set of edges $E(G) = \{e_{12}, \cdots, e_{1r_1}, \cdots, e_{1r}, e_{23}, \cdots, e_{2r}, \cdots, e_{(r-1)r}\}$.

The incidence matrix denoted by $I = (\lambda_{kd})$ is defined by $\lambda_{kd} = 1$ if v_k, $k = 1, \cdots, r_1, \cdots, r$ is a face of e_d, $d = 12, \cdots, 1r_1, \cdots, 1r, 23, \cdots, 2r, \cdots, r(r-1)$ in G, $\lambda_{kd} = 0$ if v_k is not a face of e_d in G. The matrix I has order $s \times r$, where s is the number of edges of G and r is the number of vertices in G.

Let G_1, G_2 be finite graphs and $f \in \mu(G_1, G_2)$. Then $f(G_1)$ is a subgraph of G_2. In particular, if $f \in \mu(G_1)$ with $f(G_1) = G'_1 \neq G_1$, then G'_1 is a subgraph of G_1. This suggests that the incidence matrix I' of $f(G_1) = G'_1$ is a submatrix of the incidence matrix I of G_1 possibly after rearranging its rows and columns.

We claim that the matrix I can be partition into four blocks, such that I' appears in the upper left corner block and a zero matrix O in the upper right one. The matrix R, the complement of I' will be a submatrix of I' possibly after deleting the rows and columns of I' which are not images of any of the edges $e_{1(r_1+1)}, \cdots, e_{1r}, e_{2(r_1+1)}, \cdots, e_{2r}, \cdots, e_{r_1 r}$ and the vertices $v_{r_1+1}, \cdots, v_r$, respectively.

The zero matrix O is due to the fact that non of the vertices $v_{r_1+1}, \cdots, v_r$ is incidence with any edge of the image.

$$I = \begin{array}{c} \\ \end{array}\begin{array}{cc} \overline{v}_1 & \overline{v}_2 \\ \left[\begin{array}{c|c} I' & O \\ \hline Q & R \end{array}\right] \end{array} \begin{array}{c} \overline{e}_1 \\ \overline{e}_2 \end{array}$$

where

$\overline{v}_1 = (v_1, v_2, \cdots, , v_{r_1})$,
$\overline{v}_2 = (v_{r_1+1}, \cdots, v_r)$,
$\overline{e}_1 = (e_{12}, \cdots, e_{1r_1}, e_{23}, \cdots, e_{2r_1}, \cdots, e_{(r_1-1)r_1})^T$ and
$\overline{e}_2 = (e_{1(r_1+1)}, \cdots, e_{1r}, e_{2(r_1+1)}, \cdots, e_{2r}, \cdots, e_{r_1(r_1+1)}, \cdots, e_{r_1 r})^T$.

Conversely, if the incidence matrix I of a graph G_1 can be partitioned into four blocks with a zero matrix at the right hand corner block. Then a graph folding may be defined, if there is any, as a map f of G_1 to an image $f(G_1)$ characterized by the incidence matrix I' which lie in the upper left corner of I. This map can be defined by mapping:

(i) the vertices $v_j,\ j = r_1 + 1, \cdots, r$ to the vertices $v_i, i = 1, \cdots, r_1$ if the j^{th} column in R is the same as the i^{th} column in I', after deleting the zero from i^{th} column;

(ii) the edges $e_k,\ k = 1(r_1 + 1), \cdots, r_1 r$ to the edges $e_l, l = 12, \cdots, (r_1 - 1)r_1$ if e_k and e_l are incidence.

Example 3.1 Let G be a graph whose $V(G) = \{v_1, v_2, \cdots, v_8\}$, $E(G) = \{e_{12}, e_{13}, e_{15}, e_{24}, e_{26}, e_{34}, e_{37}, e_{48}, e_{56}, e_{57}, e_{68}, e_{78}\}$ and $f: G \to G'$ be a graph folding, see Fig.1.

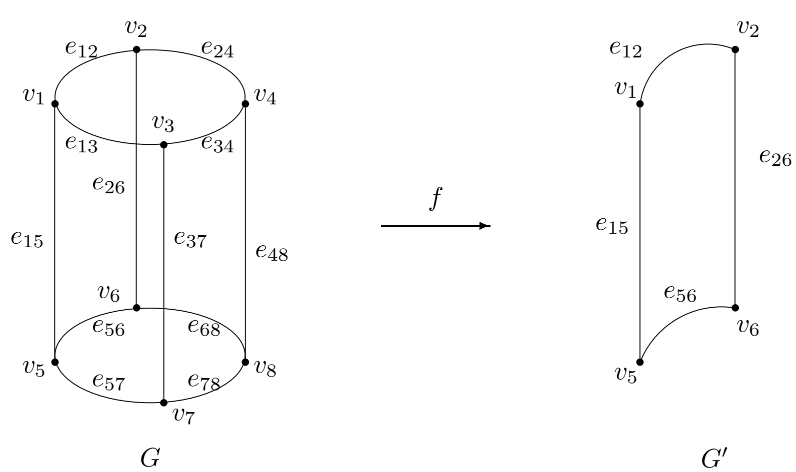


Fig.1

Then

$$I = \begin{array}{c} \\ \left[\begin{array}{cccc|cccc} v_1 & v_2 & v_5 & v_6 & v_3 & v_4 & v_7 & v_8 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ e_{12} \\ e_{13} \\ e_{15} \\ e_{24} \\ e_{26} \\ e_{34} \\ e_{37} \\ e_{48} \\ e_{56} \\ e_{57} \\ e_{68} \\ e_{78} \end{array}$$

§4. Incidence Matrices and Operations on Graph Foldings

Let G_1 and G_2 be finite graphs with $V_1 = V(G_1) = \{v_1, v_2, \cdots, v_{r_1}, v_{r_1+1}, \cdots, v_r\}$, $V_2 = V(G_2) = \{v_1, v_2, \cdots, v_{s_1}, v_{s_1+1}, \cdots, v_s\}$, $E_1 = E(G_1) = \{e_{12}, \cdots, e_{1r}, e_{23}, \cdots, e_{2r}, \cdots e_{r_1r}, \cdots, e_{(r-1)r}\}$ and $E_2 = E(G_2) = \{e_{12}, \cdots, e_{1s}, e_{23}, \cdots, e_{2s}, \cdots, e_{s_1s}, \cdots, e_{(s-1)s}\}$.

Let $f \in \mu(G_1)$ such that $f(G_1) = G_1' \neq G_1$ and $g \in \mu(G_2)$ such that $g(G_2) = G_2' \neq G_2$, and the incidence matrices $I(G_1) = (\lambda_{k_1d_1})$, where $k_1 = \{1, 2, \cdots, r_1, r_1+1, \cdots, r\}$, $d_1 = \{12, \cdots, 1r, 23, \cdots, 2r, \cdots, r_1r, \cdots, (r-1)r\}$ and $I(G_2) = (\lambda_{k_2d_2})$, where $k_2 = \{1, 2, \cdots, s_1, s_1+1, \cdots, s\}$ and $d_2 = \{12, \cdots, 1s, 23, \cdots, 2s, \cdots, s_1s, \cdots, (s-1)s\}$, respectively.

Then the graph maps $f \bigcup g :\ G_1 \bigcup G_2 \to G_1' \bigcup G_2'$ defined by

(i) $\quad \forall v \in V_1 \bigcup V_2$, $(f \bigcup g)(v) = \begin{cases} f(v) & \text{if } \ v \in V_1, \\ g(v) & \text{if } \ v \in V_2. \end{cases}$

(ii) $\ \forall e \in E_1 \bigcup E_2$, $(f \bigcup g)(e) = \begin{cases} f(e) & \text{if } \ e \in E_1, \\ g(e) & \text{if } \ e \in E_2 \end{cases}$

and $f \bigcap g :\ G_1 \bigcap G_2 \to G_1' \bigcap G_2'$ defined by:

(i) $\ \forall v \in V_1 \bigcap V_2$, $(f \bigcap g)(v) = f(v)$ or $g(v)$, where $V_1 \bigcap V_2 \neq \emptyset$.

(ii) $\forall e \in E_1 \bigcap E_2$, $(f \bigcap g)(e) = f(e)$ or $g(e)$,

are graph foldings iff f and g are graph foldings.

The incidence matrices $I(G_1 \bigcup G_2)$ and $I(G_1 \bigcap G_2)$ can be obtained from $I(G_1)$ and $I(G_2)$ as follows:

$I(G_1 \bigcup G_2) = (\lambda_{kd})$, where $k = \{1, 2, \cdots, r_1, r_1+1, \cdots, r\} \bigcup \{1, 2, \cdots, s_1, s_1+1, \cdots, s\}$ and $d = \{12, \cdots, 1r, 23, \cdots, 2r, \cdots, r_1r, \cdots, (r_1-1)r\} \bigcup \{12, \cdots, 1s, 23, \cdots, 2s, \cdots, s_1s, \cdots, (s_1 -$

$1)s\}$ such that

$\lambda_{kd}=1$ if $\lambda_{k_1d_l}=1$ in $I(G_1)$ or $\lambda_{k_2d_2}=1$ in $I(G_2)$ and $\lambda_{kd}=0$ if $\lambda_{k_1d_l}=0$ in $I(G_1)$ or $\lambda_{k_2d_2}=0$ in $I(G_2)$

and $I(G_1\bigcap G_2)=(\lambda_{kd})$, where $k=\{1,2,\cdots,r_1,r_1{+}1,\cdots,r\}\bigcap\{1,2,\cdots,s_1,s_1{+}1,\cdots,s\}$, $d=\{12,\cdots,1r,23,\cdots,2r,\cdots,r_1r,\cdots,(r_1-1)r\}\bigcap\{12,\cdots,1s,23,\cdots,2s,\cdots,s_1s,\cdots,(s_1-1)s\}$ such that

$\lambda_{kd}=1$ if $\lambda_{k_1d_l}=1$ in $I(G_1)$ and $\lambda_{k_2d_2}=1$ in $I(G_2)$ and $\lambda_{kd}=0$ if $\lambda_{k_1d_l}=0$ in $I(G_1)$ and $\lambda_{k_2d_2}=0$ in $I(G_2)$.

Example 4.1 Let G_1,G_2 be two graphs with $V(G_1)=\{v_1,v_2,v_3,v_4\}$, $E(G_1)=\{e_{12},e_{14},e_{23},e_{24},e_{34}\}$, $V(G_2)=\{v_1,v_2,v_3,v_5\}$ and $E(G_2)=\{v_1,v_2,v_3,v_5\}$ and let $f:G_1\to G_1'$, $g:G_2\to G_2'$ be graph foldings.

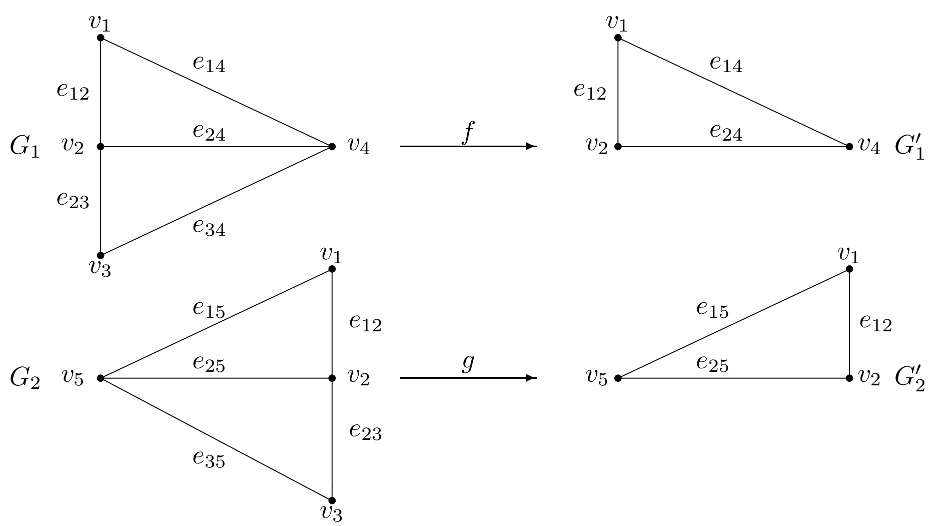


Fig.2

Then

$$I(G_1)=\begin{array}{c} \begin{matrix} v_1 & v_2 & v_4 & v_3 \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array} \begin{matrix} e_{12} \\ e_{14} \\ e_{24} \\ e_{23} \\ e_{34} \end{matrix} \qquad I(G_2)=\begin{array}{c} \begin{matrix} v_1 & v_2 & v_5 & v_3 \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array} \begin{matrix} e_{12} \\ e_{15} \\ e_{25} \\ e_{23} \\ e_{35} \end{matrix}$$

$$I(G_1\bigcup G_2)=\begin{array}{c}\begin{array}{ccccc}v_1&v_2&v_4&v_5&v_3\end{array}\\ \left[\begin{array}{cccc|c}1&1&0&0&0\\1&0&1&0&0\\1&0&0&1&0\\0&1&1&0&0\\0&1&0&1&0\\\hline 0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\end{array}\right]\end{array}\begin{array}{c}e_{12}\\e_{14}\\e_{15}\\e_{24}\\e_{25}\\e_{23}\\e_{34}\\e_{35}\end{array}\qquad I(G_1\bigcap G_2)=\begin{array}{c}\begin{array}{ccc}v_1&v_2&v_3\end{array}\\ \left[\begin{array}{cc|c}1&1&0\\\hline 0&1&1\end{array}\right]\end{array}\begin{array}{c}e_{12}\\e_{23}\end{array}$$

Let G_1, G_2 be finite graphs such that $V_1=V(G_1)=\{v_1,v_2,\cdots,v_{r_1},v_{r_1+1},\cdots,v_r\}$, $E_1=E(G_1)=\{e_{12},\cdots,e_{1r_1},\cdots,e_{1r},e_{23},\cdots,e_{2r_1},\cdots,e_{2r},\cdots,e_{r_1r},\cdots,e_{(r-1)r}\}$, $V_2=V(G_2)=\{v_{r+1},\cdots,v_{s_1},v_{s_1+1},\cdots,v_s\}$, $E_2=E(G_2)=\{e_{(r+1)(r+2)},\cdots,e_{(r+1)s_1},\cdots,e_{(r+1)s},e_{(r+2)(r+3)},\cdots,e_{(r+2)s_1},\cdots,e_{(r+2)s},\cdots,e_{s_1s},\cdots,e_{(s-1)s}\}$, where e_{ij} is the edge incidence with v_i and v_j, $e_{ij}=e_{ji}$. Let $f\in\mu(G_1)$ such that $f(G_1)=G_1'\neq G_1$ and $g\in\mu(G_2)$ such that $g(G_2)=G_2'\neq G_2$ with incidence matrices are $I(G_1)$ and $I(G_2)$ respectively. Then

(1) The join graph map $f\vee g: G_1\vee G_2\to G_1'\vee G_2'$ defined by

(i) $\forall v\in V_1\bigcup V_2$, $(f\vee g)\{v\}=\begin{cases} f\{v\} \text{ if } v\in V_1,\\ g\{v\} \text{ if } v\in V_2.\end{cases}$

(ii) $\forall e=(v_1,v_2), v_1\in V_1, v_2\in V_2$,

$$(f\vee g)\{e\}=(f\vee g)\{(v_1,v_2)\}=\{(f(v_1),f(v_2))\}\in G_1'\vee G_2'.$$

(iii) if $e=(u_1,v_1)\in E_1$, then $(f\vee g)\{e\}=(f\vee g)\{(u_1,v_1)\}=\{(f(u_1),g(v_1))\}$. Also, if $e=(u_2,v_2)\in E_2$, then $(f\vee g)\{e\}=(f\vee g)\{(u_2,v_2)\}=\{(g(u_2),g(v_2))\}$. Note that if $f\{u_1\}=f\{v_1\}$, then the image of the join graph map $(f\vee g)\{e\}$ will be a vertex of $G_1'\vee G_2'$, otherwise it will be an edge of $G_1'\vee G_2'$ ([4]), is a graph folding iff f and g are graph foldings.

The incident matrix $I(G_1\bigcup G_2)$ can be defined from $I(G_1)$ and $I(G_2)$ as follows:

$$I(G_1)=\begin{array}{c}\begin{array}{cc}\overline{v}_1&\overline{v}_1'\end{array}\\ \left[\begin{array}{c|c}I(G_1')&O\\\hline Q&R\end{array}\right]\end{array}\begin{array}{c}\overline{e}_1\\\overline{e}_1'\end{array}\qquad I(G_2)=\begin{array}{c}\begin{array}{cc}\overline{v}_2&\overline{v}_2'\end{array}\\ \left[\begin{array}{c|c}I(G_2')&O\\\hline Q&R\end{array}\right]\end{array}\begin{array}{c}\overline{e}_2\\\overline{e}_2'\end{array}$$

where, $\overline{v}_1=(v_1,v_2,\cdots,v_{r_1})$, $\overline{v}_1'=(v_{r_1+1},\cdots,v_r)$, $\overline{e}_1=\left(e_{12},\cdots,e_{1r_1},e_{23},\cdots,e_{2r_1},\cdots,e_{(r_1-1)r_1}\right)^T$, $\overline{e}_1'=\left(e_{1(r_1+1)},\cdots,e_{1r},e_{2(r_1+1)},\cdots,e_{2r},\cdots,e_{r_1(r_1+1)},\cdots,e_{r_1r}\right)^T$, $\overline{v}_2=\{v_{r+1},\cdots,v_{s_1}\}$, $\overline{v}_2'=$

$\{v_{s_1+1},\cdots,v_s\}$, $\overline{e}_1 = \big(e_{(r+1)(r+2)},\cdots,e_{(r+1)s_1},e_{(r+2)(r+3)},\cdots,e_{(r+2)s_1},\cdots,e_{(s_1-1)s_1}\big)^T$, $\overline{e}'_1 = \big(e_{(r+1)(s_1+1)},\cdots,e_{(r+1)s},e_{(r+2)(s_1+1)},\cdots,e_{(r+2)s},\cdots,e_{s_1(s_1+1)},\cdots,e_{s_1 s}\big)^T$.

Then

$$I(G_1\vee G_2) = \begin{array}{c|c} \overline{v}_\vee & \overline{v}'_\vee \\ \end{array} \left[\begin{array}{c|c} I(G'_1\vee G'_2) & O \\ \hline Q & R \end{array}\right] \begin{array}{c} \overline{e}_\vee \\ \overline{e}'_\vee \end{array}$$

where,

$\overline{v}_\vee = (v_1,v_2,\cdots,v_{r_1},v_{r_1+1},\cdots,v_{s_1})$, $\overline{v}'_\vee = \{v_{r_1+1},\cdots,v_r,v_{s_1+1},\cdots,v_s\}$,

$\overline{e}_\vee = (e_{12},\cdots,\ e_{1r_1},e_{23},\cdots,e_{2r_1},\cdots,e_{(r_1-1)r_1},e_{(r+1)(r+2)},\cdots,e_{(r+1)s_1},e_{(r+2)(r+3)},\cdots,$
$e_{(r+2)s_1},\cdots,\ e_{(s_1-1)s_1},e_{1(r+1)},\cdots,e_{1s_1},e_{2(r+1)},\cdots,e_{2s_1},\cdots,e_{r_1(r+1)},\cdots .e_{r_1 s_1}\big)^T$ and

$\overline{e}'_\vee = \big(e_{1(r_1+1)},\cdots,e_{1r},e_{2(r_1+1)},\cdots,e_{2r},\cdots,\ e_{r_1(r_1+1)},\cdots,e_{r_1 r},e_{(r+1)(s_1+1)},\cdots,e_{(r+1)s},$
$e_{(r+2)(s_1+1)},\ \cdots,e_{(r+2)s},\cdots,e_{s_1(s_1+1)},\cdots,e_{s_1 s},\ e_{1(s_1+1)},\cdots,e_{1s},e_{2(s_1+1)},\cdots,e_{2s},$
$\cdots,e_{r_1(s_1+1)},\cdots,e_{r_1 s},\ e_{(r_1+1)(r+1)},\cdots,e_{(r_1+1)s},\cdots,e_{r(r+1)},\cdots,e_{rs_1},e_{(r_1+1)(s_1+1)},$
$\cdots,e_{(r_1+1)s},\cdots,e_{r(s_1+1)},\cdots,e_{rs}\big)^T$.

Thus, $I(G_1\vee G_2) = (\lambda_{kd})$, where $k = 1,2,\cdots,r_1,r_1+1,\cdots,r,r+1,\cdots,s_1,s_1+1,\cdots,s$, $d = ij$, $i\neq j$ and $i,j = 1,2,\cdots,r_1,r_1+1,\cdots,r,r+1,\cdots,s_1,s_1+1,\cdots,s$. It is clear that if $I(G_1)$ has order $m_1\times n_1$ and $I(G_2)$ has order $m_2\times n_2$, then $I(G_1\vee G_2)$ has order $(m_1+m_2+n_1n_2)(n_1+n_2)$.

(2) The cartesian product graph map $f\times g:\ G_1\times G_2\to G'_1\times G'_2$ defined by

(i) if $v=(v_1,v_2)\in V_1\times V_2, v_1\in V_1, v_2\in V_2$, then $(f\times g)\{(v_1,v_2)\} = \{(f\{v_1\},g\{v_2\})\}\in G'_1\times G'_2$.

(ii) if $e=\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_i,\{v_2\}_k)\}, \{v_1\}_i\in V(G_1)$ and $\{v_2\}_j,\{v_2\}_k\in V(G_2)$, then

$$f\times g\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_i,\{v_2\}_k)\} = \{(\{v_1\}_i,g\{v_2\}_j),(\{v_1\}_i,g\{v_2\}_k)\}.$$

But if $e=\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_k,\{v_2\}_j)\}$, where $\{v_1\}_i,\{v_1\}_k\in V(G_1)$, $\{v_2\}_j\in V(G_2)$, then

$$(f\times g)\{(\{v_1\}_i,\{v_2\}_j),(\{v_1\}_k,\{v_2\}_j)\} = \{(f\{v_1\}_i,\{v_2\}_j),(f\{v_1\}_k,\{v_2\}_j)\}.$$

Note that if $g\{v_2\}_j = g\{v_2\}_k$, or $f\{v_1\}_i = f\{v_1\}_k$, the image of the edge e will be a vertex ([4]), is a graph folding iff f and g are graph foldings. The incidence matrix $I(G_1\times G_2)$ can be defined from $I(G_1)$ and $I(G_2)$ as follows:

$$I(G_1 \times G_2) = \begin{array}{c} \begin{array}{cc} \overline{v}_\times & \overline{v}'_\times \end{array} \\ \left[\begin{array}{c|c} I(G'_1 \vee G'_2) & O \\ \hline Q & R \end{array}\right] \end{array} \begin{array}{c} \overline{e}_\times \\ \overline{e}'_\times \end{array}$$

where,

$\overline{v}_\times = ((v_1, v_{r+1}), \cdots, (v_1, v_{s_1}), \cdots, (v_{r_1}, v_{r+1}), \cdots, (v_{r_1}, v_{s_1}))$,

$\overline{v}'_\times = ((v_1, v_{s_1+1}), \cdots, (v_1, v_s), \cdots, (v_{r_1}, v_s), (v_{r_1+1}, v_{r+1}), \cdots, (v_{r_1+1}, v_{s_1}), \cdots, (v_r, v_{s_1}))$,

$\overline{e}_\times = \big((e_{12}, v_{r+1}), \cdots, (e_{12}, v_{s_1}), \cdots, (e_{1r_1}, v_{r+1}), \cdots, (e_{1r_1}, v_{s_1}), \cdots (e_{(r_1-1)r_1}, v_{r+1}), \cdots,$
$(e_{(r_1-1)r_1}, v_{s_1}), (v_1, e_{(r+1)(r+2)}), \cdots, (v_{r_1}, e_{(r+1)(r+2)}), \cdots, (v_1, e_{(r+1)s_1}), \cdots,$
$(v_{r_1}, e_{(r+1)s_1}), \cdots, (v_1, e_{(s_1-1)s_1}), \cdots, (v_{r_1}, e_{(s_1-1)s_1})\big)^T$ and

$\overline{e}'_\times = (\ (e_{1(r_1+1)}, v_{r+1}), \cdots, (e_{1(r_1+1)}, v_{s_1}), \cdots, (e_{1r}, v_{r+1}), \cdots, (e_{1r}, v_{s_1}), \cdots, e_{r_1 r}, v_{r+1}),$
$\cdots, (e_{r_1 r}, v_{s_1}), (v_1, e_{(r+1)(s_1+1)}), \cdots, (v_{r_1}, e_{(r+1)(s_1+1)}), \cdots, (v_1, e_{(r+1)s}), \cdots,$
$(e_{(r+1)s}, v_{r_1}), \cdots, (v_1, e_{s_1 s}), \cdots, (v_{r_1}, e_{s_1 s}), (e_{12}, v_{s_1+1}), \cdots, (e_{12}, v_s), \cdots, (e_{1r}, v_{s_1+1}),$
$\cdots, (e_{1r}, v_s), \cdots, (e_{(r_1-1)r_1}, v_{s_1+1}), (e_{(r_1-1)r_1}, v_s), (v_{r_1+1}, e_{(r+1)(r+2)}), \cdots,$
$(v_r, e_{(r+1)(r+2)}), \cdots, (v_{r_1+1}, e_{(r+1)s_1}), \cdots, (v_r, e_{(r+1)s_1}), \cdots, (v_{r+1}, e_{(s_1-1)s_1}), \cdots,$
$(v_r, e_{(s_1-1)s_1}), (e_{1(r_1+1)}, v_{s_1+1}), \cdots, (e_{1(r_1+1)}, v_{s+1}), \cdots, (e_{1r}, v_{s_1+1}), \cdots, (e_{1r}, v_s),$
$\cdots, (e_{r_1 r}, v_{s_1+1}), \cdots, (e_{r_1 r}, v_s), (v_{r_1+1}, e_{(r+1)(s_1+1)}), \cdots, (v_{r_1+1}, e_{(r+1)s_1}), \cdots,$
$(v_{r_1+1}, e_{s_1 s}), \cdots, (v_r, e_{(r+1)(s_1+1)}), \cdots, (v_r, e_{(r+1)s}), \cdots, (v_r, e_{s_1 s})\big)^T$.

It is clear that if $I(G_1)$ has order $m_1 \times n_1$ and $I(G_2)$ has order $m_2 \times n_2$, then $I(G_1 \times G_2)$ has order $(m_1 n_2 + m_2 n_1) \times (n_1 n_2)$.

(3) The tensor product graph map $f \otimes g : G_1 \otimes G_2 \to G'_1 \otimes G'_2$ defined by:

(i) if $v = (v_1, v_2) \in V(G_1 \otimes G_2) = V_1 \times V_2$, then $(f \otimes g)\{(v_1, v_2)\} = \{(f\{v_1\}, g\{v_2\})\} \in V(G'_1 \otimes G'_2)$;

(ii) let $e = \{(\{v_1\}_i, \{v_2\}_j), (\{v_1\}_k, \{v_2\}_l)\}$, where $\{v_1\}_i$ is adjacent to $\{v_2\}_k$ and $\{v_2\}_j$ is adjacent to $\{v_2\}_l$, then $(f \otimes g)\{e\} = f\{(\{v_1\}_i, \{v_1\}_k)\} \otimes g\{(\{v_2\}_j, \{v_2\}_l$, i.e., $(f \otimes g)(G_1 \otimes G_2) = f(G_1) \times g(G_2)$ ([4]) is a graph folding is a graph folding iff f and g are graph foldings.

The incidence matrix $I(G_1 \otimes G_2)$ can be defined from $I(G_1)$ and $I(G_2)$ as follows:

$$I(G_1 \otimes G_2) = \begin{array}{c} \begin{array}{cc} \overline{v}_\otimes & \overline{v}'_\otimes \end{array} \\ \left[\begin{array}{c|c} I(G'_1 \otimes G'_2) & O \\ \hline Q & R \end{array}\right] \end{array} \begin{array}{c} \overline{e}_\otimes \\ \overline{e}'_\otimes \end{array}$$

where,

$$\overline{v}_{\otimes} = ((v_1, v_{r+1}), \cdots, (v_1, v_{s_1}), \cdots, (v_{r_1}, v_{r+1}), \cdots, (v_{r_1}, v_{s_1})),$$
$$\overline{v}'_{\otimes} = ((v_1, v_{s_1+1}), \cdots, (v_1, v_s), \cdots, (v_{r_1}, v_s), (v_{r_1+1}, v_{r+1}), \cdots, (v_{r_1+1}, v_{s_1}), \cdots, (v_r, v_{s_1})),$$
$$\overline{e}_{\otimes} = \big(e_{(1,r+1)(2,r+2)}, e_{1,r+2)(2,r+1)}, \cdots, e_{(1,s_1-1)(2,s_1)}, e_{(1,s_1)(2,s_1-1)}, \cdots, e_{(r_1-1,r+1)(r_1,r+2)}, e_{(r_1-1,r+2)(r_1,r+1)}, \cdots, e_{(r_1-1,s_1-1)(r_1,s_1)}, e_{(r_1-1,s_1)(r_1,s_1-1)}\big)^T \text{ and}$$
$$\overline{e}'_{\otimes} = \big(e_{(1,r+1)(2,s_1+1)}, e_{(1,s_1+1)(2,r+1)}, \cdots, e_{(1,s_1)(2,s)}, e_{(1,s)(2,s_1)}, \cdots, e_{(r_1-1,r+1)(r_1,s_1+1)}, e_{(r_1-1,s_1+1)(r_1,r+1)}, \cdots, e_{(r_1-1,s_1)(r_1,s)}, e_{(r_1-1,s)(r_1,s_1)}, e_{(1,r+1)(r_1+1,r+2)}, e_{(1,r+2)(r_1+1,r+1)}, \cdots, e_{(1,s_1-1)(r_1+1,s_1)}, e_{(1,s_1)(r_1+1,s_1-1)} \cdots, e_{(r_1,r+1)(r,r+2)}, e_{(r_1,r+2)(r,r+1)}, \cdots, e_{(r_1,s_1-1)(r,s_1)}, e_{(r_1,s_1)(r,s_1-1)}, e_{(1,r+1)(r_1+1,s_1+1)}, e_{(1,s_1+1)(r_1+1,r+1)}, \cdots, e_{(1,s_1)(r_1+1,s)}, e_{(1,s)(r_1+1,s_1)}, \cdots, e_{(r_1,r+1)(r,s_1+1)}, e_{(r_1,s_1+1)(r,r+1)}, \cdots, e_{(r_1,s_1)(r,s)}, e_{(r_1,s)(r,s_1)}\big)^T.$$

It is clear that if $I(G_1)$ has order $m_1 \times n_1$ and $I(G_2)$ has order $m_2 \times n_2$, then $I(G_1 \otimes G_2)$ has order $(2m_1m_2) \times (n_1n_2)$.

(4) The normal product graph map $f \circ g : G_1 \circ G_2 \to G'_1 \circ G_2$ defined by

(i) for any vertex $v = (v_1, v_2) \in V(G_1 \circ G_2) = V_1 \times G_2$, then

$$(f \circ g)\{(v_1, v_2)\} = \{(f\{v_1\}, g\{v_2\})\} \in V(G'_1 \circ G'_2);$$

(ii) for any edge $e = \{(\{v_1\}_i, \{v_2\}_j), (\{v_1\}_k, \{v_2\}_l)\}$, then

$$\begin{aligned}(f \circ g)\{e\} &= (f \circ g)\{(\{v_1\}_i, \{v_2\}_j), (\{v_1\}_k, \{v_2\}_l)\}\\ &= f\{(\{v_1\}_i, g\{v_2\}_j), (\{v_1\}_k, g\{v_2\}_l)\}\\ &= \{(f\{v_1\}_i, g\{v_2\}_j), (f\{v_1\}_k, g\{v_2\}_l)\}.\end{aligned}$$

Note that if $f\{v_1\}_i = f\{v_1\}_k$ and $g\{v_2\}_j = g\{v_2\}_l$, then $(f \circ g)\{e\}$ will be a vertex ([4]) is a graph folding is a graph folding if f and g are graph foldings.

The incidence matrix $I(G_1 \circ G_2)$ can be obtained from $I(G_1 \times G_2)$ and $I(G_1 \otimes G_2)$, since $G_1 \circ G_2 = (G_1 \times G_2) \bigcup (G_1 \otimes G_2)$.

It is clear that if $I(G_1)$ has order $m_1 \times n_1$ and $I(G_2)$ has order $m_2 \times n_2$, then $I_{f \circ g}$ has order $(m_1n_2 + m_2n_1 + 2m_1m_2) \times (n_1n_2)$.

(5) The composition product graph map $f[g] :\ G_1[G_2] \to G'_1[G'_2]$ defined by:

(i) if $v = (v_1, v_2) \in V(G_1[G_2]) = V1 \times V_2$, then $f[g]\{(v_1, v_2)\} = \{(f\{v_1\}, g\{v_2\})\} \in (G'_1[G'_2])$;

(ii) let $e = \{(\{v_1\}_i, \{v_2\}_j), (\{v_1\}_k, \{v_2\}_l)\}$. If $\{v_1\}_i$ is adjacent to $\{v_1\}_k$, then $f[g]\{e\} = \{(\{v_1\}_i, g\{v_2\}_j), (f\{v_1\}_k, g\{v_2\}_l)\}$. If $\{v_1\}_i = \{v_1\}_k$ and $\{v_2\}_j$ is adjacent to $\{v_2\}_l$, then $f[g]\{e\} = \{(\{v_1\}_i, g\{v_2\}_j), (f\{v_1\}_i, g\{v_2\}_l)\}$.

Note that if $f\{v_1\}_i = f\{v_1\}_k$ and $g\{v_2\}_j = g\{v_2\}_l$, then $f[g]\{e\}$ will be a vertex, also if $g\{v_2\}_j = g\{v_2\}_l$, then $f[g]$ will be a vertex ([4]) is a graph folding is a graph folding if f, g are graph foldings and the incidence matrix $I(G_1[G_2])$ can be obtained from $I(G_1)$ and $I(G_2)$ as follows:

$$I(G_1[G_2]) = \begin{array}{c} \begin{array}{cc} \overline{v}_{\square} & \overline{v}'_{\square} \end{array} \\ \left[\begin{array}{c|c} I(G'_1[G'_2]) & O \\ \hline Q & R \end{array}\right] \begin{array}{c} \overline{e}_{\square} \\ \overline{e}'_{\square} \end{array} \end{array}$$

where,

$$\overline{v}_{\square} = ((v_1, v_{r+1}), \cdots, (v_1, v_{s_1}), \cdots, (v_{r_1}, v_{r+1}), \cdots, (v_{r_1}, v_{s_1})),$$
$$\overline{v}'_{\square} = ((v_1, v_{s_1+1}), \cdots, (v_1, v_s), \cdots, (v_{r_1}, v_s), (v_{r_1+1}, v_{r+1}), \cdots, (v_{r_1+1}, v_{s_1}), \cdots, (v_r, v_{s_1})),$$
$$\begin{aligned}\overline{e}_{\square} = \big(&(v_1, e_{(r+1)(r+2)}, \cdots, (v_1, e_{(s_1-1)s_1}), \cdots, (v_{r_1}, e_{(r+1)(r+2)}), \cdots, (v_{r_1}, e_{(s_1-1)s_1}),\\ &e_{(1,r+1)(2,r+1)}, \cdots, e_{(r_1-1,r+1)(r_1,r+1)}, e_{(1,r+1)(2,r+2))}, e_{(1,r+2)(2,r+1)}, \cdots,\\ &e_{(r_1-1,r+1)(r_1,r+2)}, e_{(r_1-1,r+2)(r_1,r+1)}, \cdots, e_{(1,r+1)(2,s_1)}, e_{(1,s_1)(2,r+1)}, \cdots,\\ &e_{(r_1-1,r+1)(r_1,s_1)}, e_{(r_1-1,s_1)(r_1,r+1)}, e_{(1,r+2)(2,r+2)}, \cdots, e_{(r_1-1,r+2)(r_1,r+2)},\\ &\cdots, e_{(1,r+2)(2,s_1)}, e_{(1,s_1)(2,r+2)}, \cdots, e_{(r_1-1,r+2)(r_1,s_1)}, e_{(r_1-1,s_1)(r_1,r+2)}, \cdots,\\ &e_{(1,s_1-1)(2,s_1)}, e_{(1,s_1)(2,s_1-1)}, \cdots, e_{(r_1-1,s_1-1)(r_1,s_1)}, e_{(r_1-1,s_1)(r_1,s_1-1)}, e_{(1,s_1)(2,s_1)},\\ &\cdots, e_{(r_1-1,s_1)(r_1,s_1)}\big)^T \text{ and}\end{aligned}$$
$$\begin{aligned}\overline{e}'_{\square} = \big(&(v_1, e_{(r+1)(s_1+1)}), \cdots, (v_1, e_{s_1 s}), \cdots, (v_{r_1}, e_{(r+1)(s_1+1)}), \cdots, (v_{r_1}, e_{s_1 s}),\\ &(v_{r_1+1}, e_{(r+1)(r+2)}), \cdots, (v_{r_1+1}, e_{(r+1)(r+2)}), \cdots, (v_{r_1+1}, e_{(s_1-1)s_1}),\\ &(v_{r_1+1}, e_{(r+1)(s_1+1)}), \cdots, (v_{r_1+1}, e_{s_1 s}), \cdots, (v_r, e_{(r+1)(r+2)}), \cdots, (v_r, e_{(s_1-1)s_1}),\\ &(v_r, e_{(r+1)(s_1+1)}), \cdots, (v_r, e_{s_1 s}), e_{(1,r+1)(r_1+1,r+1)}, \cdots, e_{(r_1,r+1)(r,r+1)},\\ &e_{(1,r+1)(r_1+1,r+2)}, e_{(1,r+2)(r_1+1,r+1)}, \cdots, e_{(r_1,r+1)(r,r+2)}, e_{(r_1,r+2)(r,r+1)}, \cdots,\\ &e_{(1,r+1)(r_1+1,s_1)}, e_{(1,s_1)(r_1+1,r+1)}, \cdots, e_{(r_1,r+1)(r,s_1)}, e_{(r_1,s_1)(r,r+1)}, \cdots,\\ &e_{(1,r+1)(r_1+1,s)}, e_{(1,s)(r_1+1,r+1)}, \cdots, e_{(r_1,r+1)(r,s)}, e_{(r_1,s)(r,r+1)}, e_{(1,r+2)(r_1+1,r+2)}, \cdots,\\ &e_{(r_1,r+2)(r,r+2)}, \cdots, e_{(1,r+2)(r_1+1,s)}, e_{(1,s)(r_1+1,r+2)}, \cdots, e_{(r_1,r+2)(r,s)}, e_{(r_1,s)(r,r+2)},\\ &\cdots, e_{(1,s_1)(r_1+1,s_1)}, \cdots, e_{(r_1,s_1)(r,s_1)}, \cdots, e_{(1,s_1)(r,s_1)}, e_{(1,s)(r_1+1,s_1)}, \cdots, e_{(r_1,s_1)(r,s)},\\ &e_{(r_1,s)(r,s_1)}, \cdots, e_{(1,s)(r_1+1,s)}, \cdots, e_{(r_1,s)(r,s)}, e_{(1,r+1)(2,s_1+1)}, e_{(1,s_1+1)(2,r+1)}, \cdots,\\ &e_{(r_1-1,r+1)(r_1,s_1+1)}, e_{(r_1-1,s_1+1)(r_1,r+1)}, \cdots, e_{(1,r+1)(2,s)}, e_{(1,s)(2,r+1)}, \cdots,\\ &e_{(r_1-1,r+1)(r_1,s)}, e_{(r_1-1,s)(r_1,r+1)}, \cdots, e_{(1,s_1)(2,s)}, e_{(1,s)(2,s_1)}, \cdots, e_{(r_1-1,s_1)(r_1,s)},\\ &e_{(r_1-1,s)(r_1,s_1)}, e_{(1,s_1+1)(2,s_1+1)}, \cdots, e_{(r_1-1,s_1+1)(r_1,s_1+1)}, \cdots, e_{(1,s_1+1)(2,s)},\\ &e_{(1,s)(2,s_1+1)}, \cdots, e_{(r_1-1,s_1+1)(r_1,s)}, e_{(r_1-1,s)(r_1,s_1+1)}, \cdots, e_{(1,s)(2,s)}, \cdots, e_{(r_1-1,s)(r_1,s)}\big)^T\end{aligned}$$

It is clear that if $I(G_1)$ has order $m_1 \times n_1$ and $I(G_2)$ has order $m_2 \times n_2$, then $I(G_1[G_2])$ has order $(n_1 m_2 + n_2 m_1) \times n_1 n_2$.

Example 4.2 Let G_1 and G_2 be two graphs such that $V(G_1) = \{v_1, v_2, v_3, v_4\}$, $E(G_1) = \{e_{12}, e_{13}, e_{14}, e_{23}, e_{34}\}$, $V(G_2) = \{v_5, v_6, v_7\}$, $E(G_2) = \{e_{56}, e_{57}\}$, and $f : G_1 \to G'_1$, $g : G_2 \to G'_2$ be graph foldings, see Fig.3.

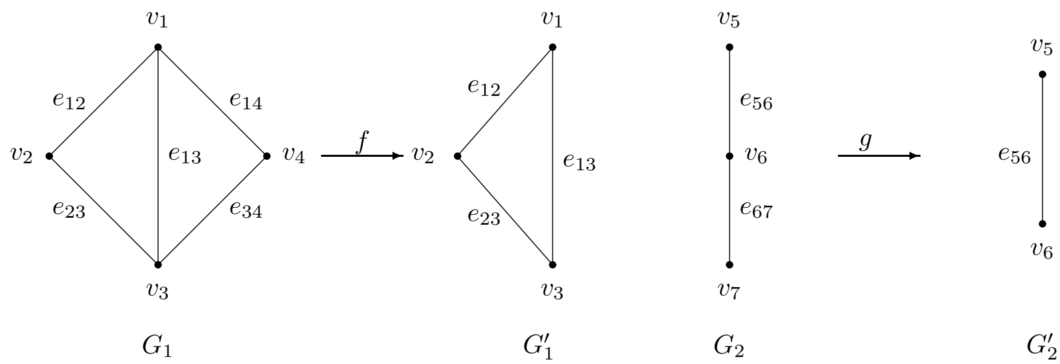


Fig.3

Their incidence matrixes are shown in the following.

$$I(G_1) = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ e_{12} \\ e_{13} \\ e_{23} \\ e_{14} \\ e_{34} \end{array} \qquad I(G_2) = \begin{array}{c} \begin{array}{ccc} v_5 & v_6 & v_7 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ \hline 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ e_{56} \\ e_{67} \end{array}$$

Then we know that $f \vee g$ is a graph folding, see Fig.4.

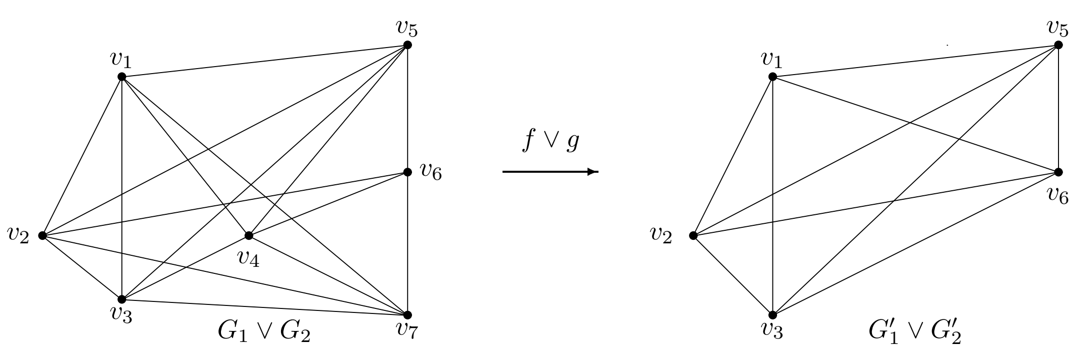


Fig.4

$$
I(G_1 \vee G_2) = \begin{array}{c} \begin{array}{ccccc|cc} v_1 & v_2 & v_3 & v_5 & v_6 & v_4 & v_7 \end{array} \\ \left[\begin{array}{ccccc|cc}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right] \end{array}
\begin{array}{c} \\ e_{12} \\ e_{13} \\ e_{23} \\ e_{56} \\ e_{15} \\ e_{16} \\ e_{25} \\ e_{26} \\ e_{35} \\ e_{36} \\ e_{14} \\ e_{34} \\ e_{67} \\ e_{17} \\ e_{27} \\ e_{37} \\ e_{45} \\ e_{46} \\ e_{47} \end{array}
$$

We also know that $f \times g$ is a graph folding, seeing Fig.5,

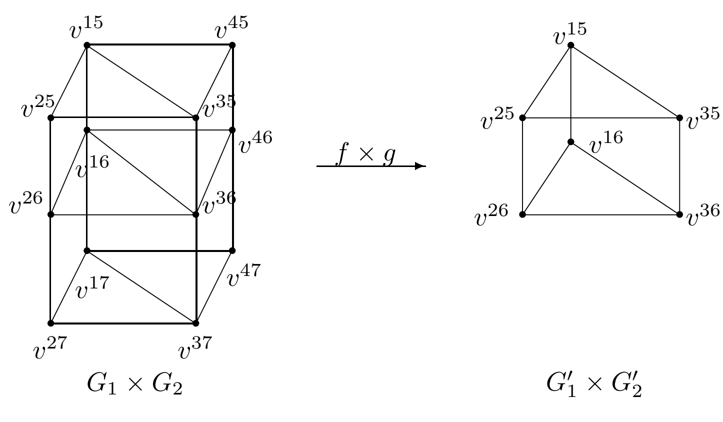


Fig.5

where $v^{15} = (v_1, v_5), v^{16} = (v_1, v_6), v^{25} = (v_2, v_5)$, $v^{16} = (v_2, v_6), v^{35} = (v_3, v_5), v^{36} = (v_3, v_6), v^{17} = (v_1, v_7), v^{27} = (v_2, v_7), v^{37} = (v_3, v_7), v^{45} = (v_4, v_5), v^{46} = (v_4, v_6), v^{47} = (v_4, v_7)$.

$$I(G_1 \times G_2) = \begin{array}{c} \begin{array}{cccccccccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} & v^{17} & v^{27} & v^{37} & v^{45} & v^{46} & v^{47} \end{array} \\ \left[\begin{array}{cccccc|cccccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ (e_{12}, v_5) \\ (e_{12}, v_6) \\ (e_{13}, v_5) \\ (e_{13}, v_6) \\ (e_{23}, v_5) \\ (e_{23}, v_6) \\ (v_1, e_{56}) \\ (v_2, e_{56}) \\ (v_3, e_{56}) \\ (e_{14}, v_5) \\ (e_{14}, v_6) \\ (e_{34}, v_5) \\ (e_{34}, v_6) \\ (v_1, e_{67}) \\ (v_2, e_{67}) \\ (v_3, e_{67}) \\ (e_{12}, v_7) \\ (e_{13}, v_7), \\ (e_{23}, v_7) \\ (v_4, e_{56}) \\ (e_{14}, v_7) \\ (e_{34}, v_7) \\ (v_4, e_{67}) \end{array}$$

Similarly, we know that $f \otimes g$, $f \circ g$ and $f[g]$ are also graph foldings, seeing Fig 6- Fig.8, where $v^{ij} = (v_i, v_j)$ for integers $1 \leq i, j \leq 7$.

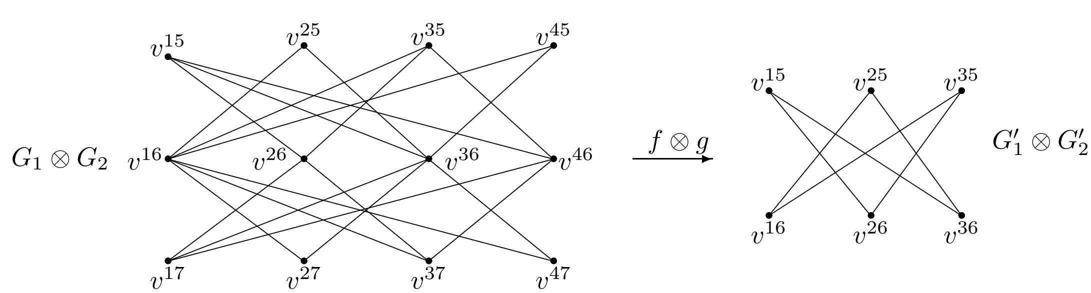
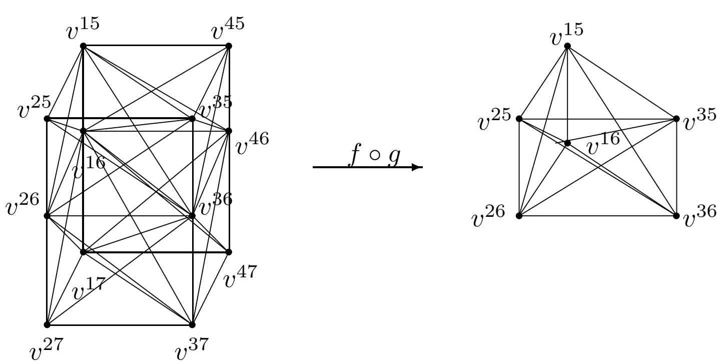


Fig.6

$$I(G_1 \otimes G_2) = \begin{array}{c} \begin{array}{cccccccccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} & v^{17} & v^{27} & v^{37} & v^{45} & v^{46} & v^{47} \end{array} \\ \left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ e_{(1,5)(2,6)} \\ e_{(1,6)(2,5)} \\ e_{(1,5)(3,6)} \\ e_{(1,6)(3,5)} \\ e_{(2,5)(3,6)} \\ e_{(2,6)(3,5)} \\ e_{(1,6)(3,7)} \\ e_{(1,7)(2,6)} \\ e_{(1,6)(3,7)} \\ e_{(1,7)(3,6)} \\ e_{(2,6)(3,7)} \\ e_{(2,7)(3,6)} \\ e_{(1,5)(4,6)} \\ e_{(1,6)(4,5)} \\ e_{(3,5)(4,6)} \\ e_{(3,6)(4,5)} \\ e_{(1,6)(4,7)} \\ e_{(1,7)(4,6)} \\ e_{(3,6)(4,7)} \\ e_{(3,7)(4,6)} \end{array}$$



$G_1 \circ G_2$ $\qquad$ $G_1' \circ G_2'$

Fig.7

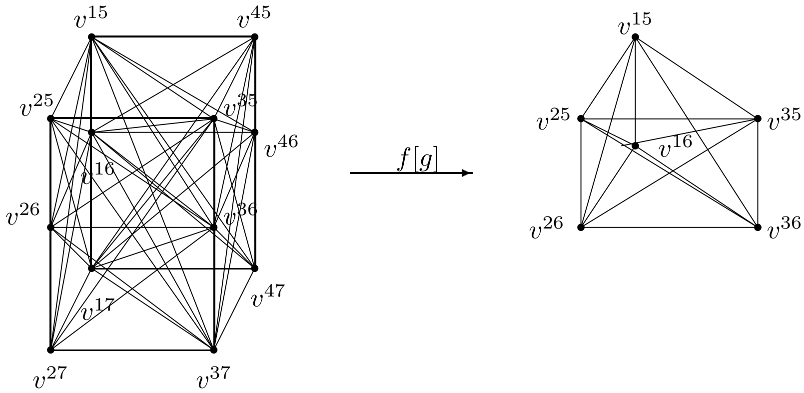
$$I(G_1 \circ G_2) = \left[\begin{array}{c|c} I(G_1' \circ G_2') & O \\ \hline Q & R \end{array}\right]$$

where,

$$I(G_1' \circ G_2') = \begin{array}{c} \begin{array}{cccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ (e_{12}, v_5) \\ (e_{12}, v_6) \\ (e_{13}, v_5) \\ (e_{13}, v_6) \\ (e_{23}, v_5) \\ (e_{23}, v_6) \\ (v_1, e_{56}) \\ (v_2, e_{56}) \\ (v_3, e_{56}) \\ e_{(1,5)(2,6)} \\ e_{(1,6)(2,5)} \\ e_{(1,5)(3,6)} \\ e_{(1,6)(3,5)} \\ e_{(2,5)(3,6)} \\ e_{(2,6)(3,5)} \end{array}$$

$$O = (0)_{15\times 6}$$

$$
Q = \begin{array}{c}
\begin{array}{cccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{array} \\
\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c}
\\
(e_{14}, v_5) \\
(e_{14}, v_6) \\
(e_{34}, v_5) \\
(e_{34}, v_6) \\
(v_1, e_{67}) \\
(v_2, e_{67}) \\
(v_3, e_{67}) \\
(e_{12}, v_7) \\
(e_{13}, v_7), \\
(e_{23}, v_7) \\
(v_4, e_{56}) \\
(e_{14}, v_7) \\
(e_{34}, v_7) \\
(v_4, e_{67}) \\
e_{(1,6)(2,7)} \\
e_{(1,7)(2,6)} \\
e_{(1,6)(3,7)} \\
e_{(1,7)(3,6)} \\
e_{(2,6)(3,7)} \\
e_{(2,7)(3,6)} \\
e_{(1,5)(4,6)} \\
e_{(1,6)(4,5)} \\
e_{(3,5)(4,6)} \\
e_{(3,6)(4,5)} \\
e_{(1,6)(4,7)} \\
e_{(1,7)(4,6)} \\
e_{(3,6)(4,7)} \\
e_{(3,7)(4,6)}
\end{array}
\qquad
R = \begin{array}{c}
\begin{array}{cccccc} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{array} \\
\left[\begin{array}{cccccc}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{array}\right]
\end{array}
\begin{array}{c}
\\
(e_{14}, v_5) \\
(e_{14}, v_6) \\
(e_{34}, v_5) \\
(e_{34}, v_6) \\
(v_1, e_{67}) \\
(v_2, e_{67}) \\
(v_3, e_{67}) \\
(e_{12}, v_7) \\
(e_{13}, v_7), \\
(e_{23}, v_7) \\
(v_4, e_{56}) \\
(e_{14}, v_7) \\
(e_{34}, v_7) \\
(v_4, e_{67}) \\
e_{(1,6)(2,7)} \\
e_{(1,7)(2,6)} \\
e_{(1,6)(3,7)} \\
e_{(1,7)(3,6)} \\
e_{(2,6)(3,7)} \\
e_{(2,7)(3,6)} \\
e_{(1,5)(4,6)} \\
e_{(1,6)(4,5)} \\
e_{(3,5)(4,6)} \\
e_{(3,6)(4,5)} \\
e_{(1,6)(4,7)} \\
e_{(1,7)(4,6)} \\
e_{(3,6)(4,7)} \\
e_{(3,7)(4,6)}
\end{array}
$$



$G_1[G_2]$ $\qquad$ $G_1'[G_2']$

Fig.8

$$I(G_1[G_2]) = \left[\begin{array}{c|c} I(G_1'[G_2']) & O \\ \hline Q & R \end{array}\right], \quad I(G_1'[G_2']) = \begin{array}{c} \begin{matrix} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}, \quad \begin{array}{c} \\ (v_1, e_{56}) \\ (v_2, e_{56}) \\ (v_3, e_{56}) \\ e_{(1,5)(2,5)} \\ e_{(1,5)(3,5)} \\ e_{(2,5)(3,5)} \\ e_{(1,5)(2,6)} \\ e_{(1,6)(2,5)} \\ e_{(1,5)(3,6)} \\ e_{(1,6)(3,5)} \\ e_{(2,5)(3,6)} \\ e_{(3,5)(2,6)} \\ e_{(1,6)(2,6)} \\ e_{(1,6)(3,6)} \\ e_{(2,6)(3,6)} \end{array}$$

$O = (0)_{15\times 6}$ and

$$Q = \begin{array}{c} \\ \end{array}\begin{array}{cccccc|c} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} & \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & (v_1, e_{67}) \\ 0 & 0 & 0 & 1 & 0 & 0 & (v_2, e_{67}) \\ 0 & 0 & 0 & 0 & 0 & 1 & (v_3, e_{67}) \\ 0 & 0 & 0 & 0 & 0 & 0 & (v_4, e_{56}) \\ 0 & 0 & 0 & 0 & 0 & 0 & (v_4, e_{67}) \\ 1 & 0 & 0 & 0 & 0 & 0 & e_{(1,5)(4,5)} \\ 0 & 0 & 0 & 0 & 1 & 0 & e_{(3,5)(4,5)} \\ 1 & 0 & 0 & 0 & 0 & 0 & e_{(1,5)(4,6)} \\ 0 & 0 & 0 & 0 & 1 & 0 & e_{(1,6)(4,5)} \\ 1 & 0 & 0 & 0 & 0 & 0 & e_{(3,5)(4,6)} \\ 0 & 0 & 0 & 0 & 1 & 0 & e_{(3,6)(4,5)} \\ 0 & 1 & 0 & 0 & 0 & 0 & e_{(1,5)(4,7)} \\ 0 & 0 & 0 & 0 & 0 & 1 & e_{(1,7)(4,5)} \\ 0 & 1 & 0 & 0 & 0 & 0 & e_{(3,5)(4,7)} \\ 0 & 0 & 0 & 0 & 0 & 1 & e_{(3,7)(4,5)} \\ 0 & 1 & 0 & 0 & 0 & 0 & e_{(1,6)(4,6)} \\ 0 & 0 & 0 & 0 & 0 & 1 & e_{(3,6)(4,6)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(1,6)(4,7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(1,7)(4,6)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(3,6)(4,7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(3,7)(4,6)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(1,7)(4,7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(3,7)(4,7)} \\ 0 & 0 & 1 & 0 & 0 & 0 & e_{(1,5)(2,7)} \\ 0 & 0 & 0 & 0 & 1 & 0 & e_{(1,7)(2,5)} \\ 0 & 0 & 0 & 0 & 1 & 0 & e_{(1,5)(3,7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(1,7)(3,5)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(2,5)(3,7)} \\ 0 & 0 & 0 & 1 & 0 & 0 & e_{(2,7)(3,7)} \\ 0 & 0 & 0 & 0 & 0 & 1 & e_{(2,7)(3,5)} \\ 0 & 0 & 0 & 0 & 0 & 1 & e_{(1,6)(2,7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(1,7)(2,6)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(1,6)(3,7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(1,7)(3,6)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(2,6)(3,7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(1,7)(2,7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(1,7)(3,7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & e_{(2,7)(3,7)} \end{array}$$

$$R = \begin{array}{cccccc|c} v^{15} & v^{16} & v^{25} & v^{26} & v^{35} & v^{36} & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & (v_1, e_{67}) \\ 0 & 1 & 0 & 0 & 0 & 0 & (v_2, e_{67}) \\ 0 & 0 & 1 & 0 & 0 & 0 & (v_3, e_{67}) \\ 0 & 0 & 0 & 1 & 1 & 0 & (v_4, e_{56}) \\ 0 & 0 & 0 & 0 & 1 & 1 & (v_4, e_{67}) \\ 0 & 0 & 0 & 1 & 0 & 0 & e_{(1,5)(4,5)} \\ 0 & 0 & 0 & 1 & 0 & 0 & e_{(3,5)(4,5)} \\ 0 & 0 & 0 & 0 & 1 & 0 & e_{(1,5)(4,6)} \\ 0 & 0 & 0 & 0 & 1 & 0 & e_{(1,6)(4,5)} \\ 0 & 0 & 0 & 0 & 0 & 1 & e_{(3,5)(4,6)} \\ 0 & 0 & 0 & 0 & 0 & 1 & e_{(3,6)(4,5)} \\ 0 & 0 & 0 & 1 & 0 & 0 & e_{(1,5)(4,7)} \\ 0 & 0 & 0 & 1 & 0 & 0 & e_{(1,7)(4,5)} \\ 0 & 0 & 0 & 0 & 1 & 0 & e_{(3,5)(4,7)} \\ 0 & 0 & 0 & 0 & 1 & 0 & e_{(3,7)(4,5)} \\ 0 & 0 & 0 & 0 & 0 & 1 & e_{(1,6)(4,6)} \\ 0 & 0 & 0 & 0 & 0 & 1 & e_{(3,6)(4,6)} \\ 1 & 0 & 0 & 1 & 0 & 0 & e_{(1,6)(4,7)} \\ 0 & 0 & 1 & 1 & 0 & 0 & e_{(1,7)(4,6)} \\ 1 & 0 & 0 & 0 & 1 & 0 & e_{(3,6)(4,7)} \\ 0 & 0 & 1 & 0 & 1 & 0 & e_{(3,7)(4,6)} \\ 1 & 0 & 0 & 0 & 0 & 1 & e_{(1,7)(4,7)} \\ 0 & 0 & 1 & 0 & 0 & 1 & e_{(3,7)(4,7)} \\ 1 & 0 & 0 & 0 & 0 & 0 & e_{(1,5)(2,7)} \\ 1 & 0 & 0 & 0 & 0 & 0 & e_{(1,7)(2,5)} \\ 0 & 1 & 0 & 0 & 0 & 0 & e_{(1,5)(3,7)} \\ 1 & 0 & 0 & 1 & 0 & 0 & e_{(1,7)(3,5)} \\ 0 & 0 & 1 & 1 & 0 & 0 & e_{(2,5)(3,7)} \\ 1 & 0 & 0 & 0 & 0 & 0 & e_{(2,7)(3,7)} \\ 1 & 0 & 0 & 0 & 0 & 0 & e_{(2,7)(3,5)} \\ 0 & 1 & 0 & 0 & 0 & 0 & e_{(1,6)(2,7)} \\ 1 & 0 & 0 & 0 & 1 & 0 & e_{(1,7)(2,6)} \\ 0 & 0 & 1 & 0 & 1 & 0 & e_{(1,6)(3,7)} \\ 1 & 1 & 0 & 0 & 0 & 0 & e_{(1,7)(3,6)} \\ 1 & 0 & 1 & 0 & 0 & 0 & e_{(2,6)(3,7)} \\ 0 & 1 & 1 & 0 & 0 & 0 & e_{(1,7)(2,7)} \\ 1 & 0 & 0 & 0 & 0 & 1 & e_{(1,7)(3,7)} \\ 0 & 0 & 1 & 0 & 0 & 1 & e_{(2,7)(3,7)} \end{array}$$

References

[1] E.M.El-Kholy and A.El-Esawy, Graph folding of some special graphs, *J. Math. Statistics* Nal., 1(2005), pp. 66-70, Sci. pub. U.S.A.

[2] E.M.El-Kholy, R.Shain and A.El-Esawy, Operation on graphs and graph folding, *Int. J. Applied Math.*, Bulgria, (inpress).

[3] J.L.Gross and T.W.Tucker, *Topological Graph Theory*, John Wiely & Sons, Inc. New York, U.S.A., 1987.

[4] R.Balakrishnan and K.Ranganathan, *Textbook of Graph Theory*, Springer-Verlag, Inst. New York, 2000.

International J.Math. Combin. Vol.2(2013), 116-117

The First International Conference on Smarandache Multispace and Multistructure was held in China

In recent decades, Smarandache's notions of multispace and multistructure were widely spread and have shown much importance in sciences around the world. Organized by Prof. Linfan Mao, a professional conference on multispaces and multistructures, named the First International Conference on Smarandache Multispace and Multistructure was held in Beijing University of Civil Engineering and Architecture of P. R. China on June 28-30, 2013, which was announced by American Mathematical Society in advance.

The Smarandache multispace and multistructure are qualitative notions, but both can be applied to metric and non-metric systems. There were 46 researchers haven taken part in this conference with 14 papers on Smarandache multispaces and geometry, birings, neutrosophy, neutrosophic groups, regular maps and topological graphs with applications to non-solvable equation systems.

Prof. Yanpei Liu reports on topological graphs

Prof.Linfan Mao reports on non-solvable systems of equations

Prof.Shaofei Du reports on regular maps with developments

Applications of Smarandache multispaces and multistructures underline a combinatorial mathematical structure and interchangeability with other sciences, including gravitational fields, weak and strong interactions, traffic network, etc.

All participants have showed a genuine interest on topics discussed in this conference and would like to carry these notions forward in their scientific works.

Progress is the activity of today and the assurance of tomorrow.

By Emerson, an American thinker.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats, viz., .tex, .dvi, .pdf, or.ps may be submitted electronically to one member of the Editorial Board for consideration in the **International Journal of Mathematical Combinatorics** (*ISSN 1937-1055*). An effort is made to publish a paper duly recommended by a referee within a period of 3 months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

June 2013

Contents

An International Journal on Mathematical Combinatorics

ISSN 1937-1055