ON SOME OF THE SMARANDACHE'S PROBLEMS

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CONTENTS:

PREFACE

In 1996 the author wrote reviews for "Zentralblatt für Mathematik" for books [1] and [2] and this was him first contact of with the Smarandache's problems.

In [1] Florentin Smarandache formulated 105 unsolved problems, while in [2] C. Dumitrescu and V. Seleacu formulated 140 unsolved problems. The second book contains almost all problems from [1], but now every one problem has unique number and by this reason the author will use the numeration of the problems from [2]. Also, in [2] there are some problems, which are not included in [1]. On the other hane, there are problems from [1], which are not included in [2]. One of them is Problem 62 from [1], which is included here under the same number.

In the summer of 1998 the author found the books in his library and for a first time tried to solve a problem from them. After some attempts one of the problems was solved and this was a power impulse for the next research. In the present book are collected the 27 problems solved by the middle of February 1999.

The bigger part of the problems discussed in the present book (22 in number) are related to different sequences. For each of them the form of the n-th member is determined and for all of them except 4 problems - the form of the n-th partial sum. Four of the problems are proved; modifications of two of the problems are formulated; counterexamples to two of the problems are constructed.

When the text was ready, the author received from "Zentralblatt für Mathematik" Charles Ashbacher's book [8] for reviewing. The author read immediatelly the book [8] and he was delighted to see that only five of the problems on which he had worked are discussed there and that the approach to these problems is different in both books. Reading [8], the author understood that there are other books related to the Smarandache's problems [9-13], which he had not known up to the moment.

The author hopes to prove some other problems from [1] and [2] in future, but there are problems, for which it is not clear whether they will be solved in the next years or will share the fate of Fermat's Last Theorem.

The author would like to express his acknowledgements to Dr. Mladen V. Vassilev - Missana and Nikolai G. Nikolov, who read and corrected the text, to his daughter Vassia K. Atanassova and his students Valentina V. Radeva and Hristo T. Aladjov for collaboration, to Prof. Vasile Seleacu and Dr. M. L. Perez who encouraged him to prepare the book, and to Prof. Florentin Smarandache for the interesting problems which were a pleasant preoccupation for the author during half an year.

March 6, 1999

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§1. ON THE 4-th SMARANDACHE'S PROBLEM [1]

The 4-th problem from [2] (see also 18-th problem from [1]) is the following:

Smarandache deconstructive sequence:

$$\underbrace{1,\ 23,\ 456,\ 789}\underbrace{1,\ 23456,\ 789}\underbrace{123,\ 456789}\underbrace{1,\ 23456789},$$

$$\underbrace{123456789},\ \underbrace{123456789}1,\ \dots$$

Let the n-th term of the above sequence be a_n. Then we can see that the first digits of the first nine members are, respectively: 1, 2, 4, 7, 2, 7, 4, 2, 1. Let us define the function ω as follows:

r	$\omega(r)$
0	1
1	1
2	2
3	4
4	7
5	2
6	7
7	4
8	2
9	1

[1] see also
K. Atanassov, On the 4-th Smarandache's problem. *Notes on Number Theory and Discrete Mathematics,* Vol. 5 (1999), No. 1, 33-35.

Here we shall use the arithmetic function ψ, discussed shortly in §16 and detailed in the author's paper [3] .

Now, we can prove that the form of the n-th member of the above sequence is

$$a_n = \overline{b_1 b_2 \dots b_n},$$

where

$$b_1 = \omega(n - [\frac{n}{9}])$$

$$b_2 = \psi(\omega(n - [\frac{n}{9}]) + 1)$$

$$\dots$$

$$b_n = \psi(\omega(n - [\frac{n}{9}]) + n - 1).$$

Every natural number n can be represented in the form

$$n = 9q + r,$$

where $q \geq 0$ is a natural number and $r \in \{1, 2, ..., 9\}$.

We shall prove by induction that the forms of nine sequential members $a_{n+1}, a_{n+2}, ..., a_{n+9}$, where $n = 9q + r$, are the following:

$$a_{9q+1} = \underbrace{\underbrace{12...9}\underbrace{12...9}...\underbrace{12...9}}_{q \text{ times}}1$$

$$a_{9q+2} = \underbrace{\underbrace{23...1}\underbrace{23...1}...\underbrace{23...1}}_{q \text{ times}}23$$

$$a_{9q+3} = \underbrace{\underbrace{45...3}\underbrace{45...3}...\underbrace{45...3}}_{q \text{ times}}456$$

$$a_{9q+4} = \underbrace{\underbrace{78...6}\underbrace{78...6}...\underbrace{78...6}}_{q \text{ times}}7891$$

$$a_{9q+5} = \underbrace{\underbrace{23...1}\underbrace{23...1}...\underbrace{23...1}}_{q \text{ times}}23456$$

$$a_{9q+6} = \underbrace{\underbrace{78...6}\underbrace{78...6}...\underbrace{78...6}}_{q \text{ times}}789123$$

$$a_{9q+7} = \underbrace{\underbrace{45...3}\underbrace{45...3}...\underbrace{45...3}}_{q \text{ times}}4567891$$

$$a_{9q+8} = \underbrace{\underbrace{23...1}\underbrace{23...1}...\underbrace{23...1}}_{q \text{ times}}23456789$$

$$a_{9q+9} = \underbrace{\underbrace{12...9}\underbrace{12...9}...\underbrace{12...9}}_{q+1 \text{ times}}$$

When $q = 0$ the validity of the above assertion is obvious. Let us assume that for some natural number q, a_{9q+1}, a_{9q+2}, ... a_{9q+9} have the above forms. Then for a_{9q+10}, a_{9q+11}, ... a_{9q+18} we obtain the following representations, taking $p = q + 1$:

$$a_{9q+10} = a_{9p+1} = \underbrace{\underbrace{12...9}\underbrace{12...9}...\underbrace{12...9}}_{p \text{ times}}1$$

$$a_{9q+11} = a_{9p+2} = \underbrace{\underbrace{23...1}\underbrace{23...1}...\underbrace{23...1}}_{p \text{ times}}23$$

$$a_{9q+12} = a_{9p+3} = \underbrace{\underbrace{45...3}\underbrace{45...3}...\underbrace{45...3}}_{p \text{ times}}456$$

$$a_{9q+13} = a_{9p+4} = \underbrace{\underbrace{78...6}\underbrace{78...6}...\underbrace{78...6}}_{p \text{ times}}7891$$

$$a_{9q+14} = a_{9p+5} = \underbrace{\underbrace{23...1}\underbrace{23...1}...\underbrace{23...1}}_{p \text{ times}}23456$$

$$a_{9q+15} = a_{9p+6} = \underbrace{\underbrace{78...6}\underbrace{78...6}...\underbrace{78...6}}_{p \text{ times}}789123$$

$$a_{9q+16} = a_{9p+7} = \underbrace{\underbrace{45...3}\underbrace{45...3}...\underbrace{45...3}}_{p \text{ times}}4567891$$

$$a_{9q+17} = a_{9p+8} = \underbrace{\underbrace{23...1}\underbrace{23...1}...\underbrace{23...1}}_{p \text{ times}}23456789$$

$$a_{9q+18} = a_{9p+9} = \underbrace{\underbrace{12...9}\underbrace{12...9}...\underbrace{12...9}}_{p+1 \text{ times}}$$

To the above sequence $\{a_n\}_{n=1}^{\infty}$ we can juxtapose the sequence $\{\psi(a_n)\}_{n=1}^{\infty}$ for which we can prove (as above) that its basis is $[1,5,6,7,2,3,4,8,9]$.

The problem can be generalized, e.g., to the following form:

Study the sequence $\{a_n\}_{n=1}^{\infty}$, *which* s*-th member has the form*

$$a_s = \overline{b_1 b_2 ... b_{s.k}},$$

where $b_1 b_2 ... b_{s.k} \in \{1, 2, ..., 9\}$ *and*

$$b_1 = \omega'(s - [\frac{s}{9}])$$

$$b_2 = \psi(\omega'(s - [\frac{s}{9}]) + 1)$$

$$...$$

$$b_{s.k} = \psi(\omega'(s - [\frac{s}{9}]) + s.k - 1),$$

and here

r	$\omega'(r)$
1	1
2	$\psi(k+1)$
3	$\psi(3k+1)$
4	$\psi(6k+1)$
5	$\psi(10k+1)$
6	$\psi(15k+1)$
7	$\psi(21k+1)$
8	$\psi(28k+1)$
9	$\psi(36k+1)$

For example, when $k = 2$:

$$\underbrace{12,\ 3456,\ 789}\underbrace{123,\ 456789}\underbrace{12,\ 3456789}\underbrace{123,\ 456789}\underbrace{123456,\ 789}\underbrace{123456789}\underbrace{12,\ 3456789}\underbrace{123456789},\ \underbrace{123456789}\underbrace{123456789},\ldots$$

To the last sequence $\{a_n\}_{n=1}^{\infty}$ we can juxtapose again the sequence $\{\psi(a_n)\}_{n=1}^{\infty}$ for which we can prove (as above) that its basis is $[3,9,3,6,3,6,9,8,9]$.

§2. ON THE 16-th SMARANDACHE'S PROBLEM [2]

The 16-th problem from [2] (see also 21-st problem from [1]) is the following:

Digital sum:

$$\underbrace{0,1,2,3,4,5,6,7,8,9},\underbrace{1,2,3,4,5,6,7,8,9,10},\underbrace{2,3,4,5,6,7,8,9,10,11},$$
$$\underbrace{3,4,5,6,7,8,9,10,11,12},\underbrace{4,5,6,7,8,9,10,11,12,13},$$
$$\underbrace{5,6,7,8,9,10,11,12,13,14},\ldots \qquad (1)$$

($d_s(n)$ is the sum of digits.)
Study this sequence.

First we shall note that function d_s is the first step of another arithmetic (digital) function φ, discussed in details in the author's paper [3] and shortly - in §16.

After applying of this function over the set of the natural numbers, or over the above sequence, we obtain the sequence

$$\underbrace{0,1,2,3,4,5,6,7,8,9},\underbrace{1,2,3,4,5,6,7,8,9},\underbrace{10,2,3,4,5,6,7,8,9},\ldots$$
$$\underbrace{10,11,3,4,5,6,7,8,9},\ldots\underbrace{10,11,12,4,5,6,7,8,9},\ldots$$

---

[2] see also
K. Atanassov, On the 16-th Smarandache's problem. *Notes on Number Theory and Discrete Mathematics*, Vol. 5 (1999), No. 1, 36-38.

On the other hand, in [3] (shortly in §16) another function (ψ) is introduced. After its applying over the set of the natural numbers, or over the above sequence, we obtain the sequence

$$\underbrace{0,1,2,3,4,5,6,7,8,9},\underbrace{1,2,3,4,5,6,7,8,9},\underbrace{1,2,3,4,5,6,7,8,9},\ldots$$

and the set [1,2,3,4,5,6,7,8,9] is called a *basis* of the set of the natural numbers about ψ.

Below we shall show the form of the general term of the sequence from the Smarandache's problem. Let its members are denoted as $a_1, a_2, \ldots, a_n, \ldots$. The form of the member a_n is:

$$a_n = n - 9. \sum_{k=1}^{\infty} [\frac{n}{10^k}]. \qquad (2)$$

The validity of (2) can be proved, e.g., by induction. It is obviously valid for $n = 1$. Let us assume that for some n (2) is true. For n there are two cases.

Case 1: $n \neq \underbrace{9\,9\,\ldots\,9}_{m \text{ times}}$ $(m \geq 1)$. Therefore

$$n + 1 \leq \underbrace{9\,9\ldots\,9}_{m \text{ times}}$$

and

$$\sum_{k=1}^{\infty} [\frac{n}{10^k}] = \sum_{k=1}^{\infty} [\frac{n+1}{10^k}],$$

from where

$$a_{n+1} = a_n + 1 = n - 9. \sum_{k=1}^{\infty} [\frac{n}{10^k}] + 1 = (n+1) - 9. \sum_{k=1}^{\infty} [\frac{n+1}{10^k}].$$

Case 2: $n = \underbrace{9\,9\,...\,9}_{m \text{ times}}$. Therefore

$$n + 1 = 1\underbrace{0\,0...\,0}_{m \text{ times}}$$

and

$$a_{n+1} = 1 = 1\underbrace{0\,0...\,0}_{m \text{ times}} - \underbrace{9\,9\,...\,9}_{m \text{ times}} = 1\underbrace{0\,0...\,0}_{m \text{ times}} - 9.(1\underbrace{0\,0\,...\,0}_{m-1 \text{ times}}$$

$$+1\underbrace{0\,0\,...\,0}_{m-2 \text{ times}} + ...1)$$

$$= 1\underbrace{0\,0...\,0}_{m \text{ times}} - 9.\sum_{k=1}^{\infty}[\frac{100...0}{10^k}] = (n+1) - 9.\sum_{k=1}^{\infty}[\frac{n+1}{10^k}].$$

Therefore (2) is true.

The second important question, which must be disccused about the sequence (1), is the validity of the equality $d_s(m) + d_s(n) = d_s(m+n)$. Obviously, it is not always valid. For example

$$d_s(2) + d_s(3) = 2 + 3 = 5 = d_s(5),$$

but

$$d_s(52) + d_s(53) = 7 + 8 = 15 \neq 6 = d_s(105).$$

The following assertion is true

$$d_s(m+n) = \begin{cases} d_s(m) + d_s(n), \text{ if } d_s(m) + d_s(n) \leq 9.\max([\frac{d_s(m)}{9}], [\frac{d_s(n)}{9}]) \\ d_s(m) + d_s(n) - 9.\max([\frac{d_s(m)}{9}], [\frac{d_s(n)}{9}]), \text{ otherwise} \end{cases}$$

The proof can be made again by the method of induction.

Let

$$R_k = k + (k+1) + ... + (k+9) = 10k + 45.$$

Obviously, R_k is the sum of the elements of the k-th group of (1).

Therefore, the sum of the first n members of (1) will be

$$S_n = \sum_{k=0}^{[\frac{n}{10}]-1} R_k + [\frac{n}{10}] + ([\frac{n}{10}]+1) + ... + ([\frac{n}{10}] + n - 10.[\frac{n}{10}] - 1)$$

$$= 5.[\frac{n}{10}].([\frac{n}{10}]+8) + (n - 10.[\frac{n}{10}]).[\frac{n}{10}] + \frac{1}{2}.(n - 10.[\frac{n}{10}])$$

$$.(n - 10.[\frac{n}{10}] - 1),$$

i.e.,

$$S_n = 5.[\frac{n}{10}].([\frac{n}{10}]+8) + (n - 10.[\frac{n}{10}]).(\frac{n-1}{2} - 4.[\frac{n}{10}]).$$

This equality can be proved directly or by induction.

§3. ON THE 22-nd, THE 23-rd, AND THE 24-th SMARANDACHE'S PROBLEMS [3]

The 22-nd problem from [2] (see also 27-th problem from [1]) is the following:

Smarandache square complements:

$$1,2,3,1,5,6,7,2,1,10,11,3,14,15,1,17,2,19,5,21,22,23,6,1,26,$$
$$3,7,29,30,31,2,33,34,35,1,37,38,39,10,41,42,43,11,5,46,47,3,$$
$$1,2,51,13,53,6,55,14,57,58,59,15,61,62,7,1,65,66,67,17,69,$$
$$70,71,2,\ldots$$

For each integer n to find the smallest integer k such that nk is a perfect square.
(All these numbers are square free.)

The 23-rd problem from [2] (see also 28-th problem from [1]) is the following:

Smarandache cubic complements:

$$1,4,9,2,25,36,49,1,3,100,121,18,169,196,225,4,289,12,361,50,$$
$$441,484,529,9,5,676,1,841,900,961,2,1089,1156,1225,6,1369,$$
$$1444,1521,25,1681,1764,1849,242,75,2116,2209,36,7,20,\ldots$$

[3] see also
K. Atanassov, On the 22-nd, the 23-rd, and the 24-th Smarandache's problems. *Notes on Number Theory and Discrete Mathematics*, Vol. 5 (1998), No. 2, 80-82.

For each integer n to find the smallest integer k such that nk is a perfect cub.
(All these numbers are cube free.)

The 24-th problem from [2] (see also 29-th problem from [1]) is the following:

Smarandache m−power complements:
For each integer n to find the smallest integer k such that nk is a perfect m−power (m ≥ 2).
(All these numbers are m−power free.)

Let us define by $c_m(n)$ the m−power complement of the natural number n. Let everywhere below

$$n = \prod_{i=1}^{k} p_i^{a_i},$$

where $p_1 < p_2 < ... < p_k$ are different prime numbers and $a_1, a_2, ...,$ $a_k \geq 1$ are natural numbers.

Each of the three problems is related to determining the form of $c_m(n)$ for an arbitrary number n. When $m = 2$, we obtain

$$c_2(n) = \prod_{i=1}^{k} p_i^{b_i},$$

where

$$b_i \equiv a_i (\text{mod } 2)$$

and $b_i \in \{0, 1\}$ for every $i = 1, 2, ..., k$.

We shall prove that the following properties hold for function c_2

(1) For every natural number n:

$$n \geq c_2(n);$$

(2) For every natural number n:

$$n = c_2(n) \text{ \underline{iff}}^{4} \quad n = \prod_{i=1}^{k} p_i,$$

for the different prime numbers $p_1 < p_2 < \dots < p_k$;
(3) For every natural number n:

$$c_2(c_2(n)) = c_2(n).$$

The validity of these assertions is checked easily.

If $n = \prod_{i=1}^{k} p_i$ for the different prime numbers $p_1 < p_2 < \dots < p_k$, then, obviously,

$$n = c_2(n).$$

On the other hand, if $n = c_2(n)$, then for every i $(1 \leq i \leq k)$:

$$a_i = b_i.$$

But $a_i \geq 1$ and $b_i \leq 1$. Therefore,

$$a_i = b_i = 1,$$

i.e., $n = \prod_{i=1}^{k} p_i$.

The check of (3) can be performed as follows. Let

$$c_2(n) = \prod_{i=1}^{k} p_i^{b_i},$$

[4] everywhere we shall write "iff" instead of "if and only if".

where $b_i \in \{0,1\}$ for every i $(1 \le i \le k)$. Let

$$c_2(c_2(n)) = \prod_{i=1}^{k} p_i^{d_i},$$

where $d_i \in \{0,1\}$ for every i $(1 \le i \le k)$.

Now, if for some i $b_i = 0$, then $d_i = 0$, too; and if for some i $b_i = 1$, then $d_i = 1$, too. Therefore,

$$c_2(c_2(n)) = \prod_{i=1}^{k} p_i^{d_i} = \prod_{i=1}^{k} p_i^{b_i} = c_2(n).$$

When $m = 3$ we obtain

$$c_3(n) = \prod_{i=1}^{k} p_i^{b_i},$$

where

$$b_i \equiv -a_i (\text{mod } 3)$$

and $b_i \in \{0,1,2\}$ for every $i = 1,2,...,k$.

Immediately it can be seen that none of the above three properties is valid for c_3. Now holds the property

(2') For every natural number n:

$$c_3(n) \neq n.$$

Indeed, for the a_1 there are three cases (the same is valid for a_2, ..., a_k, too):

Case 1: $a_1 = 3s + 1$ for some integer $s \ge 0$. Then $b_1 = 2$. If $s = 0$, then p_1 is a divisor of n, but p_1^2 is not a divisor of n, while p_1^2 is a divisor of $c_3(n)$; if $s > 0$, then p_1^3 is a divisor of n, but p_1^3 is not a divisor of $c_3(n)$;

Case 2: $a_1 = 3s+2$ for some integer $s \ge 0$. Then $b_1 = 1$. Therefore, where $b_i \in \{0,1\}$ for every i $(1 \leq i \leq k)$. Let

$$c_2(c_2(n)) = \prod_{i=1}^{k} p_i^{d_i},$$

where $d_i \in \{0,1\}$ for every i $(1 \leq i \leq k)$.

Now, if for some i $b_i = 0$, then $d_i = 0$, too; and if for some i $b_i = 1$, then $d_i = 1$, too. Therefore,

$$c_2(c_2(n)) = \prod_{i=1}^{k} p_i^{d_i} = \prod_{i=1}^{k} p_i^{b_i} = c_2(n).$$

When $m = 3$ we obtain

$$c_3(n) = \prod_{i=1}^{k} p_i^{b_i},$$

where

$$b_i \equiv -a_i (\text{mod } 3)$$

and $b_i \in \{0,1,2\}$ for every $i = 1, 2, ..., k$.

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Case 1: $a_1 = 3s + 1$ for some integer $s \geq 0$. Then $b_1 = 2$. If $s = 0$, then p_1 is a divisor of n, but p_1^2 is not a divisor of n, while p_1^2 is a divisor of $c_3(n)$; if $s > 0$, then p_1^3 is a divisor of n, but p_1^3 is not a divisor of $c_3(n)$;
Case 2: $a_1 = 3s+2$ for some integer $s \geq 0$. Then $b_1 = 1$. Therefore,

p_1^2 is a divisor of n, but p_1^2 is not a divisor of $c_3(n)$;
Case 3: $a_1 = 3s$ for some integer $s \geq 1$. Then $b_1 = 0$. Therefore, p_1 is a divisor of n, but p_1 is not a divisor of $c_3(n)$.

Finally, when $m \leq 2$ is an arbitrary natural number, then

$$c_m(n) = \prod_{i=1}^{k} p_i^{b_i},$$

where

$$b_i \equiv -a_i (\text{mod } m)$$

and $b_i \in \{0, 1, 2, ..., m-1\}$ for every $i = 1, 2, ..., k$.

If m is an even number, the above property (3) is valid. Property (1) now has the form:

(1") If for every $i = 1, 2, ..., k$ has the form $p_i = [m.s + \frac{m}{2}]$, or $p_i = [m.s + \frac{m}{2} + 1]$, or ... , or $p_i = m.s$, where $[x]$ is the integer part of the real number x, then

$$n \geq c_m(n),$$

but the opposite is not always valid.

Also, in this case the equality (2) has the form:

(2") For every natural number n:

$$n = c_m(n) \text{ \underline{iff} } m = 2s \text{ for some natural number } s \text{ and } n = \prod_{i=1}^{k} p_i^s,$$

for the different prime numbers $p_1 < p_2 < ... < p_k$.

The validity of the following equalities is easily proved:

(4) For every natural number n:

$$n^3 = c_2(n).c_3(n) \text{ \underline{iff} } n = \prod_{i=1}^{k} p_i^s,$$

and

(5) For every three natural numbers n, p, q:

$c_p(n) = c_q(n)$ <u>iff</u> for every $i = 1, 2, ..., k$: there exists a natural number s, such that $a_i = pqs$, or $a_i = pqs - 1$, or $a_i = pqs - 2$, or ..., or $a_i = pqs - min(p, q) + 1$.

§4. ON THE 37-th AND 38-th SMARANDACHE'S PROBLEMS [5]

The 37-th and 38-th problems from [2] (see also 39-th problem from [1]) are the following:

(Inferior) prime part:

$$2,3,3,5,5,7,7,7,7,11,11,13,13,13,13,17,17,19,19,19,19,23,23,$$

$$23,23,23,23,29,29,31,31,31,31,31,31,37,37,37,37,41,41,43,43,$$

$$43,43,47,47,47,47,47,47,53,53,53,53,53,53,59,\ldots$$

(For any positive real number n one defines $p_p(n)$ as the largest prime number less than or equal to n.)

(Superior) prime part:

$$2,2,2,3,5,5,7,7,11,11,11,11,13,13,17,17,17,17,19,19,23,23,23,$$

$$23,29,29,29,29,29,29,31,31,37,37,37,37,37,37,41,41,41,41,43,$$

$$43,47,47,47,47,53,53,53,53,53,53,59,59,59,59,\ldots$$

(For any positive real number n one defines $P_p(n)$ as the smallest prime number greater than or equal to n.)
Study these sequences.

[5] see also
K. Atanassov, On the 37-th and 38-th Smarandache's problems. *Notes on Number Theory and Discrete Mathematics*, Vol. 5 (1999), No. 2, 83-85.

First, we should note that in the first sequence $n \geq 2$, while in the second one $n \geq 0$. Would be better, if the first two members of the second sequence are omitted. Let everywhere below $n \geq 2$.

Second, let us denote by

$$\{p_1, p_2, p_3, ...\} = \{2, 3, 5, ...\},$$

the set of all prime numbers. Let $p_0 = 1$, and let $\pi(n)$ be the number of the prime numbers less or equal to n (see e.g., [3]).

Then the $n-$th member of the first sequence is

$$p_p(n) = p_{\pi(n)-1}$$

and of the second sequence is

$$P_p(n) = p_{\pi(n)+\mathcal{B}(n)},$$

where

$$\mathcal{B}(n) = \begin{cases} 0, & \text{if } n \text{ is a prime number} \\ 1, & \text{otherwise} \end{cases}$$

(see [5]).

The checks of these equalities are straightforward, or by induction.

Therefore, the values of the n-th partial sums

$$X_n = \sum_{k=1}^{n} p_p(k)$$

and

$$Y_n = \sum_{k=1}^{n} P_p(k)$$

of the two Smarandache's sequences are, respectively,

$$X_n = \sum_{k=2}^{\pi(n)} (p_k - p_{k-1}).p_{k-1} + (n - p_{\pi(n)} + 1).p_{\pi(n)} \qquad (1)$$

and

$$Y_n = \sum_{k=1}^{\pi(n)} (p_k - p_{k-1}) \cdot p_k + (n - p_{\pi(n)}) \cdot p_{\pi(n)+B(n)}. \qquad (2)$$

The proofs can be made by the method of induction. For example, the validity of (2) is proved as follows.

Let n = 2. Then the validity of (2) is obvious. Let us assume that (2) is valid for some natural number n. For the forms of n and $n+1$ there are three cases:

(a) n and $n+1$ are not prime numbers. Therefore,

$$\pi(n+1) = \pi(n)$$

and

$$B(n+1) = B(n) = 1,$$

and then

$$X_{n+1} = Y_n + P_p(n+1)$$

$$= \sum_{k=1}^{\pi(n)} (p_k - p_{k-1}) \cdot p_k + (n - p_{\pi(n)}) \cdot p_{\pi(n)+B(n)} + p_{\pi(n+1)+B(n+1)}$$

$$= \sum_{k=1}^{\pi(n+1)} (p_k - p_{k-1}) \cdot p_k + (n - p_{\pi(n+1)}) \cdot p_{\pi(n+1)+B(n+1)}$$

$$+ p_{\pi(n+1)+B(n+1)}$$

$$= \sum_{k=1}^{\pi(n+1)} (p_k - p_{k-1}) \cdot p_k + ((n+1) - p_{\pi(n+1)}) \cdot p_{\pi(n+1)+B(n+1)}.$$

(b) n is a prime number. Therefore, for $n > 2$ $n+1$ is not a prime number,

$$\pi(n+1) = \pi(n),$$

$$n = p_{\pi(n)},$$

$$\mathcal{B}(n) = 0,$$

$$\mathcal{B}(n+1) = 1,$$

and then

$$Y_{n+1} = Y_n + P_p(n+1)$$

$$= \sum_{k=1}^{\pi(n)} (p_k - p_{k-1}).p_k + (n - p_{\pi(n)}).p_{\pi(n)+\mathcal{B}(n)} + p_{\pi(n+1)+\mathcal{B}(n+1)}$$

(from $n - p_{\pi(n)} = 0$ and $n + 1 - p_{\pi(n+1)} = n + 1 - p_{\pi(n)} = 1$)

$$= \sum_{k=1}^{\pi(n+1)} (p_k - p_{k-1}).p_k + ((n+1) - p_{\pi(n+1)}).p_{\pi(n+1)+\mathcal{B}(n+1)}.$$

(c) $n + 1$ is a prime number. Therefore, for $n > 2$ n is not a prime number,

$$\pi(n+1) = \pi(n) + 1,$$

$$n + 1 = p_{\pi(n+1)},$$

$$\mathcal{B}(n) = 1,$$

$$\mathcal{B}(n+1) = 0,$$

and then

$$Y_{n+1} = Y_n + P_p(n+1)$$

$$= \sum_{k=1}^{\pi(n)} (p_k - p_{k-1}).p_k + (n - p_{\pi(n)}).p_{\pi(n)+\mathcal{B}(n)} + p_{\pi(n+1)+\mathcal{B}(n+1)}$$

(from $p_{\pi(n)+\mathcal{B}(n+1)} = p_{\pi(n)+1+0} = p_{\pi(n)+\mathcal{B}(n)}$)

$$= \sum_{k=1}^{\pi(n)} (p_k - p_{k-1}).p_k + ((n+1) - p_{\pi(n)}).p_{\pi(n)+\mathcal{B}(n)}$$

$$= \sum_{k=1}^{\pi(n)} (p_k - p_{k-1}).p_k + (p_{\pi(n+1)} - p_{\pi(n)}).p_{\pi(n+1)}$$

$$= \sum_{k=1}^{\pi(n+1)} (p_k - p_{k-1}).p_k + ((n+1) - p_{\pi(n+1)}).p_{\pi(n+1)+B(n+1)}$$

Therefore, (2) is valid.
The validity of (1) is proved analogically.

§5. ON THE 39-th, 40-th, 41-st AND THE 42-nd SMARANDACHE'S PROBLEMS [6]

The 39-th and 40-th problems from [2] (see also 40-th problem from [1]) are the following:

(Inferior) square part:

$$0, 1, 1, 1, 4, 4, 4, 4, 4, 9, 9, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 16, 16,$$

$$25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 25, 36, 36, 36, 36, 36, 36, 36, 36, 36,$$

$$36, 36, 36, 36, 49, 49, \ldots$$

(the largest square less than or equal to n.)

(Superior) square part:

$$0, 1, 4, 4, 4, 9, 9, 9, 9, 9, 16, 16, 16, 16, 16, 16, 16, 25, 25, 25, 25, 25, 25,$$

$$25, 25, 25, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 49, 49, \ldots$$

(the smallest square greater than or equal to n.)
Study these sequences.

The 41-st and 42-nd problems from [1] (see also 41-st problem from [1]) are the following:

[6] see also
V. Radeva and K. Atanassov, On the 40-th and 41-st Smarandache's problems. *Notes on Number Theory and Discrete Mathematics*, Vol. 4 (1998), No. 3, 101-104.

(Inferior) cube part:

$$0,1,1,1,1,1,1,1,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,27,27,$$
$$27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,$$
$$27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,64,64,64,\ldots$$

(the largest cube less than or equal to n.)

(Superior) cube part:

$$0,1,8,8,8,8,8,8,8,27,27,27,27,27,27,27,27,27,27,27,27,27,27,$$
$$27,27,27,27,27,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,$$
$$64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,64,$$
$$64,64,64,64,64,125,125,125,\ldots$$

(the smallest cube greater than or equal to n.)
Study these sequences.

Below we shall use the usual notations: $[x]$ and $\lceil x \rceil$ for the integer part of the real number x and for the least integer $\geq x$, respectively.

The n-th term of every one of the above sequences is, respectively

$$a_n = [\sqrt{n}]^2,$$

of the second -

$$b_n = \lceil \sqrt{n} \rceil^2,$$

of the third -

$$c_n = [\sqrt[3]{n}]^3,$$

and of the fourth -

$$d_n = \lceil \sqrt[3]{n} \rceil^3.$$

The checks of these equalities is direct, or by induction.

We can prove easily the validity of the following equalities:

$$\sum_{k=1}^{n} (2k-1).k^2 = \frac{n(n+1)(3n^2+n-1)}{6}, \quad (1)$$

$$\sum_{k=1}^{n} (2k+1).k^2 = \frac{n(n+1)(3n^2+5n+1)}{6}, \quad (2)$$

$$\sum_{k=1}^{n} (3k^2-3k+1).k^3 = \frac{n(n+1)(5n^4+4n^3-4n^2-n+1)}{10}, \quad (3)$$

$$\sum_{k=1}^{n} (3k^2+3k+1).k^3$$

$$= \frac{n(n+1)(5n^4+16n^3+14n^2+5n+1)}{10}. \quad (4)$$

For example,

$$\sum_{k=1}^{n} (2k-1).k^2 = 2.\sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2$$

$$= 2.\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(3n^2+n+1)}{6},$$

i.e. (1) is true.

Now using (1) - (4), we shall show the values of the n-th partial sums

$$A_n = \sum_{k=1}^{n} a_k,$$

$$B_n = \sum_{k=1}^{n} b_k,$$

$$C_n = \sum_{k=1}^{n} c_k$$

and

$$D_n = \sum_{k=1}^{n} d_k,$$

of the four Smarandache's sequences. They are, respectively,

$$A_n = \frac{[\sqrt{n}-1]([\sqrt{n}-1]+1)(3[\sqrt{n}-1]^2+5[\sqrt{n}-1]+1)}{6}$$

$$+(n-[\sqrt{n}]^2+1).[\sqrt{n}]^2, \qquad (5)$$

$$B_n = \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6}+(n-[\sqrt{n}]^2).[\sqrt{n}]^2, \quad (6)$$

$$C_n = \frac{[\sqrt[3]{n}-1]([\sqrt[3]{n}-1]+1)(5[\sqrt[3]{n}-1]^4+16[\sqrt[3]{n}-1]^3}{10}$$

$$+\frac{14[\sqrt[3]{n}-1]^2+[\sqrt[3]{n}-1]-1)}{10}$$

$$+(n-[\sqrt[3]{n}]^3+1).[\sqrt[3]{n}]^3, \qquad (7)$$

$$D_n = \frac{[\sqrt[3]{n}]([\sqrt[3]{n}]+1)(5[\sqrt[3]{n}]^4+4[\sqrt[3]{n}]^3-4[\sqrt[3]{n}]^2-[\sqrt[3]{n}]+1)}{10}$$

$$(n - [\sqrt[3]{n}]^3 + 1).\lceil\sqrt[3]{n}\rceil^3. \tag{8}$$

The proofs can be made again by induction. For example, the validity of (6) is proved as follows.

Let n = 1. Then the validity of (6) is obvious. Let us assume that (6) is valid for some natural number n. For the form of n there are three cases:

(a) n and $n+1$ are not squares. Therefore, $[\sqrt{n+1}] = [\sqrt{n}]$ and $\lceil\sqrt{n+1}\rceil = \lceil\sqrt{n}\rceil$ and then

$$B_{n+1} = B_n + b_{n+1} = B_n + \lceil\sqrt{n+1}\rceil^2 = B_n + \lceil\sqrt{n}\rceil^2$$

$$= \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2 + [\sqrt{n}] - 1)}{6} + (n - [\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2 + \lceil\sqrt{n}\rceil^2$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]+1)(3[\sqrt{n+1}]^2 + [\sqrt{n+1}] - 1)}{6}$$

$$+(n + 1 - [\sqrt{n+1}]^2).\lceil\sqrt{n+1}\rceil^2;$$

(b) n is a square (hence, $n+1$ is not a square). Therefore, $[\sqrt{n+1}] = [\sqrt{n}]$, $n = [\sqrt{n}]^2 = [\sqrt{n+1}]^2$ and $\lceil\sqrt{n+1}\rceil = \lceil\sqrt{n}\rceil + 1$ and then

$$B_{n+1} = B_n + b_{n+1} = B_n + \lceil\sqrt{n+1}\rceil^2$$

$$= \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2 + [\sqrt{n}] - 1)}{6} + (n - [\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2$$

$$+\lceil\sqrt{n+1}\rceil^2$$

$$= \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2 + [\sqrt{n}] - 1)}{6} + 0 + 1.\lceil\sqrt{n+1}\rceil^2$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]+1)(3[\sqrt{n+1}]^2+[\sqrt{n+1}]-1)}{6}$$

$$+(n+1-[\sqrt{n+1}]^2).\lceil\sqrt{n+1}\rceil^2;$$

(c) $n+1$ is a square (for $n > 1$ it follows that n is not a square). Therefore, $[\sqrt{n+1}] = [\sqrt{n}]+1$ and $\lceil\sqrt{n+1}\rceil = \lceil\sqrt{n}\rceil$ and then

$$B_{n+1} = B_n + b_{n+1} = B_n + \lceil\sqrt{n+1}\rceil^2 = B_n + \lceil\sqrt{n}\rceil^2$$

$$= \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6} + (n-[\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2 + \lceil\sqrt{n}\rceil^2.$$

From the equalities

$$n+1 = [\sqrt{n+1}]^2 = ([\sqrt{n}]+1)^2,$$

$$\lceil\sqrt{n}\rceil = [\sqrt{n}]+1$$

and

$$(n-[\sqrt{n}]^2+1).\lceil\sqrt{n}\rceil^2 = (([\sqrt{n}]+1)^2-[\sqrt{n}]^2).\lceil\sqrt{n}\rceil^2$$

$$= (2[\sqrt{n}]+1)([\sqrt{n}]+1)^2$$

it follows that

$$B_{n+1} = \frac{[\sqrt{n}]([\sqrt{n}]+1)(3[\sqrt{n}]^2+[\sqrt{n}]-1)}{6} + (2[\sqrt{n}]+1)([\sqrt{n}]+1)^2$$

$$= \frac{([\sqrt{n}]+1)([\sqrt{n}]+2)(3([\sqrt{n}]+1)^2+[\sqrt{n}])}{6}$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]+1)(3([\sqrt{n+1}]^2+[\sqrt{n+1}]-1)}{6}$$

$$+(n+1-[\sqrt{n+1}]^2).\lceil\sqrt{n+1}\rceil.$$

Therefore, (6) is valid.

The validity of formulas (5), (7) and (8) are proved analogically.

§6. ON THE 43-rd AND 44-th SMARANDACHE'S PROBLEMS [7]

The 43-rd and 44-th problems from [2] (see also 42-nd problem from [1]) are the following:

(Inferior) factorial part:

$$1, 2, 2, 2, 2, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 24, 24, 24, 24, 24,$$

$$24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24,$$

$$24, 24, 24, 24, 24, 24, 24, 24, 24, 24, \ldots$$

($F_p(n)$ is the largest factorial less than or equal to n.)

(Superior) factorial part:

$$1, 2, 6, 6, 6, 6, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24,$$

$$24, 24, 120, 120, 120, 120, 120, 120, 120, 120, \ldots$$

($f_p(n)$ is the smallest factorial greater than or equal to n.)
Study these sequences.

It must be noted immediately that p is not an index in F_p and f_p.

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[7] see also
V. Atanassova and K. Atanassov, On the 43-rd and 44-th Smarandache's problems. *Notes on Number Theory and Discrete Mathematics,* Vol. 5 (1999), No. 2, 86-88.

Below we shall use the usual notations: $[x]$ and $\lceil x \rceil$ for the integer part of the real number x and for the least integer $\geq x$, respectively.

First, we shall extend the definition of the function "factorial" (possibly, it is already defined, but the authors do not know this). It is defined only for natural numbers and for a given such number n it has the form:

$$n! = 1.2. \dots .n.$$

Let the new form of the function "factorial" be the following for the real positive number y:

$$y! = y.(y-1).(y-2)...(y-[y]+1),$$

where $[y]$ denotes the integer part of y.

Therefore, for the real number $y > 0$:

$$(y+1)! = y!.(y+1).$$

This new factorial has Γ–representation

$$y! = \frac{\Gamma(y+1)}{\Gamma(y-[y]+1)}$$

and representation by the Pochhammer symbol

$$y! = (y)_{[y]}$$

(see, e.g., [7]).

Obviously, if y is a natural number, $y!$ is the standard function "factorial".

It can be easily seen that the extended function has the properties similar to these of the standard function.

Second, we shall define a new function (possibly, it is already defined, too, but the authors do not know this). It is an inverse function of the function "factorial" and for the arbitraty positive real numbers x and y it has the form:

$$x? = y \ \underline{\text{iff}} \ y! = x. \qquad (1)$$

Let us show only one of its integer properties.

For every positive real number x:

$$[(x+1)?] = \begin{cases} [x?]+1, & \text{if there exists a natural number } n \text{ such that } n! = x+1 \\ [x?], & \text{otherwise} \end{cases}$$

From the above discussion it is clear that we can ignore the new factorial, using the definition

$$x? = y \ \underline{\text{iff}} \ (y)_{[y]} = x.$$

Practically, everywhere below y is a natural number, but in some places x will be a positive real number (but not an integer).

Then the $n-$th member of the first sequence is

$$F_p(n) = [n?]!$$

and of the second sequence it is

$$f_p(n) = \lceil n?\rceil!.$$

The checks of these equalities is direct, or by the method of induction.

Therefore, the values of the n-th partial sums

$$X_n = \sum_{k=1}^{n} F_p(k)$$

and

$$Y_n = \sum_{k=1}^{n} f_p(k)$$

of the two above Smarandache's sequences are, respectively,

$$X_n = \sum_{k=1}^{[n?]} (k! - (k-1)!).(k-1)! + (n - [n?]! + 1).[n?]! \quad (2)$$

and

$$Y_n = \sum_{k=1}^{[n?]} (k! - (k-1)!).k! + (n - [n?]! + 1).\lceil n?\rceil! \quad (3)$$

The proofs can be made by induction. For example, the validity of (2) is proved as follows.

Let n = 1. Then the validity of (2) is obvious. Let us assume that (2) is valid for some natural number n. For the form of $n+1$ there are two cases:

(a) for $n+1$ does not exist a natural number m for which $n+1 = m!$. Therefore,

$$[(n+1)?] = [n?]$$

and then

$$X_{n+1} = Y_n + F_p(n+1)$$

$$= \sum_{k=1}^{[n?]} (k! - (k-1)!).(k-1)! + (n - [n?]! + 1).[n?]! + [(n+1)?]!$$

$$= \sum_{k=1}^{[(n+1)?]} (k! - (k-1)!).(k-1)! + ((n+1) - [(n+1)?]! + 1).[(n+1)?]!.$$

(b) for $n+1$ there exists a natural number m for which $n+1 = m!$. Therefore, for $n > 2$ does not exist a natural number m for which $n = m!$,

$$[(n+1)?] = [n?] + 1,$$

$$[(n+1)?] = n + 1,$$

from (1), and then

$$X_{n+1} = Y_n + F_p(n+1)$$

$$= \sum_{k=1}^{[n?]} (k! - (k-1)!).(k-1)! + (n - [n?]! + 1).[n?]! + [(n+1)?]!$$

$$= \sum_{k=1}^{[n?]} (k! - (k-1)!).(k-1)! + ((n+1) - [n?]!).([(n+1)?] - 1)!$$

$$+[(n+1)?]!$$

$$= \sum_{k=1}^{[(n+1)?]} (k! - (k-1)!).(k-1)! + ((n+1) - [(n+1)?]! + 1)$$

$$.[(n+1)?]!.$$

Therefore, (2) is valid.
The validity of (3) is proved analogically.

§7. ON THE 61-st, THE 62-nd, AND THE 63-rd SMARANDACHE'S PROBLEM [8]

The 61-st problem from [2] (see also 66-th problem from [1]) is the following:

Smarandache exponents (of power 2):

$$0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4,0,1,0,2,0,1,0,3,0,1,0,2,0,1,$$
$$0,5,0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4,0,1,0,2,0,1,0,3,0,1,0,2,$$
$$0,1,0,6,0,1,\ldots$$

($e_2(n)$ is the largest exponent (of power 2) which divides n.)
Or, $e_2(n) = k$ if 2^k divides n but 2^{k+1} does not.

In [1] and [2] there are two misprints in the above sequence.

The 62-nd problem from [2] (see also 67-th problem from [1]) is the following:

Smarandache exponents (of power 3):

$$0,0,1,0,0,1,0,0,2,0,0,1,0,0,1,0,0,2,0,0,1,0,0,1,0,0,3,0,0,1,$$
$$0,0,1,0,0,2,0,0,1,0,0,1,0,0,2,0,0,1,0,0,1,0,0,2,0,0,1,0,0,1,$$

[8] see also

K. Atanassov, On the 61-st, 62-nd and 63-rd Smarandache's problems. *Notes on Number Theory and Discrete Mathematics*, Vol. 4 (1998), No. 4, 175-182.

and

M. Vassilev - Missana and K. Atanassov, Some representations related to $n!$. *Notes on Number Theory and Discrete Mathematics*, Vol. 4 (1998), No. 4, 148-153.

$$0, 0, 2, 0, 0, 1, 0 \ldots$$

($e_3(n)$ is the largest exponent (of power 3) which divides n.)
Or, $e_3(n) = k$ if 3^k divides n but 3^{k+1} does not.

The 63-rd problem from [2] (see also 68-th problem from [1]) is the following:

Smarandache exponents (of power p) { generalization }:
($e_p(n)$ is the largest exponent (of power p) which divides n, where p is an integer ≥ 2.)
Or, $e_p(n) = k$ if p^k divides n but p^{k+1} does not.

Let $[x]$ be the integer part of the real number x.

We can rewrite the first sequence to the form:

$0, 1,$
$0, 2, 0, 1,$
$0, 3, 0, 1, 0, 2, 0, 1,$
$0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1,$
$0, 5, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1,$
$0, 2, 0, 1,$
$0, 6, 0, 1, \ldots,$

and we can obtain formulas for the n−th member of the new sequence and of the the sum of its first n elements, but the following form of the first sequence is more suitable and the two corresponding formulas will be simpler:

0,
1, 0,
2, 0, 1, 0,
3, 0, 1, 0, 2, 0, 1, 0,
4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0,
5, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0,
2, 0, 1, 0,
6, 0, 1, ...,

Therefore, the k-th row ($k \geq 0$) contains 2^k members and let they be:

$$b_{k,1}, b_{k,2}, ..., b_{k,2^k}$$

and for every $i = 1, 2, ..., 2^{k-1}$:

$$b_{k,2i} = 0.$$

The second form of the sequence shows that for every $k \geq 1$:

$$b_{0,1} = 0,$$

$$b_{k,2i-1} = \begin{cases} k, & \text{if } i = 1 \\ b_{k-1,2i-1}, & \text{if } 2 \leq i \leq 2^{k-2} \\ b_{k-1,2i-2^{k-1}-1}, & \text{if } 2^{k-2}+1 \leq i \leq 2^{k-1} \end{cases} \tag{1}$$

Obviously, for every two natural numbers k, i there exists a natural number n: $b_{k,i} = e_2(n)$.

Let the natural number n be fixed. Therefore, we can determine the number of the row and the place in this row in which is places $e_2(n)$. They are:

$$k = [log_2 n]$$

and

$$i = n - 2^{[log_2 n]} + 1.$$

Then, from (1) and from the second form of the sequence it follows the following explicit representation:

$$b_{k,2i-1} = \begin{cases} k, & \text{if } i = 1 \\ k-1, & \text{if } i = 2^{k-2}+1 \\ k-2, & \text{if } i = 2^{k-3}+1 \text{ or } i = 2^{k-2}+2^{k-3}+1 \\ k-3, & \text{if } i = 2^{k-4}+1 \text{ or } i = 2^{k-3}+2^{k-4}+1 \\ & \text{or } i = 2^{k-4}+2.2^{k-3}+1 \\ & \text{or } i = 2^{k-4}+3.2^{k-3}+1 \\ k-4, & \text{if } i = 2^{k-5}+1 \\ & \text{or } i = 2^{k-5}+2^{k-4}+1 \\ & \text{or } i = 2^{k-5}+2.2^{k-4}+1 \\ & \text{or } i = 2^{k-5}+3.2^{k-4}+1 \\ & \text{or } i = 2^{k-5}+4.2^{k-4}+1 \\ & \text{or } i = 2^{k-5}+5.2^{k-4}+1 \\ & \text{or } i = 2^{k-5}+6.2^{k-4}+1 \\ & \text{or } i = 2^{k-5}+7.2^{k-4}+1 \\ \vdots & \qquad\qquad\qquad \vdots \\ k-s, & \text{if } i = 2^{k-s-1}+1 \\ & \text{or } i = 2^{k-s-1}+2^{k-s}+1 \\ & \text{or } i = 2^{k-s-1}+2.2^{k-s}+1 \\ & \text{or } i = 2^{k-s-1}+3.2^{k-s}+1 \text{ or } \ldots \\ & \text{or } i = 2^{k-s-1}+(2^{s-1}-1).2^{k-s}+1 \end{cases} \tag{2}$$

for $s < k$.

The validity of (2) is seen directly by our construction, or it can be proved, e.g., by induction.

Let R_k^2 is the sum of the member from k–th row. Easily it can be seen that

$$R_k^2 = 2^k - 1.$$

Now, let S_n^2 be the sum of the first n members of the sequence

$\{e_2(n)\}_{n=1}^{\infty}$, i.e.

$$S_n^2 = \sum_{i=1}^{n} e_2(i).$$

From (2) it can be seen that

$$S_n^2 = \sum_{j=1}^{[log_2 n]-1} R_j^2 + \sum_{i=1}^{n-2^{[log_2 n]}+1} b_{[log_2 n],i}$$

$$= (2^{[log_2 n]} - 1 - [log_2 n]) + [log_2 n] + \sum_{j=1}^{[log_2 n]} j.[\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}].$$

Therefore,

$$S_n^2 = 2^{[log_2 n]+1} - 1 + \sum_{j=1}^{[log_2 n]} j.[\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}]. \qquad (3)$$

The validity of (3) can be proved, e.g., by induction, using (2).

Also by induction it can be proved that

$$S_n^2 = 2^{[log_2 n]+1} - 1 + \sum_{j=1}^{[log_2 n]} j.([\frac{n - 2^{[log_2 n]}}{2^j}]$$

$$-[\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}]). \qquad (4)$$

On the other hand, it can be proved directly, that the right parts of (3) and (4) coincide. For this aim, it is enough to prove that for every natural number n and for every natural number j such that $n \geq 2^{j+1}$ the following identity is valid:

$$[\frac{n - 2^{[log_2 n]}}{2^j}] = [\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}] + [\frac{n - 2^{[log_2 n]} + 2^j}{2^{j+1}}],$$

which can be made, e.g., by induction.

Analogically, we shall rewrite the second sequence to the form:

1, 0, 0, 1, 0, 0,
2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0,
3, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0,
3, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0,
4, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0...

Therefore, the k-th row ($k \geq 1$) contains 2.3^k members and let they be:

$$b_{k,1}, b_{k,2}, ..., b_{k,2.3^k}$$

and for every $i = 1, 2, ..., 2.3^{k-1}$:

$$b_{k,3i-1} = b_{k,3i} = 0.$$

The second form of this sequence shows that for every $k \geq 1$:

$$b_{k,3i-2} = \begin{cases} k, & \text{if } i = 1 \\ b_{k-1,3i}, & \text{if } 2 \leq i \leq 2.3^{k-2} \\ b_{k-1,3i-3^{k-1}}, & \text{if } 2.3^{k-2}+1 \leq i \leq 3^{k-1} \\ b_{k-1,3i-3^k}, & \text{if } 3^{k-1}+1 \leq i \leq 2.3^{k-1} \end{cases} \quad (5)$$

As in the first case, for every two natural numbers k, i there exists a natural number n: $b_{k,i} = e_2(n)$.

Let the natural number n be fixed. Therefore, we can determine the number of the row and the place in this row in which is places $e_2(n)$. They are:

$$k = [log_3 n]$$

and

$$i = n - 3^{[log_3 n]} + 1.$$

Then, from (5) and from the second form of this sequence it follows the following explicit representation (for $s < k$):

$$b_{k,3i-2} = \begin{cases} k, & \text{if } i = 1 \text{ or } i = 3^{k-1}+1 \\ k-1, & \text{if } i = 3^{k-2}+1 \text{ or } i = 2.3^{k-2}+1 \\ & \text{or } i = 4.3^{k-2}+1 \text{ or } i = 5.3^{k-2}+1 \\ k-2, & \text{if } i = 3^{k-3}+1 \text{ or } i = 2.3^{k-3}+1 \\ & \text{or } i = 3^{k-3}+3^{k-2}+1 \\ & \text{or } i = 2.3^{k-3}+3^{k-2}+1 \\ & \text{or } i = 3^{k-3}+2.3^{k-2}+1 \\ & \text{or } i = 2.3^{k-3}+2.3^{k-2}+1 \\ & \text{or } i = 3^{k-3}+3.3^{k-2}+1 \\ & \text{or } i = 2.3^{k-3}+3.3^{k-2}+1 \\ & \text{or } i = 3^{k-3}+4.3^{k-2}+1 \\ & \text{or } i = 2.3^{k-3}+4.3^{k-2}+1 \\ & \text{or } i = 3^{k-3}+5.3^{k-2}+1 \\ & \text{or } i = 2.3^{k-3}+5.3^{k-2}+1 \\ \vdots & \vdots \\ k-s, & \text{if } i = 3^{k-s-1}+1 \text{ or } i = 2.3^{k-s-1}+1 \\ & \text{or } i = 3^{k-s-1}+3^{k-s}+1 \\ & \text{or } i = 2.3^{k-s-1}+3^{k-s}+1 \text{ or } \dots \\ & \text{or } i = 3^{k-s-1}+(2.3^{s-1}-1).3^{k-s}+1 \\ & \text{or } i = 2.3^{k-s-1}+(2.3^{s-1}-1).3^{k-s}+1 \end{cases} \qquad (6)$$

The validity of (6) is seen directly by our construction, or it can be proved, e.g., by induction.

Let R_k^3 is the sum of the members from the $k-$th row. It is easily seen that

$$R_k^3 = 3^k - 1.$$

Now, let S_n^3 be the sum of the first n members of the sequence $\{e_3(n)\}_{n=1}^{\infty}$, i.e.

$$S_n^3 = \sum_{i=1}^{n} e_3(i).$$

From (6) it can be seen, that it is valid:

$$S_n^3 = \sum_{j=1}^{[log_3 n]-1} R_j^3 + \sum_{i=1}^{n-3^{[log_3 n]}+1} b_{[log_3 n],i}$$

$$= \frac{3^{[log_3 n]}-1}{2} - [log_3 n] + \sum_{j=1}^{[log_3 n]} j.([\frac{n-3^{[log_3 n]}}{3^{j-1}}]$$

$$-[\frac{n-3^{[log_3 n]}}{3^j}]) + [log_3 n].$$

Therefore,

$$S_n^3 = \frac{3^{[log_3 n]}-1}{2} + \sum_{j=1}^{[log_3 n]} j.([\frac{n-3^{[log_3 n]}}{3^j}]$$

$$-[\frac{n-3^{[log_3 n]}}{3^{j+1}}]). \qquad (7)$$

The validity of (7) can be proved, e.g., by induction, using (6).

By analogy with the above constructions, we can write the sequence of the p-th powers, where p is a prime number of the form:

$$\underbrace{\underbrace{0,\dots,0}_{p-1},1,\underbrace{0,\dots,0}_{p-1},1}_{p-1}$$

$$\underbrace{\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\dots,\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1}}_{p-1}$$

$$\underbrace{0,\dots,0}_{p-1},3,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\underbrace{\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\dots,\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1}}_{p-1}$$

$$\underbrace{0,\dots,0}_{p-1},3,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\underbrace{\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\dots,\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1}}_{p-1}$$

$$\bullet \qquad \bullet \qquad \bullet$$

$$\underbrace{0,\dots,0}_{p-1},3,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\underbrace{\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,0}_{p-1},1}_{p-1},\dots,\underbrace{0,\dots,0}_{p-1},2,\underbrace{\underbrace{0,\dots,0}_{p-1},1,\dots,\underbrace{0,\dots,}_{p-1}1}_{p-1}}_{p-1}$$

Therefore, the k-th row ($k \geq 1$) contains $(p-1).p^k$ members and let them be:

$$b_{k,1}, b_{k,2}, ..., b_{k,(p-1).p^k}$$

and for every $i = 1, 2, ..., (p-1).p^{k-1}$:

$$b_{k,p.i-p+2} = b_{k,p.i-p+3} = ... = b_{k,p.i} = 0.$$

The second form of this sequence shows that for every $k \geq 1$:

$$b_{k,p.i-p+1} = \begin{cases} k, & \text{if } i = 1 \\ b_{k-1,p.i}, & \text{if } 2 \leq i \leq (p-1).p^{k-2} \\ b_{k-1,p.i-p^{k-2}}, & \text{if } (p-1).p^{k-2}+1 \leq i \\ & \leq p^{k-2} \\ \vdots & \vdots \\ b_{k-1,p.i-s.p^{k-1}}, & \text{if } s.p^{k-1}+1 \leq i \\ & \leq (s+1).p^{k-1} \\ & \text{for } s = 1, 2, ..., p-2 \end{cases} \tag{8}$$

As in the first case, for every two natural numbers k, i there exists a natural number n: $b_{k,i} = e_p(n)$.

Let the natural number n be fixed. Therefore, we can determine the number of the row and the place in this row where $e_p(n)$ is. They are:

$$k = [log_p n]$$

and

$$i = n - p^{[log_p n]} + 1.$$

Then, from (8) and from the second form of this sequence it follows the following explicit representation:

$$b_{k,p.i-p+1} = \begin{cases} k, & \text{if } i = 1 \text{ or } i = p^{k-1}+1 \\ & \text{or } i = 2.p^{k-1} \text{ or } i = (p-2).p^{k-1}+1 \\ k-1, & \text{if } i = p^{k-2}+1 \text{ or } i = 2.p^{k-2}+1 \\ & \text{or ... or } i = (p-1).p^{k-2}+1 \\ & \text{or } i = (p+1).p^{k-2}+1 \text{ or ...} \\ & \text{or } i = (2p-1).p^{k-2}+1 \text{ or ...} \\ & \text{or } i = ((p-2).p)+1).p^{k-2}+1 \text{ or ...} \\ & \text{or } i = ((p-1).p)-1).p^{k-2}+1 \\ \vdots & \qquad\qquad \vdots \\ k-s, & \text{if } i = p^{k-s-1}+1 \text{ or } i = 2.p^{k-s-1}+1 \\ & \text{or ... or } i = (p-1).p^{k-s-1}+1 \\ & \text{or } i = p^{k-s-1}+p^{k-s}+1 \text{ or ...} \\ & \text{or } i = (p-1).p^{k-s-1}+p^{k-s}+1 \text{ ...} \\ & \text{or } i = p^{k-s-1}+((p-1).p^{s-1}-1) \\ & .p^{k-s}+1 \text{ ...} \\ & \text{or } i = (p-1).p^{k-s-1} \\ & +((p-1).p^{s-1}-1).p^{k-s}+1 \end{cases} \tag{9}$$

for $s < k$.

The validity of (9) is seen directly by our construction, or it can be proved, e.g., by induction.

Let R_k^p is the sum of the members from the k-th row. It can be easily seen that

$$R_k^p = p^k - 1.$$

Now, let S_n^p be the sum of the first n members of the sequence $\{e_p(n)\}_{n=1}^{\infty}$, i.e.

$$S_n^p = \sum_{i=1}^{n} e_p(i).$$

From (9) it can be seen that it is valid:

$$S_n^p = \sum_{j=1}^{[log_p n]-1} R_j^p + \sum_{i=1}^{n-p^{[log_p n]}+p} b_{[log_p n],i}$$

$$= \frac{p^{[log_p n]+1}-1}{p-1} + \sum_{j=1}^{[log_p n]} j.([\frac{n-p^{[log_p n]}}{p^j}] - [\frac{n-p^{[log_p n]}}{p^{j+1}}]). \quad (10)$$

The validity of (10) can be proved, e.g., by induction, using (9).

Finally, we shall note that (10) can be used for representation of $n!$. It is

$$n! = \prod_{p\in\mathcal{P}} (\frac{p^{[log_p n]+1}-1}{p-1} + \sum_{j=1}^{[log_p n]} j.([\frac{n-p^{[log_p n]}}{p^j}]$$

$$-[\frac{n-p^{[log_p n]}}{p^{j+1}}]))$$

or

$$n! = \prod_{i=1}^{\pi(n)} (\frac{p^{[log_p n]+1}-1}{p-1} + \sum_{j=1}^{[log_p n]} j.([\frac{n-p^{[log_p n]}}{p^j}]$$

$$-[\frac{n-p^{[log_p n]}}{p^{j+1}}])),$$

where

$$\mathcal{P} = \{p_1, p_2, p_3, ...\} = \{2, 3, 5, ...\}$$

is the set of the prime numbers.

§8. ON THE 97-th, THE 98-th AND THE 99-th SMARANDACHE'S PROBLEMS [9]

The 97-th problem from [2] (see also 6-th problem from [1]) is the following:

Smarandache constructive set (of digits 1,2):

$$1, 2, 11, 12, 21, 22, 111, 112, 121, 122, 211, 212, 221, 222, 1111, 1112,$$

$$1121, 1122, 1211, 1212, 1221, 1222, 2111, 2112, 2121, 2122, 2211,$$

$$2212, 2221, 2222, \ldots$$

(Numbers formed by digits 1 and 2 only.)

Definition:
a1) 1, 2 belongs to S_2;
a2) if a, b belongs to S_2, then $\overline{ab}$ belongs to S_2 too;
a3) only elements obtained by rules a1) and a2) applied a finite number of times belong to S_2.

Remark:
- there are 2^k numbers of k digits in the sequence, for $k = 1, 2, 3, \ldots$;
- to obtain from the $k-$digits number group the $(k+1)-$digits number group, just put first the digit 1 and second the digit 2 in the front of all $k-$digits numbers.

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[9] see also
H. Aladjov and K. Atanassov, On the 97-th, 98-th and 99-th Smarandache's problems. *Notes on Number Theory and Discrete Mathematics*, Vol. 5 (1999), No. 3, 89-93.

The 98-th problem from [2] (see also 7-th problem from [1]) is the following:

Smarandache constructive set (of digits 1,2,3):

1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33, 111, 112, 113, 121, 122, 123, 131,

132, 133, 211, 212, 213, 221, 222, 223, 231, 232, 233, 311, 312, 313, 321,

322, 323, 331, 332, 333, ...

(Numbers formed by digits 1, 2, and 3 only.)

Definition:
a1) 1, 2, 3 belongs to S_3*;*
a2) if a, b *belongs to* S_3*, then* $\overline{ab}$ *belongs to* S_3 *too;*
a3) only elements obtained by rules a1) and a2) applied a finite number of times belong to S_3*.*

Remark:
- there are 3^k *numbers of k digits in the sequence, for* $k = 1, 2, 3, \ldots$*;*
- to obtain from the $k-$*digits number group the* $(k+1)-$*digits number group, just put first the digit 1, second the digit 2, and third the digit 3 in the front of all* $k-$*digits numbers.*

The 99-th problem from [2] (see also 8-th problem from [1]) is the following:

Smarandache generalized constructive set:
(Numbers formed by digits $d_1, d_2, \ldots, d_m$ *only, and* d_i *being different each other,* $1 \leq m \leq 9$*.)*

Definition:
a1) $d_1, d_2, \ldots, d_m$ *belongs to* S_m*;*
a2) if a, b *belongs to* S_2*, then* $\overline{ab}$ *belongs to* S_2 *too;*
a3) only elements obtained by rules a1) and a2) applied a finite num-

ber of times belong to S_m.

Remark:
- there are m^k numbers of k digits in the sequence, for $k = 1, 2, 3, ...$;
- to obtain from the k-digits number group the $(k+1)$-digits number group, just put first the digit d_1, second the digit d_2, ..., and the m-time digit d_m in the front of all k-digits numbers.

More general: all digits d_i can be replaced by numbers as large as we want (therefore of many digits each), and also m can be as large as we want.

As in the previous sections, we can construct new sequences for every one of the three sequences in the following forms, respectively: a new form of the first sequence

1, 2,
11, 12, 21, 22,
111, 112, 121, 122, 211, 212, 221, 222,
1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222, 2111, 2112, 2121,
2122, 2211, 2212, 2221, 2222, ...

a new form of the second sequence

1, 2, 3,
11, 12, 13, 21, 22, 23, 31, 32, 33,
111, 112, 113, 121, 122, 123, 131, 132, 133, 211, 212, 213, 221, 222,
223, 231, 232, 233, 311, 312, 313, 321, 322, 323, 331, 332, 333,
1111, 1112, 1113, ...

a new form of the third sequence

$$d_1, d_2, ..., d_m$$
$$d_1 d_1, d_1 d_2, ..., d_1 d_m, d_2 d_1, ... d_m d_m,$$
$$d_1 d_1 d_1, d_1 d_1 d_2, ..., d_m d_m d_m, ...$$

As it is noted in the beginning of the section, the number of the members of the k-th row in the first, second and third sequence in the new form will be respectively 2^k, 3^k and m^k.

Let us label the three sequences, respectively, as S_2, S_3 and S_m. Therefore, we can represent these sets, respectively, by:

$$S_2 = \bigcup_{n=1}^{\infty} \{\overline{a_1 a_2 ... a_n} \mid a_1, a_2, ..., a_n \in \{1,2\}\} \equiv \bigcup_{n=1}^{\infty} A_{2,n}$$

and, as it was mentioned above,

$$card(A_{2,n}) = 2^n,$$

where $card(X)$ is the cardinality of the set X;

$$S_3 = \bigcup_{n=1}^{\infty} \{\overline{a_1 a_2 ... a_n} \mid a_1, a_2, ..., a_n \in \{1,2,3\}\} \equiv \bigcup_{n=1}^{\infty} A_{3,n}$$

and

$$card(A_{3,n}) = 3^n;$$

$$S_m = \bigcup_{n=1}^{\infty} \{\overline{a_1 a_2 ... a_n} \mid a_1, a_2, ..., a_n \in \{d_1, d_2, ..., d_m\}\}$$

$$\equiv \bigcup_{n=1}^{\infty} A_{m,n}$$

and

$$card(A_{m,n}) = m^n.$$

In the general (third) case we shall define:

$$B_{m,n} = \sum_{x \in A_{m,n}} x.$$

Therefore,

$$B_{2,1} = 3 = 2^0.3.1,$$
$$B_{2,2} = 66 = 2^1.3.11,$$
$$B_{2,3} = 1332 = 2^2.3.111,$$
$$B_{2,4} = 26664 = 2^3.3.1111, ...$$

$$B_{3,1} = 6 = 3^0.6.1,$$
$$B_{3,2} = 198 = 3^1.6.11,$$
$$B_{3,3} = 5994 = 3^2.6.111,$$
$$B_{3,4} = 59994 = 3^3.6.1111, ...$$

It is interesting to note, for example, that

$$B_{4,1} = 4^0.10.1,$$
$$B_{4,2} = 440 = 4^1.10.11,$$
$$B_{4,3} = 17760 = 4^2.10.111, ...$$

Now we can prove by induction that

$$B_{m,n} = m^{n-1}.(\sum_{i=1}^{m} d_i).\underbrace{11...1}_{n \text{times}}. \qquad (1)$$

Indeed, for m - fixed natural number and $n = 1$ we obtain that

$$B_{m,1} = \sum_{i=1}^{m} d_i = m^0.(\sum_{i=1}^{m} d_i).1.$$

Let us assume that $B_{m,n}$ satisfies (1) for some natural number $n \geq 1$ (m is fixed). Then from the above construction it is seen that

$$B_{m,n+1} = m.(m^{n-1}.(\sum_{i=1}^{m} d_i).\underbrace{11...1}_{n\text{times}}) + m^n.10^n.(\sum_{i=1}^{m} d_i)$$

$$= m^n.(\sum_{i=1}^{m} d_i).(1\underbrace{00...0}_{n\text{times}} + \underbrace{11...1}_{n\text{times}})$$

$$= m^{(n+1)-1}.(\sum_{i=1}^{m} d_i).\underbrace{11...1}_{(n+1)\text{times}}),$$

with which (1) is proved.

Below, using the usual notation $[x]$ for the integer part of the real number x, we shall give a formula for the s-th member $x_{m,s}$ of the general (third) sequence. The validity of this formula is proved also by induction. It is:

$$x_{m,s} = \sum_{i=1}^{[log_m(s+1)(m-1)]} 10^{i-1}.(r([\frac{s - m.[\frac{m^{i-1}-1}{m-1}]}{m^{i-1}}], m) + 1), \quad (2)$$

where

$$r(p, q) = p - q.[\frac{p}{q}]$$

for every two natural numbers p and q, i.e., function r determines the remainder of the division of p by q.

When $m = 2$, (2) obtains the form

$$x_{2,s} = \sum_{i=1}^{[log_2(s+1)]} 10^{i-1}.(r([\frac{s - 2.m^i + m}{m^{i-1}}], 2) + 1),$$

and when $m = 3$, (2) obtains the form

$$x_{m,s} = \sum_{i=1}^{[log_3 2.(s+1)]} 10^{i-1}.(r([\frac{s-3.[\frac{m^{i-1}-1}{3}]}{3^{i-1}}],3)+1).$$

Using formula (2) we can show the s–th partial sum of the third sequence (and from there - of the first and the second sequences). It is

$$S_{m,s} = \sum_{i=1}^{s} x_{m,s},$$

but we can construct the following simpler formula by a calculating point of view, having in mind that the s–th member of the third sequence is placed in the $([log_m((s-2)(m-1)+1)]+1)$–th subsequence and also the sum of the members of the first $([log_m((s-2)(m-1)+1)]$ sequences can be calculated by (1):

$$S_{m,s} = \sum_{i=1}^{[log_m((s-2)(m-1)+1)]} B_{m,i} + \sum_{i=s-t+1}^{s} x_{m,s},$$

where

$$t = \frac{m^{[log_m((s-2)(m-1)+1)]}-1}{m-1}.$$

§9. ON THE 100-th, THE 101-st AND THE 102-nd SMARANDACHE'S PROBLEMS [10]

The 100-th problem from [2] (see also 80-th problem from [1]) is the following:

Square roots:

$$0,1,1,1,2,2,2,2,2,3,3,3,3,3,3,3,4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,$$
$$5,5,5,5,5,6,6,6,6,6,6,6,6,6,6,6,6,6,7,7,\ldots$$

($s_q(n)$ is the superior integer part of square root of n.)
Remark: this sequence is the natural sequence, where each number is repeated $2n+1$ times, because between n^2 (included) and $(n+1)^2$ (excluded) there are $(n+1)^2 - n^2$ different numbers.
Study this sequence.

The 101-st problem from [2] (see also 81-st problem from [1]) is the following:

Cubical roots:

$$0,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,3,3,3,3,$$
$$3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,$$
$$3,3,4,4,4,\ldots$$

---

[10] see also
K. Atanassov, On the 100-th, 101-st and 102-nd Smarandache's problems. *Notes on Number Theory and Discrete Mathematics*, Vol. 5 (1999), No. 3, 94-96.

($c_q(n)$ is the superior integer part of cubical root of n.)
Remark: this sequence is the natural sequence, where each number is repeated $3n^2 + 3n + 1$ times, because between n^3 (included) and $(n+1)^3$ (excluded) there are $(n+1)^3 - n^3$ different numbers.
Study this sequence.

The 102-nd problem from [2] (see also 82-nd problem from [1]) is the following:

m−power roots:
($m_q(n)$ is the superior integer part of m−power root of n.)
Remark: this sequence is the natural sequence, where each number is repeated $(n+1)^m - n^m$ times.
Study this sequence.

Below we shall use the usual notation: $[x]$ for the integer part of the real number x.

The author thinks that these are some of the most trivial S-marandache's problems. The n-th term of each of the above sequences is, respectively

$$x_n = [\sqrt{n}],$$

of the second -

$$y_n = [\sqrt[3]{n}],$$

and of the third -

$$z_n = [\sqrt[m]{n}].$$

The checks of these equalities is straightforward, or by induction. We can easily prove the validity of the following equalities:

$$\sum_{k=1}^{n} (2k+1).k = \frac{n(n+1)(4n+5)}{6}, \tag{1}$$

$$\sum_{k=1}^{n} (3k^2+3k+1).k = \frac{n(n+1)(3n^2+7n+4)}{6}. \qquad (2)$$

Now using (1) and (2), we shall show the values of the n-th partial sums

$$X_n = \sum_{k=1}^{n} x_k,$$

$$Y_n = \sum_{k=1}^{n} y_k$$

and

$$Z_n = \sum_{k=1}^{n} z_k,$$

of the three Smarandache's sequences. They are, respectively,

$$X_n = \frac{([\sqrt{n}]-1)[\sqrt{n}](4[\sqrt{n}]+1)}{6} + n - [\sqrt{n}]^2 + 1).[\sqrt{n}], \qquad (3)$$

$$Y_n = \frac{([\sqrt[3]{n}-1)[\sqrt[3]{n}]^2(3[\sqrt[3]{n}+1)}{4} + (n - [\sqrt[3]{n}]^3 + 1).[\sqrt[3]{n}], \qquad (4)$$

$$Z_n = \sum_{k=1}^{n} (([\sqrt[m]{k}]+1)^m - [\sqrt[m]{k}]^m)[\sqrt[m]{k}-1]^m$$

$$+(n - [\sqrt[m]{n}]^m + 1).[\sqrt[m]{n}]. \qquad (5)$$

The proofs can be made by induction. For example, the validity of (3) is proved as follows.

Let n = 1. Then the validity of (3) is obvious. Let us assume that (3) is valid for some natural number n. For the form of n there

are two cases:
(a) $n+1$ is not a square. Therefore,

$$[\sqrt{n+1}] = [\sqrt{n}]$$

and then

$$X_{n+1} = X_n + x_{n+1}$$

$$= \frac{[\sqrt{n}]([\sqrt{n}]-1)(4[\sqrt{n}]+1)}{6} + (n - [\sqrt{n}]^2 + 1).[\sqrt{n}] + [\sqrt{n+1}]$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]-1)(4[\sqrt{n+1}]+1)}{6} + (n + 1 - [\sqrt{n+1}]^2 + 1)$$

$$.[\sqrt{n+1}].$$

(b) $n+1$ is a square (for $n \geq 1$ it follows that n is not a square). Therefore,

$$[\sqrt{n+1}] = [\sqrt{n}] + 1$$

and then

$$X_{n+1} = X_n + x_{n+1}$$

$$= \frac{[\sqrt{n}]([\sqrt{n}]-1)(4[\sqrt{n}]+1)}{6} + (n - [\sqrt{n}]^2 + 1).[\sqrt{n}] + [\sqrt{n+1}]$$

$$= \frac{([\sqrt{n+1}]-1)([\sqrt{n+1}]-2)(4[\sqrt{n+1}]-3)}{6}$$

$$+(n + 1 - ([\sqrt{n+1}]-1)^2).([\sqrt{n}]-1) + [\sqrt{n+1}]$$

$$= \frac{[\sqrt{n+1}]([\sqrt{n+1}]-1)(4[\sqrt{n+1}]+1)}{6}$$

$$-([\sqrt{n+1}]-1)(2[\sqrt{n+1}]-1)$$

$$+(n+1-([\sqrt{n+1}]-1)^2).([\sqrt{n}]-1)+[\sqrt{n+1}]$$

$$=\frac{[\sqrt{n+1}]([\sqrt{n+1}]-1)(4[\sqrt{n+1}]+1)}{6}+[\sqrt{n+1}]$$

$$=\frac{[\sqrt{n+1}]([\sqrt{n+1}]-1)(4[\sqrt{n+1}]+1)}{6}$$

$$+((n+1)-[\sqrt{n+1}]^2+1).[\sqrt{n+1}].$$

Therefore, (3) is valid.

The validity of formulas (4) and (5) are proved analogically.

§10. ON THE 117-th SMARANDACHE'S PROBLEM [11]

The 117-th Smarandache's problem (see [2]) is:

Let p be an odd positive number. Then p and $p+2$ are twin primes if and only if

$$(p-1)!(\frac{1}{p}+\frac{2}{p+2})+\frac{1}{p}+\frac{1}{p+2}$$

is an integer.

Below we shall present a solution of this problem.

Let

$$A \equiv (p-1)!(\frac{1}{p}+\frac{2}{p+2})+\frac{1}{p}+\frac{1}{p+2} = \frac{(p-1)!(3p+2)+2p+2}{p(p+2)}$$

$$= \frac{B}{p(p+2)},$$

where

$$B \equiv (p-1)!(3p+2)+2p+2.$$

Hence,

$$B = 3p! + 2p + 2((p-1)!+1).$$

Therefore,

$$p|B \text{ \underline{iff} } p|((p-1)!+1) \text{ \underline{iff} } p \text{ is a prime number}$$

(from Wilson's theorem - see, e.g. [4]).

[11] see also
K. Atanassov, On the 117-th Smarandache's problem. *Notes on Number Theory and Discrete Mathematics*, Vol. 5 (1999), No. 3, 97-98.

On the other hand,

$$B = (p+2)(p-1)! + 2(p+2) + 2p! - 2$$

$$= (p+2)(p-1)! + 2(p+2) + \frac{2}{p+1}((p+1)! - (p+1))$$

$$= (p+2)(p-1)! + 2(p+2) + \frac{2}{p+1}(((p+1)! + 1) - (p+2)).$$

Therefore (from $(p+1, p+2) = 1$ for $p \geq 2$),

$(p+2)|B$ <u>iff</u> $(p+2)|((p+1)!+1)$ <u>iff</u> $p+2$ is a prime number

(from Wilson's theorem).

Hence,

$p(p+2)|B$ <u>iff</u> p and $p+2$ are twin primes.

Therefore, A is an integer if and only if p and $p+2$ are twin primes. Thus, we solved the problem.

Finally, we shall note that in [6] the following assertion is proved:

p and $p+2$ are twin primes <u>*iff*</u> $p(p+2)|C$,

where

$$C = 4(p-1)! + p + 4.$$

It is easily to see that

$$B = C + 3p(2(p-1)! + 1). \qquad (*)$$

From $(p+2)|(2(p-1)!+1)$ if and only if $(p+2)$ is a prime number, from (*) and from the above assertion from [6] we obtain another proof of the Smarandache's problem. Also, both our first proof and (*) yield another proof of the assertion from [6].

§11. ON THE 118-th SMARANDACHE'S PROBLEM [12]

The 118-th Smarandache's problem (see [2]) is:

"Smarandache criterion for coprimes":

If a, b are strictly positive integers, then: a and b are coprimes if and only if

$$a^{\varphi(b)+1} + b^{\varphi(a)+1} \equiv a + b (\mathrm{mod}\ ab),$$

where φ is Euler's totient.

For the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $p_1, p_2, ..., p_k$ are different prime numbers and $\alpha_1, \alpha_2, ..., \alpha_k \geq 1$ are natural numbers, the Euler's totient is defined by:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1).$$

Below we shall introduce a solution of one direction of this problem and we shall introduce a counterexample to the other direction of the problem.

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[12] see also

K. Atanassov, On the 118-th Smarandache's problem. *Notes on Number Theory and Discrete Mathematics*, Vol. 5 (1999), No. 3, 99.

Let a, b are strictly positive integers for which $(a,b) = 1$. Hence, from one of the Euler's theorems:

If m and n are natural numbers and $(m,n) = 1$, then

$$m^{\varphi(n)} \equiv 1 (\text{mod } n)$$

(see, e.g., [4]) it follows that

$$a^{\varphi(b)} \equiv 1 (\text{mod } b)$$

and

$$b^{\varphi(b)} \equiv 1 (\text{mod } a).$$

Therefore,

$$a^{\varphi(b)+1} \equiv a (\text{mod } ab)$$

and

$$b^{\varphi(b)+1} \equiv b (\text{mod } ab)$$

from where it follows that really

$$a^{\varphi(b)+1} + b^{\varphi(a)+1} \equiv a + b (\text{mod } ab).$$

It can be seen easily that the other direction of the Smarandache's problem is not valid. For example, if $a = 6$ and $b = 10$, and, therefore, $(a,b) = 2$, then:

$$6^{\varphi(10)+1} + 10^{\varphi(6)+1} = 6^5 + 10^3 = 7776 + 1000 = 8776 \equiv 16 (\text{mod } 60).$$

Therefore, the "Smarandache's criterion for coprimes" is valid only in the form:

If a, b are strictly positive coprime integers, then

$$a^{\varphi(b)+1} + b^{\varphi(a)+1} \equiv a + b (\text{mod } ab).$$

§12. ON THE 125-th SMARANDACHE'S PROBLEM [13]

The 125-th Smarandache's problem (see [2]) is:

To prove that

$$n! > k^{n-k+1} \prod_{i=0}^{k-1} [\frac{n-i}{k}]! \qquad (*)$$

for any non-null positive integers n and k.

Below we shall introduce a solution to the problem.

First, let us define for every negative integer $m : m! = 0$.

Let everywhere k be a fixed natural number. Obviously, if for some n: $k > n$, then the inequality (*) is obvious, because its right side is equal to 0. Also, it can be easily seen that (*) is valid for $n = 1$. Let us assume that (*) is valid for some natural number n. Then,

$$(n+1)! - k^{n-k+2} \prod_{i=0}^{k-1} [\frac{n-i+1}{k}]!$$

(by the induction assumption)

$$> (n+1).k^{n-k+1} \prod_{i=0}^{k-1} [\frac{n-i}{k}]! - k^{n-k+2} \prod_{i=0}^{k-1} [\frac{n-i+1}{k}]!$$

---

[13] see also
K. Atanassov, On the 125-th Smarandache's problem. *Notes on Number Theory and Discrete Mathematics,* Vol. 5 (1999), No. 2, 125.

$$= k^{n-k+1} \prod_{i=0}^{k-2} [\frac{n-i}{k}]!.((n+1).[\frac{n-k+1}{k}]! - k.[\frac{n+1}{k}]!) \geq 0,$$

because

$$(n+1).[\frac{n-k+1}{k}]! - k.[\frac{n+1}{k}]!$$

$$= (n+1).[\frac{n-k+1}{k}]! - k.[\frac{n-k+1}{k} + 1]!$$

$$= [\frac{n-k+1}{k}]!.(n+1-k.[\frac{n+1}{k}]) \geq 0.$$

With this the problem is solved.

Finally, we shall formulate two new problems:

1. Let $y > 0$ be a real number and let k be a natural number. Will the inequality

$$y! > k^{y-k+1} \prod_{i=0}^{k-1} [\frac{y-i}{k}]!$$

be valid again?

2. For the same y and k will the inequality

$$y! > k^{y-k+1} \prod_{i=0}^{k-1} \frac{y-i}{k}!$$

be valid?

§13. ON THE 126-th SMARANDACHE'S PROBLEM [14]

The following Smarandache's problem is formulated in [2] with the title "Smarandache divisibility theorem":

If a and m are integers, and $m > 0$, then:

$$(a^m - a)(m-1)!$$

is divisible by m.

The proof of this assertion follows directly from the Fermat's Little Theorem (see, e.g. [4]).

Really, let a and m are integers and let $m > 0$.

There are two cases for m:

(a) m is a prime number. Then from the Fermat's Little Theorem follows that $a^m - a$ is divisible by m and, therefore,

$$A \equiv (a^m - a)(m-1)!$$

is divisible by m.

(b) m is not a prime number. Then $m = p.r$ for the natural numbers r and the prime number p. If $r \neq p$ (r can be as a prime number, as well as a composite number), then $2 \leq p, r \leq m-1$ and $p, r \in \{1, 2, ..., m\}$. Therefore, p and r are different divisors of $(m-1)!$ and, hence, $(m-1)!$ is divisible by m. Hence A is divisible by m, too.

---

[14] see also

K. Atanassov, On the 126-th Smarandache's problem. *Notes on Number Theory and Discrete Mathematics,* Vol. 5 (1999), No. 1, 39-40.

The last case is $r = p$, i.e., r is a prime number and $m = p^2$. Therefore, $(m-1)! = p.B$ for some natural number B and we must prove that

$$a^{p^2} - a = p.C$$

for some natural number C.

Indeed, if $p|a$, then $p|(a^{p^2} - a)$, i.e.

$$a^{p^2} - a = p.C,$$

for some natural number C. On the other hand, if it is not valid that $p|a$, $p|(a^{p^2} - a)$, too, because of the representation

$$a^{p^2} - a = a.(D^{p-1} - 1),$$

where

$$D = p^{p+1}$$

and the fact that $p|(D^{p-1}-1)$ according to Fermat's Little Theorem, i.e., again

$$a^{p^2} - a = p.C$$

for some natural number C.

Therefore,

$$A = p^2.B.C,$$

i.e., A is divisible by m.

Therefore, the "Smarandache's divisibility theorem" is valid.

There are other ways for proving the last part of the proof. For example, Dr. Mladen Vassilev - Missana gave the following.

Let $m = p^2$. We remind the Legendre's formula

$$ord_p x! = [\frac{x}{p}] + [\frac{x}{p^2}] + [\frac{x}{p^3}] + \ldots$$

For $x = m - 1$ we obtain

$$ord_p(m-1)! = ord_p(p^2-1)! = [p - \frac{1}{p}] + [1 - \frac{1}{p^2}] = p - 1.$$

Therefore, $p^2|(m-1)!$ iff $p-1 \geq 2$, i.e., iff $p \geq 3$.

There remains only the case $m = 2^2 = 4$. In this case $2 = p|(m-1)!$ and obviously, we have

$$2 = p|(a^m - a) = a^4 - a = a(a^2-1)(a^2+1) = a(a-1)(a+1)(a^2+1),$$

so again it is fulfilled

$$p^2 = m|(a^m - 1)(m-1)!.$$

Therefore, the problem is solved.

§14. ON THE 62-nd SMARANDACHE'S PROBLEM [15]

The 62-th problem from [1] is the following:
Let $1 \leq a_1 < a_2 < \ldots$ be an infinite sequence of integers such that any three members do not constitute an arithmetic progression. Is it true that always

$$\sum_{n\geq 1} \frac{1}{a_n} \leq 2?$$

Here we shall give a counterexample.

Easily it can be seen that the set of numbers $\{1,2,4,5,10\}$ does not contain three numbers being members of an arithmetic progression. On the other hand

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{10}=2\frac{1}{20}>2.$$

Therefore, Smarandache's problem is not true in the present form, because the sum of the members of every one sequence with the above property and with first members 1, 2, 4, 5, 10 will be bigger than 2.

The sequence 1, 3, 4, 6, 10, 11, 13 is another (more complex) counterexample, because

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{10}+\frac{1}{11}+\frac{1}{13}=2\frac{158}{8580}.$$

---

[15] see also
K. Atanassov, On the 62-nd Smarandache's problem. *Notes on Number Theory and Discrete Mathematics,* Vol. 5 (1999), No. 3, 100-101.

The third counterexample is the sequence 1, 4, 5, 8, 10, 13, 14, 17, 28, 31, 32, 35, because

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{5}+\frac{1}{8}+\frac{1}{10}+\frac{1}{13}+\frac{1}{14}+\frac{1}{17}+\frac{1}{28}+\frac{1}{31}+\frac{1}{32}+\frac{1}{35}=2.009968957\ldots$$

Essentially interesting is the problem in the following form:

Let $2 \leq a_1 < a_2 < \ldots$ be an infinite sequence of integers such that any three members do not constitute an arithmetic progression. Is it true that

$$\sum_{n\geq 1} \frac{1}{a_n} \leq 2?$$

Unfortunately, neither this is so. The set

$$\{2,3,5,6,11,12,14,15,29,30,32,33,38,39,41,42,83,84,86,87,92,$$
$$93,95,96,110,111,113,114,119,120,122,123,245,246,248,249,$$
$$254,255,257,258,272,273,275,276,281,282,284,285,326,327,$$
$$329,330,335,336,338,339,353,354,356,357,362,363\}$$

is a counterexample, because

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{11}+\frac{1}{12}+\frac{1}{14}+\frac{1}{15}+\frac{1}{29}+\frac{1}{30}+\frac{1}{32}+\frac{1}{33}+\frac{1}{38}+\frac{1}{39}+\frac{1}{41}+\frac{1}{42}$$

$$+\frac{1}{83}+\frac{1}{84}+\frac{1}{86}+\frac{1}{87}+\frac{1}{92}+\frac{1}{93}+\frac{1}{95}+\frac{1}{96}+\frac{1}{110}+\frac{1}{111}+\frac{1}{113}+\frac{1}{114}+\frac{1}{119}$$

$$+\frac{1}{120}+\frac{1}{122}+\frac{1}{123}+\frac{1}{245}+\frac{1}{246}+\frac{1}{248}+\frac{1}{249}+\frac{1}{254}+\frac{1}{255}+\frac{1}{257}+\frac{1}{258}$$

$$+\frac{1}{272}+\frac{1}{273}+\frac{1}{275}+\frac{1}{276}+\frac{1}{281}+\frac{1}{282}+\frac{1}{284}+\frac{1}{285}+\frac{1}{326}+\frac{1}{327}+\frac{1}{329}$$

$$+\frac{1}{330}+\frac{1}{335}+\frac{1}{336}+\frac{1}{338}+\frac{1}{339}+\frac{1}{353}+\frac{1}{354}+\frac{1}{356}+\frac{1}{357}+\frac{1}{362}+\frac{1}{363}$$

$$= 2.00169313...$$

Some modifications of this problem will be discussed in a further research held by the author.

§15. CONCLUSION

As it is written in the Preface, the Problems from [1,2,8,14] generate the interest of the author to them. He hopes that in future he will solve some other problems.

On the other hand, the solutions of some of the described in the present book problems generate ideas for further modifications and extensions.

For example, the last problem (the 62-nd from [1]) can obtain the following form:

Let $3 \leq a_1 < a_2 < \ldots$ be an infinite sequence of integers such that any three members do not constitute an arithmetic progression. Is it true that

$$\sum_{n \geq 1} \frac{1}{a_n} \leq 2?$$

The computer check shows that it is valid. Now, we can would like to construct the sequence which has the minimal possible members; and, therefore, the maximal possible sum. As it is easily seen from the first three counterexamples, the minimal sequence, starting with 1 is 1, 2, 4, 5, 10, 11, 13, 14, ... For it we can prove easily that it has the basis $[1, 2, 4, 5]$ with a length 4 with respect to function ψ from [3] (see §16).

The fourth counterexample contains the elements of the minimal sequence, starting with 2, and it is 2, 3, 5, 6, 11, 12, 14, 15, 29, ... For it we can prove that it has the basis $[2, 3, 5, 6]$ with a length 4 with respect to function ψ from §16.

Now, we can define the number

$$S_{min}(k) = \sum_{n\geq 1} \frac{1}{a_n},$$

where the sequence $\{a_n\}_{n=1}^{\infty}$ is the minimal possible and for it

$$k \leq a_1 < a_2 < \dots$$

A further author's research will be devoted to this sequence.

The described solutions of the problems 61-st, 62-nd and 63-rd from [2] show some new possibilities for research related to function "factorial" and to the Smarandache's function S, while the new function, which is dual to the function "factorial" and which is used in the solutions of the problems 43-rd and 44-th from [2], must be studied in details. Author hopes that in the near future he will receive new results related to these problems.

Up to now the author does not know explicit formulas for the partial sums of the sequences from 4-th, 22-nd, 23-rd and 24-th problems and determining of such formulas will be an aim for him.

The author thinks that the formulation of 125-th and 126-th problems from [2] can be generalized in future, too.

Some of the problems from [8] also will be discussed in a further research of him.

§16. APPENDIX

Here we shall describe two arithmetic functions which were used in some of the previous sections, following [3] (see also [15-20].

For

$$n = \sum_{i=1}^{m} a_i.10^{m-i} \equiv \overline{a_1 a_2 ... a_m},$$

where a_i is a natural number and $0 \le a_i \le 9$ $(1 \le i \le m)$ let (see [3]):

$$\varphi(n) = \begin{cases} 0 & , \text{if } n = 0 \\ \sum_{i=1}^{m} a_i & , \text{otherwise} \end{cases}$$

and for the sequence of functions $\varphi_0, \varphi_1, \varphi_2, ...$, where ($l$ is a natural number)

$$\varphi_0(n) = n,$$

$$\varphi_{l+1} = \varphi(\varphi_l(n)),$$

let the function ψ be defined by

$$\psi(n) = \varphi_l(n),$$

in which

$$\varphi_{l+1}(n) = \varphi_l(n).$$

This function has the following (and other) properties (see [3]):

$$\psi(m + n) = \psi(\psi(m) + \psi(n)),$$

$$\psi(m.n) = \psi(\psi(m).\psi(n)) = \psi(m.\psi(n)) = \psi(\psi(m).n),$$
$$\psi(m^n) = \psi(\psi(m)^n),$$
$$\psi(n+9) = \psi(n),$$
$$\psi(9n) = 9.$$

Let the sequence $a_1, a_2, ...$ with members - natural numbers, be given and let

$$c_i = \psi(a_i) \ (i = 1, 2, ...).$$

Hence, we deduce the sequence $c_1, c_2, ...$ from the former sequence. If k and l exist, such that $l \geq 0$,

$$c_{i+l} = c_{k+i+l} = c_{2k+i+l} = ...$$

for $1 \leq i \leq k$, then we shall say that

$$[c_{l+1}, c_{l+2}, ..., c_{l+k}]$$

is a base of the sequence $c_1, c_2, ...$ with a length k and with respect to function ψ.

For example, the Fibonacci sequence $\{F_i\}_{i=0}^{\infty}$, for which

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \ (n \geq 0)$$

has a base with a length of 24 with respect to the function ψ and it is the following:

$$[1,1,2,3,5,8,4,3,7,1,8,9,8,8,7,6,4,1,5,6,2,8,1,9];$$

the Lucas sequence $\{L_i\}_{i=0}^{\infty}$, for which

$$L_2 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n \ (n \geq 0)$$

also has a base with a length of 24 with respect to the function ψ and it is the following:

$$[2,1,3,4,7,2,9,2,2,4,6,1,7,8,6,5,2,7,9,7,7,5,3,8];$$

even the Lucas-Lehmer sequence $\{l_i\}_{i=0}^{\infty}$, for which

$$l_1 = 4, l_{n+1} = l_n^2 - 2 \ (n \geq 0)$$

has a base with a length of 1 with respect to the function ψ and it is [5].

The $k-th$ triangular number t_k is defined by the formula

$$t_k = \frac{k(k+1)}{2}$$

and it has a base with a length of 9 with the form

$$[1,3,6,1,5,3,1,9,9].$$

It is directly checked that the bases of the sequences $\{n^k\}_{k=1}^{\infty}$ for $n = 1, 2, ..., 9$ are those introduced in the following table.

n	a base of a sequence $\{n^k\}_{i=1}^{\infty}$	a length of the base
1	1	1
2	2,4,8,7,5,1	6
3	9	1
4	4,7,1	3
5	5,7,8,4,2,1	6
6	9	1
7	7,4,1	3
8	8,1	2
9	9	1

On the other hand, the sequence $\{n^n\}_{n=1}^{\infty}$ has a base (with a length of 9) with the form

$$[1,4,9,1,2,9,7,1,9],$$

and the sequence $\{k^{n!}\}_{n=1}^{\infty}$ has a base with a length of 9 with the form

$$\begin{cases} [1] & \text{, if } k \neq 3m \text{ some some natural number } m \\ [9] & \text{, if } k = 3m \text{ some some natural number } m \end{cases}$$

We must note that in [3] there are some misprints, corrected here.

An obvious, but unpublished up to now result is that the sequence $\{\psi(n!)\}_{n=1}^{\infty}$ has a base with a length of 1 with respect to the function ψ and it is [9]. The first members of this sequence are

$$1, 2, 6, 6, 3, 9, 9, 9, ...$$

We shall finish with two new results related to the concept "factorial" which occur in some places in this book.

The concepts of $n!!$ is already introduced and there are some problems in [1,2] related to it. Let us define the new factorial $n!!!$ only for numbers with the forms $3k+1$ and $3k+2$:

$$n!!! = 1.2.4.5.7.8.10.11...n$$

We shall prove that the sequence $\{\psi(n!!!)\}_{n=1}^{\infty}$ has a base with a length of 12 with respect to the function ψ and it is

$$[1, 2, 8, 4, 1, 8, 8, 7, 1, 5, 8, 1].$$

Really, the validity of the assertion for the first 12 natural numbers with the above mentioned forms, i.e., the numbers

$$1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17,$$

is directly checked. Let us assume that the assertion is valid for the numbers

$$(18k+1)!!!, (18k+2)!!!, (18k+4)!!!, (18k+5)!!!, (18k+7)!!!, (18k+8)!!!,$$

$$(18k+10)!!!, (18k+11)!!!, (18k+13)!!!, (18k+14)!!!, (18k+16)!!!,$$

$$(18k+17)!!!.$$

Then

$$\psi((18k+19)!!!) = \psi((18k+17)!!!.(18k+19))$$

$$= \psi(\psi(18k+17)!!!.\psi(18k+19))$$

$$= \psi(1.1) = 1;$$

$$\psi((18k+20)!!!) = \psi((18k+19)!!!.(18k+20))$$

$$= \psi(\psi(18k+19)!!!.\psi(18k+20))$$

$$= \psi(1.2) = 2;$$

$$\psi((18k+22)!!!) = \psi((18k+20)!!!.(18k+22))$$

$$= \psi(\psi(18k+20)!!!.\psi(18k+22))$$

$$= \psi(2.4) = 8,$$

etc., with which the assertion is proved.

Having in mind that every natural number has exactly one of the forms $3k+3$, $3k+1$ and $3k+2$, for the natural number $n = 3k+m$, where $m \in \{1,2,3\}$ and $k \geq 1$ is a natural number, we can define:

$$n!_m = \begin{cases} 1.4...(3k+1), & \text{if } n = 3k+1 \text{ and } m = 1 \\ 2.5...(3k+2), & \text{if } n = 3k+2 \text{ and } m = 2 \\ 3.6...(3k+3), & \text{if } n = 3k+3 \text{ and } m = 3 \end{cases}$$

As above, we can prove that:

• for the natural number n with the form $3k+1$, the sequence $\{\psi(n!_1)\}_{n=1}^{\infty}$ has a base with a length of 3 with respect to the function ψ and it is

$$[1, \psi(3k+1), 1];$$

• for the natural number n with the form $3k+2$, the sequence $\{\psi(n!_1)\}_{n=1}^{\infty}$ has a base with a length of 6 with respect to the function ψ and it is

$$[2, \psi(6k+4), 8, 7, \psi(3k+5), 1];$$

• for the natural number n with the form $3k+3$, the sequence $\{\psi(n!_1)\}_{n=1}^{\infty}$ has a base with a length of 1 with respect to the function ψ and it is [9] and only its first member is 3.

Now we can see that

$$n!!! = \begin{cases} (3k+1)!_1.(3k-1)!_2, & \text{if } n = 3k+1 \text{ and } k \geq 1 \\ \\ (3k+1)!_1.(3k+2)!_2, & \text{if } n = 3k+2 \text{ and } k \geq 1 \end{cases}$$

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