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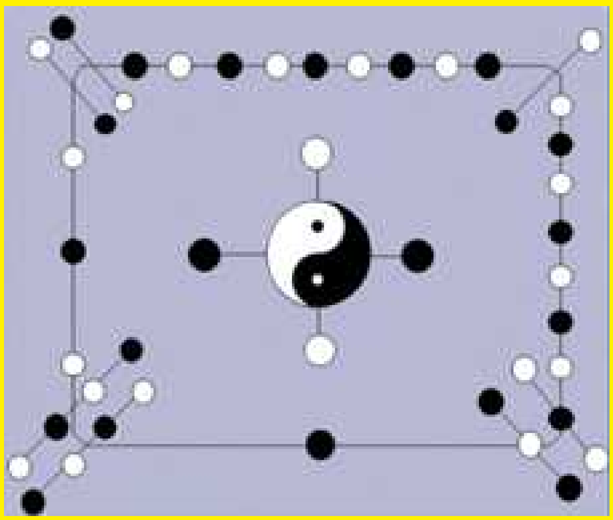

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Mathematics, rightly viewed, posses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture.

By Bertrand Russell, an England philosopher and mathematician.

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Global Stability of Non-Solvable Ordinary Differential Equations With Applications

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Abstract: Different from the system in classical mathematics, a Smarandache system is a contradictory system in which an axiom behaves in at least two different ways within the same system, i.e., validated and invalided, or only invalided but in multiple distinct ways. Such systems exist extensively in the world, particularly, in our daily life. In this paper, we discuss such a kind of Smarandache system, i.e., non-solvable ordinary differential equation systems by a combinatorial approach, classify these systems and characterize their behaviors, particularly, the global stability, such as those of sum-stability and prod-stability of such linear and non-linear differential equations. Some applications of such systems to other sciences, such as those of globally controlling of infectious diseases, establishing dynamical equations of instable structure, particularly, the n-body problem and understanding global stability of matters with multilateral properties can be also found.

Key Words: Global stability, non-solvable ordinary differential equation, general solution, G-solution, sum-stability, prod-stability, asymptotic behavior, Smarandache system, inherit graph, instable structure, dynamical equation, multilateral matter.

AMS(2010): 05C15, 34A30, 34A34, 37C75, 70F10, 92B05

§1. Introduction

Finding the exact solution of an equation system is a main but a difficult objective unless some special cases in classical mathematics. Contrary to this fact, *what is about the non-solvable case for an equation system?* In fact, such an equation system is nothing but a contradictory system, and characterized only by having no solution as a conclusion. But our world is overlap and hybrid. The number of non-solvable equations is much more than that of the solvable and such equation systems can be also applied for characterizing the behavior of things, which reflect the real appearances of things by that their complexity in our world. It should be noted that such non-solvable linear algebraic equation systems have been characterized recently by the author in the reference [7]. The main purpose of this paper is to characterize the behavior of such non-solvable ordinary differential equation systems.

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Assume $m,\ n \geq 1$ to be integers in this paper. Let

$$\dot{X} = F(X) \qquad (DES^1)$$

be an autonomous differential equation with $F : \mathbf{R}^n \to \mathbf{R}^n$ and $F(\overline{0}) = 0$, particularly, let

$$\dot{X} = AX \qquad (LDES^1)$$

be a linear differential equation system and

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0 \qquad (LDE^n)$$

a linear differential equation of order n with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix} \quad \text{and} \quad F(t,X) = \begin{bmatrix} f_1(t,X) \\ f_2(t,X) \\ \cdots \\ f_n(t,X) \end{bmatrix},$$

where all $a_i,\ a_{ij},\ 1 \leq i,j \leq n$ are real numbers with

$$\dot{X} = (\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n)^T$$

and $f_i(t)$ is a continuous function on an interval $[a,b]$ for integers $0 \leq i \leq n$. The following result is well-known for the solutions of $(LDES^1)$ and (LDE^n) in references.

Theorem 1.1([13]) *If $F(X)$ is continuous in*

$$U(X_0):\ |t - t_0| \leq a, \quad \|X - X_0\| \leq b \ \ (a > 0,\ b > 0)$$

then there exists a solution $X(t)$ of differential equation (DES^1) in the interval $|t - t_0| \leq h$, where $h = \min\{a, b/M\}$, $M = \max\limits_{(t,X)\in U(t_0,X_0)} \|F(t,X)\|$.

Theorem 1.2([13]) *Let λ_i be the k_i-fold zero of the characteristic equation*

$$\det(A - \lambda I_{n\times n}) = |A - \lambda I_{n\times n}| = 0$$

or the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

with $k_1 + k_2 + \cdots + k_s = n$. Then the general solution of $(LDES^1)$ is

$$\sum_{i=1}^{n} c_i \overline{\beta}_i(t) e^{\alpha_i t},$$

where, c_i is a constant, $\overline{\beta}_i(t)$ is an n-dimensional vector consisting of polynomials in t determined as follows

$$\overline{\beta}_1(t) = \begin{bmatrix} t_{11} \\ t_{21} \\ \cdots \\ t_{n1} \end{bmatrix}$$

$$\overline{\beta}_2(t) = \begin{bmatrix} t_{11}t + t_{12} \\ t_{21}t + t_{22} \\ \cdots\cdots\cdots \\ t_{n1}t + t_{n2} \end{bmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\overline{\beta}_{k_1}(t) = \begin{bmatrix} \frac{t_{11}}{(k_1-1)!}t^{k_1-1} + \frac{t_{12}}{(k_1-2)!}t^{k_1-2} + \cdots + t_{1k_1} \\ \frac{t_{21}}{(k_1-1)!}t^{k_1-1} + \frac{t_{22}}{(k_1-2)!}t^{k_1-2} + \cdots + t_{2k_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{t_{n1}}{(k_1-1)!}t^{k_1-1} + \frac{t_{n2}}{(k_1-2)!}t^{k_1-2} + \cdots + t_{nk_1} \end{bmatrix}$$

$$\overline{\beta}_{k_1+1}(t) = \begin{bmatrix} t_{1(k_1+1)} \\ t_{2(k_1+1)} \\ \cdots\cdots \\ t_{n(k_1+1)} \end{bmatrix}$$

$$\overline{\beta}_{k_1+2}(t) = \begin{bmatrix} t_{11}t + t_{12} \\ t_{21}t + t_{22} \\ \cdots\cdots\cdots \\ t_{n1}t + t_{n2} \end{bmatrix}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\overline{\beta}_n(t) = \begin{bmatrix} \frac{t_{1(n-k_s+1)}}{(k_s-1)!}t^{k_s-1} + \frac{t_{1(n-k_s+2)}}{(k_s-2)!}t^{k_s-2} + \cdots + t_{1n} \\ \frac{t_{2(n-k_s+1)}}{(k_s-1)!}t^{k_s-1} + \frac{t_{2(n-k_s+2)}}{(k_s-2)!}t^{k_s-2} + \cdots + t_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{t_{n(n-k_s+1)}}{(k_s-1)!}t^{k_s-1} + \frac{t_{n(n-k_s+2)}}{(k_s-2)!}t^{k_s-2} + \cdots + t_{nn} \end{bmatrix}$$

with each t_{ij} a real number for $1 \leq i, j \leq n$ such that $\det([t_{ij}]_{n\times n}) \neq 0$,

$$\alpha_i = \begin{cases} \lambda_1, & if\ 1 \leq i \leq k_1; \\ \lambda_2, & if\ k_1 + 1 \leq i \leq k_2; \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots; \\ \lambda_s, & if\ k_1 + k_2 + \cdots + k_{s-1} + 1 \leq i \leq n. \end{cases}$$

The general solution of linear differential equation (LDE^n) *is*

$$\sum_{i=1}^{s}(c_{i1}t^{k_i-1}+c_{i2}t^{k_i-2}+\cdots+c_{i(k_i-1)}t+c_{ik_i})e^{\lambda_i t},$$

with constants $c_{ij},\ 1\leq i\leq s, 1\leq j\leq k_i$.

Such a vector family $\overline{\beta}_i(t)e^{\alpha_i t},\ 1\leq i\leq n$ of the differential equation system ($LDES^1$) and a family $t^l e^{\lambda_i t},\ 1\leq l\leq k_i, 1\leq i\leq s$ of the linear differential equation (LDE^n) are called the *solution basis*, denoted by

$$\mathscr{B}=\{\ \overline{\beta}_i(t)e^{\alpha_i t}\mid 1\leq i\leq n\ \}\ \text{ or }\ \mathscr{C}=\{\ t^l e^{\lambda_i t}\mid 1\leq i\leq s, 1\leq l\leq k_i\ \}.$$

We only consider autonomous differential systems in this paper. Theorem 1.2 implies that any linear differential equation system ($LDES^1$) of first order and any differential equation (LDE^n) of order n with real coefficients are solvable. Thus a linear differential equation system of first order is non-solvable only if the number of equations is more than that of variables, and a differential equation system of order $n\geq 2$ is non-solvable only if the number of equations is more than 2. Generally, such a contradictory system, i.e., a Smarandache system [4]-[6] is defined following.

Definition 1.3([4]-[6]) *A rule* $\mathcal{R}$ *in a mathematical system* $(\Sigma;\mathcal{R})$ *is said to be Smarandachely denied if it behaves in at least two different ways within the same set* Σ*, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma;\mathcal{R})$ *is a mathematical system which has at least one Smarandachely denied rule* $\mathcal{R}$.

Generally, let $(\Sigma_1;\mathcal{R}_1)$ $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be mathematical systems, where $\mathcal{R}_i$ is a rule on Σ_i for integers $1\leq i\leq m$. If for two integers $i,j,\ 1\leq i,j\leq m$, $\Sigma_i\neq\Sigma_j$ or $\Sigma_i=\Sigma_j$ but $\mathcal{R}_i\neq\mathcal{R}_j$, then they are said to be *different*, otherwise, *identical*. We also know the conception of Smarandache multi-space defined following.

Definition 1.4([4]-[6]) *Let* $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ *be* $m\geq 2$ *mathematical spaces, different two by two. A Smarandache multi-space* $\widetilde{\Sigma}$ *is a union* $\bigcup\limits_{i=1}^{m}\Sigma_i$ *with rules* $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$ *on* $\widetilde{\Sigma}$*, i.e., the rule* $\mathcal{R}_i$ *on* Σ_i *for integers* $1\leq i\leq m$*, denoted by* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.

A Smarandache multi-space $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ inherits a combinatorial structure, i.e., a vertex-edge labeled graph defined following.

Definition 1.5([4]-[6]) *Let* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *be a Smarandache multi-space with* $\widetilde{\Sigma}=\bigcup\limits_{i=1}^{m}\Sigma_i$ *and* $\widetilde{\mathcal{R}}=\bigcup\limits_{i=1}^{m}\mathcal{R}_i$*. Its underlying graph* $G\left[\widetilde{\Sigma},\widetilde{R}\right]$ *is a labeled simple graph defined by*

$$V\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right)=\{\Sigma_1,\Sigma_2,\cdots,\Sigma_m\},$$

$$E\left(G\left[\widetilde{\Sigma},\widetilde{R}\right]\right)=\{\ (\Sigma_i,\Sigma_j)\mid\Sigma_i\bigcap\Sigma_j\neq\emptyset, 1\leq i,j\leq m\}$$

with an edge labeling

$$l^E:\ (\Sigma_i,\Sigma_j)\in E\left(G\left[\widetilde{S},\widetilde{R}\right]\right)\to l^E(\Sigma_i,\Sigma_j)=\varpi\left(\Sigma_i\bigcap\Sigma_j\right),$$

where ϖ *is a characteristic on* $\Sigma_i\bigcap\Sigma_j$ *such that* $\Sigma_i\bigcap\Sigma_j$ *is isomorphic to* $\Sigma_k\bigcap\Sigma_l$ *if and only if* $\varpi(\Sigma_i\bigcap\Sigma_j)=\varpi\left(\Sigma_k\bigcap\Sigma_l\right)$ *for integers* $1\le i,j,k,l\le m$.

Now for integers $m,\ n\ge 1$, let

$$\dot{X}=F_1(X),\ \dot{X}=F_2(X),\cdots,\dot{X}=F_m(X) \qquad (DES_m^1)$$

be a differential equation system with continuous $F_i:\mathbf{R}^n\to\mathbf{R}^n$ such that $F_i(\overline{0})=\overline{0}$, particularly, let

$$\dot{X}=A_1X,\cdots,\dot{X}=A_kX,\cdots,\dot{X}=A_mX \qquad (LDES_m^1)$$

be a linear ordinary differential equation system of first order and

$$\begin{cases} x^{(n)}+a_{11}^{[0]}x^{(n-1)}+\cdots+a_{1n}^{[0]}x=0\\ x^{(n)}+a_{21}^{[0]}x^{(n-1)}+\cdots+a_{2n}^{[0]}x=0\\ \cdots\cdots\cdots\cdots\\ x^{(n)}+a_{m1}^{[0]}x^{(n-1)}+\cdots+a_{mn}^{[0]}x=0\end{cases} \qquad (LDE_m^n)$$

a linear differential equation system of order n with

$$A_k=\begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]}\\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1}^{[k]} & a_{n2}^{[k]} & \cdots & a_{nn}^{[k]}\end{bmatrix}\quad\text{and}\quad X=\begin{bmatrix} x_1(t)\\ x_2(t)\\ \cdots\\ x_n(t)\end{bmatrix}$$

where each $a_{ij}^{[k]}$ is a real number for integers $0\le k\le m,\ 1\le i,j\le n$.

Definition 1.6 *An ordinary differential equation system* (DES_m^1) *or* $(LDES_m^1)$ *(or* (LDE_m^n)*) are called non-solvable if there are no function* $X(t)$ *(or* $x(t)$*) hold with* (DES_m^1) *or* $(LDES_m^1)$ *(or* (LDE_m^n)*) unless the constants.*

The main purpose of this paper is to find contradictory ordinary differential equation systems, characterize the non-solvable spaces of such differential equation systems. For such objective, we are needed to extend the conception of solution of linear differential equations in classical mathematics following.

Definition 1.7 *Let* S_i^0 *be the solution basis of the* i*th equation in* (DES_m^1). *The* $\vee$*-solvable,* $\wedge$*-solvable and non-solvable spaces of differential equation system* (DES_m^1) *are respectively defined by*

$$\bigcup_{i=1}^{m}S_i^0,\ \ \bigcap_{i=1}^{m}S_i^0\ \ \textit{and}\ \ \bigcup_{i=1}^{m}S_i^0-\bigcap_{i=1}^{m}S_i^0,$$

where S_i^0 *is the solution space of the* i*th equation in* (DES_m^1).

According to Theorem 1.2, the general solution of the ith differential equation in $(LDES_m^1)$ or the ith differential equation system in (LDE_m^n) is a linear space spanned by the elements in the solution basis $\mathscr{B}_i$ or $\mathscr{C}_i$ for integers $1 \leq i \leq m$. Thus we can simplify the vertex-edge labeled graph $G\left[\widetilde{\sum}, \widetilde{R}\right]$ replaced each $\sum_i$ by the solution basis $\mathscr{B}_i$ (or $\mathscr{C}_i$) and $\sum_i \bigcap \sum_j$ by $\mathscr{B}_i \bigcap \mathscr{B}_j$ (or $\mathscr{C}_i \bigcap \mathscr{C}_j$) if $\mathscr{B}_i \bigcap \mathscr{B}_j \neq \emptyset$ (or $\mathscr{C}_i \bigcap \mathscr{C}_j \neq \emptyset$) for integers $1 \leq i, j \leq m$. Such a vertex-edge labeled graph is called the *basis graph* of $(LDES_m^1)$ $((LDE_m^n))$, denoted respectively by $G[LDES_m^1]$ or $G[LDE_m^n]$ and the underlying graph of $G[LDES_m^1]$ or $G[LDE_m^n]$, i.e., cleared away all labels on $G[LDES_m^1]$ or $G[LDE_m^n]$ are denoted by $\hat{G}[LDES_m^1]$ or $\hat{G}[LDE_m^n]$.

Notice that $\bigcap\limits_{i=1}^{m} S_i^0 = \bigcup\limits_{i=1}^{m} S_i^0$, i.e., the non-solvable space is empty only if $m = 1$ in $(LDEq)$. Thus $G[LDES^1] \simeq K_1$ or $G[LDE^n] \simeq K_1$ only if $m = 1$. But in general, the basis graph $G[LDES_m^1]$ or $G[LDE_m^n]$ is not trivial. For example, let $m = 4$ and $\mathscr{B}_1^0 = \{e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}\}$, $\mathscr{B}_2^0 = \{e^{\lambda_3 t}, e^{\lambda_4 t}, e^{\lambda_5 t}\}$, $\mathscr{B}_3^0 = \{e^{\lambda_1 t}, e^{\lambda_3 t}, e^{\lambda_5 t}\}$ and $\mathscr{B}_4^0 = \{e^{\lambda_4 t}, e^{\lambda_5 t}, e^{\lambda_6 t}\}$, where λ_i, $1 \leq i \leq 6$ are real numbers different two by two. Then its edge-labeled graph $G[LDES_m^1]$ or $G[LDE_m^n]$ is shown in Fig.1.1.

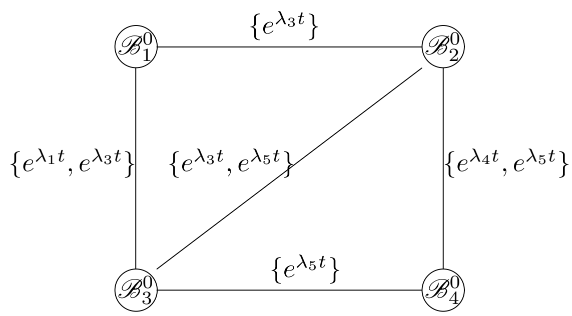


Fig.1.1

If some functions $F_i(X)$, $1 \leq i \leq m$ are non-linear in (DES_m^1), we can linearize these non-linear equations $\dot{X} = F_i(X)$ at the point $\overline{0}$, i.e., if

$$F_i(X) = F_i'(\overline{0})X + R_i(X),$$

where $F_i'(\overline{0})$ is an $n \times n$ matrix, we replace the ith equation $\dot{X} = F_i(X)$ by a linear differential equation

$$\dot{X} = F_i'(\overline{0})X$$

in (DES_m^1). Whence, we get a uniquely linear differential equation system $(LDES_m^1)$ from (DES_m^1) and its basis graph $G[LDES_m^1]$. Such a basis graph $G[LDES_m^1]$ of linearized differential equation system (DES_m^1) is defined to be the *linearized basis graph* of (DES_m^1) and denoted by $G[DES_m^1]$.

All of these notions will contribute to the characterizing of non-solvable differential equation systems. For terminologies and notations not mentioned here, we follow the [13] for differential equations, [2] for linear algebra, [3]-[6], [11]-[12] for graphs and Smarandache systems, and [1], [12] for mechanics.

§2. Non-Solvable Linear Ordinary Differential Equations

2.1 Characteristics of Non-Solvable Linear Ordinary Differential Equations

First, we know the following conclusion for non-solvable linear differential equation systems $(LDES_m^1)$ or (LDE_m^n).

Theorem 2.1 *The differential equation system* $(LDES_m^1)$ *is solvable if and only if*

$$(|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|) \neq 1$$

i.e., $(LDEq)$ *is non-solvable if and only if*

$$(|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|) = 1.$$

Similarly, the differential equation system (LDE_m^n) *is solvable if and only if*

$$(P_1(\lambda), P_2(\lambda), \cdots, P_m(\lambda)) \neq 1,$$

i.e., (LDE_m^n) *is non-solvable if and only if*

$$(P_1(\lambda), P_2(\lambda), \cdots, P_m(\lambda)) = 1,$$

where $P_i(\lambda) = \lambda^n + a_{i1}^{[0]}\lambda^{n-1} + \cdots + a_{i(n-1)}^{[0]}\lambda + a_{in}^{[0]}$ *for integers* $1 \leq i \leq m$.

Proof Let $\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in}$ be the n solutions of equation $|A_i - \lambda I_{n\times n}| = 0$ and $\mathscr{B}_i$ the solution basis of ith differential equation in $(LDES_m^1)$ or (LDE_m^n) for integers $1 \leq i \leq m$. Clearly, if $(LDES_m^1)$ $((LDE_m^n))$ is solvable, then

$$\bigcap_{i=1}^{m} \mathscr{B}_i \neq \emptyset, \quad \text{i.e.,} \quad \bigcap_{i=1}^{m} \{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in}\} \neq \emptyset$$

by Definition 1.5 and Theorem 1.2. Choose $\lambda_0 \in \bigcap_{i=1}^{m} \{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in}\}$. Then $(\lambda - \lambda_0)$ is a common divisor of these polynomials $|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|$. Thus

$$(|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|) \neq 1.$$

Conversely, if

$$(|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|) \neq 1,$$

let $(\lambda - \lambda_{01}), (\lambda - \lambda_{02}), \cdots, (\lambda - \lambda_{0l})$ be all the common divisors of polynomials $|A_1 - \lambda I_{n\times n}, |A_2 - \lambda I_{n\times n}|, \cdots, |A_m - \lambda I_{n\times n}|$, where $\lambda_{0i} \neq \lambda_{0j}$ if $i \neq j$ for $1 \leq i, j \leq l$. Then it is clear that

$$C_1 e^{\lambda_{01}} + C_2 e^{\lambda_{02}} + \cdots + C_l e^{\lambda_{0l}}$$

is a solution of $(LEDq)$ $((LDE_m^n))$ for constants $C_1, C_2, \cdots, C_l$. □

For discussing the non-solvable space of a linear differential equation system $(LEDS_m^1)$ or (LDE_m^n) in details, we introduce the following conception.

Definition 2.2 *For two integers $1 \leq i, j \leq m$, the differential equations*

$$\begin{cases} \frac{dX_i}{dt} = A_i X \\ \\ \frac{dX_j}{dt} = A_j X \end{cases} \qquad (LDES^1_{ij})$$

in $(LDES^1_m)$ or

$$\begin{cases} x^{(n)} + a^{[0]}_{i1} x^{(n-1)} + \cdots + a^{[0]}_{in} x = 0 \\ x^{(n)} + a^{[0]}_{j1} x^{(n-1)} + \cdots + a^{[0]}_{jn} x = 0 \end{cases} \qquad (LDE^n_{ij})$$

in (LDE^n_m) are parallel if $\mathscr{B}_i \bigcap \mathscr{B}_j = \emptyset$.

Then, the following conclusion is clear.

Theorem 2.3 *For two integers $1 \leq i, j \leq m$, two differential equations $(LDES^1_{ij})$ (or (LDE^n_{ij})) are parallel if and only if*

$$(|A_i| - \lambda I_{n\times n}, |A_j| - \lambda I_{n\times n}) = 1 \quad (or\ (P_i(\lambda), P_j(\lambda)) = 1),$$

where $(f(x), g(x))$ is the least common divisor of $f(x)$ and $g(x)$, $P_k(\lambda) = \lambda^n + a^{[0]}_{k1}\lambda^{n-1} + \cdots + a^{[0]}_{k(n-1)}\lambda + a^{[0]}_{kn}$ for $k = i, j$.

Proof By definition, two differential equations $(LEDS^1_{ij})$ in $(LDES^1_m)$ are parallel if and only if the characteristic equations

$$|A_i - \lambda I_{n\times n}| = 0 \quad \text{and} \quad |A_j - \lambda I_{n\times n}| = 0$$

have no same roots. Thus the polynomials $|A_i| - \lambda I_{n\times n}$ and $|A_j| - \lambda I_{n\times n}$ are coprime, which means that

$$(|A_i - \lambda I_{n\times n}, |A_j - \lambda I_{n\times n}) = 1.$$

Similarly, two differential equations (LED^n_{ij}) in (LDE^n_m) are parallel if and only if the characteristic equations $P_i(\lambda) = 0$ and $P_j(\lambda) = 0$ have no same roots, i.e., $(P_i(\lambda), P_j(\lambda)) = 1$.□

Let $f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$, $g(x) = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ with roots $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_n$, respectively. A *resultant* $R(f, g)$ of $f(x)$ and $g(x)$ is defined by

$$R(f, g) = a_0^m b_0^n \prod_{i,j} (x_i - y_j).$$

The following result is well-known in polynomial algebra.

Theorem 2.4 *Let $f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$, $g(x) = b_0x^n + b_1x^{n-1} + \cdots +$*

$b_{n-1}x + b_n$ *with roots* $x_1, x_2, \cdots, x_m$ *and* $y_1, y_2, \cdots, y_n$*, respectively. Define a matrix*

$$V(f,g) = \begin{bmatrix} a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & 0 & \cdots & 0 & 0 \\ 0 & b_0 & b_1 & \cdots & b_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & b_0 & b_1 & \cdots & b_n \end{bmatrix}$$

Then

$$R(f,g) = \det V(f,g).$$

We get the following result immediately by Theorem 2.3.

Corollary 2.5 (1) *For two integers* $1 \leq i,j \leq m$*, two differential equations* $(LDES^1_{ij})$ *are parallel in* $(LDES^1_m)$ *if and only if*

$$R(|A_i - \lambda I_{n\times n}|, |A_j - \lambda I_{n\times n}|) \neq 0,$$

particularly, the homogenous equations

$$V(|A_i - \lambda I_{n\times n}|, |A_j - \lambda I_{n\times n}|)X = 0$$

have only solution $(\underbrace{0,0,\cdots,0}_{2n})^T$ *if* $|A_i - \lambda I_{n\times n}| = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ *and* $|A_j - \lambda I_{n\times n}| = b_0\lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n$.

(2) *For two integers* $1 \leq i,j \leq m$*, two differential equations* (LDE^n_{ij}) *are parallel in* (LDE^n_m) *if and only if*

$$R(P_i(\lambda), P_j(\lambda)) \neq 0,$$

particularly, the homogenous equations $V(P_i(\lambda), P_j(\lambda))X = 0$ *have only solution* $(\underbrace{0,0,\cdots,0}_{2n})^T$.

Proof Clearly, $|A_i - \lambda I_{n\times n}|$ and $|A_j - \lambda I_{n\times n}|$ have no same roots if and only if

$$R(|A_i - \lambda I_{n\times n}|, |A_j - \lambda I_{n\times n}|) \neq 0,$$

which implies that the two differential equations $(LEDS^1_{ij})$ are parallel in $(LEDS^1_m)$ and the homogenous equations

$$V(|A_i - \lambda I_{n\times n}|, |A_j - \lambda I_{n\times n}|)X = 0$$

have only solution $(\underbrace{0,0,\cdots,0}_{2n})^T$. That is the conclusion (1). The proof for the conclusion (2) is similar. □

Applying Corollary 2.5, we can determine that an edge $(\mathscr{B}_i, \mathscr{B}_j)$ does not exist in $G[LDES_m^1]$ or $G[LDE_m^n]$ if and only if the ith differential equation is parallel with the jth differential equation in $(LDES_m^1)$ or (LDE_m^n). This fact enables one to know the following result on linear non-solvable differential equation systems.

Corollary 2.6 *A linear differential equation system $(LDES_m^1)$ or (LDE_m^n) is non-solvable if $\hat{G}(LDES_m^1) \not\simeq K_m$ or $\hat{G}(LDE_m^n) \not\simeq K_m$ for integers $m, n > 1$.*

2.2 A Combinatorial Classification of Linear Differential Equations

There is a natural relation between linear differential equations and basis graphs shown in the following result.

Theorem 2.7 *Every linear homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)) uniquely determines a basis graph $G[LDES_m^1]$ ($G[LDE_m^n]$) inherited in $(LDES_m^1)$ (or in (LDE_m^n)). Conversely, every basis graph G uniquely determines a homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)) such that $G[LDES_m^1] \simeq G$ (or $G[LDE_m^n] \simeq G$).*

Proof By Definition 1.4, every linear homogeneous differential equation system $(LDES_m^1)$ or (LDE_m^n) inherits a basis graph $G[LDES_m^1]$ or $G[LDE_m^n]$, which is uniquely determined by $(LDES_m^1)$ or (LDE_m^n).

Now let G be a basis graph. For $\forall v \in V(G)$, let the basis $\mathscr{B}_v$ at the vertex v be $\mathscr{B}_v = \{\ \overline{\beta}_i(t)e^{\alpha_i t} \mid 1 \leq i \leq n_v\}$ with

$$\alpha_i = \begin{cases} \lambda_1, & if\ 1 \leq i \leq k_1; \\ \lambda_2, & if\ k_1 + 1 \leq i \leq k_2; \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots ; \\ \lambda_s, & if\ k_1 + k_2 + \cdots + k_{s-1} + 1 \leq i \leq n_v \end{cases}$$

We construct a linear homogeneous differential equation $(LDES^1)$ associated at the vertex v. By Theorem 1.2, we know the matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n_v} \\ t_{21} & t_{22} & \cdots & t_{2n_v} \\ \cdots & \cdots & \cdots & \cdots \\ t_{n_v1} & t_{n_v2} & \cdots & t_{n_vn_v} \end{bmatrix}$$

is non-degenerate. For an integer i, $1 \leq i \leq s$, let

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix}$$

be a Jordan black of $k_i \times k_i$ and

$$A = T \begin{bmatrix} J_1 & & & O \\ & J_2 & & \\ & & \ddots & \\ O & & & J_s \end{bmatrix} T^{-1}.$$

Then we are easily know the solution basis of the linear differential equation system

$$\frac{dX}{dt} = AX \qquad (LDES^1)$$

with $X = [x_1(t), x_2(t), \cdots, x_{n_v}(t)]^T$ is nothing but $\mathscr{B}_v$ by Theorem 1.2. Notice that the Jordan black and the matrix T are uniquely determined by $\mathscr{B}_v$. Thus the linear homogeneous differential equation $(LDES^1)$ is uniquely determined by $\mathscr{B}_v$. It should be noted that this construction can be processed on each vertex $v \in V(G)$. We finally get a linear homogeneous differential equation system $(LDES^1_m)$, which is uniquely determined by the basis graph G.

Similarly, we construct the linear homogeneous differential equation system (LDE^n_m) for the basis graph G. In fact, for $\forall u \in V(G)$, let the basis $\mathscr{B}_u$ at the vertex u be $\mathscr{B}_u = \{\, t^l e^{\alpha_i t} \mid 1 \leq i \leq s, 1 \leq l \leq k_i\}$. Notice that λ_i should be a k_i-fold zero of the characteristic equation $P(\lambda) = 0$ with $k_1 + k_2 + \cdots + k_s = n$. Thus $P(\lambda_i) = P'(\lambda_i) = \cdots = P^{(k_i-1)}(\lambda_i) = 0$ but $P^{(k_i)}(\lambda_i) \neq 0$ for integers $1 \leq i \leq s$. Define a polynomial $P_u(\lambda)$ following

$$P_u(\lambda) = \prod_{i=1}^{s} (\lambda - \lambda_i)^{k_i}$$

associated with the vertex u. Let its expansion be

$$P_u(\lambda) = \lambda^n + a_{u1}\lambda^{n-1} + \cdots + a_{u(n-1)}\lambda + a_{un}.$$

Now we construct a linear homogeneous differential equation

$$x^{(n)} + a_{u1}x^{(n-1)} + \cdots + a_{u(n-1)}x' + a_{un}x = 0 \qquad (L^hDE^n)$$

associated with the vertex u. Then by Theorem 1.2 we know that the basis solution of (LDE^n) is just $\mathscr{C}_u$. Notices that such a linear homogeneous differential equation (LDE^n) is uniquely constructed. Processing this construction for every vertex $u \in V(G)$, we get a linear homogeneous differential equation system (LDE^n_m). This completes the proof. □

Example 2.8 Let (LDE^n_m) be the following linear homogeneous differential equation system

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}$ and $\dot{x} = \dfrac{dx}{dt}$. Then the solution basis of equations $(1) - (6)$ are respectively $\{e^t, e^{2t}\}$, $\{e^{2t}, e^{3t}\}$, $\{e^{3t}, e^{4t}\}$, $\{e^{4t}, e^{5t}\}$, $\{e^{5t}, e^{6t}\}$, $\{e^{6t}, e^t\}$ and its basis graph is shown in Fig.2.1.

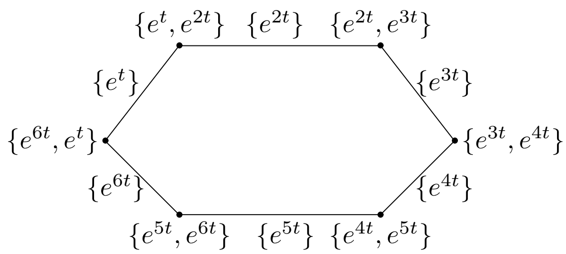


Fig.2.1 **The basis graph H**

Theorem 2.7 enables one to extend the conception of solution of linear differential equation to the following.

Definition 2.9 *A basis graph $G[LDES_m^1]$ (or $G[LDE_m^n]$) is called the graph solution of the linear homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)), abbreviated to G-solution.*

The following result is an immediately conclusion of Theorem 3.1 by definition.

Theorem 2.10 *Every linear homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)) has a unique G-solution, and for every basis graph H, there is a unique linear homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)) with G-solution H.*

Theorem 2.10 implies that one can classifies the linear homogeneous differential equation systems by those of basis graphs.

Definition 2.11 *Let $(LDES_m^1)$, $(LDES_m^1)'$ (or (LDE_m^n), $(LDE_m^n)'$) be two linear homogeneous differential equation systems with G-solutions H, H'. They are called combinatorially equivalent if there is an isomorphism $\varphi : H \rightarrow H'$, thus there is an isomorphism $\varphi : H \rightarrow H'$ of graph and labelings θ, τ on H and H' respectively such that $\varphi\theta(x) = \tau\varphi(x)$ for $\forall x \in V(H)\bigcup E(H)$, denoted by $(LDES_m^1) \overset{\varphi}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \overset{\varphi}{\simeq} (LDE_m^n)'$).*

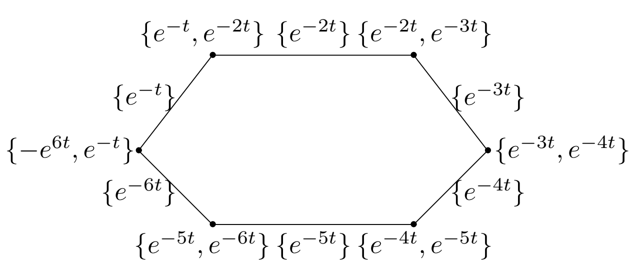


Fig.2.2 **The basis graph H'**

Example 2.12 Let $(LDE_m^n)'$ be the following linear homogeneous differential equation system

$$\begin{cases} \ddot{x}+3\dot{x}+2x=0 & (1)\\ \ddot{x}+5\dot{x}+6x=0 & (2)\\ \ddot{x}+7\dot{x}+12x=0 & (3)\\ \ddot{x}+9\dot{x}+20x=0 & (4)\\ \ddot{x}+11\dot{x}+30x=0 & (5)\\ \ddot{x}+7\dot{x}+6x=0 & (6)\end{cases}$$

Then its basis graph is shown in Fig.2.2.

Let $\varphi : H \to H'$ be determined by $\varphi(\{e^{\lambda_i t}, e^{\lambda_j t}\}) = \{e^{-\lambda_i t}, e^{-\lambda_j t}\}$ and

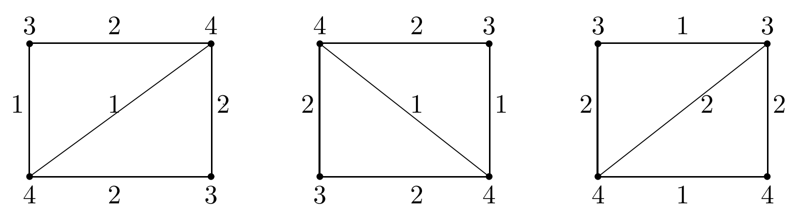
$$\varphi(\{e^{\lambda_i t}, e^{\lambda_j t}\}\bigcap\{e^{\lambda_k t}, e^{\lambda_l t}\}) = \{e^{-\lambda_i t}, e^{-\lambda_j t}\}\bigcap\{e^{-\lambda_k t}, e^{-\lambda_l t}\}$$

for integers $1 \leq i,k \leq 6$ and $j = i+1 \equiv 6(\text{mod}6)$, $l = k+1 \equiv 6(\text{mod}6)$. Then it is clear that $H \overset{\varphi}{\simeq} H'$. Thus $(LDE_m^n)'$ is combinatorially equivalent to the linear homogeneous differential equation system (LDE_m^n) appeared in Example 2.8.

Definition 2.13 *Let G be a simple graph. A vertex-edge labeled graph $\theta: G \to \mathbb{Z}^+$ is called integral if $\theta(uv) \leq \min\{\theta(u), \theta(v)\}$ for $\forall uv \in E(G)$, denoted by G^{I_θ}.*

Let $G_1^{I_\theta}$ and $G_2^{I_\tau}$ be two integral labeled graphs. They are called identical if $G_1 \overset{\varphi}{\simeq} G_2$ and $\theta(x) = \tau(\varphi(x))$ for any graph isomorphism φ and $\forall x \in V(G_1)\bigcup E(G_1)$, denoted by $G_1^{I_\theta} = G_2^{I_\tau}$.

For example, these labeled graphs shown in Fig.2.3 are all integral on K_4-e, but $G_1^{I_\theta} = G_2^{I_\tau}$, $G_1^{I_\theta} \neq G_3^{I_\sigma}$.



$G_1^{I_\theta}$ $G_2^{I_\tau}$ $G_3^{I_\sigma}$

Fig.2.3

Let $G[LDES_m^1]$ ($G[LDE_m^n]$) be a basis graph of the linear homogeneous differential equation system $(LDES_m^1)$ (or (LDE_m^n)) labeled each $v \in V(G[LDES_m^1])$ (or $v \in V(G[LDE_m^n])$) by $\mathscr{B}_v$. We are easily get a vertex-edge labeled graph by relabeling $v \in V(G[LDES_m^1])$ (or $v \in V(G[LDE_m^n])$) by $|\mathscr{B}_v|$ and $uv \in E(G[LDES_m^1])$ (or $uv \in E(G[LDE_m^n])$) by $|\mathscr{B}_u \bigcap \mathscr{B}_v|$. Obviously, such a vertex-edge labeled graph is integral, and denoted by $G^I[LDES_m^1]$ (or $G^I[LDE_m^n]$). The following result completely characterizes combinatorially equivalent linear homogeneous differential equation systems.

Theorem 2.14 *Let $(LDES_m^1)$, $(LDES_m^1)'$ (or (LDE_m^n), $(LDE_m^n)'$) be two linear homogeneous* *differential equation systems with integral labeled graphs* H, H'. *Then* $(LDES_m^1) \stackrel{\mathscr{L}}{\simeq} (LDES_m^1)'$ *(or* $(LDE_m^n) \stackrel{\mathscr{L}}{\simeq} (LDE_m^n)'$*) if and only if* $H = H'$.

Proof Clearly, $H = H'$ if $(LDES_m^1) \stackrel{\mathscr{L}}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \stackrel{\mathscr{L}}{\simeq} (LDE_m^n)'$) by definition. We prove the converse, i.e., if $H = H'$ then there must be $(LDES_m^1) \stackrel{\mathscr{L}}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \stackrel{\mathscr{L}}{\simeq} (LDE_m^n)'$).

Notice that there is an objection between two finite sets S_1, S_2 if and only if $|S_1| = |S_2|$. Let τ be a $1-1$ mapping from $\mathscr{B}_v$ on basis graph $G[LDES_m^1]$ (or basis graph $G[LDE_m^n]$) to $\mathscr{B}_{v'}$ on basis graph $G[LDES_m^1]'$ (or basis graph $G[LDE_m^n]'$) for $v, v' \in V(H')$. Now if $H = H'$, we can easily extend the identical isomorphism id_H on graph H to a $1-1$ mapping $id_H^* : G[LDES_m^1] \to G[LDES_m^1]'$ (or $id_H^* : G[LDE_m^n] \to G[LDE_m^n]'$) with labelings $\theta : v \to \mathscr{B}_v$ and $\theta'_{v'} : v' \to \mathscr{B}_{v'}$ on $G[LDES_m^1], G[LDES_m^1]'$ (or basis graphs $G[LDE_m^n], G[LDE_m^n]'$). Then it is an immediately to check that $id_H^*\theta(x) = \theta'\tau(x)$ for $\forall x \in V(G[LDES_m^1]) \bigcup E(G[LDES_m^1])$ (or for $\forall x \in V(G[LDE_m^n]) \bigcup E(G[LDE_m^n])$). Thus id_H^* is an isomorphism between basis graphs $G[LDES_m^1]$ and $G[LDES_m^1]'$ (or $G[LDE_m^n]$ and $G[LDE_m^n]'$). Thus $(LDES_m^1) \stackrel{id_H^*}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \stackrel{id_H^*}{\simeq} (LDE_m^n)'$). This completes the proof. □

According to Theorem 2.14, all linear homogeneous differential equation systems $(LDES_m^1)$ or (LDE_m^n) can be classified by G-solutions into the following classes:

Class 1. $\hat{G}[LDES_m^1] \simeq \overline{K}_m$ or $\hat{G}[LDE_m^n] \simeq \overline{K}_m$ for integers $m, n \geq 1$.

The G-solutions of differential equation systems are labeled by solution bases on $\overline{K}_m$ and any two linear differential equations in $(LDES_m^1)$ or (LDE_m^n) are parallel, which characterizes m isolated systems in this class.

For example, the following differential equation system

$$\begin{cases} \ddot{x} + 3\dot{x} + 2x = 0 \\ \ddot{x} - 5\dot{x} + 6x = 0 \\ \ddot{x} + 2\dot{x} - 3x = 0 \end{cases}$$

is of Class 1.

Class 2. $\hat{G}[LDES_m^1] \simeq K_m$ or $\hat{G}[LDE_m^n] \simeq K_m$ for integers $m, n \geq 1$.

The G-solutions of differential equation systems are labeled by solution bases on complete graphs K_m in this class. By Corollary 2.6, we know that $\hat{G}[LDES_m^1] \simeq K_m$ or $\hat{G}[LDE_m^n] \simeq K_m$ if $(LDES_m^1)$ or (LDE_m^n) is solvable. In fact, this implies that

$$\bigcap_{v \in V(K_m)} \mathscr{B}_v = \bigcap_{u,v \in V(K_m)} (\mathscr{B}_u \bigcap \mathscr{B}_v) \neq \emptyset.$$

Otherwise, $(LDES_m^1)$ or (LDE_m^n) is non-solvable.

For example, the underlying graphs of linear differential equation systems (A) and (B) in the following

$$(A) \quad \begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 \\ \ddot{x} - x = 0 \\ \ddot{x} - 4\dot{x} + 3x = 0 \\ \ddot{x} + 2\dot{x} - 3x = 0 \end{cases} \qquad (B) \quad \begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 \\ \ddot{x} - 5\dot{x} + 6x = 0 \\ \ddot{x} - 4\dot{x} + 3x = 0 \end{cases}$$

are respectively $K_4,\ K_3$. It is easily to know that (A) is solvable, but (B) is not.

Class 3. $\hat{G}[LDES_m^1] \simeq G$ or $\hat{G}[LDE_m^n] \simeq G$ with $|G| = m$ but $G \not\simeq K_m,\ \overline{K}_m$ for integers $m, n \geq 1$.

The G-solutions of differential equation systems are labeled by solution bases on G and all linear differential equation systems $(LDES_m^1)$ or (LDE_m^n) are non-solvable in this class, such as those shown in Example 2.12.

2.3 Global Stability of Linear Differential Equations

The following result on the initial problem of $(LDES^1)$ and (LDE^n) are well-known for differential equations.

Lemma 2.15([13]) *For $t \in [0, \infty)$, there is a unique solution $X(t)$ for the linear homogeneous differential equation system*

$$\frac{dX}{dt} = AX \qquad (L^hDES^1)$$

with $X(0) = X_0$ and a unique solution for

$$x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0 \qquad (L^hDE^n)$$

with $x(0) = x_0, x'(0) = x_0', \cdots, x^{(n-1)}(0) = x_0^{(n-1)}$.

Applying Lemma 2.15, we get easily a conclusion on the G-solution of $(LDES_m^1)$ with $X_v(0) = X_0^v$ for $\forall v \in V(G)$ or (LDE_m^n) with $x(0) = x_0, x'(0) = x_0', \cdots, x^{(n-1)}(0) = x_0^{(n-1)}$ by Theorem 2.10 following.

Theorem 2.16 *For $t \in [0, \infty)$, there is a unique G-solution for a linear homogeneous differential equation systems $(LDES_m^1)$ with initial value $X_v(0)$ or (LDE_m^n) with initial values $x_v(0), x_v'(0), \cdots, x_v^{(n-1)}(0)$ for $\forall v \in V(G)$.*

For discussing the stability of linear homogeneous differential equations, we introduce the conceptions of *zero G-solution* and *equilibrium point* of that $(LDES_m^1)$ or (LDE_m^n) following.

Definition 2.17 *A G-solution of a linear differential equation system $(LDES_m^1)$ with initial value $X_v(0)$ or (LDE_m^n) with initial values $x_v(0), x_v'(0), \cdots, x_v^{(n-1)}(0)$ for $\forall v \in V(G)$ is called a zero G-solution if each label $\mathscr{B}_i$ of G is replaced by $(0, \cdots, 0)$ ($|\mathscr{B}_i|$ times) and $\mathscr{B}_i \bigcap \mathscr{B}_j$ by $(0, \cdots, 0)$ ($|\mathscr{B}_i \bigcap \mathscr{B}_j|$ times) for integers $1 \leq i, j \leq m$.*

Definition 2.18 *Let* $dX/dt = A_v X$, $x^{(n)} + a_{v1}x^{(n-1)} + \cdots + a_{vn}x = 0$ *be differential equations associated with vertex* v *and* H *a spanning subgraph of* $G[LDES_m^1]$ *(or* $G[LDE_m^n]$*). A point* $X^* \in \mathbf{R}^n$ *is called a* H*-equilibrium point if* $A_v X^* = \overline{0}$ *in* $(LDES_m^1)$ *with initial value* $X_v(0)$ *or* $(X^*)^n + a_{v1}(X^*)^{n-1} + \cdots + a_{vn}X^* = \overline{0}$ *in* (LDE_m^n) *with initial values* $x_v(0), x'_v(0), \cdots, x_v^{(n-1)}(0)$ *for* $\forall v \in V(H)$.

We consider only two kind of stabilities on the zero G-solution of linear homogeneous differential equations in this section. One is the sum-stability. Another is the prod-stability.

2.3.1 **Sum-Stability**

Definition 2.19 *Let* H *be a spanning subgraph of* $G[LDES_m^1]$ *or* $G[LDE_m^n]$ *of the linear homogeneous differential equation systems* $(LDES_m^1)$ *with initial value* $X_v(0)$ *or* (LDE_m^n) *with initial values* $x_v(0), x'_v(0), \cdots, x_v^{(n-1)}(0)$. *Then* $G[LDES_m^1]$ *or* $G[LDE_m^n]$ *is called sum-stable or asymptotically sum-stable on* H *if for all solutions* $Y_v(t)$, $v \in V(H)$ *of the linear differential equations of* $(LDES_m^1)$ *or* (LDE_m^n) *with* $|Y_v(0) - X_v(0)| < \delta_v$ *exists for all* $t \geq 0$, $|\sum\limits_{v\in V(H)} Y_v(t) - \sum\limits_{v\in V(H)} X_v(t)| < \varepsilon$, *or furthermore,* $\lim\limits_{t\to 0} |\sum\limits_{v\in V(H)} Y_v(t) - \sum\limits_{v\in V(H)} X_v(t)| = 0$.

Clearly, an asymptotic sum-stability implies the sum-stability of that $G[LDES_m^1]$ or $G[LDE_m^n]$. The next result shows the relation of sum-stability with that of classical stability.

Theorem 2.20 *For a* G*-solution* $G[LDES_m^1]$ *of* $(LDES_m^1)$ *with initial value* $X_v(0)$ *(or* $G[LDE_m^n]$ *of* (LDE_m^n) *with initial values* $x_v(0), x'_v(0), \cdots, x_v^{(n-1)}(0)$*), let* H *be a spanning subgraph of* $G[LDES_m^1]$ *(or* $G[LDE_m^n]$*) and* X^* *an equilibrium point on subgraphs* H. *If* $G[LDES_m^1]$ *(or* $G[LDE_m^n]$*) is stable on any* $\forall v \in V(H)$, *then* $G[LDES_m^1]$ *(or* $G[LDE_m^n]$*) is sum-stable on* H. *Furthermore, if* $G[LDES_m^1]$ *(or* $G[LDE_m^n]$*) is asymptotically sum-stable for at least one vertex* $v \in V(H)$, *then* $G[LDES_m^1]$ *(or* $G[LDE_m^n]$*) is asymptotically sum-stable on* H.

Proof Notice that

$$|\sum_{v\in V(H)} p_v Y_v(t) - \sum_{v\in V(H)} p_v X_v(t)| \leq \sum_{v\in V(H)} p_v |Y_v(t) - X_v(t)|$$

and

$$\lim_{t\to 0} |\sum_{v\in V(H)} p_v Y_v(t) - \sum_{v\in V(H)} p_v X_v(t)| \leq \sum_{v\in V(H)} p_v \lim_{t\to 0} |Y_v(t) - X_v(t)|.$$

Then the conclusion on sum-stability follows. □

For linear homogenous differential equations $(LDES^1)$ (or (LDE^n)), the following result on stability of its solution $X(t) = \overline{0}$ (or $x(t) = 0$) is well-known.

Lemma 2.21 *Let* $\gamma = \max\{\ \mathrm{Re}\lambda \mid |A - \lambda I_{n\times n}| = 0\}$. *Then the stability of the trivial solution* $X(t) = \overline{0}$ *of linear homogenous differential equations* $(LDES^1)$ *(or* $x(t) = 0$ *of* (LDE^n)*) is determined as follows:*

(1) *if* $\gamma < 0$, *then it is asymptotically stable;*

(2) *if $\gamma > 0$, then it is unstable;*

(3) *if $\gamma = 0$, then it is not asymptotically stable, and stable if and only if $m'(\lambda) = m(\lambda)$ for every λ with $\mathrm{Re}\lambda = 0$, where $m(\lambda)$ is the algebraic multiplicity and $m'(\lambda)$ the dimension of eigenspace of λ.*

By Theorem 2.20 and Lemma 2.21, the following result on the stability of zero G-solution of $(LDES_m^1)$ and (LDE_m^n) is obtained.

Theorem 2.22 *A zero G-solution of linear homogenous differential equation systems $(LDES_m^1)$ (or (LDE_m^n)) is asymptotically sum-stable on a spanning subgraph H of $G[LDES_m^1]$ (or $G[LDE_m^n]$) if and only if $\mathrm{Re}\alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ in $(LDES^1)$ or $\mathrm{Re}\lambda_v < 0$ for each $t^{l_v}e^{\lambda_v t} \in \mathscr{C}_v$ in (LDE_m^n) hold for $\forall v \in V(H)$.*

Proof The sufficiency is an immediately conclusion of Theorem 2.20.

Conversely, if there is a vertex $v \in V(H)$ such that $\mathrm{Re}\alpha_v \geq 0$ for $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ in $(LDES^1)$ or $\mathrm{Re}\lambda_v \geq 0$ for $t^{l_v}e^{\lambda_v t} \in \mathscr{C}_v$ in (LDE_m^n), then we are easily knowing that

$$\lim_{t\to\infty} \overline{\beta}_v(t)e^{\alpha_v t} \to \infty$$

if $\alpha_v > 0$ or $\overline{\beta}_v(t) \neq$constant, and

$$\lim_{t\to\infty} t^{l_v}e^{\lambda_v t} \to \infty$$

if $\lambda_v > 0$ or $l_v > 0$, which implies that the zero G-solution of linear homogenous differential equation systems $(LDES^1)$ or (LDE^n) is not asymptotically sum-stable on H. □

The following result of Hurwitz on real number of eigenvalue of a characteristic polynomial is useful for determining the asymptotically stability of the zero G-solution of $(LDES_m^1)$ and (LDE_m^n).

Lemma 2.23 *Let $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ be a polynomial with real coefficients a_i, $1 \leq i \leq n$ and*

$$\Delta_1 = |a_1|, \quad \Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix}, \cdots \Delta_n = \begin{vmatrix} a_1 & 1 & 0 & & \cdots & & 0 & \\ a_3 & a_2 & a_1 & 0 & & \cdots & & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & & & \cdots & & & & a_n \end{vmatrix}.$$

Then $Re\lambda < 0$ for all roots λ of $P(\lambda)$ if and only if $\Delta_i > 0$ for integers $1 \leq i \leq n$.

Thus, we get the following result by Theorem 2.22 and lemma 2.23.

Corollary 2.24 *Let $\Delta_1^v, \Delta_2^v, \cdots, \Delta_n^v$ be the associated determinants with characteristic polynomials determined in Lemma 4.8 for $\forall v \in V(G[LDES_m^1])$ or $V(G[LDE_m^n])$. Then for a spanning subgraph $H < G[LDES_m^1]$ or $G[LDE_m^n]$, the zero G-solutions of $(LDES_m^1)$ and (LDE_m^n) is asymptotically sum-stable on H if $\Delta_1^v > 0, \Delta_2^v > 0, \cdots, \Delta_n^v > 0$ for $\forall v \in V(H)$.*

Particularly, if $n = 2$, we are easily knowing that $Re\lambda < 0$ for all roots λ of $P(\lambda)$ if and only if $a_1 > 0$ and $a_2 > 0$ by Lemma 2.23. We get the following result.

Corollary 2.25 *Let $H < G[LDES_m^1]$ or $G[LDE_m^n]$ be a spanning subgraph. If the characteristic polynomials are $\lambda^2 + a_1^v\lambda + a_2^v$ for $v \in V(H)$ in $(LDES_m^1)$ (or $(L^hDE_m^2)$), then the zero G-solutions of $(LDES_m^1)$ and (LDE_m^2) is asymptotically sum-stable on H if $a_1^v > 0,\ a_2^v > 0$ for $\forall v \in V(H)$.*

2.3.2 Prod-Stability

Definition 2.26 *Let H be a spanning subgraph of $G[LDES_m^1]$ or $G[LDE_m^n]$ of the linear homogeneous differential equation systems $(LDES_m^1)$ with initial value $X_v(0)$ or (LDE_m^n) with initial values $x_v(0), x_v'(0), \cdots, x_v^{(n-1)}(0)$. Then $G[LDES_m^1]$ or $G[LDE_m^n]$ is called prod-stable or asymptotically prod-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$ of the linear differential equations of $(LDES_m^1)$ or (LDE_m^n) with $|Y_v(0) - X_v(0)| < \delta_v$ exists for all $t \geq 0$, $|\prod\limits_{v\in V(H)} Y_v(t) - \prod\limits_{v\in V(H)} X_v(t)| < \varepsilon$, or furthermore, $\lim\limits_{t\to 0} |\prod\limits_{v\in V(H)} Y_v(t) - \prod\limits_{v\in V(H)} X_v(t)| = 0$.*

We know the following result on the prod-stability of linear differential equation system $(LDES_m^1)$ and (LDE_m^n).

Theorem 2.27 *A zero G-solution of linear homogenous differential equation systems $(LDES_m^1)$ (or (LDE_m^n)) is asymptotically prod-stable on a spanning subgraph H of $G[LDES_m^1]$ (or $G[LDE_m^n]$) if and only if $\sum\limits_{v\in V(H)} \mathrm{Re}\alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ in $(LDES^1)$ or $\sum\limits_{v\in V(H)} \mathrm{Re}\lambda_v < 0$ for each $t^{l_v}e^{\lambda_v t} \in \mathscr{C}_v$ in (LDE_m^n).*

Proof Applying Theorem 1.2, we know that a solution $X_v(t)$ at the vertex v has the form

$$X_v(t) = \sum_{i=1}^{n} c_i\overline{\beta}_v(t)e^{\alpha_v t}.$$

Whence,

$$\begin{aligned}\left|\prod_{v\in V(H)} X_v(t)\right| &= \left|\prod_{v\in V(H)}\sum_{i=1}^{n} c_i\overline{\beta}_v(t)e^{\alpha_v t}\right| \\ &= \left|\sum_{i=1}^{n}\prod_{v\in V(H)} c_i\overline{\beta}_v(t)e^{\alpha_v t}\right| = \left|\sum_{i=1}^{n}\prod_{v\in V(H)} c_i\overline{\beta}_v(t)\right| e^{\sum\limits_{v\in V(H)}\alpha_v t}.\end{aligned}$$

Whence, the zero G-solution of homogenous $(LDES_m^1)$ (or (LDE_m^n)) is asymptotically sum-stable on subgraph H if and only if $\sum\limits_{v\in V(H)} \mathrm{Re}\alpha_v < 0$ for $\forall\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ in $(LDES^1)$ or $\sum\limits_{v\in V(H)} \mathrm{Re}\lambda_v < 0$ for $\forall t^{l_v}e^{\lambda_v t} \in \mathscr{C}_v$ in (LDE_m^n). □

Applying Theorem 2.22, the following conclusion is a corollary of Theorem 2.27.

Corollary 2.28 *A zero G-solution of linear homogenous differential equation systems $(LDES_m^1)$*

(or (LDE_m^n)) is asymptotically prod-stable if it is asymptotically sum-stable on a spanning subgraph H of $G[LDES_m^1]$ (or $G[LDE_m^n]$). Particularly, it is asymptotically prod-stable if the zero solution $\overline{0}$ is stable on $\forall v \in V(H)$.

Example 2.29 Let a G-solution of $(LDES_m^1)$ or (LDE_m^n) be the basis graph shown in Fig.2.4, where $v_1 = \{e^{-2t}, e^{-3t}, e^{3t}\}$, $v_2 = \{e^{-3t}, e^{-4t}\}$, $v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}$, $v_4 = \{e^{-5t}, e^{-6t}, e^{-8t}\}$, $v_5 = \{e^{-t}, e^{-6t}\}$, $v_6 = \{e^{-t}, e^{-2t}, e^{-8t}\}$. Then the zero G-solution is sum-stable on the triangle $v_4v_5v_6$, but it is not on the triangle $v_1v_2v_3$. In fact, it is prod-stable on the triangle $v_1v_2v_3$.

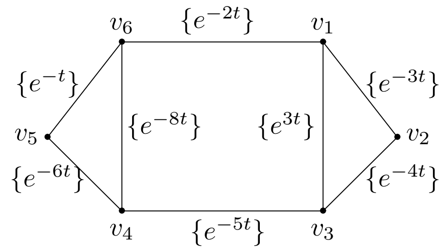


Fig.2.4 **A basis graph**

§3. Global Stability of Non-Solvable Non-Linear Differential Equations

For differential equation system (DES_m^1), we consider the stability of its zero G-solution of linearized differential equation system $(LDES_m^1)$ in this section.

3.1 Global Stability of Non-Solvable Differential Equations

Definition 3.1 *Let H be a spanning subgraph of $G[DES_m^1]$ of the linearized differential equation systems (DES_m^1) with initial value $X_v(0)$. A point $X^* \in \mathbf{R}^n$ is called a H-equilibrium point of differential equation system (DES_m^1) if $f_v(X^*) = \overline{0}$ for $\forall v \in V(H)$.*

Clearly, $\overline{0}$ is a H-equilibrium point for any spanning subgraph H of $G[DES_m^1]$ by definition. Whence, its zero G-solution of linearized differential equation system $(LDES_m^1)$ is a solution of (DES_m^1).

Definition 3.2 *Let H be a spanning subgraph of $G[DES_m^1]$ of the linearized differential equation systems (DES_m^1) with initial value $X_v(0)$. Then $G[DES_m^1]$ is called sum-stable or asymptotically sum-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$ of (DES_m^1) with $\|Y_v(0) - X_v(0)\| < \delta_v$ exists for all $t \geq 0$,*

$$\left\| \sum_{v \in V(H)} Y_v(t) - \sum_{v \in V(H)} X_v(t) \right\| < \varepsilon,$$

or furthermore,

$$\lim_{t\to 0}\left\|\sum_{v\in V(H)} Y_v(t) - \sum_{v\in V(H)} X_v(t)\right\| = 0,$$

and prod-stable or asymptotically prod-stable on H if for all solutions $Y_v(t)$, $v \in V(H)$ of (DES^1_m) with $\|Y_v(0) - X_v(0)\| < \delta_v$ exists for all $t \geq 0$,

$$\left\|\prod_{v\in V(H)} Y_v(t) - \prod_{v\in V(H)} X_v(t)\right\| < \varepsilon,$$

or furthermore,

$$\lim_{t\to 0}\left\|\prod_{v\in V(H)} Y_v(t) - \prod_{v\in V(H)} X_v(t)\right\| = 0.$$

Clearly, the asymptotically sum-stability or prod-stability implies respectively that the sum-stability or prod-stability.

Then we get the following result on the sum-stability and prod-stability of the zero G-solution of (DES^1_m).

Theorem 3.3 *For a G-solution $G[DES^1_m]$ of differential equation systems (DES^1_m) with initial value $X_v(0)$, let H_1, H_2 be spanning subgraphs of $G[DES^1_m]$. If the zero G-solution of (DES^1_m) is sum-stable or asymptotically sum-stable on H_1 and H_2, then the zero G-solution of (DES^1_m) is sum-stable or asymptotically sum-stable on $H_1 \bigcup H_2$.*

Similarly, if the zero G-solution of (DES^1_m) is prod-stable or asymptotically prod-stable on H_1 and $X_v(t)$ is bounded for $\forall v \in V(H_2)$, then the zero G-solution of (DES^1_m) is prod-stable or asymptotically prod-stable on $H_1 \bigcup H_2$.

Proof Notice that

$$\|X_1 + X_2\| \leq \|X_1\| + \|X_2\| \quad \text{and} \quad \|X_1X_2\| \leq \|X_1\|\|X_2\|$$

in $\mathbf{R}^n$. We know that

$$\begin{aligned}\left\|\sum_{v\in V(H_1)\bigcup V(H_2)} X_v(t)\right\| &= \left\|\sum_{v\in V(H_1)} X_v(t) + \sum_{v\in V(H_2)} X_v(t)\right\| \\ &\leq \left\|\sum_{v\in V(H_1)} X_v(t)\right\| + \left\|\sum_{v\in V(H_2)} X_v(t)\right\|\end{aligned}$$

and

$$\begin{aligned}\left\|\prod_{v\in V(H_1)\bigcup V(H_2)} X_v(t)\right\| &= \left\|\prod_{v\in V(H_1)} X_v(t) \prod_{v\in V(H_2)} X_v(t)\right\| \\ &\leq \left\|\prod_{v\in V(H_1)} X_v(t)\right\| \left\|\prod_{v\in V(H_2)} X_v(t)\right\|.\end{aligned}$$

Whence,

$$\left\| \sum_{v\in V(H_1)\bigcup V(H_2)} X_v(t) \right\| \le \epsilon \ \text{ or } \ \lim_{t\to 0}\left\| \sum_{v\in V(H_1)\bigcup V(H_2)} X_v(t) \right\| = 0$$

if $\epsilon = \epsilon_1 + \epsilon_2$ with

$$\left\| \sum_{v\in V(H_1)} X_v(t) \right\| \le \epsilon_1 \ \text{ and } \ \left\| \sum_{v\in V(H_2)} X_v(t) \right\| \le \epsilon_2$$

or

$$\lim_{t\to 0}\left\| \sum_{v\in V(H_1)} X_v(t) \right\| = 0 \ \text{ and } \ \lim_{t\to 0}\left\| \sum_{v\in V(H_2)} X_v(t) \right\| = 0.$$

This is the conclusion (1). For the conclusion (2), notice that

$$\left\| \prod_{v\in V(H_1)\bigcup V(H_2)} X_v(t) \right\| \le \left\| \prod_{v\in V(H_1)} X_v(t) \right\| \left\| \prod_{v\in V(H_2)} X_v(t) \right\| \le M\epsilon$$

if

$$\left\| \prod_{v\in V(H_1)} X_v(t) \right\| \le \epsilon \ \text{ and } \ \left\| \prod_{v\in V(H_2)} X_v(t) \right\| \le M.$$

Consequently, the zero G-solution of (DES_m^1) is prod-stable or asymptotically prod-stable on $H_1 \bigcup H_2$. □

Theorem 3.3 enables one to get the following conclusion which establishes the relation of stability of differential equations at vertices with that of sum-stability and prod-stability.

Corollary 3.4 *For a G-solution $G[DES_m^1]$ of differential equation system (DES_m^1) with initial value $X_v(0)$, let H be a spanning subgraph of $G[DES_m^1]$. If the zero solution is stable or asymptotically stable at each vertex $v \in V(H)$, then it is sum-stable, or asymptotically sum-stable and if the zero solution is stable or asymptotically stable in a vertex $u \in V(H)$ and $X_v(t)$ is bounded for $\forall v \in V(H) \setminus \{u\}$, then it is prod-stable, or asymptotically prod-stable on H.*

It should be noted that the converse of Theorem 3.3 is not always true. For example, let

$$\left\| \sum_{v\in V(H_1)} X_v(t) \right\| \le a + \epsilon \ \text{ and } \ \left\| \sum_{v\in V(H_2)} X_v(t) \right\| \le -a + \epsilon.$$

Then the zero G-solution $G[DES_m^1]$ of differential equation system (DES_m^1) is not sum-stable on subgraphs H_1 and H_2, but

$$\left\| \sum_{v\in V(H_1\bigcup H_2)} X_v(t) \right\| \le \left\| \sum_{v\in V(H_1)} X_v(t) \right\| + \left\| \sum_{v\in V(H_2)} X_v(t) \right\| = \epsilon.$$

Thus the zero G-solution $G[DES_m^1]$ of differential equation system (DES_m^1) is sum-stable on subgraphs $H_1 \bigcup H_2$. Similarly, let

$$\left\| \prod_{v\in V(H_1)} X_v(t) \right\| \leq \frac{\epsilon}{t^r} \quad \text{and} \quad \left\| \sum_{v\in V(H_2)} X_v(t) \right\| \leq t^r$$

for a real number r. Then the zero G-solution $G[DES_m^1]$ of (DES_m^1) is not prod-stable on subgraphs H_1 and $X_v(t)$ is not bounded for $v \in V(H_2)$ if $r > 0$. However, it is prod-stable on subgraphs $H_1 \bigcup H_2$ for

$$\left\| \prod_{v\in V(H_1\bigcup H_2)} X_v(t) \right\| \leq \left\| \prod_{v\in V(H_1)} X_v(t) \right\| \left\| \prod_{v\in V(H_2)} X_v(t) \right\| = \epsilon.$$

3.2 **Linearized Differential Equations**

Applying these conclusions on linear differential equation systems in the previous section, we can find conditions on $F_i(X),\ 1 \leq i \leq m$ for the sum-stability and prod-stability at $\overline{0}$ following. For this objective, we need the following useful result.

Lemma 3.5([13]) *Let $\dot{X} = AX + B(X)$ be a non-linear differential equation, where A is a constant $n \times n$ matrix and $\mathrm{Re}\lambda_i < 0$ for all eigenvalues λ_i of A and $B(X)$ is continuous defined on $t \geq 0,\ \|X\| \leq \alpha$ with*

$$\lim_{\|X\|\to 0} \frac{\|B(X)\|}{\|X\|} = 0.$$

Then there exist constants $c > 0,\ \beta > 0$ and $\delta,\ 0 < \delta < \alpha$ such that

$$\|X(0)\| \leq \varepsilon \leq \frac{\delta}{2c} \quad \textit{implies that} \quad \|X(t)\| \leq c\varepsilon e^{-\beta t/2}.$$

Theorem 3.6 *Let (DES_m^1) be a non-linear differential equation system, H a spanning subgraph of $G[DES_m^1]$ and*

$$F_v(X) = F_v'\left(\overline{0}\right) X + R_v(X)$$

such that

$$\lim_{\|X\|\to \overline{0}} \frac{\|R_v(X)\|}{\|X\|} = 0$$

for $\forall v \in V(H)$. Then the zero G-solution of (DES_m^1) is asymptotically sum-stable or asymptotically prod-stable on H if $\mathrm{Re}\alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v,\ v \in V(H)$ in (DES_m^1).

Proof Define $c = \max\{c_v,\ v \in V(H)\}$, $\varepsilon = \min\{\varepsilon_v,\ v \in V(H)\}$ and $\beta = \min\{\beta_v,\ v \in V(H)\}$. Applying Lemma 3.5, we know that for $\forall v \in V(H)$,

$$\|X_v(0)\| \leq \varepsilon \leq \frac{\delta}{2c} \quad \text{implies that} \quad \|X_v(t)\| \leq c\varepsilon e^{-\beta t/2}.$$

Whence,

$$\left\|\sum_{v\in V(H)} X_v(t)\right\| \leq \sum_{v\in V(H)} \|X_v(t)\| \leq |H|c\varepsilon e^{-\beta t/2}$$

$$\left\|\prod_{v\in V(H)} X_v(t)\right\| \leq \prod_{v\in V(H)} \|X_v(t)\| \leq c^{|H|}\varepsilon^{|H|}e^{-|H|\beta t/2}.$$

Consequently,

$$\lim_{t\to 0}\left\|\sum_{v\in V(H)} X_v(t)\right\| \to 0 \quad \text{and} \quad \lim_{t\to 0}\left\|\prod_{v\in V(H)} X_v(t)\right\| \to 0.$$

Thus the zero G-solution (DES_m^n) is asymptotically sum-stable or asymptotically prod-stable on H by definition. □

3.3 Liapunov Functions on G-Solutions

We have know Liapunov functions associated with differential equations. Similarly, we introduce Liapunov functions for determining the sum-stability or prod-stability of (DES_m^1) following.

Definition 3.7 *Let (DES_m^1) be a differential equation system, $H < G[DES_m^1]$ a spanning subgraph and a H-equilibrium point X^* of (DES_m^1). A differentiable function $L : \mathscr{O} \to \mathbf{R}$ defined on an open subset $\mathscr{O} \subset \mathbf{R}^n$ is called a Liapunov sum-function on X^* for H if*

(1) *$L(X^*) = 0$ and $L\left(\sum\limits_{v\in V(H)} X_v(t)\right) > 0$ if $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$;*

(2) *$\dot{L}\left(\sum\limits_{v\in V(H)} X_v(t)\right) \leq 0$ for $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$,*

and a Liapunov prod-function on X^* for H if

(1) *$L(X^*) = 0$ and $L\left(\prod\limits_{v\in V(H)} X_v(t)\right) > 0$ if $\prod\limits_{v\in V(H)} X_v(t) \neq X^*$;*

(2) *$\dot{L}\left(\prod\limits_{v\in V(H)} X_v(t)\right) \leq 0$ for $\prod\limits_{v\in V(H)} X_v(t) \neq X^*$.*

Then, the following conclusions on the sum-stable and prod-stable of zero G-solutions of differential equations holds.

Theorem 3.8 *For a G-solution $G[DES_m^1]$ of a differential equation system (DES_m^1) with initial value $X_v(0)$, let H be a spanning subgraph of $G[DES_m^1]$ and X^* an equilibrium point of (DES_m^1) on H.*

(1) *If there is a Liapunov sum-function $L : \mathscr{O} \to \mathbf{R}$ on X^*, then the zero G-solution $G[DES_m^1]$ is sum-stable on X^* for H. Furthermore, if*

$$\dot{L}\left(\sum_{v\in V(H)} X_v(t)\right) < 0$$

for $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$, *then the zero* G*-solution* $G[DES_m^1]$ *is asymptotically sum-stable on* X^* *for* H.

(2) *If there is a Liapunov prod-function* $L : \mathscr{O} \to \mathbf{R}$ *on* X^* *for* H, *then the zero* G*-solution* $G[DES_m^1]$ *is prod-stable on* X^* *for* H. *Furthermore, if*

$$\dot{L}\left(\prod_{v\in V(H)} X_v(t)\right) < 0$$

for $\prod\limits_{v\in V(H)} X_v(t) \neq X^*$, *then the zero* G*-solution* $G[DES_m^1]$ *is asymptotically prod-stable on* X^* *for* H.

Proof Let $\epsilon > 0$ be a so small number that the closed ball $B_\epsilon(X^*)$ centered at X^* with radius ϵ lies entirely in $\mathscr{O}$ and ϖ the minimum value of L on the boundary of $B_\epsilon(X^*)$, i.e., the sphere $S_\epsilon(X^*)$. Clearly, $\varpi > 0$ by assumption. Define $U = \{X \in B_\epsilon(X^*) | L(X) < \varpi\}$. Notice that $X^* \in U$ and L is non-increasing on $\sum\limits_{v\in V(H)} X_v(t)$ by definition. Whence, there are no solutions $X_v(t),\ v \in V(H)$ starting in U such that $\sum\limits_{v\in V(H)} X_v(t)$ meet the sphere $S_\epsilon(X^*)$. Thus all solutions $X_v(t),\ v \in V(H)$ starting in U enable $\sum\limits_{v\in V(H)} X_v(t)$ included in ball $B_\epsilon(X^*)$. Consequently, the zero G-solution $G[DES_m^1]$ is sum-stable on H by definition.

Now assume that

$$\dot{L}\left(\sum_{v\in V(H)} X_v(t)\right) < 0$$

for $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$. Thus L is strictly decreasing on $\sum\limits_{v\in V(H)} X_v(t)$. If $X_v(t),\ v \in V(H)$ are solutions starting in $U - X^*$ such that $\sum\limits_{v\in V(H)} X_v(t_n) \to Y^*$ for $n \to \infty$ with $Y^* \in B_\epsilon(X^*)$, then it must be $Y^* = X^*$. Otherwise, since

$$L\left(\sum_{v\in V(H)} X_v(t)\right) > L(Y^*)$$

by the assumption

$$\dot{L}\left(\sum_{v\in V(H)} X_v(t)\right) < 0$$

for all $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$ and

$$L\left(\sum_{v\in V(H)} X_v(t_n)\right) \to L(Y^*)$$

by the continuity of L, if $Y^* \neq X^*$, let $Y_v(t), v \in V(H)$ be the solutions starting at Y^*. Then for any $\eta > 0$,

$$L\left(\sum_{v\in V(H)} Y_v(\eta)\right) < L(Y^*).$$

But then there is a contradiction

$$L\left(\sum_{v\in V(H)} X_v(t_n+\eta)\right) < L(Y^*)$$

yields by letting $Y_v(0) = \sum\limits_{v\in V(H)} X_v(t_n)$ for sufficiently large n. Thus, there must be $Y_v^* = X^*$. Whence, the zero G-solution $G[DES_m^1]$ is asymptotically sum-stable on H by definition. This is the conclusion (1).

Similarly, we can prove the conclusion (2). □

The following result shows the combination of Liapunov sum-functions or prod-functions.

Theorem 3.9 *For a G-solution $G[DES_m^1]$ of a differential equation system (DES_m^1) with initial value $X_v(0)$, let H_1, H_2 be spanning subgraphs of $G[DES_m^1]$, X^* an equilibrium point of (DES_m^1) on $H_1 \bigcup H_2$ and*

$$R^+(x,y) = \sum_{i\geq 0, j\geq 0} a_{i,j}x^iy^j$$

be a polynomial with $a_{i,j} \geq 0$ for integers $i,j \geq 0$. Then $R^+(L_1,L_2)$ is a Liapunov sum-function or Liapunov prod-function on X^* for $H_1 \bigcup H_2$ with conventions for integers $i,j,k,l \geq 0$ that

$$\begin{aligned}
& a_{ij}L_1^iL_2^j\left(\sum_{v\in V(H_1\bigcup V(H_2)} X_v(t)\right) + a_{kl}L_1^kL_2^l\left(\sum_{v\in V(H_1\bigcup V(H_2)} X_v(t)\right)\\
& = a_{ij}L_1^i\left(\sum_{v\in V(H_1)} X_v(t)\right) L_2^j\left(\sum_{v\in V(H_2)} X_v(t)\right)\\
& +a_{kl}L_1^k\left(\sum_{v\in V(H_1)} X_v(t)\right) L_2^l\left(\sum_{v\in V(H_2)} X_v(t)\right)
\end{aligned}$$

if L_1, L_2 are Liapunov sum-functions and

$$\begin{aligned}
& a_{ij}L_1^iL_2^j\left(\prod_{v\in V(H_1\bigcup V(H_2)} X_v(t)\right) + a_{kl}L_1^kL_2^l\left(\prod_{v\in V(H_1\bigcup V(H_2)} X_v(t)\right)\\
& = a_{ij}L_1^i\left(\prod_{v\in V(H_1)} X_v(t)\right) L_2^j\left(\prod_{v\in V(H_2)} X_v(t)\right)\\
& +a_{kl}L_1^k\left(\prod_{v\in V(H_1)} X_v(t)\right) L_2^l\left(\prod_{v\in V(H_2)} X_v(t)\right)
\end{aligned}$$

if L_1, L_2 are Liapunov prod-functions on X^* for H_1 and H_2, respectively. Particularly, if there is a Liapunov sum-function (Liapunov prod-function) L on H_1 and H_2, then L is also a Liapunov sum-function (Liapunov prod-function) on $H_1 \bigcup H_2$.

Proof Notice that

$$\frac{d\left(a_{ij}L_1^iL_2^j\right)}{dt} = a_{ij}\left(iL_1^{i-1}\dot{L}_1L_2^j + jL_1^iL_1^{j-1}\dot{L}_2\right)$$

if $i, j \geq 1$. Whence,

$$a_{ij}L_1^iL_2^j\left(\sum_{v\in V(H_1\bigcup V(H_2)}X_v(t)\right)\geq 0$$

if

$$L_1\left(\sum_{v\in V(H_1)}X_v(t)\right)\geq 0 \text{ and } L_2\left(\sum_{v\in V(H_2)}X_v(t)\right)\geq 0$$

and

$$\frac{d(a_{ij}L_1^iL_2^j)}{dt}\left(\sum_{v\in V(H_1\bigcup V(H_2)}X_v(t)\right)\leq 0$$

if

$$\dot{L}_1\left(\sum_{v\in V(H_1)}X_v(t)\right)\leq 0 \text{ and } \dot{L}_2\left(\sum_{v\in V(H_2)}X_v(t)\right)\leq 0.$$

Thus $R^+(L_1, L_2)$ is a Liapunov sum-function on X^* for $H_1\bigcup H_2$.

Similarly, we can know that $R^+(L_1, L_2)$ is a Liapunov prod-function on X^* for $H_1\bigcup H_2$ if L_1, L_2 are Liapunov prod-functions on X^* for H_1 and H_2. □

Theorem 3.9 enables one easily to get the stability of the zero G-solutions of (DES_m^1).

Corollary 3.10 *For a differential equation system (DES_m^1), let $H < G[DES_m^1]$ be a spanning subgraph. If L_v is a Liapunov function on vertex v for $\forall v\in V(H)$, then the functions*

$$L_S^H=\sum_{v\in V(H)}L_v \ \text{ and } \ L_P^H=\prod_{v\in V(H)}L_v$$

are respectively Liapunov sum-function and Liapunov prod-function on graph H. Particularly, if $L = L_v$ for $\forall v\in V(H)$, then L is both a Liapunov sum-function and a Liapunov prod-function on H.

Example 3.11 Let (DES_m^1) be determined by

$$\left\{\begin{array}{c}dx_1/dt=\lambda_{11}x_1\\ dx_2/dt=\lambda_{12}x_2\\ \cdots\cdots\cdots\\ dx_n/dt=\lambda_{1n}x_n\end{array}\right. \quad \left\{\begin{array}{c}dx_1/dt=\lambda_{21}x_1\\ dx_2/dt=\lambda_{22}x_2\\ \cdots\cdots\cdots\\ dx_n/dt=\lambda_{2n}x_n\end{array}\right. \quad \cdots \quad \left\{\begin{array}{c}dx_1/dt=\lambda_{n1}x_1\\ dx_2/dt=\lambda_{n2}x_2\\ \cdots\cdots\cdots\\ dx_n/dt=\lambda_{nn}x_n\end{array}\right.$$

where all λ_{ij}, $1\leq i\leq m, 1\leq j\leq n$ are real and $\lambda_{ij_1}\neq\lambda_{ij_2}$ if $j_1\neq j_2$ for integers $1\leq i\leq m$. Let $L = x_1^2 + x_2^2 + \cdots + x_n^2$. Then

$$\dot{L}=\lambda_{i1}x_1^2+\lambda_{i2}x_2^2+\cdots+\lambda_{in}x_n^2$$

for integers $1 \leq i \leq n$. Whence, it is a Liapunov function for the ith differential equation if $\lambda_{ij} < 0$ for integers $1 \leq j \leq n$. Now let $H < G[LDES_m^1]$ be a spanning subgraph of $G[LDES_m^1]$. Then L is both a Liapunov sum-function and a Liapunov prod-function on H if $\lambda_{vj} < 0$ for $\forall v\in V(H)$ by Corollaries 3.10.

Theorem 3.12 *Let $L:\mathscr{O}\rightarrow\mathbf{R}$ be a differentiable function with $L(\overline{0})=0$ and $L\left(\sum\limits_{v\in V(H)}X\right)>0$ always holds in an area of its ϵ-neighborhood $U(\epsilon)$ of $\overline{0}$ for $\varepsilon>0$, denoted by $U^+(\overline{0},\varepsilon)$ such area of ε-neighborhood of $\overline{0}$ with $L\left(\sum\limits_{v\in V(H)}X\right)>0$ and $H<G[DES_m^1]$ be a spanning subgraph.*

(1) *If*

$$\left\|L\left(\sum_{v\in V(H)}X\right)\right\|\leq M$$

with M a positive number and

$$\dot{L}\left(\sum_{v\in V(H)}X\right)>0$$

in $U^+(\overline{0},\epsilon)$, and for $\forall\epsilon>0$, there exists a positive number c_1,c_2 such that

$$L\left(\sum_{v\in V(H)}X\right)\geq c_1>0\ \ implies\ \ \dot{L}\left(\sum_{v\in V(H)}X\right)\geq c_2>0,$$

then the zero G-solution $G[DES_m^1]$ is not sum-stable on H. Such a function $L:\mathscr{O}\rightarrow\mathbf{R}$ is called a non-Liapunov sum-function on H.

(2) *If*

$$\left\|L\left(\prod_{v\in V(H)}X\right)\right\|\leq N$$

with N a positive number and

$$\dot{L}\left(\prod_{v\in V(H)}X\right)>0$$

in $U^+(\overline{0},\epsilon)$, and for $\forall\epsilon>0$, there exists positive numbers d_1,d_2 such that

$$L\left(\prod_{v\in V(H)}X\right)\geq d_1>0\ \ implies\ \ \dot{L}\left(\prod_{v\in V(H)}X\right)\geq d_2>0,$$

then the zero G-solution $G[DES_m^1]$ is not prod-stable on H. Such a function $L:\mathscr{O}\rightarrow\mathbf{R}$ is called a non-Liapunov prod-function on H.

Proof Generally, if $\|L(X)\|$ is bounded and $\dot{L}(X)>0$ in $U^+(\overline{0},\epsilon)$, and for $\forall\epsilon>0$, there exists positive numbers c_1,c_2 such that if $L(X)\geq c_1>0$, then $\dot{L}(X)\geq c_2>0$, we prove that there exists $t_1>t_0$ such that $\|X(t_1,t_0)\|>\epsilon_0$ for a number $\epsilon_0>0$, where $X(t_1,t_0)$ denotes the solution of (DES_m^n) passing through $X(t_0)$. Otherwise, there must be $\|X(t_1,t_0)\|<\epsilon_0$ for $t\geq t_0$. By $\dot{L}(X)>0$ we know that $L(X(t))>L(X(t_0))>0$ for $t\geq t_0$. Combining this fact with the condition $\dot{L}(X)\geq c_2>0$, we get that

$$L(X(t))=L(X(t_0))+\int_{t_0}^{t}\frac{dL(X(s))}{ds}\geq L(X(t_0))+c_2(t-t_0).$$

Thus $L(X(t)) \to +\infty$ if $t \to +\infty$, a contradiction to the assumption that $L(X)$ is bounded. Whence, there exists $t_1 > t_0$ such that

$$\|X(t_1, t_0)\| > \epsilon_0.$$

Applying this conclusion, we immediately know that the zero G-solution $G[DES_m^1]$ is not sum-stable or prod-stable on H by conditions in (1) or (2). □

Similar to Theorem 3.9, we know results for non-Liapunov sum-function or prod-function by Theorem 3.12 following.

Theorem 3.13 *For a G-solution $G[DES_m^1]$ of a differential equation system (DES_m^1) with initial value $X_v(0)$, let H_1, H_2 be spanning subgraphs of $G[DES_m^1]$, $\overline{0}$ an equilibrium point of (DES_m^1) on $H_1 \bigcup H_2$. Then $R^+(L_1, L_2)$ is a non-Liapunov sum-function or non-Liapunov prod-function on $\overline{0}$ for $H_1 \bigcup H_2$ with conventions for*

$$a_{ij} L_1^i L_2^j \left(\sum_{v \in V(H_1 \bigcup V(H_2)} X_v(t) \right) + a_{kl} L_1^k L_2^l \left(\sum_{v \in V(H_1 \bigcup V(H_2)} X_v(t) \right)$$

and

$$a_{ij} L_1^i L_2^j \left(\prod_{v \in V(H_1 \bigcup V(H_2)} X_v(t) \right) + a_{kl} L_1^k L_2^l \left(\prod_{v \in V(H_1 \bigcup V(H_2)} X_v(t) \right)$$

the same as in Theorem 3.9 if L_1, L_2 are non-Liapunov sum-functions or non-Liapunov prod-functions on $\overline{0}$ for H_1 and H_2, respectively. Particularly, if there is a non-Liapunov sum-function (non-Liapunov prod-function) L on H_1 and H_2, then L is also a non-Liapunov sum-function (non-Liapunov prod-function) on $H_1 \bigcup H_2$.

Proof Similarly, we can show that $R^+(L_1, L_2)$ satisfies these conditions on $H_1 \bigcup H_2$ for non-Liapunov sum-functions or non-Liapunov prod-functions in Theorem 3.12 if L_1, L_2 are non-Liapunov sum-functions or non-Liapunov prod-functions on $\overline{0}$ for H_1 and H_2, respectively. Thus $R^+(L_1, L_2)$ is a non-Liapunov sum-function or non-Liapunov prod-function on $\overline{0}$. □

Corollary 3.14 *For a differential equation system (DES_m^1), let $H < G[DES_m^1]$ be a spanning subgraph. If L_v is a non-Liapunov function on vertex v for $\forall v \in V(H)$, then the functions*

$$L_S^H = \sum_{v \in V(H)} L_v \quad and \quad L_P^H = \prod_{v \in V(H)} L_v$$

are respectively non-Liapunov sum-function and non-Liapunov prod-function on graph H. Particularly, if $L = L_v$ for $\forall v \in V(H)$, then L is both a non-Liapunov sum-function and a non-Liapunov prod-function on H.

Example 3.15 Let (DES_m^1) be

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1^2 - \lambda_1 x_2^2 \\ \dot{x}_2 = \dfrac{\lambda_1}{2} x_1 x_2 \end{cases} \quad \begin{cases} \dot{x}_2 = \lambda_2 x_1^2 - \lambda_2 x_2^2 \\ \dot{x}_2 = \dfrac{\lambda_2}{2} x_1 x_2 \end{cases} \quad \cdots \quad \begin{cases} \dot{x}_1 = \lambda_m x_1^2 - \lambda_m x_2^2 \\ \dot{x}_2 = \dfrac{\lambda_m}{2} x_1 x_2 \end{cases}$$

with constants $\lambda_i > 0$ for integers $1 \leq i \leq m$ and $L(x_1, x_2) = x_1^2 - 2x_2^2$. Then $\dot{L}(x_1, x_2) = 4\lambda_i x_1 L(x_1, x_2)$ for the i-th equation in (DES_m^1). Calculation shows that $L(x_1, x_2) > 0$ if $x_1 > \sqrt{2}x_2$ or $x_1 < -\sqrt{2}x_2$ and $\dot{L}(x_1, x_2) > 4c^{\frac{3}{2}}$ for $L(x_1, x_2) > c$ in the area of $L(x_1, x_2) > 0$. Applying Theorem 3.12, we know the zero solution of (DES_m^1) is not stable for the i-th equation for any integer $1 \leq i \leq m$. Applying Corollary 3.14, we know that L is a non-Liapunov sum-function and non-Liapunov prod-function on any spanning subgraph $H < G[DES_m^1]$.

§4. Global Stability of Shifted Non-Solvable Differential Equations

The differential equation systems (DES_m^1) discussed in previous sections are all in a same Euclidean space $\mathbf{R}^n$. We consider the case that they are not in a same space $\mathbf{R}^n$, i.e., shifted differential equation systems in this section. These differential equation systems and their non-solvability are defined in the following.

Definition 4.1 *A shifted differential equation system $(SDES_m^1)$ is such a differential equation system*

$$\dot{X}_1 = F_1(X_1),\ \dot{X}_2 = F_2(X_2), \cdots, \dot{X}_m = F_m(X_m) \qquad (SDES_m^1)$$

with

$$X_1 = (x_1, x_2, \cdots, x_l, x_{1(l+1)}, x_{1(l+2)}, \cdots, x_{1n}),$$
$$X_2 = (x_1, x_2, \cdots, x_l, x_{2(l+1)}, x_{2(l+2)}, \cdots, x_{2n}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$X_m = (x_1, x_2, \cdots, x_l, x_{m(l+1)}, x_{m(l+2)}, \cdots, x_{mn}),$$

where $x_1, x_2, \cdots, x_l, x_{i(l+j)}, 1 \leq i \leq m, 1 \leq j \leq n-l$ are distinct variables and $F_s : \mathbf{R}^n \to \mathbf{R}^n$ is continuous such that $F_s(\overline{0}) = \overline{0}$ for integers $1 \leq s \leq m$.

A shifted differential equation system $(SDES_m^1)$ is non-solvable if there are integers i, j, $1 \leq i, j \leq m$ and an integer k, $1 \leq k \leq l$ such that $x_k^{[i]}(t) \neq x_k^{[j]}(t)$, where $x_k^{[i]}(t), x_k^{[j]}(t)$ are solutions $x_k(t)$ of the i-th and j-th equations in $(SDES_m^1)$, respectively.

The number $\dim(SDES_m^1)$ of variables $x_1, x_2, \cdots, x_l, x_{i(l+j)}, 1 \leq i \leq m, 1 \leq j \leq n-l$ in Definition 4.1 is uniquely determined by $(SDES_m^1)$, i.e., $\dim(SDES_m^1) = mn - (m-1)l$. For classifying and finding the stability of these differential equations, we similarly introduce the linearized basis graphs $G[SDES_m^1]$ of a shifted differential equation system to that of (DES_m^1), i.e., a vertex-edge labeled graph with

$$V(G[SDES_m^1]) = \{\mathscr{B}_i | 1 \leq i \leq m\},$$
$$E(G[SDES_m^1]) = \{(\mathscr{B}_i, \mathscr{B}_j) | \mathscr{B}_i \bigcap \mathscr{B}_j \neq \emptyset, 1 \leq i, j \leq m\},$$

where $\mathscr{B}_i$ is the solution basis of the i-th linearized differential equation $\dot{X}_i = F_i'(\overline{0})X_i$ for integers $1 \leq i \leq m$, called such a vertex-edge labeled graph $G[SDES_m^1]$ the *G-solution* of $(SDES_m^1)$ and its zero G-solution replaced $\mathscr{B}_i$ by $(0, \cdots, 0)$ ($|\mathscr{B}_i|$ times) and $\mathscr{B}_i \bigcap \mathscr{B}_j$ by $(0, \cdots, 0)$ ($|\mathscr{B}_i \bigcap \mathscr{B}_j|$ times) for integers $1 \leq i, j \leq m$.

Let $(LDES_m^1)$, $(LDES_m^1)'$ be linearized differential equation systems of shifted differential equation systems $(SDES_m^1)$ and $(SDES_m^1)$ with G-solutions H, H'. Similarly, they are called combinatorially equivalent if there is an isomorphism $\varphi: \ H \to H'$ of graph and labelings θ, τ on H and H' respectively such that $\varphi\theta(x) = \tau\varphi(x)$ for $\forall x \in V(H)\bigcup E(H)$, denoted by $(SDES^1_m) \overset{\varphi}{\simeq} (SDES^1_m)'$. Notice that if we remove these superfluous variables from $G[SDES^1_m]$, then we get nothing but the same vertex-edge labeled graph of $(LDES^1_m)$ in $\mathbf{R}^l$. Thus we can classify shifted differential similarly to $(LDES^1_m)$ in $\mathbf{R}^l$. The following result can be proved similarly to Theorem 2.14.

Theorem 4.2 *Let $(LDES^1_m)$, $(LDES^1_m)'$ be linearized differential equation systems of two shifted differential equation systems $(SDES^1_m)$, $(SDES^1_m)'$ with integral labeled graphs H, H'. Then $(SDES^1_m) \overset{\varphi}{\simeq} (SDES^1_m)'$ if and only if $H = H'$.*

The stability of these shifted differential equation systems $(SDES^1_m)$ is also similarly to that of (DES^1_m). For example, we know the results on the stability of $(SDES^1_m)$ similar to Theorems 2.22, 2.27 and 3.6 following.

Theorem 4.3 *Let $(LDES^1_m)$ be a shifted linear differential equation systems and $H < G[LDES^1_m]$ a spanning subgraph. A zero G-solution of $(LDES^1_m)$ is asymptotically sum-stable on H if and only if $\mathrm{Re}\alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ in $(LDES^1)$ hold for $\forall v \in V(H)$ and it is asymptotically prod-stable on H if and only if $\sum\limits_{v\in V(H)} \mathrm{Re}\alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ in $(LDES^1)$.*

Theorem 4.4 *Let $(SDES^1_m)$ be a shifted differential equation system, $H < G[SDES^1_m]$ a spanning subgraph and*

$$F_v(X) = F'_v\left(\overline{0}\right) X + R_v(X)$$

such that

$$\lim_{\|X\|\to\overline{0}} \frac{\|R_v(X)\|}{\|X\|} = 0$$

for $\forall v \in V(H)$. Then the zero G-solution of $(SDES^1_m)$ is asymptotically sum-stable or asymptotically prod-stable on H if $\mathrm{Re}\alpha_v < 0$ for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v,\ v \in V(H)$ in $(SDES^1_m)$.

For the Liapunov sum-function or Liapunov prod-function of a shifted differential equation system $(SDES^1_m)$, we choose it to be a differentiable function $L : \mathscr{O} \subset \mathbf{R}^{\dim(SDES^1_m)} \to \mathbf{R}$ with conditions in Definition 3.7 hold. Then we know the following result similar to Theorem 3.8.

Theorem 4.5 *For a G-solution $G[SDES^1_m]$ of a shifted differential equation system $(SDES^1_m)$ with initial value $X_v(0)$, let H be a spanning subgraph of $G[DES^1_m]$ and X^* an equilibrium point of $(SDES^1_m)$ on H.*

(1) *If there is a Liapunov sum-function $L : \mathscr{O} \subset \mathbf{R}^{\dim(SDES^1_m)} \to \mathbf{R}$ on X^*, then the zero G-solution $G[SDES^1_m]$ is sum-stable on X^* for H, and furthermore, if*

$$\dot{L}\left(\sum_{v\in V(H)} X_v(t)\right) < 0$$

for $\sum\limits_{v\in V(H)} X_v(t) \neq X^*$, then the zero G-solution $G[SDES^1_m]$ is asymptotically sum-stable on X^* for H.

(2) *If there is a Liapunov prod-function* $L : \mathscr{O} \subset \mathbf{R}^{\dim(SDES^1_m)} \to \mathbf{R}$ *on* X^* *for* H*, then the zero G-solution* $G[SDES^1_m]$ *is prod-stable on* X^* *for* H*, and furthermore, if*

$$\dot{L}\left(\prod_{v\in V(H)} X_v(t)\right) < 0$$

for $\prod\limits_{v\in V(H)} X_v(t) \neq X^*$*, then the zero G-solution* $G[SDES^1_m]$ *is asymptotically prod-stable on* X^* *for* H*.*

§5. Applications

5.1 Global Control of Infectious Diseases

An immediate application of non-solvable differential equations is the globally control of infectious diseases with more than one infectious virus in an area. Assume that there are three kind groups in persons at time t, i.e., infected $I(t)$, susceptible $S(t)$ and recovered $R(t)$, and the total population is constant in that area. We consider two cases of virus for infectious diseases:

Case 1 *There are* m *known virus* $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ *with infected rate* k_i*, heal rate* h_i *for integers* $1 \leq i \leq m$ *and an person infected a virus* $\mathscr{V}_i$ *will never infects other viruses* $\mathscr{V}_j$ *for* $j \neq i$*.*

Case 2 *There are* m *varying* $\mathscr{V}_1, \mathscr{V}_2, \cdots, \mathscr{V}_m$ *from a virus* $\mathscr{V}$ *with infected rate* k_i*, heal rate* h_i *for integers* $1 \leq i \leq m$ *such as those shown in Fig.*5.1*.*

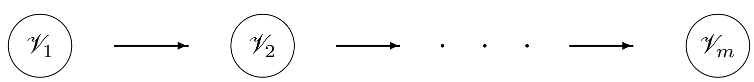


Fig.5.1

We are easily to establish a non-solvable differential model for the spread of infectious viruses by applying the SIR model of one infectious disease following:

$$\begin{cases} \dot{S} = -k_1SI \\ \dot{I} = k_1SI - h_1I \\ \dot{R} = h_1I \end{cases} \quad \begin{cases} \dot{S} = -k_2SI \\ \dot{I} = k_2SI - h_2I \\ \dot{R} = h_2I \end{cases} \quad \cdots \quad \begin{cases} \dot{S} = -k_mSI \\ \dot{I} = k_mSI - h_mI \\ \dot{R} = h_mI \end{cases} \qquad (DES^1_m)$$

Notice that the total population is constant by assumption, i.e., $S + I + R$ is constant. Thus we only need to consider the following simplified system

$$\begin{cases} \dot{S} = -k_1SI \\ \dot{I} = k_1SI - h_1I \end{cases} \quad \begin{cases} \dot{S} = -k_2SI \\ \dot{I} = k_2SI - h_2I \end{cases} \quad \cdots \quad \begin{cases} \dot{S} = -k_mSI \\ \dot{I} = k_mSI - h_mI \end{cases} \qquad (DES^1_m)$$

The equilibrium points of this system are $I = 0$, the S-axis with linearization at equilibrium points

$$\begin{cases}\dot{S}=-k_1S\\ \dot{I}=k_1S-h_1\end{cases}\quad \begin{cases}\dot{S}=-k_2S\\ \dot{I}=k_2S-h_2\end{cases}\quad \cdots \quad \begin{cases}\dot{S}=-k_mS\\ \dot{I}=k_mS-h_m\end{cases}\qquad (LDES_m^1)$$

Calculation shows that the eigenvalues of the ith equation are 0 and k_iS-h_i, which is negative, i.e., stable if $0<S<h_i/k_i$ for integers $1\leq i\leq m$. For any spanning subgraph $H<G[LDES_m^1]$, we know that its zero G-solution is asymptotically sum-stable on H if $0<S<h_v/k_v$ for $v\in V(H)$ by Theorem 2.22, and it is asymptotically sum-stable on H if

$$\sum_{v\in V(H)}(k_vS-h_v)<0 \qquad \text{i.e.,} \qquad 0<S<\sum_{v\in V(H)}h_v\Big/\sum_{v\in V(H)}k_v$$

by Theorem 2.27. Notice that if $I_i(t)$, $S_i(t)$ are probability functions for infectious viruses $\mathscr{V}_i$, $1\leq i\leq m$ in an area, then $\prod\limits_{i=1}^{m}I_i(t)$ and $\prod\limits_{i=1}^{m}S_i(t)$ are just the probability functions for all these infectious viruses. This fact enables one to get the conclusion following for globally control of infectious diseases.

Conclusion 5.1 *For m infectious viruses $\mathscr{V}_1,\mathscr{V}_2,\cdots,\mathscr{V}_m$ in an area with infected rate k_i, heal rate h_i for integers $1\leq i\leq m$, then they decline to 0 finally if*

$$0<S<\sum_{i=1}^{m}h_i\Big/\sum_{i=1}^{m}k_i\ ,$$

i.e., these infectious viruses are globally controlled. Particularly, they are globally controlled if each of them is controlled in this area.

5.2 Dynamical Equations of Instable Structure

There are two kind of engineering structures, i.e., stable and instable. An engineering structure is *instable* if its state moving further away and the equilibrium is upset after being moved slightly. For example, the structure (a) is engineering stable but (b) is not shown in Fig.5.2,

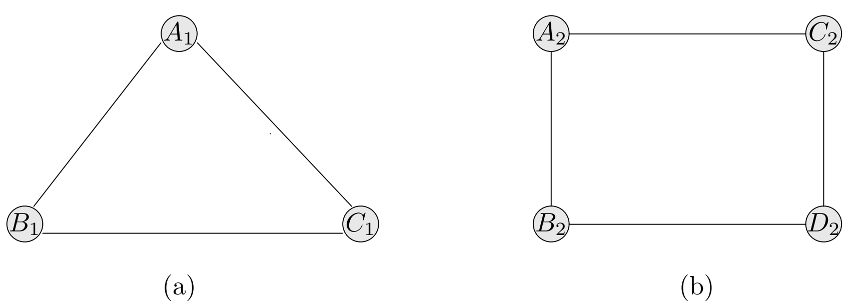


Fig.5.2

where each edge is a rigid body and each vertex denotes a hinged connection. The motion of a stable structure can be characterized similarly as a rigid body. But such a way can not be applied for instable structures for their internal deformations such as those shown in Fig.5.3.

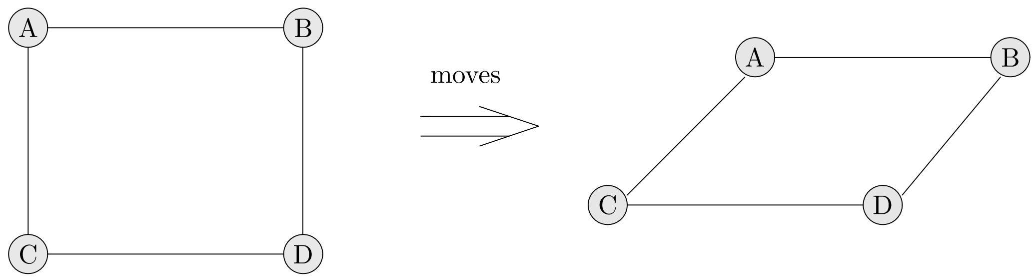


Fig.5.3

Furthermore, let $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m$ be m particles in $\mathbf{R}^3$ with some relations, for instance, the gravitation between particles $\mathscr{P}_i$ and $\mathscr{P}_j$ for $1 \leq i, j \leq m$. Thus we get an instable structure underlying a graph G with

$V(G) = \{\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m\}$;

$E(G) = \{(\mathscr{P}_i, \mathscr{P}_j) | \text{there exists a relation between } \mathscr{P}_i \text{ and } \mathscr{P}_j\}$.

For example, the underlying graph in Fig.5.4 is C_4. Assume the dynamical behavior of particle $\mathscr{P}_i$ at time t has been completely characterized by the differential equations $\dot{X} = F_i(X, t)$, where $X = (x_1, x_2, x_3)$. Then we get a non-solvable differential equation system

$$\dot{X} = F_i(X, t), \quad 1 \leq i \leq m$$

underlying the graph G. Particularly, if all differential equations are autonomous, i.e., depend on X alone, not on time t, we get a non-solvable autonomous differential equation system

$$\dot{X} = F_i(X), \quad 1 \leq i \leq m.$$

All of these differential equation systems particularly answer a question presented in [3] for establishing the graph dynamics, and if they satisfy conditions in Theorems 2.22, 2.27 or 3.6, then they are sum-stable or prod-stable. For example, let the motion equations of 4 members in Fig.5.3 be respectively

$$\text{AB} : \ddot{X}_{AB} = 0; \quad \text{CD} : \ddot{X}_{CD} = 0, \quad \text{AC} : \ddot{X}_{AC} = a_{AC}, \quad \text{BC} : \ddot{X}_{BC} = a_{BC},$$

where X_{AB}, X_{CD}, X_{AC} and X_{BC} denote central positions of members AB, CD, AC, BC and a_{AC}, a_{BC} are constants. Solving these equations enable one to get

$$X_{AB} = c_{AB}t + d_{AB}, \quad X_{AC} = a_{AC}t^2 + c_{AC}t + d_{AC},$$
$$X_{CD} = c_{CD}t + d_{CD}, \quad X_{BC} = a_{BC}t^2 + c_{BC}t + d_{BC},$$

where $c_{AB}, c_{AC}, c_{CD}, c_{BC}, d_{AB}, d_{AC}, d_{CD}, d_{BC}$ are constants. Thus we get a non-solvable differential equation system

$$\ddot{X} = 0; \quad \ddot{X} = 0, \quad \ddot{X} = a_{AC}, \quad \ddot{X} = a_{BC},$$

or a non-solvable algebraic equation system

$$X = c_{AB}t + d_{AB}, \quad X = a_{AC}t^2 + c_{AC}t + d_{AC},$$
$$X = c_{CD}t + d_{CD}, \quad X = a_{BC}t^2 + c_{BC}t + d_{BC}$$

for characterizing the behavior of the instable structure in Fig.5.3 if constants $c_{AB}, c_{AC}, c_{CD}, c_{BC}$, $d_{AB}, d_{AC}, d_{CD}, d_{BC}$ are different.

Now let $X_1, X_2, \cdots, X_m$ be the respectively positions in $\mathbf{R}^3$ with initial values $X_1^0, X_2^0, \cdots, X_m^0$, $\dot{X}_1^0, \dot{X}_2^0, \cdots, \dot{X}_m^0$ and $M_1, M_2, \cdots, M_m$ the masses of particles $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m$. If $m = 2$, then from Newton's law of gravitation we get that

$$\ddot{X}_1 = GM_2 \frac{X_2 - X_1}{|X_2 - X_1|^3}, \qquad \ddot{X}_2 = GM_1 \frac{X_1 - X_2}{|X_1 - X_2|^3},$$

where G is the gravitational constant. Let $X = X_2 - X_1 = (x_1, x_2, x_3)$. Calculation shows that

$$\ddot{X} = -G\left(M_1 + M_2\right) \frac{X}{|X|^3}.$$

Such an equation can be completely solved by introducing the spherical polar coordinates

$$\begin{cases} x_1 = r\cos\phi\cos\theta \\ x_2 = r\cos\phi\cos\theta \\ x_3 = r\sin\theta \end{cases}$$

with $r \geq 0, 0 \leq \phi \leq \pi, 0 \leq \theta < 2\pi$, where $r = \|X\|$, $\phi = \angle Xoz, \theta = \angle X'ox$ with X' the projection of X in the plane xoy are parameters with $r = \alpha/(1 + \epsilon\cos\phi)$ hold for some constants α, ϵ. Whence,

$$X_1(t) = GM_2 \int \left(\int \frac{X}{|X|^3} dt \right) dt \quad \text{and} \quad X_2(t) = -GM_1 \int \left(\int \frac{X}{|X|^3} dt \right) dt.$$

Notice the additivity of gravitation between particles. The gravitational action of particles $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m$ on $\mathscr{P}$ can be regarded as the respective actions of $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_m$ on $\mathscr{P}$, such as those shown in Fig.5.4.

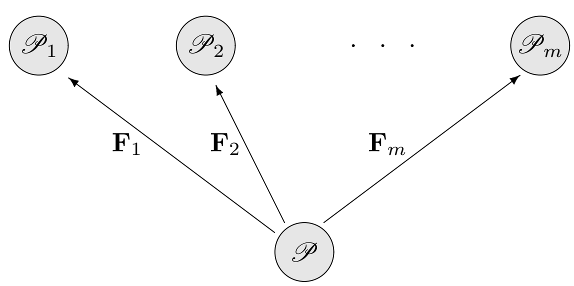


Fig.5.4

Thus we can establish the differential equations two by two, i.e., $\mathscr{P}_1$ acts on $\mathscr{P}$, $\mathscr{P}_2$ acts on $\mathscr{P}$, $\cdots$, $\mathscr{P}_m$ acts on $\mathscr{P}$ and get a non-solvable differential equation system

$$\ddot{X} = GM_i \frac{X_i - X}{|X_i - X|^3}, \qquad \mathscr{P}_i \neq \mathscr{P}, \ \ 1 \leq i \leq m.$$

Fortunately, each of these differential equations in this system can be solved likewise that of $m = 2$. Not loss of generality, assume $\widehat{X}_i(t)$ to be the solution of the differential equation in the case of $\mathscr{P}_i \neq \mathscr{P}, \ 1 \leq i \leq m$. Then

$$X(t) = \sum_{\mathscr{P}_i \neq \mathscr{P}} \widehat{X}_i(t) = G \sum_{\mathscr{P}_i \neq \mathscr{P}} M_i \int \left(\int \frac{X_i - X}{|X_i - X|^3} dt \right) dt$$

is nothing but the position of particle $\mathscr{P}$ at time t in $\mathbf{R}^3$ under the actions of $\mathscr{P}_i \neq \mathscr{P}$ for integers $1 \leq i \leq m$, i.e., its position can be characterized completely by the additivity of gravitational force.

5.3 Global Stability of Multilateral Matters

Usually, one determines the behavior of a matter by observing its appearances revealed before one's eyes. If a matter emerges more lateralities before one's eyes, for instance the different states of a multiple state matter. We have to establish different models, particularly, differential equations for understanding that matter. In fact, each of these differential equations can be solved but they are contradictory altogether, i.e., non-solvable in common meaning. Such a multilateral matter is *globally stable* if these differential equations are sum or prod-stable in all.

Concretely, let $S_1, S_2, \cdots, S_m$ be m lateral appearances of a matter $\mathscr{M}$ in $\mathbf{R}^3$ which are respectively characterized by differential equations

$$\dot{X}_i = H_i(X_i, t), \qquad 1 \leq i \leq m,$$

where $X_i \in \mathbf{R}^3$, a 3-dimensional vector of surveying parameters for $S_i, 1 \leq i \leq m$. Thus we get a non-solvable differential equations

$$\dot{X} = H_i(X, t), \qquad 1 \leq i \leq m \qquad\qquad (DES_m^1)$$

in $\mathbf{R}^3$. Noticing that all these equations characterize a same matter $\mathscr{M}$, there must be equilibrium points X^* for all these equations. Let

$$H_i(X, t) = H_i'(X^*)X + R_i(X^*),$$

where

$$H_i'(X^*) = \begin{bmatrix} h_{11}^{[i]} & h_{12}^{[i]} & \cdots & h_{1n}^{[i]} \\ h_{21}^{[i]} & h_{22}^{[i]} & \cdots & h_{2n}^{[i]} \\ \cdots & \cdots & \cdots & \cdots \\ h_{n1}^{[i]} & h_{n2}^{[i]} & \cdots & h_{nn}^{[i]} \end{bmatrix}$$

is an $n \times n$ matrix. Consider the non-solvable linear differential equation system

$$\dot{X} = H_i'(X^*)X, \qquad 1 \leq i \leq m \qquad\qquad (LDES_m^1)$$

with a basis graph G. According to Theorem 3.6, if

$$\lim_{\|X\|\to X^*}\frac{\|R_i(X)\|}{\|X\|}=0$$

for integers $1\leq i\leq m$, then the G-solution of these differential equations is asymptotically sum-stable or asymptotically prod-stable on G if each $\mathrm{Re}\alpha_k^{[i]}<0$ for all eigenvalues $\alpha_k^{[i]}$ of matrix $H_i'(X^*)$, $1\leq i\leq m$. Thus we therefore determine the behavior of matter $\mathscr{M}$ is globally stable nearly enough X^*. Otherwise, if there exists such an equation which is not stable at the point X^*, then the matter $\mathscr{M}$ is not globally stable. By such a way, if we can determine these differential equations are stable in everywhere, then we can finally conclude that M is globally stable.

Conversely, let $\mathscr{M}$ be a globally stable matter characterized by a non-solvable differential equation

$$\dot{X}=H_i(X,t)$$

for its laterality S_i, $1\leq i\leq m$. Then the differential equations

$$\dot{X}=H_i(X,t),\qquad 1\leq i\leq m \qquad (DES_m^1)$$

are sum-stable or prod-stable in all by definition. Consequently, we get a sum-stable or prod-stable non-solvable differential equation system.

Combining all of these previous discussions, we get an interesting conclusion following.

Conclusion 5.2 *Let $\mathscr{M}^{GS},\overline{\mathscr{M}}^{GS}$ be respectively the sets of globally stable multilateral matters, non-stable multilateral matters characterized by non-solvable differential equation systems and $\mathscr{DE},\overline{\mathscr{DE}}$ the sets of sum or prod-stable non-solvable differential equation systems, not sum or prod-stable non-solvable differential equation systems. then*

(1) $\forall\mathscr{M}\in\mathscr{M}^{GS}\Rightarrow\exists(DES_m^1)\in\mathscr{DE}$;

(2) $\forall\mathscr{M}\in\overline{\mathscr{M}}^{GS}\Rightarrow\exists(DES_m^1)\in\overline{\mathscr{DE}}$.

Particularly, let $\mathscr{M}$ be a multiple state matter. If all of its states are stable, then $\mathscr{M}$ is globally stable. Otherwise, it is unstable.

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m^{th}-Root Randers Change of a Finsler Metric

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Abstract: In this paper, we introduce a m^{th}-root Randers changed Finsler metric as

$$\bar{L}(x,y) = L(x,y) + \beta(x,y),$$

where $L = \{a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2} \cdots y^{i_m}\}^{\frac{1}{m}}$ is a m^{th}-root metric and β-is one form. Further we obtained the relation between the v- and hv- curvature tensor of m^{th}-root Finsler space and its m^{th}-root Randers changed Finsler space and obtained some theorems for its S3 and S4-likeness of Finsler spaces and when this changed Finsler space will be Berwald space (resp. Landsberg space). Also we obtain T-tensor for the m^{th}-root Randers changed Finsler space $\bar{F}^n$.

Key Words: Randers change, m^{th}-root metric, Berwald space, Landsberg space, S3 and S4-like Finsler space.

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§1. Introduction

Let $F^n = (M^n, L)$ be a n-dimensional Finsler space, whose M^n is the n-dimensional differentiable manifold and $L(x,y)$ is the Finsler fundamental function. In general, $L(x,y)$ is a function of point $x = (x^i)$ and element of support $y = (y^i)$, and positively homogeneous of degree one in y. In the year 1971 Matsumoto [6] introduced the transformations of Finsler metric given by

$$L^{'}(x,y) = L(x,y) + \beta(x,y)$$
$$L^{''2}(x,y) = L^2(x,y) + \beta^2(x,y),$$

where, $\beta = b_i(x)y^i$ is a one-form [1] and $b_i(x)$ are components of covariant vector which is a function of position alone. If $L(x,y)$ is a Riemannian metric, then the Finsler space with a metric $L(x,y) = \alpha(x,y) + \beta(x,y)$ is known as Randers space which is introduced by G.Randers [5]. In papers [3, 7, 8, 9], Randers spaces have been studied from a geometrical viewpoint and various theorem were obtained. In 1978, Numata [10] introduced another β-change of Finsler metric given by $L(x,y) = \mu(x,y) + \beta(x,y)$ where $\mu = \{a_{ij}(y)y^i y^j\}^{\frac{1}{2}}$ is a Minkowski metric and β as above. This metric is of similar form of Randers one, but there are different tensor

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properties, because the Riemannian space with the metric α is characterized by $C^i_{jk} = 0$ and on the other hand the locally Minkowski space with the metric μ by $R_{hijk} = 0,\ C_{hij|k} = 0$.

In the year 1979, Shimada [4] introduced the concept of m^{th} root metric and developed it as an interesting example of Finsler metrics, immediately following M.Matsumoto and S.Numatas theory of cubic metrics [2]. By introducing the regularity of the metric various fundamental quantities as a Finsler metric could be found. In particular, the Cartan connection of a Finsler space with m-th root metric could be discussed from the theoretical standpoint. In 1992-1993, the m-th root metrics have begun to be applied to theoretical physics [11, 12], but the results of the investigations are not yet ready for acceding to the demands of various applications.

In the present paper we introduce a m^{th}-root Randers changed Finsler metric as

$$\bar{L}(x,y) = L(x,y) + \beta(x,y)$$

where $L = \{a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2 \cdots y^{i_m}}\}^{\frac{1}{m}}$ is a m^{th}-root metric. This metric is of the similar form to the Randers one in the sense that the Riemannian metric is replaced with the m^{th}-root metric, due to this we call this change as m^{th}-root Randers change of the Finsler metric. Further we obtained the relation between the v-and hv-curvature tensor of m^{th}-root Finsler space and its m^{th}-root Randers changed Finsler space and obtained some theorems for its S3 and S4-likeness of Finsler spaces and when this changed Finsler space will be Berwald space (resp. Landsberg space). Also we obtain T-tensor for the m^{th}-root Randers changed Finsler space $\bar{F}^n$.

§2. The Fundamental Tensors of $\bar{F}^n$

We consider an n-dimensional Finsler space $\bar{F}^n$ with a metric $\bar{L}(x,y)$ given by

$$\bar{L}(x,y) = L(x,y) + b_i(x) y^i \tag{1}$$

where

$$L = \{a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2} \cdots y^{i_m}\}^{\frac{1}{m}} \tag{2}$$

By putting

$$\begin{aligned} &(\text{I}).\quad L^{m-1} a_i(x,y) \quad = a_{i i_2 \cdots i_m}(x) y^{i_2} y^{i_3} \cdots y^{i_m} \\ &(\text{II}).\ \ L^{m-2} a_{ij}(x,y) \ = a_{ij i_3 i_4 \cdots i_m}(x) y^{i_3} y^{i_4} \cdots y^{i_m} \\ &(\text{III}).\ L^{m-3} a_{ijk}(x,y) = a_{ijk i_4 i_5 \cdots i_m}(x) y^{i_4} y^{i_5} \cdots y^{i_m} \end{aligned} \tag{3}$$

Now differentiating equation (1) with respect to y^i, we get the normalized supporting element $\bar{l}_i = \dot{\partial}_i \bar{L}$ as

$$\bar{l}_i = a_i + b_i \tag{4}$$

where $a_i = l_i$ is the normalized supporting element for the m^{th}-root metric. Again differentiating above equation with respect to y^j, the angular metric tensor $\bar{h}_{ij} = \bar{L}\dot{\partial}_i\dot{\partial}_j\bar{L}$ is given as

$$\frac{\bar{h}_{ij}}{\bar{L}} = \frac{h_{ij}}{L} \tag{5}$$

where h_{ij} is the angular metric tensor of m^{th}-root Finsler space with metric L given by [4]

$$h_{ij} = (m-1)(a_{ij} - a_i a_j) \tag{6}$$

The fundamental metric tensor $\bar{g}_{ij} = \dot{\partial}_i \dot{\partial}_j \frac{\bar{L}^2}{2} = \bar{h}_{ij} + \bar{l}_i \bar{l}_j$ of Finsler space F^n are obtained from equations (4), (5) and (6), which is given by

$$\bar{g}_{ij} = (m-1)\tau a_{ij} + \{1-(m-1)\tau\} a_i a_j + (a_i b_j + a_j b_i) + b_i b_j \tag{7}$$

where $\tau = \frac{\bar{L}}{L}$. It is easy to show that

$$\dot{\partial}_i \tau = \frac{\{(1-\tau)a_i + b_i\}}{L}, \quad \dot{\partial}_j a_i = \frac{(m-1)(a_{ij} - a_i a_j)}{L}, \quad \dot{\partial}_k a_{ij} = \frac{(m-2)(a_{ijk} - a_{ij} a_k)}{L}$$

Therefore from (7), it follows (h)hv-torsion tensor $\bar{C}_{ijk} = \dot{\partial}_k \frac{\bar{g}_{ij}}{2}$ of the Cartan's connection $C\Gamma$ are given by

$$\begin{aligned} 2L\bar{C}_{ijk} &= (m-1)(m-2)\tau a_{ijk} + [\{1-(m-1)\tau\}(m-1)](a_{ij} a_k \\ &+ a_{jk} a_i + a_{ki} a_j) + (m-1)(a_{ij} b_k + a_{jk} b_i + a_{ki} b_j) - \\ &(m-1)(a_i a_j b_k + a_j a_k b_i + a_i a_k b_j) + (m-1)\{(2m-1)\tau - 3\} a_i a_j a_k \end{aligned} \tag{8}$$

In view of equation (6) the equation (8) may be written as

$$\bar{C}_{ijk} = \tau C_{ijk} + \frac{(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j)}{2L} \tag{9}$$

where $m_i = b_i - \frac{\beta}{L} a_i$ and C_{ijk} is the (h)hv-torsion tensor of the Cartan's connection $C\Gamma$ of the m^{th}-root Finsler metric L given by

$$2LC_{ijk} = (m-1)(m-2)\{a_{ijk} - (a_{ij} a_k + a_{jk} a_i + a_{ki} a_j) + 2 a_i a_j a_k\} \tag{10}$$

Let us suppose that the intrinsic metric tensor $a_{ij}(x, y)$ of the m^{th}-root metric L has non-vanishing determinant. Then the inverse matrix (a^{ij}) of (a_{ij}) exists. Therefore the reciprocal metric tensor $\bar{g}^{ij}$ of $\bar{F}^n$ is obtain from equation (7) which is given by

$$\bar{g}^{ij} = \frac{1}{(m-1)\tau} a^{ij} + \frac{b^2 + (m-1)\tau - 1}{(m-1)\tau(1+q)^2} a^i a^j - \frac{(a^i b^j + a^j b^i)}{(m-1)\tau(1+q)} \tag{11}$$

where $a^i = a^{ij} a_j$, $b^i = a^{ij} b_j$, $b^2 = b^i b_i$, $q = a^i b_i = a_i b^i = \beta/L$.

Proposition 2.1 *The normalized supporting element l_i, angular metric tensor h_{ij}, metric tensor g_{ij} and (h)hv-torsion tensor C_{ijk} of Finsler space with m^{th}-root Randers changed metric are given by (4), (5), (7) and (9) respectively.*

§3. The v-Curvature Tensor of $\bar{F}^n$

From (6), (10) and definition of m_i and a^i, we get the following identities

$$\begin{aligned} &a^i a_i = 1, \quad a_{ijk} a^i = a_{jk}, \quad C_{ijk} a^i = 0, \quad h_{ij} a^i = 0, \\ &m_i a^i = 0, \quad h_{ij} b^j = 3m_i, \quad m_i b^i = (b^2 - q^2) \end{aligned} \tag{12}$$

To find the v-curvature tensor of F^n, we first find (h)hv-torsion tensor $\bar{C}^i_{jk} = \bar{g}^{ir}\bar{C}_{jrk}$

$$\begin{aligned} \bar{C}^i_{jk} &= \frac{1}{m-1}C^i_{jk} + \frac{1}{2(m-1)\bar{L}}(h^i_j m_k + h^i_k m_j + h_{jk}m^i) - \\ & \frac{a^i}{\bar{L}(1+q)}\{m_j m_k + \frac{1}{(m-1)(m-2)}h_{jk}\} - \frac{1}{(m-1)(1+q)}a^i C_{jrk}b^r \end{aligned} \quad (13)$$

where $LC^i_{jk} = LC_{jrk}a^{ir} = (m-1)\{a^i_{jk} - (\delta^i_j a_k + \delta^i_k a_j + a^i a_{jk}) + 2a^i a_j a_k\}$,

$$\begin{aligned} h^i_j &= h_{jr}a^{ir} = (m-1)(\delta^i_j - a^i a_j) \\ m^i &= m_r a^{ir} = b^i - qa^i, \quad \textit{and} \quad a^i_{jk} = a^{ir}a_{jrk} \end{aligned} \quad (14)$$

From (12) and (14), we have the following identities

$$\begin{aligned} & C_{ijr}h^r_p = C^r_{ij}h_{pr} = (m-1)C_{ijp}, \quad C_{ijr}m^r = C_{ijr}b^r, \\ & m_r h^r_i = (m-1)m_i, \quad m_i m^i = (b^2 - q^2), \\ & h_{ir}h^r_j = (m-1)h_{ij}, \quad h_{ir}m^r = (m-1)m_i \end{aligned} \quad (15)$$

From (9) and (13), we get after applying the identities (15)

$$\begin{aligned} \bar{C}_{ijr}\bar{C}^r_{hk} &= \frac{\tau}{(m-1)}C_{ijr}C^r_{hk} + \frac{1}{2L}(C_{ijh}m_k + C_{ijk}m_h + C_{hjk}m_i + C_{hik}m_j) \\ & + \frac{1}{2(m-1)}(C_{ijr}h_{hk} + C_{hrk}h_{ij})b^r + \frac{1}{4(m-1)L\bar{L}}(b^2 - q^2)h_{ij}h_{hk} \\ & + \frac{1}{4L\bar{L}}(2h_{ij}m_h m_k + 2h_{kh}m_i m_j + h_{jh}m_i m_k \\ & + h_{jk}m_i m_h + h_{ih}m_j m_k + h_{ik}m_j m_h) \end{aligned} \quad (16)$$

Now we shall find the v-curvature tensor $\bar{S}_{hijk} = \bar{C}_{ijr}\bar{C}^r_{hk} - \bar{C}_{ikr}\bar{C}^r_{hj}$. The tensor is obtained from (16) and given by

$$\begin{aligned} \bar{S}_{hijk} &= \Theta_{(jk)}\{\frac{\tau}{m-1}C_{ijr}C^r_{hk} + h_{ij}m_{hk} + h_{hk}m_{ij}\} \\ &= \frac{\tau}{(m-1)}S_{hijk} + \Theta_{(jk)}\{h_{ij}m_{hk} + h_{hk}m_{ij}\} \end{aligned} \quad (17)$$

where

$$m_{ij} = \frac{1}{2(m-1)L}\{C_{ijr}b^r + \frac{(b^2 - q^2)}{4\bar{L}}h_{ij} + \frac{(m-1)}{2}\bar{L}^{-1}m_i m_j\} \quad (18)$$

and the symbol $\Theta_{(jk)}\{\cdots\}$ denotes the exchange of j, k and subtraction.

Proposition 3.1 *The v-curvature tensor $\bar{S}_{hijk}$ of m^{th}-root Randers changed Finsler space $\bar{F}^n$ with respect to Cartan's connection $C\Gamma$ is of the form (17).*

It is well known [13] that the v-curvature tensor of any three-dimensional Finsler space is of the form

$$L^2 S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}) \quad (19)$$

Owing to this fact M. Matsumoto [13] defined the S3-like Finsler space F^n $(n \geq 3)$ as such a Finsler space in which v-curvature tensor is of the form (19). The scalar S in (19) is a function of x alone.

The v-curvature tensor of any four-dimensional Finsler space may be written as [13]

$$L^2 S_{hijk} = \Theta_{(jk)}\{h_{hj}K_{ki} + h_{ik}K_{hj}\} \tag{20}$$

where K_{ij} is a (0, 2) type symmetric Finsler tensor field which is such that $K_{ij}y^j = 0$. A Finsler space $F^n(n \geq 4)$ is called S4-like Finsler space [13] if its v-curvature tensor is of the form (20).

From (17), (19), (20) and (5) we have the following theorems.

Theorem 3.1 *The m^{th}-root Randers changed S3-like or S4-like Finsler space is S4-like Finsler space.*

Theorem 3.2 *If v-curvature tensor of m^{th}-root Randers changed Finsler space $\bar{F}^n$ vanishes, then the Finsler space with m^{th}-root metric F^n is S4-like Finsler space.*

If v-curvature tensor of Finsler space with m^{th}-root metric F^n vanishes then equation (17) reduces to

$$\bar{S}_{hijk} = h_{ij}m_{hk} + h_{hk}m_{ij} - h_{ik}m_{hj} - h_{hj}m_{ik} \tag{21}$$

By virtue of (21) and (11) and the Ricci tensor $\bar{S}_{ik} = \bar{g}^{hk}\bar{S}_{hijk}$ is of the form

$$\bar{S}_{ik} = (-\frac{1}{(m-1)\tau})\{mh_{ik} + (m-1)(n-3)m_{ik}\},$$

where $m = m_{ij}a^{ij}$, which in view of (18) may be written as

$$\bar{S}_{ik} + H_1 h_{ik} + H_2 C_{ikr}b^r = H_3 m_i m_k, \tag{22}$$

where

$$H_1 = \frac{m}{(m-1)\tau} + \frac{(n-3)(b^2-q^2)}{8(m-1)\bar{L}^2},$$
$$H_2 = \frac{(n-3)}{2(m-1)\bar{L}},$$
$$H_3 = -\frac{(n-3)}{2\bar{L}^2}.$$

From (22), we have the following

Theorem 3.3 *If v-curvature tensor of m^{th}-root Randers changed Finsler space $\bar{F}^n$ vanishes then there exist scalar H_1 and H_2 in Finsler space with m^{th}-root metric $F^n(n \geq 4)$ such that matrix $\parallel \bar{S}_{ik} + H_1 h_{ik} + H_2 C_{ikr}b^r \parallel$ is of rank two.*

§4. The $(v)hv$-Torsion Tensor and hv-Curvature Tensor of $\bar{F}^n$

Now we concerned with $(v)hv$-torsion tensor P_{ijk} and hv-curvature tensor P_{hijk}. With respect to the Cartan connection $C\Gamma$, $L_{|i} = 0$, $l_{i|j} = 0$, $h_{ij|k} = 0$ hold good [13].

Taking h-covariant derivative of equation (9) and using (4) and $l_i = a_i = 0$ we have

$$\bar{C}_{ijk|h} = \tau C_{ijk|h} + \frac{b_{i|h}}{L}C_{ijk} + \frac{(h_{ij}b_{k|h} + h_{jk}b_{i|h} + h_{ki}b_{j|h})}{2L} \tag{23}$$

From equation (6) and using relation $h_{ij|h} = 0$ We have

$$a_{ij|h} = 0, \quad \text{and} \quad a_{ijk|h} = \frac{2LC_{ijk|h}}{(m-1)(m-2)} \tag{24}$$

The $(v)hv$-torsion tensor P_{ijk} and the hv-curvature tensor P_{hijk} of the Cartan connection $C\Gamma$ are written in the form, respectively

$$P_{ijk} = C_{ijk|0}, \tag{25}$$

$$P_{hijk} = C_{ijk|h} - C_{hjk|i} + P_{ikr}C^r_{jk} - P_{hkr}C^r_{ji}$$

where the subscript '0' means the contraction for the supporting element y^i. Therefore the $(v)hv$-torsion tensor $\bar{P}_{ijk}$ and the hv-curvature tensor $\bar{P}_{hijk}$ of the Cartan connection $C\Gamma$ for the Finsler space with m^{th}-root Randers metric by using (10), (23), (24) and (25) we have

$$\bar{P}_{ijk} = \frac{(m-1)(m-2)}{2L}\tau a_{ijk|0} + \frac{b_{i|0}}{L}C_{ijk} + \frac{(h_{ij}b_{k|0} + h_{jk}b_{i|0} + h_{ki}b_{j|0})}{2L} \tag{26}$$

and

$$\bar{P}_{hijk} = (m-1)(m-2)(2L)^{-1}\Theta_{(jk)}(a_{ijk|h} + \bar{P}_{ikr}\bar{C}^r_{jh}) \tag{27}$$

Definition 4.1([13]) *A Finsler space is called a Berwald space (resp. Landsberg space) if $C_{ijk|h} = 0$ (resp. $P_{ijk} = 0$) holds good.*

Consequently, from (24) and (26) we have

Theorem 4.1 *A Finsler space with the m^{th}-root Randers changed metric is a Berwald space (resp. Landsberg space), if and only if $a_{ijk|h} = 0$ (resp. $a_{ijk|0} = 0$ and $b_{i|h}$ is covariently constant.*

Proposition 4.1 *The $v(hv)$-torsion tensor and hv-curvature tensor $\bar{P}_{hijk}$ of m^{th}-root Randers changed Finsler space $\bar{F}^n$ with respect to Cartan's connection $C\Gamma$ is of the form (26) and (27).*

§5. T-Tensor of $\bar{F}^n$

Now, the T-tensor is given by [11,13]

$$T_{hijk} = LC_{hij}|_k + l_iC_{hjk} + l_jC_{hik} + l_kC_{hij} + l_hC_{ijk}$$

The above equation for m^{th}-root Randers changed Finsler space $\bar{F}^n$ is given as

$$\bar{T}_{hijk} = \bar{L}\bar{C}_{hij}|_k + \bar{l}_i\bar{C}_{hjk} + \bar{l}_j\bar{C}_{hik} + \bar{l}_k\bar{C}_{hij} + \bar{l}_h\bar{C}_{ijk} \tag{28}$$

The v-derivative of h_{ij} and L is given by [13]

$$h_{ij}|_k = -\frac{1}{L}(h_{ik}l_j + h_{jk}l_i), \ \ \textit{and} \ \ L|_i = l_i \tag{29}$$

Now using (29), the v-derivative of C_{ijk} is given as

$$\begin{aligned}\bar{L}\bar{C}_{ijk}|_h &= \tau\frac{(Lb_h-\beta l_h)}{L}C_{ijk}+\bar{L}\tau C_{ijk}|_h-\tau\frac{1}{2L}(h_{ih}l_jm_k+h_{jh}l_im_k \\ &+h_{jh}l_km_i+h_{kh}l_jm_i+h_{ih}l_km_j+h_{kh}l_im_j+h_{ij}l_hm_k \\ &+h_{jk}l_hm_i+h_{ki}l_hm_j)+\frac{\tau}{2}(h_{ij}m_k|_h+h_{jk}m_i|_h+h_{ki}m_j|_h)\end{aligned} \tag{30}$$

Using (4), (9) and (30), the T-tensor for m^{th}-root Randers changed Finsler space $\bar{F}^n$ is given by

$$\begin{aligned}\bar{T}_{hijk} &= \tau(T_{hijk}+B_{hijk})+\frac{\tau}{2L}(h_{jk}m_hl_i+h_{ik}m_hl_j+h_{ij}m_hl_k \\ &+h_{ki}m_jl_h)+\frac{1}{2L}(h_{hj}m_kb_i+h_{jk}m_hb_i+h_{kh}m_jb_i+h_{ik}m_hb_j \\ &+h_{ih}m_kb_j+h_{kh}m_ib_j+h_{ij}m_hb_k+h_{ih}m_jb_k+h_{jh}m_ib_k+h_{ij}m_kb_h \\ &+h_{ki}m_jb_h+h_{jk}m_ib_h)+\frac{\tau}{2}(h_{ij}m_k|_h+h_{jk}m_i|_h+h_{ki}m_j|_h) \\ &+\tau\frac{(Lb_h-\beta l_h)}{L}C_{ijk}\end{aligned} \tag{31}$$

where $B_{hijk}=\beta C_{hij}|_k+b_iC_{hjk}+b_jC_{hik}+b_kC_{hij}+b_hC_{ijk}$. Thus, we know

Proposition 5.1 *The T-tensor $\bar{T}_{hijk}$ for m^{th}-root Randers changed Finsler space $\bar{F}^n$ is given by (31).*

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Quarter-Symmetric Metric Connection On Pseudosymmetric Lorentzian $\alpha-$Sasakian Manifolds

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Abstract: The object of this paper is to introduce a quarter-symmetric metric connection in a pseudosymmetric Lorentzian α-Sasakian manifold and to study of some properties of it. Also we shall discuss some properties of the Weyl-pseudosymmetric Lorentzian $\alpha-$Sasakian manifold and Ricci-pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with respet to quarter-symmetric metric connection. We have given an example of pseudosymmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection.

Key Words: Lorentzian α-Sasakian manifold, quarter-symmetric metric connection, pseudosymmetric Lorentzian $\alpha-$Sasakian manifolds, Ricci-pseudosymmetric, Weyl-pseudosymmetric, $\eta-$Einstein manifold.

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§1. Introduction

The theory of pseudosymmetric manifold has been developed by many authors by two ways. One is the Chaki sense [8], [3] and another is Deszcz sense [2], [9], [11]. In this paper we shall study some properties of pseudosymmetric and Ricci-symmetric Lorentzian $\alpha-$ Sasakian manifolds with respect to quarter-symmetric metric connection in Deszcz sense. The notion of pseudo-symmetry is a natural generalization of semi-symmetry, along the line of spaces of constant sectional curvature and locally symmetric space.

A Riemannian manifold (M, g) of dimension n is said to be pseudosymmetric if the Riemannian curvature tensor R satisfies the conditions ([1]):

1. $$(R(X,Y).R)(U,V,W) = L_R[((X \wedge Y).R)(U,V,W)] \tag{1}$$

for all vector fields X, Y, U, V, W on M , where$L_R \in C^\infty(M)$, $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ and $X \wedge Y$ is an endomorphism defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \tag{2}$$

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$$\begin{aligned}
2.\ (R(X,Y).R)(U,V,W) &= R(X,Y)(R(U,V)W) - R(R(X,Y)U,V)W \\
&\quad -R(U,R(X,Y)V)W - R(U,V)(R(X,Y)W) \qquad (3)\\
3.\ ((X\wedge Y).R)(U,V,W) &= (X\wedge Y)(R(U,V)W) - R((X\wedge Y)U,V)W \\
&\quad -R(U,(X\wedge Y)V)W - R(U,V)((X\wedge Y)W). \qquad (4)
\end{aligned}$$

M is said to be pseudosymmetric of constant type if L is constant. A Riemannian manifold (M,g) is called semi-symmetric if $R.R=0$, where $R.R$ is the derivative of R by R.

Remark 1.1 We know, the $(0,k+2)$ tensor fields $R.T$ and $Q(g,T)$ are defined by

$$\begin{aligned}
(R.T)(X_1,\cdots,X_k;X,Y) &= (R(X,Y).T)(X_1,\cdots,X_k) \\
&= -T(R(X,Y)X_1,\cdots,X_k) - \cdots - T(X_1,\cdots,R(X,Y)X_k) \\
Q(g,T)(X_1,\cdots,X_k;X,Y) &= -((X\wedge Y).T)(X_1,\cdots,X_k) \\
&= T((X\wedge Y)X_1,\cdots,X_k) + \cdots + T(X_1,\cdots,(X\wedge Y)X_k),
\end{aligned}$$

where T is a $(0,k)$ tensor field ([4],[5]).

Let S and r denote the Ricci tensor and the scalar curvature tensor of M respectively. The operator Q and the $(0,2)-$tensor S^2 are defined by

$$S(X,Y) = g(QX,Y) \qquad (5)$$

and

$$S^2(X,Y) = S(QX,Y) \qquad (6)$$

The Weyl conformal curvature operator C is defined by

$$C(X,Y) = R(X,Y) - \frac{1}{n-2}[X\wedge QY + QX\wedge Y - \frac{r}{n-1}X\wedge Y]. \qquad (7)$$

If $C=0$, $n\geq 3$ then M is called conformally flat. If the tensor $R.C$ and $Q(g,C)$ are linearly dependent then M is called Weyl-pseudosymmetric. This is equivalent to

$$R.C(U,V,W;X,Y) = L_C[((X\wedge Y).C)(U,V)W], \qquad (8)$$

holds on the set $U_C = \{x\in M : C\neq 0 \text{ at } x\}$, where L_C is defined on U_C. If $R.C=0$, then M is called Weyl-semi-symmetric. If $\nabla C=0$, then M is called conformally symmetric ([6],[10]).

§2. Preliminaries

A n-dimensional differentiable manifold M is said to be a Lorentzian $\alpha-$Sasakian manifold if it admits a $(1,1)-$tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy the following conditions,

$$\phi^2 = I + \eta\otimes\xi, \qquad (9)$$

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{10}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{11}$$

$$g(X, \xi) = \eta(X) \tag{12}$$

and

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + \eta(Y)X\} \tag{13}$$

for $\forall X, Y \in \chi(M)$ and for smooth functions α on M, ∇ denotes covariant differentiation operator with respect to Lorentzian metric g ([6], [7]).

For a Lorentzian $\alpha-$Sasakian manifold,it can be shown that ([6],[7]):

$$\nabla_X \xi = \alpha\phi X, \tag{14}$$

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y). \tag{15}$$

Further on a Lorentzian $\alpha-$Sasakian manifold, the following relations hold ([6])

$$\eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{16}$$

$$R(\xi, X)Y = \alpha^2[g(Y, Z)\xi - \eta(Y)X], \tag{17}$$

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{18}$$

$$S(\xi, X) = S(X, \xi) = (n-1)\alpha^2\eta(X), \tag{19}$$

$$S(\xi, \xi) = -(n-1)\alpha^2, \tag{20}$$

$$Q\xi = (n-1)\alpha^2\xi. \tag{21}$$

The above relations will be used in following sections.

§3. Quarter-Symmetric Metric Connection on Lorentzian $\alpha-$Sasakian Manifold

Let M be a Lorentzian α-Sasakian manifold with Levi-Civita connection ∇ and $X, Y, Z \in \chi(M)$. We define a linear connection D on M by

$$D_X Y = \nabla_X Y + \eta(Y)\phi(X) \tag{22}$$

where η is $1-$form and ϕ is a tensor field of type $(1, 1)$. D is said to be quarter-symmetric connection if $\bar{T}$, the torsion tensor with respect to the connection D, satisfies

$$\bar{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{23}$$

D is said to be metric connection if

$$(D_X g)(Y, Z) = 0. \tag{24}$$

A linear connection D is said to be quarter-symmetric metric connection if it satisfies (22), (23) and (24).

Now we shall show the existence of the quarter-symmetric metric connection D on a Lorentzian $\alpha-$Sasakian manifold M.

Theorem 3.1 *Let X, Y, Z be any vectors fields on a Lorentzian $\alpha-$Sasakian manifold M and let a connection D is given by*

$$\begin{aligned} 2g(D_XY, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) \\ &\quad -g([Y, Z], X) + g([Z, X], Y) + g(\eta(Y)\phi X - \eta(X)\phi Y, Z) \\ &\quad +g(\eta(X)\phi Z - \eta(Z)\phi X, Y) + g(\eta(Y)\phi Z - \eta(Z)\phi Y, X). \end{aligned} \tag{25}$$

Then D is a quarter-symmetric metric connection on M.

Proof It can be verified that $D : (X, Y) \to D_XY$ satisfies the following equations:

$$D_X(Y + Z) = D_XY + D_XZ, \tag{26}$$

$$D_{X+Y}Z = D_XZ + D_YZ, \tag{27}$$

$$D_{fX}Y = fD_XY, \tag{28}$$

$$D_X(fY) = f(D_XY) + (Xf)Y \tag{29}$$

for all $X, Y, Z \in \chi(M)$ and for all f, differentiable function on M.

From (26), (27), (28) and (29), we can conclude that D is a linear connection on M. From (25) we have,

$$g(D_XY, Z) - g(D_YX, Z) = g([X, Y], Z) + \eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)$$

or,

$$D_XY - D_YX - [X, Y] = \eta(Y)\phi X - \eta(X)\phi Y$$

or,

$$\bar{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y \tag{30}$$

Again from (25) we get,

$$2g(D_XY, Z) + 2g(D_XZ, Y) = 2Xg(Y, Z), \quad \text{or,} \quad (D_Xg)(Y, Z) = 0.$$

This shows that D is a quarter-symmetric metric connection on M. □

§4. Curvature Tensor and Ricci Tensor with Respect to Quarter-Symmetric Metric Connection D in a Lorentzian $\alpha-$Sasakian Manifold

Let $\bar{R}(X, Y)Z$ and $R(X, Y)Z$ be the curvature tensors with respect to the quarter-symmetric metric connection D and with respect to the Riemannian connection ∇ respectively on a Lorentzian $\alpha-$ Sasakian manifold M. A relation between the curvature tensors $\bar{R}(X, Y)Z$ and $R(X, Y)Z$ on M is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + \alpha[g(\phi X, Z)\phi Y \\ &\quad -g(\phi Y, Z)\phi X] + \alpha\eta(Z)[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{31}$$

Also from (31), we obtain

$$\bar{S}(X, Y) = S(X, Y) + \alpha[g(X, Y) + n\eta(X)\eta(Y)], \tag{32}$$

where $\bar{S}$ and S are the Ricci tensors of the connections D and ∇ respectively.

Again

$$\begin{aligned}\bar{S}^2(X,Y) &= S^2(X,Y) - \alpha(n-2)S(X,Y) - \alpha^2(n-1)g(X,Y) \\ &\quad +\alpha^2 n(n-1)(\alpha-1)\eta(X)\eta(Y). \end{aligned} \tag{33}$$

Contracting (32), we get

$$\bar{r} = r, \tag{34}$$

where $\bar{r}$ and r are the scalar curvature with respect to the connection D and ∇ respectively.

Let $\bar{C}$ be the conformal curvature tensors on Lorentzian $\alpha-$ Sasakian manifolds with respect to the connections D. Then

$$\begin{aligned}\bar{C}(X,Y)Z &= \bar{R}(X,Y)Z - \frac{1}{n-2}[\bar{S}(Y,Z)X - g(X,Z)\bar{Q}Y + g(Y,Z)\bar{Q}X \\ &\quad -\bar{S}(X,Z)Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{35}$$

where $\bar{Q}$ is Ricci operator with the connection D on M and

$$\bar{S}(X,Y) = g(\bar{Q}X,Y), \tag{36}$$

$$\bar{S}^2(X,Y) = \bar{S}(\bar{Q}X,Y). \tag{37}$$

Now we shall prove the following theorem.

Theorem 4.1 *Let M be a Lorentzian $\alpha-$Sasakian manifold with respect to the quarter-symmetric metric connection D, then the following relations hold:*

$$\bar{R}(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X] + \alpha\eta(Y)[X + \eta(X)\xi], \tag{38}$$

$$\eta(\bar{R}(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{39}$$

$$\bar{R}(X,Y)\xi = (\alpha^2-\alpha)[\eta(Y)X - \eta(X)Y], \tag{40}$$

$$\bar{S}(X,\xi) = \bar{S}(\xi,X) = (n-1)(\alpha^2-\alpha)\eta(X), \tag{41}$$

$$\bar{S}^2(X,\xi) = \bar{S}^2(\xi,X) = \alpha^2(n-1)^2(\alpha-1)^2\eta(X), \tag{42}$$

$$\bar{S}(\xi,\xi) = -(n-1)(\alpha^2-\alpha), \tag{43}$$

$$\bar{Q}X = QX - \alpha(n-1)X, \tag{44}$$

$$\bar{Q}\xi = (n-1)(\alpha^2-\alpha)\xi. \tag{45}$$

Proof Since M is a Lorentzian $\alpha-$Sasakian manifold with respect to the quarter-symmetric metric connection D, then replacing $X=\xi$ in (31) and using (10) and (17) we get (38). Using (10) and (16), from (31) we get (39). To prove (40), we put $Z=\xi$ in (31) and then we use (18). Replacing $Y=\xi$ in (32) and using (19) we get (41). Putting $Y=\xi$ in (33) and using (6) and (19) we get (42). Again putting $X=Y=\xi$ in (32) and using (20) we get (43). Using (36) and (41) we get (44). Then putting $X=\xi$ in (44) we get (45). □

§5. Lorentzian $\alpha-$Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection D Satisfying the Condition $\bar{C}.\bar{S}=0$.

In this section we shall find out the characterization of Lorentzian $\alpha-$Sasakian manifold with respect to the quarter-symmetric metric connection D satisfying the condition $\bar{C}.\bar{S}=0$. We define $\bar{C}.\bar{S}=0$ on M by

$$(\bar{C}(X,Y).\bar{S})(Z,W)=-\bar{S}(\bar{C}(X,Y)Z,W)-\bar{S}(Z,\bar{C}(X,Y)W), \tag{46}$$

where $X,Y,Z,W\in\chi(M)$.

Theorem 5.1 *Let M be an $n-$dimensional Lorentzian $\alpha-$ Sasakian manifold with respect to the quarter-symmetric metric connection D. If $\bar{C}.\bar{S}=0$, then*

$$\begin{aligned}\frac{1}{n-2}\bar{S}^2(X,Y) &= [(\alpha^2-\alpha)+\frac{\bar{r}}{(n-1)(n-2)}]\bar{S}(X,Y)\\ &\quad +\frac{\alpha^2-\alpha}{n-2}[\alpha(n-1)(\alpha-n+1)-\bar{r}]g(X,Y)\\ &\quad -\alpha(n-1)(\alpha^2-\alpha)\eta(X)\eta(Y).\end{aligned} \tag{47}$$

Proof Let us consider M be an n-dimensional Lorentzian $\alpha-$Sasakian manifold with respect the quarter-symmetric metric connection D satisfying the condition $\bar{C}.\bar{S}=0$. Then from (46), we get

$$\bar{S}(\bar{C}(X,Y)Z,W)+\bar{S}(Z,\bar{C}(X,Y)W)=0, \tag{48}$$

where $X,Y,Z,W\in\chi(M)$. Now putting $X=\xi$ in (48), we get

$$\bar{S}(\bar{C}(\xi,X)Y,Z)+\bar{S}(Y,\bar{C}(\xi,X)Z)=0. \tag{49}$$

Using (35), (37), (38) and (41), we have

$$\begin{aligned}\bar{S}(\bar{C}(\xi,X)Y,Z) &= (n-1)(\alpha^2-\alpha)[\alpha^2-\frac{(n-1)(\alpha^2-\alpha)}{n-2}+\frac{\bar{r}}{(n-1)(n-2)}]\eta(Z)g(X,Y)\\ &\quad +[\alpha-\alpha^2+\frac{(n-1)(\alpha^2-\alpha)}{n-2}-\frac{\bar{r}}{(n-1)(n-2)}]\eta(Y)\bar{S}(X,Z)\\ &\quad +\alpha(\alpha^2-\alpha)(n-1)\eta(X)\eta(Y)\eta(Z)\\ &\quad -\frac{1}{n-2}[(n-1)(\alpha^2-\alpha)\eta(Z)\bar{S}(X,Y)-\bar{S}^2(X,Z)\eta(Y)]\end{aligned} \tag{50}$$

and

$$\begin{aligned}\bar{S}(Y,\bar{C}(\xi,X)Z) &= (n-1)(\alpha^2-\alpha)[\alpha^2-\frac{(n-1)(\alpha^2-\alpha)}{n-2}+\frac{\bar{r}}{(n-1)(n-2)}]\eta(Y)g(X,Z)\\ &\quad +[\alpha-\alpha^2+\frac{(n-1)(\alpha^2-\alpha)}{n-2}-\frac{\bar{r}}{(n-1)(n-2)}]\eta(Z)\bar{S}(Y,X)\\ &\quad +\alpha(\alpha^2-\alpha)(n-1)\eta(X)\eta(Y)\eta(Z)\\ &\quad -\frac{1}{n-2}[(n-1)(\alpha^2-\alpha)\eta(Y)\bar{S}(X,Z)-\bar{S}^2(X,Y)\eta(Z)].\end{aligned} \tag{51}$$

Using (50) and (51) in (49), we get

$$(n-1)(\alpha^2-\alpha)[\alpha^2-\frac{(n-1)(\alpha^2-\alpha)}{n-2}+\frac{\bar{r}}{(n-1)(n-2)}][g(X,Y)\eta(Z)$$
$$+g(X,Z)\eta(Y)]+2\alpha(\alpha^2-\alpha)(n-1)\eta(X)\eta(Y)\eta(Z)$$
$$+[\alpha-\alpha^2+\frac{(n-1)(\alpha^2-\alpha)}{n-2}-\frac{\bar{r}}{(n-1)(n-2)}][\eta(Y)\bar{S}(X,Z)+\eta(Z)\bar{S}(Y,X)]$$
$$-\frac{1}{n-2}[(n-1)(\alpha^2-\alpha)\{\eta(Z)\bar{S}(X,Y)+\eta(Y)\bar{S}(X,Z)\}$$
$$-\{\bar{S}^2(X,Z)\eta(Y)+\bar{S}^2(X,Y)\eta(Z)\}]=0. \tag{52}$$

Replacing $Z=\xi$ in (52) and using (41) and (42), we get

$$\begin{aligned}\frac{1}{n-2}\bar{S}^2(X,Y) &= [(\alpha^2-\alpha)+\frac{\bar{r}}{(n-1)(n-2)}]\bar{S}(X,Y)\\ &+\frac{\alpha^2-\alpha}{n-2}[\alpha(n-1)(\alpha-n+1)-\bar{r}]g(X,Y)\\ &-\alpha(n-1)(\alpha^2-\alpha)\eta(X)\eta(Y).\end{aligned}$$ □

An $n-$dimensional Lorentzian $\alpha-$Sasakian manifold M with the quarter-symmetric metric connection D is said to be $\eta-$Einstein if its Ricci tensor $\bar{S}$ is of the form

$$\bar{S}(X,Y)=Ag(X,Y)+B\eta(X)\eta(Y), \tag{53}$$

where A, B are smooth functions of M. Now putting $X=Y=e_i$, $i=1,2,\cdots,n$ in (53) and taking summation for $1\leq i\leq n$ we get

$$An-B=\bar{r}. \tag{54}$$

Again replacing $X=Y=\xi$ in (53) we have

$$A-B=(n-1)(\alpha^2-\alpha). \tag{55}$$

Solving (54) and (55) we obtain

$$A=\frac{\bar{r}}{n-1}-(\alpha^2-\alpha)\ \text{ and }\ B=\frac{\bar{r}}{n-1}-n(\alpha^2-\alpha).$$

Thus the Ricci tensor of an $\eta-$Einstein manifold with the quarter-symmetric metric connection D is given by

$$\bar{S}(X,Y)=[\frac{\bar{r}}{n-1}-(\alpha^2-\alpha)]g(X,Y)+[\frac{\bar{r}}{n-1}-n(\alpha^2-\alpha)]\eta(X)\eta(Y). \tag{56}$$

§6. $\eta-$Einstein Lorentzian $\alpha-$Sasakian Manifold with Respect to the Quarter-Symmetric Metric Connection D Satisfying the Condition $\bar{C}.\bar{S}=0$.

Theorem 6.1 *Let M be an $\eta-$Einstein Lorentzian $\alpha-$Sasakian manifold of dimension. Then $\bar{C}.\bar{S}=0$ iff*

$$\frac{n\alpha-2\alpha}{n\alpha^2-2\alpha}[\eta(\bar{R}(X,Y)Z)\eta(W)+\eta(\bar{R}(X,Y)W)\eta(Z)]=0,$$

where $X, Y, Z, W \in \chi(M)$.

Proof Let M be an $\eta-$Einstein Lorentzian $\alpha-$Sasakian manifold with respect to the quarter-symmetric metric connection D satisfying $\bar{C}.\bar{S} = 0$. Using (56) in (48), we get

$$\eta(\bar{C}(X,Y)Z)\eta(W) + \eta(\bar{C}(X,Y)W)\eta(Z) = 0,$$

or,

$$\frac{n\alpha - 2\alpha}{n\alpha^2 - 2\alpha}[\eta(\bar{R}(X,Y)Z)\eta(W) + \eta(\bar{R}(X,Y)W)\eta(Z)] = 0.$$

Conversely, using (56) we have

$$\begin{aligned}(\bar{C}(X,Y).\bar{S})(Z,W) &= -[\frac{\bar{r}}{n-1} - n(\alpha^2 - \alpha)][\eta(\bar{C}(X,Y)Z)\eta(W) + \eta(\bar{C}(X,Y)W)\eta(Z)] \\ &= -\frac{n\alpha - 2\alpha}{n\alpha^2 - 2\alpha}[\eta(\bar{R}(X,Y)Z)\eta(W) + \eta(\bar{R}(X,Y)W)\eta(Z)] = 0. \quad \square\end{aligned}$$

§7. Ricci Pseudosymmetric Lorentzian $\alpha-$Sasakian Manifolds with Quarter-Symmetric Metric Connection D

Theorem 7.1 *A Ricci pseudosymmetric Lorentzian α- Sasakian manifolds M with quarter-symmetric metric connection D with restriction $Y = W = \xi$ and $L_{\bar{S}} = 1$ is an $\eta-$Einstein manifold.*

Proof Lorentzian $\alpha-$Sasakian manifolds M with quarter-symmetric metric connection D is called a Ricci pseudosymmetric Lorentzian $\alpha-$Sasakian manifolds if

$$(\bar{R}(X,Y).\bar{S})(Z,W) = L_{\bar{S}}[((X \wedge Y).\bar{S})(Z,W)], \tag{57}$$

or,

$$\bar{S}(\bar{R}(X,Y)Z,W) + \bar{S}(Z,\bar{R}(X,Y)W) = L_{\bar{S}}[\bar{S}((X \wedge Y)Z,W) + \bar{S}(Z,(X \wedge Y)W)]. \tag{58}$$

Putting $Y = W = \xi$ in (58) and using (2), (38) and (41), we have

$$\begin{aligned}&L_{\bar{S}}[\bar{S}(X,Z) - (n-1)(\alpha^2 - \alpha)g(X,Z)] \\ &= (\alpha^2 - \alpha)\bar{S}(X,Z) - \alpha^2(\alpha^2 - \alpha)(n-1)g(X,Z) - \alpha(\alpha^2 - \alpha)(n-1)\eta(X)\eta(Z).\end{aligned} \tag{59}$$

Then for $L_{\bar{S}} = 1$,

$$(\alpha^2 - \alpha - 1)\bar{S}(X,Z) = (\alpha^2 - \alpha)(n-1)[(\alpha^2 - 1)g(X,Z) + \alpha\eta(X)\eta(Z)].$$

Thus M is an $\eta-$Einstein manifold. $\square$

Corollary 7.1 *A Ricci semisymmetric Lorentzian α-Sasakian manifold M with quarter-symmetric metric connection D with restriction $Y = W = \xi$ is an $\eta-$Einstein manifold.*

Proof Sine M is Ricci semisymmetric Lorentzian α-Sasakian manifolds with quarter-symmetric metric connection D, then $L_{\bar{C}} = 0$. Putting $L_{\bar{C}} = 0$ in (59) we get

$$\bar{S}(X,Z) = \alpha^2(n-1)g(X,Z) + \alpha(n-1)\eta(X)\eta(Z). \qquad \square$$

§8. Pseudosymmetric Lorentzian $\alpha-$Sasakian Manifold and Weyl-pseudosymmetric Lorentzian $\alpha-$Sasakian Manifold with Quarter-Symmetric Metric Connection

In the present section we shall give the definition of pseudosymmetric Lorentzian $\alpha-$Sasakian manifold and Weyl-pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection and discuss some properties on it.

Definition 8.1 *A Lorentzian $\alpha-$Sasakian manifold M with quarter-symmetric metric connection D is said to be pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection if the curvature tensor$\bar{R}$ of M with respect to D satisfies the conditions*

$$(\bar{R}(X,Y).\bar{R})(U,V,W) = L_{\bar{R}}[((X\wedge Y).\bar{R})(U,V,W)], \tag{60}$$

where

$$\begin{aligned}(\bar{R}(X,Y).\bar{R})(U,V,W) &= \bar{R}(X,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(X,Y)U,V)W \\ &\quad -\bar{R}(U,\bar{R}(X,Y)V)W - \bar{R}(U,V)(R(X,Y)W),\end{aligned} \tag{61}$$

and

$$\begin{aligned}((X\wedge Y).\bar{R})(U,V,W) &= (X\wedge Y)(\bar{R}(U,V)W) - \bar{R}((X\wedge Y)U,V)W \\ &\quad -\bar{R}(U,(X\wedge Y)V)W - \bar{R}(U,V)((X\wedge Y)W).\end{aligned} \tag{62}$$

Definition 8.2 *A Lorentzian $\alpha-$Sasakian manifold M with quarter-symmetric metric connection D is said to be Weyl- pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection if the curvature tensor$\bar{R}$ of M with respect to D satisfies the conditions*

$$(\bar{R}(X,Y).\bar{C})(U,V,W) = L_{\bar{C}}[((X\wedge Y).\bar{C})(U,V,W)], \tag{63}$$

where

$$\begin{aligned}(\bar{R}(X,Y).\bar{C})(U,V,W) &= \bar{R}(X,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(X,Y)U,V)W \\ &\quad -\bar{C}(U,\bar{R}(X,Y)V)W - \bar{C}(U,V)(R(X,Y)W)\end{aligned} \tag{64}$$

and

$$\begin{aligned}((X\wedge Y).\bar{C})(U,V,W) &= (X\wedge Y)(\bar{C}(U,V)W) - \bar{C}((X\wedge Y)U,V)W \\ &\quad -\bar{C}(U,(X\wedge Y)V)W - \bar{C}(U,V)((X\wedge Y)W).\end{aligned} \tag{65}$$

Theorem 8.1 *LetM be an n dimensional Lorentzian $\alpha-$Sasakian manifold. If M is Weyl-pseudosymmetric then M is either conformally flat and M is $\eta-$Einstein manifold or $L_{\bar{C}} = \alpha^2$.*

Proof Let M be an Weyl-pseudosymmetric Lorentzian $\alpha-$Sasakian manifold and $X, Y, U, V, W \in \chi(M)$. Then using (64) and (65) in (63), we have

$$\begin{aligned}
&\bar{R}(X,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(X,Y)U,V)W \\
&-\bar{C}(U,\bar{R}(X,Y)V)W - \bar{C}(U,V)(R(X,Y)W) \\
&= L_{\bar{C}}[(X\wedge Y)(\bar{C}(U,V)W) - \bar{C}((X\wedge Y)U,V)W \quad (66)\\
&-\bar{C}(U,(X\wedge Y)V)W - \bar{C}(U,V)((X\wedge Y)W)]. \quad (67)
\end{aligned}$$

Replacing X with ξ in (66) we obtain

$$\begin{aligned}
&\bar{R}(\xi,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(\xi,Y)U,V)W \\
&-\bar{C}(U,\bar{R}(\xi,Y)V)W - \bar{C}(U,V)(R(\xi,Y)W) \\
&= L_{\bar{C}}[(\xi\wedge Y)(\bar{C}(U,V)W) - \bar{C}((\xi\wedge Y)U,V)W \\
&-\bar{C}(U,(\xi\wedge Y)V)W - \bar{C}(U,V)((\xi\wedge Y)W)]. \quad (68)
\end{aligned}$$

Using (2), (38) in (67) and taking inner product of (67) with ξ, we get

$$\begin{aligned}
&\alpha^2[-\bar{C}(U,V,W,Y) - \eta(\bar{C}(U,V)W)\eta(Y) - g(Y,U)\eta(\bar{C}(\xi,V)W) \\
&+\eta(U)\eta(\bar{C}(Y,V)W) - g(Y,V)\eta(\bar{C}(U,\xi)W) + \eta(V)\eta(\bar{C}(U,Y)W) \\
&+\eta(W)\eta(\bar{C}(U,V)Y)] - \alpha[\eta(U)\eta(\bar{C}(\phi^2 Y,V)W) \\
&+\eta(V)\eta(\bar{C}(U,\phi^2 Y)W) + \eta(W)\eta(\bar{C}(U,V)\phi^2 Y)] \\
&= L_{\bar{C}}[-\bar{C}(Y,U,V,W) - \eta(Y)\eta(\bar{C}(U,V)W) - g(Y,U)\eta(\bar{C}(\xi,V)W) \\
&+\eta(U)\eta(\bar{C}(Y,V)W) - g(Y,V)\eta(\bar{C}(U,\xi)W) + \eta(V)\eta(\bar{C}(U,Y)W) \\
&+\eta(W)\eta(\bar{C}(U,V)Y)]. \quad (69)
\end{aligned}$$

Putting $Y = U$, we get

$$[L_{\bar{C}} - \alpha^2][g(U,U)\eta(\bar{C}(\xi,V)W) + g(U,V)\eta(\bar{C}(U,\xi)W)] + \alpha\eta(V)\eta(\bar{C}(\phi^2 U,V)W) = 0. \quad (70)$$

Replacing $U = \xi$ in (68), we obtain

$$[L_{\bar{C}} - \alpha^2]\eta(\bar{C}(\xi,V)W) = 0. \quad (71)$$

The formula (69) gives either $\eta(\bar{C}(\xi,V)W) = 0$ or $L_{\bar{C}} - \alpha^2 = 0$.

Now $L_{\bar{C}} - \alpha^2 \neq 0$, then $\eta(\bar{C}(\xi,V)W) = 0$, then we have M is conformally flat and which gives

$$\bar{S}(V,W) = Ag(V,W) + B\eta(V)\eta(W),$$

where

$$A = [\alpha^2 - \frac{(n-1)(\alpha^2-\alpha)}{n-2} + \frac{r}{(n-1)(n-2)}](n-2)$$

and

$$B = [\alpha^2 - \frac{2(n-1)(\alpha^2-\alpha)}{n-2} + \frac{r}{(n-1)(n-2)}](n-2),$$

which shows that M is an $\eta-$Einstein manifold. Now if $\eta(\bar{C}(\xi,V)W) \neq 0$, then $L_{\bar{C}} = \alpha^2$. □

Theorem 8.2 *LetM be an n dimensional Lorentzian $\alpha-$Sasakian manifold. If M is pseudosymmetric then eitherM is a space of constant curvature and $\alpha g(X,Y)=\eta(X)\eta(Y)$, for $\alpha\neq 0$ or $L_{\bar{R}}=\alpha^2$, for $X,Y\in\chi(M)$.*

Proof Let M be a pseudosymmetric Lorentzian $\alpha-$Sasakian manifold and $X,Y,U,V,W\in\chi(M)$. Then using (61) and (62) in (60), we have

$$\begin{aligned}&\bar{R}(X,Y)(\bar{R}(U,V)W)-\bar{R}(\bar{R}(X,Y)U,V)W\\&-\bar{R}(U,\bar{R}(X,Y)V)W-\bar{R}(U,V)(R(X,Y)W)\\&=L_{\bar{R}}[(X\wedge Y)(\bar{R}(U,V)W)-\bar{R}((X\wedge Y)U,V)W\\&-\bar{R}(U,(X\wedge Y)V)W-\bar{R}(U,V)((X\wedge Y)W)].\end{aligned}\tag{72}$$

Replacing X with ξ in (70) we obtain

$$\begin{aligned}&\bar{R}(\xi,Y)(\bar{R}(U,V)W)-\bar{R}(\bar{R}(\xi,Y)U,V)W\\&-\bar{R}(U,\bar{R}(\xi,Y)V)W-\bar{R}(U,V)(R(\xi,Y)W)\\&=L_{\bar{R}}[(\xi\wedge Y)(\bar{R}(U,V)W)-\bar{R}((\xi\wedge Y)U,V)W\\&-\bar{R}(U,(\xi\wedge Y)V)W-\bar{R}(U,V)((\xi\wedge Y)W)].\end{aligned}\tag{73}$$

Using (2), (38) in (71) and taking inner product of (71) with ξ, we get

$$\begin{aligned}&\alpha^2[-\bar{R}(U,V,W,Y)-\eta(\bar{R}(U,V)W)\eta(Y)-g(Y,U)\eta(\bar{R}(\xi,V)W)\\&+\eta(U)\eta(\bar{R}(Y,V)W)-g(Y,V)\eta(\bar{R}(U,\xi)W)+\eta(V)\eta(\bar{R}(U,Y)W)\\&+\eta(W)\eta(\bar{R}(U,V)Y)]-\alpha[\eta(U)\eta(\bar{R}(\phi^2Y,V)W)\\&+\eta(V)\eta(\bar{R}(U,\phi^2Y)W)+\eta(W)\eta(\bar{R}(U,V)\phi^2Y)]\\&=L_{\bar{R}}[-\bar{R}(Y,U,V,W)-\eta(Y)\eta(\bar{R}(U,V)W)-g(Y,U)\eta(\bar{R}(\xi,V)W)\\&+\eta(U)\eta(\bar{R}(Y,V)W)-g(Y,V)\eta(\bar{R}(U,\xi)W)+\eta(V)\eta(\bar{R}(U,Y)W)\\&+\eta(W)\eta(\bar{R}(U,V)Y)].\end{aligned}$$

Putting $Y=U$, we get

$$[L_{\bar{R}}-\alpha^2][g(U,U)\eta(\bar{R}(\xi,V)W)+g(U,V)\eta(\bar{R}(U,\xi)W)]+\alpha\eta(V)\eta(\bar{R}(\phi^2U,V)W)=0.\tag{74}$$

Replacing $U=\xi$ in (72), we obtain

$$[L_{\bar{R}}-\alpha^2]\eta(\bar{R}(\xi,V)W)=0.\tag{75}$$

The formula (73) gives either $\eta(\bar{R}(\xi,V)W)=0$ or $L_{\bar{R}}-\alpha^2=0$. Now $L_{\bar{R}}-\alpha^2\neq 0$, then $\eta(\bar{R}(\xi,V)W)=0$. We have M is a space of constant curvature and $\eta(\bar{R}(\xi,V)W)=0$ gives $\alpha g(V,W)=\eta(X)\eta(Y)$ for $\alpha\neq 0$. If $\eta(\bar{R}(\xi,V)W)\neq 0$, then we have $L_{\bar{R}}=\alpha^2$. □

§9. Examples

Let us consider the three dimensional manifold $M=\{(x_1,x_2,x_3)\in R^3: x_1,x_2,x_3\in R\}$, where (x_1,x_2,x_3) are the standard coordinates of R^3. We consider the vector fields

$$e_1=e^{x_3}\frac{\partial}{\partial x_2},\quad e_2=e^{x_3}(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2})\ \text{ and }\ e_3=\alpha\frac{\partial}{\partial x_3},$$

where α is a constant.

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and hence a basis of $\chi(M)$. The Lorentzian metric g is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = -1.$$

Then the form of metric becomes

$$g = \frac{1}{(e^{x_3})^2}(dx_2)^2 - \frac{1}{\alpha^2}(dx_3)^2,$$

which is a Lorentzian metric.

Let η be the $1-$form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$ and the $(1,1)-$tensor field ϕ is defined by

$$\phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0.$$

From the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$
$$\phi^2(X) = X + \eta(X)e_3 \quad \text{and}$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\alpha e_1, \quad [e_2, e_3] = -\alpha e_2.$$

Koszul's formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) \quad &= \quad Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Then from above formula we can calculate the followings,

$$\nabla_{e_1} e_1 = -\alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\alpha e_1,$$
$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -\alpha e_3, \quad \nabla_{e_2} e_3 = -\alpha e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

Hence the structure (ϕ, ξ, η, g) is a Lorentzian $\alpha-$Sasakian manifold [7].

Using (22), we find D, the quarter-symmetric metric connection on M following:

$$D_{e_1} e_1 = -\alpha e_3, \quad D_{e_1} e_2 = 0, \quad D_{e_1} e_3 = e_1(1 - \alpha),$$
$$D_{e_2} e_1 = 0, \quad D_{e_2} e_2 = -\alpha e_3, \quad D_{e_2} e_3 = e_2(1 - \alpha),$$
$$D_{e_3} e_1 = 0, \quad D_{e_3} e_2 = 0, \quad D_{e_3} e_3 = 0.$$

Using (23), the torson tensor $\bar{T}$, with respect to quarter-symmetric metric connection D as follows:

$$\bar{T}(e_i, e_i) = 0, \ \forall i = 1, 2, 3$$
$$\bar{T}(e_1, e_2) = 0, \ \bar{T}(e_1, e_3) = e_1, \ \bar{T}(e_2, e_3) = e_2.$$

Also $(D_{e_1} g)(e_2, e_3) = (D_{e_2} g)(e_3, e_1) = (D_{e_3} g)(e_1, e_2) = 0$. Thus M is Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection D.

Now we calculate curvature tensor $\bar{R}$ and Ricci tensors $\bar{S}$ as follows:

$$\bar{R}(e_1, e_2)e_3 = 0, \quad \bar{R}(e_1, e_3)e_3 = -(\alpha^2 - \alpha)e_1,$$
$$\bar{R}(e_3, e_2)e_2 = \alpha^2 e_3, \quad \bar{R}(e_3, e_1)e_1 = \alpha^2 e_3,$$
$$\bar{R}(e_2, e_1)e_1 = (\alpha^2 - \alpha)e_2, \quad \bar{R}(e_2, e_3)e_3 = -\alpha^2 e_2,$$
$$\bar{R}(e_1, e_2)e_2 = (\alpha^2 - \alpha)e_1.$$
$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = -\alpha \text{ and } \bar{S}(e_3, e_3) = -2\alpha^2 + (n-1)\alpha.$$

Again using (2), we get

$$(e_1, e_2)e_3 = 0, \quad (e_i \wedge e_i)e_j = 0, \quad \forall i, j = 1, 2, 3,$$
$$(e_1 \wedge e_2)e_2 = (e_1 \wedge e_3)e_3 = -e_1, \quad (e_2 \wedge e_1)e_1 = (e_2 \wedge e_3)e_3 = -e_2,$$
$$(e_3 \wedge e_2)e_2 = (e_3 \wedge e_1)e_1 = -e_3.$$

Now,

$$\bar{R}(e_1, e_2)(\bar{R}(e_3, e_1)e_2) = 0, \quad \bar{R}(\bar{R}(e_1, e_2)e_3, e_1)e_2 = 0,$$
$$\bar{R}(e_3, \bar{R}(e_1, e_2)e_1)e_2 = -\alpha^2(\alpha^2 - \alpha)e_3,$$
$$(\bar{R}(e_3, e_1)(\bar{R}(e_1, e_2)e_2) = \alpha^2(\alpha^2 - \alpha)e_3.$$

Therefore, $(\bar{R}(e_1, e_2).\bar{R})(e_3, e_1, e_2) = 0$.

Again,

$$(e_1 \wedge e_2)(\bar{R}(e_3, e_1)e_2) = 0, \quad \bar{R}((e_1 \wedge e_2)e_3, e_1)e_2 = 0,$$
$$\bar{R}(e_3, (e_1 \wedge e_2)e_1)e_2 = \alpha^2 e_3, \quad \bar{R}(e_3, e_1)((e_1 \wedge e_2)e_2) = -\alpha^2 e_3.$$

Then $((e_1, e_2).\bar{R})(e_3, e_1, e_2) = 0$. Thus $(\bar{R}(e_1, e_2).\bar{R})(e_3, e_1, e_2) = L_{\bar{R}}[((e_1, e_2).\bar{R})(e_3, e_1, e_2)]$ for any function $= L_{\bar{R}} \in C^\infty(M)$.

Similarly, any combination of e_1, e_2 and e_3 we can show (60). Hence M is a pseudosymmetric Lorentzian $\alpha-$Sasakian manifold with quarter-symmetric metric connection.

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The Skew Energy of
Cayley Digraphs of Cyclic Groups and Dihedral Groups

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Abstract: This paper is motivated by the skew energy of a digraph as vitiated by C.Adiga, R.Balakrishnan and Wasin So [1]. We introduce and investigate the skew energy of a Cayley digraphs of cyclic groups and dihedral groups and establish sharp upper bound for the same.

Key Words: Cayley graph, dihedral graph, skew energy.

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§1. Introduction

Let G be a non trivial finite group and S be an non-empty subset of G such that for $x \in S, x^{-1} \notin S$ and $I_G \notin S$, then the Cayley digraph $\Gamma = Cay(G, S)$ of G with respect to S is defined as a simple directed graph with vertex set G and arc set $E(\Gamma) = \{(g, h) | hg^{-1} \in S\}$. If S is inverse closed and doesn't contain identity then $Cay(G, S)$ is viewed as undirected graph and is simply the Cayley graph of G with respect to S. It easily follows that valency of $Cay(G, S)$ is $|S|$ and $Cay(G, S)$ is connected if and only if $\langle S \rangle = G$. For an elaborate literature on Cayley graphs one may refer [5]. A dihedral group D_{2n} is a group with $2n$ elements such that it contains an element $'a'$ of order 2 and an element $'b'$ of order n with $a^{-1}ba = b^{-1}$. Thus $D_{2n} = \langle a, b | a^2 = b^n = 1, a^{-1}ba = b^{-1} \rangle = \langle a, b | a^2 = b^n = 1, a^{-1}ba = b^{\alpha}, \alpha \not\equiv 1 (mod \quad n), \alpha^2 \equiv 1 (mod \quad n) \rangle$.

If $n = 2$, then D_4 is Abelian; for $n \geq 3$, D_{2n} is not abelian. The elements of diheral group can be explicitly listed as

$$D_{2n} = \{1, a, ab, ab^2, \cdots, ab^{n-1}, b, b^2, \cdots, b^{n-1}\}.$$

In short, its elements can be listed as $a^i b^k$ where $i = 0, 1$ and $k = 0, 1, \cdots, (n-1)$. It is easy to explicitly describe the product of any two elements $a^i b^k a^j b^l = a^r b^s$ as follows:

1. If $j = 0$ then $r = i$ and s equals the remainder of $k + l$ modulo n.
2. If $j = 1$, then r is the remainder of $i + j$ modulo 2 and s is the remainder of $k\alpha + l$ modulo n.

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The orders of the elements in the Dihedral group D_{2n} are: $o(1) = 1$, $o(ab^i) = 2$, where; $0 \leq i \leq n-1$, $o(b^i) = n$, where; $0 < i \leq n-1$ and if n is even than $o(b^{\frac{n}{2}}) = 2$.

Let Γ be a digraph of order n with vertex set $V(\Gamma) = \{v_1, \cdots, v_n\}$,and arc set $\Lambda(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. We assume that Γ does not have loops and multiple arcs, i.e.,$(v_i, v_i) \notin \Lambda(\Gamma)$ for all i , and $(v_i, v_j) \in \Lambda(\Gamma)$ implies that $(v_j, v_i) \notin \Lambda(\Gamma)$. Hence the underlying undirected graph G_Γ of Γ is a simple graph. The skew-adjacency matrix of Γ is the $n \times n$ matrix $S(\Gamma) = [s_{ij}]$, where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Lambda(\Gamma)$,$s_{ij} = -1$ whenever $(v_j, v_i) \in \Lambda(\Gamma)$, and $s_{ij} = 0$ otherwise. Because of the assumptions on Γ, $S(\Gamma)$ is indeed a skew-symmetric matrix. Hence the eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of $S(\Gamma)$ are all purely imaginary numbers, and the singular values of $S(\Gamma)$ coincide with the absolute values $\{|\lambda_1|, \cdots, |\lambda_n|\}$, of its eigenvalues. Consequently, the energy of $S(\Gamma)$, which is defined as the sum of its singular values [6], is also the sum of the absolute values of its eigenvalues. For the sake of convenience, we simply refer the energy of $S(\Gamma)$ as the skew energy of the digraph Γ. If we denote the skew energy of Γ by $\varepsilon_s(\Gamma)$ then, $\varepsilon_s(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$.

The degree of a vertex in a digraph Γ is the degree of the corresponding vertex of the underlying graph of Γ. Let $D(\Gamma) = diag(d_1, d_2, \cdots, d_n)$, the diagonal matrix with vertex degrees $d_1, d_2, \cdots, d_n$ of $v_1, v_2, \cdots, v_n$ and $S(\Gamma)$ be the skew adjacency matrix of a simple digraph Γ, possessing n vertices and m edges. Then $L(\Gamma) = D(\Gamma) - S(\Gamma)$ is called the Laplacian matrix of the digraph Γ. If $\lambda_i, i = 1, 2, \cdots, n$ are the eigenvalues of the Laplacian matrix $L(\Gamma)$ then the skew Laplacian energy of the digraph Γ is defined as $SLE(\Gamma) = \sum_{i=1}^{n} |\lambda_i - \frac{2m}{n}|$.

An $n \times n$ matrix S is said to be a circulant matrix if its entries satisfy $s_{ij} = s_{1,j-i+1}$, where the subscripts are reduced modulo n and lie in the set $\{1, 2, ..., n\}$. In other words, ith row of S is obtained from the first row of S by a cyclic shift of $i-1$ steps, and so any circulant matrix is determined by its first row. It is easy to see that the eigenvalues of S are $\lambda_k = \sum_{j=1}^{n} s_{1j}\omega^{(j-1)k}$, $k = 0, 1, \cdots, n-1$. For any positive integer n, let $\tau_n = \{\omega^k : 0 \leq k < n\}$ be the set of all nth roots of unity, where $\omega = e^{\frac{2\pi i}{n}} = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ that $i^2 = -1$. τ_n is an abelian group with respect to multiplication. A circulant graph is a graph Γ whose adjacency matrix $A(\Gamma)$ is a circulant matrix. More details about circulant graphs can be found in [3].

Ever since the concept of the energy of simple undirected graphs was introduced by Gutman in [7], there has been a constant stream of papers devoted to this topic. In [1], Adiga, et al. have studied the skew energy of digraphs. In [4], Gui-Xian Tian, gave the skew energy of orientations of hypercubes. In this paper we introduce and investigate the skew energy of a Cayley digraphs of cyclic groups and dihedral groups and establish sharp upper bound for the same.

§2. **Main Results**

First we present some facts that are needed to prove our main results.

Lemma 2.1([2]) *Let Γ is disconnected graph into the λ components $\Gamma_1, \Gamma_2, \cdots, \Gamma_\lambda$, then*

$$Spec(\Gamma) = \bigcup_{i=1}^{\lambda} Spec(\Gamma_i).$$

Lemma 2.2 *Let* $\omega = e^{\frac{2k\pi i}{n}} = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})$ *for* $1 \leq k \leq n$, *where* n *is a positive integer and* $i^2 = -1$. *Then*

(i) $\omega^t + \omega^{n-t} = 2\cos(\frac{2kt\pi}{n})$ *for* $1 \leq k \leq n$,

(ii) $\omega^t - \omega^{n-t} = 2i\sin(\frac{2kt\pi}{n})$ *for* $1 \leq k \leq n$.

Lemma 2.3([1]) *Let* n *be a positive integer. Then*

(i) $\sum_{k=1}^{\frac{n-1}{2}} \sin\frac{2k\pi}{n} = \frac{1}{2}cot\frac{\pi}{2n}$, $n \equiv 1(mod2)$,

(ii) $\sum_{k=1}^{\frac{n-2}{2}} \sin\frac{2k\pi}{n} = \cot\frac{\pi}{n}$, $n \equiv 0(mod2)$,

(iii) $\sum_{k=1}^{\frac{n-1}{2}} |\cos\frac{2k\pi}{n}| = \frac{1}{2}\csc\frac{\pi}{2n} - \frac{1}{2}$, $n \equiv 1(mod2)$,

(iv) $\sum_{k=0}^{n-1} |\cos\frac{2k\pi}{n}| = 2\cot\frac{\pi}{n}$, $n \equiv 0(mod4)$,

(iv) $\sum_{k=0}^{n-1} |\cos\frac{2k\pi}{n}| = 2\csc\frac{\pi}{n}$, $n \equiv 2(mod4)$,

(vi) $\sum_{k=1}^{\frac{n}{2}} \sin\frac{(2k-1)\pi}{n} = \csc\frac{\pi}{n}$, $n \equiv 0(mod2)$,

(vii) $\sum_{k=1}^{n-1} \sin\frac{k\pi}{n} = \cot\frac{\pi}{2n}$, $n \equiv 1(mod2)$,

(viii) $\sum_{k=1}^{n-1} |\cos\frac{2k\pi}{n}| = \csc\frac{\pi}{2n} - 1$, $n \equiv 1(mod2)$.

Lemma 2.4 *Let* n *be a positive integer. Then*

(i) $\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{2k\pi}{n}| = \cot\frac{\pi}{n} - 1$, $n \equiv 0(mod4)$,

(ii) $\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{2k\pi}{n}| = \csc\frac{\pi}{n} - 1$, $n \equiv 2(mod4)$.

Proof The proof of (i) follows directly from Lemma 2.3(iv), and (ii) is a consequence of Lemma 2.3(v). □

Lemma 2.5 *Let n be a positive integer. Then*

$$(i)\ \sum_{k=0}^{\frac{n-1}{2}} \sin\frac{2k\pi}{n} = \sum_{k=0}^{\frac{n-1}{2}} |\sin\frac{4k\pi}{n}|,\ n \equiv 1(mod 2),$$

$$(ii)\ \sum_{k=0}^{\frac{n-2}{2}} \sin\frac{2k\pi}{n} = \sum_{k=0}^{\frac{n-2}{2}} |\sin\frac{4k\pi}{n}|,\ n \equiv 2(mod 4),$$

$$(iii)\ \sum_{k=1}^{\frac{n-2}{2}} \sin\frac{2k\pi}{n} = 1 + 2\sum_{k=1}^{\frac{n-4}{4}} \sin\frac{2k\pi}{n},\ n \equiv 0(mod 4),$$

$$(iv)\ \sum_{k=1}^{\frac{n-2}{2}} |\sin\frac{4k\pi}{n}| = 2\sum_{k=1}^{\frac{n-4}{4}} |\sin\frac{4k\pi}{n}|,\ n \equiv 0(mod 4),$$

$$(v)\ \sum_{k=1}^{\frac{n-4}{4}} \sin\frac{4k\pi}{n} = \csc\frac{4\pi}{n} + \cot\frac{4\pi}{n},\ n \equiv 0(mod 8),$$

$$(vi)\ \sum_{k=1}^{\frac{n-4}{4}} \sin\frac{4k\pi}{n} = \cot\frac{2\pi}{n},\ n \equiv 4(mod 8).$$

Proof (i) Let $n \equiv 1(mod 2)$, $f(k) = \sin\frac{2k\pi}{n}$ and $g(k) = \sin\frac{4k\pi}{n}$, where $k \in \{0, 1, 2, \cdots, \frac{n-1}{2}\}$. Then it is easy to check that

$$g(k) = \begin{cases} f(2k) & \text{if} \quad 0 \le k \le \lfloor\frac{n-1}{4}\rfloor, \\ -f(n-2k) & \text{if} \quad \lfloor\frac{n-1}{4}\rfloor < k \le \frac{n-1}{2}. \end{cases}$$

This implies (i).

(ii) Let $n \equiv 2(mod 4)$, $f(k) = \sin\frac{2k\pi}{n}$ and $g(k) = \sin\frac{4k\pi}{n}$, where $k \in \{0, 1, 2, \cdots, \frac{n-2}{2}\}$. Then it follows that

$$g(k) = \begin{cases} f(2k) & \text{if} \quad 1 \le k \le \frac{n-2}{4}, \\ -f(-\frac{n}{2}+2k) & \text{if} \quad \frac{n-2}{4} < k \le \frac{n-2}{2}. \end{cases}$$

This implies (ii). Proofs of (iii) and (iv) are similar to that of (i) and (ii).

(v) Let $n \equiv 0(mod 8)$. Then $n = 8m$, $m \in \mathbb{N}$ and

$$\begin{aligned}\sum_{k=1}^{\frac{n-4}{4}} \sin\frac{4k\pi}{n} &= \sum_{k=1}^{2m-1} \sin\frac{k\pi}{2m} = \left(\sin\frac{\pi}{2m} + \sin\frac{3\pi}{2m} + \cdots + \sin\frac{(2m-1)\pi}{2m}\right) \\ &\quad + \left(\sin\frac{2\pi}{2m} + sin\frac{4\pi}{2m} + \cdots + \sin\frac{(2m-2)\pi}{2m}\right) \\ &= \csc\frac{\pi}{2m} + \cot\frac{\pi}{2m} \\ &= \csc\frac{4\pi}{n} + \cot\frac{4\pi}{n} \quad \text{(Using Lemma2.3}(vi)\text{ and }(ii)).\end{aligned}$$

(vi) Let $n \equiv 4(mod 8)$. Then $n = 8m + 4$, $m \in \mathbb{N}$ and

$$\begin{aligned}\sum_{k=1}^{\frac{n-4}{4}} \sin\frac{4k\pi}{n} &= \sum_{k=1}^{2m} \sin\frac{k\pi}{2m+1} \\ &= \cot\frac{\pi}{2(2m+1)} = \cot\frac{2\pi}{n}. \quad \text{(using Lemma 2.3}(vii))\end{aligned}$$

□

Lemma 2.6 *Let n be a positive integer. Then*

(i) $\displaystyle\sum_{k=1}^{n-1} |\sin\frac{2k\pi}{n}| = \cot\frac{\pi}{2n}$, $n \equiv 1(mod 2)$,

(ii) $\displaystyle\sum_{k=1}^{n-1} |\sin\frac{2k\pi}{n}| = 2\cot\frac{\pi}{n}$, $n \equiv 0(mod 2)$.

Now we compute skew energy of some Cayley digraphs.

Theorem 2.7 *Let $G = \{v_1 = e, v_2, \cdots, v_n\}$ be a group, $S = \{v_i\} \subset G$ with $v_i \neq v_i^{-1}$, $v_i \neq e$ and $\Gamma = Cay(G, S)$ be a Cayley digraph on G with respect to S. Suppose $H = \langle S\rangle$, $|H| = m$, $|G : H| = \lambda$. Then*

$$\varepsilon_s(\Gamma) = \begin{cases} 2\lambda cot\frac{\pi}{2m} & if \quad m \equiv 1(mod2), \\ 4\lambda cot\frac{\pi}{m} & if \quad m \equiv 0(mod2). \end{cases}$$

Proof Let $G = \{v_1 = e, v_2, v_3, \cdots, v_n\}$, $S = \{v_i\}$, $v_i \in G$ with $v_i \neq v_i^{-1}$, $v_i \neq e$ and suppose $H = \langle S\rangle$, $|H| = m$, $|G : H| = \lambda$. If $\lambda = 1$ then $G = \{e, v_i, v_i^2, \cdots, v_i^{n-1}\}$ and hence the the skew-adjacency matrix of $\Gamma = Cay(G, S)$ is a circulant matrix. Its first row is $[0, 1, 0, \cdots, 0, -1]$. So all eigenvalues of Γ are $\lambda_k = \omega^k - \omega^{kn-k} = \omega^k - \omega^{-k} = 2isin\frac{2k\pi}{n}$, $k = 0, 1, \cdots, n-1$ where $\omega = e^{\frac{2\pi i}{n}}$ and $i^2 = -1$. Applying Lemma 2.2(ii), we obtain $\lambda_k = 2isin\frac{2k\pi}{n}$, $k = 0, 1, \cdots, n-1$. Now by Lemma 2.6 we have

$$\varepsilon_s(\Gamma) = \sum_{k=0}^{n-1} |2isin\frac{2k\pi}{n}| = 2\sum_{k=1}^{n-1} |sin\frac{2k\pi}{n}| = \begin{cases} 2cot\frac{\pi}{2n} & \text{if} \quad n \equiv 1(mod2), \\ 4cot\frac{\pi}{n} & \text{if} \quad n \equiv 0(mod2). \end{cases}$$

If $\lambda > 1$, then Γ is disconnected graph in to the $\Gamma_i, i = 1, \cdots, \lambda$ components and all components are isomorphic with Cayley digraph $\Gamma_m = Cay(H, S)$ where $H = \langle v_i : \; v_i^m = 1\rangle$

and $m|n$, $S = \{v_i\}$. Since Γ is not connected, by Lemma 2.1, its energy is the sum of the energies of its connected components. Thus

$$\varepsilon_s(\Gamma) = \sum_{i=1}^{\lambda} \varepsilon_s(\Gamma_i) = \lambda\varepsilon_s(Cay(H,S)) = \begin{cases} 2\lambda cot\frac{\pi}{2m} & \text{if} \quad m \equiv 1(mod2), \\ 4\lambda cot\frac{\pi}{m} & \text{if} \quad m \equiv 0(mod2). \end{cases}$$

This completes the proof. □

Theorem 2.8 *Let $G = \{e, b, b^2, \cdots, b^{n-1}\}$ be a cyclic group of order n, and $\Gamma = Cay(G,S)$ be a Cayley digraph on G with respect to $S = \{b^i, b^j\}, 0 < i, j \le n-1, i \ne j$ $H = \langle S\rangle$, $|H| = m$, and $|G : H| = \lambda$. Then*

(i) $\varepsilon_s(\Gamma) \le 4\lambda \cot\frac{\pi}{2m}$ *if* $m \equiv 1(mod2)$,

(ii) $\varepsilon_s(\Gamma) \le 8\lambda \cot\frac{\pi}{m}$ *if* $m \equiv 2(mod4)$,

(iii) $\varepsilon_s(\Gamma) \le 4\lambda(\cot\frac{\pi}{m} + 2\csc\frac{4\pi}{m} + 2\cot\frac{4\pi}{m})$ *if* $m \equiv 0(mod8)$,

(iv) $\varepsilon_s(\Gamma) \le 4\lambda(\cot\frac{\pi}{m} + 2\cot\frac{2\pi}{m}))$ *if* $m \equiv 4(mod8)$.

Proof Let $G = \{e, b, b^2, \cdots, b^{n-1}\}$ be a cyclic group of order n and $\Gamma = Cay(G,S)$ be a Cayley digraph on G with respect to $S = \{b^i, b^j\}, 0 < i, j \le n-1, i \ne j$, $H = \langle S\rangle$, $|H| = m$, and $|G : H| = \lambda$. If $\lambda = 1$, then $G = H$ and hence the the skew-adjacency matrix of $\Gamma = Cay(G,S)$ is a circulant matrix. So all eigenvalues of Γ are$\lambda_k = \omega^k - \omega^{-k} + \omega^{2k} - \omega^{-2k}$, $k = 0, 1, \cdots, n-1$, where $\omega = e^{\frac{2\pi i}{n}}$ and $i^2 = -1$. Hence

$$\lambda_k = \omega^k - \omega^{-k} + \omega^{2k} - \omega^{-2k} = 2isin\frac{2k\pi}{n} + 2isin\frac{4k\pi}{n} = 2i(sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n})$$

for $k = 0, 1, \cdots, n-1$.

(i) Suppose $n \equiv 1(mod2)$. Then

$$\begin{aligned} \varepsilon_s(\Gamma) &= \sum_{k=0}^{n-1} |\lambda_k| = \sum_{k=0}^{n-1} |2i(sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n})| \\ &= \sum_{k=1}^{n-1} |2i(sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n})| \\ &= 4\sum_{k=1}^{\frac{n-1}{2}} |sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n}| \\ &\le 4(\sum_{k=1}^{\frac{n-1}{2}} |sin\frac{2k\pi}{n}| + \sum_{k=1}^{\frac{n-1}{2}} |sin\frac{4k\pi}{n}|) \\ &= 4(\sum_{k=1}^{\frac{n-1}{2}} sin\frac{2k\pi}{n} + \sum_{k=1}^{\frac{n-1}{2}} sin\frac{2k\pi}{n}) \quad \text{(using Lemma 2.5}(i)) \\ &= 4cot\frac{\pi}{2n} \quad \text{(applying Lemma 2.3}(i)). \end{aligned}$$

Thus (i) holds for $\lambda = 1$.

Suppose $\lambda > 1$. Then Γ is disconnected graph in to the $\Gamma_i, i = 1, \cdots, \lambda$ components and all components are isomorphic with Cayley digraph $\Gamma_m = Cay(H, S)$ where $H = \langle v_i : v_i^m = 1\rangle$ and $m|n$, $S = \{v_i\}$. Since Γ is not connected, its energy is the sum of the energies of its connected components. Thus

$$\varepsilon_s(\Gamma) = \sum_{i=1}^{\lambda} \varepsilon_s(\Gamma_i)\lambda\varepsilon_s(Cay(H, S)) \leq 4\lambda \cot \frac{\pi}{2m}.$$

Now we shall prove (ii),(iii) and (iv) only for $\lambda = 1$. For $\lambda > 1$, proofs are similar to that of (i).

(ii) If $n \equiv 2(mod4)$, then

$$\begin{aligned}
\varepsilon_s(\Gamma) &= \sum_{k=0}^{n-1}|\lambda_k| = \sum_{k=0}^{n-1}|2i(sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n})| \\
&= \sum_{k=1}^{n-1}|2i(sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n})| \\
&= 4\sum_{k=1}^{\frac{n-2}{2}}|sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n}| \\
&\leq 4(\sum_{k=1}^{\frac{n-2}{2}}|sin\frac{2k\pi}{n}| + \sum_{k=1}^{\frac{n-2}{2}}|sin\frac{4k\pi}{n}|) \\
&= 4(\sum_{k=1}^{\frac{n-2}{2}} sin\frac{2k\pi}{n} + \sum_{k=1}^{\frac{n-2}{2}} sin\frac{2k\pi}{n}) \quad \text{(using Lemma 2.5}(ii)) \\
&= 8cot\frac{\pi}{n}.
\end{aligned}$$

Here we used the Lemma 2.3(ii).

(iii) If $n \equiv 0(mod8)$, then

$$\begin{aligned}
\varepsilon_s(\Gamma) &= 4\sum_{k=1}^{\frac{n-2}{2}}|sin\frac{2k\pi}{n} + sin\frac{4k\pi}{n}| \\
&\leq 4(\sum_{k=1}^{\frac{n-2}{2}}|sin\frac{2k\pi}{n}| + \sum_{k=1}^{\frac{n-2}{2}}|sin\frac{4k\pi}{n}|) \\
&= 4(\sum_{k=1}^{\frac{n-2}{2}} sin\frac{2k\pi}{n} + 2\sum_{k=1}^{\frac{n-4}{4}} sin\frac{4k\pi}{n}) \quad \text{(using Lemma 2.5}(iv)) \\
&= 4cot\frac{\pi}{n} + 8csc\frac{4\pi}{n} + 8cot\frac{4\pi}{n}.
\end{aligned}$$

To get the last equality we have used Lemma 2.3(ii) and 2.5(v).

(iv) If $n \equiv 4(mod 8)$, then

$$\begin{aligned}\varepsilon_s(\Gamma) &\leq 4(\sum_{k=1}^{\frac{n-2}{2}} |sin\frac{2k\pi}{n}| + \sum_{k=1}^{\frac{n-2}{2}} |sin\frac{4k\pi}{n}|) \\ &= 4(\sum_{k=1}^{\frac{n-2}{2}} sin\frac{2k\pi}{n} + 2\sum_{k=1}^{\frac{n-4}{4}} sin\frac{4k\pi}{n}) \text{ (using Lemma 2.5}(iv)) \\ &= 4(cot\frac{\pi}{n} + 2cot\frac{2\pi}{n})).\end{aligned}$$

To get the last equality we have used Lemma 2.3(ii) and 2.5(vi). □

Lemma 2.9 *Let $G = \langle b : b^n = 1\rangle$ be a cyclic group and $\Gamma = Cay(G, S_t)$, $t \in \{1, \cdots, \lfloor\frac{n-1}{2}\rfloor\}$,be a Cayley digraph on G with respect to $S_t = \{b^l, b^{2l}, \cdots, b^{tl}\}$, where $l \in U(n) = \{r : 1 \leq r < n, gcd(n, r) = 1\}$. Then the eigenvalues of Γ are*

$$\lambda_k = \sum_{j=0}^{|S_t|} 2isin\frac{2kj\pi}{n}, \quad k = 0, 1, \cdots, n-1,$$

where $i^2 = -1$.

Proof The proof directly follows from the definition of cyclic group and is similar to that of Theorem 2.7. □

Lemma 2.10 *Let $G = \langle b : b^n = 1\rangle$ be a cyclic group and $\Gamma = Cay(G, S_t)$, $t \in \{1, \cdots, \lfloor\frac{n-1}{2}\rfloor\}$,be a Cayley digraph on G with respect to $S_t = \{b^l, b^{2l}, \cdots, b^{tl}\}$ where $l \in U(n) = \{r : 1 \leq r < n, gcd(n, r) = 1\}$. Also suppose $\varepsilon_s(\Gamma), SLE(\Gamma)$ denote the skew energy and the skew Laplacian energy of Γ respectively. Then $\varepsilon_s(\Gamma) = SLE(\Gamma)$.*

Proof The proof directly follows from the definition of the skew energy and the skew Laplacian energy. □

Lemma 2.11 *Let n be a positive integer. Then*

$$(i) \sum_{k=1}^{\frac{n-1}{2}} \cos\frac{4k\pi}{n} = \sum_{k=1}^{\frac{n-1}{2}} \cos\frac{2k\pi}{n},\ n \equiv 1(mod 2),$$

$$(ii) \sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{4k\pi}{n}| = \csc\frac{\pi}{n} - 1,\ n \equiv 2(mod 4),$$

$$(iii) \sum_{k=1}^{\frac{n-2}{2}} \cos\frac{4k\pi}{n} = -1,\ n \equiv 0(mod 4).$$

Proof (i) Let $n \equiv 1(mod 2)$, $f(k) = \cos\frac{2k\pi}{n}$, $g(k) = \cos\frac{4k\pi}{n}$, where $k \in \{1, 2, \cdots, \frac{n-1}{2}\}$.

It is easy to verify that

$$g(k)=\begin{cases} f(2k) & \text{if} \quad 1\le k\le \lfloor\frac{n-1}{4}\rfloor,\\ f(n-2k) & \text{if} \quad \lfloor\frac{n-1}{4}\rfloor < k\le \frac{n-1}{2}.\end{cases}$$

This implies (i).

(ii) Let $n\equiv 2(mod4)$. Then $n=4m+2$ for some $m\in\mathbb{N}$. We have

$$\begin{aligned}\sum_{k=1}^{\frac{n-2}{2}}|\cos\frac{4k\pi}{n}| &= \sum_{k=1}^{\frac{n-2}{2}}|\cos\frac{4k\pi}{n}|=\sum_{k=1}^{2m}|\cos\frac{4k\pi}{4m+2}|\\ &= \sum_{k=1}^{2m}|\cos\frac{2k\pi}{2m+1}|=\csc\frac{\pi}{2(2m+1)}-1 \ \ \text{(using Lemma 2.3}(viii))\\ &= \csc\frac{\pi}{n}-1.\end{aligned}$$

(iii) Suppose $n=4m, m\in\mathbb{N}$. Then

$$\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}=\sum_{k=1}^{2m-1}cos\frac{k\pi}{m}=\cos\frac{m\pi}{m}+\sum_{k=1}^{m-1}cos\frac{k\pi}{m}+\sum_{k=m+1}^{2m-1}cos\frac{k\pi}{m}.$$

Changing k to $k+m$ in the last summation we get $\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}=-1.$ □

Theorem 2.12 *Let $G=\langle b:b^n=1\rangle$ be a cyclic group and $\Gamma=Cay(G,S)$,be a Cayley digraph on G with respect to $S=\{b^l\}$ where $l\in U(n)=\{r:1\le r<n, gcd(n,r)=1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ , $D(\Gamma)=diag(d_1,d_2,\cdots,d_n)$, the diagonal matrix with vertex degrees $d_1,d_2,\cdots,d_n$ of $e,b,b^2,\cdots,b^{n-1}$. Suppose $L(\Gamma)=D(\Gamma)-C_s(\Gamma)$ and $\mu_1,\cdots,\mu_n$ are eigenvalues of $L(\Gamma)$. We define $\alpha(\Gamma)=\sum_{i=1}^{n}\mu_i^2$. Then*

(i) $\alpha(\Gamma)\le 2n+2csc\frac{\pi}{2n}$ *if* $n\equiv 1(mod2)$,

(ii) $\alpha(\Gamma)\le 2(n-1)+4\csc\frac{\pi}{n}$ *if* $n\equiv 2(mod4)$,

(iii) $\alpha(\Gamma)=2(n-2)$ *if* $n\equiv 4(mod0)$.

Proof Let $G=\langle b:b^n=1\rangle$ be a cyclic group and $\Gamma=Cay(G,S)$,be a Cayley digraph on G with respect to $S=\{b^l\}$ where $l\in U(n)=\{r:1\le r<n, gcd(n,r)=1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ. Note that underlying graph of Γ is a 2−regular graph. Hence $D(\Gamma)=diag(2,2,\cdots,2)$. Suppose $L(\Gamma)=D(\Gamma)-C_s(\Gamma)$ then $L(\Gamma)$ is a circulant matrix and its first row is $[2,-1,0,\cdots,0,1]$. This implies that the eigenvalues of $L(\Gamma)$ are $\mu_k=2-\omega^k+\omega^{kn-k}=2-\omega^k+\omega^{-k}=2-(\omega^k-\omega^{-k})=2-2i\sin\frac{2k\pi}{n}$, $k=0,1,\cdots,n-1$ where $\omega=e^{\frac{2\pi i}{n}}$ and $i^2=-1$.

If $n \equiv 1(mod 2)$, then

$$\begin{aligned}
\alpha(\Gamma) &= \sum_{k=0}^{n-1} \mu_k^2 \\
&= \sum_{k=0}^{n-1} (2 - 2i \sin \frac{2k\pi}{n})^2 = 4 + \sum_{k=1}^{n-1} (2 - 2i \sin \frac{2k\pi}{n})^2 \\
&= 4 + \sum_{k=1}^{\frac{n-1}{2}} (2 - 2i \sin \frac{2k\pi}{n})^2 + \sum_{k=\frac{n+1}{2}}^{n-1} (2 - 2i \sin \frac{2k\pi}{n})^2 \\
&= 4 + \sum_{k=1}^{\frac{n-1}{2}} (2 - 2i \sin \frac{2k\pi}{n})^2 + \sum_{k=1}^{\frac{n-1}{2}} (2 + 2i \sin \frac{2k\pi}{n})^2 \\
&= 4 + \sum_{k=1}^{\frac{n-1}{2}} 2(4 - 4\sin^2 \frac{2k\pi}{n}) = 4 + 8(\frac{n-1}{2}) - 8 \sum_{k=1}^{\frac{n-1}{2}} \sin^2 \frac{2k\pi}{n} \\
&= 4 + 8(\frac{n-1}{2}) - 8 \sum_{k=1}^{\frac{n-1}{2}} (\frac{1}{2} - \frac{1}{2} \cos \frac{4k\pi}{n}) = 4 + 4(\frac{n-1}{2}) + 4 \sum_{k=1}^{\frac{n-1}{2}} \cos \frac{4k\pi}{n} \\
&= 4 + 4(\frac{n-1}{2}) + 4 \sum_{k=1}^{\frac{n-1}{2}} \cos \frac{2k\pi}{n} \quad \text{(using Lemma 2.11}(i)) \\
&\leq 4 + 4(\frac{n-1}{2}) + 4 \sum_{k=1}^{\frac{n-1}{2}} |\cos \frac{2k\pi}{n}| \\
&= 4 + 4(\frac{n-1}{2}) + 4(\frac{1}{2} \csc \frac{\pi}{2n} - \frac{1}{2}) \quad \text{(using Lemma 2.3}(iii)) \\
&= 2n + 2 \csc \frac{\pi}{2n}.
\end{aligned}$$

If $n \equiv 0(mod 2)$, then

$$\begin{aligned}
\alpha(\Gamma) &= \sum_{k=0}^{n-1} \mu_k^2 \\
&= \sum_{k=0}^{n-1} (2 - 2i \sin \frac{2k\pi}{n})^2 = 4 + \sum_{k=1}^{n-1} (2 - 2i \sin \frac{2k\pi}{n})^2 \\
&= 4 + \sum_{k=1}^{\frac{n-2}{2}} (2 - 2i \sin \frac{2k\pi}{n})^2 + (2 - 2i \sin \frac{2(\frac{n}{2})\pi}{n})^2 + \sum_{k=\frac{n+2}{2}}^{n-1} (2 - 2i \sin \frac{2k\pi}{n})^2 \\
&= 8 + \sum_{k=1}^{\frac{n-2}{2}} (2 - 2i \sin \frac{2k\pi}{n})^2 + \sum_{k=1}^{\frac{n-2}{2}} (2 + 2i \sin \frac{2k\pi}{n})^2 \\
&= 8 + \sum_{k=1}^{\frac{n-2}{2}} 2(4 - 4\sin^2 \frac{2k\pi}{n}) = 4 + 8(\frac{n-2}{2}) - 8 \sum_{k=1}^{\frac{n-2}{2}} \sin^2 \frac{2k\pi}{n} \\
&= 4 + 8(\frac{n-2}{2}) - 8 \sum_{k=1}^{\frac{n-2}{2}} (\frac{1}{2} - \frac{1}{2} \cos \frac{4k\pi}{n}) = 4 + 4(\frac{n-2}{2}) + 4 \sum_{k=1}^{\frac{n-2}{2}} \cos \frac{4k\pi}{n}.
\end{aligned}$$

If $n \equiv 2(mod4)$, then

$$\begin{aligned}\alpha(\Gamma) &\leq 4+4(\frac{n-2}{2})+4\sum_{k=1}^{\frac{n-2}{2}}|\cos\frac{4k\pi}{n}| \\ &= 4+4(\frac{n-1}{2})+4(\csc\frac{\pi}{n}-1) \quad (\text{using Lemma2.11}(ii)) \\ &= 2(n-1)+4\csc\frac{\pi}{n}.\end{aligned}$$

This completes the proof of (ii).

If $n \equiv 0(mod4)$, then

$$\begin{aligned}\alpha(\Gamma) &= 4+4(\frac{n-2}{2})+4\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n} \\ &= 4+4(\frac{n-2}{2})+4(-1) \quad (\text{using Lemma2.11}(iii)) \\ &= 2(n-2).\end{aligned}$$

This completes the proof of (iii). □

Lemma 2.13 *Let n be a positive integer. Then*

(i) $\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{6k\pi}{n}| = \frac{3}{2}\csc\frac{3\pi}{2n}+\frac{1}{2}$, $n \equiv 3(mod6)$,

(ii) $\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{6k\pi}{n}| = \sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{2k\pi}{n}|$, $n \equiv 1(mod6)$,

(iii) $\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{6k\pi}{n}| = \sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{2k\pi}{n}|$, $n \equiv 5(mod6)$,

(iv) $\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{6k\pi}{n} = 0$, $n \equiv 0(mod6)$,

(v) $\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{6k\pi}{n} = \sum_{k=1}^{\frac{n-2}{2}}\cos\frac{2k\pi}{n}$, $n \equiv 2(mod6)$,

(vi) $\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{6k\pi}{n} = \sum_{k=1}^{\frac{n-2}{2}}\cos\frac{2k\pi}{n}$, $n \equiv 4(mod6)$,

vii) $\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{8k\pi}{n} = \sum_{k=1}^{\frac{n-1}{2}}\cos\frac{4k\pi}{n}$, $n \equiv 1(mod2)$,

(vii) $\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{8k\pi}{n} = \sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}$, $n \equiv 2(mod4)$,

$$(viii)\ \sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{8k\pi}{n}| = -1 + 2\csc\frac{2\pi}{n},\ n \equiv 4(mod 8),$$

$$(ix)\ \sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{8k\pi}{n}| = -1 + 4\cot\frac{4\pi}{n},\ n \equiv 0(mod 16),$$

$$(x)\ \sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{8k\pi}{n}| = -1 + 4\csc\frac{4\pi}{n},\ n \equiv 8(mod 16) \text{ and } n \equiv 2 \text{ or } 4(mod 6).$$

Proof (i) Suppose $n = 6m+3$, then

$$\begin{aligned}
\sum_{k=1}^{\frac{n-1}{2}} |\cos\frac{6k\pi}{n}| &= \sum_{k=1}^{3m+1} |\cos\frac{2k\pi}{2m+1}| \\
&= \sum_{k=1}^{2m} |\cos\frac{2k\pi}{2m+1}| + \sum_{k=2m+1}^{3m+1} |\cos\frac{2k\pi}{2m+1}| \\
&= \sum_{k=1}^{2m} |\cos\frac{2k\pi}{2m+1}| + \sum_{k=0}^{m} |\cos\frac{2k\pi}{2m+1}| \\
&\quad \text{(changing k to k + (2m + 1) in the last summation)} \\
&= \frac{3}{2}\csc\frac{3\pi}{2n} + \frac{1}{2} \quad \text{(using Lemma 2.3}(viii), (iii)).
\end{aligned}$$

(ii) Let $n = 6m+1$, $f(k) = \cos\frac{2k\pi}{n}$, $g(k) = \cos\frac{6k\pi}{n}$, where $k \in \{1, 2, \cdots, \frac{n-1}{2}\}$. Then we have

$$g(k) = \begin{cases} f(3k) & \text{if} \quad 1 \le k \le m, \\ f(n-3k) & \text{if} \quad m < k \le 2m, \\ f(3k-n) & \text{if} \quad 2m < k \le 3m. \end{cases}$$

This implies (ii).

The proofs of (iii), (iv), (v), (vi), (vii) and $(viii)$ are similar to the proof of (ii).

Suppose $n = 4m$ then

$$\begin{aligned}
\sum_{k=1}^{\frac{n-2}{2}} |\cos\frac{8k\pi}{n}| &= \sum_{k=1}^{2m-1} |\cos\frac{2k\pi}{m}| \\
&= 1 + \sum_{k=1}^{m-1} |\cos\frac{2k\pi}{m}| + \sum_{k=m+1}^{2m-1} |\cos\frac{2k\pi}{m}| \\
&= 1 + 2\sum_{k=1}^{m-1} |\cos\frac{2k\pi}{m}| = -1 + 2\sum_{k=0}^{m-1} |\cos\frac{2k\pi}{m}|
\end{aligned}$$

$$= \begin{cases} -1+2\csc\frac{\pi}{2m} \text{ if } & m\equiv 1(mod2) \\ -1+4\cot\frac{\pi}{m} \text{ if } & m\equiv 0(mod4) \\ -1+4\csc\frac{\pi}{m} \text{ if } & m\equiv 2(mod4) \end{cases}$$

$$= \begin{cases} -1+2\csc\frac{2\pi}{n} \text{ if } & n\equiv 4(mod8), \\ -1+4\cot\frac{4\pi}{n} \text{ if } & n\equiv 0(mod16), \\ -1+4\csc\frac{4\pi}{n} \text{ if } & n\equiv 8(mod16). \end{cases}$$

This completes the proof of $(viii)$,(ix),(x). □

Theorem 2.14 *Let $G=\langle b : b^n = 1\rangle$ be a cyclic group and $\Gamma = Cay(G,S)$,be a Cayley digraph on G with respect to $S=\{b^l, b^{2l}\}$ where $l\in U(n)=\{r : 1\le r < n, gcd(n,r)=1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ , $D(\Gamma)=diag(d_1,d_2,\cdots,d_n)$, the diagonal matrix with vertex degrees $d_1,d_2,\cdots,d_n$ of $e,b,b^2,\cdots,b^{n-1}$. Suppose $L(\Gamma)=D(\Gamma)-C_s(\Gamma)$ and $\mu_1,\cdots,\mu_n$ are eigenvalues of $L(\Gamma)$. Define $\alpha(\Gamma)=\sum_{i=1}^{n}\mu_i^2$. Then*

(i) $\alpha(\Gamma)\le 4(3n+2)-12\csc\frac{3\pi}{2n}$ *if* $n\equiv 3(mod6)$.

(ii) $\alpha(\Gamma)\le 4(3n+2)-4\csc\frac{\pi}{2n}$ *if* $n\equiv 1$ *or* $5(mod6)$.

(iii) $\alpha(\Gamma)\le 4(3n-2)+16\csc\frac{\pi}{n}$ *if* $n\equiv 2(mod4)$ *and* $n\equiv 0(mod6)$.

(iv) $\alpha(\Gamma)\le 4(3n-2)+24\csc\frac{\pi}{n}$ *if* $n\equiv 2(mod4)$*and* $n\equiv 2$ *or* $4(mod6)$.

(v) $\alpha(\Gamma)\le 4(3n-2)+8\cot\frac{\pi}{n}+8\csc\frac{2\pi}{n}$ *if* $n\equiv 4(mod8)$ *and* $n\equiv 0(mod6)$.

(vi) $\alpha(\Gamma)\le 4(3n-4)+16\cot\frac{\pi}{n}+8\csc\frac{2\pi}{n}$ *if* $n\equiv 4(mod8)$ *and* $n\equiv 2$ *or* $4(mod6)$.

(vii) $\alpha(\Gamma)\le 4(3n-2)+8\cot\frac{\pi}{n}+16\cot\frac{4\pi}{n}$ *if* $n\equiv 0(mod16)$, *and* $n\equiv 0(mod6)$.

$(viii)$ $\alpha(\Gamma)\le 2(n-8)+16\cot\frac{\pi}{n}+16\cot\frac{4\pi}{n}$ *if* $n\equiv 0(mod16)$ *and* $n\equiv 2$ *or* $4(mod6)$.

(ix) $\alpha(\Gamma)\le 4(3n-2)+8\cot\frac{\pi}{n}+16\csc\frac{4\pi}{n}$ *if* $n\equiv 8(mod16)$ *and* $n\equiv 0(mod6)$.

(x) $\alpha(\Gamma)\le 4(3n-4)+16\cot\frac{\pi}{n}+16\csc\frac{4\pi}{n}$ *if* $n\equiv 8(mod16)$ *and* $n\equiv 2$ *or* $4(mod6)$.

Proof Let $G=\langle b : b^n = 1\rangle$ be a cyclic group and $\Gamma = Cay(G,S)$,be a Cayley digraph on G with respect to $S=\{b^l, b^{2l}\}$ where $l\in U(n)=\{r : 1\le r < n, gcd(n,r)=1\}$ and $C_s(\Gamma)$ be the skew-adjacency matrix of Γ. Note that underlying graph of Γ is a $4-$regular graph. Hence $D(\Gamma)=diag(4,4,\cdots,4)$. Suppose $L(\Gamma)=D(\Gamma)-C_s(\Gamma)$ then $L(\Gamma)$ is circulant matrix and its first row is $[4,-1,-1,\cdots,0,1,1]$. This implies that the eigenvalues of $L(\Gamma)$ are

$$\mu_k = 4-\omega^k-\omega^{2k}+\omega^{-2k}+\omega^{-k} = 4-2i(\sin\frac{2k\pi}{n}+\sin\frac{4k\pi}{n}),\quad k=0,1,\cdots,n-1,$$

where $\omega = e^{\frac{2\pi i}{n}}$ and $i^2=-1$. It is clear that

$$\mu_{n-k} = 4+2i(\sin\frac{2k\pi}{n}+\sin\frac{4k\pi}{n}) \text{ and } \mu_k+\mu_{n-k}=8$$

for $k = 1, 2, \cdots, \lfloor\frac{n-1}{2}\rfloor$. So

$$\begin{aligned}
\mu_k^2 + \mu_{n-k}^2 &= 64 - 2\mu_k\mu_{n-k} \\
&= 64 - 2(4 - 2i(\sin\frac{2k\pi}{n} + \sin\frac{4k\pi}{n}))(4 + 2i(\sin\frac{2k\pi}{n} + \sin\frac{4k\pi}{n})) \\
&= 64 - 2(16 + 4(\sin\frac{2k\pi}{n} + \sin\frac{4k\pi}{n})^2) \\
&= 24 - 8\cos\frac{2k\pi}{n} + 4\cos\frac{4k\pi}{n} - 8\cos\frac{6k\pi}{n} + 4\cos\frac{8k\pi}{n}
\end{aligned}$$

for $k = 1, 2, \cdots, \lfloor\frac{n-1}{2}\rfloor$. Let $n \equiv 1(mod 2)$, then

$$\begin{aligned}
\alpha(\Gamma) &= \sum_{k=0}^{n-1}\mu_k^2 \\
&= \mu_0^2 + \sum_{k=1}^{n-1}\lambda_k^2 \\
&= 16 + \sum_{k=1}^{\frac{n-1}{2}}(\mu_k^2 + \mu_{n-k}^2) \\
&= 16 + \sum_{k=1}^{\frac{n-1}{2}}(24 - 8\cos\frac{2k\pi}{n} + 4\cos\frac{4k\pi}{n} - 8\cos\frac{6k\pi}{n} + 4\cos\frac{8k\pi}{n}) \\
&= 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n} + 4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{4k\pi}{n} - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{6k\pi}{n} + 4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{8k\pi}{n} \\
&= 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n} + 4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n} - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{6k\pi}{n} + 4\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{2k\pi}{n} \\
&\qquad \text{(using Lemma2.11}(i), 2.13(vii)) \\
&= 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}\cos\frac{6k\pi}{n} \le 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{6k\pi}{n}|. \qquad (2.1)
\end{aligned}$$

(i) If $n \equiv 3(mod 6)$ then using Lemma 2.13(i) in above inequelity, we get

$$\alpha(\Gamma) \le 4(3n+1) - 8(\frac{3}{2}\csc\frac{3\pi}{2n} - \frac{1}{2}) = 4(3n+2) - 12\csc\frac{3\pi}{2n}.$$

This completes the proof of (i).

(ii) If $n \equiv 1$ or $5(mod 6)$ and using Lemma 2.13(ii) and (iii), we get

$$\begin{aligned}
\alpha(\Gamma) &\le 4(3n+1) - 8\sum_{k=1}^{\frac{n-1}{2}}|\cos\frac{2k\pi}{n}| \\
&= 4(3n+1) - 8(\frac{1}{2}\csc\frac{\pi}{2n} - \frac{1}{2}) = 4(3n+2) - 4\csc\frac{\pi}{2n}.
\end{aligned}$$

Let $n \equiv 0(mod2)$. Then

$$\begin{aligned}\alpha(\Gamma) &= \sum_{k=0}^{n-1}\mu_k^2 = \mu_0^2+\mu_{\frac{n}{2}}^2+\sum_{k=1,k\neq\frac{n}{2}}^{n-1}\mu_k^2 = 16+16+\sum_{k=1}^{\frac{n-2}{2}}(\mu_k^2+\mu_{n-k}^2)\\ &= 32+\sum_{k=1}^{\frac{n-2}{2}}(24-8\cos\frac{2k\pi}{n}+4\cos\frac{4k\pi}{n}-8\cos\frac{6k\pi}{n}+4\cos\frac{8k\pi}{n})\\ &= 4(3n+2)-8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{2k\pi}{n}+4\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}-8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{6k\pi}{n}+4\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{8k\pi}{n}. \quad (2.2)\end{aligned}$$

If $n \equiv 2(\text{mod } 4)$, then employing Lemma 2.13(viii) in (2.2), we get

$$\begin{aligned}\alpha(\Gamma) &= 4(3n+2)-8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{2k\pi}{n}+4\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}-8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{6k\pi}{n}+4\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}\\ &= 4(3n+2)-8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{2k\pi}{n}+8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}-8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{6k\pi}{n}. \quad (2.3)\end{aligned}$$

(iii) If $n \equiv 2(mod4)$ and $n \equiv 0(\text{mod } 6)$, then using Lemma 2.13$(iv)$ in (2.3) we deduce that

$$\begin{aligned}\alpha(\Gamma) &\leq 4(3n+2)+8\sum_{k=1}^{\frac{n-2}{2}}|\cos\frac{2k\pi}{n}|+8\sum_{k=1}^{\frac{n-2}{2}}|\cos\frac{4k\pi}{n}|\\ &= 4(3n+2)+16(\csc\frac{\pi}{n}-1) = 4(3n-2)+16\csc\frac{\pi}{n}\end{aligned}$$

by using Lemma 2.4(ii) and 2.11(ii).

(iv) If $n \equiv 2(mod4)$ and $n \equiv 2$ or $4(mod6)$, then using Lemma 2.13(v) and (vi) in (2.3) we see that

$$\begin{aligned}\alpha(\Gamma) &= 4(3n+2)-8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{2k\pi}{n}+8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{4k\pi}{n}-8\sum_{k=1}^{\frac{n-2}{2}}\cos\frac{2k\pi}{n}\\ &\leq 4(3n+2)+16\sum_{k=1}^{\frac{n-2}{2}}|\cos\frac{2k\pi}{n}|+8\sum_{k=1}^{\frac{n-2}{2}}|\cos\frac{4k\pi}{n}|\\ &\leq 4(3n+2)+24(\csc\frac{\pi}{n}-1) = 4(3n-2)+24\csc\frac{\pi}{n}.\end{aligned}$$

Similarly we can prove (v) to (x). □

We give few interesting results on the skew energy of Cayley digraphs on dihedral groups D_{2n}.

Theorem 2.15 *Let $D_{2n} = \langle a,b|a^2=b^n=1, a^{-1}ba=b^{-1}\rangle$ the dihedral group of order $2n$ and $\Gamma = Cay(D_{2n},S)$ be a Cayley digraph on D_{2n} with respect to $S=\{b^i\}$, $1 \leq i \leq n-1$, and*

$H = \langle S\rangle$, $|H| = m$, $|D'_{2n} : H| = \lambda$ *that,* D'_{2n} *is the commutator subgroup of* D_{2n}*. Then*

$$\varepsilon_s(\Gamma) = \begin{cases} 4\lambda cot\frac{\pi}{2m} & if \quad m \equiv 1(mod2), \\ 8\lambda cot\frac{\pi}{m} & if \quad m \equiv 0(mod2). \end{cases}$$

Proof The proof of Theorem 2.15 directly follows from the definition of dihedral group and Theorem 2.7. □

Theorem 2.16 *Let* $D_{2n} = \langle a, b|a^2 = b^n = 1, a^{-1}ba = b^{-1}\rangle$ *the dihedral group of order* $2n$ *and* $\Gamma = Cay(D_{2n}, S)$ *be a Cayley digraph on* D_{2n} *with respect to* $S = \{b^i, b^j\}$, $1 \le i, j \le n-1, i \ne j$, *and* $H = \langle S\rangle$, $|H| = m$, $|D'_{2n} : H| = \lambda$ *Then* $\Gamma = Cay(D_{2n}, S)$ *is a circulant digraph and its skew energy*

(*i*) $\varepsilon_s(\Gamma) \le 8\lambda cot\frac{\pi}{2m}$ *if* $m \equiv 1(mod2)$,

(*ii*) $\varepsilon_s(\Gamma) \le 16\lambda cot\frac{\pi}{m}$ *if* $m \equiv 2(mod4)$,

(*iii*) $\varepsilon_s(\Gamma) \le 8\lambda(cot\frac{\pi}{m} + 2csc\frac{4\pi}{m} + 2cot\frac{4\pi}{m})$ *if* $m \equiv 0(mod8)$,

(*iv*) $\varepsilon_s(\Gamma) \le 8\lambda(cot\frac{\pi}{m} + 2cot\frac{2\pi}{m}))$ *if* $m \equiv 4(mod8)$.

Proof The proof of Theorem 2.16 directly follows from the definition of dihedral group and Theorem 2.8. □

Theorem 2.17 *Let* $D_{2n} = \langle a, b|a^2 = b^n = 1, a^{-1}ba = b^{-1}\rangle$ *the dihedral group of order* $2n$ *and* $\Gamma = Cay(D_{2n}, S)$ *be a Cayley digraph on* D_{2n} *with respect to* $S = \{b^l\}$ *where* $l \in U(n) = \{r : 1 \le r < n, gcd(n, r) = 1\}$ *and* $C_s(\Gamma)$ *be the skew-adjacency matrix of* Γ, $D(\Gamma)$ *is the* $n \times n$ *matrix such that* $d_{ij} = 2$ *whenever* $i = j$ *otherwise* $d_{ij} = 0$. *Suppose* $L(\Gamma) = D(\Gamma) - C_s(\Gamma)$ *and* $\lambda_1, \cdots, \lambda_n$ *are eigenvalues of* $L(\Gamma)$. *Define* $\alpha(\Gamma) = \sum\limits_{i=1}^{n} \lambda_i^2$. *Then*

(*i*) $\alpha(\Gamma) \le 4n + 4\csc\frac{\pi}{2n}$ *if* $n \equiv 1(\mathrm{mod}\ 2)$,

(*ii*) $\alpha(\Gamma) \le 4(n-1) + 8\csc\frac{\pi}{n}$ *if* $n \equiv 2(\mathrm{mod}\ 4)$

(*iii*) $\alpha(\Gamma) = 4(n-2)$ *if* $n \equiv 0(\mathrm{mod}\ 4)$

Proof The proof of Theorem 2.17 directly follows from the definition of dihedral group and Theorem 2.12. □

Theorem 2.18 *Let* $D_{2n} = \langle a, b|a^2 = b^n = 1, a^{-1}ba = b^{-1}\rangle$ *the dihedral group of order* $2n$ *and* $\Gamma = Cay(D_{2n}, S)$ *be a Cayley digraph on* D_{2n} *with respect to* $S = \{b^l, b^{2l}\}$ *where* $l \in U(n) = \{r : 1 \le r < n, gcd(n, r) = 1\}$ *and* $C_s(\Gamma)$ *be the skew-adjacency matrix of* Γ , $D(\Gamma)$ *is the* $n \times n$ *matrix such that* $d_{ij} = 4$ *whenever* $i = j$ *otherwise* $d_{ij} = 0$. *Suppose* $L(\Gamma) = D(\Gamma) - C_s(\Gamma)$ *and* $\lambda_1, \cdots, \lambda_n$ *are eigenvalues of* $L(\Gamma)$. *Define* $\alpha(\Gamma) = \sum\limits_{i=1}^{n} \lambda_i^2$. *Then*

(i) $\alpha(\Gamma) \leq 8(3n+2) - 24\csc\frac{3\pi}{2n}$ *if* $n \equiv 3(mod6)$.

(ii) $\alpha(\Gamma) \leq 8(3n+2) - 8\csc\frac{\pi}{2n}$ *if* $n \equiv 1$ *or* $5(mod6)$.

(iii) $\alpha(\Gamma) \leq 8(3n-2) + 32\csc\frac{\pi}{n}$ *if* $n \equiv 2(mod4)$ *and* $n \equiv 0(mod6)$.

(iv) $\alpha(\Gamma) \leq 8(3n-2) + 48\csc\frac{\pi}{n}$ *if* $n \equiv 2(mod4)$*and* $n \equiv 2$ *or* $4(mod6)$.

(v) $\alpha(\Gamma) \leq 8(3n-2) + 16\cot\frac{\pi}{n} + 16\csc\frac{2\pi}{n}$ *if* $n \equiv 4(mod8)$ *and* $n \equiv 0(mod6)$.

(vi) $\alpha(\Gamma) \leq 8(3n-4) + 32\cot\frac{\pi}{n} + 16\csc\frac{2\pi}{n}$ *if* $n \equiv 4(mod8)$ *and* $n \equiv 2$ *or* $4(mod6)$.

(vii) $\alpha(\Gamma) \leq 8(3n-2) + 16\cot\frac{\pi}{n} + 32\cot\frac{4\pi}{n}$ *if* $n \equiv 0(mod16)$, *and* $n \equiv 0(mod6)$.

(viii) $\alpha(\Gamma) \leq 4(n-8) + 32\cot\frac{\pi}{n} + 32\cot\frac{4\pi}{n}$ *if* $n \equiv 0(mod16)$ *and* $n \equiv 2$ *or* $4(mod6)$.

(ix) $\alpha(\Gamma) \leq 8(3n-2) + 16\cot\frac{\pi}{n} + 32\csc\frac{4\pi}{n}$ *if* $n \equiv 8(mod16)$ *and* $n \equiv 0(mod6)$.

(x) $\alpha(\Gamma) \leq 8(3n-4) + 32\cot\frac{\pi}{n} + 32\csc\frac{4\pi}{n}$ *if* $n \equiv 8(mod16)$ *and* $n \equiv 2$ *or* $4(mod6)$.

Proof The proof of Theorem 2.18 directly follows from the definition of dihedral group and Theorem 2.14. □

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Equivalence of Kropina and Projective Change of Finsler Metric

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Abstract: A change of Finsler metric $L(x,y) \to \overline{L}(x,y)$ is called Kropina change if $\overline{L}(x,y) = \dfrac{L^2}{\beta}$, where $\beta(x,y) = b_i(x)\, y^i$ is a one-form on a smooth manifold M^n. The change $L \to \overline{L}$ is called projective change if every geodesic of one space is transformed to a geodesic of the other. The purpose of the present paper is to find the necessary and sufficient condition under which a Kropina change becomes a projective change.

Key Words: Kropina change, projective change, Finsler space.

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§1. Preliminaries

Let $F^n = (M^n, L)$ be a Finsler space equipped with the fundamental function $L(x,y)$ on the smooth manifold M^n. Let $\beta = b_i(x)y^i$ be a one-form on the manifold M^n, then $L \to \dfrac{L^2}{\beta}$ is called Kropina change of Finsler metric [5]. If we write $\overline{L} = \dfrac{L^2}{\beta}$ and $\overline{F}^n = (M^n, \overline{L})$, then the Finsler space $\overline{F}^n$ is said to be obtained from F^n by Kropina change. The quantities corresponding to $\overline{F}^n$ are denoted by putting bar on those quantities.

The fundamental metric tensor g_{ij}, the normalized element of support l_i and angular metric tensor h_{ij} of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad l_i = \frac{\partial L}{\partial y^i} \qquad \text{and} \qquad h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - l_i l_j.$$

We shall denote the partial derivative with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$$L_i = \dot{\partial}_i L, \qquad L_{ij} = \dot{\partial}_i \dot{\partial}_j L, \qquad L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L.$$

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Thus

$$L_i = l_i, \qquad L^{-1}\, h_{ij} = L_{ij}$$

The geodesic of F^n are given by the system of differential equations

$$\frac{d^2x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where $G^i(x,y)$ are positively homogeneous of degree two in y^i and is given by

$$2G^i = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \qquad F = \frac{L^2}{2}$$

where g^{ij} are the inverse of g_{ij}.

The well known Berwald connection $B\Gamma = \{G^i_{jk}, G^i_j\}$ of a Finsler space is constructed from the quantity G^i appearing in the equation of geodesic and is given by [6]

$$G^i_j = \dot{\partial}_j G^i, \qquad G^i_{jk} = \dot{\partial}_k G^i_j.$$

The Cartan's connection $C\Gamma = \{F^i_{jk}, G^i_j, C^i_{jk}\}$ is constructed from the metric function L by the following five axioms [6]:

(i) $g_{ij|k} = 0$; (ii) $g_{ij}|_k = 0$; (iii) $F^i_{jk} = F^i_{kj}$; (iv) $F^i_{0k} = G^i_k$; (v) $C^i_{jk} = C^i_{kj}$.

where $_{|k}$ and $|_k$ denote h and $v-$covariant derivatives with respect to $C\Gamma$. It is clear that the $h-$covariant derivative of L with respect to $B\Gamma$ and $C\Gamma$ are same and vanishes identically. Furthermore the $h-$covariant derivatives of L_i, L_{ij} with respect to $C\Gamma$ are also zero.

We denote

$$2r_{ij} = b_{i|j} + b_{j|i}, \qquad 2s_{ij} = b_{i|j} - b_{j|i}.$$

§2. **Kropina Change of Finsler Metric**

The Kropina change of Finsler metric L is given by

(2.1) $$\overline{L} = \frac{L^2}{\beta}, \qquad \text{where} \qquad \beta(x,y) = b_i(x)\, y^i.$$

We may put

(2.2) $$\overline{G}^i = G^i + D^i.$$

Then $\overline{G}^i_j = G^i_j + D^i_j$ and $\overline{G}^i_{jk} = G^i_{jk} + D^i_{jk}$, where $D^i_j = \dot{\partial}_j D^i$ and $D^i_{jk} = \dot{\partial}_k D^i_j$. The tensors D^i, D^i_j and D^i_{jk} are positively homogeneous in y^i of degree two, one and zero respectively.

To find D^i we deal with equations $L_{ij|k} = 0$ [2], where $L_{ij|k}$ is the $h-$covariant derivative of $L_{ij} = h_{ij}/L$ with respect to Cartan's connection $C\Gamma$. Then

(2.3) $$\partial_k L_{ij} - L_{ijr} G^r_k - L_{rj} F^r_{ik} - L_{ir} F^r_{jk} = 0.$$

Since $\dot{\partial}_i \beta = b_i$, from (2.1), we have

$$(2.4)\quad (a)\ \overline{L}_i = \frac{2L}{\beta}L_i - \frac{L^2}{\beta^2}b_i;$$

$$(b)\ \overline{L}_{ij} = \frac{2L}{\beta}L_{ij} + \frac{2}{\beta}L_iL_j - \frac{2L}{\beta^2}(L_ib_j + L_jb_i) + \frac{2L^2}{\beta^3}b_ib_j;$$

$$(c)\ \partial_k\overline{L}_i = \frac{2L}{\beta}(\partial_kL_i) + \left(\frac{2}{\beta}L_i - \frac{2L}{\beta^2}b_i\right)\partial_kL + \left(\frac{2L^2}{\beta^3}b_i - \frac{2L}{\beta^2}L_i\right)\partial_k\beta - \frac{L^2}{\beta^2}(\partial_kb_i);$$

$$\begin{aligned}(d)\ \partial_k\overline{L}_{ij} &= \frac{2L}{\beta}(\partial_kL_{ij}) + \left[\frac{2}{\beta}L_{ij} - \frac{2}{\beta^2}(L_ib_j + L_jb_i) + \frac{4L}{\beta^3}b_ib_j\right](\partial_kL)\\ &- \left[\frac{2L}{\beta^2}L_{ij} + \frac{2}{\beta^2}L_iL_j + \frac{6L^2}{\beta^4}b_ib_j - \frac{4L}{\beta^3}(L_ib_j + L_jb_i)\right](\partial_k\beta)\\ &+ \left[\frac{2}{\beta}L_j - \frac{2L}{\beta^2}b_j\right](\partial_kL_i) + \left[\frac{2}{\beta}L_i - \frac{2L}{\beta^2}b_i\right](\partial_kL_j)\\ &+ \left[\frac{2L^2}{\beta^3}b_j - \frac{2L}{\beta^2}L_j\right](\partial_kb_i) + \left[\frac{2L^2}{\beta^3}b_i - \frac{2L}{\beta^2}L_i\right](\partial_kb_j)\end{aligned}$$

$$\begin{aligned}(e)\ \overline{L}_{ijk} &= \frac{2L}{\beta}L_{ijk} + \frac{2}{\beta}(L_jL_{jk} + L_jL_{ik} + L_kL_{ij}) - \frac{2L}{\beta}(L_{ij}b_k + L_{ik}b_j + L_{jk}b_i)\\ &-\frac{2}{\beta^2}(L_iL_jb_k + L_iL_kb_j + L_jL_kb_i) + \frac{4L}{\beta^3}(b_ib_jL_k + b_ib_kL_j + b_jb_kL_i)\\ &-\frac{6L^2}{\beta^4}b_ib_jb_k.\end{aligned}$$

Since $\overline{L}_{ij|k} = 0$ in $\overline{F}^n$, after using (2.2), we have

$$\partial_k\overline{L}_{ij} - \overline{L}_{ijr}(G^r_k + D^r_k) - \overline{L}_{rj}(F^r_{ik} + {}^cD^r_{ik}) - \overline{L}_{ir}(F^r_{jk} + {}^cD^r_{jk}) = 0,$$

where $\overline{F}^i_{jk} - F^i_{jk} = {}^cD^i_{jk}$.

Using equations (2.3) and (2.4)(b), (d), (e), the above equation may be written as

$$\begin{aligned}&-\frac{2L}{\beta}[L_{ijr}D^r_k + L_{rj}\,{}^cD^r_{ik} + L_{ir}\,{}^cD^r_{jk}] + \left[\frac{2}{\beta}L_{ij} - \frac{2}{\beta^2}(L_ib_j + L_jb_i)\right.\\ &+\left.\frac{4L}{\beta^3}b_ib_j\right]L_rG^r_k - \left[\frac{2L}{\beta^2}L_{ij} + \frac{2}{\beta^2}L_iL_j + \frac{6L^2}{\beta^4}b_ib_j - \frac{4L}{\beta^3}(L_ib_j + L_jb_i)\right]\times\\ &(r_{0k} + s_{ok} + b_rG^r_k) + \left(\frac{2}{\beta}L_j - \frac{2L}{\beta^2}b_j\right)(L_{ir}G^r_k + L_rF^r_{ik}) + \left(\frac{2}{\beta}L_i - \frac{2L}{\beta^2}b_i\right)\times\\ &(L_{jr}G^r_k + L_rF^r_{jk}) + \left(\frac{2L^2}{\beta^3}b_j - \frac{2L^2}{\beta^2}L_j\right)(r_{ik} + s_{ik} + b_rF^r_{ik})\\ &+\left(\frac{2L^2}{\beta^3}b_i - \frac{2L}{\beta^2}L_i\right)(r_{jk} + s_{jk} + b_rF^r_{jk}) + \left\{\frac{2L}{\beta^2}(L_{ij}b_r + L_{ir}b_j + L_{jr}b_i)\right.\\ &+\frac{2}{\beta^2}(L_iL_jb_r + L_iL_rb_j + L_jL_rb_i) - \frac{2}{\beta}(L_iL_{jr} + L_jL_{ir} + L_rL_{ij})\\ &-\left.\frac{4L}{\beta^3}(b_ib_jL_r + b_jb_rL_i + b_ib_rL_j) + \frac{6L^2}{\beta^4}b_ib_jb_r\right\}(G^r_k + D^r_k)\\ &+\left\{\frac{2L}{\beta^2}(L_rb_j + L_jb_r) - \frac{2}{\beta}L_rL_j - \frac{2L^2}{\beta^3}b_rb_j\right\}(F^r_{ik} + {}^cD^r_{ik})\\ &+\left\{\frac{2L}{\beta^2}(L_ib_r + L_rb_i) - \frac{2}{\beta}L_iL_r - \frac{2L^2}{\beta^3}b_ib_r\right\}(F^r_{jk} + {}^cD^r_{jk}) = 0,\end{aligned}\tag{2.5}$$

where '0' stands for contraction with respect to y^i viz. $r_{0k} = r_{ik}y^i$, $r_{00} = r_{ij}y^iy^j$.

Contracting (2.5) with y^k, we get

$$\begin{aligned}
&2\left\{\frac{2L}{\beta}L_{ijr} - \frac{2L}{\beta^2}(L_{ij}b_r + L_{ir}b_j + L_{jr}b_i) - \frac{2}{\beta^2}(L_iL_jb_r + L_iL_rb_j + L_jL_rb_i)\right.\\
&\left.+\frac{2}{\beta}(L_iL_{jr} + L_jL_{ir} + L_rL_{ij}) + \frac{4L}{\beta^3}(b_ib_jL_r + b_jb_rL_i + b_ib_rL_j) - \frac{6L^2}{\beta^4}b_ib_jb_r\right\}D^r\\
&+\left\{\frac{2L}{\beta^2}L_{ij} + \frac{2}{\beta^2}L_iL_j + \frac{6L^2}{\beta^4}b_ib_j - \frac{4L}{\beta^3}(L_ib_j + L_jb_i)\right\}r_{00}\\
&+\left\{\frac{2L}{\beta}L_{rj} - \frac{2L}{\beta^2}(L_rb_j + L_jb_r) + \frac{2}{\beta}L_rL_j + \frac{2L^2}{\beta^3}b_rb_j\right\}D^r_i\\
&+\left\{\frac{2L}{\beta}L_{ir} - \frac{2L}{\beta^2}(L_ib_r + L_rb_i) + \frac{2}{\beta}L_iL_r + \frac{2L^2}{\beta^3}b_ib_r\right\}D^r_j\\
&+\left(\frac{2L}{\beta^2}L_j - \frac{2L^2}{\beta^3}b_j\right)(r_{i0} + s_{i0}) + \left(\frac{2L}{\beta^2}L_i - \frac{2L^2}{\beta^3}b_i\right)(r_{j0} + s_{j0}) = 0,
\end{aligned} \tag{2.6}$$

where we have used the fact that $D^i_{jk}y^j = {}^cD^i_{jk}y^j = D^i_k$ [3].

Next, we deal with $\overline{L}_{i|j} = 0$, that is $\partial_j\overline{L}_i - \overline{L}_{ir}\overline{G}^r_j - \overline{L}_r\overline{F}^r_{ij} = 0$. Then

$$\partial_j\overline{L}_i - \overline{L}_{ir}(G^r_j + D^r_j) - \overline{L}_r(F^r_{ij} + {}^cD^r_{ij}) = 0. \tag{2.7}$$

Putting the values of $\partial_j\overline{L}_i$, $\overline{L}_{ir}$ and $\overline{L}_r$ from (2.4) in (2.7) and using equation

$$L_{i|j} = \partial_jL_i - L_{ir}G^r_j - L_rF^r_{ij} = 0,$$

we get

$$\begin{aligned}
-\frac{L^2}{\beta^2}b_{i|j} = &\left[\frac{2L}{\beta}L_{ir} + \frac{2}{\beta}L_iL_r - \frac{2L}{\beta^2}(L_ib_r + L_rb_i) + \frac{2L^2}{\beta^3}b_ib_r\right]D^r_j\\
&+\left(\frac{2L}{\beta^2}L_i - \frac{2L^2}{\beta^3}b_i\right)(r_{j0} + s_{j0}) + \left[\frac{2L}{\beta}L_r - \frac{L^2}{\beta^2}b_r\right]{}^cD^r_{ij},
\end{aligned}$$

where $b_{i|k} = \partial_kb_i - b_rF^r_{ik}$.

Since $2r_{ij} = b_{i|j} + b_{j|i}$, $2s_{ij} = b_{i|j} - b_{j|i}$, the above equation gives

$$\begin{aligned}
-\frac{2L^2}{\beta^2}r_{ij} = &\left[\frac{2L}{\beta}L_{ir} + \frac{2}{\beta}L_iL_r - \frac{2L}{\beta^2}(L_ib_r + L_rb_i) + \frac{2L^2}{\beta^3}b_ib_r\right]D^r_j\\
&+\left[\frac{2L}{\beta}L_{jr} + \frac{2}{\beta}L_jL_r - \frac{2L}{\beta^2}(L_jb_r + L_rb_j) + \frac{2L^2}{\beta^3}b_jb_r\right]D^r_i\\
&+\left(\frac{2L}{\beta^2}L_i - \frac{2L^2}{\beta^3}b_i\right)(r_{j0} + s_{j0}) + \left(\frac{2L}{\beta^2}L_j - \frac{2L^2}{\beta^3}b_j\right)(r_{i0} + s_{i0})\\
&+2\left[\frac{2L}{\beta}L_r - \frac{L^2}{\beta^2}b_r\right]{}^cD^r_{ij}
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
-\frac{2L^2}{\beta^2}s_{ij} = &\left[\frac{2L}{\beta}L_{ir} + \frac{2}{\beta}L_iL_r - \frac{2L}{\beta^2}(L_ib_r + L_rb_i) + \frac{2L^2}{\beta^3}b_ib_r\right]D^r_j\\
&-\left[\frac{2L}{\beta}L_{jr} + \frac{2}{\beta}L_jL_r - \frac{2L}{\beta^2}(L_jb_r + L_rb_j) + \frac{2L^2}{\beta^3}b_jb_r\right]D^r_i\\
&+\left(\frac{2L}{\beta^2}L_i - \frac{2L^2}{\beta^3}b_i\right)(r_{j0} + s_{j0}) - \left(\frac{2L}{\beta^2}L_j - \frac{2L^2}{\beta^3}b_j\right)(r_{i0} + s_{i0}).
\end{aligned} \tag{2.9}$$

Subtracting (2.8) from (2.6) and re-arranging the terms, we get

$$\begin{aligned}
&\left\{\frac{2L}{\beta}L_{ijr}-\frac{2L}{\beta^2}(L_{ij}b_r+L_{ir}b_j+L_{jr}b_i)-\frac{2}{\beta^2}(L_iL_jb_r+L_iL_rb_j+L_jL_rb_i)\right.\\
&\left.+\frac{2}{\beta}(L_iL_{jr}+L_jL_{ir}+L_rL_{ij})+\frac{4L}{\beta^3}(b_ib_jL_r+b_jb_rL_i+b_ib_rL_j)-\frac{6L^2}{\beta^4}b_ib_jb_r\right\}D^r\\
&+\left[\frac{L}{\beta^2}L_{ij}+\frac{1}{\beta^2}L_iL_j+\frac{3L^2}{\beta^4}b_ib_j-\frac{2L}{\beta^3}(L_ib_j+L_jb_i)\right]r_{00}-\frac{L^2}{\beta^2}r_{ij}\\
&=\left[\frac{2L}{\beta}L_r-\frac{L^2}{\beta^2}b_r\right]{}^cD^r_{ij}.
\end{aligned}\tag{2.10}$$

Contracting (2.10) by y^i, we obtain

$$\begin{aligned}
&\left[-\frac{2L}{\beta}L_{jr}+\frac{2L}{\beta^2}(L_jb_r+L_rb_j)-\frac{2}{\beta}L_jL_r-\frac{2L^2}{\beta^3}b_jb_r\right]D^r\\
&+\left[\frac{L^2}{\beta^3}b_j-\frac{L}{\beta^2}L_j\right]r_{00}-\frac{L^2}{\beta^2}r_{0j}=\left[\frac{2L}{\beta}L_r-\frac{L^2}{\beta^2}b_r\right]D^r_j.
\end{aligned}\tag{2.11}$$

Subtracting (2.9) from (2.6) and re-arranging the terms, we get

$$\begin{aligned}
&\left\{\frac{2L}{\beta}L_{ijr}-\frac{2L}{\beta^2}(L_{ij}b_r+L_{ir}b_j+L_{jr}b_i)-\frac{2}{\beta^2}(L_iL_jb_r+L_iL_rb_j+L_jL_rb_i)\right.\\
&\left.+\frac{2}{\beta}(L_iL_{jr}+L_jL_{ir}+L_rL_{ij})+\frac{4L}{\beta^3}(b_ib_jL_r+b_jb_rL_i+b_ib_rL_j)-\frac{6L^2}{\beta^4}b_ib_jb_r\right\}D^r\\
&+\left[\frac{L}{\beta^2}L_{ij}+\frac{1}{\beta^2}L_iL_j+\frac{3L^2}{\beta^4}b_ib_j-\frac{2L}{\beta^3}(L_ib_j+L_jb_i)\right]r_{00}\\
&+\left(\frac{2L}{\beta^2}L_i-\frac{2L^2}{\beta^3}b_i\right)(r_{j0}+s_{j0})+\left[\frac{2L}{\beta}L_{ir}-\frac{2L}{\beta^2}(L_ib_r+L_rb_i)+\frac{2}{\beta}L_iL_r\right.\\
&\left.+\frac{2L^2}{\beta^3}b_ib_r\right]D^r_j=-\frac{L^2}{\beta^2}s_{ij}.
\end{aligned}\tag{2.12}$$

Contracting (2.11) and (2.12) by y^j, we get

$$\left[\frac{2L}{\beta}L_r-\frac{L^2}{\beta^2}b_r\right]D^r=-\frac{L^2}{2\beta^2}r_{00}\tag{2.13}$$

and

$$\left[\frac{2L}{\beta}L_{ir}-\frac{2L}{\beta^2}(L_ib_r+L_rb_i)+\frac{2}{\beta}L_iL_r+\frac{2L^2}{\beta^3}b_ib_r\right]D^r=-\frac{L^2}{\beta^2}s_{i0}+\left(\frac{L^2}{\beta^3}b_i-\frac{L}{\beta^2}L_i\right)r_{00}.\tag{2.14}$$

In view of $LL_{ir}=g_{ir}-L_iL_r$, the equation (2.14) can be written as

$$\frac{2}{\beta}g_{ir}D^r+\left[\frac{2L^2}{\beta^3}b_i-\frac{2L}{\beta^2}L_i\right](b_rD^r)-\frac{2L}{\beta^2}b_i(L_rD^r)=-\frac{L^2}{\beta^2}s_{i0}+\left(\frac{L^2}{\beta^3}b_i-\frac{L}{\beta^2}L_i\right)r_{00}.\tag{2.15}$$

Contracting (2.15) by $b^i=g^{ij}b_j$, we get

$$2b^2L^2(b_rD^r)-2b^2\beta L(L_rD^r)=-\beta L^2s_0+(L^2b^2-\beta^2)r_{00},\tag{2.16}$$

where we have written s_0 for $s_{r0}b^r$.

Equation (2.13) can be written as

$$-2L^2(b_rD^r) + 4\beta L(L_rD^r) = -L^2 r_{00}. \tag{2.17}$$

The equation (2.16) and (2.17) constitute the system of algebraic equations in L_rD^r and b_rD^r. Solving equations (2.16) and (2.17) for b_rD^r and L_rD^r, we get

$$b_rD^r = \frac{1}{2L^2b^2}[(b^2L^2 - 2\beta^2)r_{00} - 2\beta L^2 s_0] \tag{2.18}$$

and

$$L_rD^r = -\frac{1}{2L^2b^2}[L^3 s_0 + \beta L r_{00}]. \tag{2.19}$$

Contracting (2.15) by g^{ji} and re-arranging terms, we obtain

$$D^j = \left[\frac{2\beta L(b_rD^r) - \beta L r_{00}}{2\beta^2}\right] l^j + \left[\frac{L^2 r_{00} + 2\beta L(L_rD^r) - 2L^2(b_rD^r)}{2\beta^2}\right] b^j - \frac{L^2}{2\beta} s_0^j \tag{2.20}$$

Putting the values of b_rD^r and L_rD^r from equations (2.18) and (2.19) respectively in (2.20), we get

$$D^i = \left(\frac{\beta r_{00} + L^2 s_0}{2b^2\beta}\right) b^i - \left(\frac{\beta r_{00} + L^2 s_0}{b^2L^2}\right) y^i - \frac{L^2}{2\beta} s_0^i, \quad \text{where} \quad l^i = \frac{y^i}{L}. \tag{2.21}$$

Proposition 2.1 *The difference tensor $D^i = \overline{G}^i - G^i$ of Kropina change of Finsler metric is given by (2.21).*

§3. Projective Change of Finsler Metric

The Finsler space $\overline{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\overline{F}^n$. Thus the change $L \to \overline{L}$ is projective if $\overline{G}^i = G^i + P(x,y)y^i$, where $P(x,y)$ is a homogeneous scalar function of degree one in y^i, called projective factor [4].

Thus from (2.2) it follows that $L \to \overline{L}$ is projective iff $D^i = Py^i$. Now we consider that the Kropina change $L \to \overline{L} = \dfrac{L^2}{\beta}$ is projective. Then from equation (2.21), we have

$$Py^i = \left(\frac{\beta r_{00} + L^2 s_0}{2b^2\beta}\right) b^i - \left(\frac{\beta r_{00} + L^2 s_0}{b^2L^2}\right) y^i - \frac{L^2}{2\beta} s_0^i. \tag{3.1}$$

Contracting (3.1) by $y_i\,(= g_{ij}\,y^j)$ and using the fact that $s_0^i y_i = 0$ and $y_i\,y^i = L^2$, we get

$$P = -\frac{1}{2b^2L^2}(\beta r_{00} + L^2 s_0). \tag{3.2}$$

Putting the value of P from (3.2) in (3.1), we get

$$\left(\frac{\beta r_{00} + L^2 s_0}{2b^2L^2}\right) y^i = \left(\frac{\beta r_{00} + L^2 s_0}{2b^2\beta}\right) b^i - \frac{L^2}{2\beta} s_0^i. \tag{3.3}$$

Transvecting (3.3) by b_i, we get

$$r_{00} = -\frac{\beta s_0}{\triangle}, \quad \text{where} \quad \triangle = \left(\frac{\beta}{L}\right)^2 - b^2 \neq 0. \tag{3.4}$$

Substituting the value of r_{00} from (3.4) in (3.2), we get

$$P = \frac{1}{2\triangle} s_0. \tag{3.5}$$

Eliminating P and r_{00} from (3.5), (3.4) and (3.2), we get

$$s^i_0 = \left[\frac{\beta}{L^2} y^i - b^i\right] \frac{s_0}{\triangle}. \tag{3.6}$$

The equations (3.4) and (3.6) give the necessary conditions under which a Kropina change becomes a projective change.

Conversely, if conditions (3.4) and (3.6) are satisfied, then putting these conditions in (2.21), we get

$$D^i = \frac{s_0}{2\triangle} y^i \ \text{ i.e. } \ D^i = Py^i, \quad \text{where} \quad P = \frac{s_0}{2\triangle}.$$

Thus $\overline{F}^n$ is projective to F^n.

Theorem 3.1 *The Kropina change of a Finsler space is projective if and only if (3.4) and (3.6) hold and then the projective factor P is given by $P = \frac{s_0}{2\triangle}$, where $\triangle = \left(\frac{\beta}{L}\right)^2 - b^2$.*

§4. A Particular Case

Let us assume that L is a metric of a Riemannian space i.e. $L = \sqrt{a_{ij}(x)y^i y^j} = \alpha$. Then $\overline{L} = \dfrac{\alpha^2}{\beta}$ which is the metric of Kropina space. In this case $b_{i|j} = b_{i;j}$ where $;j$ denotes the covariant derivative with respect to Christoffel symbols constructed from Riemannian metric α. Thus r_{ij} and s_{ij} are functions of coordinates only and in view of theorem (3.1) it follows that the Riemannian space is projective to Kropina space iff $r_{00} = -\dfrac{\beta}{\triangle} s_0$ and $s^i_0 = \left(\dfrac{\beta}{\alpha^2} y^i - b^i\right) \frac{s_0}{\triangle}$, where $\triangle = \left(\dfrac{\beta}{\alpha}\right)^2 - b^2 \neq 0$. These equations may be written as

(4.1) (a) $r_{00}\beta^2 = \alpha^2(b^2 r_{00} - \beta s_0)$; (b) $s^i_0(\beta^2 - b^2\alpha^2) = (\beta^2 y^i - \alpha^2 b^i)s_0$.

From (4.1)(a), it follows that if $\alpha^2 \not\equiv o(\text{mod } \beta)$ i.e. β is not a factor of α^2, then there exists a scalar function $f(x)$ such that

(4.2) (a) $b^2 r_{00} - \beta s_0 = \beta^2 f(x)$; (b) $r_{00} = \alpha^2 f(x)$.

From (4.2)(b), we get $r_{ij} = f(x)a_{ij}$ and therefore (4.2)(a) reduces to

$$\beta s_0 = (b^2\alpha^2 - \beta^2)f(x).$$

This equation may be written as

$$b_i s_j + b_j s_i = 2(b^2 a_{ij} - b_i b_j) f(x)$$

which after contraction with b^j gives $b^2 s_i = 0$. If $b^2 \neq 0$ then we get $s_i = 0$, i.e. $s_{ij} = 0$.

Hence equation (4.1) holds identically and (4.2)(a) and (b) give

$$(b^2\alpha^2 - \beta^2) f(x) = 0 \quad \text{i.e.} \quad f(x) = 0 \quad \text{as} \quad b^2\alpha^2 - \beta^2 \neq 0.$$

Thus $r_{00} = 0$, i.e. $r_{ij} = 0$. Hence $b_{i;j} = 0$, i.e. the pair (α, β) is parallel pair.

Conversely, if $b_{i;j} = 0$, the equation (4.1)(a) and (4.1)(b) hold identically. Thus we get the following theorem which has been proved in [1], [7].

Theorem 4.1 *The Riemannian space with metric α is projective to a Kropina space with metric $\frac{\alpha^2}{\beta}$ iff the (α, β) is parallel pair.*

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Geometric Mean Labeling
Of Graphs Obtained from Some Graph Operations

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Abstract: A function f is called a geometric mean labeling of a graph $G(V,E)$ if $f : V(G) \to \{1,2,3,\ldots,q+1\}$ is injective and the induced function $f^* : E(G) \to \{1,2,3,\ldots,q\}$ defined as

$$f^*(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor,\ \forall uv \in E(G),$$

is bijective. A graph that admits a geometric mean labeling is called a geometric mean graph. In this paper, we have discussed the geometric meanness of graphs obtained from some graph operations.

Key Words: Labeling, geometric mean labeling, geometric mean graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [3]. For a detailed survey on graph labeling, we refer [2].

Cycle on n vertices is denoted by C_n and a path on n vertices is denoted by P_n. A tree T is a connected acyclic graph. Square of a graph G, denoted by G^2, has the vertex set as in G and two vertices are adjacent in G^2 if they are at a distance either 1 or 2 apart in G. A graph obtained from a path of length m by replacing each edge by C_n is called as mC_n-snake, for $m \geq 1$ ad $n \geq 3$.

The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if either they are adjacent vertices of G or adjacent edges of G or one is a vertex of G and the other one is an edge incident on it. The graph Tadpoles $T(n,k)$ is obtained by identifying a vertex of the cycle C_n to an end vertex of the path P_k. The H-graph is obtained from two paths $u_1,u_2,\ldots,u_n$ and $v_1,v_2,\cdots,v_n$ of equal length by joining an edge $u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}$ when n is odd and $u_{\frac{n+2}{2}}v_{\frac{n}{2}}$ when n is even. An arbitrary supersubdivision $P(m_1,m_2,\cdots,m_{n-1})$ of a path P_n is a graph obtained by replacing each i^{th} edge of P_n by identifying its end vertices of the edge with a partition of K_{2,m_i} having 2 elements, where m_i is

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any positive integer. $G \odot K_1$ is the graph obtained from G by attaching a new pendant vertex to each vertex of G.

The study of graceful graphs and graceful labeling methods was first introduced by Rosa [5]. The concept of mean labeling was first introduced by S.Somasundaram and R.Ponraj [6] and it was developed in [4,7]. S.K.Vaidya et al. [11] have discussed the mean labeling in the context of path union of cycle and the arbitrary supersubdivision of the path P_n. S.K.Vaidya et al. [8-10] have discussed the mean labeling in the context of some graph operations. In [1], A.Durai Baskar et al. introduced geometric mean labeling of graph.

A function f is called a geometric mean labeling of a graph $G(V,E)$ if $f : V(G) \rightarrow \{1,2,3,\cdots,q+1\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1,2,3,\cdots,q\}$ defined as

$$f^*(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor, \quad \forall uv \in E(G),$$

is bijective. A graph that admits a geometric mean labeling is called a geometric mean graph.

In this paper we have obtained the geometric meanness of the graphs, union of two cycles C_m and C_n, union of the cycle C_m and a path P_n, P_n^2, mC_n-snake for $m \geq 1$ and $n \geq 3$, the total graph $T(P_n)$ of P_n, the Tadpoles $T(n,k)$, the graph obtained by identifying a vertex of any two cycles C_m and C_n, the graph obtained by identifying an edge of any two cycles C_m and C_n, the graph obtained by joining any two cycles C_m and C_n by a path P_k, the H-graph and the arbitrary supersubdivision of a path $P(1,2,\cdots,n-1)$.

§2. Main Results

Theorem 2.1 *Union of any two cycles C_m and C_n is a geometric mean graph.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycles C_m and C_n respectively. We define $f : V(C_m \cup C_n) \rightarrow \{1,2,3,\cdots,m+n+1\}$ as follows:

$$f(u_i) = \begin{cases} i & \text{if } 1 \leq i \leq \left\lfloor \sqrt{m+2} \right\rfloor - 1 \\ i+1 & \text{if } \left\lfloor \sqrt{m+2} \right\rfloor \leq i \leq m-1, \end{cases}$$

$$f(u_m) = m+2 \text{ and}$$

$$f(v_i) = \begin{cases} m+n+3-2i & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ m+1 & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ m-n+2i & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} i & \text{if } 1 \leq i \leq \left\lfloor \sqrt{m+2} \right\rfloor - 1 \\ i+1 & \text{if } \left\lfloor \sqrt{m+2} \right\rfloor \leq i \leq m-1, \end{cases}$$

$$f^*(u, u_m) = \left\lfloor \sqrt{m+2} \right\rfloor,$$

$$f^*(v_iv_{i+1}) = \begin{cases} m+n+1-2i & \text{if } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ m+1 & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ m+2 & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ m-n+2i & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n-1 \end{cases}$$

and $$f^*(v_1v_n) = m+n.$$

Hence, f is a geometric mean labeling of the graph $C_m \cup C_n$. Thus the graph $C_m \cup C_n$ is a geometric mean graph, for any $m, n \ge 3$. □

A geometric mean labeling of $C_7 \cup C_{10}$ is shown in Fig.1.

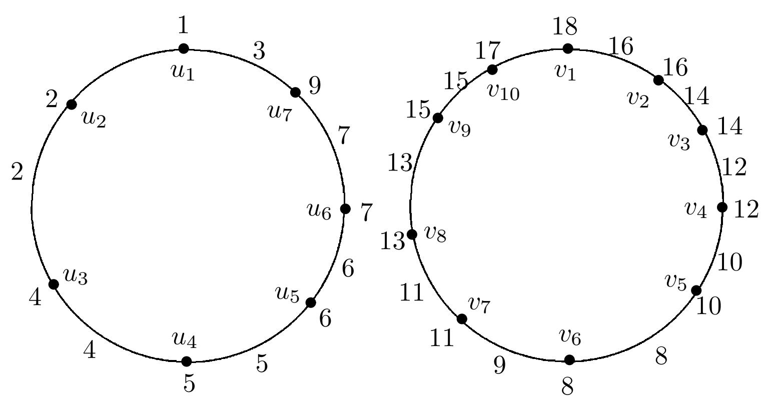


Fig.1

The graph $C_m \cup nT, n \ge 2$ cannot be a geometric mean graph. But the graph $C_m \cup T$ may be a geometric mean graph.

Theorem 2.2 *The graph $C_m \cup P_n$ is a geometric mean graph.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_m and the path P_n respectively. We define $f : V(C_m \cup P_n) \to \{1, 2, 3, \cdots, m+n\}$ as follows:

$$f(u_i) = \begin{cases} m+n+2-2i & \text{if } 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor \\ n & \text{if } i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \\ n-m-1+2i & \text{if } \left\lfloor \frac{m}{2} \right\rfloor + 2 \le i \le m, \end{cases}$$

$f(v_i) = i$, for $1 \le i \le n-1$ and

$f(v_n) = n+1.$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} m+n-2i & \text{if } 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor \\ n & \text{if } i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is odd} \\ n+1 & \text{if } i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is even} \\ n-m-1+2i & \text{if } \left\lfloor \frac{m}{2} \right\rfloor + 2 \le i \le m-1, \end{cases}$$

$$f^*(u_1u_m) = m+n-1 \text{ and}$$
$$f^*(v_iv_{i+1}) = i, \text{ for } 1 \le i \le n-1.$$

Hence, f is a geometric mean labeling of the graph $C_m \cup P_n$. Thus the graph $C_m \cup P_n$ is a geometric mean graph, for any $m \geq 3$ and $n \geq 2$. □

A geometric mean labeling of $C_{12} \cup P_7$ is shown in Fig.2.

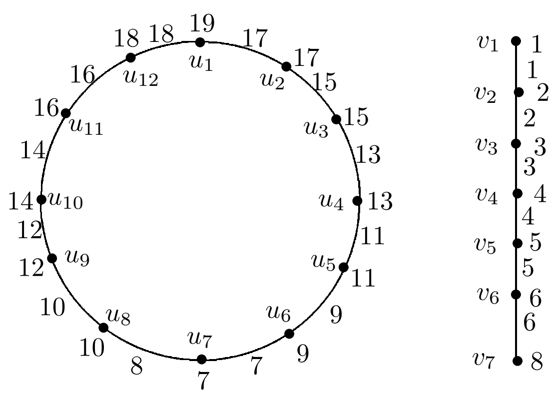


Fig.2

The T-graph T_n is obtained by attaching a pendant vertex to a neighbor of the pendant vertex of a path on $(n-1)$ vertices.

Theorem 2.3 *For a T-graph T_n, $T_n \cup C_m$ is a geometric mean graph, for $n \geq 2$ and $m \geq 3$.*

Proof Let $u_1, u_2, \cdots, u_{n-1}$ be the vertices of the path P_{n-1} and u_n be the pendant vertex identified with u_2. Let $v_1, v_2, \cdots, v_m$ be the vertices of the cycle C_m.

$$V(T_n \cup C_m) = V(C_m) \cup V(P_n) \cup \{u_n\} \text{ and}$$
$$E(T_n \cup C_m) = E(C_m) \cup E(P_n) \cup \{u_2u_n\}.$$

We define $f : V(T_n \cup C_m) \to \{1, 2, 3, \cdots, m+n\}$ as follows:

$$f(u_i) = i+1, \text{ for } 1 \le i \le n-2,$$
$$f(u_{n-1}) = n-1,$$
$$f(u_n) = 1,$$

$$f(v_i) = \begin{cases} m+n+2-2i & \text{if } 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor \\ n & \text{if } i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \\ n-m-1+2i & \text{if } \left\lfloor \frac{m}{2} \right\rfloor + 2 \le i \le m. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = i+1, \text{ for } 1 \le i \le n-2,$$
$$f^*(u_2u_n) = 1,$$
$$f^*(v_iv_{i+1}) = \begin{cases} m+n-2i & \text{if } 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor \\ n & \text{if } i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is odd} \\ n+1 & \text{if } i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is even} \\ n-m-1+2i & \text{if } \left\lfloor \frac{m}{2} \right\rfloor + 2 \le i \le m-1 \end{cases} \quad \text{and}$$
$$f^*(v_1v_m) = m+n-1.$$

Hence f is a geometric mean labeling of $T_n \cup C_m$. Thus the graph $T_n \cup C_m$ is a geometric mean graph, for $n \ge 2$ and $m \ge 3$. □

A geometric mean labeling of $T_7 \cup C_6$ is as shown in Fig.3.

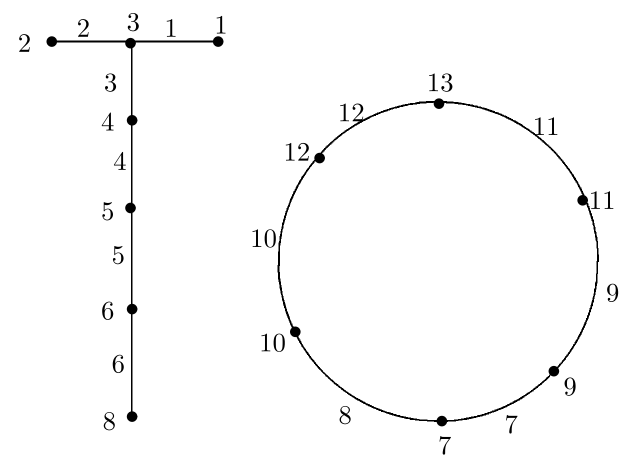


Fig.3

Theorem 2.4 *P_n^2 is a geometric mean graph, for $n \ge 3$.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n. We define $f : V(P_n^2) \to \{1, 2, 3, \cdots, 2(n-1)\}$ as follows:

$$f(v_i) = 2i-1, \text{ for } 1 \le i \le n-1 \text{ and}$$
$$f(v_n) = 2(n-1).$$

The induced edge labeling is as follows:

$$f^*(v_iv_{i+1}) = 2i-1, \text{ for } 1 \leq i \leq n-1 \text{ and}$$
$$f^*(v_iv_{i+2}) = 2i, \text{ for } \ 1 \leq i \leq n-2.$$

Hence, f is a geometric mean labeling of the graph P_n^2. Thus the graph P_n^2 is a geometric mean graph, for $n \geq 3$. □

A geometric mean labeling of P_9^2 is shown in Fig.4.

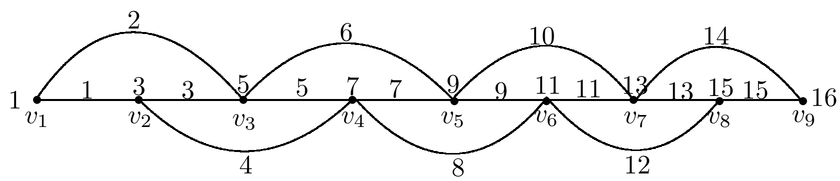


Fig.4

Theorem 2.5 *mC_n-snake is a geometric mean graph, for any $m \geq 1$ and $n = 3, 4$.*

Proof The proof is divided into two cases.

Case 1 $n = 3$.

Let $v_1^{(i)}, v_2^{(i)}$ and $v_3^{(i)}$ be the vertices of the i^{th} copy of the cycle C_3, for $1 \leq i \leq m$. The mC_3-snake G is obtained by identifying $v_3^{(i)}$ and $v_1^{(i+1)}$, for $1 \leq i \leq m-1$. We define $f : V(G) \to \{1, 2, 3 \cdots, 3m+1\}$ as follows:

$$f(v_1^{(i)}) = 3i-2, \text{ for } 1 \leq i \leq m$$
$$f(v_2^{(i)}) = 3i, \text{ for } 1 \leq i \leq m \text{ and}$$
$$f(v_3^{(i)}) = 3i+1, \text{for } 1 \leq i \leq m.$$

The induced edge labeling is as follows:

$$f^*(v_1^{(i)}v_2^{(i)}) = 3i-2, \text{ for } 1 \leq i \leq m,$$
$$f^*(v_2^{(i)}v_3^{(i)}) = 3i, \text{ for } 1 \leq i \leq m \text{ and}$$
$$f^*(v_1^{(i)}v_3^{(i)}) = 3i-1, \text{ for } 1 \leq i \leq m.$$

Hence, f is a geometric mean labeling of the graph mC_3-snake. For example, a geometric mean labeling of $6C_3$-snake is shown in Fig.5.

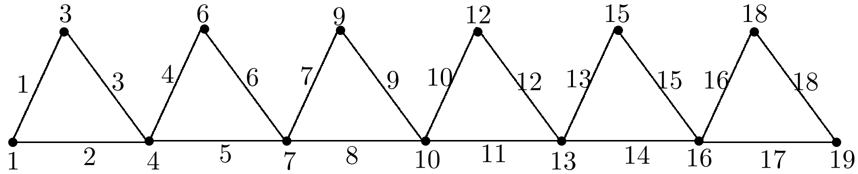


Fig.5

Case 2 $n = 4$.

Let $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}$ and $v_4^{(i)}$ be the vertices of the i^{th} copy of the cycle C_4, for $1 \leq i \leq m$. The mC_4-snake G is obtained by identifying $v_4^{(i)}$ and $v_1^{(i+1)}$, for $1 \leq i \leq m-1$. We define $f : V(G) \to \{1, 2, 3, \cdots, 4m+1\}$ as follows:

$$f(v_1^{(i)}) = 4i - 3, \text{ for } 1 \leq i \leq m,$$
$$f(v_2^{(i)}) = 4i - 1, \text{ for } 1 \leq i \leq m,$$
$$f(v_3^{(i)}) = 4i, \text{ for } 1 \leq i \leq m \text{ and}$$
$$f(v_4^{(i)}) = 4i + 1, \text{for } 1 \leq i \leq m.$$

The induced edge labeling is as follows:

$$f^*(v_1^{(i)} v_2^{(i)}) = 4i - 3, \text{ for } 1 \leq i \leq m,$$
$$f^*(v_2^{(i)} v_3^{(i)}) = 4i - 1, \text{ for } 1 \leq i \leq m$$
$$f^*(v_3^{(i)} v_4^{(i)}) = 4i, \text{ for } 1 \leq i \leq m \text{ and}$$
$$f^*(v_1^{(i)} v_4^{(i)}) = 4i - 2, \text{ for } 1 \leq i \leq m.$$

Hence, f is a geometric mean labeling of the graph mC_4-snake. □

A geometric mean labeling of $5C_4$-snake is shown in Fig.6.

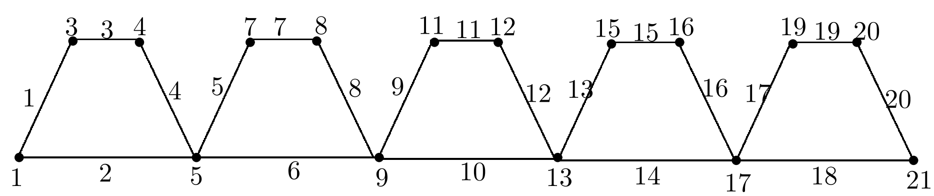


Fig.6

Theorem 2.6 *$T(P_n)$ is a geometric mean graph, for $n \geq 2$.*

Proof Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1}; 1 \leq i \leq n-1\}$ be the vertex set and edge set of the path P_n. Then

$$V(T(P_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \cdots, e_{n-1}\} \text{ and}$$
$$E(T(P_n)) = \{v_i v_{i+1}, e_i v_i, e_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i e_{i+1}; 1 \leq i \leq n-2\}.$$

We define $f : V(T(P_n)) \to \{1, 2, 3, \cdots, 4(n-1)\}$ as follows:

$$f(v_i) = 4i - 3, \text{ for } 1 \leq i \leq n-1,$$
$$f(v_n) = 4n - 4 \text{ and}$$
$$f(e_i) = 4i - 1, \text{ for } 1 \leq i \leq n-1.$$

The induced edge labeling is as follows:

$$\begin{aligned} f^*(v_iv_{i+1}) &= 4i-2, \text{ for } 1 \leq i \leq n-1, \\ f^*(e_ie_{i+1}) &= 4i, \text{ for } 1 \leq i \leq n-2, \\ f^*(e_iv_i) &= 4i-3, \text{ for } 1 \leq i \leq n-1 \text{ and} \\ f^*(e_iv_{i+1}) &= 4i-1, \text{ for } 1 \leq i \leq n-1. \end{aligned}$$

Hence, f is a geometric mean labeling of the graph $T(P_n)$. Thus the graph $T(P_n)$ is a geometric mean graph, for $n \geq 2$. □

A geometric mean labeling of $T(P_5)$ is shown in Fig.7.

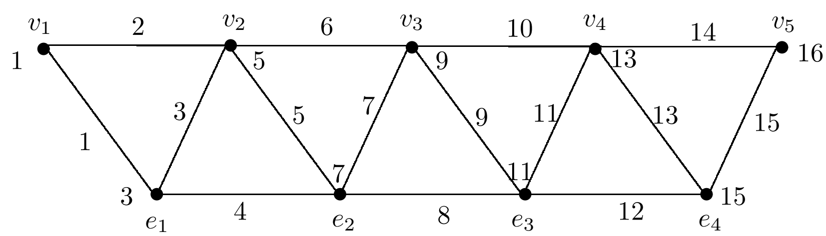


Fig.7

Theorem 2.7 *Tadpoles $T(n,k)$ is a geometric mean graph.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_k$ be the vertices of the cycle C_n and the path P_k respectively. Let $T(n,k)$ be the graph obtained by identifying the vertex u_n of the cycle C_n to the end vertex v_1 of the path P_k. We define $f : V(T(n,k)) \to \{1, 2, 3, \cdots, n+k\}$ as follows:

$$f(u_i) = \begin{cases} i & \text{if } 1 \leq i \leq \left\lfloor \sqrt{n+1} \right\rfloor - 1 \\ i+1 & \text{if } \left\lfloor \sqrt{n+1} \right\rfloor \leq i \leq n \end{cases} \quad \text{and}$$

$$f(v_i) = n+i, \text{ for } 2 \leq i \leq k.$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} i & \text{if } 1 \leq i \leq \left\lfloor \sqrt{n+1} \right\rfloor - 1 \\ i+1 & \text{if } \left\lfloor \sqrt{n+1} \right\rfloor \leq i \leq n-1, \end{cases}$$

$$f^*(u_1u_n) = \left\lfloor \sqrt{n+1} \right\rfloor \text{ and}$$

$$f^*(v_iv_{i+1}) = n+i, \text{ for } 1 \leq i \leq k-1.$$

Hence, f is a geometric mean labeling of the graph $T(n,k)$. Thus the graph $T(n,k)$ is a geometric mean graph. □

A geometric mean labeling of the Tadpoles $T(7,5)$ is shown in Fig.8.

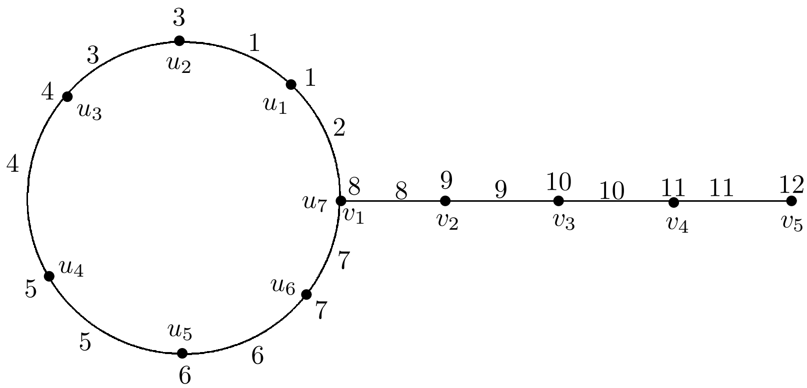


Fig.8

Theorem 2.8 *The graph obtained by identifying a vertex of any two cycles C_m and C_n is a geometric mean graph.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycles C_m and C_n respectively. Let G be the resultant graph obtained by identifying the vertex u_m of the cycle C_m to the vertex v_n of the cycle C_n. We define $f : V(G) \to \{1, 2, 3, \cdots, m+n+1\}$ as follows:

$$f(u_i) = \begin{cases} i & \text{if } 1 \leq i \leq \lfloor\sqrt{m+1}\rfloor - 1 \\ i+1 & \text{if } \lfloor\sqrt{m+1}\rfloor \leq i \leq m \end{cases} \quad \text{and}$$

$$f(v_i) = \begin{cases} m+1+i & \text{if } 1 \leq i \leq \left\lfloor\sqrt{(m+1)(m+n+1)}\right\rfloor - m - 2 \\ m+2+i & \text{if } \left\lfloor\sqrt{(m+1)(m+n+1)}\right\rfloor - m - 1 \leq i \leq n-1. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} i & \text{if } 1 \leq i \leq \lfloor\sqrt{m+1}\rfloor - 1, \\ i+1 & \text{if } \lfloor\sqrt{m+1}\rfloor \leq i \leq m-1, \end{cases}$$

$$f^*(v_iv_{i+1}) = \begin{cases} m+1+i & \text{if } 1 \leq i \leq \left\lfloor\sqrt{(m+1)(m+n+1)}\right\rfloor - m - 2, \\ m+2+i & \text{if } \left\lfloor\sqrt{(m+1)(m+n+1)}\right\rfloor - m - 1 \leq i \leq n-2, \end{cases}$$

$$f^*(u_1u_m) = \lfloor\sqrt{m+1}\rfloor,$$

$$f^*(v_{n-1}v_n) = \left\lfloor\sqrt{(m+1)(m+n+1)}\right\rfloor \text{ and}$$

$$f^*(v_1v_n) = m+1.$$

Hence, f is a geometric mean labeling of the graph G. Thus the resultant graph G is a geometric mean graph. □

A geometric mean labeling of the graph G obtained by identifying a vertex of the cycles C_8 and C_{12}, is shown in Fig.9.

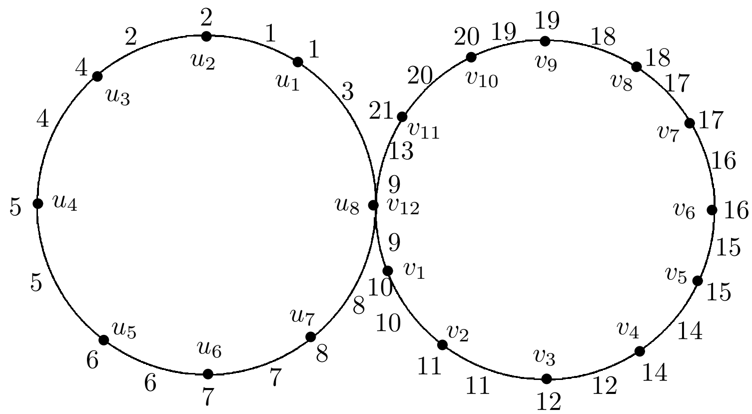


Fig.9

Theorem 2.9 *The graph obtained by identifying an edge of any two cycles C_m and C_n is a geometric mean graph.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycles C_m and C_n respectively. Let G be the resultant graph obtained by identifying an edge $u_{m-1}u_m$ of cycle C_m with an edge $v_{n-1}v_n$ of the cycle C_n. We define $f : V(G) \to \{1, 2, 3, \cdots, m+n\}$ as follows:

$$f(u_i) = \begin{cases} i & \text{if } 1 \leq i \leq \left\lfloor \sqrt{m+1} \right\rfloor - 1 \\ i+1 & \text{if } \left\lfloor \sqrt{m+1} \right\rfloor \leq i \leq m \end{cases} \quad \text{and}$$

$$f(v_i) = \begin{cases} m+1+i & \text{if } 1 \leq i \leq \left\lfloor \sqrt{m(m+n)} \right\rfloor - m - 2 \\ m+2+i & \text{if } \left\lfloor \sqrt{m(m+n)} \right\rfloor - m - 1 \leq i \leq n-2. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} i & \text{if } 1 \leq i \leq \left\lfloor \sqrt{m+1} \right\rfloor - 1 \\ i+1 & \text{if } \left\lfloor \sqrt{m+1} \right\rfloor \leq i \leq m-1, \end{cases}$$

$$f^*(v_iv_{i+1}) = \begin{cases} m+1+i & \text{if } 1 \leq i \leq \left\lfloor \sqrt{m(m+n} \right\rfloor - m - 2 \\ m+2+i & \text{if } \left\lfloor \sqrt{m(m+n)} \right\rfloor - m - 1 \leq i \leq n-3, \end{cases}$$

$$f^*(u_1u_m) = \left\lfloor \sqrt{m+1} \right\rfloor,$$

$$f^*(v_1v_n) = m+1 \text{ and}$$

$$f^*(v_{n-2}v_{n-1}) = \left\lfloor \sqrt{m(m+n)} \right\rfloor.$$

Hence, f is a geometric mean labeling of the graph G. Thus the resultant graph G is a geometric mean graph. □

A geometric mean labeling of the graph G obtained by identifying an edge of the cycles C_{10} and C_{13}, is shown in Fig.10.

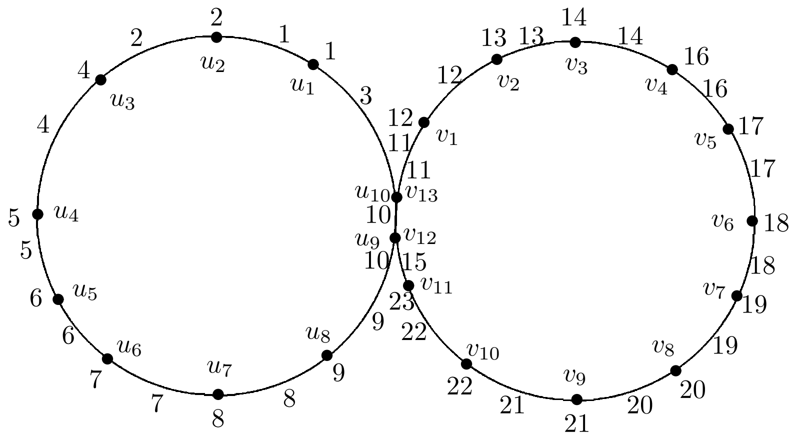


Fig.10

Theorem 2.10 *The graph obtained by joining any two cycles C_m and C_n by a path P_k is a geometric mean graph.*

Proof Let G be a graph obtained by joining any two cycles C_m and C_n by a path P_k. Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycles C_m and C_n respectively. Let $w_1, w_2, \ldots, w_k$ be the vertices of the path P_k with $u_m = w_1$ and $w_k = v_n$. We define $f : V(G) \to \{1, 2, 3, \cdots, m+k+n\}$ as follows:

$$f(u_i) = \begin{cases} i & \text{if } 1 \le i \le \lfloor \sqrt{m+1} \rfloor - 1 \\ i+1 & \text{if } \lfloor \sqrt{m+1} \rfloor \le i \le m, \end{cases}$$

$$f(w_i) = m+i, \text{ for } 2 \le i \le k \text{ and}$$

$$f(v_i) = \begin{cases} m+k+i & \text{if } 1 \le i \le \left\lfloor \sqrt{(m+k)(m+k+n)} \right\rfloor - m - k - 1 \\ m+k+1+i & \text{if } \left\lfloor \sqrt{(m+k)(m+k+n)} \right\rfloor - m - k \le i \le n-1. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} i & \text{if } 1 \le i \le \lfloor \sqrt{m+1} \rfloor - 1 \\ i+1 & \text{if } \lfloor \sqrt{m+1} \rfloor \le i \le m-1, \end{cases}$$

$$f^*(w_iw_{i+1}) = m+i, \text{ for } 1 \le i \le k-1,$$

$$f^*(v_iv_{i+1}) = \begin{cases} m+k+i & \text{if } 1 \le i \le \left\lfloor \sqrt{(m+k)(m+k+n)} \right\rfloor - m - k - 1 \\ m+k+1+i & \text{if } \left\lfloor \sqrt{(m+k)(m+k+n)} \right\rfloor - m - k \le i \le n-2, \end{cases}$$

$$f^*(u_1u_m) = \lfloor \sqrt{m+1} \rfloor,$$

$$f^*(v_nv_{n-1}) = \left\lfloor \sqrt{(m+k)(m+k+n)} \right\rfloor \text{ and}$$

$$f^*(v_1v_n) = m+k.$$

Hence, f is a geometric mean labeling of the graph G. Thus the resultant graph G is a geometric mean graph. □

A geometric mean labeling of the graph G obtained by joining two cycles C_7 and C_{10} by a path P_4, is shown in Fig.11.

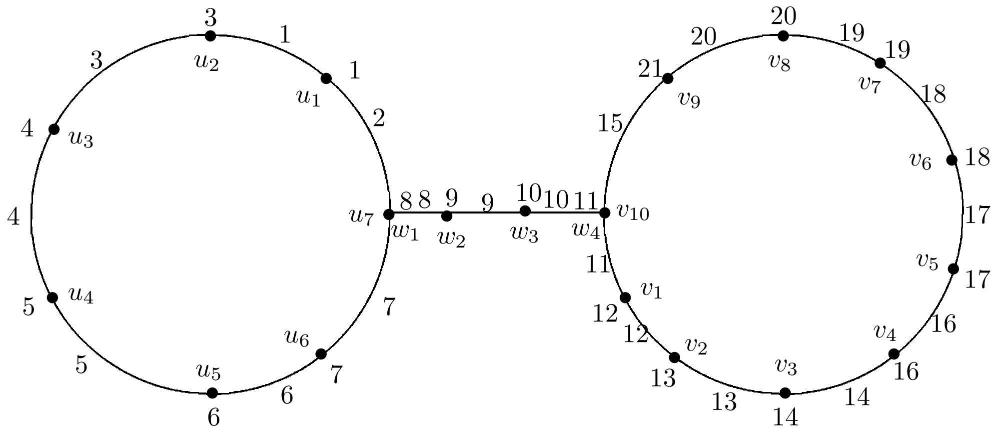


Fig.11

Theorem 2.11 *Any H-graph G is a geometric mean graph.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices on the paths of length n in G.

Case 1 n is odd.

We define $f : V(G) \to \{1, 2, 3, \cdots, 2n\}$ as follows:

$$f(u_i) = i, \text{ for } 1 \le i \le n \text{ and}$$

$$f(v_i) = \begin{cases} n + 2i & \text{if } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ n + 2i - 1 & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 3n + 1 - 2i & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_i u_{i+1}) = i, \text{ for } 1 \le i \le n - 1,$$

$$f^*(u_i v_i) = n, \text{ for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and}$$

$$f^*(v_i v_{i+1}) = \begin{cases} n + 2i & \text{if } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 3n - 1 - 2i & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n - 1. \end{cases}$$

Case 2 n is even.

We define $f : V(G) \to \{1, 2, 3, \cdots, 2n\}$ as follows:

$$f(u_i) = i, \text{ for } 1 \le i \le n \text{ and}$$

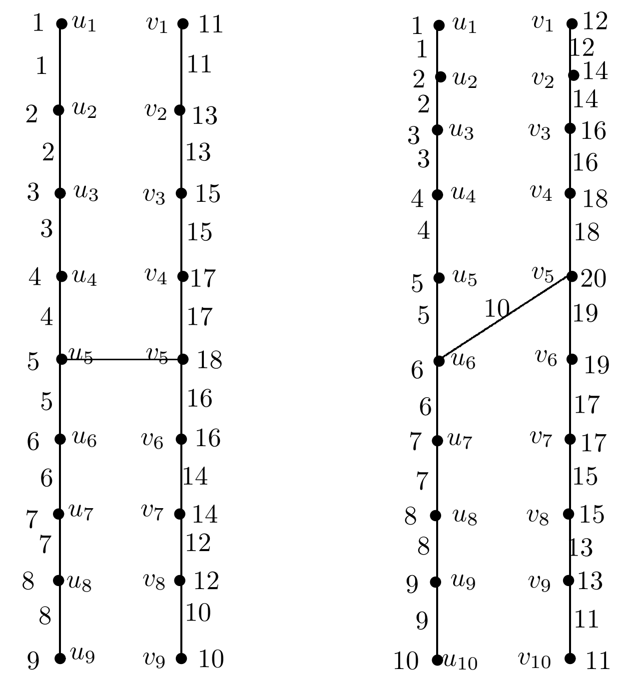
$$f(v_i) = \begin{cases} n + 2i & \text{if } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 3n + 1 - 2i & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(u_iu_{i+1}) = i, \text{ for } 1 \le i \le n-1,$$
$$f^*(u_{i+1}v_i) = n, \text{ for } i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and}$$
$$f^*(v_iv_{i+1}) = \begin{cases} n+2i & \text{if } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 3n-1-2i & \text{if } \left\lfloor \frac{n}{2} \right\rfloor \le i \le n-1. \end{cases}$$

Hence, H-graph admits a geometric mean labeling. □

A geometric mean labeling of H-graphs G_1 and G_2 are shown in Fig.12.



G_1 G_2

Fig.12

Theorem 2.12 *For any $n \ge 2, P(1,2,3,\cdots,n-1)$ is a geometric mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n and let u_{ij} be the vertices of the partition of K_{2,m_i} with cardinality $m_i, 1 \le i \le n-1$ and $1 \le j \le m_i$. We define $f: V(P(1,2,\cdots,n-1)) \to \{1,2,3,\ldots,n(n-1)+1\}$ as follows:

$$f(v_i) = i(i-1)+1, \text{ for } 1 \le i \le n \text{ and}$$
$$f(u_{ij}) = i(i-1)+2j, \text{ for } 1 \le j \le i \text{ and } 1 \le i \le n-1.$$

The induced edge labeling is as follows:

$$f^*(v_iu_{ij}) = i(i-1)+j, \text{ for } 1 \le j \le i \text{ and } 1 \le i \le n-1$$
$$f^*(u_{ij}v_{i+1}) = i^2+j, \text{ for } 1 \le j \le i \text{ and } 1 \le i \le n-1.$$

Hence, f is a geometric mean labeling of the graph $P(1, 2, \cdots, n-1)$. Thus the graph $P(1, 2, \cdots, n-1)$ is a geometric mean graph. □

A geometric mean labeling of $P(1, 2, 3, 4, 5)$ is shown in Fig.13.

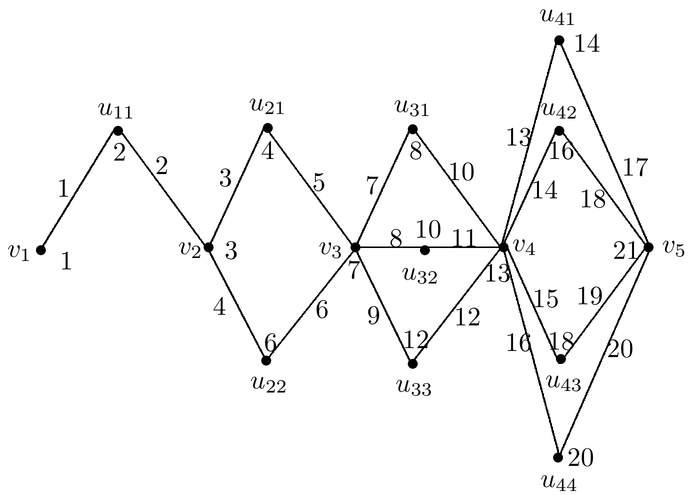


Fig.13

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4-Ordered Hamiltonicity of the Complete Expansion Graphs of Cayley Graphs

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Abstract: In this paper, we prove that the Complete expansion graph $And(k)$ $(k \geq 6)$ is 4-ordered hamiltonian graph by the method of classification discuss.

Key Words: Andrásfai graph, complete expansion graph, k-ordered hamiltonian graph.

AMS(2010): 05C38 05C45 05C70

§1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let C be a cycle with given orientation in graph X, $\vec{C}(\vec{C} = C)$ with anticlockwise direction and $\overleftarrow{C}$ with clockwise direction. If $x \in V(C)$, then we use x^+ to denote the successor of x on C and x^- to denote its predecessor. Use $C[x,y]$ denote (x,y)-path on C; $C(x,y)$ denote (x,y)-path missing x,y on C. Any undefined notation follows that of [1, 2].

Definition 1.1([1]) *Let G be a group and let C be a subset of G that is closed under taking inverses and does not contain the identity, then the Cayley graph $X(G,C)$ is the graph with vertex set G and edge set $E(X(G,C)) = \{gh : hg^{-1} \in C\}$.*

For a Cayley graph G, it may not be a hamiltonian graph, but a Cayley graph of Abelian group is a hamiltonian graph. $And(k)$ is a family of Cayley graph, which is named by the Hungarian mathematician Andrásfai, it is a k-regular graph with the order $n = 3k-1$ and it is a hamiltonian graph.

Definition 1.2([1]) *For any integer $k \geq 1$, let $G = Z_{3k-1}$ denote the additive group of integer modulo $3k-1$ and let C be the subset of Z_{3k-1} consisting of the elements congruent to 1 modulo 3. Then we denote the Cayley graph $X(G,C)$ by $And(k)$.*

For convenience, we note $Z_{3k-1} = \{u_0, u_1, \ldots, u_{3k-2}\}$. For $u_i, u_j \in V[And(k)], u_i \sim u_j$ if and only if $j - i \equiv \pm 1 mod 3$. The result are directly by the definition of Andrásfai graph.

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Lemma 1.3 *Let C be any hamiltonian cycle in $And(k)(k \geq 2)$.*

(1) *If $\forall u, x \in V(And(k))$, $u \sim x$ is a chord of C, then $u^- \sim x^-$, $u^+ \sim x^+$.*

(2) *If $\forall u, x, y \in V(And(k))$, $u \sim x$, $u \sim y$ are two chords of C, then $x \sim y^+$.*

The definition of k-ordered hamiltonian graph was given in 1997 by Lenhard as follows.

Definition 1.4([3]) *A hamiltonian graph G of order ν is k-ordered, $2 \leq k \leq \nu$, if for every sequence $(v_1, v_2, \ldots, v_k)$ of k distinct vertices of G, there exists a hamiltonian cycle that encounters $(v_1, v_2, \ldots, v_k)$ in this order.*

Faudree developed above definition into a k-ordered graph.

Definition 1.5([4]) *For a positive integer k, a graph G is k-ordered if for every ordered set of k vertices, there is a cycle that encounters the vertices of the set in the given order.*

It has been shown that $And(k)(k \geq 4)$ is 4-ordered hamiltonian graph by in [5]. The concept of expansion transformation graph of a graph was given in 2009 by A Yongga at first. Then an equivalence definition of complete expansion graph was given by her, that is, the method defined by Cartesian product in [6] as follows.

Definition 1.6([6]) *Let G be any graph and $L(G)$ be the line graph of G. Non-trivial component of $G \square L(G)$ is said complete expansion graph (CEG for short) of G, denoted by $\vartheta(G)$, said the map ϑ be a complete expansion transformation of G.*

The proof of main result in this paper is mainly according to the following conclusions.

Theorem 1.7([1]) *The Cayley graph $X(G, C)$ is vertex transitive.*

Theorem 1.8([5]) *$And(k)(k \geq 4)$ is 4-ordered hamiltonian graph.*

Theorem 1.9([7]) *Every even regular graph has a 2-factor.*

The notations following is useful throughout the paper. For $u \in V(G)$, the clique with the order $d_G(u)$ in $\vartheta(G)$ by u is denoted as $\vartheta(u)$. All cliques are the cliques in $\vartheta(And(k))$ determined by the vertices in $And(k)$, that is maximum Clique. For u, $v \in V(G)$, $\vartheta(u) \sim \vartheta(v)$ means there exist $x \in V(\vartheta(u))$, $y \in V(\vartheta(v))$, $\mathtt{s.t.} x \sim y$ in $V(\vartheta(G))$, edge (x, y) is said an edge stretching out from $\vartheta(u)$. Use $G_{\vartheta(u)}[x, y; s, t]$ to denote (x, y)-longest path missing s, t in $\vartheta(u)$, where $x, y, s, t \in V(\vartheta(u))$.

§2. Main Results with Proofs

We consider that whether $\vartheta(And(k))$ $(k \geq 4)$ is 4-ordered hamiltonian graph or not in this section.

Theorem 2.1 *$\vartheta(And(k))$ $(k \geq 6)$ is a 4-ordered hamiltonian graph.*

The following lemmas are necessary for the proof of Theorem 2.1.

Lemma 2.2 *For any $u \in V(And(k))(k \geq 2)$, $\forall x, y \in N(u)$, there exists a hamiltonian cycle C in $And(k)$, s.t. $ux \in E(C)$ and $uy \in E(C)$.*

Proof Let C_0 is a hamiltonian cycle $u_0 \sim u_1 \sim u_2 \sim \ldots \sim u_{3k-2} \sim u_0$ in $And(k)(k \geq 2)$. For $u \in V(And(k))(k \geq 2)$, $\forall x, y \in N(u)$, then we consider the following cases.

Case 1 $x \sim u \sim y$ on $\overleftarrow{C}_0$. Then $C = C_0$ is that so, since C_0 is a hamiltonian cycle.

Case 2 $x \sim u$ and $u \not\sim y$ on $\overleftarrow{C}_0$ or $x \not\sim u$ and $u \sim y$ on $\overleftarrow{C}_0$. If $x \sim u$ and $u \not\sim y$ on $\overleftarrow{C}_0$, then we can find a hamiltonian cycle C in $And(k)(k \geq 2)$ according to Lemma 1, that is,

$$C = u \sim x \sim \overrightarrow{C}_0(x, y^-) \sim y^- \sim u^- \sim \overleftarrow{C}_0(u^-, y) \sim y \sim u;$$

If $x \not\sim u$ and $u \sim y$ on $\overleftarrow{C}_0$, then we can find a hamiltonian cycle C in $And(k)(k \geq 2)$ according to Lemma 1.3, that is,

$$C = u \sim x \sim \overleftarrow{C}_0(x, u^+) \sim u^+ \sim x^+ \sim \overrightarrow{C}_0(x^+, y) \sim y \sim u.$$

Case 3 $x \not\sim u \not\sim y$ on $\overleftarrow{C}_0$. Then we can find a hamiltonian cycle C in $And(k)(k \geq 2)$ according to Lemma 1.3, that is,

$$C = u \sim x \sim y^+ \sim \overrightarrow{C}_0(y^+, u^-) \sim u^- \sim x^- \sim \overleftarrow{C}_0(x^-, u^+) \sim u^+ \sim \overrightarrow{C}_0[x^+, y] \sim u.$$

For any $u \in V(And(k))(k \geq 2)$, Lemma 2.2 is true since $And(k)$ is vertex transitive. □

Corollary 2.3 *For any two edges which stretch out from any Clique, there exists a hamiltonian cycle in $\vartheta(And(k))$ containing them.*

Lemma 2.4 *If k is an odd number, then $And(k)(k \geq 3)$ can be decomposed into one 1-factor and $\dfrac{k-1}{2}$ 2-factors.*

Proof $3k-1$ is an even number, since k is an odd number. There exists one 1-factor M in $And(k)$ by the definition of $And(k)$. According to Theorem 1.9 and the condition of Lemma 2.4 for integers $k \geq 3$, $And(k) - E(M)$ is a $(k-1)$-regular graph with a hamiltonian cycle C_1, $And(k) - E(M) - E(C_1)$ is a $(k-3)$-regular graph with a hamiltonian cycle C_2, $\cdots$, $And(k) - E(M) - \sum\limits_{i=1}^{\frac{k-1}{2}} E(C_i)$ is an empty graph.

Assume $k = 2r+1 (r \in Z^+)$, since k is an odd number. First we shall prove the result for $r = 1$, and then by induction on r. If $r = 1$ $(k = 3)$, it is easy to see that $And(k) - E(M)$ is a hamiltonian cycle C_1 by Theorem 1.9 and the analysis form of Lemma 2.4 , so the result is clearly true.

Now, we assume that the result is true if $r = n(r \geq 1, k = 2n+1)$, that is, $And(2n+1)$ can be decomposed into one 1-factor and n 2-factors. Considering the case of $r = n+1(k = 2n+3$, we know $And(2n+3)(And[2(n+1)+1])$ can be decomposed into one 1-factor and $n+1$ 2-factors according to the induction.

Thus, if k is an odd number, then $And(k)(k \geq 3)$ can be decomposed into one 1-factor and $\dfrac{k-1}{2}$ 2-factors. □

Proof of Theorem 2.1

$\vartheta(And(k))$ is a hamiltonian graph, since $And(k)$ is a hamiltonian graph. So there exists a hamiltonian cycle C_0 in $\vartheta(And(k))$ and a hamiltonian cycle $C_0{}'$ in $And(k)$, such that $C_0 = \vartheta(C_0{}')$, without loss of generality

$$C_0{}' = u_0 u_1 \ldots u_{3k-2} u_0,$$

then

$$C_0 = u_{0,1} u_{0,2} \ldots u_{0,k} u_{1,1} u_{1,2} \ldots u_{1,k} \ldots u_{3k-2,1} u_{3k-2,2} \ldots u_{3k-2,k} u_{0,1},$$

where $u_{i,j} \in V(\vartheta(u_i))$, $u_i \in V(And(k))$, $d_{And(k)}(u_i) = k \geq 6$, $i = 0, 1, 2, \ldots, 3k-2$, and $u_{i,1}^-$, $u_{i,k}^+ \notin V(\vartheta(u_i))$, $u_{i,j}^+ = u_{i,j+1}$ $(1 \leq j \leq k-1)$ and $u_{i,l}^- = u_{i,l-1}$ $(2 \leq l \leq k)$. There are three cyclic orders $\forall u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h} \in V[\vartheta(And(k))]$ according to the definition of the ring arrangement of the second kind, as follows: $(u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h})$, $(u_{a,b}, u_{e,f}, u_{c,d}, u_{g,h})$, $(u_{a,b}, u_{c,d}, u_{g,h}, u_{e,f})$(see Fig.1). Let $S = \{(u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h}), (u_{a,b}, u_{e,f}, u_{c,d}, u_{g,h}), (u_{a,b}, u_{c,d}, u_{g,h}, u_{e,f})\}$.

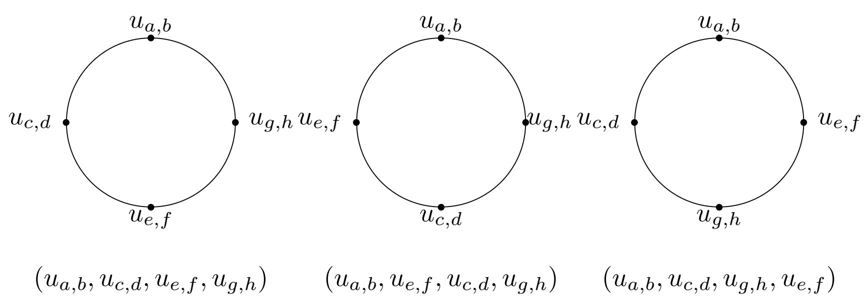

Fig.1 Three cyclic orders

Now, we show that 4-ordered hamiltonicity of $\vartheta(And(k))$ $(k \geq 6)$. In fact, we need to prove that $\alpha \in S$, there exists a hamiltonian cycle containing α. Without loss of generality, hamiltonian cycle C_0 encounters $(u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h})$ in this order. So we just prove: $\forall \beta \in S \setminus (u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h})$, there exists a hamiltonian cycle containing β.

According to the Pigeonhole principle, we consider following cases.

Case 1 If these four vertices $u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h}$ are contained in distinct four Cliques of $\vartheta(And(k))$, respectively. And Theorem 2.1 is true by the result in [5].

Case 2 If these four vertices $u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h}$ are contained in a same Clique of $\vartheta(And(k))$, then $a = c = e = g, b < d < f < h$. Let $S = \{(u_{a,b}, u_{a,d}, u_{a,f}, u_{a,h}), (u_{a,b}, u_{a,f}, u_{a,d}, u_{a,h}), (u_{a,b}, u_{a,d}, u_{a,h}, u_{a,f})\}$.

(1) For $(u_{a,b}, u_{a,d}, u_{a,f}, u_{a,h}) \in S$. C_0 is the hamiltonian cycle that encounters $(u_{a,b}, u_{a,d}, u_{a,f}, u_{a,h})$ in this order, clearly.

(2) For $(u_{a,b}, u_{a,f}, u_{a,d}, u_{a,h}) \in S$. We can find a hamiltonian cycle

$$C = u_{a,b}\overrightarrow{C}_0(u_{a,b}, u^-_{a,d})u^-_{a,d}u_{a,f}\overleftarrow{C}_0(u_{a,f}, u_{a,d})u_{a,d}u^+_{a,f}\overrightarrow{C}_0(u^+_{a,f}, u_{a,h})u_{a,h}\overrightarrow{C}_0(u_{a,h}, u_{a,b})u_{a,b}$$

that encounters $(u_{a,b}, u_{a,f}, u_{a,d}, u_{a,h})$ in this order.

(3) For $(u_{a,b}, u_{a,d}, u_{a,h}, u_{a,f}) \in S$. We can find a hamiltonian cycle

$$C = u_{a,1}\overrightarrow{C}_0(u_{a,1}, u^-_{a,f})u^-_{a,f}u_{a,k}\overleftarrow{C}_0(u_{a,k}, u_{a,f})u_{a,f}\vartheta(C_1)(u_{a,f}, u_{a,1})u_{a,1}$$

that encounters $(u_{a,b}, u_{a,d}, u_{a,h}, u_{a,f})$ in this order by Lemma 2.2 and Corollary 2.3 (see Fig.2).

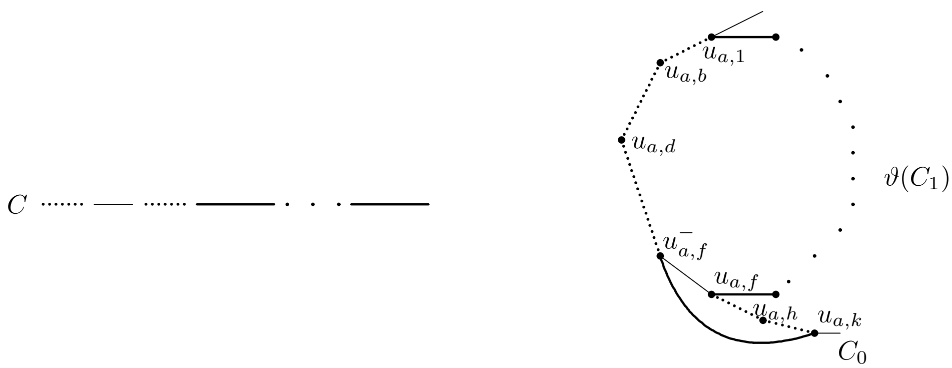

Fig.2 $C = u_{a,1}\overrightarrow{C}_0(u_{a,1}, u^-_{a,f})u^-_{a,f}u_{a,k}\overleftarrow{C}_0(u_{a,k}, u_{a,f})u_{a,f}\vartheta(C_1)(u_{a,f}, u_{a,1})u_{a,1}$

Case 3 If these four vertices $u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h}$ are contained in distinct two Cliques of $\vartheta(And(k))$, without loss of generality, we assume that $u_{a,b}, u_{c,d} \in V(\vartheta(u_a))$ and $u_{e,f}, u_{g,h} \in V(\vartheta(u_e))$ in $\vartheta(And(k))$ or $u_{a,b}, u_{c,d}, u_{e,f} \in V(\vartheta(u_a))$ and $u_{g,h} \in V(\vartheta(u_g))$ in $\vartheta(And(k))$ according to the notations. Let $S = \{(u_{a,b}, u_{a,d}, u_{e,f}, u_{e,h}), (u_{a,b}, u_{e,f}, u_{a,d}, u_{e,h}), (u_{a,b}, u_{a,d}, u_{e,h}, u_{e,f})\}$.

Subcase 3.1 $u_{a,b}, u_{c,d} \in V(\vartheta(u_a))$ and $u_{e,f}, u_{g,h} \in V(\vartheta(u_e))$ in $\vartheta(And(k))$.

(1) For $(u_{a,b}, u_{a,d}, u_{e,f}, u_{e,h}) \in S$. C_0 is the hamiltonian cycle that encounters $(u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h})$ in this order, clearly.

(2) For $(u_{a,b}, u_{e,f}, u_{a,d}, u_{e,h}) \in S$. Let C_1 is a hamiltonian cycle in $And(k)$ or $And(k) - E(M)$, C_2 is a hamiltonian cycle in $And(k) - E(C_1)$ or $And(k) - E(M) - E(C_1)$(see Fig.3). Use $A(C_1)$ to denote a cycle that only through two vertices in $\vartheta(u_i)(i = 1, 2, \ldots, 3k-2)$ and related with $\vartheta(C_1)$, and use $A(C_2)$ to denote the longest cycle missing the vertex on $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k)) - M$(see Fig.3). We suppose that $P_1 = [x, y]$, $P_2 = [p, u_{a,b}]$ on cycle $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k)) - M$ and $P_3 = [m, n]$, $P_4 = [s, t]$ on cycle $A(C_2)$ in $\vartheta(And(k)) - A(C_1)$ or $\vartheta(And(k)) - M - A(C_1)$ by Theorem 3[7], the analysis of Lemma 2.4 and the definition of CEG (see appendix). Now, we have a discussion about the position of vertex x, y, p, s and n in $\vartheta(And(k))$.

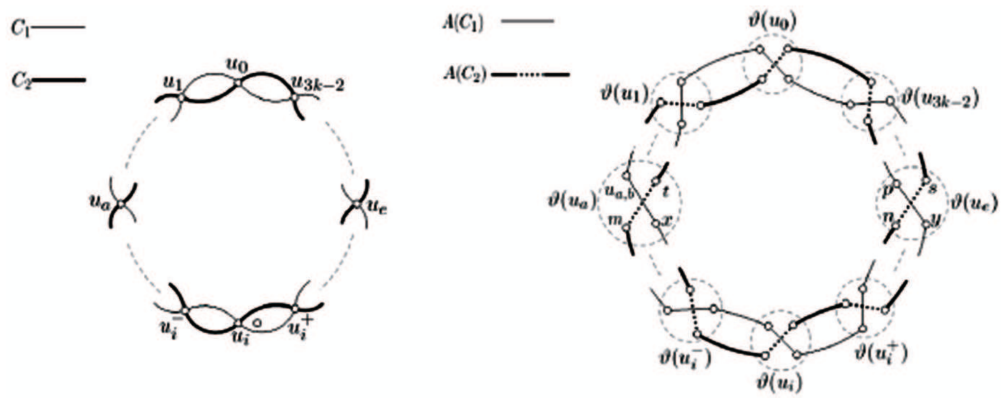

Fig.3

In where, C_1 in $And(k)$ or $And(k)-E(M)$, C_2 in $And(k)-E(C_1)$ or $And(k)-E(M)-E(C_1)$, $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k))-M$, $A(C_2)$ in $\vartheta(And(k))-A(C_1)$ or $\vartheta(And(k))-A(C_1)-M$.

$$x \neq u_{a,b}, u_{a,d}, \begin{cases} y = u_{e,f}, \begin{cases} p = u_{e,h}, \cdots\cdots (1) \\ p \neq u_{e,h}, \begin{cases} s, n \neq u_{e,h}, \cdots\cdots (2) \\ s = u_{e,h}, \cdots\cdots (3) \\ n = u_{e,h}, \cdots\cdots (4) \end{cases} \end{cases} \\ y = u_{e,h}, \begin{cases} p = u_{e,f}, \cdots\cdots (5) \\ p \neq u_{e,f}, \begin{cases} s, n \neq u_{e,f}, \cdots\cdots (6) \\ s = u_{e,f}, \cdots\cdots (7) \\ n = u_{e,f}, \cdots\cdots (8) \end{cases} \end{cases} \\ y \neq u_{e,f}, u_{e,h}, \begin{cases} p = u_{e,f}, \begin{cases} s, n \neq u_{e,h}, \cdots\cdots (9) \\ s = u_{e,h}, \cdots\cdots (10) \\ n = u_{e,h}, , \cdots\cdots (11) \end{cases} \\ p = u_{e,h}, \begin{cases} s, n \neq u_{e,f}, \cdots\cdots (12) \\ s = u_{e,f}, \cdots\cdots (13) \\ n = u_{e,f}, \cdots\cdots (14) \end{cases} \\ p \neq u_{e,f}, u_{e,h}, \begin{cases} s = u_{e,f}, \cdots\cdots (15) \\ n = u_{e,f}, \cdots\cdots (16) \\ s = u_{e,h}, \cdots\cdots (17) \\ n = u_{e,h}, \cdots\cdots (18) \\ s = u_{e,f}, n = u_{e,h}, \cdots\cdots (19) \\ s = u_{e,h}, n = u_{e,f}, \cdots\cdots (20) \\ s \neq u_{e,f}, n \neq u_{e,h}, \cdots\cdots (21) \end{cases} \end{cases} \end{cases}$$

For cases (1) and (2), we can find a hamiltonian cycle

$$u_{a,b}xP_1(x,y)ysP_4(s,t)tG_{\vartheta(u_a)}[t,m;u_{a,b},x]mP_3(m,n)nG_{\vartheta(u_e)}[n,p;y,s]pP_2(p,u_{a,b})u_{a,b}$$

that encounters $(u_{a,b},u_{e,f},u_{a,d},u_{e,h})$ in this order.

For cases (3)-(21), we can find a hamiltonian cycle that encounters $(u_{a,b},u_{e,f},u_{a,d},u_{e,h})$ in this order according to the method of (1) and (2).

(3)For cases (2)-(11) and (15)-(21), we can find a hamiltonian cycle that encounters $(u_{a,b},u_{a,d},u_{e,h},u_{e,f})$ in this order according to the method of Case3.1(2).

For case (1), we can find a hamiltonian cycle

$$u_{a,b}G_{\vartheta(u_a)}[u_{a,b},m;t]mP_3'(m,n)nG_{\vartheta(u_e)}[n,p;y,s]pysP_4'(s,t)tu_{a,b}$$

that encounters $(u_{a,b},u_{a,d},u_{e,h},u_{e,f})$ in this order. P_i' is the path which through the all vertices in $\vartheta(u_i)(i=a,\ldots,e)$ and related with $P_i(i=3,4)$ in $\vartheta(And(k))-A(C_1)$ or $\vartheta(And(k))-M-A(C_1)$(see Fig.4).

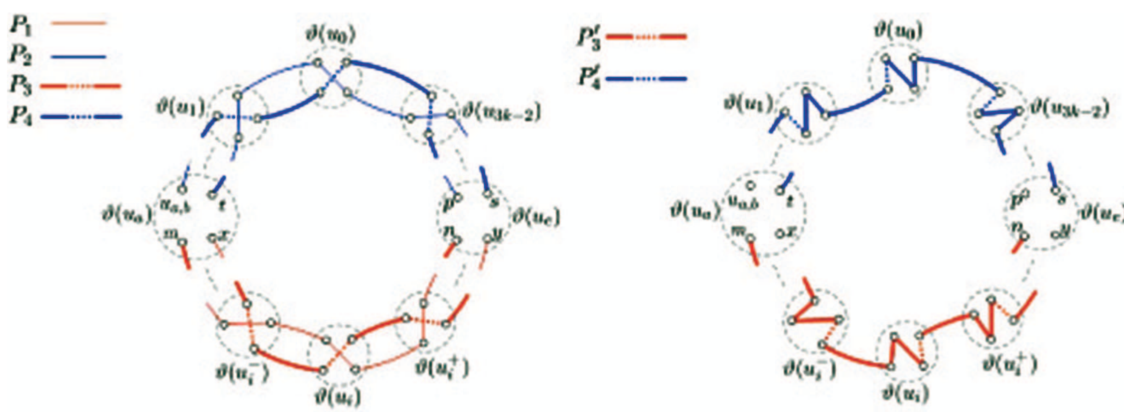

Fig.4

In where, P_1, P_2 in $\vartheta(And(k))$ or $\vartheta(And(k))-M$, P_3, P_4 in $\vartheta(And(k))-A(C_1)$ or $\vartheta(And(k))-M-A(C_1)$, P_3', P_4' related with P_3, P_4 in $\vartheta(And(k))-A(C_1)$ or $\vartheta(And(k))-M-A(C_1)$.

For 12-14, we can find a hamiltonian cycle that encounters $(u_{a,b},u_{a,d},u_{e,h},u_{e,f})$ in this order according to the method of 1.

Subcase 3.2 $u_{a,b},u_{c,d},u_{e,f}\in V(\vartheta(u_a))$ and $u_{g,h}\in V(\vartheta(u_g))$ in $\vartheta(And(k))$. For all condition , we see the result is proved by the method of Subcase 3.1.

Case 4 If these four vertices $u_{a,b},u_{c,d},u_{e,f},u_{g,h}$ are contained in distinct three Cliques of $\vartheta(And(k))$. Without loss of generality, we assume that $u_{a,b},u_{c,d}\in V(\vartheta(u_a))$, $u_{e,f}\in V(\vartheta(u_e))$ and $u_{g,h}\in V(\vartheta(u_g))$ in $\vartheta(And(k))$.

(1) For $(u_{a,b},u_{a,d},u_{e,f},u_{g,h})\in S$, C_0 is the hamiltonian cycle that encounters $(u_{a,b},u_{a,d},u_{e,f},u_{g,h})$ in this order, clearly.

(2) For $(u_{a,b},u_{e,f},u_{a,d},u_{g,h})\in S$. Let C_1 is a hamiltonian cycle in $And(k)$ or $And(k)-E(M)$, C_2 is a hamiltonian cycle in $And(k)-E(C_1)$ or $And(k)-E(M)-E(C_1)$, C_3 is a

hamiltonian cycle in $And(k)-E(C_1)-E(C_2)$ or $And(k)-E(M)-E(C_1)-E(C_2)$(see Fig.5). Use $A(C_j)$ to denote a cycle that only through two vertices in $\vartheta(u_i)(i=1,2,\ldots,3k-2)$ and related with $\vartheta(C_j)(j=1,2)$, and use $A(C_3)$ to denote the longest cycle missing the vertex on $A(C_1)$ and $A(C_2)$ in $\vartheta(And(k))$ or $\vartheta(And(k))-M$(see Figure5). We can suppose that $P_1=[u_{c,d},x]$, $P_2=[y,u_{a,b}]$ on cycle $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k))-M$, $P_3=[m,n]$, $P_4=[p,q]$ on cycle $A(C_2)$ in $\vartheta(And(k))-A(C_1)$ or $\vartheta(And(k))-M-A(C_1)$ and $P_5=[s,t]$, $P_6=[w,z]$ on $A(C_3)$ in $\vartheta(And(k))-\sum\limits_{i=1}^{2}A(C_i)$ or $\vartheta(And(k))-M-\sum\limits_{i=1}^{2}A(C_i)$ by Theorem 3[7], the analysis of Lemma 3 and the definition of CEG (see appendix). Now, we have a discussion about the position of vertex m, q, x, y, p and n in $\vartheta(And(k))$.

$$m,q\neq u_{a,b},u_{a,d},\ \begin{cases} x=u_{g,h},y\neq u_{g,h},\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)\\ x\neq u_{g,h},\ \begin{cases} y=u_{g,h},\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)\\ y\neq u_{g,h};\ \begin{cases} p=u_{g,h},\cdots\cdots\cdots\cdots (3)\\ n=u_{g,h},\cdots\cdots\cdots\cdots (4)\\ p,n\neq u_{g,h}.\ \cdots\cdots\cdots (5)\end{cases}\end{cases}\end{cases}$$

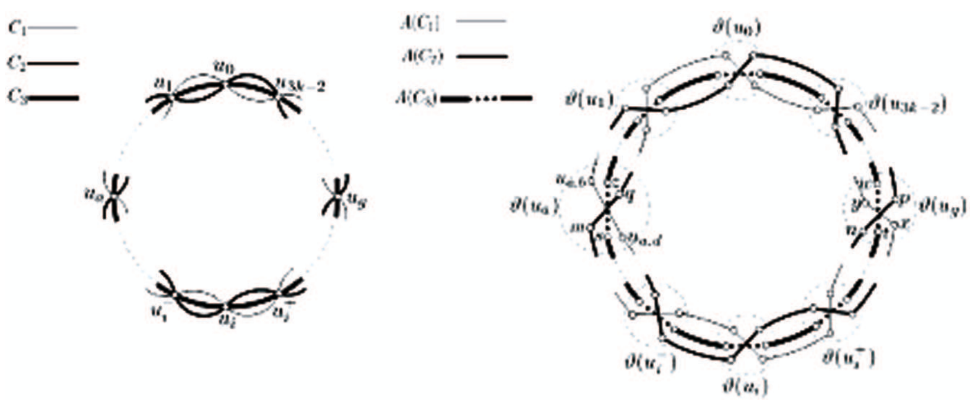

Fig.5

In where, C_1 in $And(k)$ or $And(k)-E(M)$, C_2 in $And(k)-E(C_1)$ or $And(k)-E(M)-E(C_1)$, C_3 in $And(k)-E(C_1)-E(C_2)$ or $And(k)-E(M)-E(C_1)-E(C_2)$, $A(C_1)$ in $\vartheta(And(k))$ or $\vartheta(And(k))-M$, $A(C_2)$ in $\vartheta(And(k))-A(C_1)$ or $\vartheta(And(k))-A(C_1)-M$, $A(C_3)$ in $\vartheta(And(k))-A(C_1)-A(C_2)$ or $\vartheta(And(k))-A(C_1)-A(C_2)-M$.

For case (1), if $u_{e,f}\in V(P_i)$ $(i=2,3,4)$, we can find a hamiltonian cycle that encounters $(u_{a,b},u_{e,f},u_{a,d},u_{g,h})$ in this order according to the method of Subcase 3.1,(2).

If $u_{e,f}\in V(P_1)$, we can find a hamiltonian cycle

$$u_{a,b}qP_4'(q,p)pnP_3'(n,m)mG_{\vartheta(u_a)}[m,s;u_{a,b},q]sP_5'(s,t)tG_{\vartheta(u_g)}[t,y;p,n,t]yP_2'(y,u_{a,b})u_{a,b}\ \textit{or}$$

$$u_{a,b}mP_3'(m,n)npP_4'(q,p)qG_{\vartheta(u_a)}[q,s;u_{a,b},m]sP_5'(s,t)tG_{\vartheta(u_g)}[t,y;p,n,t]yP_2'(y,u_{a,b})u_{a,b}$$

that encounters $(u_{a,b}, u_{e,f}, u_{a,d}, u_{g,h})$ in this order. There exist some vertices which belong to a same Clique on P_1, P_i and $P_j(i = 3, 4; j = 5, 6)$. And $u_{e,f} \in V(P_i')(i = 3$ or $4)$. P_i' is the path which through the all vertices in $\vartheta(u_i)(i = a, \ldots, g)$ and related with $P_i(i = 5, 6)$ in $\vartheta(And(k)) - \sum_{i=1}^{2} A(C_i)$ or $\vartheta(And(k)) - M - \sum_{i=1}^{2} A(C_i)$, and missing the vertex on P_3', P_4'(refers to Figure4).

For cases (2)-(5), we can find a hamiltonian cycle that encounters $(u_{a,b}, u_{e,f}, u_{a,d}, u_{g,h})$ in this order according to the method of (1).

(3) For cases (1)-(5), we can find a hamiltonian cycle that encounters $(u_{a,b}, u_{a,d}, u_{g,h}, u_{e,f})$ in this order according to the method of Case 4(2). □

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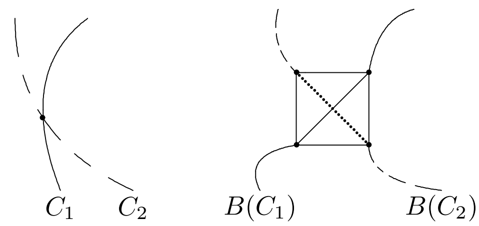
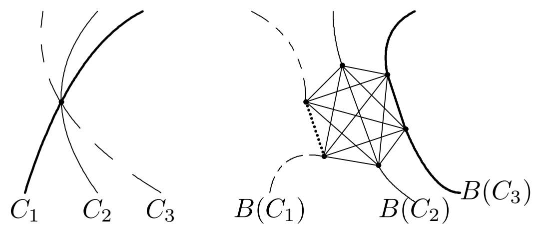
Appendix

By the theorem 1.9, the analysis of Lemma 2.4, the definition of CEG, $And(k)$ and the parity of $k(s \in Z^+)$, we know that

$k = 2s$	$And(k)$	$\vartheta(And(k))$
$s = 1$	C_5	C_{10}
$s = 2$	$And(4) - E(C_1) = C_2$	$\vartheta(And(4)) - B(C_1) = B(C_2)$
$s = 3$	$And(6) - E(C_1) - E(C_2) = C_3$	$\vartheta(And(6)) - B(C_1) - B(C_2) = B(C_3)$
$\vdots$	$\vdots$	$\vdots$
$s = n$	$And(2n) - \sum_{i=1}^{n-1} E(C_i) = C_n$	$\vartheta(And(2n)) - \sum_{i=1}^{n-1} B(C_i) = B(C_n)$



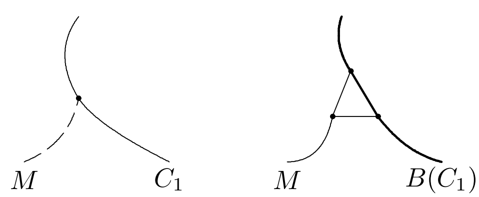
s=2



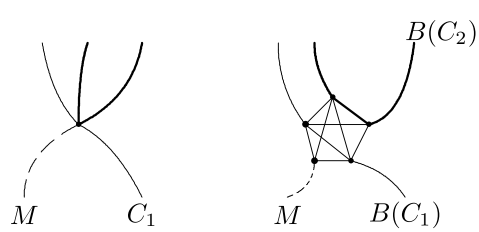
s=3

If k is odd, it should be illustrated that the M's selection method, that is, M satisfy condition $u_{a,b}, u_{c,d}, u_{e,f}, u_{g,h} \notin V(M)$ in $\vartheta(And(k))$. It can be done, because $k \geq 7$.

$k = 2s+1$	$And(k)$	$\vartheta(And(k))$
$s = 1$	$And(3) - E(M) = C_1$	$\vartheta(And(3)) - M = B(C_21)$
$s = 2$	$And(5) - E(M) - E(C_1) = C_2$	$\vartheta(And(5)) - E(M) - B(C_1) = B(C_2)$
$s = 3$	$And(7) - E(M) - E(C_1) - E(C_2) = C_3$	$\vartheta(And(7)) - M - B(C_1) - B(C_2) = B(C_3)$
$\vdots$	$\vdots$	$\vdots$
$s = n$	$And(2n+1) - E(M) - \sum\limits_{i=1}^{n-1} E(C_i) = C_n$	$\vartheta(And(2n+1)) - M - \sum\limits_{i=1}^{n-1} B(C_i) = B(C_n)$



s=1



s=2

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On Equitable Coloring of Weak Product of Odd Cycles

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Abstract: In this article, we present algorithms for equitable weak product graph of cycles C_m and C_n, $C_m \times C_n$ such that it has an equitable chromatic value, $\chi_=(C_m \times C_n) = 3$, with mn odd and m or n is not a multiple of 3.

Key Words: Equitable coloring, equitable chromatic number, weak product, direct product, cross product.

AMS(2010): 05C78

§1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A k-coloring on G is a function $f : V(G) \to [1,k] = \{1,2,\cdots,k\}$, such that if $uv \in E(G), u, v \in V(G)$ then $f(u) \neq f(v)$. A value $\chi(G) = k$, the chromatic number of G is the smallest positive integer for which G is k-colorable. G is said to be equitably k-colorable if for a proper k-coloring of G with vertex color class $V_1, V_2 \cdots V_k$, then $|(|V_i| - |V_j|)| \leq 1$ for all $i, j \in [i,k]$. Suppose n is the smallest integer such that G is equitably k-colorable, then n is the equitable chromatic number, $\chi_=(G)$, of G.

The notion of equitable coloring of a graph was introduced in [6] by Meyer. Notable work on the subject includes [7] where outer planar graphs were considered and [8] where general planar graphs were investigated. In [1] equitable coloring of the product of trees was considered. Chen et al. in [2] showed that for $m, n \geq 3$, $\chi_=(C_m \times C_n = 2)$ if mn is even and $\chi_=(C_m \times C_n = 3)$ if mn is odd. Recent work include [4], [5]. Furmanczyk in [3] discussed the equitable coloring of product graphs in general, following [2], where the authors separated the proofs of mn into various parts including the following:

1. m, n odd with $n = 0 \mod 3$
2. m, n odd, with
   (a) either m or n, say n satisfying $n - 1 \equiv 0 \mod 3$

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(b) either m or n, say n satisfying $n-2 \equiv 0 \bmod 3$.

In this paper we present equitable coloring schemes which

1. improve the proof in (b) above and
2. can be employed in developing the equitable 3-coloring for $C_m \times C_n$ with mn odd.

§2. Preliminaries

Let G_1 and G_2 be two graphs with $V(G_1)$ and $E(G_1)$ as the vertex and edge sets for G_1 respectively and $V(G_2)$ and $E(G_2)$ as the vertex and edge sets of G_2 respectively. The weak product of G_1 and G_2 is the graph $G_1 \times G_2$ such that $V(G_1 \times G_2) = \{(u,v) = u \in V(G) \text{and } u \in V(G_2)\}$ and $E(G_1 \times G_2) =$
$\{(u_1v_1)(u_2v_2) : u_1u_2 \in E(G_1) \text{and } v_1v_2 \in E(G_2)\}$. A graph $P_m = u_0u_1u_2\cdots u_{m-1}$ is a path of length $m-1$ if for all $u_i, v_j \in V(P_m), i \neq j$. A graph $C_m = u_0u_1u_2\cdots u_{m-1}$ is a cycle of length m if for all $u_i, v_j \in V(C_m), i \neq j$ and $u_0u_{m-1} \in E(C_m)$.

The following results due to Chen et al gives the equitable chromatic numbers of product of cycles.

Theorem 2.1([2]) *Let $m, n \geq 3$. Then*

$$\chi_=(C_m \times C_n) = \begin{cases} 2 & \text{if } mn \quad \text{is} \quad \text{even} \\ 3 & \text{if } mn \quad \text{is} \quad \text{odd.} \end{cases}$$

We require the following lemma in the main result.

Lemma 2.2 *Let n be any odd integer and let $n-1 \equiv 0 \bmod 3$. Then $n-1 \equiv 0 \bmod 6$.*

Proof Since n is odd, then there exists a positive integer m, such that $n = 2m+1$. Now since n is odd then, $n-1$ is even. Let $2m \equiv 0 \bmod 3$. Clearly, $n \geq 3$. Now $2m = 3k$ where k is an even positive integer. Thus $2m = 3(2k')$ for some positive integer k' and thus $2m = 6k'$. Hence $n-1 = 6k'$. □

§3. Main Results

In this section, we present the algorithms for the equitable 3-coloring of $C_m \times C_n$ with where m and n are odd with say $n-1 \equiv 0 \bmod 3$ and $n-2 \equiv 0 \bmod 3$.

Algorithm 1 Let $C_m \times C_n$ be product graph and let mn be odd, with $n-1 = 0 \bmod 3$.

Step 1 Define the following coloring for $u_iv_j \in V(C_m \times C_n)$.

$$f(u_iv_j) = \begin{cases} \alpha_2 & \text{for } \{u_iv_j : j \in [n-1]; j \geq 5; j+1 = 0 \bmod 3\} \\ \alpha_1 & \text{for } \{u_iv_j : j \in [n-1]; j+2 = 0 \bmod 3\} \cup \{u_iv_2 : i \in [m-1]\} \\ \alpha_3 & \text{for } \{u_iv_j : j \in [n-1]; j \geq 6; j = 0 \bmod 3\} \cup \{u_iv_1, i \in [m-1]\}. \end{cases}$$

Step 2 For all $u_iv_0; i \in [2]$, define the following coloring:

(a)

$$f(u_iv_0) = \begin{cases} \alpha_1 & \text{for } i = 1 \\ \alpha_2 & \text{for } i = 0, 2 \end{cases}$$

(b)

$$f(u_iv_3) = \begin{cases} \alpha_2 & \text{for } i = 1, 2 \\ \alpha_3 & \text{for } i = 0 \end{cases}$$

Step 3 Repeat Step 2(a) and Step 2(b) for all u_iv_0 and u_iv_3 for each $i \in [x, x+2]$ where $x = 0 \mod 3$.

Proof of Algorithm 1 Suppose n is odd and $n - 1 = 0 \mod 3$. From Lemma 2.2 above, $n - 1 = 0 \mod 6$ and consequently, $n - 4 = 0 \mod 3$. Suppose $\frac{n-4}{3} = n'$, where n is a positive integer. Let $P_m \times P_{n-4}$ be a subgraph of $P_m \times P_n$, where $P_{n-4} = v_4v_5 \cdots v_{n-1}$.. For all $u_iv_j \in V(P_m \times P_{n-4})$, let

$$f(u_iv_j) = \begin{cases} \alpha_1 & \text{for } \{u_iv_j : j \in [n-1], j+2 = 0 \mod 3\} \\ \alpha_2 & \text{for } \{u_iv_j : j \in [n-1], j \geq 5; j+1 = 0 \mod 3\} \\ \alpha_3 & \text{for } \{u_iv_j : j \in [n-1]; j \geq 6; j = 0 \mod 3\}. \end{cases}$$

From $f(u_iv_j)$ defined above, we see that $P_m \times P_{n-4}$ is equitably 3-colorable with color set $\{\alpha_1, \alpha_2, \alpha_3\} \equiv [1, 3]$, where $|V_{\alpha_1}| = |V_{\alpha_2}| = |V_{\alpha_3}| = mn'$. Next we show that there exists a 3-coloring of $P_m \times P_4$ that merges with $P_m \times P_{n-4}$ whose 3-coloring is defined by $f(u_iv_j)$ above. First, let $F(P_3 \times P_4)$ be the 3- coloring such that

$$F(P_3 \times P_4) = \begin{matrix} \alpha_2 & \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_1 & \alpha_3 \end{matrix}$$

From $F(P_3 \times P_4)$ we observe for all $j \in [3]$, that for $F(u_0v_j) \subset F(P_3 \times P_4), |V_{\alpha_1}| = 1, |V_{\alpha_2}| = 1, |V_{\alpha_3}| = 2$; for $F(u_1v_j) \subset F(P_3 \times P_4), |V_{\alpha_1}| = 2, |V_{\alpha_2}| = 1, |V_{\alpha_3}| = 1$; and for $F(u_0v_j) \subset F(P_3 \times P_4), |V_{\alpha_1}| = 1, |V_{\alpha_2}| = 2, |V_{\alpha_3}| = 1$.

We observe, over all, that for $F(P_3 \times P_4), |V_{\alpha_1}| = |V_{\alpha_2}| = |V_{\alpha_3}| = 4$. These confirm that $P_3 \times P_4$ is equitably 3-colorable at every stage of $i \in [2]$ and that $F(P_2 \times P_4) \subset F(P_3 \times P_4)$ is an equitable 3-coloring of $P_2 \times P_4$ for both $P_2 \times P_4 \subset P_3 \times P_4$. Now the equitable 3-coloring of $P_m \times P_4$ is now obtainable by repeating $F(P_3 \times P_4)$ at each interval $[x, x+2]$, where $x = 0 \mod 3$, until we reach m. Clearly, $F(u_iv_3) \cap F(u_iv_4) = \emptyset$ since $\alpha_1 \notin F(u_iv_3)$. Thus $P_m \times P_n$ is equitably 3-colorable based on the colorings defined earlier. Likewise, $F(u_iv_0) \cap F(u_iu_{n-1}) = \emptyset$ since $\alpha_3 \notin F(u_iv_0)$. Thus $P_m \times C_n$ is equitably 3-colorable based on the coloring defined above for $P_m \times P_n$.

Finally, for any $m \geq 3$, the equitable 3-coloring of $P_m \times P_{n-4}$ with respect to $F(P_m \times P_{n-4})$ above is equivalent to the equitable 3-coloring of $C_m \times C_{n-4}$ since $u_iv_ju_iv_{j+1} \notin E(P_m \times P_{m-4})$ for all $j \in [n-5]$. Also, for $m \geq 3$ the equitable 3-coloring of $P_m \times P_4$ with respect to

$F(P_m \times P_4)$ above is equivalent to the equitably 3- coloring of $C_m \times C_4$ by mere observation. Thus, $C_m \times C_n$ is equitably 3- colorable or all positive integer m and odd positive integer n such that $n-1=0 \mod 3$.

Algorithm 2 Let m or n, say n be odd such that $n-2=0 \mod 3$.

Step 1 Define the following coloring:

$$f(u_iv_j)=\begin{cases}\alpha_1 & \text{for } \{u_iv_j : j\in[n-1], j+1=0 \mod 3\}\\ \alpha_2 & \text{for } \{u_iv_j : j\in[n-1], j=0 \mod 3\}\\ \alpha_3 & \text{for } \{u_iv_j : j\in[n-1], j-1=0 \mod 3\}\end{cases}$$

Step 2(a) For all $i\in[2]$, let $f(u_iv_0)=\alpha_1,\alpha_2,\alpha_1$ respectively $\alpha_1,\alpha_2\in[2]$.

Step 2(b) For all $i\in[2]$, let $f(u_iv_1)=\alpha_3,\alpha_2,\alpha_3$ respectively, $\alpha_3\in[2]$.

Step 3 Repeat step 2(a) and Step 2(b) above for all $i\in[x,x+2]$, where x is a positive integer and $x=0 \mod 3$.

Proof of Algorithm 2 Let n be odd and let $n-2=0 \mod 3$. By $f(u_iv_j)$ in step 1, $P_m\times P_{n-2}$, where $P_{n-2}=v_2v_3\cdots v_{n-1}$, is equitably 3-colorable with $|V_{\alpha_1}|=|V_{\alpha_2}|=|V_{\alpha_3}|=mn''$ where $n''=\frac{n-2}{3}$ and $F(u_iv_2)\cap F(u_iv_{n-1})=\emptyset$ for all $i\in[m-1]$. Now, let

$$F(P_3\times P_2)=\begin{matrix}\alpha_1 & \alpha_3\\ \alpha_2 & \alpha_2\\ \alpha_1 & \alpha_3\end{matrix}$$

It is clear that $F(P_3\times P_2)$ above follows from the coloring defined in step 2 of the algorithm and that $F(P_3\times P_2)$ is an equitable 3-coloring of $P_3\times P_2$ where $|V_{\alpha_1}|=|V_{\alpha_2}|=|V_{\alpha_3}|=2$. It is also clear that $F(P_3\times P_2)$ has an equitable coloring at $P_1\times P_2$ with $|V_{\alpha_1}|=1, |V_{\alpha_2}|=0, |V_{\alpha_3}|=1$ and at $P_2\times P_2$ with $|V_{\alpha_1}|=1, |V_{\alpha_2}|=2, |V_{\alpha_3}|=1$. Now, let with $x=0 \mod 3$. For all $x\in[m-1]$, let $f(u_xv_j)=\alpha_1,\alpha_3$ for both $j=0,1$ respectively; for $x+1\in[m-1]$, let $f(u_{x+1}v_j)=\alpha_2$, for $j=0,1$ and for $x+2\in[m-1]$, let $f(u_{x+2}v_j)=\alpha_1,\alpha_3$ for $j=0,1$. With this last scheme, we have $P_m\times P_2$ that has an equitable 3- coloring for any value of m.

Finally, we can see that $P_m\times P_2$, for any m, so equitably, 3-colored merges with $P_m\times P_{n-2}$ that is equitably 3-colored earlier by $f(u_iv_j)$, such that $F(u_iv_1)\cap F(u_iv_2)=\emptyset$ for all $i\in[m-1]$ (by a similar argument as in the proof of Algorithm 1) and $F(u_iv_0)\cap F(u_iv_{n-1})=\emptyset$ for all $i\in[m-1]$ (by a similar argument as in the proof of Algorithm 1). □

Likewise $C_m\times C_n$ is equitable 3-colorable (by a similar argument as in the proof of Algorithm 1). Therefore, $C_m\times C_n$ is equitably 3-colorable for any $m\geq 3$ and odd n, such that $n-2=0 \mod 3$.

§4. Examples

In Fig.1, we demonstrate how our algorithms equitably color graphs $C_5\times C_5$ and $C_5\times C_7$, which are two cases that illustrate $n-2=0 \mod 3$ and $n-1=0 \mod 3$ respectively. In the first case, we see that $\chi_=(C_5 \times C_5) = 3$, with $|V_1| = 8$ $|V_2| = 9$ and $|V_3| = 8$ and in the second case, $\chi_=(C_5 \times C_7) = 3$, with $|V_1| = 12$ $|V_2| = 11$ and $|V_3| = 12$. (Note that the first coloring takes care of the third instance in subcase 2.4 of [2] where it is a special case.)

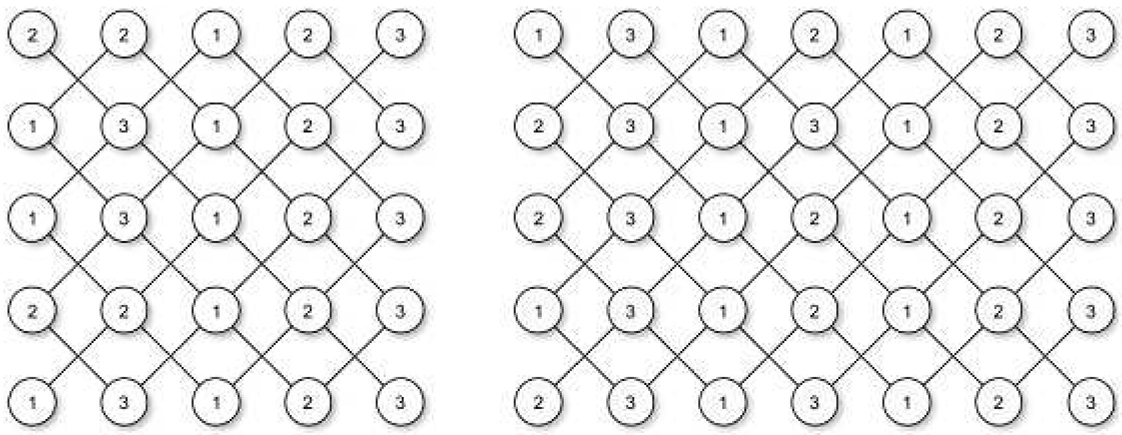

Fig.1 Equitable coloring of graphs $C_5 \times C_5$ and $C_5 \times C_7$

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Corrigendum: *On Set-Semigraceful Graphs*

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In this short communication we rectify certain errors which are in the paper, On Set-Semigraceful Graphs, *International J. Math. Combin.*, Vol.2(2012), 59-70. The following are the correct versions of the respective results.

Remark 3.2 (5) The Double Stars $ST(m, n)$ where $|V|$ is not a power of 2, are set-semigraceful by Theorem 2.13.

Remark 3.5 (3) The Double Stars $ST(m, n)$ where m is odd and $m + n + 2 = 2^l$, are not set-semigraceful by Theorem 2.12.

Delete the following sentence below Remark 3.9: "In fact the result given by Theorem 3.3 holds for any set-semigraceful graph as we see in the following".

Theorem 4.8([3]) *Every graph can be embedded as an induced subgraph of a connected set-graceful graph.*

Since every set-graceful graph is set-semigraceful, from the above theorem it follows that

Theorem 4.8A *Every graph can be embedded as an induced subgraph of a connected set-semigraceful graph.*

However, below we prove:

Theorem 4.8B *Every graph can be embedded as an induced subgraph of a connected set-semigraceful graph which is not set-graceful.*

Proof Any graph H with $o(H) \leq 5$ and $s(H) \leq 2$ and the graphs P_4, $P_4 \cup K_1$, $P_3 \cup K_2$ and P_5 are induced subgraphs of the set-semigraceful cycle C_{10} which is not set-graceful. Again any

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graph H' with $3 \le o(H') \le 5$ and $3 \le s(H') \le 9$ can be obtained as an induced subgraph of $H_1 \vee K_1$ for some graph H_1 with $o(H_1) = 5$ and $3 \le s(H_1) \le 9$. Then $3 < \log_2(|E(H_1 \vee K_1)| + 1) < 4$, since $8 \le s(H_1 \vee K_1) < 15$ and hence $H_1 \vee K_1$ is not set-graceful. By Theorem 2.4,

$$\begin{aligned} 4 &= \lceil \log_2(|E(H_1 \vee K_1)| + 1) \rceil \le \gamma(H_1 \vee K_1) \\ &\le \gamma(K_6) \quad \text{(by Theorem 2.5)} \\ &= 4 \quad \text{(by Theorem 2.19)} \end{aligned}$$

So that $H_1 \vee K_1$ is set-semigraceful. Further, note that K_5 is set-semigraceful but not set-graceful.

Now let $G = (V, E)$; $V = \{v_1, \ldots, v_n\}$ be a graph of order $n \ge 6$. Consider a set-indexer g of G with indexing set $X = \{x_1, \ldots, x_n\}$ defined by $g(v_i) = \{x_i\}$; $1 \le i \le n$. Let $S = \{g(e) : e \in E\} \cup \{g(v) : v \in V\}$. Note that $|S| = |E| + n$. Now take a new vertex u and join with all the vertices of G. Let m be any integer such that $2^{n-1} < m < 2^n - (|E| + n + 1)$. Since $|E| \le \dfrac{n(n-1)}{2}$ and $n \ge 6$, such an integer always exists. Take m new vertices $u_1, \ldots, u_m$ and join all of them with u. A set-indexer f of the resulting graph G' can be defined as follows:

$$f(u) = \emptyset, \quad f(v_i) = g(v_i); \quad 1 \le i \le n.$$

Besides, f assigns the vertices $u_1, \ldots, u_m$ with any m distinct elements of $2^X \setminus (S \cup \emptyset)$. Thus, $\gamma(G') \le n$. But we have $2^n > |E| + n + m + 1 > m > 2^{n-1}$ so that $\gamma(G') \ge n$, by Theorem 2.4. Hence,

$$\log_2(|E(G')| + 1) < \lceil \log_2(|E(G')| + 1) \rceil = n = \gamma(G').$$

This shows that G' is set-semigraceful, but not set-graceful. □

Corollary 4.16 *The double fan $P_k \vee K_2$ where $k = 2^n - m$ and $2^n \ge 3m$; $n \ge 3$ is set-semigraceful.*

Proof Let $G = P_k \vee K_2$; $K_2 = (u_1, u_2)$. By Theorem 2.4, $\gamma(G) \ge \lceil \log_2(|E| + 1) \rceil = \lceil \log_2(3(2^n - m) + 1) \rceil = n + 2$. But, $3m \le 2^n \Rightarrow m < 2^{n-1} - 1$. Therefore,

$$\begin{aligned} & 2^n - (2^{n-1} - 2)) \le 2^n - m < 2^n - 1 \\ & \Rightarrow 2^{n-1} + 1 \le 2^n - m - 1 < 2^n - 2 \\ & \Rightarrow 2^{n-1} + 1 \le k - 1 < 2^n - 2; \quad k = 2^n - m \\ & \Rightarrow 2^{n-1} + 1 \le |E(P_k)| < 2^n \\ & \Rightarrow \lceil \log_2(|E(P_k)| + 1) \rceil = n \\ & \Rightarrow \gamma(P_k) = n \end{aligned}$$

since P_k is set-semigraceful by Remark 3.2(3). □

Let f be a set-indexer of P_k with indexing set $X = \{x_1, \ldots, x_n\}$. Define a set-indexer g of G with indexing set $Y = X \cup \{x_{n+1}, x_{n+2}\}$ as follows:

$g(v) = f(v)$ for every $v \in V(P_k)$, $g(u_1) = \{x_{n+1}\}$ and $g(u_2) = \{x_{n+2}\}$.

Corollary 4.17 *The graph $K_{1,2^n-1} \vee K_2$ is set-semigraceful.*

Proof The proof follows from Theorems 4.15 and 2.33. □

Theorem 4.18 *Let C_k where $k = 2^n - m$ and $2^n + 1 > 3m$; $n \geq 2$ be set-semigraceful. Then the graph $C_k \vee K_2$ is set-semigraceful.*

Proof Let $G = C_k \vee K_2$; $K_2 = (u_1, u_2)$. By theorem 2.4, $\gamma(G) \geq \lceil \log_2(|E| + 1) \rceil = \lceil \log_2(3(2^n - m) + 2) \rceil = n + 2$. But, $3m \leq 2^n + 1 \Rightarrow m < 2^{n-1}$. Therefore,

$$\begin{aligned}
&2^n - (2^{n-1} - 1)) \leq 2^n - m < 2^n \\
&\Rightarrow 2^{n-1} + 1 \leq k < 2^n; \quad k = 2^n - m \\
&\Rightarrow 2^{n-1} + 1 \leq |E(C_k)| < 2^n \\
&\Rightarrow \lceil \log_2(|E(C_k)| + 1) \rceil = n \\
&\Rightarrow \gamma(C_k) = n
\end{aligned}$$

since C_k is set-semigraceful. □

Let f be a set-indexer of C_k with indexing set $X = \{x_1, \ldots, x_n\}$. Define a set-indexer g of G with indexing set $Y = X \cup \{x_{n+1}, x_{n+2}\}$ as follows:

$g(v) = f(v)$ for every $v \in V(C_k)$, $g(u_1) = \{x_{n+1}\}$ and $g(u_2) = \{x_{n+2}\}$.

Corollary 4.21 *W_n where $2^m - 1 \leq n \leq 2^m + 2^{m-1} - 2$; $m \geq 3$ is set-semigraceful.*

Proof The proof follows from Theorem 3.15 and Corollary 4.20. □

Theorem 4.22 *If W_{2k} where $\frac{2^{n-1}}{3} \leq k < 2^{n-2}$; $n \geq 4$ is set-semigraceful, then the gear graph of order $2k + 1$ is set-semigraceful.*

Proof Let G be the gear graph of order $2k + 1$. Then by theorem 2.4,

$$\begin{aligned}
&\lceil \log_2(3k + 1) \rceil \leq \gamma(G) \leq \gamma(W_{2k}) \quad \text{(by Theorem 2.5)} \\
&= \lceil \log_2(4k + 1) \rceil \quad (\text{since } \mathrm{W_{2k}} \text{ is set} - \text{semigraceful}) \\
&= \lceil \log_2(3k + 1) \rceil
\end{aligned}$$

since

$$\begin{aligned}
\frac{2^{n-1}}{3} \leq k < 2^{n-2} \qquad &\Rightarrow 2^{n-1} \leq 3k < 4k < 2^n \\
&\Rightarrow 2^{n-1} + 1 \leq 3k + 1 < 4k + 1 \leq 2^n.
\end{aligned}$$

Thus

$$\gamma(G) = \lceil \log_2(|E| + 1) \rceil.$$

So that G is set-semigraceful. □

Your time is limited, so don't waste it living someone else's life.

By Steve Jobs.

First International Conference

On Smarandache Multispace and Multistructure

Organized by Dr.Linfan Mao, Academy of Mathematics and Systems, Chinese Academy of Sciences, Beijing 100190, P.R.China. In American Mathematical Society's Calendar website:

http://www.ams.org/meetings/calendar/2013_jun28-30_beijing100190.html

June 28-30, 2013, Send papers by June 1, 2013 to Dr.Linfan Mao by regular mail to the above postal address, or by email to maolinfan@163.com.

A *Smarandache multispace* (or *S-multispace*) with its *multistructure* is a finite or infinite (countable or uncountable) union of many spaces that have various structures. The spaces may overlap, which were introduced by Smarandache in 1969 under his idea of hybrid science: *combining different fields into a unifying field*, which is closer to our real life world since we live in a heterogeneous space. Today, this idea is widely accepted by the world of sciences.

The S-multispace is a qualitative notion, since it is too large and includes both metric and non-metric spaces. It is believed that the smarandache multispace with its multistructure is the best candidate for 21st century *Theory of Everything* in any domain. It unifies many knowledge fields. A such multispace can be used for example in physics for the *Unified Field Theory* that tries to unite the gravitational, electromagnetic, weak and strong interactions. Or in the parallel quantum computing and in the mu-bit theory, in multi-entangled states or particles and up to multi-entangles objects. We also mention: the algebraic multispaces (multi-groups, multi-rings, multi-vector spaces, multi-operation systems and multi-manifolds, geometric multispaces (combinations of Euclidean and non-Euclidean geometries into one space as in Smarandache geometries), theoretical physics, including the relativity theory, the M-theory and the cosmology, then multi-space models for p-branes and cosmology, etc.

The multispace and multistructure were first used in the Smarandache geometries (1969), which are combinations of different geometric spaces such that at least one geometric axiom behaves differently in each such space. In paradoxism (1980), which is a vanguard in literature, arts, and science, based on finding common things to opposite ideas, i.e. combination of contradictory fields. In neutrosophy (1995), which is a generalization of dialectics in philosophy, and takes into consideration not only an entity $< A >$ and its opposite $< Anti\mathrm{A} >$ as dialectics does, but also the neutralities ¡neutA¿ in between. Neutrosophy combines all these three $< A >$, $< Anti\mathrm{A} >$, and $< neut\mathrm{A} >$ together. Neutrosophy is a metaphilosophy, including *neutrosophic logic*, *neutrosophic set* and *neutrosophic probability* (1995), which have, behind the classical values of truth and falsehood, a third component called indeterminacy (or neutrality, which is neither true nor false, or is both true and false simultaneously - again a combination of opposites: true and false in indeterminacy). Also used in Smarandache algebraic structures (1998), where some algebraic structures are included in other algebraic structures.

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