# Canonical gauge-invariant variables for scalar perturbations in synchronous coordinates

Zdzisław A. Golda, Andrzej Woszczyna and Karolina Zawada

Astronomical Observatory, Jagellonian University
ul. Orla 171, 30–244 Kraków, Poland

**Abstract**

Under an appropriate change of the perturbation variable Lifshitz-Khalatnikov propagation equations for the scalar perturbation reduce to d'Alembert equation. The change of variables is based on the Darboux transform.

## 1 Introduction

The gauge-invariant perturbation theories efficiently eliminate nonphysical perturbations, yet they provide the propagation equations in noncanonical form — different for each theory. In this case, pure artefacts of the choice of the reference system, gauge or the perturbation variable, are likely to be confused with new dynamical phenomena yet unknown in the laboratory-scale physics. This particularly concerns the large scale limit, where some "non-oscillatory behaviour" outside the particle horizon is commonly expected.

In this paper we show that the definition of the gauge-invariant variables still possesses a freedom to chose the canonical variables, i.e. variables which satisfy d'Alembert equation in its standard form. To show that we have used the classical Lifshitz-Khalatnikov formalism, which after being appropriately extended is equivalent to other "manifestly" gauge invariant descriptions. A systematic construction of the gauge-invariant canonical variables is provided. Consequently, the scalar perturbation propagates like a massless scalar field in the Roberson-Walker space-time, and therefore there is a close analogy between the perturbation theory (the acoustics of the expanding universes) and the field theory in the curved space-time. Propagation of the sound waves in the early universe may be also considered as an example of the acoustic geometry in the sense of Unruh [1].

The paper is organized as follows: In the second Section we remind the method of elimination of spurious modes in the Lifshitz-Khalatnikov theory [2]. This Section is a generalization of the procedure given in [3] to the case of arbitrary space curvature and arbitrary density-dependent pressure $p = p(\epsilon)$.

In Section 3 we present the general method of construction of gauge-invariant quantities in the synchronous system of reference. We show that the procedure is not unique, and therefore, there exist vast classes of physically relevant gauge-invariant variables. The remaining freedom to choose between them may be used to better describe the perturbation dynamical properties, or to construct more adequate observables. A specific choice depends on the researcher motivation, and on the character of the problem to be solved. Finally in Section 4 we adopt the previously defined methods to obtain a canonical form of the perturbation equations — the d'Alembert equation.

# 2 Synchronous system of reference — techniques of reduction of the gauge freedom

Consider the Robertson-Walker universe of arbitrary space curvature ($K = -1, 0, +1$)

$$g_{\mu\nu} = a^2(\eta)\ \mathrm{diag}\left[-1,\ 1,\ \frac{\sin^2(\sqrt{K}\chi)}{K},\ \frac{\sin^2(\sqrt{K}\chi)}{K}\sin^2\theta\right], \tag{1}$$

with the hydrodynamic energy-momentum tensor

$$T^{\mu\nu} = (\epsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu} \tag{2}$$

and the barotropic equation of state $p = p(\epsilon)$. We investigate small perturbations $\delta g_{\mu\nu}$ of the metric (1). We limit ourselves to scalar (density) perturbations, therefore, $\delta g_{\mu\nu}$ is defined by two scalar functions $\lambda_{(k)}(\eta)$ and $\mu_{(k)}(\eta)$

$$\begin{aligned} \delta g_{\mu 0} &= 0, && (3)\\ \delta g_m{}^n &= \sum_k \left(\lambda_{(k)}(\eta) P_m{}^n + \mu_{(k)}(\eta) Q_m{}^n\right) + \text{c.c.}, && (4) \end{aligned}$$

where $P_m{}^n$ and $Q_m{}^n$ stand for scalar harmonics [2],

$$\begin{aligned} Q_m{}^n &= \frac{1}{3} Q_{(k)}(\mathbf{x})\delta_m{}^n, && (5)\\ P_m{}^n &= \frac{1}{k^2 - K} Q_{;m}{}^{;n} + Q_m{}^n, && (6) \end{aligned}$$

$Q_{(k)}(\mathbf{x})$ is the complex solution of the Helmholtz equation $Q_{(k)}(\mathbf{x})_{;m}{}^{;m} = -(k^2 - K)Q_{(k)}(\mathbf{x})$, $\mathbf{x} = \{\chi, \theta, \varphi\}$, and the amplitudes $\lambda_{(k)}(\eta)$ and $\mu_{(k)}(\eta)$ satisfy the system of two second order equations convoluted to each other

$$\lambda''_{(k)}(\eta) + 2\frac{a'(\eta)}{a(\eta)}\lambda'_{(k)}(\eta) - \frac{k^2 - K}{3}\left[\lambda_{(k)}(\eta) + \mu_{(k)}(\eta)\right] = 0, \tag{7}$$

$$\begin{aligned} &\mu''_{(k)}(\eta) + \left[2 + 3c_s^2(\eta)\right]\frac{a'(\eta)}{a(\eta)}\mu'_{(k)}(\eta)\\ &\qquad + \frac{k^2 - 4K}{3}\left[1 + 3c_s^2(\eta)\right]\left[\lambda_{(k)}(\eta) + \mu_{(k)}(\eta)\right] = 0. \end{aligned} \tag{8}$$

$c_s(\eta)$ denotes the sound velocity: $c_s(\eta) = \sqrt{p'(\eta)/\epsilon'(\eta)} \neq 0$. Then the density contrast[1] $\delta = \delta\epsilon/\epsilon$ is a linear combination [2]

$$\delta(\eta, \mathbf{x}) = \sum_k \delta_{(k)}(\eta)\, Q_{(k)}(\mathbf{x}) + \text{c.c.}, \tag{9}$$

and

$$\delta_{(k)}(\eta) = \frac{\mathcal{A}_{(k)}}{3\epsilon(\eta)a^2(\eta)} \left[ (k^2 - 4K)(\lambda_{(k)}(\eta) + \mu_{(k)}(\eta)) + 3\frac{a'(\eta)}{a(\eta)}\mu'_{(k)}(\eta) \right], \tag{10}$$

where $\mathcal{A}_{(k)}$ are arbitrary complex numbers. The system (7–8) defines the four dimensional phase-space, which means that the system possesses two physical and two gauge degrees of freedom. Pure gauge solutions are known, and for $\lambda_{(k)}(\eta)$ they respectively read [2]

$$G_1(\eta) = 1, \tag{11}$$

$$G_2(\eta) = -(k^2 - K) \int \frac{1}{a(\eta)} \mathrm{d}\eta. \tag{12}$$

Difficulties with fixing the gauge freedom have stimulated elaboration of the gauge-invariant theories [4, 5, 6, 7, 8, 9]. After the famous Bardeen's paper [10] these theories were intensively developed for almost two decades. Yet, the original Lifshitz-Khalatnikov formalism, when appropriately extended, provides an equally good description of inhomogeneities. The procedure is as follows: we reduce (7–8) system to the single fourth-order equation for $\lambda_{(k)}(\eta)$.

$$\begin{aligned}
&{\lambda_{(k)}}^{(4)}(\eta) + a(\eta)H(\eta)\left[4 + 3c_s^2(\eta)\right]{\lambda_{(k)}}^{(3)}(\eta) \\
&+ \left\{ -5K + a^2(\eta)\left[ 2\left(\frac{\epsilon(\eta)}{3} - p(\eta)\right) + \left(9H^2(\eta) - \epsilon(\eta)\right)c_s^2(\eta) \right] \right. \\
&+ \left. \left(k^2 - K\right)c_s^2(\eta) \right\} \lambda''_{(k)}(\eta) + a(\eta)H(\eta)\left[-a^2(\eta)\left[\epsilon(\eta) + 3p(\eta)\right]\right. \\
&+ \left. 2a^2(\eta)H^2(\eta)\left[1 + 3c_s^2(\eta)\right] + \left(k^2 - K\right)c_s^2(\eta)\right]\lambda'_{(k)}(\eta) = 0,
\end{aligned} \tag{13}$$

where $H(\eta)$ stands for the Hubble parameter $H(\eta) = a'(\eta)/a^2(\eta)$. The knowledge of the gauge solutions (11–12) enables one to extract the gauge space from the space of all solutions. First, we write the solutions in the form

$$\lambda_{(k)}(\eta) = f_1\lambda_{1(k)}(\eta) + f_2\lambda_{2(k)}(\eta) + g_1G_{1(k)}(\eta) + g_2G_{2(k)}(\eta) \tag{14}$$

with explicitly separated linear subspace $g_1G_{1(k)}(\eta) + g_2G_{2(k)}(\eta)$ carrying all the gauge freedom ($g_1, g_2$ two arbitrary coefficients). Subsequently, we adopt the Darboux transform[2] [12] of $\lambda_{(k)}(\eta)$ to express the two remaining (physical)

[1]In more rigorous notation one should write $\delta = \delta\epsilon/\epsilon_0$, where $\epsilon_0$ denotes background, unperturbed energy density. For the sake of simplicity we skip the subscript "0" in formulas.

[2]Darboux transform of a function $f(x)$ is defined as $\widehat{f}(x) = A(x)f(x) + B(x)f'(x)$ (compare also formula (18) in section 3) with arbitrary $x$-dependent coefficients $A$ and $B$. The same transform is extensively exploited in the soliton theory [11].

degrees of freedom. A new perturbation variable $B_{(k)}(\eta)$ appears

$$B_{(k)}(\eta) = a(\eta)\frac{\mathrm{d}}{\mathrm{d}\,\eta}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\,\eta}\frac{\lambda_{(k)}(\eta)}{G_{1(k)}(\eta)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}\,\eta}\frac{G_{2(k)}(\eta)}{G_{1(k)}(\eta)}\right)^{-1}\right] = a(\eta)\frac{\mathrm{d}}{\mathrm{d}\,\eta}\frac{\frac{\mathrm{d}}{\mathrm{d}\,\eta}\lambda_{(k)}(\eta)}{\frac{\mathrm{d}}{\mathrm{d}\,\eta}G_{2(k)}(\eta)}. \tag{15}$$

On strength of (13) the variable $B_{(k)}(\eta)$ satisfies the second order equation

$$\begin{aligned} &B''_{(k)}(\eta) - \left[1 - 3c_\mathrm{s}^2(\eta)\right] a(\eta)H(\eta)\, B'_{(k)}(\eta) \\ &- \left[\left(\frac{1}{3} + c_\mathrm{s}^2(\eta)\right) a^2(\eta)\epsilon(\eta) - \left(k^2 - K\right) c_\mathrm{s}^2(\eta)\right] B_{(k)}(\eta) = 0. \end{aligned} \tag{16}$$

Equation (16) does not contain gauge the coefficients, therefore, the variable $B_{(k)}(\eta)$ is gauge-invariant. Now, the function $\lambda_{(k)}(\eta)$ can be found by the inverse Darboux transform of the two linearly independent solutions to equation (16)[3]

$$\begin{aligned} \lambda_{(k)}(\eta) = f_1(K-k^2)\int\frac{1}{a(\eta)}\left[\int\frac{B_{1(k)}(\eta)}{a(\eta)}\mathrm{d}\eta\right]\mathrm{d}\eta \\ + f_2(K-k^2)\int\frac{1}{a(\eta)}\left[\int\frac{B_{2(k)}(\eta)}{a(\eta)}\mathrm{d}\eta\right]\mathrm{d}\eta + g_1G_{1(k)}(\eta) + g_2G_{2(k)}(\eta). \end{aligned} \tag{17}$$

On can insert $\lambda_{(k)}(\eta)$ to (7) and solve it algebraically to obtain the second unknown function $\mu_{(k)}(\eta)$. As a result, by fixing four numbers $\{f_1, f_2, g_1, g_2\}$ one uniquely determines the metric correction $\delta g_{\mu\nu}$. The metric correction is not gauge-invariant and cannot be such in any perturbation formalism. Quantities containing time integrals of $B_{(k)}(\eta)$ would exhibit similar properties, while those containing $B_{(k)}(\eta)$ and derivatives are obviously gauge-invariant.

## 3 Families of gauge-invariant variables in the synchronous reference system

The metric perturbation can be written in the form (17) if the solutions of the equation (16) are explicitly known. However, even in the case when these solutions are not known one may employ the equation (16) to "convert" some non-gauge-invariant perturbation variables into gauge-invariant ones. This particularly refers to perturbation measures based on hydrodynamic quantities, since they involve the metric derivatives and are free of the metric elements themselves. The procedure is following:
1) For the non-gauge-invariant perturbation variable $X$ we take the linear combination of $X$ and its time derivative $X'$ (the Darboux transform)

$$\widehat{X}(\eta, \mathbf{x}) = c_1(\eta)\left[\frac{\partial}{\partial\eta}X(\eta,\mathbf{x}) + c_2(\eta)X(\eta,\mathbf{x})\right], \tag{18}$$

[3] The integration constants in the inverse Darboux transform can be set arbitrary, because the gauge freedom is already guaranteed by free choice of the coefficients $g_1$ and $g_2$.

which means that

$$\widehat{X}(\eta, \mathbf{x}) = \sum_k \widehat{X}_{(k)}(\eta)\, Q_{(k)}(\mathbf{x}) + \text{c.c.}, \tag{19}$$

and

$$\widehat{X}_{(k)}(\eta) = c_1(\eta)\left[\frac{\mathrm{d}}{\mathrm{d}\,\eta} X_{(k)}(\eta) + c_2(\eta) X_{(k)}(\eta)\right]. \tag{20}$$

2) We apply the equation (16) to formally express the coefficients $\widehat{X}_{(k)}(\eta)$ in the form

$$\widehat{X}_{(k)}(\eta) = F_1(\eta)\frac{\mathrm{d}}{\mathrm{d}\,\eta} B_{(k)}(\eta) + F_2(\eta) B_{(k)}(\eta) + F_3(\eta)\int \frac{B_{(k)}(\eta)}{a(\eta)}\,\mathrm{d}\eta. \tag{21}$$

3) We look for $c_1(\eta)$ and $c_2(\eta)$ such that $\widehat{X}_{(k)}(\eta)$ contains solely function $B_{(k)}(\eta)$ and its derivatives but not integrals. (Integral over $\eta$ in (21) introduces arbitrary constant $c_{(k)}$, which depends on the wave vector $\mathbf{k}$. The variable $\widehat{X}(\eta, \mathbf{x})$ (formula (19)) would contain then terms being arbitrary functions of the space coordinate $\mathbf{x}$. In this way integration constant $c_{(k)}$ restores one degree of the gauge freedom.) The relevant condition $F_3(\eta) = 0$ leads to a linear equation for $c_2(\eta)$, and leaves $c_1(\eta)$ free. Therefore, the new gauge-invariant variable $\widehat{X}(\eta, \mathbf{x})$ is actually a $c_1(\eta)$-dependent family of variables, where $c_1(\eta)$ is an arbitrary function of time. Below we show some examples.

## 3.1 The density contrast

On strength of (9), (7–8) and (17) the density contrast $\delta_{(k)}(\eta)$ can be expressed by use of the gauge invariant function $B_{(k)}(\eta)$.

$$\begin{aligned}\delta_{(k)}(\eta) \quad = \quad & -3\frac{H(\eta)}{a^3(\eta)\epsilon(\eta)} B'_{(k)}(\eta) - \frac{1}{a^2(\eta)}\left[\frac{k^2 - K}{a^2(\eta)\epsilon(\eta)} - 1\right] B_{(k)}(\eta) \\ & + \frac{3}{2}\left[p(\eta) + \epsilon(\eta)\right]\frac{H(\eta)}{\epsilon(\eta)}\int \frac{B_{(k)}(\eta)}{a(\eta)}\mathrm{d}\eta. \end{aligned} \tag{22}$$

The contrast $\delta_{(k)}(\eta)$ is not gauge-invariant, as it is well known [10]. The gauge freedom is caused by the last term in (22) containing the integral of $B_{(k)}(\eta)$. Let us introduce a new variable — the “modified density contrast”

$$\widehat{\delta}_{(k)}(\eta) = c_1(\eta)\left[\frac{\mathrm{d}}{\mathrm{d}\,\eta}\delta_{(k)}(\eta) + c_2(\eta)\delta_{(k)}(\eta)\right]. \tag{23}$$

With the aid of (16) one obtains

$$\widehat{\delta}_{(k)}(\eta) = F_1(\eta)\frac{\mathrm{d}}{\mathrm{d}\,\eta} B_{(k)}(\eta) + F_2(\eta) B_{(k)}(\eta) + F_3(\eta)\int \frac{B_{(k)}(\eta)}{a(\eta)}\mathrm{d}\eta, \tag{24}$$

where the function $F_3(\eta)$ reads[4]

$$F_3(\eta) = \frac{3}{2} c_1(\eta) H(\eta) \left[1 + \frac{p(\eta)}{\epsilon(\eta)}\right] \left[\frac{\mathrm{d}}{\mathrm{d}\,\eta} \ln\left(\frac{1}{a(\eta)} \frac{\epsilon'(\eta)}{\epsilon(\eta)}\right) + c_2(\eta)\right]. \tag{25}$$

By setting

$$c_2(\eta) = -\frac{\mathrm{d}}{\mathrm{d}\,\eta} \ln\left(\frac{1}{a(\eta)} \frac{\epsilon'(\eta)}{\epsilon(\eta)}\right), \tag{26}$$

one eliminates the integral of $B_{(k)}(\eta)$ ($F_3(\eta) = 0$), hence $\widehat{\delta}_{(k)}(\eta)$ becomes a gauge independent quantity

$$\widehat{\delta}_{(k)}(\eta) = c_1(\eta) \frac{\epsilon'(\eta)}{a(\eta)\epsilon(\eta)} \frac{\mathrm{d}}{\mathrm{d}\,\eta} \left[\frac{a(\eta)\epsilon(\eta)}{\epsilon'(\eta)} \delta_{(k)}(\eta)\right]. \tag{27}$$

This variable is still defined up to the factor $c_1(\eta)$ being an arbitrary function of time.

## 3.2 The expansion rate contrast

Identically we proceed with inhomogeneities in the expansion rate[5] $\delta\vartheta(\eta, \mathbf{x})$.

$$\delta\vartheta(\eta, \mathbf{x}) = \sum_k \delta\vartheta_{(k)}(\eta)\, Q_{(k)}(\mathbf{x}) + \text{c.c.}, \tag{28}$$

where

$$\begin{aligned} \delta\vartheta_{(k)}(\eta) \quad = \quad & \frac{1}{a^3(\eta)} \left[\frac{k^2 - K}{a^2(\eta)(p(\eta) + \epsilon(\eta))} - \frac{3}{2}\right] B'_{(k)}(\eta) \\ & - \frac{H(\eta)}{a^2(\eta)} \left[\frac{k^2 - K}{a^2(\eta)(p(\eta) + \epsilon(\eta))} - \frac{3}{2}\right] B_{(k)}(\eta) \\ & + \frac{3}{2} \left[\frac{1}{2}\left(p(\eta) + \epsilon(\eta)\right) - \frac{K}{a^2(\eta)}\right] \int \frac{B_{(k)}(\eta)}{a(\eta)} \mathrm{d}\eta\,. \end{aligned} \tag{29}$$

We introduce $\widehat{\delta\vartheta}_{(k)}(\eta)$ defined as a linear combination

$$\widehat{\delta\vartheta}_{(k)}(\eta) = c_1(\eta) \left[\frac{\mathrm{d}}{\mathrm{d}\,\eta} \delta\vartheta_{(k)}(\eta) + c_2(\eta) \delta\vartheta_{(k)}(\eta)\right], \tag{30}$$

and eliminate the integral of $B_{(k)}(\eta)$. We finally obtain

$$\widehat{\delta\vartheta}_{(k)}(\eta) = c_1(\eta) \left[\frac{\mathrm{d}}{\mathrm{d}\,\eta} \delta\vartheta_{(k)}(\eta) - \frac{\mathrm{d}}{\mathrm{d}\,\eta} \ln\left[\frac{H'(\eta)}{a(\eta)}\right] \delta\vartheta_{(k)}(\eta)\right]. \tag{31}$$

[4] $F_1(\eta)$ and $F_2(\eta)$ are irrelevant to the problem discussed here, therefore we skip writing them explicitly.

[5] $\vartheta \equiv u^{\mu}{}_{;\mu}$ is the expansion [13].

### 3.3 Other variables

By employing the same procedure one may define gauge-invariant variables which combine inhomogeneities in Hubble flow with inhomogeneities of the energy density

$$\widehat{\mathsf{r}}(\eta, \mathbf{x}) = c_1(\eta)\left[\delta\epsilon(\eta, \mathbf{x}) - \frac{1}{3}\frac{\epsilon'(\eta)}{H'(\eta)}\delta\vartheta(\eta, \mathbf{x})\right]. \qquad (32)$$

It is clear that functions $F(\widehat{\delta}, \widehat{\delta\vartheta}, \widehat{\mathsf{r}}, \ldots, \epsilon, H, \ldots)$ of the gauge-invariant variables $\{\widehat{\delta}, \widehat{\delta\vartheta}, \widehat{\mathsf{r}}, \ldots\}$ and the parameters of the background universe $\{\epsilon, H, \ldots\}$ are also gauge-invariant. None of these variables is preferred by the perturbation theory itself. The choice of the variable depends on its physical meaning and usefulness, its relation to other physical concepts and theories, and eventually on the ability to construct relevant observables.

# 4 Perturbation equations in the canonical form

In all the examples above we constructed the gauge-invariant variables which refer to the synchronous system of reference. The function $c_2(\eta)$ was uniquely determined for each $\widehat{\delta}(\eta)$, $\widehat{\vartheta}(\eta)$ and $\widehat{X}(\eta)$ by the demand of their gauge-invariance. The freedom to make an arbitrary choice of the time-dependent factor $c_1(\eta)$ still remains, and therefore, we are now able to construct gauge-invariant variables with particular dynamical properties. Consider the variable $\widehat{\delta}(\eta)$ with $c_1(\eta)$ chosen as

$$c_1(\eta) = a^2(\eta)H^2(\eta)\frac{\epsilon(\eta)}{\epsilon'(\eta)}. \qquad (33)$$

We obtain the gauge-invariant variable[6]

$$\widehat{\delta}_{(k)}(\eta) = a(\eta)H^2(\eta)\frac{\mathrm{d}}{\mathrm{d}\,\eta}\left[a(\eta)\frac{\epsilon(\eta)}{\epsilon'(\eta)}\delta_{(k)}(\eta)\right], \qquad (34)$$

which satisfies the propagation equation of the form

$$\widehat{\delta}''_{(k)}(\eta) + \left[2\frac{\mathsf{a}'(\eta)}{\mathsf{a}(\eta)} - \frac{c_\mathrm{s}'(\eta)}{c_\mathrm{s}(\eta)}\right]\widehat{\delta}'_{(k)}(\eta) + c_\mathrm{s}^2(\eta)(k^2 - K)\widehat{\delta}_{(k)}(\eta) = 0, \qquad (35)$$

with the function $\mathsf{a}(\eta)$ defined as

$$\mathsf{a}(\eta) = a(\eta)\sqrt{\frac{1}{c_\mathrm{s}(\eta)}\frac{\epsilon(\eta) + p(\eta)}{3H^2(\eta)}}. \qquad (36)$$

Now we introduce the new time parameter $\mathrm{d}\xi = c_\mathrm{s}(\eta)\mathrm{d}\eta$ (acoustic conformal time [14]). After this reparametrization the propagation equation reduces to d'Alembert equation

$$\nabla^\mu\nabla_\mu\,\widehat{\delta}(\xi, \mathbf{x}) = 0 \qquad (37)$$

[6]Please note the misprint in paper [14] formula (7), already corrected in gr-qc/0302091.

in the Robertson-Walker space-time with coordinates $\{\xi, \chi, \theta, \varphi\}$ and the metric form

$$\mathsf{g}_{\mu\nu} = \mathsf{a}^2(\xi)\ \mathrm{diag}\left[-1,\ 1,\ \frac{\sin^2(\sqrt{K}\chi)}{K},\ \frac{\sin^2(\sqrt{K}\chi)}{K}\sin^2\theta\right], \tag{38}$$

Therefore, the modified density contrast $\widehat{\delta}(\xi, \mathbf{x})$ propagates in the Robertson-Walker space-time with the scale factor $\mathsf{a}(\xi)$ in the same manner as minimally coupled scalar field in the Robertson-Walker space-time with the scale factor $a(\eta)$ [15]. All the classical results obtained in the field theory on the curved space-times apply to the density perturbations in the expanding universe. One can introduce the Lagrangian

$$\mathcal{L} = \frac{1}{2}\mathsf{g}^{\mu\nu}\widehat{\delta}_{,\mu}\widehat{\delta}_{,\nu} \tag{39}$$

for field $\widehat{\delta}(\xi, \mathbf{x})$ and derive the equation of motion (37) by use of the Euler-Lagrange equations

$$\nabla_\mu \frac{\partial}{\partial\widehat{\delta}_{,\mu}}\mathcal{L} - \frac{\partial}{\partial\widehat{\delta}}\mathcal{L} = 0. \tag{40}$$

An appropriate change of the perturbation variables [14] reduces the propagation equation of any gauge-invariant theory to the form of the equation (16). By use of the procedure discussed above one can construct canonical variables, and as a consequence, each of these formalism can be expressed in the Lagrange-Hamilton language.

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