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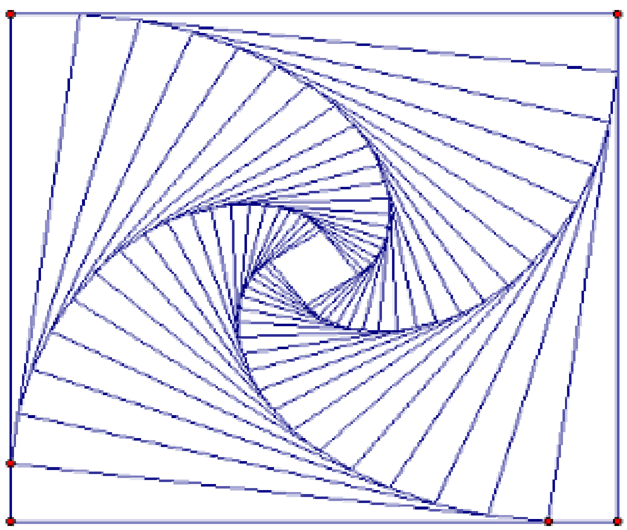
LI JUNZHUANG, LIU DUANSEN
(EDITORS)

RESEARCH ON SMARANDACHE PROBLEMS IN NUMBER THEORY (VOL. II)

PROCEEDINGS OF THE FIRST NORTHWEST CONFERENCE ON NUMBER THEORY



Hexis
Phoenix, AZ
2005

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Contents

This book is dedicated to Professor Florentin Smarandache, who listed many new and unsolved problems in number theory.

Preface

Arithmetic is where numbers run across your mind looking for the answer.
Arithmetic is like numbers spinning in your head faster and faster until you blow up with the answer.
KABOOM!!
Then you sit back down and begin the next problem.

Alexander Nathanson

Mathematics is often referred to as a servant of science and number theory as a queen of mathematics, by which we are to understand that number theory should contribute to the development of science by continuously supplying challenging problems and illustrative examples of its own discipline in anticipation of later practical use in the real world.

The research on Smarandache Problems plays a key role in the development of number theory. Therefore, many mathematicians show their interest in the Smarandache problems and they conduct much research on them. Under such circumstances, we published the book <RESEARCH ON SMARANDACHE PROBLEMS IN NUMBER THEORY>, Vol. I, in September, 2004. That book stimulated more Chinese mathematicians to pay attention to Smarandache conjectures, open and solved problems in number theory.

The First Northwest Number Theory Conference was held in Shangluo Teacher's College, China, in March 2005. One of the sessions was dedicated to the Smarandache problems. In that session, several professors gave a talk on Smarandache problems and many participants lectured on Smarandache problems both extensively and intensively.

This book includes 34 papers, most of which were written by participants of the above mentioned conference. All these papers are original and have been refereed. The themes of these papers range from the mean value or hybrid mean value of Smarandache type functions, the mean value of some famous number theoretic functions acting on the Smarandache sequences, to the convergence property of some infinite series involving the Smarandache type sequences.

We sincerely thank all the authors and the referees for their important contributions. Thanks are also due to Dr. Xu Zhefeng for his effort of making files of LaTeX style. The last, but not the least, thanks are due to the teachers and students of Shangluo Teacher's College in China for their great help in our successful conference.

August 10, 2005
Zhang Wenpeng, Li Junzhuang, Liu Duansen

ON THE SMARANDACHE M-TH POWER RESIDUES*

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Abstract For any positive integer n, let $\phi(n)$ be the Euler function, and $a_m(n)$ denotes the Smarandache m-th power residues function of n. The main purpose of this paper is using the elementary method to study the number of the solutions of the equation $\phi(n) = a_m(n)$, and give all solutions for this equation.

Keywords: Arithmetical function; Equation; Solutions.

§1. Introduction

For any positive integer n, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers. The Smarandache m-th power residues function $a_m(n)$ are defined as

$$a_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}, \quad \beta_i = \min\{m-1, \alpha_i\}, i = 1, 2. \cdots, k.$$

In problem 65 of [1], Professor F.Smarandache asked us to study the properties of this function. Let $\phi(n)$ denotes the Euler function. That is, $\phi(n)$ denotes the number of all positive integers not exceeding n which are relatively prime to n. It is clear that $\phi(n)$ and $a_m(n)$ both are multiplicative functions. In this paper, we shall use the elementary method to study the solutions of the equation involving these two functions, and give all solutions for it. That is, we shall prove the following:

Theorem. *Let m be a fixed integer with $m \geq 2$. Then the equation $\phi(n) = a_m(n)$ have $m+1$ solutions, namely*

$$n = 1, 2^m, 2^\alpha 3^m, \quad \alpha = 1, 2, \cdots, m-1.$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers, then from the definitions of $\phi(n)$ and $a_m(n)$ we have

$$a_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}, \quad \beta_i = \min\{m-1, \alpha_i\}, i = 1, 2. \cdots, k \quad (1)$$

*This work is supported by N.S.F. of P.R.China (10271093, 60472068)

and

$$\phi(n) = p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)\cdots p_k^{\alpha_k-1}(p_k-1). \tag{2}$$

It is clear that $n = 1$ is a solution of the equation $\phi(n) = a_m(n)$. If $n > 1$, then we will discuss the problem in four cases:

(i) If $\exists\, \alpha_i > m$, then we have $\alpha_i - 1 \geq m$, $\max\limits_{1\leq j\leq k}\{\beta_j\} \leq m-1$. Combining (1) and (2), we know that $\phi(n) \neq a_m(n)$. That is, there is no any solution satisfied $\phi(n) = a_m(n)$ in this case;

(ii) If $\alpha_1 = \alpha_2 = \cdots = \alpha_k = m$, then we have

$$a_m(n) = p_1^{m-1}p_2^{m-1}\cdots p_k^{m-1}$$

and

$$\phi(n) = p_1^{m-1}(p_1-1)p_2^{m-1}(p_2-1)\cdots p_k^{m-1}(p_k-1).$$

It is clear that only $n = 2^m$ is a solution of the equation in this case;

(iii) If $\max\limits_{1\leq i\leq k}\{\alpha_i\} < m$, then from (1) and (2), and noting that $\beta_k = \alpha_k > \alpha_k - 1$, we know that $\phi(n) \neq a_m(n)$ for all n in this case;

(iv) If $\exists\, i, j$ such that $\alpha_i = m$ and $\alpha_j < m$, then from (1), (2) and equation $\phi(n) = a_m(n)$, we get $p_1 = 2$. If not, then $\phi(n)$ is an even number, but $a_m(n)$ is an odd number. Noting that if $\alpha_k < m$, then $\phi(n) \neq a_m(n)$, so we have $\alpha_k = m$ and

$$\begin{aligned}\phi(n) &= 2^{\alpha_1-1}p_2^{\alpha_2-1}(p_2-1)\cdots p_k^{m-1}(p_k-1)\\ &= a_m(2^{\alpha_1})a_m(p_2^{\alpha_2})\cdots a_m(p_k^m) = a_m(n).\end{aligned}$$

If $\alpha_1 = m$, then $n = 2^m$ is the case of (ii). Hence, $\alpha_1 < m$. Now from the equation, we have

$$p_2^{\alpha_2-1}(p_2-1)\cdots p_k^{m-1}(p_k-1) = 2a_m(p_2^{\alpha_2})\cdots a_m(p_k^m). \tag{3}$$

Noting that $2|(p_i-1)$ if $i > 1$, from (3) we can deduce that only one term with the form $p_i - 1$ in the left side of (3). That is,

$$n = 2^\alpha p^m.$$

From

$$\phi(2^\alpha p^m) = a_m(2^\alpha p^m) \Longleftrightarrow p^{m-1}(p-1) = 2p^{m-1},$$

we get $n = 2^\alpha 3^m, 1 \leq \alpha < m$. That is, $n = 2^\alpha 3^m, \quad (1 \leq \alpha < m)$ are all the solutions of $\phi(n) = a_m(n)$ in this case.

Now combining the above four cases we may immediately get all $m+1$ solutions of equation $\phi(n) = a_m(n)$, namely

$$n = 1, 2^m, 2^\alpha 3^m, \quad \alpha = 1, 2.\cdots, m-1.$$

This completes the proof of Theorem.

References

[1] F. Smarandache. Only Problems, Not Solutions. Chicago: Xiquan Publishing House, 1993, pp54.

[2] Tom M. Apostol. Introduction to Analytic Number Theory. New York: Springer-Verlag, 1976.

AN EQUATION INVOLVING EULER'S FUNCTION*

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Abstract In this paper, we use the elementary methods to study the number of the solutions of an equation involving the Euler's function $\phi(n)$, and give an interesting identity.

Keywords: Multiplicative function; Arithmetical property; Identity.

§1. Introduction

For any positive integer $n \geq 1$, the Euler function $\phi(n)$ is defined to be the number of all positive integers not exceeding n which are relatively prime to n. We also define $J(n)$ as the number of all primitive Dirichlet's characters mod n. Let A denotes the set of all positive integers n satisfying the equation $\phi^2(n) = nJ(n)$. In this paper, we using the elementary methods to study the convergent properties of a new Dirichlet's series involing the solutions of the equation $\phi^2(n) = nJ(n)$, and give an interesting identity for it. That is, we shall prove the following conclusion:

Theorem. *For any real number $s > \frac{1}{2}$, we have the identity*

$$\sum_{\substack{n=1 \\ n \in A}}^{\infty} \frac{1}{n^s} = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)},$$

where $\zeta(s)$ is the Riemann zeta-function.

Taking $s = 1$ and 2, and note that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(12) = 691\pi^{12}/638512875$, we may immidiately deduce the following identities:

$$\sum_{\substack{n=1 \\ n \in A}}^{\infty} \frac{1}{n} = \frac{315}{2\pi^4}\zeta(3) \quad \text{and} \quad \sum_{\substack{n=1 \\ n \in A}}^{\infty} \frac{1}{n^2} = \frac{15015}{1382}\frac{1}{\pi^2}.$$

*This work is supported by N.S.F. of P.R.China (10271093)

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. First note that both $\phi(n)$ and $J(n)$ are multiplicative functions, and if $n = p^\alpha$, then $J(p) = p-2$, $\phi(p) = p-1$, $J(p^\alpha) = p^{\alpha-2}(p-1)^2$ and $\phi(p^\alpha) = p^{\alpha-1}(p-1)$ for all positive integer $\alpha > 1$ and prime p. So we will discuss the solutions of the equation $\phi^2(n) = nJ(n)$ into three cases.

(a) It is clear that $n = 1$ is a solution of the equation $\phi^2(n) = nJ(n)$.

(b) If $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the prime powers decomposition of n with all $\alpha_i \geq 2$ ($i = 1, 2, \cdots\cdots, k$), then we have

$$J(n) = p_1^{\alpha_1 - 2}(p_1 - 1)^2 \cdots p_k^{\alpha_k - 2}(p_k - 1)^2$$

and

$$\phi^2(n) = \left(p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)\cdots p_k^{\alpha_k-1}(p_k-1)\right)^2.$$

For this case, n is also a solution of the equation $\phi^2(n) = nJ(n)$.

(c) If $n = p_1 p_2 \cdots p_r p_{r+1}^{\alpha_{r+1}} \cdots p_k^{\alpha_k}$ with $p_1 < p_2 < \cdots < p_r$ and $\alpha_j > 1$, $j = r+1,\ r+2,\ \cdots,\ k$, then from (b) and the definition of $\phi(n)$ and $J(n)$ we have $\phi^2(n) = nJ(n)$ if and only if

$$\phi^2(p_1 p_2 \cdots p_r) = p_1 p_2 \cdots p_r J(p_1 p_2 \cdots p_r)$$

or

$$(p_1-1)^2(p_2-1)^2\cdots(p_r-1)^2 = p_1 p_2 \cdots p_r (p_1-2)(p_2-2)\cdots(p_r-2).$$

It is clear that p_r can not divide $(p_1-1)^2(p_2-1)^2\cdots(p_r-1)^2$. So the equation $\phi^2(n) = nJ(n)$ has no solution in this case.

Combining the above three cases we may immediately obtain the set of all solutions of the equation $\phi^2(n) = nJ(n)$ is $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ with all $\alpha_i > 1$ and 1. That is, A is the set of all square-full numbers and 1.

Now we define the arithmetical function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & if \quad n \in A, \\ 0, & if \quad otherwise. \end{cases}$$

For any real number $s > 0$, it is clear that

$$\sum_{\substack{n=1 \\ n \in A}}^{\infty} \frac{1}{n^s} < \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent if $s > 1$. So for $s \geq 2$, from the Euler product formula (see [2]),we have

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^s} &= \prod_p\left(1+\frac{a(p^2)}{p^{2s}}+\frac{a(p^3)}{p^{3s}}+\cdots\right)\\
&= \prod_p\left(1+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots\right)\\
&= \prod_p\left(1+\frac{1}{p^{2s}(1-\frac{1}{p^s})}\right)\\
&= \prod_p\frac{p^{2s}-p^s+1}{p^{2s}-p^s}\\
&= \prod_p\frac{p^{3s}+1}{p^s(p^{2s}-1)}\\
&= \prod_p\frac{p^{6s}-1}{p^s(p^{2s}-1)(p^{3s}-1)}\\
&= \prod_p\frac{1-\frac{1}{p^{6s}}}{(1-\frac{1}{p^{2s}})(1-\frac{1}{p^{3s}})}\\
&= \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)},
\end{aligned}
$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ denotes the product over all primes.

This completes the proof of the theorem.

References

[1] F.Smaradache. Only problems, not solutions, Xiquan Publishing House, Chicago, 1993.

[2] Tom M. Apostol. Introduction to Analytic Number Theory. New York: Springer-Verlag, 1976.

ON THE FUNCTION $\varphi(N)$ AND $\delta_K(N)$ *

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Abstract The main purpose of this paper is using the elementary method to study the divisibility of $\delta_k(n)$ by $\varphi(n)$, and find all the n such that $\varphi(n) \mid \delta_k(n)$.

Keywords: Euler function; Arithmetical function; Divisibility.

§1. Introduction and main results

For a fixed positive integer k and any positive integer n, we define a new arithmetic function as following:

$$\delta_k(n) = \max\{d \mid d \mid n, (d,k) = 1\}$$

If $n \geq 1$ the Euler function $\varphi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n; thus,

$$\varphi(n) = \sum_{k=1}^{n}{}' 1$$

where $\sum_{k=1}^{n}{}'$ indicates that the sum is extended over those k relatively prime to n.

In this paper, we shall study the divisibility of $\delta_k(n)$ by $\varphi(n)$, and find all the n such that $\varphi(n) \mid \delta_k(n)$. In fact, we shall prove the following result:

Theorem. *$\varphi(n) \mid \delta_k(n)$ if and only if $n = 2^\alpha 3^\beta$, where $\alpha > 0, \beta \geq 0, \alpha, \beta \in N$.*

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. First we need the following Lemma.

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Lemma *For $n \geq 1$, we have*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proof. See Theorem 2.4 in reference [2].

Now we use the above lemma to complete the proof of Theorem. We will discuss it in two cases.

I. $(n, k) = 1$, then $\delta_k(n) = n$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ is the factorization of n into prime powers. From the Lemma, we can write

$$\varphi(n) = p_1^{\alpha_1 - 1}(p_1 - 1) p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_s^{\alpha_s - 1}(p_s - 1),$$

If $\varphi(n) \mid \delta_k(n)$, then

$$p_1^{\alpha_1 - 1}(p_1 - 1) p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_s^{\alpha_s - 1}(p_s - 1) \mid n,$$

that is

$$(p_1 - 1)(p_2 - 1) \cdots (p_s - 1) \mid p_1 p_2 \cdots p_s,$$

from the above formula, we are sure of

i) $p_1 = 2$, if not, $(p_1 - 1)(p_2 - 1) \cdots (p_s - 1)$ is even but $p_1 p_2 \cdots p_s$ odd and

ii) $s \leq 2$, since $2|(p_i - 1) \quad (i = 2, 3, \cdots, s)$ but $p_2 p_3 \cdots p_s$ can not be divided by 2.

If $s = 1$, then $n = 2^\alpha, \alpha \geq 1$.

If $s = 2$, then $n = 2^\alpha 3^\beta, \alpha \geq 1, \beta > 1$.

Thus, we can obtain that $n = 2^\alpha 3^\beta, \alpha \geq 1, \beta \geq 0$ such that $\varphi(n) \mid \delta_k(n)$ when $(n, k) = 1$.

II. $(n, k) \neq 1$, we can write $n = n_1 \cdot n_2$, where $(n_1, k) = 1$ and $(n_1, n_2) = 1$, then

$$\delta_k(n) = n_1,$$

and

$$\varphi(n) = \varphi(n_1)\varphi(n_2),$$

If $\varphi(n) \mid \delta_k(n)$, that means

$$\varphi(n_1)\varphi(n_2) \mid n_1,$$

that is

$$\varphi(n_1) \mid n_1 \tag{1}$$

$$\varphi(n_2) \mid n_1 \tag{2}$$

from (1), we can get

$$n_1 = 2^{\alpha_1} 3^{\beta_1}, \tag{3}$$

where $\alpha_1 \geq 1, \beta_1 \geq 0, \alpha_1, \beta_1 \in N$.
combining (3)and(4), we can easily get

$$\varphi(n_2) \mid 2^{\alpha_1} 3^{\beta_1},$$

this means

$$\varphi(n_2) = 1.$$

Otherwise $(n_1, n_2) \neq 1$.
So

$$n_2 = 1$$

Altogether, whether $(n, k) = 1$ or not, we can obtain $n = 2^\alpha 3^\beta, \alpha \geq 1, \beta \geq 0$ such that $\varphi(n) \mid \delta_k(n)$.

This completes the proof of the theorem.

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[3] F.Mertens. Ein Beitrag Zur Analytischen Zahlentheorie, Crelle's Journal, 78(1874), 46-62.

ON THE MEAN VALUE OF THE DIRICHLET'S DIVISOR FUNCTION IN SOME SPECIAL SETS*

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Abstract The main purpose of this paper is using the analytic methods to study the mean value properties of the Dirichlet's divisor function in some special sets, and give several interesting asymptotic formulae for them.

Keywords: Divisor function; Mean value; Asymptotic formula.

§1. Introduction

For any positive integer n and $k \geq 2$, the k-th power complement function $b_k(n)$ of n is the smallest positive integer such that $nb_k(n)$ is a perfect k-th power. In problem 29 of reference [1], Professor F.Smarandache asked us to study the properties of this function. About this problem, some people had studied it before, and obtained some interesting conclusions, see references [4] and [5]. In this paper, we define two new sets $B = \{n \in N, b_k(n) \mid n\}$ and $C = \{n \in N, n \mid b_k(n)\}$. Then we use the analytic methods to study the mean value properties of the Dirichlet's divisor function $d(n)$ acting on these two special sets, and obtain two interesting asymptotic formulae for them. That is, we shall prove the following :

Theorem 1 . For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n \leq x \\ n \in B}} d(n) = \frac{mx^{\frac{1}{m}}}{\zeta^{m+1}(2)} R\left(p^{\frac{1}{m}}\right) \cdot f(\log x) + +O\left(x^{\frac{1}{2m}+\varepsilon}\right),$$

where

$$R\left(p^{\frac{1}{m}}\right) = \prod_p \left(1 + \frac{p^m\left((p^{\frac{1}{m}}-1)(m+1)+p^{\frac{1}{m}}\right)}{(p+1)^{m+1}(p^{\frac{1}{m}}-1)^2} - \frac{\left(p^{\frac{1}{m}}-1\right)^2 \sum_{i=2}^{m+1}\binom{m+1}{i}p^{m+1-i}}{(p+1)^{m+1}(p^{\frac{1}{m}}-1)^2}\right),$$

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$f(y)$ is a polynomial of y with degree $m = [\frac{k+1}{2}]$, and ε is any fixed positive number.

Theorem 2 . For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in C}} d(n) = \frac{x\log x}{\zeta(l+1)} \prod_p \left(1 - \frac{(l+1)(p-1)}{p^{l+2}-p}\right) + Ax + +O(x^{\frac{1}{2}+\varepsilon}),$$

where $l = \left[\frac{k}{2}\right]$, A is a constant, and ε is any fixed positive number.

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. In fact, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers. Then it is clear that $b_k(n) = b_k(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}) = b_k(p_1^{\alpha_1})b_k(p_2^{\alpha_2})\cdots b_k(p_s^{\alpha_s})$. That is, $b_k(n)$ is a multiplicative function. So we first study the problem in the case $n = p^\alpha$.

(1) If $\alpha \geq k$, then from the definition of $b_k(n)$ we know that $b_k(n) \mid n$. Therefore $n \in B$.

(2) If $\alpha \leq k$, then $b_k(n) = p^{k-\alpha}$. So combining (1), (2) and the definition of B and C, we deduce that $n \in B$ if $\alpha \geq [\frac{k+1}{2}]$, and $n \in C$ if $\alpha \leq [\frac{k}{2}]$. Now we prove the theorem 1 and theorem 2 respectively. First let

$$f(s) = \sum_{n\in B} \frac{d(n)}{n^s}.$$

Then from the definition of $b_k(n)$ and B, the properties of the Dirichlet's divisor function and the Euler product formula [2], we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{d(p^m)}{p^{ms}} + \frac{d(p^{m+1})}{p^{(m+1)s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{m+1}{p^{ms}} + \frac{m+2}{p^{(m+1)s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{m+1}{p^{ms}} + \frac{m+1}{p^{ms}(p^s-1)} + \frac{p^s}{p^{ms}(p^s-1)^2}\right) \\ &= \frac{\zeta^{m+1}(ms)}{\zeta^{m+1}(2ms)} \prod_p \left(1 + \frac{p^{m^2s}((p^s-1)(m+1)+p^s)}{(p^{ms}+1)^{m+1}(p^s-1)^2}\right. \qquad (1) \\ &\qquad \left. - \frac{(p^s-1)^2 \sum_{i=2}^{m+1} \binom{m+1}{i} p^{m(m+1-i)s}}{(p^{ms}+1)^{m+1}(p^s-1)^2}\right), \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function and $m = [\frac{k+1}{2}]$.

Obviously, we have

$$|d(n)| \leq n, \qquad \left|\sum_{n=1}^{\infty} \frac{d(n)}{n^\sigma}\right| \leq \frac{1}{\sigma - 1 - \frac{1}{m}},$$

where $\sigma > 1 - \frac{1}{m}$ is the real part of s. Therefore by Perron formula [3], with $s_0 = 0$, $b = \frac{2}{m}$, $T = x^{\frac{3}{2m}}$, we have

$$\sum_{\substack{n \leq x \\ n \in B}} d(n) = \frac{1}{2\pi i} \int_{\frac{2}{m}-iT}^{\frac{2}{m}+iT} \frac{\zeta^{m+1}(ms)}{\zeta^{m+1}(2ms)} R(s) \frac{x^s}{s} ds + O(x^{\frac{1}{2m}+\varepsilon}),$$

where

$$R(s) = \prod_p \left(1 + \frac{p^{m^2 s}\left((p^s-1)(m+1)+p^s\right) - (p^s-1)^2 \sum\limits_{i=2}^{m+1} \binom{m+1}{i} p^{m(m+1-i)s}}{(p^{ms}+1)^{m+1}(p^s-1)^2} \right).$$

To estimate the main term

$$\frac{1}{2\pi i} \int_{\frac{2}{m}-iT}^{\frac{2}{m}+iT} \frac{\zeta^{m+1}(ms)}{\zeta^{m+1}(2ms)} R(s) \frac{x^s}{s} ds,$$

we move the integral line from $s = \frac{2}{m} \pm iT$ to $s = \frac{1}{2m} \pm iT$, then the function

$$\frac{\zeta^{m+1}(ms)}{\zeta^{m}(2ms)} R(s) \frac{x^s}{s}$$

have one $m+1$ order pole point at $s = \frac{1}{m}$ with residue

$$\begin{aligned}
& \lim_{s \to 1} \frac{1}{m!} \left((ms-1)^{m+1} \zeta^{m+1}(ms) \frac{R(s)x^s}{\zeta^{m+1}(2ms)s} \right)^{(m)} \\
= \; & \lim_{s \to 1} \frac{1}{m!} \tbinom{m}{0} \left((ms-1)^{m+1} \zeta^{m+1}(ms) \right)^{(m)} \frac{R(s)x^s}{\zeta^{m+1}(2ms)s} + \\
& + \lim_{s \to 1} \frac{1}{m!} \tbinom{m}{1} \left((ms-1)^{m+1} \zeta^{m+1}(ms) \right)^{(m-1)} \left(\frac{R(s)x^s}{\zeta^{m+1}(2ms)s} \right)' + \cdots \\
& + \lim_{s \to 1} \frac{1}{m!} \tbinom{m}{m} (ms-1)^{m+1} \zeta^{m+1}(ms) \left(\frac{R(s)x^s}{\zeta^{m+1}(2ms)s} \right)^{(m)} \\
= \; & \frac{m x^{\frac{1}{m}}}{\zeta^{m+1}(2)} R\left(\frac{1}{m}\right) f(\log x) + O\left(x^{\frac{1}{2m}+\epsilon}\right),
\end{aligned}$$

where $f(y)$ is a polynomial of y with degree k, and ε is any fixed positive number.

So that we can get

$$\begin{aligned}&\frac{1}{2\pi i}\left(\int_{\frac{2}{m}-iT}^{\frac{2}{m}+iT}+\int_{\frac{2}{m}+iT}^{\frac{1}{2m}+iT}+\int_{\frac{1}{2m}+iT}^{\frac{1}{2m}-iT}+\int_{\frac{1}{2m}-iT}^{\frac{2}{m}-iT}\right)\frac{\zeta^{m+1}(ms)x^s}{\zeta^{m+1}(2ms)s}R(s)ds\\ =\ &\frac{mx^{\frac{1}{m}}}{\zeta^{m+1}(2)}R\left(\frac{1}{m}\right)f(\log x)+O(x^{\frac{1}{2m}+\epsilon}).\end{aligned}$$

It is easy to estimate

$$\left|\frac{1}{2\pi i}\int_{\frac{2}{m}+iT}^{\frac{1}{2m}+iT}\frac{\zeta^{m+1}(ms)x^s}{\zeta^{m+1}(2ms)s}R(s)ds\right|\ll x^{\frac{1}{2m}+\varepsilon},$$

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2m}-iT}^{\frac{2}{m}-iT}\frac{\zeta^{m+1}(ms)x^s}{\zeta^{m+1}(2ms)s}R(s)ds\right|\ll x^{\frac{1}{2m}+\varepsilon},$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2m}+iT}^{\frac{1}{2m}-iT}\frac{\zeta^{m+1}(ms)x^s}{\zeta^{m+1}(2ms)s}R(s)ds\right|\ll x^{\frac{1}{2m}+\varepsilon}.$$

Therefore, we have

$$\sum_{\substack{n\leq x\\ n\in B}}d(n)=\frac{mx^{\frac{1}{m}}}{\zeta^{m+1}(2)}R\left(p^{\frac{1}{m}}\right)\cdot f(\log x)++O\left(x^{\frac{1}{2m}+\varepsilon}\right),$$

where

$$R\left(p^{\frac{1}{m}}\right)=\prod_p\left(1+\frac{p^m\left((p^{\frac{1}{m}}-1)(m+1)+p^{\frac{1}{m}}\right)-\left(p^{\frac{1}{m}}-1\right)^2\sum\limits_{i=2}^{m+1}\binom{m+1}{i}p^{m+1-i}}{(p+1)^{m+1}(p^{\frac{1}{m}}-1)^2}\right).$$

This completes the proof of Theorem 1.

For any integer $k\geq 2$ and any real number $s>1$, let

$$g(s)=\sum_{n\in C}\frac{d(n)}{n^s}.$$

Then from the Euler product formula [3] we have

$$\begin{aligned}g(s)\ &=\ \sum_{n\in C}\frac{d(n)}{n^s}\\ &=\ \prod_p\left(1+\frac{d(p)}{p^s}+\frac{d(p^2)}{p^{2s}}+\cdots+\frac{d(p^l)}{p^{ms}}\right)\end{aligned}$$

$$
\begin{aligned}
&= \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \cdots + \frac{l+1}{p^{ls}}\right) \\
&= \prod_p \left(\frac{1}{1-\frac{1}{p^s}}\right)\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots + \frac{1}{p^{ls}} - \frac{l+1}{p^{(l+1)s}}\right) \\
&= \zeta(s)\prod_p \left(\frac{1-\frac{1}{p^{(l+1)s}}}{1-\frac{1}{p^s}} - \frac{l+1}{p^{(l+1)s}}\right) \\
&= \frac{\zeta^2(s)}{\zeta((l+1)s)}\prod_p \left(1 - \frac{(l+1)(p^s-1)}{p^{(l+2)s}-p^s}\right),
\end{aligned}
$$

where $l = [\frac{k}{2}]$. So by Perron formula [3] and the methods of proving Theorem 1 we can easily get Theorem 2.

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A NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE*

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Abstract Let p and q be two distinct primes, $e_{pq}(n)$ denote the largest exponent of power pq which divides n. In this paper, we study the properties of the sequence $e_{pq}(n)$, and give an interesting asymptotic formula for the mean value $\sum_{n\leq x} e_{pq}(n)$.

Keywords: Largest exponent; Asymptotic formula; Mean value.

§1. Introduction

Let p and q be two distinct primes, $e_{pq}(n)$ denote the largest exponent of power pq which divides n. In problem 68 of [1], Professor F. Smarandache asked us to study the properties of the sequence $e_p(n)$. About this problem, some people had studied it, and obtained a series of interesting results (See references [4]). In this paper, we use the analytic method to study the properties of the sequence $e_{pq}(n)$, and give a sharp asymptotic formula for its mean value $\sum_{n\leq x} e_{pq}(n)$. That is, we shall prove the following:

Theorem. Let p and q be two distinct primes, then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} e_{pq}(n) = \frac{x}{pq-1} + O\left(x^{1/2+\varepsilon}\right),$$

where ε is any fixed positive number.

§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. For any complex s, we define the function

$$f(s) = \sum_{n=1}^{\infty} \frac{e_{pq}(n)}{n^s}.$$

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Note that the definition of $e_{pq}(n)$, and applying the Euler product formula (See Theorem 11.6 of [2]), we may get

$$\begin{aligned} f(s) &= \sum_{\alpha=0}^{\infty}\sum_{t=1}^{\infty}\sum_{\substack{n_1=1\\(n_1,pq)=1}}^{\infty}\frac{\alpha}{(p^{\alpha+t}q^{\alpha}n_1)^s} \\ &\quad+\sum_{\alpha=0}^{\infty}\sum_{t=1}^{\infty}\sum_{\substack{n_1=1\\(n_1,pq)=1}}^{\infty}\frac{\alpha}{(p^{\alpha}q^{\alpha+t}n_1)^s}+\sum_{\alpha=0}^{\infty}\sum_{\substack{n_1=1\\(n_1,pq)=1}}^{\infty}\frac{\alpha}{(p^{\alpha}q^{\alpha}n_1)^s} \\ &= \sum_{\alpha=0}^{\infty}\frac{\alpha}{(pq)^{\alpha s}}\sum_{t=1}^{\infty}\frac{1}{p^{ts}}\sum_{\substack{n_1=1\\(n_1,pq)=1}}^{\infty}\frac{1}{n_1{}^s} \\ &\quad+\sum_{\alpha=0}^{\infty}\frac{\alpha}{(pq)^{\alpha s}}\sum_{t=1}^{\infty}\frac{1}{q^{ts}}\sum_{\substack{n_1=1\\(n_1,pq)=1}}^{\infty}\frac{1}{n_1{}^s}+\sum_{\alpha=0}^{\infty}\frac{\alpha}{(pq)^{\alpha s}}\sum_{\substack{n_1=1\\(n_1,pq)=1}}^{\infty}\frac{1}{n_1{}^s} \\ &= \frac{\zeta(s)}{(pq)^s-1}, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

Obviously, we have

$$e_{pq}(n)\le \log_{pq} n\le \ln n \qquad \left|\sum_{n=1}^{\infty}\frac{e_{pq}(n)}{n^{\sigma}}\right|\le\frac{1}{\sigma-1},$$

where σ is the real part of s. Therefore by Parron's formula (See reference [3]) we can get

$$\sum_{n\le x}\frac{e_{pq}(n)}{n^{s_0}}=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{\zeta(s+s_0)}{(pq)^{s+s_0}-1}\frac{x^s}{s}ds+O\left(\frac{x^bB(b+\sigma_0)}{T}\right)$$

$$+O\left(x^{1-\sigma_0}H(2x)\min\left\{1,\frac{\log x}{T}\right\}\right)+O\left(x^{-\sigma_0}H(N)\min\left\{1,\frac{x}{\|x\|}\right\}\right)$$

where N is the nearest integer to x, $\|x\|=|x-N|$. Taking $s_0=0, b=\frac{3}{2}, H(x)=\ln x, B(\sigma)=\frac{1}{\sigma-1}$, we have

$$\sum_{n\le x}e_{pq}(n)=\frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}\frac{\zeta(s)}{(pq)^s-1}\frac{x^s}{s}ds+O(\frac{x^{\frac{3}{2}+\varepsilon}}{T}).$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}\frac{\zeta(s)}{(pq)^s-1}\frac{x^s}{s}ds,$$

we move the integral line from $s = \frac{3}{2} \pm iT$ to $s = \frac{1}{2} \pm iT$. This time, the function

$$g(s) = \frac{\zeta(s)}{(pq)^s - 1}\frac{x^s}{s}$$

has a simple pole point at $s = 1$, and the residue is $\frac{x}{pq-1}$. So we have

$$\frac{1}{2\pi i}\left(\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta(s)}{(pq)^s - 1}\frac{x^s}{s}ds = \frac{x}{pq-1}.$$

Taking $T = x$, and note that

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\left(\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta(s)}{(pq)^s - 1}\frac{x^s}{s}ds\right| \\
&\ll \int_{\frac{1}{2}}^{\frac{3}{2}}\left|\zeta(\sigma + iT)\frac{1}{(pq)^\sigma - 1}\frac{x^{\frac{3}{2}}}{T}\right|d\sigma \\
&\ll \frac{x^{\frac{3}{2}+\varepsilon}}{T} = x^{\frac{1}{2}+\varepsilon}
\end{aligned}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\frac{\zeta(s)}{(pq)^s - 1}\frac{x^s}{s}ds\right| \ll \int_0^T|\zeta(\frac{1}{2}+it)\frac{1}{(pq)^{\frac{1}{2}} - 1}\frac{x^{\frac{1}{2}}}{t}|dt \ll x^{\frac{1}{2}+\varepsilon},$$

we may immediately get the asymptotic formula

$$\sum_{n\leq x} e_{pq}(n) = \frac{x}{pq-1} + O\left(x^{1/2+\varepsilon}\right).$$

This completes the proof of Theorem.

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ON THE ADDITIVE K-TH POWER COMPLEMENTS*

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Abstract The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the additive k-th power complements, and give some interesting asymptotic formulae for it.

Keywords: Additive k-th power complements; Mean value; Asymptotic formula.

§1. Introduction

For any positive integer n, the Smarandache k-th power complements $b_k(n)$ is the smallest positive integer such that $nb_k(n)$ is a complete k-th power, see problem 29 of [1]. Similar to the Smarandache k-th power complements, the additive k-th power complements $a_k(n)$ is defined as follows: $a_k(n)$ is the smallest nonnegative integer such that $a_k(n) + n$ is a perfect k-th power. For example, if $k = 2$, we have the additive square complements sequence $\{a_2(n)\}$ $(n = 1, 2, \cdots)$ as follows: $a_2(1) = 0, a_2(2) = 2, a_2(3) = 1, a_2(4) = 0, a_2(5) = 4, a_2(6) = 3, a_2(7) = 2, a_2(8) = 1, a_2(9) = 0, \cdots$. About this problem, many authors have studied it before, and obtained some interesting results. For example, Z.F. Xu [4] studied the mean value properties of the additive k-th power complements, and gave the following:

Proposition . *For any real number $x \geq 3$ and fixed positive integer $k \geq 2$, we have the asymptotic formula:*

$$\sum_{n\leq x} a_k(n) = \frac{k^2}{4k-2} x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right).$$

For any fixed positive integer m, the definition of the arithmetical function $\delta_m(n)$ is

$$\delta_m(n) = \begin{cases} \max\{d \in N \quad | \quad d|n, (d,m) = 1\}, & if \quad n \neq 0, \\ 0, & if \quad n = 0. \end{cases}$$

In this paper, we shall use the elementary and analytic methods to study the mean value properties of the new arithmetical function $\delta_m(a_k(n))$, and give

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an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. *For any real number $x \geq 3$ and positive integer m, we have the asymptotic formula:*

$$\sum_{n\leq x} \delta_m(a_k(n)) = \frac{k^2}{2(2k-1)} x^{2-\frac{1}{k}} \prod_{p|m} \frac{p}{p+1} + O\left(x^{2-\frac{2}{k}}\right), \quad \text{if} \quad k > 3,$$

where $\prod_{p|m}$ denotes the product over all prime divisors p of m, and ϵ is any fixed positive number.

Taking $k = 2, 3$ in our Theorem, we may immediately deduce the following:

Corollary. *For any real number $x \geq 1$, we have the asymptotic formulae*

$$\sum_{n\leq x} \delta_m(a_2(n)) = \frac{2}{3} x^{\frac{3}{2}} \prod_{p|m} \frac{p}{p+1} + O\left(x^{\frac{5}{4}+\epsilon}\right)$$

and

$$\sum_{n\leq x} \delta_m(a_3(n)) = \frac{9}{10} x^{\frac{5}{3}} \prod_{p|m} \frac{p}{p+1} + O\left(x^{\frac{4}{3}+\epsilon}\right).$$

§2. Some lemmas

To complete the proof of the theorem, we need following Lemmas:

Lemma 1. *For any real number $x > 1$ and positive integer m, we have the asymptotic formula*

$$\sum_{n\leq x} \delta_m(n) = \frac{x^2}{2} \prod_{p|m} \frac{p}{p+1} + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where ϵ is any fixed positive number.

Proof. Let $s = \sigma + it$ be a complex number and $f(s) = \sum\limits_{n=1}^{\infty} \frac{\delta_m(n)}{n^s}$. Note that $\delta_m(n) \ll n$, so it is clear that $f(s)$ is a Dirichlet series absolutely convergent for Re$(s)>2$, by the Euler product formula [2] and the definition of $\delta_m(n)$ we get

$$\begin{aligned} f(s) = \sum_{n=1}^{\infty} \frac{\delta_m(n)}{n^s} &= \prod_{p} \left(1 + \frac{\delta_m(p)}{p^s} + \frac{\delta_m(p^2)}{p^{2s}} + \cdots + \frac{\delta_m(p^{2n})}{p^{ns}} + \cdots\right) \\ &= \prod_{p|m} \left(1 + \frac{\delta_m(p)}{p^s} + \frac{\delta_m(p^2)}{p^{2s}} + \cdots + \frac{\delta_m(p^{2n})}{p^{ns}} + \cdots\right) \end{aligned}$$

$$
\begin{aligned}
&\times \prod_{p\dagger m}\left(1+\frac{\delta_m(p)}{p^s}+\frac{\delta_m(p^2)}{p^{2s}}+\cdots+\frac{\delta_m(p^{2n})}{p^{ns}}+\cdots\right)\\
=\ & \prod_{p|m}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{ns}}+\cdots\right)\\
&\times \prod_{p\dagger m}\left(1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\cdots+\frac{p^{2n}}{p^{ns}}+\cdots\right)\\
=\ & \prod_{p|m}\left(\frac{1}{1-\frac{1}{p^s}}\right)\prod_{p\dagger m}\left(\frac{1}{1-\frac{1}{p^{s-1}}}\right)\\
=\ & \zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right), \qquad (1)
\end{aligned}
$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ denotes the product over all primes.

From (1) and Perron's formula [3], we have

$$
\sum_{n\le x}\delta_m(n)=\frac{1}{2\pi i}\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds+O\left(\frac{x^{\frac{5}{2}+\epsilon}}{T}\right), \quad (2)
$$

where ϵ is any fixed positive number.

Now we move the integral line in (2) from $s=\frac{5}{2}\pm iT$ to $s=\frac{3}{2}\pm iT$. This time, the function $\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}$ has a simple pole point at $s=2$ with residue

$$
\frac{x^2}{2}\prod_{p|m}\frac{p}{p+1}. \quad (3)
$$

Hence, we have

$$
\begin{aligned}
&\frac{1}{2\pi i}\left(\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT}+\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}+\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}\right)\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds\\
=\ &\frac{x^2}{2}\prod_{p|m}\frac{p}{p+1}. \qquad (4)
\end{aligned}
$$

We can easily get the estimate

$$
\left|\frac{1}{2\pi i}\left(\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT}+\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT}\right)\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds\right|\ll\frac{x^{\frac{5}{2}+\epsilon}}{T} \quad (5)
$$

and

$$
\left|\frac{1}{2\pi i}\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}\zeta(s-1)\prod_{p|m}\left(\frac{p^s-p}{p^s-1}\right)\cdot\frac{x^s}{s}ds\right|\ll x^{\frac{3}{2}+\epsilon}. \quad (6)
$$

Taking $T = x$, combining (2), (4), (5) and (6) we deduce that

$$\sum_{n\leq x} \delta_m(n) = \frac{x^2}{2}\prod_{p|m}\frac{p}{p+1} + O(x^{\frac{3}{2}+\epsilon}). \tag{7}$$

This completes the proof of Lemma 1.

Lemma 2. *For any real number $x \geq 3$ and any nonnegative arithmetical function $f(n)$ with $f(0) = 0$, we have the asymptotic formula:*

$$\sum_{n\leq x} f(a_k(n)) = \sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} \sum_{n\leq g(t)} f(n) + O\left(\sum_{n\leq g\left(\left[x^{\frac{1}{k}}\right]\right)} f(n)\right),$$

where $[x]$ denotes the greatest integer not exceeding x and $g(t) = \sum_{i=1}^{k-1}\binom{i}{k}t^i$.

Proof. See reference [4].

§3. Proof of the theorem

In this section, we will complete the proof of the theorem. From the definition of $\delta_m(a_k(n))$, Lemma 1 and Lemma 2, we have

$$\begin{aligned}
&\sum_{n\leq x} \delta_m(a_k(n))\\
=& \sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} \sum_{n\leq g(t)} \delta_m(n) + O\left(\sum_{n\leq g\left(\left[x^{\frac{1}{k}}\right]\right)} \delta_m(n)\right)\\
=& \sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1}\left(\frac{k^2t^{2k-2}}{2}\prod_{p|m}\frac{p}{p+1} + O(t^{2k-3})\right) + O\left(x^{1-\frac{1}{k}}\ln x\right) \qquad \text{if} \qquad k > 3\\
=& \frac{k^2}{2}\prod_{p|m}\frac{p}{p+1}\sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} t^{2k-2} + O(x^{2-\frac{2}{k}})\\
=& \frac{k^2}{2(2k-1)}\prod_{p|m}\frac{p}{p+1}x^{2-\frac{1}{k}} + O(x^{2-\frac{2}{k}}).
\end{aligned}$$

For the cases of $k = 2, 3$, we can also prove the results by Lemma 2. For example,

$$\sum_{n\leq x} \delta_m(a_2(n))$$

$$= \sum_{t=1}^{\left[x^{\frac{1}{2}}\right]-1} \sum_{n\le 2t} \delta_m(n) + O\left(\sum_{n\le g\left(\left[x^{\frac{1}{2}}\right]\right)} \delta_m(n) \right)$$

$$= \sum_{t=1}^{\left[x^{\frac{1}{2}}\right]-1} \left(2t^2 \prod_{p|m} \frac{p}{p+1} + O(t^{\frac{3}{2}+\epsilon}) \right) + O\left(x^{1-\frac{1}{2}} \ln x\right)$$

$$= \frac{2x^{\frac{3}{2}}}{3} \prod_{p|m} \frac{p}{p+1} + O(x^{\frac{5}{4}+\epsilon}).$$

This completes the proof of the theorem.

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ON THE SMARANDACHE-RIEMANN ZETA SEQUENCE

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Abstract The main purpose of this paper is using the elementary method to study the properties of the Smarandache-Riemann zeta sequence, and solve a conjecture posed by Murthy.

Keywords: Smarandache-Riemann zeta sequence; Murthy's conjecture; Elementary method.

§1. Introduction

For any positive integer n, we let $a(2n, 2)$ denotes the sum of the base 2 digits of $2n$. That is, if $2n = a_1 2^{\alpha_1} + a_2 2^{\alpha_2} + \ldots + a_s 2^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \ldots > \alpha_1 \geq 0$, where $a_i = 0$ or 1, $i = 1, 2, \ldots, s$, then $a(2n, 2) = \sum_{i=1}^{s} a_i$.
For any complex number s with $\text{Re}(s) > 1$, we define Riemann-zeta function as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For any positive integer n, let T_n be a number such that

$$\zeta(2n) = \frac{\pi^{2n}}{T_n},$$

where π is ratio of the circumference of a circle to its diameter. Then the sequence $T = \{T_n\}_{n=1}^{\infty}$ is called the Smarandache-Riemann zeta sequence. In [1], Murthy believed that T_n is a sequence of integers. Simultaneous, he proposed the following:

Conjecture. *No two terms of T_n are relatively prime.*

In this paper, we shall prove the following conclusion:

Theorem. *There exists infinite positive integers n such that T_n is not an integer.*

From this Theorem we know that the Murthy's conjecture is not correct, because there exists infinite positive integers n such that T_n is not an integer.

§2. Some simple lemmas

Before the proof of the theorem, some simple lemmas will be useful.

Lemma 1. *If $ord(2,(2n)!) < 2n-2$, then T_n is not an integer, where $ord(2,(2n)!)$ denotes the order of prime 2 in $(2n)!$.*

Proof. See reference [2].

Lemma 2. *For any positive integer $n \geq 1$, we have the identity:*

$$\alpha_2(2n) \equiv \sum_{i=1}^{+\infty} \left[\frac{2n}{2^i}\right] = 2n - a(2n,2),$$

where $[x]$ denotes the greatest integer not exceeding x.

Proof. From the properties of $[x]$ we know that

$$\begin{aligned} \left[\frac{2n}{2^i}\right] &= \left[\frac{a_1 2^{\alpha_1} + a_2 2^{\alpha_2} + \ldots + a_s 2^{\alpha_s}}{2^i}\right] \\ &= \begin{cases} \sum\limits_{j=k}^{s} a_j 2^{\alpha_j - i}, & \text{if} \quad \alpha_{k-1} < i \leq \alpha_k; \\ 0, & \text{if} \quad i > \alpha_s. \end{cases} \end{aligned}$$

So from this formula we have

$$\begin{aligned} \alpha_2(2n) &\equiv \sum_{i=1}^{+\infty} \left[\frac{2n}{2^i}\right] = \sum_{i=1}^{+\infty} \left[\frac{a_1 2^{\alpha_1} + a_2 2^{\alpha_2} + \cdots + a_s 2^{\alpha_s}}{2^i}\right] \\ &= \sum_{j=1}^{s} \sum_{k=1}^{\alpha_j} a_j 2^{\alpha_j - k} = \sum_{j=1}^{s} a_j (1 + 2 + 2^2 + \cdots + 2^{\alpha_j - 1}) \\ &= \sum_{j=1}^{s} a_j \cdot \frac{2^{\alpha_j} - 1}{2 - 1} = \sum_{j=1}^{s} (a_j 2^{\alpha_j} - a_j) \\ &= 2n - a(2n,2) \end{aligned}$$

This completes the proof of Lemma 2.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. From Lemma 2 and the Theorem 3.14 of [3] we know that

$$ord(2,(2n)!) = \alpha_2(2n) \equiv \sum_{i=1}^{+\infty} \left[\frac{2n}{2^i}\right] = 2n - a(2n,2).$$

Now it is clear that $ord(2,(2n)!) < 2n-2$ if and only if

$$2n - a(2n,2) < 2n - 2.$$

That is

$$a(2n,2) \geq 3.$$

In fact, there exists infinite positive integers n such that $a(2n,2) \geq 3$. For example, taking $n = 2^{\alpha_1} + 2^{\alpha_2} + 2^{\alpha_3} + 2^{\alpha_4}$ with $\alpha_4 > \alpha_3 > \alpha_2 > \alpha_1 \geq 0$, then $a(2n,2) \geq 3$. For these n, from Lemma 1 we know that T_n is not an integer. Since there are infinite positive integers $\alpha_1, \alpha_2, \alpha_3$ and α_4, it means that there exists infinite positive integers n such that $a(2n,2) \geq 3$. Therefore, there exists infinite positive integers n such that T_n is not an integer.

This completes the proof of Theorem.

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ON THE PROPERTIES OF THE HEXAGON-NUMBERS

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Abstract In this paper, we shall use the elementary method to study the properties of the hexagon-numbers, and give an interesting identity involving the hexagon-numbers.

Keywords: Hexagon-number; Arithmetical property; Identity.

§1. Introduction

For any positive integer m, if $n = m(2m-1)$, then we call such an integer n as hexagon-number (see reference [1]). The hexagon-numbers $1, 6, 15, 28, \cdots$ are closely related to the hexagons. Moreover, the hexagon-numbers are the partial sums of the terms in the arithmetic progression

$$1, 5, 9, 13, \cdots, 4n+1, \cdots.$$

For any positive integer n, let m be the largest positive integer that satisfy the inequality

$$m(2m-1) \leq n < (m+1)(2m+1).$$

Now we define $a(n) = m(2m-1)$, and call $a(n)$ as the hexagon-number part of n. In this paper, we shall use the elementary method to study the convergent properties of the Dirichlet's series $f(s) = \sum\limits_{n=1}^{\infty} \frac{1}{a^s(n)}$, and give an interesting identity for the case $s = 2$. That is, we shall prove the following:

Theorem. *For any real number $s > 1$, the infinity series $f(s)$ is convergent, and*

$$f(2) = \sum_{n=1}^{\infty} \frac{1}{a^2(n)} = \frac{5}{3}\pi^2 - 4\ln 2.$$

§2. Proof of Theorem

In this section, we shall complete the proof of Theorem. First from the definition of $a(n)$ we know that there exists

$$(m+1)(2m+1) - m(2m-1) = 4m+1$$

solutions for the equation $a(n) = m(2m-1)$. So we can easy deduce that

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{a^s(n)} = \sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ a(n)=m(2m-1)}}^{\infty} \frac{1}{a^s(n)} = \sum_{m=1}^{\infty} \frac{4m+1}{m^s(2m-1)^s}.$$

Thus we may conclude that $f(s)$ is convergent if $2s-1 > 1$. That is, $s > 1$.

Now we shall use the elementary method to calculate the exact value of $f(2)$. Applying the above identity we have

$$\begin{aligned}
f(2) &= \sum_{m=1}^{\infty} \frac{4m+1}{m^2(2m-1)^2} \\
&= \sum_{m=1}^{\infty} \frac{1}{m^2} + \sum_{m=1}^{\infty} \frac{12}{(2m-1)^2} - \sum_{m=1}^{\infty} \frac{8}{m(2m-1)} \\
&= \sum_{m=1}^{\infty} \frac{1}{m^2} + 12\left(\sum_{m=1}^{\infty} \frac{1}{m^2} - \sum_{m=1}^{\infty} \frac{1}{4m^2}\right) - \sum_{m=1}^{\infty} \frac{8}{m(2m-1)} \\
&= \zeta(2) + 12\left(\zeta(2) - \frac{1}{4}\zeta(2)\right) - \sum_{m=1}^{\infty} \frac{8}{m(2m-1)} \\
&= 10\zeta(2) - \sum_{m=1}^{\infty} \frac{8}{m(2m-1)} \\
&= 10\zeta(2) - 4\left(\sum_{m=1}^{\infty} \frac{1}{2m-1} - \sum_{m=1}^{\infty} \frac{1}{2m}\right) \\
&= 10\zeta(2) - 4\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

From the Taylor's expansion (see reference [2]) for $\ln(1+x)$ we know that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \cdots.$$

Using this identity with $x = 1$ we may immediately get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

Combing all the above, and note that $\zeta(2) = \frac{\pi^2}{6}$ we may obtain

$$f(2) = 10\zeta(2) - 4\ln 2 = \frac{5}{3}\pi^2 - 4\ln 2.$$

This completes the proof of Theorem.

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ON THE MEAN VALUE OF THE K-TH POWER PART RESIDUE FUNCTION

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Abstract Similar to the Smarandache k-th power complements, we define the k-th power part residue function $f_k(n)$ is the smallest nonnegative integer such that $n - f_k(n)$ is a perfect k-th power. The main purpose of this paper is using the elementary methods to study the mean value properties of $\frac{1}{f_k(n)+1}$, and give an interesting asymptotic formula for it.

Keywords: k-th power part residue function; Mean value; Asymptotic formula.

§1. Introduction and results

For any positive integer n, the Smarandache k-th power complements $b_k(n)$ is the smallest positive integer such that $nb_k(n)$ is a perfect k-th power (see problem 29 of [1]). Similar to the Smarandache k-th power complements, Xu Zhefeng in [2] defined the additive k-th power complements $a_k(n)$ as follows: $a_k(n)$ is the smallest nonnegative integer such that $n + a_k(n)$ is a perfect k-th power.

As a generalization of [2], we will define the k-th power part residue function $f_k(n)$ as the smallest nonnegative integer such that $n - f_k(n)$ is a perfect k-th power. For example, if $k = 2$, we have the square part residue sequence $\{f_2(n)\}$ $(n = 1, 2, \ldots)$ as following: $0, 1, 2, 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 5, 6, 0, 1, 2, \ldots$. Meanwhile, let p be a prime, $e_p(n)$ denotes the largest exponent of power p which divides n. About the relations between $e_p(n)$ and $f_k(n)$, it seems that none had studied them before, at least we couldn't find any reference about it.

In this paper, we use the elementary methods to study the mean value properties of $\frac{1}{f_k(n)+1}$ and $e_p(f_k(n))$, and obtain two sharper asymptotic formulae for them. That is, we will prove the following conclusions:

Theorem 1. For any real number $x \geq 3$, we have the asymptotic formula

$$\sum_{n\leq x} \frac{1}{f_k(n)+1} = \frac{k-1}{k} x^{\frac{1}{k}} \ln x + (\ln x + \gamma - k + 1)x^{\frac{1}{k}} + O(\ln x),$$

where γ is the Euler constant.

Theorem 2. For any real number $x \geq 3$, we have the asymptotic formula

$$\sum_{n \leq x} e_p(f_k(n)) = \frac{1}{p-1} x + O\left(\frac{k}{p-1} x^{1-\frac{1}{k}}\right).$$

§2. Some Lemmas

To complete the proof of the above theorems, we need following several Lemmas.

Lemma 1. For any real number $x \geq 1$, we have

$$\sum_{n \leq x} e_p(n) = \frac{1}{p-1} x + O(\ln^2 x).$$

Proof. See reference [3].

Lemma 2. Let $h(n)$ be a nonnegative arithmetical function with $h(0) = 0$. Then, for any real number $x \geq 1$ we have the asymptotic formula:

$$\sum_{n \leq x} h(f_k(n)) = \sum_{t=1}^{M-1} \sum_{n \leq g(t)} h(n) + O\left(\sum_{n \leq g(M)} h(n)\right),$$

where $g(t) = \sum_{i=1}^{k-1} \binom{k}{i} t^i$ and $M = [x^{\frac{1}{k}}]$, $[x]$ denotes the greatest integer not exceeding x.

Proof. For any real number $x \geq 1$, let M be a fixed positive integer such that

$$M^k \leq x < (M+1)^k.$$

Noting that if n pass through the integers in the interval $[t^k, (t+1)^k)$, then $f_k(n)$ pass through all integers in the interval $[0, (t+1)^k - t^k)$, so we can deduce that

$$\begin{aligned}
\sum_{n \leq x} h(f_k(n)) &= \sum_{t=1}^{M-1} \sum_{t^k \leq n < (t+1)^k} h(f_k(n)) + \sum_{M^k \leq n \leq x} h(f_k(n)) \\
&= \sum_{t=1}^{M-1} \sum_{n \leq g(t)} h(n) + \sum_{0 \leq n < x - M^k} h(n) \\
&= \sum_{t=1}^{M-1} \sum_{n \leq g(t)} h(n) + O\left(\sum_{0 \leq n \leq (M+1)^k - M^k} h(n)\right) \\
&= \sum_{t=1}^{M-1} \sum_{n \leq g(t)} h(n) + O\left(\sum_{n \leq g(M)} h(n)\right).
\end{aligned}$$

This proves the Lemma 2.

Lemma 3. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\le x}\frac{1}{n}=\ln x+\gamma+O\left(\frac{1}{x}\right),$$

where γ is the Euler constant.

Proof. See reference [4].

Lemma 4 For any real number $x \ge 3$, we have the asymptotic formula

$$\sum_{n\le x} f_k(n)=\frac{k^2}{2(2k-1)}x^{2-\frac{1}{k}}+O\left(x^{2-\frac{2}{k}}\right).$$

Proof. Let $h(n) = n$ and $M = [x^{\frac{1}{k}}]$, then from Lemma 2 and Euler summation formula (see reference [4]) we obtain

$$\begin{aligned}\sum_{n\le x} f_k(n) &= \sum_{t=1}^{M-1}\sum_{n\le g(t)} n+O\left(\sum_{n\le g(M)} n\right)\\ &= \frac{1}{2}\sum_{t=1}^{[x^{\frac{1}{k}}]-1} k^2t^{2k-2}+O(x^{2-\frac{2}{k}})\\ &= \frac{k^2}{2(2k-1)}x^{2-\frac{1}{k}}+O(x^{2-\frac{2}{k}}).\end{aligned}$$

This proves Lemma 4.

§3. Proof of the theorems

In this section, we shall complete the proof of Theorems. First we prove Theorem 1. Let $h(n) = n$ and $M = [x^{\frac{1}{k}}]$, then from Lemma 2 and Lemma 3 we obtain

$$\begin{aligned}&\sum_{n\le x}\frac{1}{f_k(n)+1}\\ =\ & \sum_{t=1}^{M-1}\sum_{n\le g(t)}\frac{1}{n+1}+O\left(\sum_{n\le g(M)}\frac{1}{n+1}\right)\\ =\ & \sum_{t=1}^{M-1}\left(\ln(kt^{k-1})+\ln\left(1+O\left(\frac{1}{t}\right)\right)+\gamma+O\left(\frac{1}{g(t)}\right)\right)+O(\ln x)\\ =\ & (k-1)\ln((M-1)!)+(\ln k+\gamma)(M-1)+O(\ln x)\\ =\ & (k-1)\left(([x^{\frac{1}{k}}]-1)\ln([x^{\frac{1}{k}}]-1)-([x^{\frac{1}{k}}]-1)\right)\\ & +(\ln k+\gamma)([x^{\frac{1}{k}}]-1)+O(\ln x)\\ =\ & \frac{k-1}{k}x^{\frac{1}{k}}\ln x+(\ln x+\gamma-k+1)x^{\frac{1}{k}}+O(\ln x).\end{aligned}$$

This completes the proof of Theorem 1.

The proof of Theorem 2. Note that the definition of $e_p(n)$, from Lemma 1 and Lemma 4 we have

$$\begin{aligned}
&\sum_{n\le x} e_p(f_k(n))\\
=\ &\sum_{t=1}^{M} \sum_{(t-1)^k\le n<t^k} e_p(f_k(n)) + \sum_{M^k\le n<x} e_p(f_k(n))\\
=\ &\sum_{t=1}^{M} \sum_{j=0}^{(t+1)^k-t^k} e_p(j) + O\left(\sum_{M^k\le n<(M+1)^k} e_p(f_k(n))\right)\\
=\ &\sum_{t=1}^{M}\left(\frac{1}{p-1}\left((t+1)^k-t^k\right)+O\left(\ln^2((t+1)^k-t^k)\right)\right)\\
&+O\left(\sum_{0\le n<(M+1)^k-M^k} e_p(n)\right)\\
=\ &\frac{1}{p-1}\left((M+1)^k-1\right)+O(\ln^2(kM^k-1))+O\left(\frac{g(M)}{p-1}\right)\\
=\ &\frac{1}{p-1}M^k+O\left(\frac{k}{p-1}M^{k-1}\right).
\end{aligned}$$

Taking $M=[x^{\frac{1}{k}}]$, we easily get

$$\sum_{n\le x} e_p(f_k(n)) = \frac{1}{p-1}x + O\left(\frac{k}{p-1}x^{1-\frac{1}{k}}\right).$$

This completes the proof of Theorem 2.

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ON THE INTEGER PART OF THE M-TH ROOT AND THE K-TH POWER FREE NUMBER

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Abstract The main purpose of this paper is using the elementary methods to study the mean value properties of a special arithmetical function involving the integer part of the m-th root of an integer and the k-th power free numbers, and give an interesting asymptotic formula for it.

Keywords: Integer part of the m-th root; k-th power free numbers; Asymptotic formula.

§1. Introduction and results

Let m and k are two fixed positive integers with $k \geq 2$. For any positive integer n, we define arithmetical function $b_m(n)$ as the integer part of the m-th root of n. That is, $b_m(n) = [n^{\frac{1}{m}}]$, where $[x]$ denotes the greatest integer $\leq x$. For example, $b_2(1) = 1$, $b_2(2) = 1$, $b_2(3) = 1$, $b_2(4) = 2$, $b_2(5) = 2$, $b_2(6) = 2$, $b_2(7) = 2$, $b_2(8) = 2$, $b_2(9) = 3$, $\cdots\cdots$. In reference [1], Professor F.Smarandache asked us to study the properties of the sequences $\{b_k(n)\}$. About this problem, I do not know whether there exists any progress. But none study the mean value properties of $\{b_k(n)\}$ over the k-th power free numbers, here we call a positive integer n as k-th power free number, if $p^k \dagger n$ for any prime p. For convenience, we let $\mathcal{A}_k$ denotes the set of all k-th power free numbers. In this paper, we shall use the elementary methods to study the mean value properties of $b_m(n)$ over the set $\mathcal{A}_k$, and give an interesting formula for it. That is, we shall prove the following result:

Theorem. *Let m and k are two fixed positive integers with $k \geq 2$. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}_k}} b_m(n) = \frac{1}{\zeta(k)} \frac{m}{m+1} x^{\frac{m+1}{m}} + O(x),$$

where $\zeta(k)$ is the Riemann zeta-function.

From this theorem we may immediately deduce the following Corollaries:

Corollary 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}_2}} b_2(n) = \frac{4}{\pi^2}x^{\frac{3}{2}} + O\left(x\right) \quad \text{and} \quad \sum_{\substack{n\leq x\\ n\in\mathcal{A}_2}} b_3(n) = \frac{9}{2\pi^2}x^{\frac{4}{3}} + O\left(x\right).$$

Corollary 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}_4}} b_4(n) = \frac{64}{\pi^4}x^{\frac{5}{4}} + O\left(x\right) \quad \text{and} \quad \sum_{\substack{n\leq x\\ n\in\mathcal{A}_4}} b_5(n) = \frac{75}{\pi^4}x^{\frac{6}{5}} + O\left(x\right).$$

§2. Some Lemmas

Before completing the proof of the theorem, we need the following lemmas.

Lemma 1. *Let m and k are two fixed positive integers with $k \geq 2$. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} n^{\frac{1}{m}} = \frac{m}{m+1}x^{\frac{m+1}{m}} + O(x) \quad \text{and} \quad \sum_{n\leq x}\frac{\mu(n)}{n^k} = \frac{1}{\zeta(k)} + O\left(x^{-k+1}\right),$$

where $\mu(n)$ is the Möbius function, and $\zeta(k)$ is the Riemann zeta-function.

Proof. Using the Euler's summation formula (see Theorem 3.1 of [2]) we can easily deduce these conclusions.

Lemma 2. *Let $a(n)$ denotes the character function of the k-th power free numbers (That is, if n is a k-th power free number, then $a(n) = 1$; if n is not a k-th power free number, then $a(n) = 0$). Then for any positive integer n, we have the identity*

$$a(n) = \sum_{d^k|n}\mu(d),$$

where $\mu(d)$ is the $M\ddot{o}bius$ function.

Proof. From the properties of $M\ddot{o}bius$ function we know that for any positive integer n,

$$\sum_{d|n}\mu(d) = \left[\frac{1}{n}\right].$$

Let u^k denotes the greatest k-th power divisor of n. That is, $n = u^k v$, where v is a k-th power free number. It is clear that n is a k-th power free number if and only if $u = 1$. In this case,

$$\sum_{d^k|n}\mu(d) = \sum_{d^k|u^k}\mu(d) = \sum_{d|u}\mu(d) = \mu(1) = 1 = a(n).$$

If $u > 1$, then from the above formula we have

$$\sum_{d^k|n}\mu(d) = \sum_{d^k|u^k}\mu(d) = \sum_{d|u}\mu(d) = \left[\frac{1}{u}\right] = 0 = a(n).$$

This proves Lemma 2.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem. For any real number $x \geq 1$ and positive integer $k \geq 2$, from Lemma 1 and Lemma 2 and the definition of $b_m(n)$ we have

$$
\begin{aligned}
& \sum_{\substack{n \leq x \\ n \in \mathcal{A}_k}} b_m(n) \\
= & \sum_{n \leq x} a(n) b_m(n) \\
= & \sum_{n \leq x} \sum_{d^k \mid n} \mu(d) \left[n^{\frac{1}{m}} \right] \\
= & \sum_{d^k t \leq x} \mu(d) \left[(d^k t)^{\frac{1}{m}} \right] \\
= & \sum_{d^k \leq x} \mu(d) d^{\frac{k}{m}} \sum_{t \leq x/d^k} t^{\frac{1}{m}} + O\left(\sum_{d^k \leq x} \sum_{t \leq x/d^k} 1 \right) \\
= & \sum_{d^k \leq x} \mu(d) d^{\frac{k}{m}} \left(\frac{m}{m+1} x^{\frac{m+1}{m}} \frac{1}{d^{\frac{k(m+1)}{m}}} + O\left(\left(\frac{x}{d^k} \right)^{\frac{1}{m}} \right) \right) + O\left(x \sum_{d^k \leq x} \frac{1}{d^k} \right) \\
= & \frac{m}{m+1} x^{\frac{m+1}{m}} \sum_{d^k \leq x} \frac{\mu(d)}{d^k} + O\left(x^{\frac{1}{m}} \sum_{d^k \leq x} 1 \right) + O(x) \\
= & \frac{1}{\zeta(k)} \frac{m}{m+1} x^{\frac{m+1}{m}} + O(x).
\end{aligned}
$$

This completes the proof of Theorem.

Now the Corollaries follows from $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$.

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A FORMULA FOR SMARANDACHE LCM RATIO SEQUENCE

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Abstract In this paper, a reduction formula for Smarandache LCM ratio sequences SLR(5) is given.

Keywords: Smarandache LCM ratio sequences; Reduction formula.

§1. Introduction

Let $(x_1, x_2, ..., x_t)$ and $[x_1, x_2, ..., x_t]$ denote the greatest common divisor and the least common multiple of any postive integers $x_1, x_2, ..., x_t$ respectively. Let r be a positive integer with $r > 1$. For any positive integer n, let

$$T(r,n) = \frac{[n, n+1, ..., n+r-1]}{[1, 2, ...r]},$$

then the sequences $SLR(r) = \{T(r,n)\}_\infty$ is called Smarandache LCM ratio sequences of degree r. In reference $[1]$, Maohua Le studied the properties of $SLR(r)$, and gave two reduction formulas for SLR(3) and SLR(4). In this paper, we study the calculating problem of $SLR(5)$, and prove the following conclusion.

Theorem. For any postive integer n, we have the calculating formula :

$$T(5,n) = \begin{cases} \frac{1}{1440} n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 0, 8 \bmod 12; \\ \frac{1}{120} n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 1, 7 \bmod 12; \\ \frac{1}{720} n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 2, 6 \bmod 12; \\ \frac{1}{360} n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 3, 5, 9, 11 \bmod 12; \\ \frac{1}{480} n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 4 \bmod 12; \\ \frac{1}{240} n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 10 \bmod 12. \end{cases}$$

§2. Proof of the theorem

To complete the proof of Theorem, we need following several simple Lemmas.

Lemma 1. For any postive integer a and b, we have $(a, b)[a, b] = ab$.

Lemma 2. For any postive integer s with $s < t$, we have

$$(x_1, x_2, ..., x_t) = ((x_1, ..., x_s), (x_{s+1}, ..., x_t))$$

and

$$[x_1, x_2, ..., x_t] = [[x_1, ..., x_s], [x_{s+1}, ..., x_t]].$$

Lemma 3. For any postive integer n, we have

$$T(4, n) = \begin{cases} \frac{1}{24} n(n+1)(n+2)(n+3), & \text{if } n \equiv 1, 2 \bmod 3; \\ \frac{1}{72} n(n+1)(n+2)(n+3), & \text{if } n \equiv 0 \bmod 3. \end{cases}$$

The proof of Lemma 1 and Lemma 2 can be found in [3], Lemma 3 is proved in reference [1].

Now we use these Lemmas to complete the proof of our Theorem. In fact, for any positive integer n, from the properties of the least common multiple of any positive integers we know that

$$\begin{aligned} [n, n+1, n+2, n+3, n+4] &= [[n, n+1, n+2, n+3], n+4] \\ &= \frac{[n, n+1, n+2, n+3](n+4)}{([n, n+1, n+2, n+3], n+4)}. \end{aligned}$$

Note that $[1, 2, 3, 4, 5] = 60$, $[1, 2, 3, 4] = 12$ and

$$([n, n+1, n+2, n+3], n+4) = \begin{cases} 4, & \text{if } n \equiv 0, 4 \bmod 12; \\ 1, & \text{if } n \equiv 1, 3, 5, 7, 9 \bmod 12; \\ 6, & \text{if } n \equiv 2 \bmod 12; \\ 2, & \text{if } n \equiv 6, 10 \bmod 12; \\ 12, & \text{if } n \equiv 8 \bmod 12; \\ 3, & \text{if } n \equiv 11 \bmod 12, \end{cases}$$

Combining Lemma 3, we and easily get the theorem.

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ON THE M-TH POWER COMPLEMENTS SEQUENCE*

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Abstract The main purpose of this paper is using the elementary and analytic methods to study the asymptotic properties of the m-th power complements sequence, and give several interesting asymptotic formulae for it.

Keywords: M-th power complements; Riemann zeta-function; Asymptotic formula.

§1. Introduction and results

For any positive integer $n \geq 2$, let $b_m(n)$ denote the m-th power complements sequence. That is, $b_m(n)$ denotes the smallest positive integer such that $nb_m(n)$ is a complete m-th power. In problem 29 of the reference [1], Professor F.Smarandache asked us to study the properties of this sequence. About this problem, some authors had studied it before, and obtained some interesting results, see reference [4] and [5]. The main purpose of this paper is using the elementary and analytic methods to study the asymptotic properties of the m-th power complements sequence, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem . *For any real number $x > 1$ and any fixed positive integers m and k, we have the asymptotic formula*

$$\begin{aligned} & \sum_{n\leq x} \frac{1}{\delta_k(b_m(n))} \\ = & \frac{x^2}{2\zeta(m)} \prod_{p\nmid k} \frac{p^m(p^m - p^{m-1}+1)}{(p^{2m}-1)} \prod_{p|k} \left(\frac{p^{m+1}}{(p+1)(p^m-1)} \right) + O\left(x^{\frac{3}{2}+\epsilon}\right), \end{aligned}$$

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where $\delta_k(n)$ defined as following:

$$\delta_k(n) = max\{d \in N \quad | \quad d|n, \quad (d,k)=1\},$$

$\prod_p$ *denotes the product over all prime numbers.*

From this theorem, we may immediately get the following:

Corollary 1. *Let $a(n)$ be the square complements sequence, then for any real number $x > 1$ and any fixed positive integer k, we have the asymptotic formula*

$$\sum_{n\le x} \frac{1}{\delta_k(a(n))} = \frac{3x^2}{\pi^2} \prod_{p\dagger k} \frac{p^4 - p^3 + p^2}{(p^4-1)} \prod_{p|k} \left(\frac{p^3}{(p+1)(p^2-1)} \right) + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

Corollary 2. *Let $b(n)$ be the cubic complements sequence, then for any real number $x > 1$ and any fixed positive integer k, we have the asymptotic formula*

$$\sum_{n\le x} \frac{1}{\delta_k(b(n))} = \frac{x^2}{2\zeta(3)} \prod_{p\dagger k} \frac{p^6 - p^5 + p^3}{(p^6-1)} \prod_{p|k} \left(\frac{p^4}{(p+1)(p^3-1)} \right) + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

§2. Proof of the theorem

In this section, we will complete the proof of Theorem.

Let $s = \sigma + it$ be a complex number and $f(s) = \sum\limits_{n=1}^{\infty} \frac{\frac{1}{\delta_k(b_m(n))}}{n^s}$. Note that $\frac{1}{\delta_k(b_m(n))} \ll 1$, so it is clear that $f(s)$ is a Dirichlet series absolutely convergent for Re$(s)> 1$. By the Euler product formula [2] and the multiplicative property of $\delta_k(b_m(n))$ we get

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{\frac{1}{\delta_k(b_m(n))}}{n^s} \qquad (1) \\ &= \prod_p \left(1 + \frac{\frac{1}{\delta_k(b_m(p))}}{p^s} + \frac{\frac{1}{\delta_k(b_m(p^2))}}{p^{2s}} + \cdots + \frac{\frac{1}{\delta_k(b_m(p^m))}}{p^{ms}} + \cdots \right) \\ &= \prod_p \left(1 + \frac{\frac{1}{\delta_k(p^{m-1})}}{p^s} + \cdots + \frac{\frac{1}{\delta_k(1)}}{p^{ms}} + \frac{\frac{1}{\delta_k(p^{m-1})}}{p^{(m+1)s}} + \cdots + \frac{\frac{1}{\delta_k(1)}}{p^{2ms}} + \cdots \right) \\ &= \prod_{p\dagger k} \left(1 + \frac{p^{m(s-1)-1}}{(p^{s-1}-1)(p^{ms}-1)} \right) \prod_{p|k} \left(\frac{1}{1-\frac{1}{p^s}} \right) \\ &= \frac{\zeta(s-1)}{\zeta(m(s-1))} \prod_{p\dagger k} \left(\frac{(p^{s-1}-1)p^{(m-1)(s-1)}}{p^{m(s-1)}-1} + \frac{p^{(m-1)(s-1)}}{(p^{ms}-1)} \right) \end{aligned}$$

$$\times \prod_{p|k} \left(\frac{p^s(p^{m(s-1)} - p^{(m-1)(s-1)})}{(p^s-1)(p^{m(s-1)}-1)} \right), \tag{2}$$

where $\zeta(s)$ is the Riemann zeta-function and $\prod_p$ denotes the product over all prime numbers.

From (1) and Perron's formula [3], we have

$$\sum_{n \leq x} \frac{1}{\delta_k(b_m(n))} \tag{3}$$

$$= \frac{1}{2\pi i} \int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} \frac{\zeta(s-1)}{\zeta(m(s-1))} \prod_{p \dagger k} \left(\frac{(p^{s-1}-1)p^{(m-1)(s-1)}}{(p^{m(s-1)}-1)} + \frac{p^{(m-1)(s-1)}}{(p^{ms}-1)} \right)$$

$$\times \prod_{p|k} \left(\frac{p^s(p^{m(s-1)} - p^{(m-1)(s-1)})}{(p^s-1)(p^{m(s-1)}-1)} \right) \cdot \frac{x^s}{s} ds + O\left(\frac{x^{\frac{3}{2}+\epsilon}}{T} \right), \tag{4}$$

where ϵ is any fixed positive number.

Now we move the integral line in (2) from $s = \frac{5}{2} \pm iT$ to $s = \frac{3}{2} \pm iT$. This time, the function

$$\frac{\zeta(s-1)}{\zeta(m(s-1))} \prod_{p \dagger k} \left(\frac{(p^{s-1}-1)p^{(m-1)(s-1)}}{(p^{m(s-1)}-1)} + \frac{p^{(m-1)(s-1)}}{(p^{ms}-1)} \right)$$

$$\times \prod_{p|k} \left(\frac{p^s(p^{m(s-1)} - p^{(m-1)(s-1)})}{(p^s-1)(p^{m(s-1)}-1)} \right) \cdot \frac{x^s}{s}$$

has a simple pole point at $s = 2$ with residue

$$\frac{x^2}{2\zeta(m)} \prod_{p \dagger k} \frac{p^m(p^m - p^{m-1}+1)}{(p^{2m}-1)} \prod_{p|k} \left(\frac{p^{m+1}}{(p+1)(p^m-1)} \right). \tag{5}$$

Hence, we have

$$\frac{1}{2\pi i} \left(\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT} + \int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} + \int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} \right) \frac{\zeta(s-1)}{\zeta(m(s-1))}$$

$$\times \prod_{p \dagger k} \left(\frac{(p^{s-1}-1)p^{(m-1)(s-1)}}{(p^{m(s-1)}-1)} + \frac{p^{(m-1)(s-1)}}{(p^{ms}-1)} \right) \tag{6}$$

$$\times \prod_{p|k} \left(\frac{p^s(p^{m(s-1)} - p^{(m-1)(s-1)})}{(p^s-1)(p^{m(s-1)}-1)} \right) \cdot \frac{x^s}{s} ds$$

$$= \frac{x^2}{2\zeta(m)} \prod_{p \dagger k} \frac{p^m(p^m - p^{m-1}+1)}{(p^{2m}-1)} \prod_{p|k} \left(\frac{p^{m+1}}{(p+1)(p^m-1)} \right). \tag{7}$$

We can easily get the estimate

$$\left|\frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT} f(s)\cdot\frac{x^s}{s}ds\right| \ll \frac{x^{\frac{5}{2}+\epsilon}}{T}, \tag{8}$$

$$\left|\frac{1}{2\pi i}\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT} f(s)\cdot\frac{x^s}{s}ds\right| \ll \frac{x^{\frac{5}{2}+\epsilon}}{T}, \tag{9}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} f(s)\cdot\frac{x^s}{s}ds\right| \ll x^{\frac{3}{2}+\epsilon}. \tag{10}$$

Taking $T = x$, combining (2), (4), (5), (6) and (7) we may deduce that

$$\begin{aligned}&\sum_{n\le x}\frac{1}{\delta_k(b_m(n))}\\ =\ &\frac{x^2}{2\zeta(m)}\prod_{p\dagger k}\frac{p^m(p^m-p^{m-1}+1)}{(p^{2m}-1)}\prod_{p|k}\left(\frac{p^{m+1}}{(p+1)(p^m-1)}\right)+O\left(x^{\frac{3}{2}+\epsilon}\right).\end{aligned}$$

This completes the proof of Theorem.

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ON THE SMARANDACHE CEIL FUNCTION AND THE DIRICHLET DIVISOR FUNCTION

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Abstract The main purpose of this paper is using the elementary methods to study the mean value properties of the composite function involving Dirichlet divisor function and Smarandache ceil function, and give an interesting asymptotic formula for it.

Keywords: Smarandache Ceil function; Dirichlet divisor function; Mean value; Asymptotic formula.

§1. Introduction

For a fixed positive integer k and any positive integer n, the Smarandache ceil function $S_k(n)$ is defined as following:

$$S_k(n) = \min\{m \in N : n | m^k\}.$$

This function was introduced by Professor Smarandeche (see reference [1]). About this function, many scholars studied its properties (see reference [2] and [3]). In reference [2], Ibstedt presented the following property:

$$(\forall a, b \in N)(a, b) = 1 \Longrightarrow S_k(ab) = S_k(a)S_k(b).$$

That is, $S_k(n)$ is a multiplicative function.

In this paper, we use elementary methods to study the mean value properties of the composite function involving $d(n)$ and $S_k(n)$, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. Let k be a given positive integer with $k \geq 2$. Then for any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{n \leq x} d(S_k(n)) = \frac{6\zeta(k) x \ln x}{\pi^2} \prod_p \left(1 - \frac{1}{p^k + p^{k-1}}\right) + Cx + O(x^{\frac{1}{2}+\varepsilon}).$$

where C is a computable constant, and ε is any fixed positive number.

Taking $k = 2$ and $k = 4$ in Theorem 1, noting that:

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90},$$

we may immediately deduce the following:

Corollary. For any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{n\leq x} d(S_2(n)) = x\ln x \prod_p \left(1 - \frac{1}{p^2+p}\right) + C_1 x + O(x^{\frac{1}{2}+\varepsilon}),$$

$$\sum_{n\leq x} d(S_4(n)) = \frac{\pi^2 x \ln x}{15} \prod_p \left(1 - \frac{1}{p^4+p^3}\right) + C_2 x + O(x^{\frac{1}{2}+\varepsilon}),$$

where C_1, C_2 are computable constants.

If $k = 1$, then function $S_k(n)$ turns into the identical transformation and the composite function $d(S_k(n))$ turns into $d(n)$. We have the following asymptotic formula:

$$\sum_{n\leq x} d(S_1(n)) = \sum_{n\leq x} d(n) = x\ln x + (2\gamma - 1)x + O(x^{\frac{1}{2}}),$$

where γ is the Euler's Constant.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{d(S_k(n))}{n^s}.$$

From the Euler product formula (see reference [5]) and the multiplicative property of $S_k(n)$, we have

$$\begin{aligned}
f(s) &= \prod_p \left(1 + \frac{d(S_k(p))}{p^s} + \frac{d(S_k(p^2))}{p^{2s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{d(p)}{p^s} + \cdots + \frac{d(p)}{p^{ks}} + \frac{d(p^2)}{p^{(k+1)s}} + \cdots + \frac{d(p^2)}{p^{2ks}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{2}{p^s} + \cdots + \frac{2}{p^{ks}} + \frac{3}{p^{(k+1)s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1 - \frac{1}{p^{ks}}}{1 - \frac{1}{p^s}} \times \left(\frac{2}{p^s} + \frac{3}{p^{(k+1)s}} + \cdots\right)\right) \\
&= \zeta(s) \prod_p \left(1 + \frac{1}{p^s} \times \left(1 + \frac{1}{p^{ks}} + \frac{1}{p^{2ks}} + \cdots\right)\right) \\
&= \zeta(s)\zeta(ks) \prod_p \left(1 - \frac{1}{p^{ks}} + \frac{1}{p^s}\right) \\
&= \zeta(s)\zeta(ks) \frac{\zeta(s)}{\zeta(2s)} \prod_p \left(1 - \frac{1}{p^{ks} + p^{(k-1)s}}\right),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. It is obvious that

$$|d(S_k(n))| \leq n, \qquad \left|\sum_{n=1}^{\infty} \frac{d(S_k(n))}{n^\sigma}\right| \leq \frac{1}{\sigma - 1},$$

where $\sigma > 1$ is the real part of s. By Perron formula (see reference [4]), let $s_0 = 0, b = \frac{3}{2}, T = x$, then we have

$$\sum_{n\leq x} d(S_k(n)) = \frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s}ds + O(x^{\frac{1}{2}+\varepsilon}),$$

where $R(s) = \prod_p \left(1 - \frac{1}{p^{ks} + p^{(k-1)s}}\right)$, and ε is any fixed positive number.

Now we calculate the term

$$\frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s}ds.$$

We move the integral line from $\frac{3}{2} \pm iT$ to $\frac{1}{2} \pm iT$. Then the function

$$\frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s}$$

has a second order pole at $s = 1$ with residue

$$\begin{aligned} &\lim_{s\to 1}\left((s-1)^2\frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s}\right)' \\ = &\lim_{s\to 1}\left(\left((s-1)^2\frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\right)'\frac{x^s}{s}\right. \\ &\left. +(s-1)^2\frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{sx^s\ln x - x^s}{s^2}\right) \\ = &\frac{6\zeta(k)x\ln(x)}{\pi^2}\prod_p\left(1 - \frac{1}{p^k + p^{k-1}}\right) + Cx, \end{aligned}$$

where C is a computable constant. So we can obtain

$$\begin{aligned} &\frac{1}{2\pi i}\left(\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s}ds \\ = &\frac{6\zeta(k)x\ln(x)}{\pi^2}\prod_p\left(1 - \frac{1}{p^k + p^{k-1}}\right) + Cx. \end{aligned}$$

Taking $T = x$, we have the estimate

$$\begin{aligned} &\left|\frac{1}{2\pi i}\left(\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s}ds\right| \\ << &\frac{x^{\frac{3}{2}+\varepsilon}}{T} = x^{\frac{1}{2}+\varepsilon} \end{aligned}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)}R(s)\frac{x^s}{s}ds\right| << x^{\frac{1}{2}+\varepsilon}.$$

So we may immediately get the asymptotic formula

$$\sum_{n\leq x} d(S_k(n)) = \frac{6\zeta(k)x\ln(x)}{\pi^2}\prod_p\left(1-\frac{1}{p^k+p^{k-1}}\right)+Cx+O(x^{\frac{1}{2}+\varepsilon}).$$

This completes the proof of Theorem.

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ON A DUAL FUNCTION OF THE SMARANDACHE CEIL FUNCTION

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Abstract Let n and k are any two positive integers, $\overline{S_k}(n)$ denotes the largest positive integer x satisfying $x^k|n$. The main purpose of this paper is using the elementary methods to study the mean value properties of the arithmetical function $d(\overline{S_k}(n))$, where $d(n)$ is the Dirichlet divisor function, and give an interesting asymptotic formula for it.

Keywords: Arithmetical function; Mean value; Asymptotic formula.

§1. Introduction and results

Let k be a fixed positive integer. For any positive integer n, the Smarandache ceil function of order k is denoted by $S_k(n)$ and has the following definition:

$$S_k(n) = \min\{x \in N : n|x^k\}.$$

This arithmetical function is a multiplicative function, and has many interesting properties, so it had be studied by many people, see reference [2]. Now we introduce a dual function of $S_k(n)$ as follows:

$$\overline{S_k}(n) = \max\{x \in N : x^k|n\}.$$

It is clear that $\overline{S_k}(n)$ is also a multiplicative function, but about its arithmetical properties, we know very little at present. In this paper, we use the elementary methods to study the mean value properties of $d(\overline{S_k}(n))$, where $d(n)$ is the Dirichlet divisor function, and obtain a sharper asymptotic formula for it. That is, we will prove the following:

Theorem. Let $k \geq 2$ be a fixed integer. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} d(\overline{S_1}(n)) = x\ln x + (2\gamma - 1)x + O\left(x^{\frac{1}{3}}\right)$$

and

$$\sum_{n\leq x} d(\overline{S_k}(n)) = \zeta(k)x + \zeta\left(\frac{1}{k}\right)x^{\frac{1}{k}} + O\left(x^{\frac{1}{k+1}}\right),$$

where γ is the Euler constant, and $\zeta(n)$ is the Riemann zeta-function.

From this Theorem we may immediately deduce the following

Corollary 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} d(\overline{S_2}(n)) = \frac{\pi^2}{6}x + \zeta\left(\frac{1}{2}\right)x^{\frac{1}{2}} + O\left(x^{\frac{1}{3}}\right).$$

Corollary 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} d(\overline{S_4}(n)) = \frac{\pi^4}{90}x + \zeta\left(\frac{1}{4}\right)x^{\frac{1}{4}} + O\left(x^{\frac{1}{5}}\right).$$

§2. Proof of the theorem

In this section, we will give the proof of the theorem. The following Lemma is necessary.

Lemma. If $x \geq 1$ and $s \geq 0$, $s \neq 1$, we have

$$\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}),$$

where

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{t-[t]}{t^{s+1}}dt.$$

Proof. This Lemma can be easily proved by using the Euler summation formula, see Theorem 3.2(b) of [3].

Now we come to the proof of the theorem. It is obvious that $\overline{S_1}(n) = n$, and this deduces the first part of the theorem immediately by the classical result on Dirichlet divisor problem, see reference [4]. Now assume $k \geq 2$, we have

$$\sum_{n\leq x} d(\overline{S_k}(n)) = \sum_{n\leq x}\sum_{d|\overline{S_k}(n)} 1,$$

from the definition of $\overline{S_k}(n)$ we know that $d|\overline{S_k}(n) \Leftrightarrow d^k|n$, hence

$$\sum_{n\leq x} d(\overline{S_k}(n)) = \sum_{n\leq x}\sum_{d^k|n} 1 = \sum_{d^k l\leq x} 1.$$

Let $\delta = x^{\frac{1}{k+1}}$, applying the above formula and Theorem 3.17 of [3] we have

$$\sum_{n\leq x} d(\overline{S_k}(n)) = \sum_{d^k l\leq x} 1$$

$$
\begin{aligned}
&= \sum_{1\le d^k\le \delta^k}\sum_{1\le l\le x/d^k} 1+\sum_{1\le l\le \delta}\sum_{1\le d^k\le x/l} 1-\sum_{1\le d^k\le \delta^k} 1\sum_{1\le l\le \delta} 1\\
&= \sum_{1\le d\le \delta}\sum_{1\le l\le x/d^k} 1+\sum_{1\le l\le \delta}\sum_{1\le d^k\le x/l} 1-[\delta]^2\\
&= \sum_{1\le d\le \delta}\left[\frac{x}{d^k}\right]+\sum_{1\le l\le \delta}\left[\left(\frac{x}{l}\right)^{1/k}\right]-(\delta^2-2\delta\{\delta\}+\{\delta\}^2).
\end{aligned}
$$

Using the above Lemma we get

$$
\begin{aligned}
&\sum_{n\le x} d(\overline{S_k}(n))\\
=\ & x\sum_{1\le d\le \delta}\frac{1}{d^k}+x^{1/k}\sum_{1\le l\le \delta}\frac{1}{l^{1/k}}+O(\delta)-(\delta^2+O(\delta))\\
=\ & x\left(\frac{\delta^{1-k}}{1-k}+\zeta(k)+O(\delta^{-k})\right)+x^{1/k}\left(\frac{\delta^{1-\frac{1}{k}}}{1-\frac{1}{k}}+\zeta\left(\frac{1}{k}\right)+O(\delta^{-\frac{1}{k}})\right)\\
&-\delta^2+O(\delta).
\end{aligned}
$$

Notice that $x=\delta^{k+1}$, we have

$$
\begin{aligned}
&\sum_{n\le x} d(\overline{S_k}(n))\\
=\ & \frac{\delta^2}{1-k}+\zeta(k)x+\frac{\delta^2}{1-\frac{1}{k}}+\zeta\left(\frac{1}{k}\right)x^{\frac{1}{k}}-\delta^2+O(\delta)\\
=\ & \zeta(k)x+\zeta\left(\frac{1}{k}\right)x^{\frac{1}{k}}+O(\delta)\\
=\ & \zeta(k)x+\zeta\left(\frac{1}{k}\right)x^{\frac{1}{k}}+O(x^{\frac{1}{k+1}}).
\end{aligned}
$$

This completes the proof of Theorem.

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ON THE LARGEST M-TH POWER NOT EXCEEDING N^*

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Abstract In this paper, some elementary methods are used to study the properties of the largest m-th power not exceeding n, and give an identity about it.

Keywords: The largest m-th power not exceding n ; Dirichlet series; Identity.

§1. Introduction and results

Let n is a positive integer. It is clear that there exists an integer k such that

$$k^m \leq n < (k+1)^m.$$

Now we define $b_m(n) = k^m$. That is, $b_m(n)$ is the largest m-th power not exceeding n. In problems 40 and 41 of reference [1], Professor F.Smarandache asked us to study the properties of the sequences $\{b_2(n)\}$ and $\{b_3(n)\}$. About these problems, some people had studied them, and obtained some interesting results, see references [2] and [3]. In this paper, we using the elementary methods to study the convergent properties of the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{1}{{b_m}^s(n)}$, and give an interesting identity. That is, we shall prove the following:

Theorem. *Let m be a fixed positive integer. Then for any real number $s > 1$, the Dirichlet series $f(s)$ is convergent and*

$$f(s) = C_m^1\zeta(ms-m+1)+C_m^2\zeta(ms-m+2)+\cdots+C_m^{m-1}\zeta(ms-1)+\zeta(ms),$$

where $C_m^n = \frac{m!}{n!(m-n)!}$, and $\zeta(s)$ is the Riemann zeta-function.

From this theorem we may immediately deduce the following:

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Corollary 1. *Taking $m = 2$ and $s = 3/2$ or $m = s = 2$ in the above Theorem, then we have the identities*

$$\sum_{n=1}^{\infty} \frac{1}{{b_2}^{\frac{3}{2}}(n)} = \frac{\pi^2}{3} + \zeta(3) \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{1}{{b_2}^2(n)} = 2\zeta(3) + \frac{\pi^4}{90}.$$

Corollary 2. *Taking $m = 3$ and $s = 2$ or $m = 2$ and $s = 3$ in the above theorem, we have the identities*

$$\sum_{n=1}^{\infty} \frac{1}{{b_3}^2(n)} = \frac{\pi^4}{30} + 3\zeta(5) + \frac{\pi^6}{945} \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{1}{{b_2}^3(n)} = 2\zeta(5) + \frac{\pi^6}{945}.$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. For any positive integer n, let $b_m(n) = k^m$. It is clear that there are exactly $(k+1)^m - k^m$ integer n such that $b_m(n) = k^m$. So we may get

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{1}{b_m^s(n)} = \sum_{k=1}^{\infty} \frac{(k+1)^m - k^m}{k^{ms}} \\ &= \sum_{k=1}^{\infty} \frac{C_m^1 k^{m-1} + C_m^2 k^{m-2} + \cdots + C_m^{m-1} k + 1}{k^{ms}}. \end{aligned}$$

From the integral criterion, we know that $f(s)$ is convergent if $ms-(m-1) > 1$. That is, $s > 1$. If $s > 1$, note that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, we have

$$f(s) = C_m^1 \zeta(ms - m + 1) + C_m^2 \zeta(ms - m + 2) + \cdots + C_m^{m-1} \zeta(ms - 1) + \zeta(ms).$$

This completes the proof of Theorem.
It's easy to compute that

$$\begin{aligned} f(2) &= C_m^1 \zeta(m+1) + C_m^2 \zeta(m+2) + \cdots + C_m^{m-1} \zeta(2m-1) + \zeta(2m), \\ f(3) &= C_m^1 \zeta(2m+1) + C_m^2 \zeta(2m+2) + \cdots + C_m^{m-1} \zeta(3m-1) + \zeta(3m). \end{aligned}$$

Now the corollaries follows from $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$ (see reference [4]) and the above theorem.

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ON THE SMARANDACHE BACK CONCATENATED ODD SEQUENCES *

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Abstract The main purpose of this paper is to study the arithmetical properties of the Smarandache back concatenated odd sequences, and give several simple properties involving the recursion formula and exact expressions for the general term.

Keywords: The Smarandache back concatenated odd sequence; Arithmetical properties; Recursion formula.

§1. Introduction and main results

The famous Smarandache back concatenated odd sequence $\{b_n\}$ is defined as following $1,\ 31,\ 531, 7531, 97531, 1197531, 131197531, 15131197531, 1715131197531, \cdots$. In problem 3 of reference [1], Professor Mihaly Bencze and Lucian Tutescu asked us to study the arithmetical properties about this sequence. It is interesting for us to study this problem. But it's a pity that none had studied it before. At least we haven't seen such a paper yet. In this paper, we shall use the elementary methods to study the arithmetical properties of the Smarandache back concatenated odd sequences, and give several simple properties involving the recursion formula, exact expressions for the general term, and so on. That is, we shall prove the following:

Theorem 1. *Let $n \geq 2$ be any positive integer with $(2n-3)$ has k digits. Then for the Smarandache back concatenated odd sequences $\{b_n\}$, we have the following recursion formula*

$$b_n = b_{n-1} + (2n-1) \times 10^{\frac{10-10^k}{18} + k \times (n-1)},$$

where $b_1 = 1$.

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Theorem 2. *Let $(2n-3)$ has k digits. Then the b_{n-1}-th term in the Smarandache back concatenated odd sequences $\{b_n\}$ has $\frac{10-10^k}{18}+k\times(n-1)$ digits.*

Theorem 3. *Let $S_{k(m,n)} = \sum_{i=m}^{n}(2i-1)\times 10^{k(i-1)}$, then we have the following exact expression for the general term. That is,*

$$b_n = 1+S_{1(2,6)}+10^{-5}\times S_{2(7,51)}+10^{-55}\times S_{3(52,501)}+\cdots+10^{\overbrace{-5\cdots5}^{-(k-1)}}\times S_{k(m,n)}.$$

Theorem 4. *Let S_{50} denotes the summation for the first 50 terms in the Smarandache back concatenated odd sequences $\{b_n\}$, then we have*

$$\begin{aligned} S_{50} &= \frac{11\times 45\times 10^6-201\times 50}{9}+\frac{-81\times 10^6+970}{9^2} \\ &\quad +\frac{-4\times 10^6+4\times 100}{9^3}+\frac{99\times 10^{95}-13\times 44\times 10^7}{99} \\ &\quad +\frac{95\times 10^{95}+73\times 10^9}{99^2}+\frac{-4\times 10^{95}+4\times 10^{11}}{99^3}. \end{aligned}$$

§2. Some lemmas

Lemma 1. *Let k, m, n are positive integers with $m \le n$, then we have*

$$S_{k(m,n)} = \frac{(2m-1)\times 10^{k(m-1)}}{1-10^k}+2\times\frac{10^{km}[1-10^{k(n-m)}]}{(1-10^k)^2}+\frac{(2n-1)\times 10^{kn}}{1-10^k}.$$

Proof. From the definition of $S_{k(m,n)}$, we have

$$\begin{aligned} S_{k(m,n)} &= \sum_{i=m}^{n}(2i-1)\times 10^{k(i-1)} \\ &= (2m-1)\times 10^{k(m-1)}+(2m+1)\times 10^{km} \\ &\quad +\cdots+(2n-1)\times 10^{k(n-1)} \end{aligned}$$

and

$$10^k S_{k(m,n)} = (2m-1)\times 10^{km}+(2m+1)\times 10^{k(m+1)}+\cdots+(2n-1)\times 10^{kn}.$$

Thus we can get

$$(1-10^k)S_{k(m,n)} = (2m-1)\times 10^{k(m-1)}+2\times[10^{km}+10^{k(m+1)}$$

$$
\begin{aligned}
& +\cdots+10^{k(n-1)}]-(2n-1)\times 10^{kn}\\
= \; & (2m-1)\times 10^{k(m-1)}+2\times\frac{10^{km}[1-10^{k(n-m)}]}{1-10^{k}}\\
& -(2n-1)\times 10^{kn}.
\end{aligned}
$$

So we have

$$S_{k(m,n)}=\frac{(2m-1)\times 10^{k(m-1)}}{1-10^{k}}+2\times\frac{10^{km}[1-10^{k(n-m)}]}{(1-10^{k})^{2}}+\frac{(2n-1)\times 10^{kn}}{1-10^{k}}.$$

This proves Lemma 1.

Lemma 2. *Let* $S'_{k(n,m)}=\sum_{i=m}^{n} i^2\times 10^{-ki}$. *Then for any positive integers* k, m, n, *we have*

$$
\begin{aligned}
S'_{k(m,n)} \;=\; & \frac{m^2\times 10^{-km}}{1-10^{-k}}+\frac{(2m+1)\times 10^{-k\times(m+1)}}{(1-10^{-k})^{2}}\\
& +2\times\frac{10^{-k(m+2)}[1-10^{-k(n-m-1)}]}{(1-10^{-k})^{3}}\\
& -\frac{(2n-1)\times 10^{-k\times(n+1)}}{(1-10^{-k})^{2}}-\frac{n^2\times 10^{-k(n+1)}}{1-10^{-k}}.
\end{aligned}
$$

Proof. From the definition of $S'_{k(m,n)}$, we have

$$
\begin{aligned}
& S'_{k(m,n)}\\
= \; & m^2\times 10^{-k\times m}+(m+1)^2\times 10^{-k\times(m+1)}+(m+2)^2\times 10^{-k\times(m+2)}\\
& +\cdots+[m+(n-m)]^2\times 10^{-k[m+(n-m)]}
\end{aligned}
$$

and

$$
\begin{aligned}
10^{-k}S'_{k(m,n)} \;=\; & m^2\times 10^{-k\times(m+1)}+(m+1)^2\times 10^{-k\times(m+2)}\\
& +\cdots+[m+(n-m)]^2\times 10^{-k[m+(n-m)+1]}.
\end{aligned}
$$

Thus we can get

$$
\begin{aligned}
& (1-10^{-k})S'_{k(m,n)}\\
& = m^2\times 10^{-k\times m}+(2m+1)\times 10^{-k\times(m+1)}+(2m+3)\times 10^{-k\times(m+2)}\\
& \quad +\cdots+\{2[m+(n-m)]-1\}\times 10^{-k[m+(n-m)]}\\
& \quad -[m+(n-m)]^2\times 10^{-k[m+(n-m)]+1}.
\end{aligned}
$$

So we have

$$\begin{aligned}
S'_{k(m,n)} &= \frac{m^2 \times 10^{-km}}{1-10^{-k}} + \frac{\sum\limits_{i=m+1}^{n} (2i-1) \times 10^{-k\times i}}{1-10^{-k}} + \frac{n^2 \times 10^{-k(n+1)}}{1-10^{-k}} \\
&= \frac{m^2 \times 10^{-km}}{1-10^{-k}} + \frac{(2m+1) \times 10^{-k\times(m+1)}}{(1-10^{-k})^2} \\
&\quad +2 \times \frac{10^{-k(m+2)}[1-10^{-k(n-m-1)}]}{(1-10^{-k})^3} \\
&\quad -\frac{(2n-1) \times 10^{-k\times(n+1)}}{(1-10^{-k})^2} - \frac{n^2 \times 10^{-k(n+1)}}{1-10^{-k}}.
\end{aligned}$$

This proves Lemma 2.

Lemma 3. *Let* $S''_{k(n,m)} = \sum\limits_{i=m}^{n} i \times 10^{-ki}$. *Then for any positive integers* $k, m, n,$ *we have*

$$S''_{k(m,n)} = \frac{m^2 \times 10^{-km}}{1-10^{-k}} + \frac{10^{-k(m+1)}[1-10^{-k(n-m)}]}{(1-10^{-k})^2} - \frac{n \times 10^{-k(n+1)}}{(1-10^{-k})}.$$

Proof. From the definition of $S''_{k(m,n)}$, we have

$$\begin{aligned}
S''_{k(m,n)} &= m \times 10^{-km} + (m+1) \times 10^{-k(m+1)} \\
&\quad + \cdots + [m+(n-m)] \times 10^{-k[m+(n-m)]}
\end{aligned}$$

and

$$\begin{aligned}
10^{-k} S''_{k(m,n)} &= m \times 10^{-k(m+1)} + (m+1) \times 10^{-k(m+2)} \\
&\quad + \cdots + [m+(n-m)] \times 10^{-k[m+(n-m)+1]}.
\end{aligned}$$

Thus we can get

$$\begin{aligned}
(1-10^{-k}) S''_{k(m,n)} &= m \times 10^{-km} + 10^{-k(m+1)} + \cdots + 10^{-k[m+(n-m)]} \\
&\quad -[m+(n-m)] \times 10^{-k[m+(n-m)+1]}.
\end{aligned}$$

So we have

$$S''_{k(m,n)} = \frac{m \times 10^{-km}}{1-10^{-k}} + \times \frac{10^{-k(m+1)}[1-10^{-k(n-m)}]}{(1-10^{-k})^2} - \frac{n \times 10^{-k(n+1)}}{(1-10^{-k})}.$$

This proves Lemma 3.

§3. Proof of the theorems

In this section, we shall use the above lemmas to complete the proof of the theorems. First we use mathematical induction to prove Theorem 1.

(i) If $n = 2, 3$, it is obvious that Theorem 1 is true.

(ii) Assuming that Theorem 1 is true for $n = m$. That is, the following recursion formula

$$b_m = b_{m-1} + (2m-1) \times 10^{\frac{10-10^k}{18} + k\times(m-1)}$$

is true for the Smarandache back concatenated odd sequence $\{b_n\}$.

Comparing the difference between b_m and b_{m-1}, we know that b_{m-1} has $\frac{10-10^k}{18} + k \times (m-1)$ digits. If $n = m+1$, we can divide it into two cases:

(i) If $2m-1$ still has k digits, then we know that b_m has $[\frac{10-10^k}{18} + k \times (m-1) + k]$ digits. Comparing the difference between b_{m+1} and b_m, we may immediately deduce that

$$\frac{b_{m+1} - b_m}{2m+1} = 10^{\frac{10-10^k}{18} + k\times(m-1)+k}.$$

That is,

$$b_{m+1} = b_m + (2m+1) \times 10^{\frac{10-10^k}{18} + k\times m}.$$

(ii) If $2m-1$ has $k+1$ digits, then we know that b_m has $[\frac{10-10^k}{18} + k \times (m-1) + k + 1]$ digits. Recall that $(2m-3)$ has k digits and $(2m-1)$ has $k+1$ digits, which exist only in the case that

$$2m-3 = 10^k - 1, 2m-1 = 10^k + 1.$$

That is, $m = \frac{10^k}{2} + 1$. Therefore, we have

$$\begin{aligned}\frac{10-10^k}{18} + k \times (m-1) + (k+1) &= \frac{10-10^k}{18} + k \times \frac{10^k}{2} + (k+1)\\ &= \frac{10-10^{k+1}}{18} + (k+1) \times m\end{aligned}$$

Comparing the difference between b_{m+1} and b_m, we can deduce that

$$\frac{b_{m+1} - b_m}{2m+1} = 10^{\frac{10-10^k}{18} + k\times(m-1)+(k+1)} = 10^{\frac{10-10^{k+1}}{18} + (k+1)\times m}.$$

That is,

$$b_{m+1} = b_m + (2m+1) \times 10^{\frac{10-10^{k+1}}{18} + (k+1)\times m}.$$

Combining (i) and (ii), Theorem 1 is true for any positive integer n. This completes the proof of Theorem 1.

By using the result of Theorem 1, and note that the difference between b_n and b_{n-1}, we can immediately get the result of Theorem 2.

By Theorem 1, when $(2n-3)$ has k digits, let $(2m-3)$ be the least positive integer which has k odd digits (here $m=2$ when $k=1$, and $m=\frac{10^{k-1}}{2}+2$). Then we have

$$\begin{aligned}
b_n &= 1+(2\times 2-1)\times 10^{\frac{10-10^1}{18}+1\times(2-1)} \\
&\quad +(2\times 3-1)\times 10^{\frac{10-10^1}{18}+1\times(3-1)} \\
&\quad +(2\times 4-1)\times 10^{\frac{10-10^1}{18}+1\times(4-1)}+(2\times 5-1)\times 10^{\frac{10-10^1}{18}+1\times(5-1)} \\
&\quad +(2\times 6-1)\times 10^{\frac{10-10^1}{18}+1\times(6-1)}+(2\times 7-1)\times 10^{\frac{10-10^2}{18}+2\times(7-1)} \\
&\quad +(2\times 8-1)\times 10^{\frac{10-10^2}{18}+2\times(8-1)} \\
&\quad +\cdots+(2\times 51-1)\times 10^{\frac{10-10^2}{18}+2\times(51-1)} \\
&\quad +(2\times 52-1)\times 10^{\frac{10-10^3}{18}+3\times(52-1)} \\
&\quad +(2\times 53-1)\times 10^{\frac{10-10^3}{18}+3\times(53-1)} \\
&\quad +\cdots+(2\times m-1)\times 10^{\frac{10-10^k}{18}+k\times(m-1)} \\
&\quad +\cdots+(2\times n-1)\times 10^{\frac{10-10^k}{18}+k\times(n-1)}+1+10^{\frac{10-10^1}{18}}\times S_{1(2,6)} \\
&\quad +10^{\frac{10-10^2}{18}}\times S_{2(7,51)}+10^{\frac{10-10^3}{18}}\times S_{3(52,501)} \\
&\quad +\cdots+10^{\frac{10-10^k}{18}}\times S_{k(m,n)} \\
&\quad +1+10^{\frac{10-10^1}{18}}\times S_{1(2,6)}+10^{\frac{10-10^2}{18}}\times S_{2(7,51)}+10^{\frac{10-10^3}{18}}\times S_{3(52,501)} \\
&\quad +\cdots+10^{\frac{10-10^k}{18}}\times S_{k(m,n)} \\
&= 1+S_{1(2,6)}+10^{-5}\times S_{2(7,51)}+10^{-55}\times S_{3(52,501)} \\
&\quad +\cdots+10^{\overbrace{-5\cdots5}^{-(k-1)}}\times S_{k(m,n)}.
\end{aligned}$$

Applying the result of Lemma 1, we can deduce Theorem 3.

Now we come to prove Theorem 4. By using the result of Theorem 1, we have

$$\begin{aligned}
S_{50} &= b_1+b_2+b_3+\cdots+b_{49}+b_{50} \\
&= 50+49\times(2\times 2-1)\times 10^{\frac{10-10^1}{18}+1\times(2-1)} \\
&\quad +48\times(2\times 3-1)\times 10^{\frac{10-10^1}{18}+1\times(3-1)} \\
&\quad +47\times(2\times 4-1)\times 10^{\frac{10-10^1}{18}+1\times(4-1)} \\
&\quad +46\times(2\times 5-1)\times 10^{\frac{10-10^1}{18}+1\times(5-1)} \\
&\quad +45\times(2\times 6-1)\times 10^{\frac{10-10^1}{18}+1\times(6-1)}
\end{aligned}$$

$$
\begin{aligned}
& +44 \times (2 \times 7 - 1) \times 10^{\frac{10-10^2}{18} + 2\times(7-1)} \\
& +43 \times (2 \times 8 - 1) \times 10^{\frac{10-10^2}{18} + 2\times(8-1)} \\
& + \cdots + 3 \times (2 \times 48 - 1) \times 10^{\frac{10-10^2}{18} + 2\times(48-1)} \\
& +2 \times (2 \times 49 - 1) \times 10^{\frac{10-10^2}{18} + 2\times(49-1)} \\
& +1 \times (2 \times 50 - 1) \times 10^{\frac{10-10^2}{18} + 2\times(50-1)} \\
= \; & 50 \times [2 \times (51 - 50) - 1] \times 10^{\frac{10-10^1}{18} + 1\times(50-50)} \\
& +49 \times [2 \times (51 - 49) - 1] \times 10^{\frac{10-10^1}{18} + 1\times(50-49)} \\
& +48 \times [2 \times (51 - 48) - 1] \times 10^{\frac{10-10^1}{18} + 1\times(50-48)} \\
& +47 \times [2 \times (50 - 47) - 1] \times 10^{\frac{10-10^1}{18} + 1\times(50-47)} \\
& + \cdots + 45 \times [2 \times (51 - 45) - 1] \times 10^{\frac{10-10^1}{18} + 1\times(51-45)} \\
& +44 \times [2 \times (50 - 44) - 1] \times 10^{\frac{10-10^2}{18} + 2\times(50-44)} \\
& + \cdots + 2 \times [2 \times (51 - 2) - 1] \times 10^{\frac{10-10^2}{18} + 2\times(50-2)} \\
& +1 \times [2 \times (51 - 1) - 1] \times 10^{\frac{10-10^2}{18} + 2\times(50-1)} \\
= \; & 101 \times [50 \times 10^{\frac{10-10^1}{18} + 1\times 50 - 1\times 50} + 49 \times 10^{\frac{10-10^1}{18} + 1\times 50 - 1\times 49} \\
& + \cdots + 45 \times 10^{\frac{10-10^1}{18} + 1\times 50 - 1\times 45} \\
& +44 \times 10^{\frac{10-10^2}{18} + 2\times 50 - 2\times 44} \\
& + \cdots + 2 \times 10^{\frac{10-10^2}{18} + 2\times 50 - 2\times 2} \\
& +1 \times 10^{\frac{10-10^2}{18} + 2\times 50 - 2\times 1}] \\
& -2 \times [50^2 \times 10^{\frac{10-10^1}{18} + 1\times 50 - 1\times 50} \\
& +49^2 \times 10^{\frac{10-10^1}{18} + 1\times 50 - 1\times 49} \\
& + \cdots + 45^2 \times 10^{\frac{10-10^1}{18} + 1\times 50 - 1\times 45} \\
& +44^2 \times 10^{\frac{10-10^2}{18} + 2\times 50 - 2\times 44} \\
& + \cdots + 2^2 \times 10^{\frac{10-10^2}{18} + 2\times 50 - 2\times 2} \\
& +1^2 \times 10^{\frac{10-10^2}{18} + 2\times 50 - 2\times 1}] \\
= \; & 101 \times \sum_{i=45}^{50} i \times 10^{\frac{10-10^1}{18} + 1\times 50 - 1\times i} \\
& +101 \times \sum_{i=1}^{44} i \times 10^{\frac{10-10^2}{18} + 2\times 50 - 2\times i}
\end{aligned}
$$

$$
\begin{aligned}
& -2 \times \sum_{i=45}^{50} i^2 \times 10^{\frac{10-10^1}{18}+1\times 50-1\times i} \\
& -2 \times \sum_{i=1}^{44} i^2 \times 10^{\frac{10-10^2}{18}+2\times 50-2\times i} \\
= \quad & 101 \times 10^{50} \times \sum_{i=45}^{50} i \times 10^{-i} + 101 \times 10^{95} \times \sum_{i=1}^{44} i \times 10^{-2\times i} \\
& -2 \times 10^{50} \times \sum_{i=45}^{50} i^2 \times 10^{-i} - 2 \times 10^{95} \times \sum_{i=1}^{44} i^2 \times 10^{-2\times i}.
\end{aligned}
$$

Applying the results of Lemma 2 and Lemma 3, we can get

$$
\begin{aligned}
S_{50} \quad = \quad & \frac{11 \times 45 \times 10^6 - 201 \times 50}{9} + \frac{-81 \times 10^6 + 970}{9^2} \\
& + \frac{-4 \times 10^6 + 4 \times 100}{9^3} + \frac{99 \times 10^{95} - 13 \times 44 \times 10^7}{99} \\
& + \frac{95 \times 10^{95} + 73 \times 10^9}{99^2} + \frac{-4 \times 10^{95} + 4 \times 10^{11}}{99^3}.
\end{aligned}
$$

This completes the proof of Theorem 4.

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ON THE ADDITIVE HEXAGON NUMBERS COMPLEMENTS*

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Abstract In this paper, similar to the Smarandache k-th power complements, we defined the hexagon numbers complements. Using the elementary method, we studied the mean value properties of the additive hexagon numbers complements, and obtained some interesting asymptotic formulae for it.

Keywords: Additive hexagon numbers complements; Mean value; Asymptotic formula.

§1. Introduction and results

Let n be a positive integer. If there exists a positive integer m such that $n = m(2m-1)$, then we call n as a hexagon number. For any positive integer n, the Smarandache k-th power complements $b_k(n)$ is the smallest positive integer such that $nb_k(n)$ is complete k-th power, see problem 29 of [1]. Similar to the Smarandache k-th power complements, we define the additive hexagon numbers complements $a(n)$ as follows: $a(n)$ is the smallest nonnegative integer such that $a(n) + n$ is a hexagon number. For example, if $n = 1, 2, \cdots 15$, we have the additive hexagon number sequences $\{a(n)\}$ $(n = 1, 2, \cdots 15)$ as follows: $0, 4, 3, 2, 1, 0, 8, 7, 6, 5, 4, 3, 2, 1, 0$. In this paper, we study the mean value properties of the composite arithmetic function $d(a(n))$ (where $d(n)$ is the Dirichlet divisor function), and give some interesting asymptotic formulae for it. That is, we shall prove the following conclusions:

Theorem 1. *For any real number $x \geq 3$, we have the asymptotic formula:*

$$\sum_{n\leq x} a(n) = \frac{2\sqrt{2}}{3} x^{\frac{3}{2}} + O(x).$$

Theorem 2. *For any real number $x \geq 3$, we have the asymptotic formula:*

$$\sum_{n\leq x} d(a(n)) = \frac{1}{2} x \log x + \left(\frac{3}{2}\log 2 + (2\gamma - 1) - \frac{1}{2}\right) x + O(x^{\frac{2}{3}}),$$

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where γ is the Euler constant.

§2. Some Lemmas

Before the proof of the theorems, some Lemmas will be useful.

Lemma 1. *For any real number $x \geq 3$, we have the asymptotic formula:*

$$\sum_{n \leq x} d(x) = x \ln x + (2\gamma - 1)x + O\left(x^{\frac{1}{3}}\right),$$

where γ is the Euler constant.

Proof. See reference [2].

Lemma 2. *For any real number $x \geq 3$ and any nonnegative arithmetical function $f(n)$ with $f(0) = 0$, we have the asymptotic formula:*

$$\sum_{n \leq x} f(a(n)) = \sum_{m=1}^{\left[\frac{x^{\frac{1}{2}}}{2}\right]} \sum_{i \leq 4m} f(i) + O\left(\sum_{i \leq 4\left[\frac{x^{\frac{1}{2}}}{2}\right]} f(i)\right),$$

where $[x]$ denotes the greatest integer less than or equal to x.

Proof. For any real number $x \geq 1$, let M be a fixed positive integer such that

$$M(2M-1) \leq x < (M+1)(2M+1).$$

Noting that if n pass through the integers in the interval

$$[m(2m-1), (m+1)(2m+1)),$$

then $a(n)$ pass through the integers in the interval $[0, 4m]$ and $f(0) = 0$, we can write

$$\begin{aligned} \sum_{n \leq x} f(a(n)) &= \sum_{n \leq M(2M-1)} f(a(n)) + \sum_{M(2M-1) < n \leq x} f(a(n)) \\ &= \sum_{m=1}^{M} \sum_{i \leq 4m} f(i) + O\left(\sum_{i \leq x - M(2M-1)} f(i)\right), \end{aligned}$$

Since $x - M(2M-1) < (M+1)(2M+1) - M(2M-1) = 4M+1$ and $M = \left[\frac{x^{\frac{1}{2}}}{2}\right]$, we have

$$\sum_{n \leq x} f(a(n)) = \sum_{m=1}^{\left[\frac{x^{\frac{1}{2}}}{2}\right]} \sum_{i \leq 4m} f(i) + O\left(\sum_{i \leq 4\left[\frac{x^{\frac{1}{2}}}{2}\right]} f(i)\right).$$

This proves Lemma 2.

Note: This Lemma is very useful. Because if we have the mean value formula of $f(n)$, then from this Lemma, we can easily get the mean value formula of $\sum_{n\le x} f(a(n))$.

§3. Proof of the theorems

In this section, we will complete the proof of the theorems. First, we prove Theorem 1. From the definition of $a(n)$, and the Euler summation formula (see [3]), we have

$$\begin{aligned}\sum_{n\le x} a(n) &= \sum_{n\le M(2M-1)} a(n) + \sum_{M(2M-1)<n\le x} a(n)\\ &= \sum_{m=1}^{M}\sum_{i\le 4m} i + \sum_{i\le x-M(2M-1)} i\\ &= \sum_{m=1}^{M} 2m(4m+1) + O\left(\frac{(4M)^2}{2}\right)\\ &= \frac{4}{3}M(M+1)(2M+1) + O(M^2)\\ &= \frac{2\sqrt{2}}{3}x^{\frac{3}{2}} + O(x).\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. From Lemma 2, Lemma 1 and the Abel's identity (see [3]), we have

$$\begin{aligned}&\sum_{n\le x} d(a(n))\\ &= \sum_{m=1}^{M}\sum_{i\le 4m} d(i) + O\left(\sum_{i\le 4M} d(i)\right)\\ &= \sum_{m=1}^{M}\left(4m\log 4m + (2\gamma-1)4m + O\left((4m)^{\frac{1}{3}}\right)\right)\\ &\quad +O\left(4M\log 4M + (2\gamma-1)4M + O\left((4M)^{\frac{1}{3}}\right)\right)\\ &= (8\log 2 + 4(2\gamma-1))\sum_{m\le M} m + 4\sum_{m\le M} m\log m\\ &\qquad +O\left(\sum_{m\le M} m^{\frac{1}{3}}\right) + O\left(4M\log 4M\right)\\ &= (8\log 2 + 4(2\gamma-1))\left(\frac{1}{2}M^2 + O(M)\right)\end{aligned}$$

$$
\begin{aligned}
& +4\left(\frac{1}{2}M^2\log M - \frac{1}{4}(M^2-1) + O(M\log M)\right) + O(M^{\frac{4}{3}}) \\
= \quad & 2M^2\log M + (4\log 2 + 2(2\gamma-1)-1)M^2 + O(M^{\frac{4}{3}}) \\
= \quad & \frac{1}{2}x\log x + \left(\frac{3}{2}\log 2 + (2\gamma-1) - \frac{1}{2}\right)x + O(x^{\frac{2}{3}}).
\end{aligned}
$$

This completes the proof of the theorems.

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ON THE MEAN VALUE OF A NEW ARITHMETICAL FUNCTION*

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Abstract The main purpose of this paper is using the elementary methods to study the mean value properties of a new arithmetical function $\sigma(S(n))$, and give an interesting mean value formula for it.

Keywords: Smarandache function; Mean value; Asymptotic formula.

§1. Introduction and results

For any positive integer n, the Smarandache function $S(n)$ is defined as follows:

$$S(n) = \min\{m \in N : n|m!\}.$$

This function was introduced by American-Romanian number theorist Professor F.Smarandache, see reference [1]. About its arithmetical properties, many scholars had studied it, and obtained some interesting conclusions, see reference [2] and [3].

In this paper, we shall use the elementary methods to study the mean value properties of a new arithmetical function $\sigma(S(n))$, where $\sigma(n) = \sum_{d|n} d$, and give an interesting mean value formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \geq 3$, we have the asymptotic formula*

$$\sum_{n \leq x} \sigma(S(n)) = \frac{\pi^2}{12} \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

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§2. Some simple lemmas

To complete the proof of Theorem, we need some simple Lemmas. For convenience, we denotes the greatest prime divisor of n by $p(n)$. Then we have

Lemma 1. *If n is a square free number or $p(n) > \sqrt{n}$, then $S(n) = p(n)$.*
Proof. If n be a square-free number, let $n = p_1 p_2 \cdots p_r p(n)$, then $p_i < p(n)$ for $i = 1, 2, \cdots, r$. Thus

$$p_i | p(n)!, \qquad i = 1, 2, \cdots, r.$$

So $n | p(n)!$, but $p(n) \dagger (p(n) - 1)!$ and $n \dagger (p(n) - 1)!$. In this case, we have $S(n) = p(n)$;
If $p(n) > \sqrt{n}$, then $p^2(n) \dagger n$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p(n)$, so we have

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} < \sqrt{n} < p(n).$$

It is clear that

$$p_i^{\alpha_i} | p(n)!, \qquad i = 1, 2, \cdots, r.$$

So $n | (p(n))!$, but $p(n) \dagger (p(n) - 1)!$, in this case, we can also deduce that $S(n) = p(n)$.
This proves Lemma 1.

Lemma 2. *Let p be a prime, then we have the asymptotic formula*

$$\sum_{\sqrt{x} \le p \le x} p = \frac{x^2}{2 \ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Proof. Let $\pi(x)$ denotes the number of the primes up to x. Noting that (see [4])

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right),$$

from the Abel's summation formula [5], we have

$$\begin{aligned} \sum_{\sqrt{x} \le p \le x} p &= \pi(x)x - \pi(\sqrt{x})\sqrt{x} - \int_{\sqrt{x}}^{x} \pi(t)dt \\ &= \frac{x^2}{\ln x} - \frac{x^2}{2 \ln x} + O\left(\frac{x^2}{\ln^2 x}\right) \\ &= \frac{x^2}{2 \ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \end{aligned}$$

This proves Lemma 2.

§3. Proof of the theorem

In this section, we shall complete the proof of Theorem. First we define the sets $\mathcal{A}$ and $\mathcal{B}$ as following:

$$\mathcal{A} = \{n | n \le x, p(n) \le \sqrt{n}\} \ \ and \ \ \mathcal{B} = \{n | n \le x, p(n) > \sqrt{n}\}.$$

Note that $S(n) \ll p(n)\ln n$ and $\sigma(n) \ll n \ln\ln n$, we have the estimate

$$\begin{aligned} \sum_{n\in\mathcal{A}} \sigma(S(n)) &\ll \sum_{n\in\mathcal{A}} S(n)\ln\ln(S(n)) \\ &\ll \sum_{n\le x} \sqrt{n}\ln n \ln\ln n \ll x^{\frac{3}{2}}\ln x \ln\ln x. \end{aligned} \tag{1}$$

Then using the above Lemmas we may immediately get

$$\begin{aligned} \sum_{n\in\mathcal{B}} \sigma(S(n)) &= \sum_{\substack{n\le x \\ p(n)>\sqrt{n}}} \sigma(p(n)) = \sum_{n\le\sqrt{x}} \sum_{\sqrt{n}\le p\le\frac{x}{n}} \sigma(p) \\ &= \sum_{n\le\sqrt{x}} \sum_{\sqrt{x}\le p\le\frac{x}{n}} (p+1) + O\left(\sum_{n\le\sqrt{x}} \sum_{\sqrt{n}\le p\le\sqrt{x}} p\right) \\ &= \sum_{n\le\ln^2 x} \frac{x^2}{2n^2\ln\frac{x}{n}} + \sum_{\ln^2 x\le n\le\sqrt{x}} \frac{x^2}{2n^2\ln\frac{x}{n}} + O(x^{\frac{3}{2}}\ln x) \\ &= \frac{\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \end{aligned} \tag{2}$$

Combining (1) and (2) we obtain

$$\begin{aligned} \sum_{n\le x} \sigma(S(n)) &= \sum_{n\in\mathcal{A}} \sigma(S(n)) + \sum_{n\in\mathcal{B}} \sigma(S(n)) \\ &= \frac{\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right). \end{aligned}$$

This completes the proof of Theorem.

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ON THE MEAN VALUE OF A NEW ARITHMETICAL FUNCTION*

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Abstract The main purpose of this paper is using the elementary methods to study the convergent properties of a new Dirichlet's series involving the triangular numbers, and give an interesting identity for it.

Keywords: Dirichlet's series; Convergence; Identity.

§1. Introduction and results

For any positive integer m, it is clear that $m(m+1)/2$ is a positive integer, and we call it as the triangular number, because there are close relations between these numbers and the geometry. Now for any positive integer n, we define arithmetical function $c(n)$ as follows:

$$c(n) = \max\left\{m(m+1)/2 \ : \ m(m+1)/2 \leq n, \ m \in N\right\}.$$

That is, $c(n)$ is the greatest triangular number $\leq n$. From the definition of $c(n)$ we can easily deduce that $c(1) = 1$, $c(2) = 1$, $c(3) = 3$, $c(4) = 3$, $c(5) = 3$, $c(6) = 6$, $c(7) = 6$, $c(8) = 6$, $c(9) = 6$, $c(10) = 10$, $\cdots\cdots$. About this function, it seems that none had studied it before, even we do not know its arithmetical properties. In this paper, we introduce a new Dirichlet's series $f(s)$ involving the sequences $\{c(n)\}$, i.e.,

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{c^s(n)}.$$

Then we using the elementary methods to study the convergent properties of $f(s)$, and obtain an interesting identity. That is, we shall prove the following result:

Theorem *Let s be any positive real number. Then the Dirichlet's series $f(s)$ is convergent if and only if $s > 1$. Especially for $s = 2$, 3 and 4, we have*

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the identities:

$$\sum_{n=1}^{\infty} \frac{1}{c^2(n)} = \frac{2\pi^2}{3} - 4;$$
$$\sum_{n=1}^{\infty} \frac{1}{c^3(n)} = 8\zeta(3) - 4\pi^2 + 32;$$
$$\sum_{n=1}^{\infty} \frac{1}{c^4(n)} = \frac{\pi^4}{5} - 48\zeta(3) + \frac{160}{3}\pi^2 - 384;$$
$$6f(3) + f(4) = \frac{\pi^4}{5} + \frac{88}{3}\pi^2 - 192.$$

where $\zeta(k)$ is the Riemann zeta-function.

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. It is easy to see that if $\frac{m(m+1)}{2} \leq n < \frac{(m+1)(m+2)}{2}$, then $c(n) = \frac{m(m+1)}{2}$. So the same numbers $\frac{m(m+1)}{2}$ repeated $\frac{(m+1)(m+2)}{2} - \frac{m(m+1)}{2} = m+1$ times in the sequences $\{c(n)\}$. Hence, we can write

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{1}{c^s(n)} \\ &= \sum_{m=1}^{\infty} \frac{m+1}{(\frac{m(m+1)}{2})^s} \\ &= \sum_{m=1}^{\infty} \frac{2^s}{m^s(m+1)^{s-1}}. \end{aligned}$$

It is clear that $f(s)$ is convergent if $s > 1$, divergent if $s \leq 1$. Specially if $s = 2$, we can write

$$\begin{aligned} f(2) &= 4\sum_{m=1}^{\infty} \frac{1}{m^2(m+1)} \\ &= 4\sum_{m=1}^{\infty} \left(\frac{1}{m^2} - \frac{1}{m} + \frac{1}{m+1}\right) \\ &= 4\zeta(2) - 4. \end{aligned}$$

Using the same method, we can also obtain

$$f(3) = 8\zeta(3) - 24\zeta(2) + 32;$$
$$f(4) = 16\zeta(4) - 48\zeta(3) + 320\zeta(2) - 384.$$

Now the theorem follows from the identities (see reference [2]) $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. This complete the proof of the theorem.

Note: In fact, for any positive integer $s \geq 2$, using our methods we can express $f(s)$ as the Riemann zeta-function.

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ON THE SECOND CLASS PSEUDO-MULTIPLES OF 5 SEQUENCES

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Abstract The main purpose of this paper is to study the mean value properties of the second class pseudo-multiples of 5 sequences, and give an interesting asymptotic formula for it.

Keywords: Second class pseudo-multiples of 5 sequences; Mean value; Asymptotic formula.

§1. Introduction and results

A positive integer is called pseudo-multiple of 5 if some permutation of its digits is a multiple of 5, including the identity permutation. In reference [1], Professor F.Smarandache asked us to study the properties of the pseudo-multiple of 5 sequences. About this problem, Wang Xiaoying [2] had studied it, and obtained some interesting results. Now, we define the second class pseudo-multiples of 5 numbers as following:

A positive integer is called the second class pseudo-multiple of 5 if it is not a multiple of 5, but its some permutation of its digits is a multiple of 5.

For convenience, let $\mathcal{A}$ denotes the set of all pseudo-multiple of 5 sequences, and $\mathcal{B}$ denotes the set of all second class pseudo-multiple of 5 sequences.

In this paper, we shall use the elementary methods to study the mean value properties of the second class pseudo-multiple of 5 sequences, and obtain an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n\in\mathcal{B}\\ n\leq x}} d(n) = \frac{16}{25}x\left(\ln x + 2\gamma - 1 + \frac{\ln 5}{2}\right) + O\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right),$$

where $d(n)$ is the Dirichlet's divisor function, γ is the Euler constant, and ε denotes any fixed positive number.

§2. Some Lemmas

To complete the proof of the above Theorem, we need the following two Lemmas:

Lemma 1. Let q be a fixed prime or $q = 1$. Then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} d(qn) = \left(2 - \frac{1}{q}\right) x \left(\ln x + 2\gamma - 1 + \frac{\ln q}{2q-1}\right) + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where γ is the Euler constant, and ϵ denotes any fixed positive number.

Proof. If $q = 1$, then from Theorem 3.3 of [3] we know that Lemma 1 is correct. Now for any prime q and real number $s > 1$, let $f(s) = \sum_{n=1}^{\infty} \frac{d(qn)}{n^s}$. Note that $d(n)$ is a multiplicative function, so by the Euler product formula [3] we have

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{d(qn)}{n^s} = \sum_{\alpha=0}^{\infty} \sum_{\substack{n_1=1 \\ (n_1,q)=1}}^{\infty} \frac{d(q^{\alpha+1}n_1)}{q^{s\alpha} n_1^s} \\ &= \sum_{\alpha=0}^{\infty} \frac{2+\alpha}{q^{s\alpha}} \sum_{\substack{n_1=1 \\ (n_1,q)=1}}^{\infty} \frac{d(n_1)}{n_1^s} \\ &= \left(\frac{1}{1-\frac{1}{q^s}} + \frac{1}{\left(1-\frac{1}{q^s}\right)^2}\right) \\ &\quad \times \prod_{\substack{p \\ (p,q)=1}} \left(1 + \frac{d(p)}{p^s} + \frac{d(p^2)}{p^{2s}} + \cdots + \frac{d(p^k)}{p^{ks}} + \cdots\right) \\ &= \left(\frac{1}{1-\frac{1}{q^s}} + \frac{1}{\left(1-\frac{1}{q^s}\right)^2}\right) \zeta^2(s) \left(1 - \frac{1}{q^s}\right)^2 \\ &= \zeta^2(s) \left(2 - \frac{1}{q^s}\right). \end{aligned}$$

Then from this identity and the Perron's formula [4] we may immediately deduce the asymptotic formula

$$\sum_{n\leq x} d(qn) = \left(2 - \frac{1}{q}\right) x \left(\ln x + 2\gamma - 1 + \frac{\ln q}{2p-1}\right) + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

This completes the proof of Lemma 1.

Lemma 2. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} d(n) = x\ln x + (2\gamma - 1)x + O\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right).$$

Proof. For any real number $x \geq 1$, it is clear that there exists a nonnegative integer k such that $10^k \leq x < 10^{k+1}$. That is, $k \leq \log x < k+1$. According to the definition of $\mathcal{A}$ we know that the largest number of digits $(\leq x)$ not in $\mathcal{A}$ is 8^{k+1}. In fact, there are 8 one digit, they are $1, 2, 3, 4, 6, 7, 8, 9$; There are 8^2 two digits ; $\cdots\cdots$; The number of k digits are 8^k. Since

$$8^k \leq 8^{\log x} = x^{\frac{\ln 8}{\ln 10}},$$

we have

$$\sum_{\substack{n\notin A\\ n\leq x}} 1 \leq 8 + 8^2 + 8^3 + \ldots + 8^k \leq \frac{8^{k+2}}{7} \leq \frac{64}{7}8^k \leq \frac{64}{7}x^{\frac{\ln 8}{\ln 10}}.$$

Note that $d(n) \ll n^{\varepsilon}$ and $\frac{\ln 8}{\ln 10} > \frac{1}{2}$, applying Lemma 1 with $q = 1$ we have

$$\begin{aligned}
\sum_{\substack{n\in\mathcal{A}\\ n\leq x}} d(n) &= \sum_{n\leq x} d(n) - \sum_{\substack{n\notin\mathcal{A}\\ n\leq x}} d(n)\\
&= \sum_{n\leq x} d(n) + O\left(\sum_{\substack{n\notin\mathcal{A}\\ n\leq x}} x^{\varepsilon}\right)\\
&= \sum_{n\leq x} d(n) + O\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right)\\
&= x\ln x + (2\gamma - 1)x + O\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right).
\end{aligned}$$

This proves Lemma 2.

§3. Proof of the theorem

In this section, we complete the proof of Theorem. From the definition of set $\mathcal{A}$ and set $\mathcal{B}$, we know the relationship between them: $\mathcal{A} - \mathcal{B}$={multiples of 5}. Combining the above Lemmas we may immediately get

$$\begin{aligned}
\sum_{\substack{n\in\mathcal{B}\\ n\leq x}} d(n) &= \sum_{\substack{n\in\mathcal{A}\\ n\leq x}} d(n) - \sum_{5n\leq x} d(5n)\\
&= x\ln x + (2\gamma - 1)x + O\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right)\\
&\quad -\frac{9}{25}x\left(\ln x - \ln 5 + 2\gamma - 1 + \frac{\ln 5}{9}\right)\\
&= \frac{16}{25}x\left(\ln x + 2\gamma - 1 + \frac{\ln 5}{2}\right) + O\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right).
\end{aligned}$$

This completes the proof of Theorem.

Notes: In fact, we may use the similar method to study other arithmetical functions on the pseudo-multiples of 5 sequences and the second class pseudo-multiples of 5 sequences.

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A CLASS OF DIRICHLET SERIES AND ITS IDENTITIES

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Abstract In this paper, we using the elementary methods to study the convergent properties of one class Dirichlet series involving a special sequences, and give an interesting identity for it.

Keywords: Convergent property; Dirichlet series; Identity.

§1. Introduction and results

For any positive integer n and $m \geq 2$, we define the m-th power complement $b_m(n)$ is the smallest positive integer such that $nb_m(n)$ is a complete m-th power, see problem 29 of [1]. Now for any positive integer k, we also define an arithmetic function $\delta_k(n)$ as follows:

$$\delta_k(n) = \begin{cases} \max\{d \in N \quad | \quad d|n, (d,k) = 1\}, & if \quad n \neq 0, \\ 0, & if \quad n = 0. \end{cases}$$

Let $\mathcal{A}$ denotes the set of all positive integers n such that the equation $\delta_k(n) = b_m(n)$. That is, $\mathcal{A} = \{n : n \in N, \delta_k(n) = b_m(n)\}$. In this paper, we using the elementary methods to study the convergent properties of the Dirichlet series involving the set $\mathcal{A}$, and give an interesting identity for it. That is, we shall prove the following conclusion:

Theorem. *Let m be a positive even number. Then for any real number $s > 1$ and positive integer k, we have the identity:*

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{1}{n^s} = \frac{\zeta\left(\frac{m}{2}s\right)}{\zeta(ms)} \prod_{p|k} \frac{p^{\frac{3}{2}ms}}{(p^{ms}-1)\left(p^{\frac{1}{2}ms}-1\right)},$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_p$ denotes the product over all primes.

From this Theorem we may immediately deduce the following corollaries:

Corollary 1. *Let $\mathcal{B} = \{n : n \in N, \delta_k(n) = b_2(n)\}$, then we have*

$$\sum_{\substack{n=1\\ n\in\mathcal{B}}}^{\infty} \frac{1}{n^2} = \frac{15}{\pi^2} \prod_{p|k} \frac{p^6}{(p^4-1)(p^2-1)}.$$

Corollary 2. *Let $\mathcal{C} = \{n : n \in N, \delta_k(n) = b_4(n)\}$, then we have*

$$\sum_{\substack{n=1\\ n\in\mathcal{C}}}^{\infty} \frac{1}{n^2} = \sum_{\substack{n=1\\ n\in\mathcal{B}}}^{\infty} \frac{1}{n^4} = \frac{105}{\pi^4} \prod_{p|k} \frac{p^{12}}{(p^8-1)(p^4-1)}.$$

Corollary 3. *Let $\mathcal{C} = \{n : n \in N, \delta_k(n) = b_4(n)\}$, then we have*

$$\sum_{\substack{n=1\\ n\in\mathcal{C}}}^{\infty} \frac{1}{n^3} = \frac{675675}{691} \frac{1}{\pi^6} \prod_{p|k} \frac{p^{18}}{(p^{12}-1)(p^6-1)}.$$

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. For any real number $s > 0$, it is clear that

$$\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} < \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent if $s > 1$, thus $\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s}$ is also convergent if $s > 1$. Now we find the set $\mathcal{A}$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers. First from the definitions of $\delta_k(n)$ and $b_m(n)$ we konw that $\delta_k(n)$ and $b_m(n)$ both are multiplicative functions. So in order to find $\mathcal{A}$, we only discuss the problem in case $n = p^\alpha$. If $n = p^\alpha$ and $(p, k) = 1$, then we have $\delta_k(p^\alpha) = p^\alpha$; $b_m(p^\alpha) = p^{m-\alpha}$, if $1 \le \alpha \le m$; $b_m(p^\alpha) = p^{m+m[\frac{\alpha}{m}]-\alpha}$, if $\alpha > m$ and $\alpha \ne m$; $b_m(p^\alpha) = 1$, if $\alpha = rm$, where r is any positive integer, and $[x]$ denotes the greatest integer $\le x$. So in this case $\delta_k(p^\alpha) = b_m(p^\alpha)$ if and only if $\alpha = \frac{m}{2}$.

If $n = p^\alpha$ and $(p, k) \ne 1$, then $\delta_k(p^\alpha) = 1$, so in this case the equation $\delta_k(p^\alpha) = b_m(p^\alpha)$ has solution if and only if $n = p^{rm}$, $r = 0, 1, 2, \cdots$. Now from the Euler product formula (see [2]) and the definition of $\mathcal{A}$, we have

$$\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} = \prod_{p\dagger k} \left(1 + \frac{1}{p^{\frac{m}{2}s}}\right) \prod_{p|k} \left(1 + \frac{1}{p^{ms}} + \frac{1}{p^{2ms}} + \frac{1}{p^{3ms}} + \cdots\right)$$

$$= \prod_{p}\left(1+\frac{1}{p^{\frac{m}{2}s}}\right)\prod_{p|k}\frac{1}{1-\frac{1}{p^{ms}}}\prod_{p|k}\left(1+\frac{1}{p^{\frac{m}{2}s}}\right)^{-1}$$

$$= \frac{\zeta\left(\frac{m}{2}s\right)}{\zeta(ms)}\prod_{p|k}\frac{p^{\frac{3}{2}ms}}{(p^{ms}-1)\left(p^{\frac{1}{2}ms}+1\right)},$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_{p}$ denotes the product over all primes.

This completes the proof of Theorem.

The Corollaries follows from $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$, $\zeta(6)=\pi^8/945$, $\zeta(8)=\pi^8/9450$ and $\zeta(12)=\frac{691\pi^8}{638512875}$.

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AN ARITHMETICAL FUNCTION AND THE PERFECT K-TH POWER NUMBERS

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Abstract For any primes p and q with $(p, q) = 1$, the arithmetical function $e_{pq}(n)$ defined as the largest exponent of power pq which divides n. In this paper, we use the elementary methods to study the mean value properties of $e_{pq}(n)$ acting on the perfect k-th power number sequences, and give an interesting asymptotic formula for it.

Keywords: the perfect k-th power number; Asymptotic formula; Mean value.

§1. Introduction

For any prime p, let $e_p(n)$ denotes the largest exponent of power p which divides n. In problem 68 of reference [1], Professor F.Smarandache asked us to study the properties of this arithmetical function. About this problem, many scholars showed great interest in it, and obtained some interesting results, see references [2] and [3].

Similarly, we will define arithmetical function $e_{pq}(n)$ as follows: for any two primes p and q with $(p, q) = 1$, let $e_{pq}(n)$ denotes the largest exponent of power pq which divides n. That is,

$$e_{pq}(n) = \max\{\alpha \ : \ (pq)^{\alpha} \mid n, \alpha \in N^{+}\}.$$

According to [1], a number n is called a perfect k-th power number if it satisfied $k|\alpha$ for all $p^{\alpha}\|n$, where $p^{\alpha}\|n$ denotes $p^{\alpha}|n$, but $p^{\alpha+1} \dagger n$. Let $\mathcal{A}$ denotes the set of all the perfect k-th power numbers. It seems that no one knows the relations between these two arithmetical functions before. The main purpose of this paper is using the elementary methods to study the mean value properties of $e_{pq}(n)$ acting on the set $\mathcal{A}$, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. Let p and q are two primes with $(p, q) = 1$, then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}}} e_{pq}(n) = C_{p,q} k x^{\frac{1}{k}} + \left(x^{\frac{1}{2k}+\epsilon}\right),$$

where

$$C_{p,q} = \frac{(p-1)(q-1)}{pq} \sum_{n=1}^{\infty} \frac{n}{(pq)^n}$$

is a computable positive constant, and ϵ denotes any fixed positive number.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we define arithmetical function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is a perfect } k\text{-th power number;} \\ 0, & \text{otherwise .} \end{cases}$$

In order to complete the proof of Theorem, we need the following:

Lemma. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n \leq x \\ (n,pq)=1}} a(n) = x^{\frac{1}{k}} \frac{(p-1)(q-1)}{pq} + O\left(x^{\frac{1}{2k}+\epsilon}\right).$$

Proof. Let

$$f(s) = \sum_{\substack{n=1 \\ (n,pq)=1}}^{\infty} \frac{a(n)}{n^s}.$$

where $\mathrm{Re}(s) > 1$. From the Euler product formula [4] and the multiplicative properties of $a(n)$, we have

$$\begin{aligned} f(s) &= \prod_{\substack{P \\ (P,pq)=1}} \left(1 + \frac{a(P^k)}{P^{ks}} + \frac{a(P^{2k})}{P^{2ks}} + \cdots\right) \\ &= \prod_{P} \left(1 + \frac{1}{P^{ks}} + \frac{1}{P^{2ks}} + \cdots\right) \times \left(1 - \frac{1}{p^{ks}}\right) \times \left(1 - \frac{1}{q^{ks}}\right) \\ &= \zeta(ks)\left(1 - \frac{1}{p^{ks}}\right) \times \left(1 - \frac{1}{q^{ks}}\right), \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function, and $\prod_{P}$ denotes the product over all primes.

Now by Perron formula [5] with $s_0 = 0$, $b = \frac{1}{k} + \frac{1}{\log x}$, $T = x^{\frac{1}{2k}}$, $H(x) = x$ and $B(\sigma) = \frac{1}{\sigma - \frac{1}{k}}$, we have

$$\sum_{\substack{n \leq x \\ (n,pq)=1}} a(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta(ks) \frac{(p^{ks}-1)(q^{ks}-1)}{(pq)^{ks}} \frac{x^s}{s} ds + O(x^{\frac{1}{2k}+\varepsilon}).$$

To estimate the main term, we move the integral line from $b = \frac{1}{k} + \frac{1}{\log x}$ to $a = \frac{1}{2k} + \frac{1}{\log x}$ Therefore,

$$\frac{1}{2\pi i}\left(\int_{b-iT}^{b+iT} + \int_{b+iT}^{a+iT} + \int_{a+iT}^{a-iT} + \int_{a-iT}^{b-iT}\right) f(s)\frac{x^s}{s}ds = \text{Res}\left[f(s)\frac{x^s}{s}, \frac{1}{k}\right].$$

Note that $\lim_{s\to 1}\zeta(s)(s-1) = 1$, we may immediately get

$$\text{Res}\left[f(s)\frac{x^s}{s}, \frac{1}{k}\right] = x^{\frac{1}{k}}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right).$$

Now from the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{b+iT}^{a+iT} + \int_{a+iT}^{a-iT} + \int_{a-iT}^{b-iT}\right) f(s)\frac{x^s}{s}ds\right| \ll x^{\frac{1}{2k}+\epsilon},$$

we can easily get

$$\sum_{\substack{n\le x\\(n,pq)=1}} a(n) = x^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq} + O\left(x^{\frac{1}{2k}+\epsilon}\right).$$

This proves the lemma.

Now we prove the theorem. From the properties of geometrical series and the definition of $e_{pq}(n)$, combining the lemma we have

$$\begin{aligned}
&\sum_{\substack{n\le x\\ n\in\mathcal{A}}} e_{pq}(n)\\
=\ & \sum_{\substack{\alpha\le\log_{pq}x\\ k|\alpha}} \alpha \sum_{\substack{n\le\frac{x}{(pq)^\alpha}\\(n,pq)=1}} a(n)\\
=\ & \sum_{\alpha\le\frac{\log_{pq}x}{k}} k\alpha\left(\left(\frac{x}{(pq)^{k\alpha}}\right)^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq} + O\left(\left(\frac{x}{(pq)^{k\alpha}}\right)^{\frac{1}{2k}+\epsilon}\right)\right)\\
=\ & kx^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq}\left(\sum_{n=1}^{\infty}\frac{n}{(pq)^n} - \sum_{\alpha>\frac{\log_{pq}x}{k}}\frac{\alpha}{(pq)^\alpha}\right) + O\left(x^{\frac{1}{2k}+\epsilon}\right)\\
=\ & kx^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq}\left(\sum_{n=1}^{\infty}\frac{n}{(pq)^n} - \frac{1}{(pq)^{\left[\frac{\log_{pq}x}{k}\right]}}\sum_{\alpha=1}^{\infty}\frac{\alpha+\left[\frac{\log_{pq}x}{k}\right]}{(pq)^\alpha}\right)\\
& +O\left(x^{\frac{1}{2k}+\epsilon}\right)
\end{aligned}$$

$$
\begin{aligned}
&= kx^{\frac{1}{k}}\frac{(p-1)(q-1)}{pq}\left(a_{p,q}+O(x^{-\frac{1}{k}}\log x)\right)+O\left(x^{\frac{1}{2k}+\epsilon}\right)\\
&= \frac{(p-1)(q-1)}{pq}a_{p,q}kx^{\frac{1}{k}}+\left(x^{\frac{1}{2k}+\epsilon}\right),
\end{aligned}
$$

where

$$a_{p,q}=\sum_{n=1}^{\infty}\frac{n}{(pq)^n}$$

is a computable positive constant.

This completes the proof of Theorem.

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AN ARITHMETICAL FUNCTION AND ITS MEAN VALUE FORMULA

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Abstract In this paper, we use the elementary methods to study the mean value properties of an arithmetical function, and give an interesting asymptotic formula for it.

Keywords: Arithmetical function; Mean value; Asymptotic formula.

§1. Introduction

Let k be a fixed positive integer. For any positive integer n, we define the arithmetical function $b_k(n)$ as follows:

$$b_k(n) = \max \left\{ m | \sum_{i=1}^{m} i^k \leq n, n \in N \right\}.$$

That is, $b_k(n)$ is the greatest positive integer m such that $\sum\limits_{i=1}^{m} i^k \leq n$. For example, $b_2(1) = 1$, $b_2(2) = 1$, $b_2(3) = 1$, $b_2(4) = 1$, $b_2(5) = 2$, $\cdots\cdots$. In fact, from the definition of $b_k(n)$ we know that if $1 \leq n < (1 + 2^k)$, then $b(n) = 1$; if $1 + 2^k \leq n < 1 + 2^k + 3^k$, then $b(n) = 2$; $\cdots\cdots$, if $\sum\limits_{i=1}^{m} i^k \leq n < \sum\limits_{i=1}^{m+1} i^k$, then $b(n) = m$. So the same positive integer m repeated $(m+1)^k$ times in the sequence $\{b_k(n)\}$ $(n = 1, 2, 3, 4, \cdots)$. About this arithmetical function, it seems that none had studied it, at least we have not see any related references. In this paper, we shall use the elementary methods to study the mean value properties of $b_k(n)$, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} b_k(n) = \frac{(k+1)^{\frac{k+2}{k+1}}}{k+2} x^{\frac{k+2}{k+1}} + O(x).$$

From the Theorem, we may immediately deduce the following two Corollaries

Corollary 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} b_1(n) = \frac{2^{\frac{3}{2}}}{3} x^{\frac{3}{2}} + O(x).$$

Corollary 2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} b_2(n) = \frac{3^{\frac{4}{3}}}{4} x^{\frac{4}{3}} + O(x).$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we define $f_k(n) = \sum_{i=1}^{n} i^k$. For any real number $x > 1$, it is clear that there exists one and only one positive integer N such that $f_k(N) \leq x < f_k(N+1)$. For any fixed positive integer k, from the Euler's summation formula (see Theorem 3.1 of [2]) we have

$$f_k(N) = \sum_{i=1}^{N} i^k = \frac{N^{k+1}}{k+1} + O\left(N^k\right) = \frac{(N+1)^{k+1}}{k+1} + O\left(N^k\right) = f_k(N+1).$$

So that from the inequality $f_k(N) \leq x < f_k(N+1)$ we have

$$x = \frac{N^{k+1}}{k+1} + O\left(N^k\right) \quad \text{or} \quad (k+1)^{\frac{1}{k+1}} x^{\frac{1}{k+1}} = N + O(1).$$

Then from the Euler's summation formula and the definition of $b_k(n)$, we have

$$\begin{aligned}
\sum_{n\leq x} b_k(n) &= \sum_{n=1}^{N-1} \sum_{f_k(n)\leq j<f_k(n+1)} b_k(j) + \sum_{f_k(N)\leq n<x} b_k(n) \\
&= \sum_{n=1}^{N-1} (n+1)^k n + O(N^{k+1}) = \sum_{n=2}^{N} n^k(n-1) + O(N^{k+1}) \\
&= \frac{N^{k+2}}{k+2} - \frac{N^{k+1}}{k+1} + O(N^{k+1}) = \frac{N^{k+2}}{k+2} + O(N^{k+1}) \\
&= \frac{(k+1)^{\frac{k+2}{k+1}}}{k+2} x^{\frac{k+2}{k+1}} + O(x).
\end{aligned}$$

This completes the proof of the theorem.

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ON THE SQUARE COMPLEMENTS FUNCTION OF $N!$

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Abstract For any positive integer n, let $b(n)$ denote the square complements function of n. That is, $b(n)$ denotes the smallest positive integer such that $n \cdot b(n)$ is a perfect square number. In this paper, we use the elementary method to study the asymptotic properties of $b(n!)$, and give an interesting asymptotic formula for $\ln(b(n!))$.

Keywords: Square complements function; Standard factorization; Asymptotic formula.

§1. Introduction

For any positive integer n, let $b(n)$ denote the square complements function of n. That is, $b(n)$ denotes the smallest positive integer k such that nk is a perfect square. For example, $b(1) = 1$, $b(2) = 2$, $b(3) = 3$, $b(4) = 1$, $b(5) = 5$, $b(6) = 6$, $b(7) = 7$, $b(8) = 2$, $\cdots$. In problem 27 of [1], Professor F. Smarandache ask us to study the properties of $b(n)$. About this problem, some authors had studied it before, and obtained some interesting results, see references [5] and [6]. In this paper, we use the elementary method to study the asymptotic properties of the square complements function $b(n!)$ of n, and give an interesting asymptotic formula for $\ln(b(n!))$. That is, we shall prove the following:

Theorem . *For any positive integer n, we have the asymptotic formula*

$$\ln b(n!) = n \ln 2 + O\left(n \exp\left(\frac{-A \ln^{\frac{3}{5}} n}{(\ln \ln n)^{\frac{1}{5}}}\right)\right),$$

where $A > 0$ is a constant.

§2. Two simple lemmas

To complete the proof of the theorem, we need the following two simple lemmas:

Lemma 1. *Let $n! = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers. Then we have the calculate formula*

$$\begin{aligned} b(n!) &= b(p_1^{\alpha_1}) \cdot b(p_2^{\alpha_2}) \cdot \cdots \cdot b(p_k^{\alpha_k}) \\ &= p_1^{ord(p_1)} \cdot p_2^{ord(p_2)} \cdot \cdots \cdot p_2^{ord(p_k)}, \end{aligned}$$

where the ord function is defined as:

$$ord(p_i) = \begin{cases} 1 \quad , & if \quad \alpha_i \ is \ odd, \\ 0 \quad , & if \quad \alpha_i \ is \ even. \end{cases}$$

Proof. See reference [2].

Lemma 2. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\begin{aligned} \theta(x) &= \sum_{p \leq x} \ln p \\ &= x + O\left(x \exp\left(\frac{-A \ln^{\frac{3}{5}} x}{(\ln \ln x)^{\frac{1}{5}}} \right) \right). \end{aligned}$$

Proof. See references [3] or [4].

§3. Proof of the theorem

In this section, we shall complete the proof of Theorem. First from Lemma 1 we have

$$\begin{aligned} \ln b(n!) &= \ln \left(b(p_1^{\alpha_1}) \cdot b(p_2^{\alpha_2}) \cdot \cdots \cdot b(p_k^{\alpha_k}) \right) \\ &= \sum_{\substack{p \leq n \\ 2 \dagger ord(p)}} \ln p \\ &= \sum_{\frac{n}{2} < p \leq n} \ln p + \sum_{\frac{n}{4} < p \leq \frac{n}{3}} \ln p + \sum_{\frac{n}{6} < p \leq \frac{n}{5}} \ln p + \cdots + O(1). \end{aligned}$$

Let n be a positive integer large enough, if a prime factor p of $n!$ in the interval $(\frac{n}{2}, n]$, then the power of p is 1 in the standard factorization of $n!$. Similarly, if a prime factor p of $n!$ in the interval $(\frac{n}{3}, \frac{n}{2}]$, then the power of p is 2 in the standard factorization of $n!$; If a prime factor p of $n!$ in the interval $(\frac{n}{4}, \frac{n}{3}]$, then the power of p is 3 in the standard factorization of $n!$, $\cdots$.

On the other hand, note that the identity

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2,$$

then from Lemma 2 we have

$$\begin{aligned} & \ln b(n!) \\ = & \ \theta(n) - \theta\left(\frac{n}{2}\right) + \theta\left(\frac{n}{3}\right) - \theta\left(\frac{n}{4}\right) + \theta\left(\frac{n}{5}\right) - \theta\left(\frac{n}{6}\right) + \cdots + O(1) \\ = & \ n\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) + O\left(n \exp\left(\frac{-A \ln^{\frac{3}{5}} n}{(\ln\ln n)^{\frac{1}{5}}}\right)\right) \\ = & \ n \ln 2 + O\left(n \exp\left(\frac{-A \ln^{\frac{3}{5}} n}{(\ln\ln n)^{\frac{1}{5}}}\right)\right). \end{aligned}$$

This completes the proof of Theorem.

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ON THE SMARANDACHE FUNCTION

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Abstract For any positive integer n, let $S(n)$ denotes the Smarandache function. In this paper, we study the mean value properties of $\frac{S(n)}{n}$, and give an interesting asymptotic formula for it.

Keywords: Smarandache function; Mean value; Asymptotic formula.

§1. Introduction

For any positive integer n, let $S(n)$ denotes the Smarandache function. That is, $S(n)$ is the smallest positive integer m such that $n|m!$. From the definition of $S(n)$, we can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$S(n) = \max_{1 \le i \le k} \{f(p_i^{\alpha_i})\}.$$

About the arithmetical properties of $S(n)$, many scholars had studied it before (see reference [1]). The main purpose of this paper is to study the mean value properties of $\frac{S(n)}{n}$, and obtain an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number $x \ge 2$, we have the asymptotic formula

$$\sum_{n \le x} \frac{S(n)}{n} = \frac{\pi^2}{6} \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

§2. Some lemmas

To complete the proof of Theorem, we need the following several Lemmas:

Lemma 1. For any positive integer n, if n has the prime powers factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} P(n)$ such that $P(n) > \sqrt{n}$, then we have the identity

$$S(n) = P(n).$$

where $P(n)$ denotes the greatest prime divisor of n.

Proof. From the prime powers factorization of n, we may immediately get

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} < \sqrt{n}.$$

Then we have

$$p_i^{\alpha_i}|P(n)! \quad i=1,2,\cdots r.$$

Thus we can easily obtain $n|P(n)!$. But $P(n) \dagger (P(n)-1)!$, so we have

$$S(n)=P(n).$$

This completes the proof of Lemma 1.

Lemma 2. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x} S(n) = \frac{\pi^2}{12}\frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Proof. First we define two sets $\mathcal{A}$ and $\mathcal{B}$ as following:

$$\mathcal{A} = \{n|n \leq x, P(n) \leq \sqrt{n}\}$$

and

$$\mathcal{B} = \{n|n \leq x, P(n) > \sqrt{n}\}.$$

Using the Euler's summation formula (see reference [2]), we may get

$$\begin{aligned}\sum_{n\in\mathcal{A}} S(n) &\ll \sum_{n\leq x}\sqrt{n}\ln n\\ &= \int_1^x \sqrt{t}\ln t dt + \int_1^x (t-[t])(\sqrt{t}\ln t)' dt + \sqrt{x}\ln x(x-[x])\\ &\ll x^{\frac{3}{2}}\ln x.\end{aligned}$$

Similarly, from the Abel's summation forumula we also have

$$\begin{aligned}\sum_{n\in\mathcal{B}} S(n) &= \sum_{\substack{n\leq x\\ P(n)>\sqrt{n}}} P(n)\\ &= \sum_{n\leq\sqrt{x}}\sum_{n\leq p\leq\frac{x}{n}} p\\ &= \sum_{n\leq\sqrt{x}}\sum_{\sqrt{x}\leq p\leq\frac{x}{n}} p + O\left(\sum_{n\leq\sqrt{x}}\sum_{n\leq p\leq\frac{x}{n}}\sqrt{x}\right)\\ &= \sum_{n\leq\sqrt{x}}\left(\frac{x}{n}\pi\left(\frac{x}{n}\right) - \sqrt{x}\pi(\sqrt{x}) - \int_{\sqrt{x}}^{\frac{x}{n}}\pi(s)ds\right) + O\left(x^{\frac{3}{2}}\ln x\right),\end{aligned}$$

where $\pi(x)$ denotes all the numbers of primes which is not exceeding x. Note that

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

Using the above asymptotic formula, we have

$$\begin{aligned}\sum_{\sqrt{x}\le p\le \frac{x}{n}} p &= \frac{x}{n}\pi\left(\frac{x}{n}\right)-\sqrt{x}\pi(\sqrt{x})-\int_{\sqrt{x}}^{\frac{x}{n}}\pi(s)ds\\ &= \frac{1}{2}\frac{x^2}{n^2\ln x/n}-\frac{1}{2}\frac{x}{\ln\sqrt{x}}+O\left(\frac{x^2}{n^2\ln^2 x/n}\right)\\ &\quad +O\left(\frac{x}{\ln^2\sqrt{x}}\right)+O\left(\frac{x^2}{n^2\ln^2 x/n}-\frac{x}{\ln^2\sqrt{x}}\right).\end{aligned}$$

Considering the following

$$\sum_{n\le\sqrt{x}}\frac{1}{n^2}=\zeta(2)+O(\frac{1}{x}).$$

Hence we have

$$\begin{aligned}\sum_{n\le\sqrt{x}}\frac{x^2}{n^2\ln\frac{x}{n}} &= \sum_{n\le\ln^2 x}\frac{x^2}{n^2\ln\frac{x}{n}}+O\left(\sum_{\ln^2 x\le n\le\sqrt{x}}\frac{x^2}{n^2\ln\frac{x}{n}}\right)\\ &= \frac{\pi^2}{6}\frac{x^2}{\ln x}+O\left(\frac{x^2}{\ln^2 x}\right)\end{aligned}$$

and

$$\sum_{n\le\sqrt{x}}\frac{x^2}{n^2\ln^2\frac{x}{n}}=O\left(\frac{x^2}{\ln^2 x}\right).$$

Combining all the above, we may immediately deduce that

$$\begin{aligned}\sum_{n\le x}S(n) &= \sum_{n\in\mathcal{A}}S(n)+\sum_{n\in\mathcal{B}}S(n)\\ &= \frac{\pi^2x^2}{12\ln x}+O\left(\frac{x^2}{\ln^2 x}\right).\end{aligned}$$

This completes the proof of Lemma 2.

§3. Proof of the theorem

In this section, we shall complete the proof of Theorem. First applying the Abel's summation, and note that the results of Lemma 1 and Lemma 2, we may have

$$\sum_{n\le x}\frac{S(n)}{n} = \frac{1}{x}\sum_{n\le x}S(n)+\int_1^x\frac{1}{t^2}\left(\sum_{n\le t}S(t)\right)dt$$

$$= \frac{1}{x}\left(\frac{\pi^2 x^2}{12\ln x} + O\left(\frac{x^2}{\ln^2 x}\right)\right) + \int_1^x \frac{1}{t^2}\left(\frac{\pi^2 t^2}{12\ln t} + O\left(\frac{t^2}{\ln^2 t}\right)\right) dt$$
$$= \frac{\pi^2}{12}\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right) + \int_1^x \frac{\pi^2}{12\ln t}dt + O\left(\int_1^x \frac{1}{\ln^2 t}dt\right)$$
$$= \frac{\pi^2}{6}\frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).$$

This completes the proof of Theorem.

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MEAN VALUE OF THE K-POWER COMPLEMENT SEQUENCES

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Abstract For any prime $p \geq 3$ and any fixed integer $k \geq 2$, let $e_p(n)$ denote the largest exponent of power p which divides n, $a(n,k)$ denotes the k-power complement number of n. In this paper, we study the properties of the sequence $e_p(a(n,k))$, and give an interesting asymptotic formula for it.

Keywords: Largest exponent; Mean value; Asymptotic formula.

§1. Introduction

Let $p \geq 3$ be a prime, $e_p(n)$ denote the largest exponent of power p which divides n. For any fixed integer $k \geq 2$, let $a(n,k)$ denote k-power complement sequence of n. That is, $a(n,k)$ is the smallest positive integer such that $na(n,k)$ is a perfect k-power. In problem 29 of reference [1], professor F. Smarandach asked us to study the properties of this sequences. About this problem, some people had studied it before, and made some progress, see reference [2]. The main purpose of this paper is using the analytic method to study the properties of the sequence $e_p(a(n,k))$, and give an interesting asymptotic formula for its mean value. That is, we shall prove the following:

Theorem. Let $p \geq 3$ be a prime and $k \geq 2$ a fixed positive integer. Then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} e_p(a(n,k)) = \frac{(k-1)p^k - kp^{k-1} + 1}{(p^k - 1)(p-1)} x + O\left(x^{1/2+\varepsilon}\right),$$

where ε denotes any fixed positive number.

From this Theorem we may immediately deduce the following two corollaries:

Corollary 1. Let $p \geq 3$ be a prime. Then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} e_p(a(n,2)) = \frac{1}{p-1} x + O\left(x^{1/2+\varepsilon}\right).$$

Corollary 2. Let $p \geq 3$ be a prime. Then for any real number $x \geq 1$, we have

$$\sum_{n \leq x} e_p(a(n,3)) = \frac{2p+1}{p^2+p+1}x + O\left(x^{1/2+\varepsilon}\right).$$

§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. In fact, for any complex number s with $\mathrm{Re}(s) > 1$, we define the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{e_p(a(n,k))}{n^s}.$$

From the definitions of $e_p(n)$ and $a(n,k)$, and applying the Euler product formula (See reference [4]) we have

$$\begin{aligned}
& f(s) \\
= & \sum_{\alpha=0}^{\infty}\sum_{\beta=1}^{k-1} \frac{k-\beta}{p^{(\alpha k+\beta)s}} \sum_{\substack{n_1=1 \\ (n_1,p)=1}}^{\infty} \frac{1}{(n_1)^s} \\
= & k\sum_{\alpha=0}^{\infty}\sum_{\beta=1}^{k-1} \frac{1}{p^{(\alpha k+\beta)s}} \sum_{\substack{n_1=1 \\ (n_1,p)=1}}^{\infty} \frac{1}{(n_1)^s} - \sum_{\alpha=0}^{\infty} \frac{1}{p^{\alpha k s}} \sum_{\beta=1}^{k-1} \frac{\beta}{p^{\beta s}} \sum_{\substack{n_1=1 \\ (n_1,p)=1}}^{\infty} \frac{1}{(n_1)^s} \\
= & \frac{k}{1-\frac{1}{p^{ks}}}\left(\frac{\frac{1}{p^s}-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}\right)\zeta(s)\left(1-\frac{1}{p^s}\right) - \frac{1}{1-\frac{1}{p^{ks}}}\left(\frac{\frac{1}{p^s}-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}} - \frac{k-1}{p^{ks}}\right)\zeta(s) \\
= & \frac{\left(p^{(k-1)s}-1\right)[(k-1)p^s-k]}{\left(p^{ks}-1\right)\left(p^s-1\right)}\zeta(s) + \frac{k-1}{p^{ks}-1}\zeta(s) \\
= & \frac{(k-1)p^{ks}-kp^{(k-1)s}+1}{(p^{ks}-1)(p^s-1)}\zeta(s),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. Obviously, we have

$$e_p(a(n,k)) \leq k\log_p n \leq k\ln n \qquad \left|\sum_{n=1}^{\infty} \frac{e_p(a(n,k))}{n^\sigma}\right| \leq \frac{k}{\sigma-1},$$

where σ is the real part of s. Therefore by Parron's formula (See reference [3]) we can get

$$\sum_{n \leq x} \frac{e_p(a(n,k))}{n^{s_0}} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right)$$

$$+O\left(x^{1-\sigma_0}H(2x)\min\left\{1, \frac{\ln x}{T}\right\}\right) + O\left(x^{-\sigma_0}H(N)\min\left\{1, \frac{x}{\|x\|}\right\}\right)$$

where N is the nearest integer to x, $\|x\| = |x - N|$.

Taking $s_0 = 0, b = \frac{3}{2}, H(x) = k \ln x, B(\sigma) = \frac{k}{\sigma - 1}$, we have

$$\sum_{n \le x} e_p(a(n,k)) = \frac{1}{2\pi i} \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \zeta(s) R(s) \frac{x^s}{s} ds + O(\frac{x^{\frac{3}{2}+\varepsilon}}{T}),$$

where

$$R(s) = \frac{(k-1)p^{ks} - kp^{(k-1)s} + 1}{(p^{ks} - 1)(p^s - 1)}.$$

To estimate the main term

$$\frac{1}{2\pi i} \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \zeta(s) R(s) \frac{x^s}{s} ds,$$

we move the integral line from $s = \frac{3}{2} \pm iT$ to $s = \frac{1}{2} \pm iT$.

This time, the function

$$g(s) = \zeta(s) R(s) \frac{x^s}{s}$$

has a simple pole point at $s = 1$, and the residue is

$$\frac{(k-1)p^k - kp^{k-1} + 1}{(p^k - 1)(p - 1)} x.$$

So we have

$$\frac{1}{2\pi i} \left(\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT} \right) \zeta(s) R(s) \frac{x^s}{s} ds$$

$$= \frac{\left((k-1)p^k - kp^{k-1} + 1 \right) x}{(p^k - 1)(p - 1)}.$$

Note that the estimate

$$\left| \frac{1}{2\pi i} \left(\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} \right) \zeta(s) \frac{(k-1)p^{ks} - kp^{(k-1)s} + 1}{(p^{ks} - 1)(p^s - 1)} \frac{x^s}{s} ds \right|$$
$$\ll \frac{x^{\frac{3}{2}+\varepsilon}}{T}.$$

Taking $T = x$, from the above formula we may immediately get the asymptotic formula

$$\sum_{n \le x} e_p(a(n,k)) = \frac{\left((k-1)p^k - kp^{k-1} + 1 \right) x}{(p^k - 1)(p - 1)} + O\left(x^{1/2+\varepsilon} \right).$$

This completes the proof of Theorem.

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ON THE M-TH POWER FREE NUMBER SEQUENCES

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Abstract In this paper, we using the elementary methods to study the arithmetical properties of the m-th power free number sequences, and give some interesting identities for it.

Keywords: m-free number sequence; Dirichlet's series; Identity.

§1. Introduction

For any positive integer n and m with $m \geq 2$, we define the m-th power free number function $a_m(n)$ of n as follows: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ is the prime powers decomposition of n, then the m-th power free number function of n is the function: $a_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}$, where $\beta_i = \alpha_i$, if $\alpha_i \leq m-1$, and $\beta_i = 0$, if $\alpha_i \geq m$.

For any positive integer k, we also define the arithmetical function $\delta_k(n)$ as follows:

$$\delta_k(n) = \begin{cases} \max\{d \in N \quad | \quad d|n, (d,k) = 1\}, & if \quad n \neq 0, \\ 0, & if \quad n = 0. \end{cases}$$

Let $\mathcal{A}$ denotes the set of all the positive integers n such that the equation $a_m(n) = \delta_k(n)$. That is, $\mathcal{A} = \{n \in N, a_m(n) = \delta_k(n)\}$. In this paper, we using the elementary methods to study the convergent properties of the Dirichlet series involving the set $\mathcal{A}$, and give an interesting identity. That is, we shall prove the following conclusion:

Theorem. *Let $m \geq 2$ be a positive integer. Then for any real number $s > 1$, we have the identity:*

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{1}{n^s} = \frac{\zeta(s)}{\zeta(ms)} \prod_{p|k} \frac{p^{ms} - p^{(m-1)s} + 1}{p^{ms} - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

From this theorem we may immediately deduce the following:

Corollary . *Let $\mathcal{B} = \{n \in N, a_2(n) = \delta_k(n)\}$ and $\mathcal{C} = \{n \in N, a_3(n) = \delta_k(n)\}$, then we have the identities:*

$$\sum_{\substack{n=1\\ n\in\mathcal{B}}}^{\infty} \frac{1}{n^2} = \frac{15}{\pi^2} \prod_{p|k} \frac{p^4 - p^2 + 1}{p^4 - 1}$$

and

$$\sum_{\substack{n=1\\ n\in\mathcal{C}}}^{\infty} \frac{1}{n^2} = \frac{305}{2} \frac{1}{\pi^4} \prod_{p|k} \frac{p^6 - p^4 + 1}{p^6 - 1}.$$

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. First, we define the arithmetical function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & if \quad n \in \mathcal{A}, \\ 0, & if \quad otherwise. \end{cases}$$

For any real number $s > 0$, it is clear that

$$\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} < \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent if $s > 1$, thus $\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s}$ is also convergent if $s > 1$. Now let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization n into prime powers. Then from the definition of $a_m(n)$ and $\delta_k(n)$, we know that if $\alpha_j \geq m$ for some j, then $a_m(p_j^{\alpha_j}) = 1$. So only when $\alpha_i \leq m-1$ and $(p_i, k) = 1$ for all i, the equation $a_m(n) = \delta_k(n)$ has solution. If $(p_j, k) \neq 1$ for some j, then the equation $a_m(p_j^{\alpha_j}) = \delta_k(p_j^{\alpha_j})$ has solution if and only if $\alpha_j \geq m$. So from the Euler product formula (see [1]), we have

$$\begin{aligned}
\sum_{\substack{n=1\\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^s} &= \prod_{p} \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \cdots + \frac{a(p^{m-1})}{p^{(m-1)s}} + \cdots\right) \\
&= \prod_{p \dagger k} \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \cdots + \frac{a(p^{m-1})}{p^{(m-1)s}}\right) \\
&\quad \times \prod_{p|k} \left(1 + \frac{a(p)}{p^{ms}} + \frac{a(p^2)}{p^{(m+1)s}} + \frac{a(p^3)}{p^{(m+2)s}} + \cdots\right) \\
&= \prod_{p \dagger k} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{(m-1)s}}\right)
\end{aligned}$$

$$\times \prod_{p|k}\left(1+\frac{1}{p^{ms}}+\frac{1}{p^{(m+1)s}}+\frac{1}{p^{(m+2)s}}+\cdots\right)$$
$$= \frac{\zeta(s)}{\zeta(ms)}\prod_{p|k}\frac{p^{ms}-p^{(m-1)s}+1}{p^{ms}-1},$$

where $\zeta(s)$ is the Riemann zeta-function and $\prod_{p}$ denotes the product over all primes.

This completes the proof of Theorem.

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A NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE

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Abstract Let p and q are two distinct primes, $e_{pq}(n)$ denotes the largest exponent of power pq which divides n, and let $b_{pq}(n) = \sum_{t|n} e_{pq}\left(\frac{n}{t}\right) e_{pq}(t)$. In this paper, we using the analytic methods to study the mean vlaue properties of the $b_{pq}(n)$, and give an interesting asymptotic formula for it.

Keywords: Largest exponent; Mean value; Asymptotic formula.

§1. Introduction

Let p and q are two distinct primes, $e_{pq}(n)$ denotes the largest exponent of power pq which divides n. In problem 68 of [3], professor F.Smarandache asked us to study the properties of the sequence $e_p(n)$. About this problem, some people had studied it, and obtained a series interesting results, see references [2]. In this paper, we define arithmetical function $b_{pq}(n) = \sum_{t|n} e_{pq}\left(\frac{n}{t}\right) e_{pq}(t)$, then we using the analytic methods to study the mean value properties of $b_{pq}(n)$, and give give a sharp asymptotic formula for it. That is, we shall prove the following:

Theorem. Let p and q are two distinct primes, then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} b_{pq}(n) = \frac{x \ln x}{(pq-1)^2} + \frac{1 - 2\gamma - pq + 2pq\gamma - 2pq\ln(pq)}{(pq-1)^3} x + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where ε is any fixed positive number, and γ is the Euler constant.

§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. For any complex s, we define the Dirichlet's series

$$f(s) = \sum_{n=1}^{\infty} \frac{e_{pq}(n)}{n^s} \qquad \text{and} \qquad g(s) = \sum_{n=1}^{\infty} \frac{b_{pq}(n)}{n^s}.$$

It is clear that

$$g(s) = f^2(s).$$

Ren Ganglian has given (in a paper to appear) $f(s) = \frac{\zeta(s)}{(pq)^s-1}$, where $\zeta(s)$ is the Riemann zeta-function. So we can obtain that $g(s) = \frac{\zeta^2(s)}{((pq)^s-1)^2}$. Obviously, we have

$$b_{pq}(n) \le \log_{pq} n \le n \ln n \qquad \left|\sum_{n=1}^{\infty} \frac{b_{pq}(n)}{n^\sigma}\right| \le \frac{1}{\sigma-2},$$

where $\sigma(\ge 2)$ is the real part of s. Therefore, by Parron's formula (See reference [4]) we can get

$$\sum_{n\le x} \frac{b_{pq}(n)}{n^{s_0}} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \zeta^2(s+s_0)R(s+s_0)\frac{x^s}{s}ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right)$$
$$+O\left(x^{1-\sigma_0}H(2x)\min\left\{1, \frac{\log x}{T}\right\}\right)+O\left(x^{-\sigma_0}H(N)\min\left\{1, \frac{x}{\|x\|}\right\}\right),$$

where N be the nearest integer to x, $\|x\| = |x-N|$. Let

$$R(s) = \frac{1}{((pq)^s-1)^2}.$$

Taking $s_0 = 0, b = \frac{5}{2}, H(x) = x\ln x, B(\sigma) = \frac{1}{\sigma-2}$, we have

$$\sum_{n\le x} b_{pq}(n) = \frac{1}{2\pi i}\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} \zeta^2(s)R(s)\frac{x^s}{s}ds + O(\frac{x^{\frac{3}{2}+\varepsilon}}{T}).$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} \zeta^2(s)R(s)\frac{x^s}{s}ds,$$

we move the integral line from $s = \frac{5}{2} \pm iT$ to $s = \frac{1}{2} \pm iT$. This time, the function

$$\zeta^2(s)R(s)\frac{x^s}{s}$$

has a second order pole point at $s = 1$, and the residue is

$$\frac{x\ln x}{(pq-1)^2} + \frac{1-2\gamma-pq+2pq\gamma-2pq\ln(pq)}{(pq-1)^3}x,$$

where γ is Euler constant.

So we have

$$\frac{1}{2\pi i}\left(\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} + \int_{\frac{5}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{\frac{5}{2}-iT}\right)\zeta^2(s)R(s)\frac{x^s}{s}ds$$
$$= \frac{x\ln x}{(pq-1)^2} + \frac{1-2\gamma-pq+2pq\gamma-2pq\ln(pq)}{(pq-1)^3}x.$$

Taking $T = x^2$, we can easily obtain

$$\left|\frac{1}{2\pi i}\left(\int_{\frac{5}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{\frac{5}{2}-iT}\right)\zeta^2(s)R(s)\frac{x^s}{s}ds\right| \ll x^{\frac{1}{2}+\varepsilon}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\zeta^2(s)R(s)\frac{x^s}{s}ds\right| \ll x^{\frac{1}{2}+\varepsilon}.$$

So from the above formula, we may immediately get the asymptotic formula

$$\sum_{n\leq x} b_{pq}(n) = \frac{x\ln x}{(pq-1)^2} + \frac{1-2\gamma-pq+2pq\gamma-2pq\ln(pq)}{(pq-1)^3}x + O\left(x^{1/2+\varepsilon}\right).$$

This completes the proof of Theorem.

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ON THE ADDITIVE K-TH POWER PART RESIDUE FUNCTION

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Abstract Similar to the Smarandache k-th power complements, we define the additive k-th power part residue $f_k(n)$ of n is the smallest nonnegative integer such that $n - f_k(n)$ is a perfect k-th power. The main purpose of this paper is using the elementary methods to study the mean value properties of $f_k(n)$ and $d(f_k(n))$, and give two interesting asymptotic formulae for them.

Keywords: Additive k-th power part residue function; Mean value; Asymptotic formula.

§1. Introduction and results

For any positive integer n, the Smarandache k-th power complements $b_k(n)$ is the smallest positive integer $b_k(n)$ such that $nb_k(n)$ is a complete k-th power (see problem 29 of [1]). Similar to the Smarandache k-th power complements, Xu Zhefeng in [2] defined the additive k-th power complements $a_k(n)$ as follows: $a_k(n)$ is the smallest nonnegative integer such that $n + a_k(n)$ is a complete k-th power.

Similarly, we will define the additive k-th power part residue $f_k(n)$ as following: for any positive integer n,

$$f_k(n) = \min\{r | 0 \leq r = n - m^k, m \in N\}.$$

For example, if $k = 2$, we have the additive square part residue sequences $\{f_2(n)\}$ $(n = 1, 2, \ldots)$ as following:

$$0, 1, 2, 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 5, 6, 0, 1, 2, \ldots.$$

In this paper, we use the elementary methods to study the mean value properties of $f_k(n)$ and $d(f_k(n))$ (where $d(n)$ is the Dirichlet divisor function), and obtain some interesting asymptotic formulae for them. That is, we will prove the following conclusions:

Theorem 1. For any real number $x \geq 3$ and integer $k \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} f_k(n) = \frac{k^2}{2(2k-1)} x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right).$$

Theorem 2. For any real number $x \geq 3$ and integer $k \geq 2$, we also have the asymptotic formula

$$\sum_{n \leq x} d(f_k(n)) = \left(1 - \frac{1}{k}\right) x \ln x + \left(2\gamma + \ln k - 2 + \frac{1}{k}\right) x + O\left(x^{1-\frac{1}{k}} \ln x\right),$$

where γ is the Euler constant.

§2. Some Lemmas

To complete the proof of the theorems, we need following two Lemmas. First we have

Lemma 1. For any real number $x \geq 3$, we have the asymptotic formula

$$\sum_{n \leq x} d(n) = x \ln x + (2\gamma - 1)x + O(x^{\frac{1}{2}}),$$

where γ is the Euler constant.

Proof. See reference [3].

Lemma 2. For any real number $x \geq 3$ and any nonnegative arithmetical function $h(n)$ with $h(0) = 0$, we have the asymptotic formula

$$\sum_{n \leq x} h(f_k(n)) = \sum_{t=1}^{M-1} \sum_{n \leq g(t)} h(n) + O\left(\sum_{x \leq g(M)} h(n)\right),$$

where $g(t) = \sum_{i=1}^{k-1} \binom{k}{i} t^i$ and $M = [x^{\frac{1}{k}}]$, $[x]$ denotes the greatest integer not exceeding x.

Proof. For any real number $x \geq 1$, let M be a fixed positive integer such that

$$M^k \leq x < (M+1)^k.$$

Noting that if n pass through the integers in the interval $[t^k, (t+1)^k)$, then $f_k(n)$ pass through the integers in the interval $[0, (t+1)^k - t^k - 1]$ and $h(0) = 0$, we can deduce that

$$\begin{aligned}
\sum_{n \leq x} h(f_k(n)) &= \sum_{t=1}^{M-1} \sum_{t^k \leq n < (t+1)^k} h(f_k(n)) + \sum_{M^k \leq n \leq x} h(f_k(n)) \\
&= \sum_{t=1}^{M-1} \sum_{n \leq g(t)} h(n) + \sum_{0 \leq n < x - M^k} h(n) \\
&= \sum_{t=1}^{M-1} \sum_{n \leq g(t)} h(n) + O\left(\sum_{0 \leq n \leq (M+1)^k - M^k} h(n)\right) \\
&= \sum_{t=1}^{M-1} \sum_{n \leq g(t)} h(n) + O\left(\sum_{n \leq g(M)} h(n)\right),
\end{aligned}$$

where $g(t) = \sum_{i=1}^{k-1} \binom{k}{i} t^i$ and $M = [x^{\frac{1}{k}}]$.

This completes the proof of Lemma 2.

§3. Proof of the theorems

In this section, we shall use the elementary method to complete the proof of the theorems. First we prove Theorem 1. Let $h(n) = n$ and $M = [x^{\frac{1}{k}}]$, note that $x^{\frac{1}{k}} - M = O(1)$, then from Lemma 2 and the Euler summation formula (see [4]) we obtain

$$\begin{aligned}
\sum_{n \le x} f_k(n) &= \sum_{t=1}^{M-1} \sum_{n \le g(t)} n + O\left(\sum_{n \le g(M)} n\right) \\
&= \frac{1}{2} \sum_{t=1}^{M-1} k^2 t^{2k-2} + O(x^{2-\frac{2}{k}}) \\
&= \frac{k^2}{2(2k-1)} x^{2-\frac{1}{k}} + O(x^{2-\frac{2}{k}}).
\end{aligned}$$

This completes the proof of Theorem 1.

Now we prove Theorem 2. From Lemma 1 and Lemma 2 we have

$$\begin{aligned}
&\sum_{n \le x} d(f_k(n)) \\
= &\sum_{t=1}^{M-1} \sum_{n \le g(t)} d(n) + O\left(\sum_{n \le g(M)} d(n)\right) \\
= &\sum_{t=1}^{M-1} \left[kt^{k-1} \ln\left(kt^{k-1}\right) + (2\gamma - 1)kt^{k-1} + O\left((kt^{k-1})^{\frac{1}{2}}\right)\right] \\
&+O\left(x^{1-\frac{1}{k}} \ln x\right) \\
= &\sum_{t=1}^{M-1} \left(k(k-1)t^{k-1} \ln t + (2\gamma + \ln k - 1)kt^{k-1} + O(t^{\frac{k-1}{2}})\right) \\
&+O\left(x^{1-\frac{1}{k}} \ln x\right) \\
= &\ k(k-1) \sum_{t=1}^{M-1} t^{k-1} \ln t + (2\gamma + \ln k - 1)k \sum_{t=1}^{M-1} t^{k-1} \\
&+O\left(x^{1-\frac{1}{k}} \ln x\right).
\end{aligned}$$

Then from the Euler summation formula, we can easily get

$$\sum_{n\leq x} d(f_k(n)) = \left(1-\frac{1}{k}\right) x\ln x + \left(2\gamma + \ln k - 2 + \frac{1}{k}\right) x + O\left(x^{1-\frac{1}{k}}\ln x\right).$$

This completes the proof of the theorems.

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AN ARITHMETICAL FUNCTION AND THE K-TH POWER COMPLEMENTS

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Abstract The main purpose of this paper is using the analytic methods to study the asymptotic properties of an arithmetical function acting on the k-th power complements, and give an interesting asymptotic formula for it.

Keywords: Arithmetical function; Asymptotic formula; k-th power complements.

§1. Introduction and results

For any positive integer n, let $D(n)$ denotes the number of the solutions of the equation $n = n_1 n_2$ with $(n_1, n_2) = 1$. That is,

$$D(n) = \sum_{\substack{d|n \\ (d,\frac{n}{d})=1}} 1.$$

It is clear that $D(n)$ is a multiplicative function, and has many interesting arithmetical properties. Now we define another arithmetical function $A_k(n)$. For any fixed positive integer k, let $A_k(n)$ denotes the k-th power complements. That is, $A_k(n)$ denotes the smallest positive integer such that $nA_k(n)$ is a perfect k-th power. For example, $A_2(1) = 1$, $A_2(2) = 2$, $A_2(3) = 3$, $A_2(4) = 1$, $A_2(5) = 5$, $A_2(6) = 6$, $A_2(7) = 7$, $A_2(8) = 2$, $\cdots\cdots$. In reference [1], professor F.Smarandache asked us to study the properties of the sequences $\{A_k(n)\}$. About this problem, some people had studied it before, and obtained some interesting results, see references [4] and [5]. In this paper, we use the analytic methods to study the mean value properties of the arithmetical function $D(n)$ acting on the set $\{A_k(n)\}$, and obtain a sharper asymptotic formula. That is, we shall prove the following:

Theorem . For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} D(A_k(n)) = \frac{6\zeta(k)x\ln x}{\pi^2}\prod_p\left(1-\frac{2}{p^k+p^{k-1}}\right) + C(k)x + O(x^{\frac{1}{2}+\epsilon}),$$

where $C(k)$ is a computable constant, $\zeta(k)$ is the Riemann zeta-function, ϵ denotes any fixed positive number, and $\prod_p$ denotes the product over all primes.

From this theorem we may immediately deduce the following:
Corollary. For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} D(A_k(n)) = \frac{\pi^2}{15} x \ln x \prod_p \left(1 - \frac{2}{p^4+p^3}\right) + C(4)x + O(x^{\frac{1}{2}+\epsilon}).$$

§2. A Lemma

Before the proof of the theorem, a Lemma will be useful.
Lemma . *For any positive integer number $n \geq 1$, we have the identity*

$$D(n) = 2^{v(n)},$$

where $v(n)$ denotes the number of all distinct prime divisors of n, i.e.

$$v(n) = \begin{cases} 0, & if\ n = 1; \\ k, & if\ n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}. \end{cases}$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers. Note that $D(n)$ is a multiplicative function and $D(p^\alpha) = 2$, so from the definition of $v(n)$, we have

$$D(n) = \sum_{\substack{d|n \\ (d,\frac{n}{d})=1}} 1 = C_k^0 + C_k^1 + \cdots + C_k^k = 2^{v(n)}.$$

This proves the lemma.

§3. Proof of the theorem

In this section, we shall complete the proof of the theorem.
Let Dirichlet's series

$$f(s) = \sum_{n=1}^{\infty} \frac{D(A_k(n))}{n^s}.$$

For any real number $s > 1$, it is clear that $f(s)$ is absolutely convergent. So from the lemma and the Euler's product formula [2] we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{D(A_k(p))}{p^s} + \frac{D(A_k(p^2))}{p^{2s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{D(p^{k-1})}{p^s} + \frac{D(p^{k-2})}{p^{2s}} + \cdots + \frac{D(1)}{p^{ks}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{2^{v(p^{k-1})}}{p^s} + \frac{2^{v(p^{k-1})}}{p^{2s}} + \cdots + \frac{2^{v(1)}}{p^{ks}} + \cdots\right) \end{aligned}$$

$$
\begin{aligned}
&= \prod_p \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \cdots + \frac{1}{p^{ks}} + \frac{2}{p^{(k+1)s}} + \cdots\right) \\
&= \zeta(s)\zeta(ks) \prod_p \left(1 + \frac{1}{p^s} - \frac{2}{p^{ks}}\right) \\
&= \frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} \prod_p \left(1 - \frac{2}{p^{ks} + p^{(k-1)s}}\right),
\end{aligned}
$$

where $\zeta(s)$ is the Riemann zeta-function.

Therefore by Perron's formula [3] with $s_0 = 0,\ b = 2,\ T = \frac{3}{2}$, we have

$$\sum_{n\le x} D(A_k(n)) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s}ds + O(x^{\frac{1}{2}+\varepsilon}),$$

where

$$R(s) = \prod_p \left(1 - \frac{2}{p^{ks} + p^{(k-1)s}}\right).$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s}ds,$$

we move the integral line from $s = 2 \pm iT$ to $s = \frac{1}{2} \pm iT$, then the function

$$\frac{\zeta^2(s)\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s}$$

have a second order pole point at $s = 1$ with residue

$$\frac{\zeta(k)}{\zeta(2)} x\ln x \prod_p \left(1 - \frac{2}{p^k + p^{k-1}}\right) + C(k)x,$$

where $C(k)$ is a computable constant. So we have

$$
\begin{aligned}
&\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT} + \int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{2+iT} + \int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)\zeta(ks)R(s)\frac{x^s}{s}ds \\
= \ &\frac{6\zeta(k)x\ln x}{\pi^2}\prod_p \left(1 - \frac{2}{p^k + p^{k-1}}\right) + C(k)x.
\end{aligned}
$$

We can easily get the estimate

$$\frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{2+iT} + \int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)\zeta(ks)R(s)\frac{x^s}{s}ds \ll x^{\frac{1}{2}+\varepsilon}.$$

Now note that $\zeta(2) = \pi^2/6$, we may immediately obtain the asymptotic formula

$$\sum_{n\le x} D(A_k(n)) = \frac{6}{\pi^2}\zeta(k)x\ln x \prod_p \left(1 - \frac{2}{p^k + p^{k-1}}\right) + C(k)x + O(x^{\frac{1}{2}+\epsilon}),$$

where $C(k)$ is a computable constant.

This completes the proof of the theorem.

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ON THE TRIANGLE NUMBER PART RESIDUE OF A POSITIVE INTEGER

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Abstract For any positive integer n, let $a(n)$ denotes the triangle number part residue of n. That is, $a(n) = n - \frac{k(k+1)}{2}$, where k is the greatest positive integer such that $\frac{k(k+1)}{2} \leq n$. In this paper, we using the elementary methods to study the mean value properties of the sequences $\{a(n)\}$, and give two interesting asymptotic formulae for it.

Keywords: The triangle number part residue; Mean value; Asymptotic formula.

§1. Introduction and results

For any positive integer n, let $a(n)$ denotes the triangle number part residue of n. That is, $a(n) = n - \frac{k(k+1)}{2}$, where k is the greatest positive integer such that $\frac{k(k+1)}{2} \leq n$. For example, $a(1) = 0$, $a(2) = 1$, $a(3) = 0$, $a(4) = 1$, $a(5) = 2$, $a(6) = 0$, $\cdots\cdots$. In reference [1], American-Romanian number theorist Professor F. Smarandache asked us to study the arithmetical properties of this sequences. About this problem, it seems that none had studied it, at least we have not seen any related papers before. In this paper, we using the elementary methods to study the mean value properties of this sequences, and give two interesting asymptotic formulae for it. That is, we shall prove the following:

Theorem 1. *For any real number $x \geq 3$, we have the asymptotic formula*

$$\sum_{n \leq x} a(n) = \frac{\sqrt{2}}{3} x^{\frac{3}{2}} + O(x).$$

Theorem 2. *For any real number $x \geq 3$, we also have the asymptotic formula*

$$\sum_{n \leq x} d(a(n)) = \frac{1}{2} x \ln x + \left(2\gamma + \frac{\ln 2 - 3}{2}\right) x + O(x^{\frac{2}{3}}),$$

where $d(n)$ is the Dirichlet divisor function (provide $d(0) = 0$), and γ is the Euler constant.

Note: It is clear that if there exists a mean value formula for any arithmetical function $f(n)$, then using our methods we can also obtain an asymptotic formula for $\sum_{n\le x} f(a(n))$.

§2. Proof of the theorems

In this section, we shall complete the proof of the theorems. First we prove Theorem 1. For any real number $x \ge 3$, let M be a fixed positive integer such that

$$\frac{M(M+1)}{2} \le x < \frac{(M+1)(M+2)}{2}.$$

Then from the definition of $a(n)$, we have

$$\begin{aligned}
\sum_{n\le x} a(n) &= \sum_{k=1}^{M} \sum_{\frac{k(k+1)}{2}\le n<\frac{(k+1)(k+2)}{2}} a(n) - \sum_{x<n<\frac{(M+1)(M+2)}{2}} a(n) \\
&= \sum_{k=1}^{M} \sum_{i\le\left(\frac{(k+1)(k+2)}{2}-1-\frac{k(k+1)}{2}\right)} i + O\left(\sum_{s\le\left(\frac{(M+1)(M+2)}{2}-\frac{M(M+1)}{2}\right)} s\right) \\
&= \sum_{k=1}^{M}\sum_{i=0}^{k} i + O\left(M^2\right) \\
&= \frac{1}{2}\sum_{k=1}^{M} k(k+1) + O\left(M^2\right) \\
&= \frac{1}{6}M^3 + O\left(M^2\right). \qquad (1)
\end{aligned}$$

On the other hand, note that the estimates

$$\frac{M}{2} \le x - \frac{M^2}{2} < \frac{3}{2}M + 1. \qquad (2)$$

Combing (1) and (2), we have

$$\sum_{n\le x} a(n) = \frac{\sqrt{2}}{3}x^{\frac{3}{2}} + O(x).$$

This completes the proof of Theorem 1.

Now we prove Theorem 2. Using the similar method of proving Theorem 1, we have

$$\sum_{n\le x} d(a(n)) = \sum_{k=1}^{M}\sum_{i=0}^{k} d(i) + \sum_{\frac{M(M+1)}{2}<n\le x} d(a(n)).$$

Note that the asymptotic formula (see reference [2])

$$\sum_{n\le x} d(n) = x\ln x + (2\gamma - 1)x + O\left(x^{\frac{1}{3}}\right),$$

where γ is the Euler constant.

We have

$$\begin{aligned}
&\sum_{n\le x} d(a(n)) \\
=\ & \sum_{k=1}^{M}\left(k\ln k + (2\gamma-1)k + O\left(k^{\frac{1}{3}}\right)\right) + O\left(\sum_{i\le x-\frac{M(M+1)}{2}} d(i)\right) \\
=\ & \frac{1}{2}M^2\ln M - \frac{1}{4}(M^2-1) + \frac{1}{2}(2\gamma-1)M^2 + O(M^{\frac{4}{3}}).
\end{aligned} \tag{3}$$

Now combining (2) and (3) we may immediately get

$$\sum_{n\le x} d(a(n)) = \frac{1}{2}x\ln x + \left(2\gamma + \frac{\ln 2 - 3}{2}\right)x + O(x^{\frac{2}{3}}).$$

This completes the proof of Theorem 2.

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AN ARITHMETICAL FUNCTION AND THE K-FULL NUMBER SEQUENCES

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Abstract The main purpose of this paper is using the elementary methods to study the mean value properties of an arithmetical function acting on the k-full number sequences, and give an interesting asymptotic formula for it.

Keywords: k-full number sequence; Asymptotic formula; Mean value.

§1. Introduction

For any prime p, let $e_p(n)$ denotes the largest exponent (of power p) which divides n. In problem 68 of reference [1], Professor F.Smarandache asked us to study the properties of this arithmetical function. About this problem, many scholars showed great interest in it and obtained some interesting results. For example, Lv Chuan [2] used the elementary methods to studied the asymptotic properties of the mean value $\sum_{n\leq x} e_p^m(n)$, and gave the following asymptotic formula:

$$\sum_{n\leq x} e_p^m(n) = \frac{p-1}{p}Ax + O(\log^{m+1} x),$$

where A is a computable constant.

Professor Zhang [3] studied the mean value of $e_p(S_p(n))$, where $S_p(n)$ denotes the smallest integer m such that $p^n|m!$, and obtained

$$\sum_{n\leq x} e_p(S_p(n)) = \frac{p+1}{(p-1)^2}x + O(\ln^3 x).$$

In this paper, we shall use the elementary methods to study the mean value properties of $e_p(n)$ acting on the k-full number sequences, and give an interesting asymptotic formula for it. For convenience, first we give the definition of the k-full number. In fact, a number n is called a k-full number if $p|n \iff p^k|n$ for any prime divisor p of n. Let $\mathcal{A}$ denotes the set of all the k-full numbers. It seems that no one had studied the relations between the arithmetical function $e_p(n)$ and the k-full numbers. In this paper, we shall prove the following:

Theorem. Let p be a prime, then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} e_p(n) = C(p,k)a_p(k)x^{\frac{1}{k}} + \left(x^{\frac{1}{2k}+\varepsilon}\right),$$

where

$$C(p,k) = \frac{6k}{\pi^2}\left(1-\frac{1}{p-p^{\frac{k-1}{k}}+1}\right)\prod_q\left(1+\frac{1}{(q+1)(q^{\frac{1}{k}}-1)}\right),$$

$a_p(k)$ is a computable positive constant, $\varepsilon > 0$ is any fixed real number, and $\prod_q$ denotes the product over all prime q.

§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. Let $a(n)$ denotes the character function of k-full number. That is,

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is a } k\text{-full number;} \\ 0, & \text{otherwise .} \end{cases}$$

It is clear that

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}\\ (n,p)=1}} 1 = \sum_{\substack{n\leq x\\ (n,p)=1}} a(n).$$

Let Dirichlet series

$$f(s) = \sum_{\substack{n=1\\ (n,p)=1}}^{\infty} \frac{a(n)}{n^s}.$$

It is clear that this series is convergent if Re $s > 1$. From the Euler product formula [4] and the definition of $a(n)$, we have

$$\begin{aligned} & f(s) \\ = & \prod_{q\neq p}\left(1+\frac{a(q^k)}{q^{ks}}+\frac{a(q^{k+1})}{q^{(k+1)s}}+\cdots\right) \\ = & \prod_{q\neq p}\left(1+\frac{1}{q^{ks}}\cdot\frac{1}{1-\frac{1}{q^s}}\right) \\ = & \frac{\zeta(ks)}{\zeta(2ks)}\left(1-\frac{1}{p^{ks}-p^{(k-1)s}+1}\right)\prod_q\left(1+\frac{1}{(q^{ks}+1)(q^s-1)}\right), \quad (1) \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function.

Obviously, we have

$$|a(n)| \leq 1, \qquad \left|\sum_{n=1}^{\infty} \frac{a(n))}{n^{\sigma}}\right| \leq \frac{1}{\sigma - \frac{1}{k}},$$

where σ is the real part of s. Therefore, by the Perron's formula [4] we have

$$\sum_{\substack{n\leq x\\(n,p)=1}} a(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{\zeta(ks)}{\zeta(2s)} R(s)\frac{x^s}{s} ds + O(x^{\frac{1}{2k}+\varepsilon}),$$

where $R(s) = \left(1 - \frac{1}{p^{ks} - p^{(k-1)s} + 1}\right)\prod_q \left(1 + \frac{1}{(q^{ks}+1)(q^s-1)}\right)$.

Now moving the integral line from b to $a = \frac{1}{2k}$, we have

$$\frac{1}{2\pi i}\left(\int_{b-iT}^{b+iT} + \int_{b+iT}^{a+iT} + \int_{a+iT}^{a-iT} + \int_{a-iT}^{b-iT}\right) f(s)\frac{x^s}{s} ds = \text{Res}\left[f(s)\frac{x^s}{s}, \frac{1}{k}\right]. \tag{2}$$

Note that $\lim_{s\to\frac{1}{k}} \zeta(ks)\left(s - \frac{1}{k}\right) = 1$, we may immediately get

$$\text{Res}\left[f(s)\frac{x^s}{s}, \frac{1}{k}\right] = \frac{kx^{\frac{1}{k}}}{\zeta(2)} R\left(\frac{1}{k}\right). \tag{3}$$

Combining (1), (2), (3) and the following estimates

$$\left|\frac{1}{2\pi i}\int_{a+iT}^{a-iT} f(s)\frac{x^s}{s} ds\right| \ll x^{\frac{1}{2k}+\epsilon}$$

and

$$\left|\frac{1}{2\pi i}\int_{b+iT}^{a+iT} + \int_{a-iT}^{b-iT} f(s)\frac{x^s}{s} ds\right| \ll x^{\frac{1}{2k}+\epsilon}$$

we can easily get

$$\sum_{\substack{n\leq x\\(n,p)=1}} a(n) = C(p,k)x^{\frac{1}{k}} + O\left(x^{\frac{1}{2k}+\epsilon}\right),$$

where

$$C(p,k) = \frac{6k}{\pi^2}\left(1 - \frac{1}{p - p^{\frac{k-1}{k}} + 1}\right)\prod_q \left(1 + \frac{1}{(q+1)(q^{\frac{1}{k}} - 1)}\right).$$

Based on the definition of $e_p(n)$ and the above estimate, we have

$$\sum_{\substack{n\leq x\\n\in\mathcal{A}}} e_p(n) = \sum_{k\leq\alpha\leq\log_p x} \alpha \sum_{\substack{n\leq\frac{x}{p^{\alpha}}\\(n,p)=1}} a(n)$$

$$
\begin{aligned}
&= \sum_{k\le\alpha\le\log_p x} \alpha\left(\left(\frac{x}{p^\alpha}\right)^{\frac{1}{k}} C(p,k) + O\left(\left(\frac{x}{p^\alpha}\right)^{\frac{1}{2k}+\epsilon}\right)\right) \\
&= C(p,k)x^{\frac{1}{k}} \sum_{k\le\alpha\le\log_p x} \frac{\alpha}{p^{\frac{\alpha}{k}}} + O\left(x^{\frac{1}{2k}+\epsilon} \sum_{k\le\alpha\le\log_p x} \frac{\alpha}{p^{\frac{\alpha}{k}}}\right) \\
&= C(p,k)x^{\frac{1}{k}} \left(\sum_{n=1}^{\infty} \frac{n}{p^{\frac{n}{k}}} - \sum_{\alpha>\log_p x} \frac{\alpha}{p^{\frac{\alpha}{k}}} - \sum_{\alpha<k} \frac{\alpha}{p^{\frac{\alpha}{k}}}\right) + O\left(x^{\frac{1}{2k}+\epsilon}\right) \\
&= C(p,k)x^{\frac{1}{k}} \left(\sum_{n=1}^{\infty} \frac{n}{p^{\frac{n}{k}}} - \frac{1}{p^{\frac{1}{k}[\log_p x]}} \sum_{\alpha=1}^{\infty} \frac{\alpha+\log_p x}{p^{\frac{\alpha}{k}}} - \sum_{\alpha<k} \frac{\alpha}{p^{\frac{\alpha}{k}}}\right) + O\left(x^{\frac{1}{2k}+\epsilon}\right) \\
&= C(p,k)x^{\frac{1}{k}} \left(a_p(k) + O(x^{-\frac{1}{k}}\log x)\right) + O\left(x^{\frac{1}{2k}+\epsilon}\right) \\
&= C(p,k)a_p(k)x^{\frac{1}{k}} + \left(x^{\frac{1}{2k}+\epsilon}\right),
\end{aligned}
$$

where

$$C(p,k) = \frac{6k}{\pi^2}\left(1 - \frac{1}{p - p^{\frac{k-1}{k}} + 1}\right)\prod_q \left(1 + \frac{1}{(q+1)(q^{\frac{1}{k}} - 1)}\right),$$

$$a_p(k) = \sum_{n=1}^{\infty} \frac{n}{p^{\frac{n}{k}}} - \sum_{\alpha<k} \frac{\alpha}{p^{\frac{\alpha}{k}}}$$

is a computable positive constant.

This completes the proof of Theorem.

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ON THE HYBRID MEAN VALUE OF SOME SPECIAL SEQUENCES*

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Abstract The main purpose of this paper is to define a new arithmetic function by the m-th power complement numbers and the k-th power free numbers, and use the analytic methods to obtain some interesting asymptotic formulae for them.

Keywords: m-th power complement numbers; k-th power free numbers; Asymptotic formula; Arithmetic function.

§1. Introduction and main results

Let m be a fixed positive integer with $m \geq 2$. For any positive integer n, we define the m-th power complement numbers $a_m(n)$ of n as the smallest positive integer such that $na_m(n)$ is a perfect m-th power. We also call a positive integer n as k-th power free number if it can not be divided by any p^m, where p be a prime. We denotes the k-th power free number by $c_k(n)$. It is clear that $c_k(n) = n \sum\limits_{d^k|n} \mu(d)$, where $\mu(d)$ is the Möbius function.

In reference [1], professor F.Smarandache asked us to study the properties of the m-th power complement numbers and the k-th power free number sequences. About these sequences, some people had studied them, and obtained many interesting results, see references [2], [3], [4] and [5].

In this paper, we introduce a new sequences $f(n) = a_m(n)c_k(n)$, then we use the analytic method to study the mean value properties of this sequences, and obtain some sharp asymptotic formulae. That is, we shall prove the following:

Theorem 1. For any real number$x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} a_m(n)c_k(n) = \frac{6x^{m+1}}{(m+1)\pi^2} R(m+1) + O(x^{m+\frac{1}{2}+\varepsilon}),$$

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where ε denotes any fixed positive number, and

$$R(m+1) = \prod_p \left(1 + \frac{p^{(k-2)(m+1)} - 1}{(p+1)(p^{(k-1)(m+1)} - p^{(k-2)(m+1)})}\right),$$

if $m \geq k$;

$$R(m+1) = \prod_p \left(1 + \frac{(p^{m^2-1} - 1) + (p^{m(m+1)} - 1)\sum_{j=1}^{i-1} \frac{p^{jm+m+1}}{p^{(jm+1)(m+1)}}}{(p+1)(p^{m(m+1)} - p^{m^2-1})} + \frac{\sum_{j=im+1}^{k-1} \frac{p^{im+m+1}}{p^{j(m+1)}}}{p+1}\right),$$

if $m < k$.

Theorem 2. For any real number$x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} \phi(a_m(n)c_k(n)) = \frac{6x^{m+1}}{(m+1)\pi^2} R^*(m+1) + O(x^{m+\frac{1}{2}+\varepsilon}),$$

where $\phi(n)$ is the Euler function, ε denotes any fixed positive number, and

$$R^*(m+1) = \prod_p \left(1 + \frac{p(p^{(k-2)(m+1)} - 1) - (p^{(k-1)(m+1)} - 1)}{p(p+1)(p^{(k-1)(m+1)} - p^{(k-2)(m+1)})}\right),$$

If $m \geq k$;

$$R^*(m+1) = \prod_p \left(\frac{(p^{m^2-1} - 1) + (p^{m(m+1)} - 1)H(m+1)}{(p+1)(p^{m(m+1)} - p^{m^2-1})} + \frac{G(m+1)}{p+1}\right),$$

if $m < k$;

$$H(m+1) = \sum_{j=1}^{i-1} \left(\frac{p^{jm+m+2} - p^{jm+m+1}}{p^{(jm+1)(m+1)}} - \frac{1}{p}\right)$$

and

$$G(m+1) = \sum_{j=im+1}^{k-1} \frac{p^{im+m+1} - p^{im+m}}{p^{j(m+1)}}.$$

Taking $k = 2$ in Theorem 1 and Theorem 2, we may immediately deduce the following two Corollaries:

Corollary 1. For any real number $x \geq 1$ and integer $m \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} a_m(n)c_2(n) = \frac{6}{(m+1)\pi^2} x^{m+1} + O(x^{m+\frac{1}{2}+\varepsilon}).$$

Corollary 2. For any real number$x \geq 1$ and integer $m \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} \phi(a_m(n)c_2(n)) = \frac{6x^{m+1}}{(m+1)\pi^2} R^*(m+1) + O(x^{m+\frac{1}{2}+\varepsilon}),$$

where $R^*(m+1) = \prod_p \left(1 - \frac{1}{p(p+1)}\right)$.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorems. First we prove Theorem 1. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_m(n)c_k(n)}{n^s},$$

It is clear that $a_m(n)$ and $c_k(n)$ are multiplicative functions of n, so $a_m(n)c_k(n)$ is also a multiplicative function of n. If the real part of s is large enough, then the Dirichlet's series $f(s)$ is absolutely convergent. So for $m \geq k$, from the Eurler's product formula [6] we have

$$\begin{aligned}
&f(s) \\
&= \prod_p \left(1 + \frac{a_m(p)c_k(p)}{p^s} + \frac{a_m(p^2)c_k(p^2))}{p^{2s}} + \cdots + \frac{a_m(p^{k-1})c_k(p^{k-1}))}{p^{(k-1)s}}\right) \\
&= \prod_p \left(1 + \frac{1}{p^{s-m}} + \cdots + \frac{1}{p^{(2s-2)}}(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(k-3)s}})\right) \\
&= \frac{\zeta(s-m)}{\zeta(2(s-m))} \prod_p \left(1 + \frac{p^{(k-2)s} - 1}{(p^{(s-m)}+1)(p^{(k-1)s} - p^{(k-2)s})}\right);
\end{aligned}$$

if $m \leq k$ $(im \leq k < (i+1)m, i > 1)$, then

$$\begin{aligned}
&f(s) \\
&= \prod_p \left(1 + \frac{a_m(p)c_k(p)}{p^s} + \cdots + \frac{a_m(p^m)c_k(p^m)}{p^{ms}}\right. \\
&\quad + \tfrac{a_m(p^{m+1})c_k(p^{m+1})}{p^{(m+1)s}} + \cdots + \frac{a_m(p^{2m})c_k(p^{2m})}{p^{2ms}} \\
&\quad \left. + \cdots + \frac{a_m(p^{im+1})c_k(p^{im+1})}{p^{(im+1)s}} + \cdots + \frac{a_m(p^{k-1})c_k(p^{k-1})}{p^{(k-1)s}}\right) \\
&= \prod_p \left(1 + \frac{1}{p^{s-m}} + \frac{1}{p^{2s-m}}(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(m-2)s}})\right. \\
&\quad + \frac{1}{p^{(m+1)s-2m}}(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(m-1)s}}) \\
&\quad \left. + \cdots + \tfrac{1}{p^{((i-1)m+1)s-im}} + \textstyle\sum_{j=im+1}^{k-1} \tfrac{p^{(i+1)m}}{p^{js}}\right) \\
&= \frac{\zeta(s-m)}{\zeta(2(s-m))} \prod_p \left(1 + \frac{(p^{(m-1)s} - 1) + (p^{ms} - 1) \displaystyle\sum_{j=1}^{i-1} \frac{p^{jm+s}}{p^{(jm+1)s}}}{(p^{s-m}+1)(p^{ms} - p^{(m-1)s})}\right. \\
&\quad \left. + \frac{\displaystyle\sum_{j=im+1}^{k-1} \frac{p^{im+s}}{p^{js}}}{p^{s-m}+1}\right).
\end{aligned}$$

Obviously, we have the inequality

$$|a_m(n)c_k(n)| \leq n^2, \qquad \left|\sum_{n=1}^{\infty} \frac{a_m(n)c_k(n)}{n^\sigma}\right| < \frac{1}{\sigma - m - 1},$$

where $\sigma > m + 1$ is the real part of s. So by Perron formula [7] we have:

$$\begin{aligned}
\sum_{n \leq x} \frac{a(n)}{n^{s_0}} &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s + s_0) \frac{x^s}{s} ds + O\left(\frac{x^b B(b + \sigma_0)}{T}\right) \\
&\quad + O\left(x^{1-\sigma_0} H(2x) \min\left(1, \frac{\log x}{T}\right)\right) \\
&\quad + O\left(x^{-\sigma_0} H(N) \min\left(1, \frac{x}{T||x||}\right)\right).
\end{aligned}$$

when N is the nearest integer to x, $N = x - \frac{1}{2}$, $\|x\| = |x - N|$. Taking $s_0 = 0$, $b = m + 2$, $T = x^{\frac{3}{2}}$, $H(x) = x^2$, $B(\sigma) = \dfrac{1}{\sigma - m - 1}$, we have:

$$\sum_{n\leq x} a_m(n)c_k(n) = \frac{1}{2i\pi}\int_{m+2-iT}^{m+2+iT} \frac{\zeta(s-m)}{\zeta(2(s-m))}R(s)\frac{x^s}{s}ds + O(x^{m+\frac{1}{2}+\varepsilon}),$$

where

$$R(s) = \begin{cases} \displaystyle\prod_p \left(1 + \frac{p^{(k-2)s} - 1}{(p^{s-m}+1)(p^{(k-1)s} - p^{(k-2)s})}\right), & m \geq k; \\ \displaystyle\prod_p \left(1 + \frac{(p^{(m-1)s} - 1) + (p^{ms} - 1)\displaystyle\sum_{j=1}^{i-1}\frac{p^{jm+s}}{p^{(jm+1)s}}}{(p^{s-m}+1)(p^{ms} - p^{(m-1)s})} + \frac{\displaystyle\sum_{j=im+1}^{k-1}\frac{p^{im+s}}{p^{js}}}{p^{s-m}+1}\right), & m < k. \end{cases}$$

To estimate the main term

$$\frac{1}{2i\pi}\int_{m+2-iT}^{m+2+iT} \frac{\zeta(s-m)}{\zeta(2(s-m))}R(s)\frac{x^s}{s}ds + O(x^{m+\frac{1}{2}+\varepsilon}),$$

we move the integral line from $s = m + 2 \pm iT$ to $s = \frac{1}{2} \pm iT$. This time, the function

$$f(s) = \frac{\zeta(s-m)x^s}{\zeta(2(s-m))s}R(s)$$

have a simple pole point at $s = m + 1$ with residue $\dfrac{x^{m+1}}{(m+1)\zeta(2)}R(m+1)$. So we have

$$\frac{1}{2i\pi}\left(\int_{m+2-iT}^{m+2+iT} + \int_{m+2+iT}^{m+\frac{1}{2}+iT} + \int_{m+\frac{1}{2}+iT}^{m+\frac{1}{2}-iT} + \int_{m+\frac{1}{2}-iT}^{m+2-iT}\right)\frac{\zeta(s-m)x^s}{\zeta(2(s-m))s}R(s)ds$$
$$= \frac{x^{m+1}}{(m+1)\zeta(2)}R(m+1).$$

Taking $T = x^{3/2}$, we can easy get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{m+2+iT}^{m+\frac{1}{2}+iT} + \int_{m+\frac{1}{2}-iT}^{m+2-iT}\right)\frac{\zeta(s-m)x^s}{\zeta(2(s-m))s}R(s)ds\right|$$
$$\ll \int_{m+\frac{1}{2}}^{m+2}\left|\frac{\zeta(\sigma-m+iT)}{\zeta(2(\sigma-m+iT))}R(s)\frac{x^2}{T}\right|d\sigma \ll \frac{x^{m+2}}{T} = x^{m+\frac{1}{2}}$$

and

$$\left|\frac{1}{2\pi i}\int_{m+\frac{1}{2}+iT}^{m+\frac{1}{2}-iT}\frac{\zeta(s-m)x^s}{\zeta(2(s-m))s}R(s)ds\right|$$
$$\ll \int_0^T\left|\frac{\zeta(1/2+it)}{\zeta(1+2it)}\frac{x^{m+\frac{1}{2}}}{t}\right|dt \ll x^{m+\frac{1}{2}+\varepsilon}.$$

Note that $\zeta(2)=\dfrac{\pi^2}{6}$, from the above estimates we have

$$\sum_{n\le x}a_m(n)c_k(n)=\frac{6x^{m+1}}{(m+1)\pi^2}R(m+1)+O(x^{m+\frac{1}{2}+\varepsilon}).$$

This completes the proof of Theorem 1.

Now we prove Theorem 2. Let

$$f_1(s)=\sum_{n=1}^{\infty}\frac{\phi(a_m(n)c_k(n))}{n^s},$$

Then from Euler product formula [6] and the definition of $\phi(n)$, we also have

if $m\ge k$, then

$$\begin{aligned}
&f_1(s)\\
=&\prod_p\left(1+\frac{\varphi(p^{m-1}\cdot p)}{p^s}+\frac{\varphi(p^{m-2}\cdot p^2)}{p^{2s}}+\cdots+\frac{\varphi(p^{m-(k-1)}\cdot p^{k-1})}{p^{(k-1)s}}\right)\\
=&\prod_p\left(1+\frac{p^m-p^{m-1}}{p^s}+\frac{p^m-p^{m-1}}{p^{2s}}+\cdots+\frac{p^m-p^{m-1}}{p^{(k-1)s}}\right)\\
=&\prod_p\left(1+\frac{1}{p^{s-m}}+\frac{1}{p^{2s-m}}(1+\frac{1}{p^s}+\cdots+\frac{1}{p^{(k-3)s}})\right.\\
&\left.-\tfrac{1}{p^{s-m+1}}(1+\tfrac{1}{p^s}+\cdots+\tfrac{1}{p^{(k-2)s}})\right)\\
=&\frac{\zeta(s-m)}{\zeta(2(s-m))}\prod_p\left(1+\frac{p(p^{(k-2)s}-1)-(p^{(k-1)s}-1)}{p(p^{s-m}+1)(p^{(k-1)s}-p^{(k-2)s})}\right);
\end{aligned}$$

if $m < k$ $(im \le k < (i+1)m, i > 1)$, then

$$\begin{aligned}
f_1(s) &= \prod_p \left(1 + \frac{p^m - p^{m-1}}{p^s} + \cdots + \frac{p^m - p^{m-1}}{p^{ms}}\right.\\
&\quad + \tfrac{p^{2m}-p^{2m-1}}{p^{(m+1)s}} + \cdots + \frac{p^{2m} - p^{2m-1}}{p^{2ms}}\\
&\quad \left. + \cdots + \tfrac{p^{(i+1)m}-p^{(i+1)m-1}}{p^{(im+1)s}} + \cdots + \tfrac{p^{(i+1)m}-p^{(i+1)m-1}}{p^{(k-1)s}}\right)\\
&= \prod_p \left(1 + \frac{1}{p^{s-m}} + \frac{1}{p^{2s-m}}\left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(m-2)s}}\right)\right.\\
&\quad - \tfrac{1}{p^{s-m+1}}\left(1 + \tfrac{1}{p^s} + \cdots + \tfrac{1}{p^{(m-1)s}}\right)\\
&\quad + \cdots + \frac{1}{p^{(m+1)s-2m}}\left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(m-1)s}}\right)\\
&\quad - \tfrac{1}{p^{(m+1)s-2m+1}}\left(1 + \tfrac{1}{p^s} + \cdots + \tfrac{1}{p^{(m-1)s}}\right)\\
&\quad + \cdots + \frac{1}{p^{((i-1)m+1)s-im}}\left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(m-1)s}}\right)\\
&\quad - \frac{1}{p^{((i-1)m+1)s-im+1}}\left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(m-1)s}}\right)\\
&\quad \left. + \textstyle\sum_{j=im+1}^{k-1} \tfrac{p^{(i+1)m}-p^{(i+1)m-1}}{p^{js}}\right)\\
&= \frac{\zeta(s-m)}{\zeta(2(s-m))}\prod_p \left(1 + \frac{(p^{(m-1)s}-1) + (p^{ms}-1)H(s)}{(p^{s-m}+1)(p^{ms}-p^{(m-1)s})} + \frac{G(s)}{p^{s-m}+1}\right),
\end{aligned}$$

where

$$H(s) = \sum_{j=1}^{i-1}\left(\frac{p^{jm+s+1} - p^{jm+s}}{p^{(jm+1)s}} - \frac{1}{p}\right)$$

and

$$G(s) = \sum_{j=im+1}^{k-1} \frac{p^{im+s} - p^{im+s-1}}{p^{js}}.$$

By Perron formula [7] and the method of proving Theorem 1, we also have

$$\sum_{n \le x} \phi(a_m(n)c_k(n)) = \frac{1}{2i\pi}\int_{m+2-iT}^{m+2+iT} \frac{\zeta(s-m)}{\zeta(2(s-m))} R^*(s)\frac{x^s}{s}ds + O(x^{m+\frac{1}{2}+\varepsilon}),$$

where

$$R^*(s) = \begin{cases} \displaystyle\prod_p \left(1 + \frac{p(p^{(k-2)s}-1) - (p^{(k-1)s}-1)}{p(p^{s-m}+1)(p^{(k-1)s}-p^{(k-2)s})}\right), & m \ge k;\\ \displaystyle\prod_p \left(1 + \frac{(p^{(m-1)s}-1) + (p^{ms}-1)H(s)}{(p^{s-m}+1)(p^{ms}-p^{(m-1)s})}\right. \\ \displaystyle\qquad \left. + \frac{G(s)}{p^{s-m}+1}\right), & m < k. \end{cases}$$

This completes the proof of Theorem 2.

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ON THE MEAN VALUE OF A NEW ARITHMETICAL FUNCTION

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Abstract The main purpose of this paper is using the analytic method to study the mean value properties of $\delta_m(f(n))$, and obtain an interesting asymptotic formula for it.

Keywords: Arithmetic function; k-power free numbers; Mean value.

§1. Introduction and main results

For any fixed positive integer q and any positive integer n, we define the arithmetical function

$$f(n) = f(q,n) = (q,n),$$

where (q,n) denote the greatest common divisor of q and n. Obviously it is a multiplicative arithmetical function. Another famous multiplicative function is defined as

$$\delta_t(n) = \max\{d: \ d \mid n, (d,t) = 1\},$$

where t is any fixed positive integer. In reference [1], Professor J.Herzog and T.Maxsein studied the mean value of the error term

$$E_t(n) = \sum_{n\leq x} \delta_t(n) - \frac{tx^2}{2\sigma(t)},$$

where $\sigma(t) = \sum\limits_{d|t} d$, and proved that

$$\sum_{n\leq x} E_t(n) = \frac{tx^2}{4\sigma(t)} + O\left(x \ln^{\omega(t)} x\right),$$

where $\omega(t)$ denotes the number of all different prime divisors of n.

In this paper, we want to study the mean value properties of $\delta_m(f(n))$, and obtain an interesting asymptotic formula for it. First we need to introduce a special number: k-power free number. A positive integer n is called a k-power free number if it can not be divided by any p^k, where p is a prime number.

In reference [2], Professor F. Smarandache asked us to study the properties of the k-power free number sequence. About this problem, many scholars had studied it before (see reference [3] and [4]).

Furthermore, there exists an interesting identity $\delta_t(n) = \delta_{\alpha(t)}(n)$, where $\alpha(t)$ is a square free number. In this paper, we shall use the analytic method to prove the following conclusion.

Theorem. Let A denotes the set of all k-power free numbers $(k > 1)$. Then for any fixed positive integer m and any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\ n\in A}} \delta_m(f(n)) = \frac{R(1)}{\zeta(k)}x + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $\zeta(k)$ is the Riemann-zeta function, ε denotes any fixed positive number, and

$$\begin{aligned} &R(s)\\ = &\prod_{\substack{p|q\\ p\dagger m}}\left(\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}\right)\prod_{\substack{p^\beta\|q\\ p\dagger m\\ \beta<k}}\left(1+\frac{1}{p^{s-1}}+\cdots+\frac{1}{p^{\beta(s-1)}}+\frac{p^\beta-p^{\beta-(k-\beta-1)s}}{p^{(\beta+1)s}-p^{\beta s}}\right)\\ &\times\prod_{\substack{p^\beta\|q\\ p\dagger m\\ \beta\geq k}}\left(1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\cdots+\frac{p^{k-1}}{p^{(k-1)s}}\right), \end{aligned}$$

where $p^\beta\|q$ denotes $p^\beta|q$ and $p^{\beta+1}\dagger q$.

From this theorem we may immediately deduce the following three Corollaries:

Corollary 1 Let q be a square free number, then for any fixed positive integer m and any real number $x \geq 1$, we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} \delta_m(f(n)) = \frac{x}{\zeta(k)}\prod_{\substack{p|q\\ p\dagger m}}\left(\frac{p^k-1}{p^k-p^{k-1}}\right)\left(2+\frac{1-p^{2-k}}{p-1}\right)+O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Corollary 2 Let m be any fixed positive integer with $(q, m) = 1$. If q be a k-power free number, then we have

$$\begin{aligned} &\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} \delta_m(f(n))\\ = &\frac{x}{\zeta(k)}\prod_{p|q}\left(\frac{p^k-1}{p^k-p^{k-1}}\right)\prod_{p^\beta\|q}\left(\beta+1+\frac{1-p^{\beta+1-k}}{p-1}\right)+O\left(x^{\frac{1}{2}+\varepsilon}\right). \end{aligned}$$

Corollary 3 For any fixed positive integer q and any real number $x \geq 1$, we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} \delta_q(f(n)) = \frac{x}{\zeta(k)} + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. For convenience, we define a new arithmetical function $a(n)$ as follows:

$$a(n) = \begin{cases} n, & if \text{n=1}, or \text{n} is a \text{k}-free number, \\ 0, & otherwise. \end{cases}$$

It is clear that

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} n = \sum_{n\leq x} a(n).$$

Let

$$F(s) = \sum_{n=1}^{\infty} \frac{\delta_m(f(a(n)))}{n^s}.$$

From the Euler product formula [5] and the definition of $\delta_m(f(a(n)))$ we have

$$\begin{aligned} F(s) &= \prod_p \left(1 + \frac{\delta_m(f(p))}{p^s} + \frac{\delta_m(f(p^2))}{p^{2s}} + \cdots + \frac{\delta_m(f(p^{k-1}))}{p^{(k-1)s}}\right) \\ &= \prod_p \left(1 + \frac{\delta_m((q,p))}{p^s} + \frac{\delta_m((q,p^2))}{p^{2s}} + \cdots + \frac{\delta_m((q,p^{k-1}))}{p^{(k-1)s}}\right) \\ &= \prod_{p\dagger q} \left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(k-1)s}}\right) \prod_{p|(m,q)} \left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(k-1)s}}\right) \\ &\quad \times \prod_{\substack{p|q\\ p\dagger m}} \left(1 + \frac{\delta_m((m,p))}{p^s} + \cdots + \frac{\delta_m((m,p^{k-1}))}{p^{(k-1)s}}\right) \\ &= \frac{\zeta(s)}{\zeta(ks)} \prod_{\substack{p|q\\ p\dagger m}} \left(\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}\right) \prod_{\substack{p^\beta\|q\\ p\dagger m\\ \beta\geq k}} \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots + \frac{p^{k-1}}{p^{(k-1)s}}\right) \\ &\quad \times \prod_{\substack{p^\beta\|q\\ p\dagger m\\ \beta<k}} \left(1 + \frac{p}{p^s} + \cdots + \frac{p^\beta}{p^{\beta s}} + \frac{p^\beta}{p^{(\beta+1)s}} + \cdots + \frac{p^\beta}{p^{(k-1)s}}\right) \end{aligned}$$

$$= \frac{\zeta(s)}{\zeta(ks)} \prod_{\substack{p|q \\ p\dagger m}} \left(\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}} \right) \prod_{\substack{p^\beta \| q \\ p\dagger m \\ \beta \geq k}} \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots + \frac{p^{k-1}}{p^{(k-1)s}} \right)$$

$$\times \prod_{\substack{p^\beta \| q \\ p\dagger m \\ \beta < k}} \left(1 + \frac{1}{p^{s-1}} + \cdots + \frac{1}{p^{\beta(s-1)}} + \frac{p^\beta - p^{\beta-(k-\beta-1)s}}{p^{(\beta+1)s} - p^{\beta s}} \right).$$

Obviously, we have inequality

$$|\delta_m(f(a(n)))| \leq n, \qquad \left| \sum_{n=1}^{\infty} \frac{\delta_m(f(a(n)))}{n^\sigma} \right| < \frac{1}{\sigma - 1},$$

where $\sigma > 1$ is the real part of s. So by Perron formula **?** we have

$$\sum_{n \leq x} \frac{a(n)}{n^{s_0}} = \frac{1}{2i\pi} \int_{b-iT}^{b+iT} f(s+s_0) \frac{x^s}{s} ds + O\left(\frac{x^b B(b+\sigma_0)}{T} \right)$$
$$+ O\left(x^{1-\sigma_0} H(2x) \min(1, \frac{\log x}{T}) \right)$$
$$+ O\left(x^{-\sigma_0} H(N) \min(1, \frac{x}{||x||}) \right),$$

where N is the nearest integer to x, $\|x\| = |x - N|$. Taking $s_0 = 0$, $b = 2$, $T = x^{\frac{3}{2}}$, $H(x) = x^2$, $B(\sigma) = \frac{1}{\sigma - 1}$, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}}} \delta_m(f(n)) = \frac{1}{2i\pi} \int_{2-iT}^{2+iT} \frac{\zeta(s)}{\zeta(ks)} R(s) \frac{x^s}{s} ds + O(x^{\frac{1}{2}+\varepsilon}),$$

where

$$R(s) = \prod_{\substack{p|q \\ p\dagger m}} \left(\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}} \right) \prod_{\substack{p^\beta \| q \\ p\dagger m \\ \beta \geq k}} \left(1 + \frac{p}{p^s} + \frac{p^2}{p^{2s}} + \cdots + \frac{p^{k-1}}{p^{(k-1)s}} \right)$$

$$\times \prod_{\substack{p^\beta \| q \\ p\dagger m \\ \beta < k}} \left(1 + \frac{1}{p^{s-1}} + \cdots + \frac{1}{p^{\beta(s-1)}} + \frac{p^\beta - p^{\beta-(k-\beta-1)s}}{p^{(\beta+1)s} - p^{\beta s}} \right).$$

To estimate the main term

$$\frac{1}{2i\pi} \int_{2-iT}^{2+iT} \frac{\zeta(s)}{\zeta(ks)} R(s) \frac{x^s}{s} ds,$$

we move the integral line from $s = 2 \pm iT$ to $s = \frac{1}{2} \pm iT$. This time, the function

$$f(s) = \frac{\zeta(s)x^s}{\zeta(ks)s} R(s)$$

has a simple pole point at $s = 1$ with residue $\frac{R(1)}{\zeta(k)} x$. So we have

$$\frac{1}{2i\pi}\left(\int_{2-iT}^{2+iT} + \int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{2-iT}\right)\frac{\zeta(s)x^s}{\zeta(ks)s}ds = \frac{R(1)x}{\zeta(k)}.$$

We can easily get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{2-iT}\right)\frac{\zeta(s)x^s}{\zeta(ks)s}ds\right|$$
$$\ll \int_{\frac{1}{2}}^{2}\left|\frac{\zeta(\sigma - 1 + iT)}{\zeta(k(\sigma - 1 + iT))}\frac{x^2}{T}\right| d\sigma \ll \frac{x^2}{T} = x^{\frac{1}{2}}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\frac{\zeta(s)x^s}{\zeta(ks)s}ds\right| \ll \int_0^T \left|\frac{\zeta(1/2 + it)}{\zeta(k/2 + kit)}\frac{x^{\frac{1}{2}}}{t+1}\right| dt \ll x^{\frac{1}{2}+\varepsilon}.$$

Combining the above estimates we have

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}} \delta_m(f(n)) = \frac{R(1)}{\zeta(k)}x + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

This completes the proof of Theorem.

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This book contains 34 papers, most of which were written by participants to the First Northwest Number Theory Conference held in Shangluo Teacher's College, China, in March, 2005. In this Conference, several professors gave a talk on Smarandache Problems and many participants lectured on them both extensively and intensively.

All these papers are original and have been refereed. The themes of these papers range from the mean value or hybrid mean value of Smarandache type functions, the mean value of some famous number theoretic functions acting on the Smarandache sequences, to the convergence property of some infinite series involving the Smarandache type sequences.

(The Editors)

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