Cut finite element error estimates for a class of nonlinear elliptic PDEs

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Abstract

In the contexts of fluid–structure interaction and reduced order modeling for parametrically–dependent domains, immersed and embedded methods compare favorably to standard FEMs, providing simple and efficient schemes for the numerical approximation of PDEs in both cases of static and evolving geometries. In this note, the a priori analysis of unfitted numerical schemes with cut elements is extended beyond the realm of linear problems. More precisely, we consider the discretization of semilinear elliptic boundary value problems of the form $-\Delta u + f_1(u) = f_2$ with polynomial nonlinearity via the cut finite element method. Boundary conditions are enforced, using a Nitsche–type approach. To ensure stability and error estimates that are independent of the position of the boundary with respect to the mesh, the formulations are augmented with additional boundary zone ghost penalty terms. These terms act on the jumps of the normal gradients at faces associated with cut elements. A–priori error estimates are derived, while numerical examples illustrate the implementation of the method and validate the theoretical findings.

Key words: *Cut finite element method; Elliptic; Semilinear; Error estimates*

1 Introduction

The overall objective of this note is to extend the a–priori analysis of cutFEM beyond the realm of linear problems. To this end, we propose an unfitted framework for the numerical solution of a semilinear elliptic boundary value problem with a polynomial nonlinearity. Our approach is based on classical arguments for the p–Laplacian [3] and on key results from [2] for a stabilized unfitted method for the Poisson problem. We start by introducing the model problem and the necessary notation in Section 2. Then, Section 3 focuses on the derivation of the a–priori error estimates and a numerical experiment is reported in Section 4, verifying the theoretical convergence rates and showcasing the accuracy of the method. The paper concludes with a brief discussion of our contributions and suggestions for future work in Section 5.

2 The model problem and preliminaries

As a model problem, we consider a semilinear elliptic boundary value problem of the form

$$\begin{aligned} -\Delta u + f_1(u) &= f_2 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a simply connected open domain with boundary $\Gamma = \partial\Omega$. The nonlinearity is assumed to be of type $f_1(u) = |u|^{p-2} u$. Such equations have been studied previously in the context of problems with critical exponents [4] and are referred to in the theory of boundary layers of viscous fluids [9] as Emden–Fowler equations. The weak formulation

$$\int_\Omega (\nabla u \cdot \nabla v + f_1(u)v) = \int_\Omega f_2 v, \quad \text{for every } v \in H^1_0(\Omega) \tag{2.2}$$

of (2.1) clearly admits a weak solution $u \in H^1_0(\Omega)$. Assuming $f_2 \in H^{-1}(\Omega)$, the a–priori error bound $\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^p_{L^p(\Omega)} \leq \frac{1}{2}\|f_2\|^2_{H^{-1}(\Omega)}$ readily follows.

Implementation of an unfitted FEM for the discretization of (2.2) requires a fixed background domain $\mathcal{B}$ which contains Ω; let $\mathcal{B}_h$ a corresponding shape–regular mesh and its *active* (unfitted

to the boundary Γ) part $\mathcal{T}_h = \{T \in \mathcal{B}_h : T \cap \Omega \neq \emptyset\}$. Considering the *extended* domain $\Omega_\mathcal{T} = \bigcup_{T \in \mathcal{T}_h} T$ and the finite element space $V_h := \{w_h \in C^0(\bar{\Omega}_T) : w_h|_T \in \mathcal{P}^1(T), T \in \mathcal{T}_h\}$, we define a discrete counterpart to the continuous bilinear form in (2.2), setting

$$a_h(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h - \int_{\Gamma_D} v_h (\mathbf{n_\Gamma} \cdot \nabla u_h) - \int_{\Gamma_D} u_h (\mathbf{n_\Gamma} \cdot \nabla v_h) + \gamma_D h^{-1} \int_{\Gamma_D} u_h v_h, \quad (2.3)$$

for $u_h, v_h \in V_h$. Here, $\mathbf{n_\Gamma}$ denotes the outward pointing unit normal vector on the boundary Γ. The cutFEM discretization scheme reads as follows: find a discrete state $u_h \in V_h$, such that

$$a_h(u_h, v_h) + j_h(u_h, v_h) + \int_\Omega f_1(u_h) v_h = \int_\Omega f_2 v_h, \text{ for all } v_h \in V_h, \quad (2.4)$$

where the stabilization term $j_h(u_h, v_h) = \sum_{F \in \mathcal{F}} \gamma_1 h \int_F [\![\mathbf{n_F} \cdot \nabla u_h]\!][\![\mathbf{n_F} \cdot \nabla v_h]\!]$ acts on the gradient jumps $[\![\mathbf{n_F} \cdot \nabla u_h]\!] := \mathbf{n_F} \cdot \nabla u_h\big|_K - \mathbf{n_F} \cdot \nabla u_h\big|_{K'}$ of u_h over element faces $F = K \cap K'$ in the interface zone $\mathcal{F} := \{F : F \text{ is a face of } T \in \mathcal{T}_h \text{ with } T \cap \Gamma \neq \emptyset, F \notin \partial\Omega_\mathcal{T}\}$ and is included in the bilinear form to extend its coercivity from the physical domain Ω to $\Omega_\mathcal{T}$ [2]. The quantities γ_D and γ_1 in the definitions of a_h and j_h are positive penalty parameters; see Lemma 3.2 below.

3 Norms, approximation properties and a–priori analysis

The convergence analysis of the method (2.4) is based on the following mesh–dependent norms:

$$\|v\|_*^2 = \|\nabla v\|_{L^2(\Omega)}^2 + \left\|h^{-1/2}\gamma_D^{1/2} v\right\|_{L^2(\Gamma)}^2, \quad \|v\|_h^2 = \|\nabla v\|_{L^2(\Omega_\mathcal{T})}^2 + \left\|h^{-1/2}\gamma_D^{1/2} v\right\|_{L^2(\Gamma)}^2 + j_h(v, v),$$

which satisfy $\|v_h\|_* \leq C_* \|v_h\|_h$. Some preliminary lemmata and our basic error estimate follow:

Lemma 3.1. [2, Lemma 5] *Let* $\mathcal{E} : H^2(\Omega) \to H^2(\Omega_\mathcal{T})$ *a linear* H^2*–extension operator on* $\Omega_\mathcal{T}$, *such that* $\mathcal{E}\phi|_\Omega = \phi|_\Omega$, $\mathcal{E}\phi|_\Gamma = \phi|_\Gamma$, $\|\mathcal{E}\phi\|_{H^2(\Omega_\mathcal{T})} \lesssim \|\phi\|_{H^2(\Omega)}$ *and* $\Pi_h : H^1(\Omega) \to V_h$ *the Clément-type extended interpolation operator defined by* $\Pi_h\phi = \Pi_h^* \mathcal{E}\phi$, *where* $\Pi_h^* : H^1(\Omega_\mathcal{T}) \to V_h$ *is the standard Clément interpolant. Then, the estimate*

$$\|u - \Pi_h u\|_* + j(\Pi_h u, \Pi_h u)^{1/2} \leq Ch|u|_{H^2(\Omega)} \quad (3.1)$$

holds for every $u \in H^2(\Omega)$.

Lemma 3.2 (Coercivity and continuity of $a_h + j_h$). [2, Lemmata 6 and 7] *Defining the method* (2.4) *with sufficiently large parameter* γ_D *and* $\gamma_1 = 1$, *then*

$$c_{bil}\|u_h\|_h^2 \leq a_h(u_h, u_h) + j_h(u_h, u_h), \quad a_h(u_h, v_h) + j_h(u_h, v_h) \leq C_{bil}\|u_h\|_h\|v_h\|_h, \quad (3.2)$$

for every $u_h, v_h \in V_h$, *and*

$$a_h(v, v_h) \leq c_a\|v\|_*\|v_h\|_*, \quad \text{for every } v \in \left[H^2(\Omega) + V_h\right] \text{ and } v_h \in V_h, \quad (3.3)$$

independently of h *and of the way in which the boundary* Γ *intersects the background mesh.*

Lemma 3.3 (Galerkin orthogonality). *Let* $u \in H_0^1(\Omega)$ *be the solution to the semilinear problem* (2.2) *and* $u_h \in V_h$ *its finite element approximation in* (2.4). *Then,*

$$a_h(u_h - u, v_h) = \int_\Omega [f_1(u) - f_1(u_h)] v_h - j_h(u_h, v_h), \quad \text{for every } v_h \in V_h. \quad (3.4)$$

Proposition 3.4 (Optimality with respect to interpolation). *Let* $u \in H_0^1(\Omega)$ *be the solution to the semilinear problem* (2.2) *and* $u_h \in V_h$ *its finite element approximation in* (2.4). *Then, there exists a constant* $C > 0$, *independent of* u, u_h, *such that*

$$\|u_h - \Pi_h u\|_h^2 + \|u - u_h\|_{L^p(\Omega)}^p \leq C \left(\left[\|u - \Pi_h u\|_h + j_h(\Pi_h u, \Pi_h u)^{1/2}\right]^2 + \|u - \Pi_h u\|_{L^q(\Omega)}^q \right),$$
$$(3.5)$$

where q *is the conjugate index of the power* p *in the nonlinear term* $f_1(u) = |u|^{p-2} u$.

Proof. As in the proof of [3, Thm. 5.3.3, p. 319] for the p–Laplacian, there exists $c > 0$, such that

$$\int_\Omega f_1(u - u_h)(u - u_h) \leq c \int_\Omega \left[f_1(u) - f_1(u_h) \right] (u - u_h). \tag{3.6}$$

Then, denoting $e_h := u_h - \Pi_h u$, we successively apply the coercivity estimate (3.2), (3.6) and the Galerkin orthogonality (3.4) to estimate

$$c_{bil} \| e_h \|_h^2 + \frac{1}{c} \| u - u_h \|_{L^p(\Omega)}^p \leq [a_h + j_h](e_h, e_h) + \frac{1}{c} \int_\Omega f_1(u - u_h)(u - u_h)$$

$$= a_h(u - \Pi_h u, e_h) + a_h(u_h - u, e_h) + j_h(e_h, e_h) +$$

$$+ \int_\Omega \left[f_1(u) - f_1(u_h) \right] (u - u_h)$$

$$= a_h(u - \Pi_h u, e_h) + j_h(-\Pi_h u, e_h) + \int_\Omega \left[f_1(u) - f_1(u_h) \right] (u - \Pi_h u).$$

A bound for the leading two terms is readily implied by the continuity estimate (3.3), the Cauchy–Schwarz inequality and (3.1):

$$a_h(u - \Pi_h u, e_h) + j_h(-\Pi_h u, e_h) \leq c_a \| \| u - \Pi_h u \| \|_* \| \| e_h \| \|_* + j_h(\Pi_h u, \Pi_h u)^{1/2} j_h(e_h, e_h)^{1/2}$$

$$\leq \left[c_a C_* \| \| u - \Pi_h u \| \|_* + j_h(\Pi_h u, \Pi_h u)^{1/2} \right] \| \| e_h \| \|_h$$

$$\leq \frac{\max\{c_a C_*, 1\}^2}{2 c_{bil}} \left[\| \| u - \Pi_h u \| \|_* + j_h(\Pi_h u, \Pi_h u)^{1/2} \right]^2 + \frac{c_{bil}}{2} \| \| e_h \| \|_h^2,$$

while the third term is estimated by

$$\int_\Omega \left[f_1(u) - f_1(u_h) \right] (u - \Pi_h u) \leq C_{f_1} \| u - u_h \|_{L^p(\Omega)} \| u - \Pi_h u \|_{L^q(\Omega)}$$

$$\leq \frac{1}{2c} \| u - u_h \|_{L^p(\Omega)}^p + \left(\frac{p}{2c} \right)^{-q/p} \frac{C_{f_1}}{q} \| u - \Pi_h u \|_{L^q(\Omega)}^q.$$

Hence, the assertion (3.5) already follows for $C = \min\left\{ \frac{c_{bil}}{2}, \frac{1}{2c} \right\}^{-1} \max\left\{ \frac{\max\{c_a C_*, 1\}^2}{2 c_{bil}}, \left(\frac{p}{2c} \right)^{-q/p} \frac{C_{f_1}}{q} \right\}$.
□

Theorem 3.5 (Optimal convergence). *Let $u \in H_0^1(\Omega) \cap H^2(\Omega) \cap W^{2,q}(\Omega)$ be the solution to the semilinear problem (2.2) and $u_h \in V_h$ its finite element approximation in (2.4). Then, $\| \| u - u_h \| \|_* = \mathcal{O}(h)$.*

Proof. We decompose the total error $\| \| u - u_h \| \|_*$ into its discrete–error and projection–error components; i.e., $\| \| u - u_h \| \|_* \leq \| \| u - \Pi_h u \| \|_* + C_* \| \| \Pi_h u - u_h \| \|_h$. The desired estimate for the first term is already provided by (3.1), while the latter is bounded by Proposition 3.4. Indeed, by (3.1) and the properties of the Clément interpolant [5, p.69], estimate (3.5) yields

$$\| \| u_h - \Pi_h u \| \|_h^2 \leq \hat{C} \left(h^2 |u|_{H^2(\Omega)}^2 + h^{2q} |u|_{W^{2,q}(\Omega)}^q \right)$$

for $\hat{C} > 0$. Recalling $q = \frac{p}{p-1}$ is the conjugate index of p, clearly $\min\{1, q\} = 1$ and the bound is optimal.
□

4 Numerical validation

Let the two–dimensional test case of (2.1) for $p = 4$ with manufactured exact solution and right–hand side force defined respectively by $u(x, y) = \frac{1}{2}(1 - x^2 - y^2)$ and $f(x, y) = \frac{1}{8} \left(1 - x^2 - y^2 \right)^3 + 2$ in $\Omega = \mathcal{D}(0, 1)$; i.e., the unit disc centered at the origin.

Table 1: Errors and experimental orders of convergence (EOC) for H^1 and L^2 norms.

$h_{\max}$	$\|u - u_h\|_{H^1(\Omega)}$	EOC	$\|u - u_h\|_{L^2(\Omega)}$	EOC
0.15	7.74620e-2		2.47468e-3	
0.075	3.90601e-2	0.988	5.83351e-4	2.085
0.0375	1.93383e-2	1.014	1.33451e-4	2.128
0.01875	9.63082e-3	1.006	3.34143e-5	1.999
0.009375	4.80627e-3	1.003	8.12293e-6	2.040
0.0046875	2.40450e-3	0.999	2.01406e-6	2.012
Mean		1.002		2.049

We embed Ω in the background domain $\mathcal{B} = [-1.5, 1.5]^2$ and consider a corresponding sequence of successively refined tessellations $\{\mathcal{B}_{h_\ell}\}_{\ell \geq 0}$ with mesh parameters $h_\ell = 0.15 \times 2^{-\ell}$ ($\ell = 0, \ldots, 6$). Taking $\gamma_D = 1$ and $\gamma_1 = 0.1$, the theoretically predicted rates of convergence from Theorem 3.5 are verified by the numerical findings in Table 1.

5 Conclusions

The present note concentrated on the derivation of an a–priori error estimate for a cut finite element approximation of a semilinear model problem. To the authors' best knowledge, this is one of the few instances in the literature that such an analysis has been carried out beyond a linear context. Future work will delve more deeply in the analysis of unfitted FEMs for general time–dependent problems with nonlinearities. From a computational point of view, the effect of preconditioning on the performance of the method will be assessed in the spirit of [1]. Finally, the method seems promising for controlling nonlinear PDEs with uncertainties, involving large deformations and/or topological changes [6, 7].

Acknowledgments

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