An Upwind Vertex Centred Finite Volume Algorithm for Computational Contact Mechanics

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Abstract

In the current work, a second order finite volume computational framework established upon a system of first order conservation laws was employed. In this case, the classical linear momentum conservation equation is solved in conjunction with a geometric conservation equation for the deformation gradient tensor. Taking advantage of this formalism, the continuity of the normal components of the velocity and traction at contact interface was explicitly enforced by means of the Rankine-Hugoniot jump conditions. More specifically, the normal traction will be enforced in the usual linear momentum equation, whereas the normal velocity will be enforced in the geometric conservation equation without resorting to an iterative scheme. Additionally, a Total Variation Diminishing shock capturing technique was easily incorporated in order to improve dramatically the performance of the algorithm at the vicinity of contact. No ad-hoc algorithmic regularisation procedures are needed. Finally, a numerical example which considered the impact of two (nearly incompressible) elastic bars was presented in order to demonstrate the accuracy and robustness of the methodology compared to standard linear Finite Elements. It was observed that oscillations were present in the solution when using linear triangle Finite Elements while the proposed method produced results that agreed well with bilinear quadrilateral Finite Elements. In conclusion, the mixed based system of first order conservation laws effectively alleviate non-physical zero energy modes and spurious pressure instabilities when modelling the contact between two (nearly incompressible) deformable solids.

Key words: *Contact; Riemann Solver; Conservation Laws; Solid Dynamics; Shocks*

1 Introduction

Many problems in the simulation of prototype tests and manufacturing processes involve contact and impact phenomena, typically characterised by a non-smooth response associated with transitions between contact and separation modes and between stick and slip modes. Such problems are posed mathematically by demanding the usual satisfaction of linear momentum balance equation and initial and boundary conditions for each body separately, while imposing additional set of kinematic and kinetic conditions that govern the interaction of these bodies with each other. When considering frictionless model of contact, these conditions act to preclude interpenetration of the bodies (kinematic condition) and to insure compressive interaction normal to the interface (kinetic condition). One challenging aspect is that impenetrability cannot be expressed as an evolution (or algebraic) equation and so requires special numerical treatment [1]. The most common techniques addressing this include penalty method, Lagrange multiplier method, or a combination of both. In the penalty method, the impenetrability constraint is enforced as a penalty normal traction along the contact surface. This allows unpredictable amount of interpenetration and, potentially, can generate ill-conditioned systems that may require extremely small time steps for stability. For Lagrange mutiplier method, impenetrability is weakly enforced and, in general, requires computationally demanding iterative solvers [1]. In this work, an alternative approach to explicitly enforce those interface conditions is proposed through the use of a system of first order conservation equations [2].

2 Reversible Elastodynamics

Consider the three dimensional deformation of an elastic body of material density ρ_R moving from its initial undeformed configuration occupying a volume Ω_R, of boundary $\partial\Omega_R$, to a time dependent deformed configuration occupying a volume $\Omega_R(t)$, of boundary $\partial\Omega_R(t)$ at time t. The motion of the body is defined through a deformation mapping $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X}, t)$ which satisfies the following first order system of balance equations [2, 3]

$$\frac{\partial(\rho_R \boldsymbol{v})}{\partial t} - \text{DIV}\boldsymbol{P} = \boldsymbol{f}_R; \tag{1a}$$

$$\frac{\partial \boldsymbol{F}}{\partial t} - \boldsymbol{\nabla}_0 \boldsymbol{v} = \boldsymbol{0}. \tag{1b}$$

To guarantee the existence of a single-valued mapping, expression (1b) requires the satisfaction of a set of compability conditions, that is

$$\text{CURL}\boldsymbol{F} = \boldsymbol{0}. \tag{2}$$

Here, $\boldsymbol{v}$ represents the velocity field, $\boldsymbol{f}_R$ is the body force, $\boldsymbol{F}$ is the deformation gradient tensor, $\boldsymbol{P}$ is the first Piola–Kirchhoff stress tensor, DIV and CURL represent the material divergence and curl operators, and $\boldsymbol{\nabla}_0$ is the material gradient operator defined as $[\boldsymbol{\nabla}_0]_I = \frac{\partial}{\partial X_I}$.
Equation (1a) represents the usual linear momentum balance equation, whereas equation (1b) represents additional geometric conservation equation for deformation gradient tensor. When considering the interaction between two bodies, additional set of contact interface conditions (also known as Karush-Kuhn-Tucker conditions) must be suitably enforced. Taking advantage of the above formalism (1a,b), it is interesting to notice that kinetic conservation condition can be ensured in (1a), whereas the kinematic impenetrability condition can be easily enforced in (1b).
In the presence of non-smooth solutions, above equations (1a,b) are accompanied by appropriate Rankine Hugoniot jump conditions across a discontinuous surface (defined by a material unit normal vector $\boldsymbol{N}$) propagating with speed U in the reference space. This can be described as [2, 3]

$$U\,\rho_R [\![\boldsymbol{v}]\!] = -[\![\boldsymbol{P}]\!]\boldsymbol{N}; \qquad U[\![\boldsymbol{F}]\!] = -[\![\boldsymbol{v}]\!] \otimes \boldsymbol{N}, \tag{3}$$

with $[\![\cdot]\!] = [\cdot]^R - [\cdot]^L$ being defined as the jump between the right state R and the left state L of a discontinuous interface.

3 Contact Interface Conditions

When two separate bodies are in contact, they must follow certain interface conditions, whereas, when the two bodies are not in contact they can deform independently. In the case of contact-stick, one crucial interface condition is the complete continuity of the velocity and traction, that is the velocity and traction are continuous in both the normal and tangential directions of the surface. Another interface condition is the contact surface cannot support tension. In this work, these interface conditions are enforced by solving a Riemann-like problem [4] at contact region.
To achieve this, it is necessary to evaluate both the velocity and traction vectors at the contact point immediately following the impact. Note first that the impact will generate two types of shock waves (e.g. volumetric wave c_p and shear wave c_s) travelling from the contact point into each of the two bodies. Specifically, the generated shock waves will travel with volumetric speed and from equation (3a) , the velocity jump across the left and right shock waves can be deduced as

$$c_p^L \rho_R^L [\![\boldsymbol{v}]\!] = -[\![\boldsymbol{P}]\!](-\boldsymbol{N}^L); \qquad c_p^R \rho_R^R [\![\boldsymbol{v}]\!] = -[\![\boldsymbol{P}]\!](-\boldsymbol{N}^R). \tag{4}$$

Multiplying the above equations by a unique common normal vector $\boldsymbol{n}$, and after some algebraic manipulations, gives

$$t_n^C = \frac{\rho_R^R c_p^R t_n^L - \rho_R^L c_p^L t_n^R}{\rho_R^L c_p^L + \rho_R^R c_p^R} + \frac{\rho_R^L \rho_R^R c_p^L c_p^R}{\rho_R^L c_p^L + \rho_R^R c_p^R}\left(v_n^R - v_n^L\right); \qquad v_n^C = \frac{\rho_R^L c_p^L v_n^L + \rho_R^R c_p^R v_n^R}{\rho_R^L c_p^L + \rho_R^R c_p^R} - \frac{t_n^L + t_n^R}{\rho_R^L c_p^L + \rho_R^R c_p^R}, \tag{5}$$

where $t_n^L = \boldsymbol{n} \cdot \left(\boldsymbol{P}^L \boldsymbol{N}^L\right)$, $t_n^R = \boldsymbol{n} \cdot \left(\boldsymbol{P}^R \boldsymbol{N}^R\right)$, $v_n^L = \boldsymbol{n} \cdot \boldsymbol{v}^L$ and $v_n^R = \boldsymbol{n} \cdot \boldsymbol{v}^R$.

In the situation where friction is present to a sufficient degree to prevent relative sliding, a similar derivation for the common tangential components of the velocity and traction can now be followed. This is described as

$$\boldsymbol{t}_{\boldsymbol{t}}^{C}=\frac{\rho_R^R c_s^R \boldsymbol{t}_{\boldsymbol{t}}^L-\rho_R^L c_s^L \boldsymbol{t}_{\boldsymbol{t}}^R}{\rho_R^L c_s^L+\rho_R^R c_s^R}+\frac{\rho_R^L \rho_R^R c_s^L c_s^R}{\rho_R^L c_s^L+\rho_R^R c_s^R}\left(\boldsymbol{v}_{\boldsymbol{t}}^R-\boldsymbol{v}_{\boldsymbol{t}}^L\right); \qquad \boldsymbol{v}_{\boldsymbol{t}}^{C}=\frac{\rho_R^L c_s^L \boldsymbol{v}_{\boldsymbol{t}}^L+\rho_R^R c_s^R \boldsymbol{v}_{\boldsymbol{t}}^R}{\rho_R^L c_s^L+\rho_R^R c_s^R}-\frac{\boldsymbol{t}_{\boldsymbol{t}}^L+\boldsymbol{t}_{\boldsymbol{t}}^R}{\rho_R^L c_s^L+\rho_R^R c_s^R}. \tag{6}$$

Finally, the complete common velocity and traction vectors post-impact at the contact point can be evaluated as

$$\boldsymbol{t}^C=t_n^C\boldsymbol{n}+\boldsymbol{t}_t^C; \qquad \boldsymbol{v}^C=v_n^C\boldsymbol{n}+\boldsymbol{v}_t^C. \tag{7}$$

Furthermore, when considering contact-slip, it is physically possible to postulate that the common tangential traction components vanish, that is $\boldsymbol{t}_t^C=\boldsymbol{0}$. With this at hand, the common tangential components of velocity then become

$$\boldsymbol{v}_t^C=\boldsymbol{v}_t^L-\frac{\boldsymbol{t}_t^L}{\rho_R^L c_s^L}. \tag{8}$$

In the case of contact-slip, it is clear that the velocity and traction are continuous normal to the surface, but are discontinuous in the tangential direction of the surface.

4 Numerical Scheme

From spatial discretisation standpoint, a very efficient vertex centred finite volume algorithm is employed. Following the work of [2], the semi-discrete form of the underlying system (1) reads

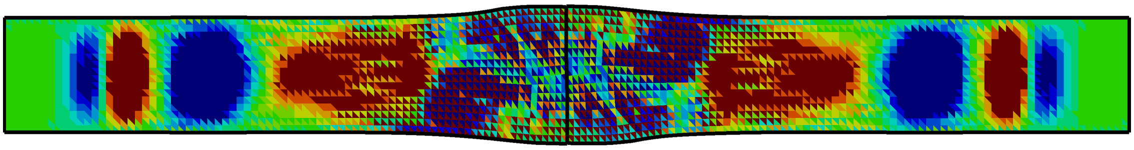
$$\Omega_R^a\frac{d\left(\rho_R^a\boldsymbol{v}_a\right)}{dt}=\left(\sum_{b\in\Lambda_a}\boldsymbol{t}^C\left\|\boldsymbol{C}_{ab}\right\|+\sum_{\gamma\in\Gamma_a^B}\boldsymbol{t}_\gamma^C\left\|\boldsymbol{C}_\gamma\right\|\right)+\Omega_R^a\boldsymbol{f}_R^a; \tag{9a}$$

$$\Omega_R^a\frac{d\boldsymbol{F}_a}{dt}=\left(\sum_{b\in\Lambda_a}\frac{1}{2}\left(\boldsymbol{v}_a+\boldsymbol{v}_b\right)\otimes\boldsymbol{C}_{ab}+\sum_{\gamma\in\Gamma_a^B}\boldsymbol{v}_\gamma^C\boldsymbol{C}_\gamma\right). \tag{9b}$$

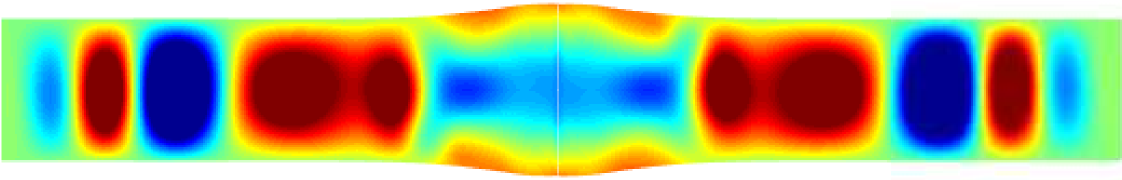
The terms within the parenthesis in (9a,b) correspond to the evaluation of the control volume interface and boundary fluxes. Such evaluation comprises a summation over edges (first term in the parenthesis) and a summation over boundary surfaces (second term in the parenthesis). Notice that the contact interface conditions presented in Section 3 can be viewed as a boundary condition that couples Lagrangian meshes together by enforcing contact-impact physics. It is important to emphasise that kinetic conservation condition is ensured in (9a), whereas, unlike the classical displacement based formulation, the kinematic impenetrability condition in this case is explicitly enforced in (9b) without resorting to computationally expensive iterative scheme. From time discretisation standpoint, an explicit type of Runge-Kutta time integrator is used.

5 Numerical Example

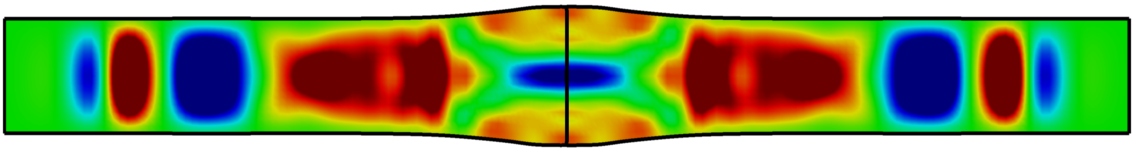
In order to show the robustness of the algorithm, two (nearly incompressible) elastic bars travelling at equal but opposite velocity of 50m/s are considered. The bars are initially set with a gap of 8mm such that the bars impact at $80\mu s$. Both bars are modelled using a neo-Hookean model with a Young's modulus of 585MPA, density of 8930kg/m^3 and Poisson's ratio of 0.495. The problem is initially analysed using the standard Finite Element Method (FEM) available in Abaqus, and then with the proposed approach. Figure 1 shows the pressure contour of post-impact bars at time $t=100\mu s$. Comparing these results, pressure oscillations is clearly observed for standard linear triangles (FEM). Remarkably, the pressure profile obtained with the proposed method compares well with those obtained from the quadrilateral element provided in Abaqus.



(a) Abaqus discretised with standard linear triangles



(b) Proposed finite volume algorithm discretised with linear triangles



(c) Abaqus discretised with bilinear quadrilaterals via mean dilatation

Figure 1: Pressure contour of post-impact bars at $t = 100\mu$s.

6 Conclusions

The paper presents a vertex centred finite volume method for the solution of fast transient computational contact mechanics, where a mixed based system of first order conservation laws is solved. It has been shown that non-physical zero energy modes and spurious pressure instabilities can be effectively alleviated when attempting to model the contact between two (nearly incompressible) deformable solids. The consideration of thermoelasticity within the current computational framework is the next step of our work.

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