Exploring pre-service mathematics teachers' knowledge of logarithm in one of the universities in Kwazulu-Natal

A thesis submitted in complete fulfilment of the academic requirements for the degree of Master of Education in mathematics education in the school of Education of the College of Humanities

By

Izuchukwu Okoye-Ogbalu

School of Education, Mathematics, Statistics and Computer Sciences cluster, University of KwaZulu Natal, Edgewood Campus

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Supervisor: Dr Themba Mthethwa **Date of submission: 2019**

PREFACE

The work done in this thesis was carried out in the College of Humanities, School of Education, University of KwaZulu - Natal, Edgewood Campus from January 2018 to January 2019 under the supervision of Dr. Themba Mthethwa.

This study represents original work done by me and has not been submitted in any form for any degree to any tertiary institution. The work used from others were properly acknowledged.

Student: ___________________________ ________________________

Izuchukwu Okoye-Ogbalu Date

Supervisor: _____________________________ ________________________

Dr. Themba Mthethwa Date

DEDICATION

This project is dedicated to my family.

ABSTRACT

The use of logarithm, an important tool for calculus and beyond, has been reduced to symbol manipulation without understanding in most entry-level college algebra courses. In most secondary school mathematics curriculum, particularly in South Africa, logarithm just occurs as the inverse function of an exponential function without a detailed explanation of the logarithm itself. The primary aim of this research, therefore, was to explore pre-service mathematics teachers' knowledge of logarithm through the use of a research task designed to observe what pre-service mathematics teachers know as they solve the problems and through the use of interview to understand how they solve the problems. Constructivism theory was used as a framework for the analysis and the interpretation of how pre-service teachers conceptualize logarithm. Constructivism theory is a useful theoretical framework for studying and explaining conceptual development through prior knowledge. This is a qualitative study conducted in one of the universities in the province of KwaZulu-Natal. The findings of this study reveal that most pre-service mathematics teachers do not have a good knowledge of logarithm. This is why they have difficulties in solving problems involving logarithm. The findings also reveal that pre-service mathematics teachers do not have good knowledge of logarithm because of how the concept of logarithm was introduced to them in secondary school.

The study concludes by recommending that lecturers in the mathematics discipline should try to design teaching material that targets the development of conceptual understanding and pre-service mathematics teachers need to develop sufficient sense of dealing with more abstract concepts in order to do justice in the teaching of logarithm at the secondary school level.

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CHAPTER ONE

1. INTRODUCTION

1.1 Overview of chapter one

This chapter presents the research process un 7dertaken in the study of the exploration of pre-service mathematics teachers' knowledge of logarithm. It also presents the study overview, with the background and purpose detailed first. Furthermore, the chapter also discusses in detail the concept of learning of mathematics; the contribution this study aimed to make in mathematics education; and the research questions that the study addresses. At the later stage in this chapter, a summary of successive chapters is provided.

1.2 Teaching and learning of mathematics

The nature of teaching and learning of mathematics has been the concern of mathematics education research. The better we understand the nature of knowledge, the better we can plan the instructional methods and activities that will enhance the learning of mathematics as a subject. Much of mathematics education research studies had looked at the means and ways we can improve the teaching and learning of mathematics at school and at tertiary institutions. Research into the teaching and learning of mathematics in general has revealed that students have difficulties in conceptualising mathematical concepts. First year mathematics students rely on rules and algorithms, they do not enjoy mathematics and are demotivated (K. Naidoo & Naidoo, 2007). As a result, they struggle to solve unfamiliar problems. Aziz, Meerah & Tambychic (as cited in Ndlovu, 2016) argue that the difficulties in the learning of mathematics are manifested in various situations such as the poor application of mathematical concepts, poor achievement in mathematics, deficiency in mathematics skills and inefficiency in mathematical problem-solving.

There has been a growing interest in research in the learning and teaching of undergraduate mathematics over the past few years. Many scholars (Breen & O'Shea, 2016; García & Cano, 2018; Meletiou-Mavrotheris & Prodromou, 2016; Nthontho, 2018; Pretorius, 2017; So, 2016) have conducted studies focusing on advanced mathematical thinking and pedagogies in the teaching and learning of mathematics. The body of research around the area of algebra has also grown extensively in recent years. Despite this, little research has been carried out in the mathematics education literature regarding the pre-service mathematics teachers' knowledge of logarithm. It is thus necessary to mention that the vast literature focuses on the area of linear algebra, but less is known about how pre-service mathematics teachers conceptualise logarithm. Furthermore, there is a shortage of research about the understanding of logarithm in South Africa. This study therefore aims to contribute to the empirical and theoretical work in this area of mathematics education by exploring pre-service mathematics teachers' knowledge of logarithm.

1.3 Background of the study

According to the research carried out by Bansilal, Mkhwanazi, and Brijlall (2014), which investigated teachers' knowledge of the subject they are teaching, the results show that teachers' average mark on the past matriculation mathematics papers was 57% and a quarter of them[1] got below 39%. From their observation, an average of 26% of the teachers tested managed to answer level 4 questions[2]. To further explore this, one might ask, what is the content knowledge (CK) of a high school mathematics teacher or what kind of CK do high school mathematics teachers' need? This question is highly relevant for the design of mathematics teacher education programs as well as for investigating teachers' professional knowledge. Teachers' knowledge is considered one of the most important predictors of student achievement (Venkat, Rollnick, Loughran, & Askew, 2014). This research, however, will explore the knowledge of logarithm of pre-service mathematics teacher's knowledge of logarithm.

The logarithm is one of the topics covered in Mathematics for Educators module (EDMA119) at this university at which this study was conducted. In this course, mathematics education students are exposed to the concepts of the logarithm. The concept of the logarithm in South Africa is first dealt with in ordinary secondary schools from Grade 12, dealing with it as an inverse of an exponential function and in Technical schools from Grade 11. At these grades, learners are introduced to the basic rules of logarithm and how to apply logarithm in financial mathematics (DBE, 2012). Then at the university level, the concept of proof of logarithmic properties is firstly dealt with. University students are expected to generalize their knowledge of school algebra in understanding logarithm.

[1] These were the qualified teachers enrolled in a university advanced certificate in education.
[2] Level 4 questions are questions that require an investigation, time to think and process multiple conditions of the problem.

1.4 Problem statement

> *Seeing there is nothing . . . that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplication, division, square and cubical extraction of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in mind by what certain and ready art I might remove those hindrances.* (Napier, p. 238).

How did a topic that historically represented a major contribution to computational mathematics become so meaningless to secondary school mathematics learners? The discovery of logarithms supported the massive calculations needed for astronomy and navigation; however, mathematicians took notice and logarithms that were once used only as computational tools took on a life of their own. A method that began as strictly a computational device later was shown to have a significant impact on understanding inverse-function relationships, and it became clear that logarithms held the “vital key in the new mathematics of calculus” (Smith, 2000, p. 773). An important tool for calculus and beyond, the use of logarithms has been reduced to symbol manipulation without understanding in most entry-level college algebra courses. In other words, what we end up with is an “easily reproducible mental experience of a mark or character strings with no other mental activity or structure beyond this primitive experience” (Harel & Kaput, 2002, p. 89).

Euler (2012), in one of his most notable publications, entitled “*Elements of Algebra*”, provides a definition of logarithmic functions: Resuming the equation $a^b = c$, we shall begin by remarking that, in the doctrine of Logarithms, we assume for the root a, a certain number taken at pleasure, and suppose this root to preserve invariably its assumed value. This being laid down, we take the exponent *b* such that the power a^b becomes equal to a given number *c*; in which case this exponent is said to be the logarithm of the number *c* . . . . We see, then, that the value of the root *a* being once established, the logarithm of any number, *c*, is nothing more than the exponent of that power of *a*, which is equal to *c*; so that *c* being equal a^b, *b* is the logarithm of the power. (Euler, 2012, p. 91).

Most school textbooks begin their discussion of logarithmic notation after exponential and inverse functions have been introduced. Typically, a definition similar to Euler’s, stating that every exponential function of the form $g(x) = b^x$ is a one-to-one function and therefore has an inverse function, is given. After the variables x and y have been interchanged, the student has successfully created the implicit form of the inverse function; however, the inverse function needs to be defined explicitly. The discussion then continues along the line of something like this: "Noting that previous algebraic skills are no longer adequate to solve the equation $x = b^y$ for the exponent y, a "new" procedure must be developed" (Hauk, Powers, & Segalla, 2015, p. 62). Students are next told that a compact notation is needed to represent this procedure; hence, the rule $log_b x = y$ if and only if $b^y = x$ is given. Next, the logarithmic function is defined as $f(x) = log_b x$ followed with a statement telling the student this is the inverse function of $g(x) = b^x$.

Prior instruction has fostered the notion that procedures are computational rules to follow; however; students are now being asked to develop an entirely new type of mental image. Logarithmic functions belong to a class of functions referred to as "transcendental." Logarithmic functions along with exponential and trigonometric functions transcend algebra in the sense that these functions cannot be expressed in terms of a finite sequence of algebraic operations. In other words, it is generally not possible to relate the value of $f(x)$ to its input x by a finite number of algebraic operations. Without a clearly defined set of algebraic rules to follow, students struggle to make sense of the concepts. What exactly is the nature of the concept image that students develop in regard to the symbolism after instruction? Is it in conflict with existing knowledge structures? Development of a concept image permits the researcher to examine how pre-service mathematics teachers conceptualize logarithm. Is the difficulty students encounter in concept acquisition intertwined with the notation that "log" is the procedure? Do they see this as a word, a variable name, or a procedure? Has their "loose" attention to the definition of the symbols involved in the algebraic notion compromised mathematical meaning? If students' conceptual structures do not reflect conceptual understanding, how can the teaching of logarithmic concepts be improved?

Although the original motivation for the teaching and learning of logarithms has all but disappeared from today's mathematics curriculum, students and teachers are left wondering: what are logarithms used for, and why are they still on the syllabus? Calculations that once proved tedious for mathematicians are no longer problematic. The development of tools to make computation easier, more accurate, and faster has predicated a change in the approach to teaching this topic; however, for most students, "log" is a mysterious button on their calculator.

The National Council of Teachers of Mathematics (NCTM) released *An Agenda for Action* in 1980 recommending that technology is made available to all students so that difficulties encountered with pencil-and-paper activities would not interfere with the learning of problem-solving strategies (D. Klein, 2003). What implications does this have for the learning and teaching of logarithms? Furthermore, the 2006 document warned, “It is dangerous to assume that skills from one era will suffice for another” (Mathematics, 2006, p. 6). Before calculators were introduced into the classroom, pencil-and-paper computation was the only accepted procedure available for use by the student and the teacher. The de-emphasis of pencil-and-paper calculations signaled important changes in classroom behavior and structure. Calculators, properly used, can act as a scaffolding agent to enable the learner to bridge minor gaps in background knowledge to reach higher levels of mathematical understandings. But how do we use this tool to bridge the ever-widening gap between students’ procedural knowledge and conceptual understanding of a logarithm? Because of the technology introduced in the late 1970s, understanding of logarithmic concepts and their associated properties has plummeted (Espedal, 2015). In light of its changed role in the curriculum, what does it mean to understand logarithms? Some may speculate on the usefulness of this topic in the high school curriculum. Still others will question its role at the postsecondary level. If it is not used as a computational tool, is it necessary that we continue to teach this topic for non-calculus bound students? In light of the push for quantitative literacy, it seems unlikely that logarithms will disappear from the curriculum as they have many useful real-world applications.

1.5 Rationale

The study rationale is to firstly address the gap in this field of research and to bringing new knowledge to the mathematics community since logarithm is one of the concepts that pre-service mathematics teachers have to learn, and therefore it has implications for teacher education. Secondary, it also based on the researchers’ experience with the in-service mathematics teachers.

1.5.1 Addressing the gap

Although a notable amount of study has been carried out in the teaching and learning of algebra at the undergraduate level, research literature in the area of the logarithm is limited, especially in the context of South Africa. This might be because logarithm is not examinable in South African high school matriculation examinations. It is done only in technical school which is few in South Africa. Much of the body of research in South Africa in the past and present in mathematics education focuses on improving and developing the teaching and learning of mathematics at the school level (Ndlovu, 2016). Surveying different journals and other collections on logarithm both nationally and internationally between 2013 and 2018 by the researcher, there have been more than 600 research articles in the teaching and learning of undergraduate mathematics. However, most of these studies were conducted internationally. It appears that the body of research in the area of the logarithm in South Africa, is very limited or non-existent at all.

1.5.2 The researchers' experiences in teaching and learning of logarithm

The researcher (as a mathematician himself) interacts with many local mathematics teachers who teach in high schools in Durban and in Pietermaritzburg. In the discussion with these high school mathematics teachers, we share experiences in teaching and learning of mathematics particularly those critical sections in school mathematics. To the researcher's surprise, most of these teachers have limited knowledge of the logarithm table or how it works. This started when the researcher asked, during one of our discussions, for another method of calculating the values while multiplying or dividing irrational numbers. All they could respond was to use a calculator and to convert to whole numbers and apply the long multiplication and division method. This made the researcher bring up the use of logarithm and how it can equally be applied in calculating that. The researcher then asked further questions if they have encountered logarithm before, and what area of logarithm they covered. Whilst the conversation was going on, the researcher realized that what these teachers know about logarithm is that it is the inverse of an exponential function. This showed that these mathematics teachers lack in-depth knowledge of logarithm.

The observation of the way in which the in-service teachers know little about logarithm motivated the researcher to undertake a study with the aim to explore the content knowledge of pre-service mathematics teachers in logarithm because the researcher believes that much content knowledge teachers have they have acquired through pre-service training. In addition, the researcher is concerned that research in teaching and learning of logarithm is limited in the South African context.

1.6 Beliefs about mathematics

In the teaching and learning of mathematical concepts, individual beliefs about the nature of mathematics play a vital role in the conceptualization of mathematical concepts. In most cases, especially at the high school level, mathematics is presented as a set of rules that needs to be learnt. This evidence can be seen in high school mathematics textbooks. This set up of textbooks creates a view among learners that mathematics is an external body of knowledge. This view is not totally wrong, but it does not provide learners with the opportunity to formulate larger ideas about the essence of mathematics.

Mathematics is universal, a body of knowledge (Povey, Adams, & Jackson, 2016). It encompasses knowledge that is focused on concepts such as quantity, structure, space and change (J. Naidoo, 2011). It can be integrated with many fields such as Natural Sciences, Engineering, Medicine and Social Sciences. J. Naidoo (2011) asserted that, mathematics is immersed in the science of patterns and that these patterns are not only found in numbers, but also in space, science, computers and imaginary abstractions. For this reason, it is our duty as mathematicians to explore these patterns so as to formulate conjectures and establish truths. It is at this point where mathematics students will be able to understand how mathematical concepts are linked. As a result, they will see mathematics beyond just the application of rules. The language of mathematics plays a vital role in the conceptualization of mathematics. According to Sahin and Soylu (2011), mathematics is a universal language, which has been formed as a result of the studies of scientists, which have unique rules and provide communication between all people in the world regardless of whether they practice in the field of mathematics or not.

1.7 Objectives of the study

The main aim of the research was to explore pre-service mathematics teachers' knowledge of logarithm. The following objectives are set out to be achieved:

1. To explore what it means for pre-service mathematics teacher to have a good knowledge logarithm.
2. To identify the difficulties the pre-service mathematics teachers, encounter with logarithm if any.
3. To investigate the way in which pre-service mathematics teachers conceptualize logarithm.

1.8 Research questions

In order to address these objectives, the study serves to answer the following research questions.

1. What does the pre-service mathematics teacher know about logarithm?
2. What are the difficulties pre-service mathematics teachers encountered with logarithm?
3. How do pre-service mathematics teachers conceptualize logarithm?

The study adopted the use of research task in collecting data and the pre-service mathematics teachers' responses were the main source of data, together with semi-structured interviews which were used to understand how the participants answer their questions. The full details in this regard are presented in chapters 3 and 4.

1.9 The significance of the study to mathematics education

The literature on the teaching and learning of logarithm reveals that this concept is relatively unexamined in the South African context. This study, through the use of constructivist's theory, draws attention to how pre-service mathematics teachers conceptualize logarithm. The study is expected to bring about what the pre-service mathematics teachers know about logarithm and the difficulties they encounter with logarithm in the South African context. This research study explores the pre-service mathematics teachers' knowledge of logarithm through the use of the interpretive paradigm within this qualitative study. The theoretical work of researchers such as (Asiala et al., 1997; Dubinsky, 2002; Duffy & Jonassen, 2013; Fosnot & Perry, 1996; Packer & Goicoechea, 2000) within the field of constructivism was explored. The study highlights the usefulness of such theories, which is the extension of the Piaget theory on reflective abstraction in the conceptualization of mathematical concepts. At the same time, it broadens the scope of how these theories overlap and shape the learning of advanced mathematics.

1.10 The scope of the study

The logarithm is one of the topics introduced to students in their first year of university study particularly those who are doing mathematics education with the intention to become mathematics teachers after completing a bachelor's degree. It is part of the South African high school's curriculum, but it is introduced as an inverse of an exponential function. The high school knowledge of logarithm and logarithmic functions is needed as the foundation to build more sections on logarithm. International literature such as Espedal (2015) has indicated that many students have difficulties understanding this concept. The study on this phenomenon has not been done in the South African context. Therefore, this in-depth study has been undertaken in order to explore pre-service mathematics teachers' knowledge of logarithm in the South African context.

This research study was limited to a group of undergraduate mathematics students who have enrolled at the university and has registered as a high school mathematics teacher. A group of 19 pre-service mathematics teachers of mixed abilities, mixed race, mixed gender and mixed culture participated in this study, details of the participants are provided in chapter 3.

1.11 Definition of key terms

To support the reader, the keywords used in the title are defined as follows:

Learners – People who are studying in secondary school.

Students - People who are studying at any institution of higher learning.

Pre-service mathematics teachers: These are undergraduate students who are enrolled in a mathematics teacher preparatory program and working towards mathematics teacher Bachelor of Education.

Knowledge of Logarithm: The Knowledge of logarithm for this study is limited to the high school curriculum content of logarithm. This is because pre-service mathematics teachers are trained to be able to teach this topic in high schools.

1.12 Outline of the chapters of this study

In determining the appropriate approach to the dissertation, the following structure has been used. The thesis comprises of five chapters, a bibliography and its appendices. The chapters are organized as follows:

Chapter One introduces the background and purpose of the study as well as the rationale for doing this research. In addition to this, the chapter presents the background of the study and the problem statement. Furthermore, it introduces the objectives of the study and the critical research questions. Moreover, the outline of the whole study was introduced.

Chapter Two presents the two main parts of the chapter. Part 1 presents the relevant literature reviewed based on the area of the logarithm. The development of earlier solutions of the logarithm, the importance of logarithm, educational background of the logarithm, students' difficulties and misconceptions of the logarithm, and alternative instruction proposals that could be applied in logarithm classes, are described. This part also described the teachers' knowledge of mathematics, the difference between academic and school mathematics and lastly discussed teacher content knowledge. Part 2 presents the theoretical framework of the study.

Chapter Three presents the research design, research methodology and procedures undertaken for this study. This chapter outlines the research instruments employed. The preliminary process involved with respect to the pilot study, research paradigm and how these fits within the study are then presented. The data collection and analysis together with the sampling and location of the study are discussed. The limitation of the study as well as the reliability of the study is presented.

Chapter Four focuses on the research instrument used and the analysis of pre-service mathematics teachers' responses from the research tasks and interviews. Furthermore, this chapter explores the findings of this research study.

Chapter Five presents the discussion and conclusions that were drawn based on the overall study. This chapter also explores and respond to critical questions of the study. The limitations and recommendations made in the study are also presented in this chapter.

CHAPTER TWO

2. LITERATURE REVIEW AND THEORETICAL FRAMEWORK

The previous chapter presented an introduction to the study in which objectives and research questions guiding this study were presented. In this chapter, I present relevant literature review and framework underpinning this study. This chapter is thus presented into two parts, part 1 is literature review and part 2 is theoretical framework.

2.1 PART 1: LITERATURE REVIEW

In this part, the literature review is presented under two categories. The first category is the literature review on logarithm which discusses the importance of logarithm, students' difficulties, alternative instructions to the teaching of logarithm and the research on the misconceptions with logarithm while the second category is the literature review on teachers' knowledge of mathematics.

2.1.1 Introduction

The literature review focuses on the pre-service teacher's knowledge of logarithm. The review is presented in two sections. The first section focuses on the development earliest solution of the logarithm, educational background of the logarithm in relation to the place of the logarithm in the high school curriculum and in algebra textbooks. It also reflects the importance of logarithm and gives some reasons why the high school textbook definition of logarithm could be responsible for many difficulties and mistakes. This section equally looks at students' difficulties and mistakes in logarithm, research on misconception with logarithm and learning of logarithm through the process, objects, as a function and in contextual problems. The second section explores the teachers' knowledge of mathematics which gives an insight into the mathematical knowledge of high school mathematics teacher and then the difference between academic and school mathematics.

It is important to highlight that since much research has not been carried out in logarithm both in high school and universities in South Africa, there are limited resources on the subject. Therefore, continuous reference is taken from Mathematics Educational journals, Mathematics journals and international research carried out in the areas of the logarithm, exponential functions and logarithmic functions in another part of the world.

2.1.2 The development of the earliest solution of the logarithm

The goal of education determines its contents and structure (Vansteenkiste, Lens, & Deci, 2006). To acknowledge educational goals in learning logarithm, it is of great significance to know what logarithm are, how they came to be and why they are important. For this, I conferred with the historical development of logarithm. The development of a concept by an individual does not necessarily follow the same path as the historical development. There is much to be gained from the knowledge of the historical development of a mathematical concept. In particular, in the study of mathematical understanding, knowledge of the historical development of mathematical ideas provides us with another perspective on students' activities. Commonalities that occur in the way a student's understanding of mathematical concept develops and the way it developed historically are attributable to commonalities in the mechanism of development and to the preservation of the historical meaning of the terminology (Czarnocha & Baker, 2017).

The history of the development of logarithm coupled with the power of the logarithmic function to model various situations and solve practical problems contributes to the continued effort to support students' understanding of logarithm as critical today as it was when slide rules and logarithmic tables were commonly used for computation (Berezovski, 2004). In the work of Confrey and Smith (1994), they outline the historical development of the concept of logarithm and note the consistency of the development with students' action. These consistencies were observed during teaching interviews designed to investigate how students learn about exponential functions. Since the development of the logarithmic function followed by the development of the exponential function, Confrey and Smith investigated the historical development of logarithm in search for explanations for students' action (Confrey & Smith, 1994). They explain how the early work of Archimedes and that of Napier form a consistent whole that illustrates the development of what they call the multiplicative units, and then propose three ways of the understanding rate of change in relation to exponential functions.

To appreciate the importance of logarithm today, it is useful to have some idea of the historical background that led to their development. According to V. J. Katz (1997), the development of logarithm can be traced back to at least the sixteenth century. In this age, astronomers were making increasingly sophisticated computations with large numbers and realized that the number of errors made could be greatly reduced by replacing multiplication and division by addition and subtraction. A different motivation was a table developed at the time, relating powers of 2 to their exponents and demonstrated that multiplication in one column corresponds to addition to the other (Cairns, 1928). This was what motivates John Napier (1550-1617) to develop the first logarithm tables, and the first list of tables was published in 1614 (Wiik, 2017). In this work, he used a different approach than the modern logarithm (the value of $\log 10\,000\,000$ was zero). It took him 20 years to develop a vast list of numbers. Later, he decided to take a different approach with $\log 1 = 0$ and $\log 10 = 1$. Then, the basic, familiar properties $\log(xy) = \log x + \log y$ and $\log \frac{x}{y} = \log x - \log y$ apply, and a close relation to the modern notion of standard form can be used; $\log(a \times 10^n) = \log a + n,\ 1 \leq a < 10$. John Napier deceased before he could compile a new table based on these principles. Henry Briggs (1561-1630), with whom Napier worked closely on this problem, constructed the new table from scratch(Wiik, 2017). This is the familiar Briggsian Logarithm base 10. He made calculations to 30 decimal places. Briggs table was completed by Adrian Vlacq in 1628 and was the foundation for nearly all logarithm tables into the twentieth century. Logarithm became an overwhelming calculation tool for astronomers because it transforms a multiplication into simpler addition, thereby making computation easier and less prone to error.

The Jesuit Alfonso Antonio de Sarasa (1618-1667) discovered the connection to the hyperbolic integral in 1649, and Nicolaus Mercator discovered the power series for the natural logarithm in 1668 (Wiik, 2017). Isaac Newton rediscovered this same power series and used it to calculate many values to over 50 decimal places. Using the laws of logarithm and clever arithmetic, he calculated the natural logarithm of many small positive integers. Leonard Euler (1707-1783) was the first to define the logarithm in terms of the exponential function. After Descartes introduced the modern symbolism a^n for powers of numbers in 1637, it was recognized that logarithm could also be interpreted as exponents (Wiik, 2017). Euler was one of the first to use the exponential property as a definition (Cajori, 1913). In his paper, he wrote that “Resuming the equation $a^b = c$, [...] we take the exponent b such that the power a^b becomes equal to a given number c; in which case this exponent b is said to be the logarithm of the number c” (Euler, 2012). For Euler, the logarithm is a particular exponent. The logarithm of c to base a is the exponent by which a must be raised to yield c. Rather than stating directly what logarithm is, Euler proceeds indirectly by conceiving it in terms of its inverse, raising to a power (with reference to the exponent). Today’s collections of formulas follow this indirect definition when they define the logarithm by the equivalence relation $\log_a b = x :\leftrightarrow a^x = b$. From the perspective of mathematics, this reading of logarithm simplifies the deductive structure of the earlier conceptualization, which is why it is widely used. Using Sfard’s terminology (Sfard, 1991), it expresses a structural conceptualization, as it is a static relation that refers to another concept. Euler used this to derive the basic properties, and he derived a power series applicable to multiple bases. The focus on power series is because it provided an accurate method for computing specific values (V. J. Katz, 1997). According to C. Weber (2016), logarithm tables were in common use until the onset of digital calculators in the late twentieth century. What is important to note is that in all cases, logarithm was a tool to ease calculations. Thus, in former times, learning logarithm was relevant for easing complex calculations. However, after the emergence of pocket calculators, logarithm no longer has this usefulness (C. Weber, 2016).

2.1.3 Importance of logarithm

Logarithm possesses a rich mathematical content that has had value all the way from the time of their invention to the recent diversity of their applications. For instance, in the eighteenth century, Ernest Weber (1795 – 1878) suggested that the sensitivity of senses decreases as the magnitude of the stimulus increases and later, Gustav Fechner (1801 – 1887), formulated the law that says, "The response of the senses varies as the logarithm of the stimulus". Mathematically, if y denotes the sensation (the effect that a stimulus produces on our senses) produced by stimulus x, then x and y are related by a law in the form of $y = log_b kx$, where constants k and b depend on the particular situation at hand. Only a few years later, Fechner used this law to find the method of measuring sensation. This method shows that when stimulus invades our senses, our body only takes in its logarithm and sends this logarithm to the brain to create a sensation (Shirali, 2002).

In line with its uses in the measuring of human senses, such as sound, light, taste, smell, etc., logarithm helped develop ideas that reduce the time to complete extensive and complex calculations (Shirali, 2002). Although information technologies such as calculators and computer have taken over this computational role, logarithm remains a great tool for calculation in mathematics and sciences (Stoll, 2006). Supplementary, logarithm are central concepts for many college mathematics courses, including calculus, differential equations and complex analysis.

However, Napier's approaches to logarithm were different from the form used today (Espedal, 2015). Students of algebra are often introduced to powerful reasoning tool with applications in many different fields. Logarithm which is an aspect of algebra is no exception to the benefits algebra offers. Logarithm can be used to serve the purpose of comparison, measuring, forecasting, explaining, illustrating and interpreting. The Richter scale was developed to measure the pressure of sound that our ears can accommodate or bear, as well as the magnitude of an earthquake through the use of logarithm (Clark & Montelle, 2010). According to Watters and Watters (2006), the logarithm is applied in studying the dynamics involved in the areas such as population growth, radioactive decay, and compound interest. These areas are part of the topics in the upper high school curriculum and the school curriculum will be discussed in detail in the next section.

2.1.4 Educational background of the logarithm

In this section, I will look at several aspects of the teaching and learning of logarithm. But I will first start with, the place of the logarithm in school curricula and algebra textbooks. This will be followed by logarithm in the university curriculum. It should be noted that logarithm as numbers or operators must be distinguished both mathematically and epistemologically from logarithm as functions (Confrey & Smith, 1994). Although both aspects of logarithm appear in secondary school level curricula. The difference between academic and school mathematics were equally discussed in detail with what mathematics do high school teachers need to know.

2.1.4.1 Logarithm in secondary school curricula

Ever since Euler (1707 – 1788), logarithm has been included in school curricula. They were used to simplify complicated manual calculations and remained relevant in schools for centuries until the introduction of pocket calculators, which have to a large extent put an end to this tradition. In the German-speaking countries, for example, the subject has been dropped entirely from curricula for lower achieving school learners. However, at the upper secondary level, the logarithm is still of some importance. They usually follow powers (with non-natural exponents) and exponential functions in grades 10–11 (i.e. age 16–17) (C. Weber, 2016). Besides the objective of manipulating expressions, the logarithm is taught to solve specific equations, particularly in the context of exponential growth and decay, and sometimes to model mathematical and real-life problems. The properties and rules that determine how logarithm behave are very important here (logarithmic laws, change of base theorem). With the broad circulation of the Elements of Algebra (originally published in German in 1765), the indirect definition $\log_a b = x :\Leftrightarrow a^x = b$ became the standard. Even today, many high school textbooks explain the logarithm with this formal relation. As a brief (and unsystematic) examination of some modern algebra textbooks in German and English suggests, they all use this equivalence relation to introduce logarithm as inverse exponents (Gallin, 2011; Griesel, Postel, Suhr, Ladenthin, & Lösche, 2016) or as inverse exponential functions (Holliday, 2005;

Murdock, Kamischke, & Kamischke, 2004; Neill, Neill, & Quadling, 2000). Some of the textbooks examined, in order to motivate learners, do first introduce to a "real-life" phenomenon that can be solved by logarithm (Hirsch, 2008), while others interpret points of a graph of exponential growth in "reverse order," i.e. by starting with some y-values and reading off their x-values (Holliday, 2005). Even in those instances where a textbook presents such an interpretation in the introductory section, it rarely makes use of it for argumentation, for instance, to make the logarithmic laws plausible (Hoon, Singh, & Ayop, 2010).

The mathematics curriculum of South African high school has been categorized into core mathematics, mathematics literacy and technical mathematics. The core mathematics is offered to all the senior high school learners while the mathematics literacy and technical mathematics are offered in Further Education and Training (FET) phase. A logarithm is a topic treated in core mathematics and technical mathematics. The logarithm taught under core mathematics covers a little perspective and it is not much detailed, compared to the NATED 550[3] which is the previous South African curriculum (Grussendorf, Booyse, & Burroughs, 2014). In most core mathematics grade 12 textbooks that I have looked at (eg. Everything mathematics text book, Platinum mathematics textbook, Classroom mathematics and Maths handbook and study guide), the logarithm is introduced as the inverse of an exponential function. While going through the Curriculum Assessment Policy Statement (CAPS), I observed that the curriculum is structured in such a way that logarithm is treated only in grade 12. It also placed emphases on the exponents as a pre-requisite knowledge to the logarithm. Exponents give a solid algebraic notation for repeated multiplication ($b^4 = b \times b \times b \times b$) and the inverse of it leads to logarithm introduction. In technical mathematics, the logarithm is taught in the grade 11 after the introduction of the exponent (Estapa & Nadolny, 2015). It further extends to conversion from logarithm form to exponential form and vice versa. It equally involves the application of laws of the logarithm to solving equations that involve exponents. From 2014 core mathematics matriculation examination, questions on logarithm have been all about the logarithmic function, of which most of them are as the inverse of an exponential function. In short, one gets the impression that the primary interpretation of logarithm conveyed by textbooks is the indirect conception proposed by Euler.

[3] NATED stands for National Department of Education.

2.1.4.2 Logarithm in the university curriculum

The transition (‘gap’) between secondary and tertiary education in mathematics is a complex phenomenon covering a vast array of problems and issues. Although there is evidence of similar ‘gaps’ in other disciplines in science and beyond, it seems that the transition in mathematics is by far the most serious and the most problematic. Compared to other subjects, mathematics in elementary and high schools enjoys a unique position, in terms of time devoted to it – both in the classroom and outside. There has been no single reform of (elementary or high school) education anywhere that has not, in one way or another, affected the way mathematics is taught. However, in spite of all efforts and energy ventured into the pre-tertiary mathematics education, the knowledge and skills of incoming university students are far from satisfactory (Holton & Artigue, 2001).

The transitional stage in education is just one instance in the sequence of major changes that every person experience in her/his life. These changes – known in anthropology as rites of passage – are events that, in a major way, influence one’s decisions about the future. This is similar when it comes to the logarithm. Logarithm is part of algebra which is mostly taught in calculus class. Logarithm at the university level goes deeper to cater differentiation and integration of logarithm with little or no link with the secondary school logarithm. Most of the Science and Engineering courses thus requires knowledge of logarithms. The logarithm tasks here are about integration, either integrating an expression involving logarithms or resulting in logarithms. Overall, the curriculum emphasizes calculations and says little about contexts in which those calculations are useful.

2.1.5 Students’ difficulties and mistakes when dealing with the logarithm

The mathematics education research literature on logarithm (in English and German) identifies and analyses two different sorts of difficulties faced by students: specific mistakes in manipulating logarithmic expressions, and more general problems in understanding the meaning of the logarithmic concept (Hirsch, 2008).

a) Mistakes in manipulating logarithmic expressions

The studies that investigate the handling of logarithm by students generally present the mistakes made as episodic observations, in a non-exhaustive fashion and without any theoretical grounding. To date, there seems to be no systematic empirical research in this context (Van Dooren, De Bock, Janssens, & Verschaffel, 2008). Thus, all that can be provided here is a list of some incorrect calculations mentioned in the literature:

i. The expression $log(x + y)$ is re-written as $log\ x + log\ y$ (Kaur & Sharon, 1994; Yen, 1999), $log\ x + log\ y$ as $x \times y$ (Berezovski, 2004), $log\ x - log\ y$ as $logxlogy$ (Lee & Heyworth, 1999), $log(xy)$ as $x\ log\ y$ (Chua & Wood, 2005), $logxlogy$ as $logxy$ (Kaur & Sharon, 1994) or as xy (Chua & Wood, 2005), etc.
ii. When treating logarithmic expressions or solving logarithmic equations, the logarithm is eliminated or "canceled out" by dividing by the symbol "log" (Espedal, 2015; Kenney, 2005; Yen, 1999).
iii. In order to calculate the value of $logab$, the root $\sqrt{ba}$ is extracted (Leopold & Edgar, 2008; K. Weber, 2002a).

b) Difficulties in understanding the meaning of the logarithmic concept

In addition to the kinds of abortive algebraic manipulations listed above, some authors also report on misconceptions in meanings. Even students capable of correctly handling logarithmic expressions may labour under such misconceptions:

i. Students conceive logarithm as a "button [...] on my calculator" (Watters & Watters, 2006), as a special "number as Pi" or simply as a "maths machine" (C. Weber, 2013).
ii. Students experience the problem of determining $\log_2 8$ without a calculator as more difficult than that of writing 8 as a power of 2 (Andelfinger, 1985).
iii. Expressions like $\log_a 1$ and $\log_b 1$ (Kenney, 2005) are thought to be different, while, conversely, expressions like $\log_{10} x$ and lnx (Kenney, 2005) or $log(x + 3)$ and $logx \times log3$ (Andelfinger, 1985; Senk & Thompson, 2006) are not recognized as being different.
iv. A logarithm is not recognized as a suitable tool for mathematical decisions and modeling. For instance, when asked which is the larger of the two numbers 25^{625} and 26^{620}, students may reason as follows: "25^{625} is bigger because it has a bigger exponent" (Berezovski, 2004). Conversely, some students are unable to articulate the implications for an actual (chemical, biological, physical) situation where two quantities are logarithmically connected (DePierro, Garafalo, & Toomey, 2008; Watters & Watters, 2006).

Having difficulties of this nature illustrates what failing to understand logarithm can mean. It goes beyond not being able to apply the knowledge of logarithm in solving a problem to being able to know what logarithm is all about.

c) Various underlying causes of the difficulties

These difficulties are given different interpretations in the research literature. In the literature surveyed for the purposes of this study, three different causes could be identified:

i. Students' prior knowledge (mainly the concept of powers or exponents) may be insufficient, or the new concept may not be adequately integrated into it (Chua & Wood, 2005; Kenney, 2005). More detailed analyses identify misconceptions behind students' algebraic mistakes such as $log(x + y) = logx + logy$ and regard them as an overgeneralization of rules, with a distinction being made between the overgeneralization of logarithmic laws (Chua & Wood, 2005), the overgeneralization of linearity (Van Dooren et al., 2008) or the overgeneralization of the distributive law of numbers to operations with the symbol of "log" (Matz, 1982). Other authors focus on the students' visual perception or the visual characteristics of the algebraic expressions. They explain overgeneralizations of the above type by the fact that students "misperceive the problem situation" (Lee & Heyworth, 1999) or by the "visual salience" of the algebraic transformations, i.e. a "visual coherence that seems to make the left- and right-hand sides appear naturally related to one another" (Kirshner & Awtry, 2004).
ii. The teachers may not make the background of the symbol sufficiently explicit (Kenney, 2005), or even evince an insufficient understanding of the logarithm themselves (Berezovski, 2004). For example, Berezovski (2004) concludes in her case study that many of the investigated pre-service secondary teachers had insufficient subject matter knowledge and limited pedagogical content knowledge regarding logarithm.
iii. The dominance of the indirect definition (Confrey & Smith, 1994; Espedal, 2015; Fermsjö, 2014; Mulqueeny, 2012; K. Weber, 2002a; Williams, 2011). In one case study, Williams (2011) reports that her students used the Eulerian definition as a kind of a one-way transferal to exit the new context of the logarithm, arguing and answering in the better-known context of exponents and not transferring the answer back.

2.1.6 Alternative instruction proposals

Because of the many difficulties this topic can cause for many students, it is sometimes discussed in teachers' journals and explored in mathematics education publications. These propose a number of alternative instructional approaches to supplement or replace traditional instruction and which aim to avoid difficulties and improve accessibility. It would appear that few of the alternative approaches have undergone a systematic quantitative evaluation or qualitative investigation, so there is little empirical data regarding the learning of students who were taught using these approaches. They will now be classified into five groups which are using history, using alternative language notation, manual calculation, using different conceptualization and by the application.

a) Using history

Most of the proposals go back to the historical roots of the concept, drawing on Napier and Bürgi. They suggest introducing and explaining logarithm with two juxtaposed progressions in the form of number lines or the columns of a table, and teaching students how to reason on the basis of this model (Clark & Montelle, 2010; Confrey & Smith, 1995; Fermsjö, 2014; J. Katz, Menezes, Van Oorschot, & Vanstone, 1996; Mulqueeny, 2012). Only a few case studies have been conducted on the impact of this kind of logarithm instruction has on students (Fermsjö, 2014; Mulqueeny, 2012). (Fermsjö, 2014), for instance, reports that this approach was helpful and that difficulties in manipulating logarithmic expressions were rarer. However, his students struggled with some new systematic mistakes. (K. Weber, 2002a) chooses a slightly different approach, introducing the logarithm $\log_a b$ as the number of factors a in b. In a pilot study with two groups of university students, he explained the logarithm to one group as the number of factors, while the other group received traditional instruction (K. Weber, 2002a). Both groups were then asked to solve tasks requiring basic computations (e.g. "What is $\log_x x$?") and rules (e.g. "$\log_a x^r$) can be simplified to what? Why?"). The treatment group performed better on these tasks than the control group. Similarly, Espedal (2015) developed some teaching material based on repeated division. This material was presented to learners of one high-school class, while another high-school class received traditional instruction. The "repeated division" group solved tasks such as "Why is $\log 1 = 0$?" or "Solve $2^{x+1} - 3 = 5$" significantly better than the control group (p. 56). However, several of the known mistakes in manipulating logarithmic expressions still occurred in both groups.

b) Using alternative language and notation

As already mentioned, it is not immediately clear what is meant by "logarithm" or "$\log_a b$". The second group of proposals, therefore, involves new language and notation. Examples include the "index" and the "exponent seeker" (Bennhardt, 2009) or the "liftoff function" (Hurwitz, 1999). A form of notation like $a^{\blacksquare}(b)$ would borrow from propaedeutic algebra, where variables are initially represented in the form of placeholders (Hammack and Lyons 1995). One could also think about introducing the "tree notation," a notation adapted from linguistic theory and artificial intelligence (Kirshner & Awtry, 2004).

c) Manual calculation

One pitfall of defining a logarithm as an exponent is that students cannot draw on any pre-existing algebraic procedure to solve an equation such as $2^x = 10$. Students mostly solve tasks like these with the "guess and check" strategy; the teacher's "solution" $x = \log_2 10$ can feel like a cop-out and may easily become entrenched as the "button on my calculator" idea (Watters & Watters, 2006). To avoid problems like these, the third group of proposals suggests having students calculate logarithmic values manually. This can be done using a sequence of nested intervals, going back to Euler's demonstration of how $\log 5$ can be calculated manually (Sandifer, 2014). Other proposals suggest utilizing slide rules or logarithm tables (Tetyana Berezovski, 2004; Ostler, 2013).

d) Using different conceptualizations

Yet another group of proposals suggests utilizing definitional properties other than the Eulerian definition of a logarithm. Some authors suggest starting with Cauchy's property, introducing logarithm as (continuous) functions that fulfill the property $\log(ab) = \log(a) + \log(b)$ and deducing all the properties and logarithmic laws step by step(Seebeck & Hummel, 1959). Other authors use the geometric fact that the area under the hyperbola is a logarithm (Panagiotou, 2011), which is the same as what was proposed by (F. Klein, 2004).

e) Applications

A final group comprises suggestions for applied, "real-life" examples. Teachers present realistic situations; which logarithm can help to calculate (Kluepfel, 1981) or which are to be represented and explored with logarithmic scales (Kirshner & Awtry, 2004; Rahn & Berndes, 1994; Wood, 2005). More elaborated approaches like the Dutch Realistic Mathematics Education use realistic problem contexts to introduce new concepts, formalizing only gradually. Thus, to introduce logarithm, students are first presented with the concept of exponential growth in several realistic contexts. Only then are logarithm presented by interpreting the graph of a realistic exponential function in reverse order (Webb, Van der Kooij, & Geist, 2011). Following a teaching experiment at a college, students claimed to have understood the meaning of logarithm. Whether they also had a better grasp of algebra is not reported.

2.1.7 Research on misconceptions with logarithm

As mentioned before, in the curriculum, a logarithm is defined as an exponent inverse and a logarithmic function as the inverse of an exponential function. Not only do students struggle with exponent laws but research has confirmed that students struggle with the topic of the logarithm (Tetyana Berezovski, 2007; Tanya Berezovski & Zazkis, 2006; Chua & Wood, 2005; Gamble, 2005; Wood, 2005). Tanya Berezovski and Zazkis (2006), in their studies, realized that students' over-dependence on the algorithm and inefficient use of the digital tool in the algorithmic approach of logarithm contribute to the problem. The situation propelled them to think about: what actually necessitates the students' choice of a particular method, the level of students' understanding of the logarithm and how that facilitates reasoning rather than just the use of the digital tool in solving logarithm. Berezovski & Zazkis realized that students' ability to deal with logarithmic expression does not imply they understand their operational meaning. K. Weber (2002a), also emphasized the inadequacy of students understanding and the difficulties they encounter in learning the concept. Many times, students memorize procedures to help them with logarithm, but they lack the meaning of concepts. Tetyana Berezovski (2007) found that not only do students lack a conceptual understanding of logarithm but so do pre-service teachers. A teachers' mathematical knowledge has a strong impact on learners' understanding so it is important to try and mend this learning gap and find where learners are making mistakes when it comes to logarithm and adapts how they teach the topic. Gamble (2005), Panagiotou (2011), and (Wood, 2005) provide different ways teachers can introduce and teach logarithm to students. In order to strengthen students' conceptual understanding of logarithm, it is important to examine where the students' weaknesses lie.

Additionally, (Chua & Wood, 2005) found that students lack a solid understanding of logarithm. He broke his data into three separate categories: knowledge or computation, understanding, and application. The knowledge or computation category contained "routine questions requiring not only direct recall or application of the definition and laws of the logarithm but simple manipulation or computation with answers obtained within two to three steps as well". The overall success rate for this category was 86%. This suggests that students understood the fundamental concepts of the logarithm. One of the highest success rates was for students to convert logarithmic equations into equivalent exponential equations. One of the lowest success rates was for students to calculate the value of $\log 100$. The study found that many students gave the response of 10 rather than the correct response of 2.

Chua and Wood's second category, understanding, had a lower success rate of 66%. Students did fairly well when they were asked for the value of 2^x when $x = log_2(log_3 5)$, however when asked to express $log_6 a$ in terms of m given that $a^m = 36$, 17 out of 79 students left the question blank and there were only 27 correct responses. Chua and Wood found some misconceptions with the problem "simplify $\frac{log_2 27}{log_2 9}$ " (p. 3). "Incorrect responses that were relatively common in this item included $log_2 3$ (about 23%) and 3 (about 14%)… The first arises probably from participants thinking that $log_2 27 \div log_2 9 = log_2(27 \div 9)$ whereas in the second response, they are possibly treating log_2 as a variable common to both the numerator and denominator which can be cancelled out" (p. 3).

Chua and Wood's third category, application items, had the lowest overall success rate at 39%. The application questions required higher level thinking and a deeper conceptual understanding of the logarithm which students did not possess. We can see that students can typically evaluate terms such as $log_2 8$, but they do not have a deep understanding of what that means. Chua and Wood recommends that in the beginning of learning logarithm to have the students put into words the logarithmic expression and explain its meaning before evaluating it.

K. Weber (2002b) indicated that learners understand exponential and logarithmic functions through exponentiation as an action and process, exponential expressions are the results of the process and generalization. Learners being able to view exponentiation as action and process, are the ones who can compute bx as b x times and when they repeat the action and reflect upon it, they interiorize that action as the process (Dubinsky, 2002). Terms such as 24 can be interpreted as an external prompt for the student to compute $2 \times 2 \times 2 \times 2$, which is a product of four factors of 2. Research indicates that students are not capable of viewing 2^4 in this way (Sfard, 1991). In generalization, learners have a full understanding of exponential functions which involve interpreting situations where a number to be evaluated is a fraction (K. Weber, 2002a).

In another study by de Gracia (2016), it showed that most students liked to skip certain important steps when working with logarithm. Moreover, among the laws of the logarithm, the first and second laws emerged to rank second and third respectively, with the most frequent number of mistakes committed by the student-respondents. In their study of the logarithm, students mixed up exponential and logarithmic rules. In particular, students often write $\log x - \log y = \frac{\log x}{\log y}$ instead of the correct expression $\log \frac{x}{y}$. Students also linearize rules and produce such as $log(a + b) = loga + logb$ or $log(2x) = 2log\ x$. de Gracia (2016) elaborated that when students are solving the logarithmic equation, they forget to check the answer is in the domain. If students got two answers and the first one checks, they tend to automatically eliminate the second choice.

Another study carried out by C. Weber (2016) concerning the graphing of logarithmic function reveals that students have difficulty with graphing of logarithmic function which might be caused by the standard interpretation of logarithm as inverse exponents. Functions, in general, can be challenging to students, for instance when it comes to interpreting graphs verbally, or deciding whether or not a given graph represents a certain function (Qi & Li, 2015; Vaninsky, 2015). In the case of logarithmic functions, the graph of a logarithmic function can be confused with the "combined graph" (Kastberg, 2002), i.e. with the exponential and the logarithmic graph both merged into a single image. Other empirical research findings show that some students view the graphs of $y = \log 2x$ and $y = \log_2 x$ as being "exactly the same" (Williams, 2011). Or, as a variant of the misconception according to which any function should be linear (Sfard, 2008), they consider logarithmic functions to be proportional. As elaborated earlier, misconceptions like these might be caused by the standard introduction of logarithmic functions as inverse exponential functions (C. Weber, 2016). But even when focusing on graphs and graphing only, several problems can arise: Firstly, determining the domain of a function is a known issue (Vaninsky, 2015). As an illustration, two students' graphs of the logarithmic function $y = \log_2(x) - 3y$: Student A's graph extrapolated to the left, intersects the x-axis and thus exceeds the domain. Moreover, his x -values form an arithmetic progression, which is not optimal in terms of the corresponding y -values. Secondly, graphs can be thought of as isolated points (Qi & Li, 2015), or the supporting points may be interpolated with a straight line (Vaninsky, 2015). And thirdly, the graph will not, or rarely, be extended beyond the range of the supporting points, or there may be an extrapolation to one side which suggests a progressive growth of the logarithmic function.

Students' difficulties like these give rise to the following question: What could the mathematical knowledge for teaching logarithm look like to endow learners with a solid basis for understanding and argumentation?

2.1.8 Learning of logarithm through process or object

According to (Sfard, 1991), numbers and functions have two different perspectives to be viewed at:

a) Structural conception – as objects: is one's ability to solve the problem completely or to recognize mathematics steps to the solution as a holistic entity. Tall, Thomas, Davis, Gray, and Simpson (1999) and Cottrill et al. (1996), see object-oriented thinking as one's ability to recognize a mathematical procedure or process as an entity without performing the procedure. The object is mostly linked to the product of the process.
b) Operationally – as processes oriented: the sequential actions that are maximized when solving mathematical problems.

Kastberg (2002) reported that students failed to see logarithmic expressions as objects. The students in her study perceived "log" as a command to operate rather than part of the expression. She found out that students sometimes correctly remembered rules and sometimes incorrectly remembered them, but they tended to believe that a problem was not finished until it was in decimal form. Kenney (2005) asked students to solve for x in the equation $log_5 x + log_5(x+4) = 1$. Both students interviewed believed that they should "cancel out" the logs because the logs were of the same base, leaving them with $x + x + 4 = 1$. Even though they had been recently tested on logarithm, both students failed to recognize that adding logarithmic expressions is equivalent to multiplying the expressions inside the logs (x and $x + 4$). This indicates a misunderstanding of logarithmic expressions as objects, because they believed that they had to get rid of the logarithm before performing any operations. Students with an object conception of logarithm ought to be able to operate on logarithm, using the rules of the logarithm, without the "removing the logarithm" first. With regards to the rules of the logarithm, K. Weber (2002a) wrote, "as time passes, one's knowledge of symbolic rules will generally decay. If one has a deep understanding of the concepts involved, these rules can be reconstructed. If not, the rules cannot be recovered" (p.101). If we take Weber's assertion to be true, then misremembering rules and failing to check them for validity (perhaps because they do not know how) could indicate a lack of understanding. Weber found that students in a pilot study who were taught in a way that focused on concepts could reconstruct rules such as, while students enrolled in a more traditional class could not reconstruct such rules, and misremembered them without correction. Kastberg (2002) found that students who were successful with computational logarithm problems misremembered rules a few weeks later, such as remembering $log_a b = \frac{log\ a}{log\ b}$ instead of correctly remembering $log_a b = \frac{log\ b}{log\ a}$ or remembering $\log a + \log b = \log(a + b)$ instead of correctly remembering $\log a + \log b = \log(ab)$. Although being proficient at the rules of the logarithm is an important part of understanding the logarithm as objects, it is insufficient for the students to simply memorize the rules. If students do not understand the rules of logarithm they will probably not remember them or be able to reconstruct them once they forget exactly what the rules are.

In examining student understanding of logarithm and logarithmic expressions as numbers, Tanya Berezovski and Zazkis (2006) expressed doubt that facility with calculating logarithmic expressions involving only numbers either with a calculator or by hand indicates an understanding of logarithm as numbers. In their findings, they suggested that students may have learned a procedure when presented with such types of problems, but that these students may not recognize that, for example, $-log_2 3$ is a number and does not need to be operated on in order to become a number. The research conducted by Kastberg (2002) supports the idea that students who can solve problems do not necessarily perceive logarithmic expressions as numbers. For example, one student referred to the process of finding a numeric value for the expression as solving an equation (p. 101). The student then correctly computed a decimal approximation for $log_4 5$, but did not seem to recognize that $log_4 5$ was already a number, instead labeling it an equation. Students considering operational steps as an object helps them to develop the structural concept. For example, in expressing $2log_b 3 + 3log_b 5$ as objects, one has to represent $2log_b 3$ by $log_b 3^2 = log_b 9$ and $3log_b 5$ by $log_b 5^3 = log_b 125$ before rephrasing it by the use of the property: $\log a + \log b = \log(ab)$ to give $log_b(9 \times 125)$ (Wood, 2005).

In some cases, a process orientation is the most helpful way to view logarithm, as demonstrated by the following vignette, Tanya Berezovski and Zazkis (2006), Carraher and Schliemann (2007), Clark and Montelle (2010) and Shirali (2002). In this instance, learners were trying to find the whole number equivalent to $5log_3 9$. After some discussion about how to use the change of base rule and input the values correctly on the calculator, one student explained that you just need to convert 9 to $3^2$and the problem becomes much simpler. This student demonstrated that she had some understanding of logarithm as processes even though her class did not.

In viewing logarithm as processes, Confrey and Smith (1994) wrote that logarithm is built from multiplication as a primitive structure in itself, not multiplication as extrapolated from the addition. They called this primitive structure "splitting" and claimed that by providing learners with contextual problems based on the splitting concept, they were able to demystify some of the rules of logarithm for learners (Confrey & Smith, 1995). They explained that if you view multiplication as a structure parallel to, instead of building from, addition, then rules like $\log a + \log b = \log(ab)$ are grounded in the understanding that addition in one structure is equivalent to multiplication in the other. While Confrey and Smith (1995) moved for less extrapolation (logarithm founded on multiplication, which is a primitive structure), Hurwitz (1999) moved for more: logarithm is founded on the exponential function (as its inverse) which is founded on multiplication which, in turn, is founded on addition. Hurwitz claimed that if students are shown that the exponential function "puts on an exponent," then the idea that logarithm, as the inverse of the exponential function, "lift off the exponent" will build upon previous students' knowledge and give students a foundation from which to build. Hurwitz explained "lifting off" as, for example, in $g_8(8^{4/3})$, applying the "liftoff function" gives $4/3$, because you have lifted off the exponent. She also reinforced her method through notation by writing **(l)**ift**(o)**ff functions ($\boldsymbol{g}_{b(x)}$), circling the l, o, and $\boldsymbol{g}_{b(x)}$.

The misconception of mathematical structural ideas (objects) contributes to some of the mistakes students make in mathematics. Yen (1999) mentioned that some students perceive "ln" as a variable in an equation like $ln(7x - 12)$, thus "ln" is a common factor where it can be expanded and become $ln(7x) - ln(12)$. Others when given the logarithmic equation $\log y = \log 100$, they divide both sides of the logarithmic equation by "log" to get 100 which implies that $y = 100$ (Lopez-Real, 2002) as a result of a misconception of object ideas. Mathematics as an object has a great beneficial effect which helps in making abstract ideas clear and assigns meaning to it. The empirical evidence from researches conducted by many mathematicians verifies that in the midst of acquiring a new mathematical concept, the object comes after the process (Sfard, 1991; Sfard & Linchevski, 1994). The process-based thinking augments object-based thinking, thus, the steps or sequential actions followed in solving problem support the authenticity of the final answer.

To go further with the review of the conception of the logarithm, it will be necessary for this research to look at logarithm as a function and in contextual problems.

2.1.9 Learning of logarithm as functions and in contextual problems

Learners sometimes struggle to see logarithm as functions. (Hurwitz, 1999) suggested this may be due in part to the notation because $\log x$ does not look like many of the common functions, such as polynomials. A student named Jamie, in the research conducted by Kasburg in Canada also commented on the fact that just seeing "logarithm" confused her and believed that the fact that it was a word, instead of a number, was what threw her and others off (Kastberg, 2002). Another student in the same study also drew the graph of the logarithm as including both the logarithmic function and the exponential function and believed that the two graphs together made up the graph of the logarithmic function. This student was a straight A student in her mathematics classes, yet she did not seem to recognize that her graph could not possibly be a function because there were x-values that corresponded to more than one y-value. It may also be that if asked if such a graph was a function, she would say no, it doesn't pass the vertical line test, and she just does not conceive of the logarithmic function as a function.

Tanya Berezovski and Zazkis (2006) posed the question "Which number is larger, 25^{625} or 26^{620}?" and found that more than half of the learners (who had just completed a unit on logarithm) did not attempt to use a logarithm to solve this problem. This seems to indicate a lack of understanding of logarithm in context because one of the primary purposes of using logarithm in contexts is to make extremely large numbers more usable. Bennett, Briggs, and Badalamenti (2008) observed that learners "have a particularly difficult time relating to" logarithm (p. 167). He suggested this may result from a lack of true application problems and suggested several real-world applications that teachers might use to help learners relate better to the logarithm, such as the decibel scale, the Richter scale, and stock analysis. Watters and Watters (2006) found that neither freshmen enrolled in biochemistry nor upper-level learners in the same program were very successful at solving pH problems that required the ability to reason with logarithm. This is the only study I could find that tested logarithmic understanding of upper-level college students who ought to have been able to solve problems with logarithm. On the other hand, Kastberg (2002) found that her subjects (college algebra students) were usually able to problem-solve their way through logarithmic problems in context, as long as they didn't know the problem involved logarithm. The students did not recognize that logarithm could be used to solve such problems, so they solved them by relating the problems to exponents (which they were more comfortable with than logarithm) and were successful, if not efficient, in solving the problems. In order to have a good understanding of logarithm in context, I believe that students ought to recognize that logarithm will help them solve the problems (as the students in the first two studies did, but Kastberg's did not), and to be able to solve the problems correctly (as the students in the first two studies did not, but Kastberg's did).

Having looked at the literature review on logarithm, it is necessary for one to know what has been reviewed concerning teachers' knowledge of mathematics.

2.2 The literature on teachers' knowledge of mathematics

Teachers' knowledge of mathematics is central in this study because this study explores pre-service mathematics teachers' knowledge of logarithm. Below I discussed teachers' knowledge as presented in the literature sources reviewed. I started with the difference between academic and school mathematics and then presented the question of what mathematics do high school teachers need and conclude with teacher content knowledge.

2.2.1 The difference between academic and school mathematics

Studying mathematics usually makes one realize that the kind of mathematics taught at university is apparently different from the kind of mathematics taught at high school (Deng, 2018; Fraser, 2016). It is well known that at the beginning of the twentieth century, Felix Klein as cited in (F. Klein, 2016) emphasized that there is a discrepancy between the mathematics taught at schools and the mathematics taught at university. At that time in Germany, the gap between school mathematics and academic mathematics was understood mainly in terms of content, since at school, the merely algebraic analysis was taught, whereas in university courses, the focus was exclusively on infinitesimal calculus (Allmendinger, 2016). In this spirit, (F. Klein, 2016) criticized that "the teacher manages to get along still with the cumbersome algebraic analysis, in spite of its difficulties and imperfections, and avoids any smooth infinitesimal calculus" and that "the university frequently takes little trouble to make a connection with what has been taught at schools but builds up its own system". However, Klein also saw differences between the mathematics taught at school and the mathematics taught at university that goes beyond aspects of content. He characterized school mathematics as being "intuitive and genetic, i.e., the entire structure is gradually erected on the basis of familiar, concrete things, in marked contrast to the customary logical and systematic method in higher education" (F. Klein, 2016). Even though infinitesimal calculus has been—at least to a certain degree—included in upper secondary school mathematics in the meantime, differences between the kinds of mathematics taught at school and at university remain. These have been illustrated for instance by (Wu, 2011). One of his examples was the topic of fractions, which is central in lower secondary mathematics: When fractions are taught in university mathematics courses, usually is defined as a set of equivalence classes of ordered pairs of integers. Addition and multiplication on this set are subsequently defined such that the axioms of a ring are satisfied, and it is routinely checked that these definitions are compatible with the equivalence relation. Hence, the rational numbers are introduced in an axiomatic-deductive way, which is typical for how academic mathematics is taught. This introduction is characterized by a high level of abstraction as well as a symbolic mathematical language and it illustrates what Klein called the "customary *logical* and *systematic* method in higher education" (F. Klein, 2016). When fractions are taught in school mathematics, the introduction normally does not start with a definition, but with a context. In order to present fractions as parts of a whole, often "familiar and concrete things" (F. Klein, 2016) like pizzas or chocolate bars are used. Since it is not defined what a "whole" is, the pizza is used as a prototypical "whole" (Wu, 2011). Also, the way in which addition and multiplication of fractions works is justified in a different manner compared to the university course: If the learners are not just asked to learn the calculation rules without any reasoning, then, usually, contexts like pizza and chocolate bars are used to make sense of why the rules should work like this. However, at this point it is not enough to interpret a fraction as a part of a whole and students are thus asked to understand fractions as different things at the same time (e. g., an operator or a ratio). There is generally not much reasoning about why fractions can be all these things at the same time and sometimes it is even said that $\frac{3}{4}$ is "3 divided by 4," which is not mathematically coherent with the students' understanding of division, as argued by (Wu, 2011). In the context of the mathematics taught at university, however, Wu pointed out that given suitable definitions of "part of a whole" and of " $m \div n$ for arbitrary integers m and n $(n \neq 0)$", it is a provable theorem that, indeed, $\frac{m}{n} = m \div n$. This illustrates that the kinds of mathematics taught at school and at university differ also in terms of rigor and in the necessity that is seen for justification

To sum up this, (Wu, 2011) show that these two kinds of mathematics typically differ in the following aspects: Mathematics as the scientific discipline taught at the university has an axiomatic-deductive structure and focuses on the rigorous establishment of theory in terms of definitions, theorems, and proofs. It usually deals with objects that are not bound to reality and it is often characterized by a high level of abstraction and a symbolic mathematical language (McCulloch, Lovett, & Edgington, 2017; Oktaviyanthi, Herman, & Dahlan, 2018; Wu, 2011).

Of course, it should be noted that mathematics as a scientific discipline does not always work in an axiomatic-deductive manner. Taking the example of fractions, it is obvious that fractions were introduced and used in mathematics before the discipline had its axiomatic structure. Also, when new concepts are found in mathematical research, the concept formation does not usually happen deductively. However, when mathematical results are reported in journals or books, and when mathematics is taught to university students, it is usually presented in an axiomatic-deductive way. Since this is the kind of mathematics that pre-service teachers, as well as future mathematicians, are confronted with during the course of their university studies, there is a need to explore what the pre-service mathematics teachers know about high school mathematical topics.

On the other hand, mathematics as a school subject usually places its main focus on applying mathematics as a tool for describing as well as understanding reality, and for facilitating everyday live (Jablonka, 2015). Consequently, mathematical objects are often introduced in an empirical manner and bound to a certain context. The term the concept formation in mathematics classrooms in the high school is, accordingly, often done in an inductive way by means of prototypes (Dreher, Lindmeier, & Heinze, 2016; Wu, 2011). Mostly, intuitive and context-related reasoning is more in the focus than rigorous proofs.

Discrepancies between a school subject and the related academic discipline do not only exist in the case of mathematics but are a more general phenomenon, as Dreher et al. (2016) pointed out that the contents of teaching are not simply the propaedeutical basics of the respective science. Just as the contents to be learned in German lessons are not simplified German studies, but represent a canon of knowledge of their own, the contents of learning mathematics are not just simplifications of mathematics as it is taught in universities. The school subjects have a "life of their own" with their own logic; that is, the meaning of the concepts taught cannot be explained simply by the logic of the respective scientific disciplines. Rather, goals about school (e.g., concepts of general education) are integrated into the meanings of the subject-specific concepts.

Taking a look back at the descriptions of academic mathematics and school mathematics given above, one recognizes in these explanations by Dewey the reasons for the major differences between these two kinds of mathematics programs. However, academic discipline and school subject are also dialectically related as Deng stated that the former supplies the guidance and direction for the latter and reveals the possibilities of growth inherent in the experience of the learners. The latter is considered as the means of leading the learner toward the realization of these possibilities (Deng, 2018). Therefore, in a sense, academic mathematics precedes school mathematics, as it functions as a frame of reference for the structure of school mathematics. However, in another sense, school mathematics precedes academic mathematics, since it provides the path for getting to know academic mathematics.

In view of these major differences between academic and school mathematics, the question arises as to what kind of mathematics high school teachers need to know and what kind of mathematics prospective mathematics teachers should be taught. Is it school mathematics? Or academic mathematics? Or both? Or something else? Questions like these have already been raised by (Dreher, Lindmeier, Heinze, & Niemand, 2018) where he asked; How in particular is his [the mathematics teacher's] knowledge related to the content of the mathematics school curriculum and to mathematics as a science? This can also be related to the gap between the teaching and learning of logarithm in both secondary schools and that of the universities. This is evident as the curriculum of South African secondary schools do not cover all the basics the secondary school learner needs to know concerning logarithm which will be the foundation for what they will learn at the university level.

2.2.2 What mathematics do high school teachers need to know?

Since academic mathematics is different from what mathematics teachers teach at the school, one could argue that Content Knowledge (CK) in mathematics teacher education should mainly focus on school mathematics. There is, however, a broad consensus among scholars and researchers in mathematics education that mathematics teachers in general—and in particular those teaching at a high school level—need to have insight into academic mathematics (Dreher et al., 2018; Stein, Engle, Smith, & Hughes, 2015; Wake, 2014; Winsløw & Grønbæk, 2014). Klein (1932) as cited in (F. Klein, 2016) already pointed out that "the teacher's knowledge should be far greater than that which he presents to his pupils. He must be familiar with the cliffs and the whirlpools in order to guide his pupils safely past them". Consequently, in many countries, teacher education for high schools includes large parts of academic mathematics, especially if there is a focus on the upper high school level (König, Blömeke, Paine, Schmidt, & Hsieh, 2014; Speer, King, & Howell, 2015). This often means that prospective mathematics teachers take largely the same courses as their fellows studying mathematics as a scientific discipline. This approach usually ensures that these prospective teachers know far more mathematics than their future students, but it does not guarantee that they can guide them safely past "the cliffs and the whirlpools" in the mathematics classroom. The gap between the academic mathematics taught at university and school mathematics is often too wide, so that prospective mathematics teachers are not able to make connections. Based on the frequently cited quote of Felix Klein (1932), this problem is well known as "double discontinuity." About 100 years after Klein, Wu (2011) argued even more critically: Teaching high school teachers the same advanced mathematics as prospective mathematics researchers and expecting "the Intellectual Trickle-Down-Theory to work overtime to give these teachers the mathematical content knowledge they need in the school classroom" is as ridiculous as teaching future French teacher's Latin instead of French.

Hence, it appears to be neither sufficient to teach prospective high school mathematics teachers school mathematics nor does academic mathematics alone ensure that pre-service teachers have the CK needed in the mathematics classroom (Buchholtz et al., 2013; Öhman, 2015). Against this background, F. Klein (2016) suggested that prospective high school mathematics teachers should be taught elementary mathematics from a higher standpoint. In his corresponding lecture series for pre-service high school mathematics teachers, which required knowledge of the main fields of academic mathematics as a prerequisite, he focused on relations between academic mathematics and school mathematics by taking an academic-mathematical perspective on school mathematics (Allmendinger, 2016).

Having discussed what, the high school mathematics teacher needs to know, we need to review the content knowledge of a high school teacher.

2.2.3 Teacher content knowledge

In any profession, there is a specialized professional knowledge that makes it unique and distinct with striking features entirely different from other professions. One of the characteristics of good teachers is that they possess a substantial amount of that specialized knowledge. The teaching of Mathematics is a multifaceted human endeavor, involving a complex, moment-by-moment interplay of different categories of knowledge. Teachers' mathematical knowledge, pedagogical competence and reasoning are key to improving students' mathematical achievement. Traditionally, the teaching of Mathematics is about telling or providing clear, step-by-step explanations of procedures while students learn by listening and practicing these procedures. It has been revealed that having a flexible, thoughtful and conceptual understanding of subject matter is critical to effective teaching (Depaepe et al., 2015; Dunekacke, Jenßen, & Blömeke, 2015). The substantial amount of knowledge required by teachers is known as pedagogical content knowledge (PCK), which is the intersection between pedagogy and content (Shulman, 2013).

What kind of CK do pre-service mathematics teachers need? How can a profession-specific mathematical CK be characterized? These questions are highly relevant for the design of mathematics teacher education programs as well as for investigating teachers' professional knowledge. As the field of mathematics education encompasses different research traditions, such central questions can be considered from different perspectives. Bishop (1992) distinguished three different research traditions—pedagogue tradition, empirical scientist tradition, and scholastic philosopher tradition—which provide a means to structure different perspectives concerning these questions. In the pedagogue tradition, the goal of inquiry is the direct improvement of practice (Bishop, 1992). Regarding the issue of pre-service mathematics teachers' CK, this would mean that the design of specific teacher education programs and courses is paramount, which is the case for practice-oriented development projects such as "Thinking mathematics in new ways" that restructured the teacher education program for the higher high school level. However, in order to investigate systematically what kind of learning opportunities in teacher education are effective or whether there is an interrelation between preservice teachers' CK and their instructional quality or student learning, it is necessary to have a corresponding model of teacher professional knowledge, a conceptualization of high school mathematics teachers' CK, and a corresponding operationalization. Such research, which has the aim of explaining educational reality by means of objective data, can be seen in the empirical scientist tradition (Bishop, 1992).

Especially during the past 15 years, this kind of research on teachers' professional knowledge has received a lot of attention among researchers in mathematics education. As a result, there is a broad base of research on how to conceptualize and capture the professional knowledge of mathematics teachers (Koehler, Mishra, Kereluik, Shin, & Graham, 2014), on effects of professional knowledge on student learning outcomes (Baumert et al., 2010), and on the development of professional knowledge during the course of mathematics teacher education (Blömeke, Hsieh, Kaiser, & Schmidt, 2014). Most of these studies are based on models for teachers' professional knowledge which draw on the categories "content knowledge" and "pedagogical content knowledge" identified by Shulman (2013). However, since Shulman's model is quite general, it is, for instance, not clear how to conceptualize and operationalize the construct of teachers' professional CK. Consequently, existing studies show wide discrepancies regarding this construct, which is usually conceptualized based on school subject knowledge and refers to academic mathematics to a greater or lesser extent (Heinze, Dreher, Lindmeier, & Niemand, 2016). Accordingly, as early as in the 1970s, Fletcher (1975) pointed out the need to specify such a profession specific mathematical knowledge. He said "The mathematics teacher requires a general knowledge of mathematics in order to be able to communicate with other mathematicians and also to establish his credentials; but he also requires special knowledge of certain areas of mathematics, in the way that an engineer or an astronomer requires special knowledge. [...] It is part of our problem that the teacher's special mathematical knowledge is inadequately defined and insufficiently esteemed".

Teachers' mathematical knowledge, pedagogical competence and insight into the development of students' mathematical ideas and reasoning are key to improving students' mathematical achievement (Deng, 2018). As (Ball, Hill, & Bass, 2005) argue that little improvement is possible without direct attention to the practice of teaching; that how well teachers know Mathematics is central, which explains why recently there has been a considerable discussion and research on teachers' subject matter knowledge, pedagogical content knowledge and mathematical knowledge for teaching. The perspective of some Mathematics Educators is that to teach a school subject like Mathematics effectively, necessitate knowledge of Mathematics that goes beyond the subject matter per se to the dimension of subject matter knowledge for teaching or what Ball and Bass (2002) term as mathematical knowledge for teaching.

However, studies conducted in the past have not adequately taken into account mathematical problems which arose in daily mathematical learning situations when analyzing teachers' pedagogical content knowledge. On the contrary, some studies found and revealed that some teachers who acquired more mathematical knowledge facilitated their students' learning and thereby improve problem-solving performance (Deng, 2018). In this regard, the conception of mathematical knowledge is a critical aspect of teachers' knowledge before they are able to help students learn.

2.2.4 Conclusion

The mathematical concepts of logarithm and logarithmic functions play an important role in advanced mathematics courses. In recent years, several research studies reported on students' understanding and misunderstanding of logarithm at the high school and undergraduate mathematics level. It is also common knowledge in the field of mathematics education that teachers' mathematical knowledge for teaching has a strong impact on students' understanding. However, there is no significant body of research that focuses on pre-service teachers' mathematical knowledge of logarithm and logarithmic functions and this is what this research seeks to look at.

2.3 PART 2: THEORETICAL FRAMEWORK

2.3.1 Introduction

This research study focuses on exploring pre-service mathematics teachers' knowledge of logarithm. The study is grounded in a constructivist conception of the learning of science. According to this conception, learning occurs when students make sense of new information by relating it to their prior knowledge (Duffy & Jonassen, 2013). Constructivist perspectives on learning have been central to much of recent empirical and theoretical work in mathematics education (Ball & Bass, 2002; Kathleen Dunaway, 2011) and as a result, have contributed to shaping mathematics reform efforts. Constructivism has provided mathematics educators with useful ways to understand learning and learners. The task of reconstructing mathematics pedagogy on the basis of a constructivist view of learning is a considerable challenge, one that the mathematics education community has only begun to tackle. This study is framed through constructivism theory for it is relevant in a study which aims at understanding the construction of knowledge and understanding.

2.3.2 Constructivism

The theory of constructivism, in the field of learning, comes under the broad heading of cognitive science. The term constructivism refers to the idea that learners construct knowledge for themselves (Bada & Olusegun, 2015). Each learner individually and socially constructs meaning as he or she learns. Constructivism is divided into social constructivism and cognitive constructivism. Although terms such as "radical constructivism" and "social constructivism" provide some orientation, there is a diversity of epistemological perspectives even within these categories (Steffe & Gale, 1995). For the purpose of this study, cognitive constructivism will be referred to as constructivism.

Constructivism is an approach to teaching and learning based on the premise that cognition (learning) is the result of "mental construction" (Juvova, Chudy, Neumeister, Plischke, & Kvintova, 2015). In other words, students learn by fitting new information together with what they already know. Constructivists believe that learning is affected by the context in which an idea is taught as well as by students' beliefs and attitudes. Constructivism is a learning theory found in psychology which explains how people might acquire knowledge and learn. It, therefore, has a direct application to education. The theory suggests that humans construct knowledge and meaning from their experiences. Constructivists view learning as the result of mental construction. That is, learning takes place when new information is built into and added to an individual's current structure of knowledge, understanding, and skills. The widespread interest in constructivism among mathematics education theorists, researchers, and practitioners has led to a plethora of different meanings for constructivism." Therefore, it seems important to describe briefly the constructivist perspective on which this study is based.

In recent years, the development of the constructivist view of learning has resulted in modifications of teaching design in many science classes (Bhattacharjee, 2015; Duit, 2016; Schcolnik, Kol, & Abarbanel, 2016). The modifications not only involve a change of teaching methods but are more likely to bring about a revolution in classroom culture, including the roles of teachers and students, as well as the course goals (Jack, 2017). In other words, an innovative constructivist teaching program normally implies a modification of teaching tasks and strategies, learning tasks and strategies, and the criteria for learning achievements. It is suggested that the teacher's role shifts from knowledge provider to learning facilitator and that the student's role shifts from information collector to the active practitioner (Kalamas Hedden, Worthy, Akins, Slinger-Friedman, & Paul, 2017; Li & Guo, 2015; Peschl, Bottaro, Hartner-Tiefenthaler, & Rötzer, 2014). The focus of learning achievement may be broadened from mere knowledge accumulation to personal development, including attitudes of learning and adoption of learning strategies (Chao, Chen, & Chuang, 2015; Lai & Hwang, 2016; Peterson, Rubie-Davies, Osborne, & Sibley, 2016).

Cognitive psychologists (Anderson, 2005; Sawyer, 2006) believes that learning is most likely to occur when an individual can associate new learning with previous knowledge. Learners work independently or in cooperation with others to internally generate unique knowledge structures. To solve a problem, students have to search for their knowledge structure for knowledge that can be used to develop a solution pathway. An individual's knowledge is self-organized through various mental associations and structure. These organized pieces of knowledge have been classified by Anderson and Krathwohi (2001) into four types: factual, conceptual, procedural and metacognitive knowing.

i. Factual (or declarative) knowledge

This type of knowledge consists of the basic elements the students must know to be acquainted with the discipline or solve problems in it. It is the knowledge that can be declared, through words and symbol systems of all kind. It includes knowledge of terminologies, e.g. that a logarithmic equation can be transformed into an exponential equation and vice versa, that $log_2 3$ is an irrational number which does not need further simplifications.

ii. Conceptual Knowledge

Knowledge of concepts is often referred to as conceptual knowledge (Fiedler, Tröbst, & Harms, 2017; Ninaus, Kiili, McMullen, & Moeller, 2016; Rittle-Johnson & Schneider, 2014). This knowledge is usually not tied to particular problem types. It can be implicit or explicit, and thus does not have to be verbalizable (Poortman, 2017; Rittle-Johnson & Schneider, 2014). It is a knowledge-rich in relationship and understanding (Woolfolk, 2010). The National Research Council adopted a definition in its review of the mathematics education research literature, defining it as 'comprehension of mathematical concepts, operations, and relations' (Groth, 2017). This type of knowledge is sometimes also called conceptual understanding or principled knowledge. At times, mathematics education researchers have used a more constrained definition. Maciejewski and Star (2016) noted that the term conceptual knowledge has come to encompass not only what is known but also the way that concepts can be known (e.g. deeply and with rich connections). This definition is based on Hiebert and LeFevre's definition in the seminal book edited by Hiebert and Lefevre (1986): "Conceptual knowledge is characterized most clearly as the knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network" (p. 3). After interviewing several mathematics education researchers, (Baroody, Feil, & Johnson, 2007) suggested that conceptual knowledge should be defined as 'knowledge about facts, and principles' (p. 107), without requiring that the knowledge be richly connected. Empirical support for this notion comes from research on conceptual change that shows that (1) novices' conceptual knowledge is often fragmented and needs to be integrated over the course of learning and (2) experts' conceptual knowledge continues to expand and become better organized (DiSessa, 2014; Schneider & Stern, 2009). Thus, there is a consensus that conceptual knowledge should be defined as knowledge of concepts. A more constrained definition requiring that the knowledge be richly connected has sometimes been used in the past, but more recent thinking views the richness of connections as a feature of conceptual knowledge that increases with expertise.

Conceptual knowledge has been described as the ability of one knowing the facts and the why of it (Frederick & Kirsch, 2011). Conceptual knowledge goes beyond just a response to the test items. The essence of it is to probe into students' result more than just the correct answer. It cannot be learned by rote. It must be learned through thoughtful, reflective learning. Fiedler et al. (2017) explained conceptual knowledge as the acquisition of enough concepts and skills to reflect, reassess and reformulate the already acquired knowledge. According to Rittle-Johnson and Schneider (2014), a student's ability to establish a relationship between pieces of information is an indication of attaining conceptual knowledge. He further explained that conceptual knowledge can be developed through the student's ability to establish a relationship between the old knowledge acquired and the new knowledge being acquiring. It takes conceptual knowledge for a student to recognize that simplifying logarithmic expression requires the application of laws of the logarithm. According to Poortman (2017), conceptual knowledge is developed through discovery learning. Kilpatrick et al in (Council, 2001) explanation outlines and summaries of what other researcher has described conceptual knowledge to be. According to them, it constitutes (a) comprehension of mathematical concepts (b) operations or process and (c) relations. According to Skemp (1978), the likelihood of a concept becoming part of students with a clear understanding is certain compared to those who memorized a procedure. In other words, developing conceptual knowledge of a concept is better retained and applied than memorizing it. Conceptual knowledge —the ability of the student to demonstrate a clear understanding of a concept—helps students to demonstrate their understanding of logarithm as Object and logarithm as Process, as stated by Sfard (1991). Conceptual knowledge differs from the factual knowledge that is applicable only to certain situations. If conceptual understanding is gained, then a person can reconstruct a procedure that may have been forgotten. On the other hand, if procedural knowledge is the limit of a person's learning, there is no way to reconstruct a forgotten procedure.

iii. Procedural knowledge

Procedural knowledge includes knowledge of how to perform certain activities, like solving a problem. Knowledge of procedures is often termed procedural understanding (Rittle-Johnson & Schneider, 2014; Tseng, 2012). For example, 'Procedural knowledge… is 'understanding how', or the understanding of the steps required to attain various goals. Procedures have been characterized using such constructs as skills, strategies, productions, and interiorized actions' (Maciejewski & Star, 2016, p. 307). The procedures can be (1) algorithms—a predetermined sequence of actions that will lead to the correct answer when executed correctly, or (2) possible actions that must be sequenced appropriately to solve a given problem (e.g. equation-solving steps). This knowledge develops through problem-solving practice and thus is tied to particular problem types. Further, 'It is the clearly sequential nature of procedures that probably sets them most apart from other forms of understanding' (Hiebert & Lefevre, 1986, p. 6).

As with conceptual knowledge, the definition of procedural understanding has sometimes included additional constraints. Within mathematics education, Maciejewski and Star (2016) noted that sometimes: 'the term procedural knowledge indicates not only what is known (knowledge of procedures) but also one way that procedures (algorithms) can be known (e.g. superficially and without rich connections)' Baroody et al. (2007, p. 408) acknowledged that: "some mathematics educators have indeed been guilty of oversimplifying their claims and loosely or inadvertently equating "knowledge memorized by rote … with computational skill or procedural understanding". Mathematics Education Researchers (MERs) usually define procedural understanding, however, in terms of understanding type—as sequential or "step-by-step on how to complete tasks" (Hiebert & Lefevre, 1986, p. 6). Thus, historically, procedural understanding has sometimes been defined more narrowly within mathematics education, but there appears to be agreement that it should not be. However, at least in mathematical problem solving, people often know and use procedures that are not automatized, but rather require conscious selection, reflection, and sequencing of steps (e.g. solving complex algebraic equations), and this knowledge of procedures can be verbalized (Maciejewski & Star, 2016). Overall, there is a consensus that procedural knowledge is the ability to execute action sequences (i.e. procedures) to solve problems. Possession of factual and conceptual knowledge reflects abstract understanding rather than practical understanding which indicates procedural knowledge. Conceptual understanding in problem-solving task alongside with procedural skill is much more effective than procedural skills alone (Samuels, 2015).

iv. Metacognitive (or regulatory) knowledge

This knowledge is a multi-faceted construct described by Haberkorn, Lockl, Pohl, Ebert, and Weinert (2014, p. 248) as "… knowledge of how to use available information to achieve a goal; ability to judge the cognitive demands of a particular task; knowledge of what strategies to use for what purpose; and assessment of one's progress both during and after the performance". As conceived by Artelt and Schneider (2015), metacognitive knowledge is knowing how and when to use factual, conceptual and procedural knowledge. For several students, this type of knowledge is a barrier because at many occasions, they have facts and can perform the procedure, but they find it difficult on how to apply what they know at the appropriate time (Özsoy & Ataman, 2017). It takes metacognitive knowledge to know when to calculate the number of payments in a future value annuity question using logarithm.

2.3.3 Summary of the theoretical framework

An effective classroom, where teachers and students are communicating optimally, is dependent on using constructivist strategies, tools, and practices. It is possible to understand and apply constructivist teaching strategies and practices in the classroom. There is a tight connection between how the teacher instructs and what the students learn. One of the factors that determine the effectiveness of classroom instruction is the teacher's knowledge of student's prior knowledge in all subject areas. Many theorists discuss advantages and disadvantages, but the actual process of learning with meaning and students constructing concepts to create knowledge is common to both types. I have explored constructivism and various mental associations and structure of an individual's knowledge to ensure an effective understanding of a concept. Knowledge has to be built on existing knowledge and one's background and experience contribute to this process.

In the next chapter, I present the research methodology and procedures undertaken for this study.

CHAPTER THREE

3. RESEARCH METHODOLOGY

3.1 Introduction

The literature review, which was discussed in the previous chapter, explored the educational background of the logarithm in relation to the place of the logarithm in the high school curriculum, and equally looks at pre-service mathematics teachers' difficulties and mistakes in logarithm. The literature equally explores the teachers' knowledge of mathematics which gives an insight on mathematical knowledge of high school mathematics teacher and then the difference between academic and school mathematics. The last section of the previous chapter provided an argument for the use of constructivism as the theoretical framework for this study. As discussed in that section constructivism provides an ideal framework for describing what pre-service mathematics teacher knows about logarithm. This theory is relevant to this study because it has provided mathematics educators with useful ways to understand learning and learners. With that in mind, the research questions and research instruments were designed.

In this chapter, the researcher describes the methodology for this research, which was used specifically to examine the mental constructions which pre-service mathematics teachers might make in the learning of logarithmic concepts. This chapter re-caps on the critical research questions and describes the limitations governing the research. It then goes on to discuss the methodology of the research by describing the paradigm within which the study was located, the research design adopted, the methods employed to conduct the study, and the separate stages (1 and 2) of the study. Additionally, the chapter concludes with a summary of the methodology.

3.2 Critical research questions in relation to methodology

The study explored pre-service mathematics teachers' knowledge of logarithm. It was designed to explore what it means for a pre-service mathematics teacher to have a good knowledge of logarithm. It equally identifies the difficulties pre-service mathematics teachers encounter with logarithm. There was a further aim to examine how pre-service mathematics teachers conceptualize logarithm. This study used the constructivist approach to exploring pre-service mathematics teachers' knowledge of logarithm.

3.3 Research Paradigm

This study was concerned with the knowledge of pre-service mathematics teachers' in logarithm. The main aim was to explore what it means for a pre-service mathematics teacher to have a good knowledge of logarithm. Considering the aim of the study, the interpretive paradigm is the most suited paradigm underpinning the methodological framework of this study. Interpretive paradigm aims to understand and interpret the world (Christiansen, Bertram, & Land, 2010). Scotland (2012) asserts that interpretivists believe that there is no one particular right or exact approach to knowledge. This suggests that there is no specific answer, but answers are subject to people's experiences. According to (Cohen, Manion, & Morrison, 2011), the interpretive paradigm is characterized by a concern for the individual and is used to understand the subjective world of human experience. Interpretive researchers begin with individuals and set out to understand their interpretation of the world around them (Cohen et al., 2011). According to Neuman (as cited in Ndlovu, 2016), interpretive researchers study meaningful social action, not just the external or observable behavior. The same is echoed by Cohen et al. (2011), who note that interpretive researchers are set to examine situations through the eyes of the participants rather than through those of the researcher.

In this study, the data were generated from pre-service teacher's understanding of some mathematical problems, and their solutions to the given problems. It is through analyzing the generated data that the researcher can understand how the pre-service mathematics teachers conceptualize logarithm. Neuman (2013) asserts that interpretive paradigms like a functionalist paradigm belong to the sociology of regulation. Cohen et al. (2011) argue that in interpretive research, the theory should be grounded in data generated, and interpretive researchers work directly with experience and understanding to build their theory. According to Schultz (as cited in Ndlovu, 2016), interpretive researchers believe reality is constructed inter-subjectively, through meanings, and that understanding is developed socially and experientially. This means that interpretive researchers aim to understand the learners' experiences from the individuals' point of view. Thorne (2016) concurs with Schultz (1962) and further claimed that 'interpretive' assumes that researchers' values are inherent in all stages of the interview and that the truth is negotiated right through the interview process.

Cohen et al. (2011) argued that interpretive inquiry interprets and discovers the perspectives of the participants in the study, and answers to the inquiry are practically dependent on the context. This study investigates the knowledge the participants have concerning logarithm. To obtain the answers to the inquiry, the researcher gave a written task in a classroom setting which he considers to be the natural setting of learning for the participants. The knowledge constructed is discovered and interpreted in natural settings. According to Lewis (2015), in an interpretive approach, the researcher presents experiences as they become constructed, and collects multiple stories when planning to group stories around a common theme. Therefore, the interpretive approach can be described as the "systematic analysis of socially meaningful action through the direct detailed observations of people in natural settings to arrive at understanding and interpretation of how people create and maintain their social world" (Neuman as cite in (as cited in Ndlovu, 2016, p. 98). Lewis (2015) argues that in the interpretive approach, the interpretation should be based on material that comes from the world of lived experiences, and which incorporates prior understanding into the interpretation. This means that the interpretive approach is more concerned with giving detailed descriptions of the phenomena. It focuses specifically on concepts that require an in-depth understanding of how the participants construct their meaning.

3.3.1 How the paradigms fit with this study

According to Neuman (as cited in Ndlovu, 2016), the goal of the interpretive paradigm is to develop an understanding of social life and to discover how people construct meaning in a natural setting. In this study, the goal was to see how pre-service mathematics teachers conceptualize logarithm to explore what it means for a preservice mathematics teacher to have a good knowledge of logarithm. The interpretive inquiry is concerned with the way in which individuals collaborate, experience the world, and the settings in which these collaborations occur. This paradigm is applicable in this study since it examined the individual conceptual understanding of the concept of the logarithm. It was of importance in the study that participants were given a written task to solve individually first.

Thorne (2016) argues that interpretive approaches rely on naturalistic methods, such as interviewing, observation and analyzing existing texts. He also asserted that these approaches ensure an adequate dialogue between the researcher and those with whom he/she interacts, so as to construct a meaningful reality and to derive meanings from the research process. An interpretive researcher studies a text, such as a conversation, to draw out elusive verbal communications in order to discover embedded meanings (Pillay, as cited in Ndlovu, 2016). In this study, it was of importance to analyze pre-service mathematics teachers' responses to written tasks in order to reveal their mathematical thinking in the context of the logarithm. It was also of significance to use interviews to understand how the participants conceptualized the concept of the logarithm, and to draw out embedded meanings. This was done with the hope that engaging in a dialogue would shed more light in the understanding of the concept of the logarithm. Through an interpretive paradigm, the researcher is able to observe different approaches to solving problems and use multiple ways to understand how participants conceptualize logarithm. The format of research questions in this study indicates interpretive research designs.

3.4 Research design and methodology

The research methodology describes the selected design and sampling method used in this study. A qualitative approach was adopted to answer the research questions. The qualitative approach was adopted because it provides multiple ways of understanding the inherent complexity and variability of human behavior and experience (Neuman, 2013). Therefore, qualitative research provides an opportunity to understand peoples' perception in their natural settings. According to Taylor, Bogdan, and DeVault (2015), qualitative researchers have a desire to step beyond the known and enter into the world of participants, to see the world from their perspectives. In doing so, they make discoveries that contribute to the development of empirical knowledge. This study aimed to explore pre-service mathematics teachers' knowledge of logarithm and to understand how they conceptualize logarithm.

By its nature, a qualitative research methodology allows one to use different research strategies to collect data. According to Merrian (as cited in Ndlovu, 2016), four qualities of qualitative research was described by her: (1) qualitative research elicits participation accounts of meanings, experience or perception about concepts; (2) it produces descriptive data; (3) qualitative approaches allow for more diversity in responses as well as capacity to adapt to new development or issues; and (4) in qualitative methods, forms of data collected can include interviews, group discussions, observations, various texts, pictures and other materials. This study makes use of a variety of methods to collect data as it used text from pre-service mathematics teachers' responses and interviews. Asiala et al. (1997) mentioned two aspects of the qualitative approach that needs to be addressed, namely: (1) the theoretical perspective taken by researchers using that approach; and (2) the actual methods by which data are collected and analyzed. In this study, the theoretical framework used informs the qualitative methodological framework that was taken by the researcher. Also, the methods used, align with the theoretical framework, which then allows the researcher to use the theoretical framework as the analytic tool.

The idea of discovering patterns of behavior or thoughts in a set of texts can be linked to qualitative research. Creswell, Hanson, Clark Plano, and Morales (2007) agree with this notion and has stated that the researcher establishes patterns and searches out the correspondence between two or more categories. Since the study was based on a qualitative approach, both inductive and deductive analyses were used. This was done by coding the written responses of all the participants. Thereafter, the categories were determined, and patterns and trends that emerged were further analyzed.

The theoretical perspectives of this study focused on what it means to learn and understand something in mathematics. Logarithm as an aspect of algebra is considered to be less abstract, however, to develop a conceptual understanding of these concepts goes beyond a mere application of logarithmic laws. Pre-service mathematics teachers' need to be able to construct and reconstruct the knowledge learnt in order to move beyond the urge to do mathematics to construct processes leading to thinking about mathematics. This would then assist them in dealing with more abstract concepts in linear algebra.

This study is qualitative in nature, and therefore, it explores pre-service mathematics teachers' knowledge of logarithm. According to Cohen et al. (2011), a case study provides a unique example of real people in real situations, enabling the reader to understand the events more clearly than simply presenting them with abstract theories or principles. The pre-service mathematics teachers must have encountered the concept of logarithm from their high school or in MATH 110 which is done in their university. Their experiences of learning the concept and the way in which they make meaning were unique. The data collected generated a new understanding of the mathematics community about how pre-service mathematics teachers conceptualized the concept of logarithm and some difficulties that they encounter. Nilson (2016) asserted that a case study should take the reader into the case situation and experience. It is imperative that the pre-service mathematics teachers' experiences of the concept are understood. According to Tellis as cited in (Ndlovu, 2016), a case study is an ideal methodology when the holistic, in-depth investigation is needed and is designed to bring out details from the viewpoint of the participants by using multiple sources of data. The data collection of this study was done in three stages. First, data was collected from the questionnaire which was distributed to all the undergraduate mathematics teachers to identify potential participants and to understand their general view about logarithm. Secondly, data were collected from pre-service mathematics teachers' responses to the logarithm research task given to them. Once the responses were analyzed, the semi-structured interviews were used to verify and clarify pre-service mathematics teachers' knowledge of logarithmic concepts.

Case studies tend to be selective, focusing on one or two issues that are fundamental to understanding the system being examined (Tellis as cited in Ndlovu, 2016). This is supported by Guthrie (2010), as he asserted that case studies are not a representation of the entire population, therefore the results are not generalized, but if appropriately selected, findings could be used in other settings. In this study, the researcher did not intend to generalize the findings, and as a result, specific choices were made as to who the participants of the study were, regardless of whether they were representative of the whole population or not.

3.5 Gaining access

The purpose of the study was to explore the pre-service mathematics teachers' knowledge of logarithm. For this study, the researcher needed to conduct research at the university. Since the researcher is a student and a specialist tutor at the university and tutor most undergraduate students, he decided to conduct the study with a group of pre-service mathematics teachers at the university. The researcher was required to obtain permission from the institution to conduct research as well as to obtain the consent from the pre-service mathematics teachers sought out to participate. To gain access to conduct the study, permission needs to be obtained from the Research Office, and the Registrar. A copy of the letter from the Registrar is attached in Appendix G and ethical clearance certificate no HSS/0347/018M from the Research Office may be found in Appendix H too.

3.5.1. Informed consent

When conducting research, ethical consideration is important, therefore the researcher had to take into consideration the following factors: informed consent; the right to withdraw; confidentiality; methodological rigor; and fairness. Before the researcher proceeded with this study, he provided all classes visited with an introductory letter. This letter discussed and defined informed consent, the right to withdraw, and confidentiality. The letter provided each participant with the reasons and purpose of the study. Each participant was required to provide their signed consent. The researcher also explained the procedures that would be followed during the research process, provided timeframes, and relevant contact details of personnel at the University. A copy of this letter may be found in Appendix F.

3.6 The context of the study

The study was conducted in a South African university with a combination of all the undergraduate mathematics education students, who were training to become mathematics teachers. In this university, the logarithm is taught in semester two module to students who plan to be mathematics teachers. The university has a diverse student body. The group that participated in the study mainly consisted of African. To major in mathematics, a student must have achieved 60% or above in mathematics for their matric. Any student who achieved level 4 (50% to 59%) in their matric results, but wishes to be a mathematics teacher, needs first to do a foundational module in mathematics and achieve 60% or above.

3.6.1 Tutor/Researcher

In my interactions with pre-service mathematics teachers, the researcher played two roles, as a tutor and a researcher, which brought both opportunities and pitfalls. Yin (2017) suggests that such an approach "offers the researcher a role in creating the phenomenon to be investigated, coupled with the capacity to examine it from the inside, to learn that which is less visible" (p. 178). In assuming both roles in this study, the researcher gained considerable inside knowledge that helped in designing problems that yielded the necessary results. Being a researcher and tutor at the same time also helped me to get to know pre-service mathematics teachers much better, in a way observing them from the back of the class would not have afforded.

Speer et al. (2015) has pointed out that many of the pitfalls of being a teacher/researcher arise when the purpose of the research is to study teaching, and that the main problem is gaining sufficient objectivity to ensure the reliability of observations and the validity of conclusions about one's own thoughts and actions. Since this study focused on learning, such pitfalls were not present. The most challenging issue in this study was that of power dynamics. Yin (2017) highlights such challenges, where pre-service mathematics teachers might have some reservations as to what they should or should not say. Although it was not that evident in this study, the researcher decided to clearly explain the purpose of the study before it commenced, and during the study itself. The goal of the study was communicated to be exclusively aimed to explore their understanding of logarithm and the difficulties they are encountering.

3.7 Sample and sampling procedure

The study was conducted with pre-service mathematics education teachers in one particular university in the province of KwaZulu-Natal. The sample and the sampling procedure are described below.

3.7.1 Sample and its characteristics

The quality of a piece of research stands or falls not only by the appropriateness of methodology and instrumentation but also by the suitability of the sampling strategy that has been adopted (Creswell et al., 2007). Sampling is an activity or process used in selecting a segment of the population for the research study (Bell, Bryman, & Harley, 2018). Christiansen et al. (2010) assert that sampling is a process of deciding which group of people, location, actions or behavior to observe or study. Researchers accumulate a sample that is suitable for their specific needs. Palinkas et al. (2015) suggest that, in qualitative research, the size of the sample should be sufficient to generate thick descriptions and rich data. It should not be too large to overload the data and not so small to prevent achievement and data redundancy (Cohen et al., 2011).

3.7.2 Purposive Sampling

The study used purposive sampling. In qualitative research, the choice of participants merely depends on relevance to the research topic, rather than on representativeness (Etikan, Musa, & Alkassim, 2016). Etikan, Musa and Alkassim (2016) further assert that qualitative researchers select cases gradually, with the specific content of a case. In this study, the researcher does not intend to generalize the findings to other universities, therefore purposive sampling is considered suitable for this study. The choice to use this group of students is due to ease of access to the participants, due to the fact that the researcher is a student at the institution. Therefore, the study was conducted during school time. According to Orcher (2016), purposive sampling means that the researcher makes specific choices about which people to include. Cohen et al. (2011) argue that purposive sampling can be used to access those who have in-depth knowledge about a particular issue. The researcher hoped that undergraduate mathematics education students would provide rich information about their knowledge of logarithm. Gaining a deep understanding of how they conceptualized logarithm will help in understanding the misconceptions and other difficulties pre-service mathematics teachers have in the learning of logarithm. Purposive sampling is a sampling strategy for a case study (Maharaj, 2018). As mentioned earlier, this study is qualitative by nature.

3.8 Research methods

The study was composed of three stages. Stage 1 was the distribution of the questionnaire to undergraduate mathematics education students that I had access to. The main focus here was to select the pre-service mathematics teachers who want to participate in the research and to ascertain how many pre-service mathematics teachers who like logarithm and can teach it after graduation. However, the analysis was based on the responses of all the 231 pre-service mathematics teachers who returned their questionnaire. In this stage of the study, the analysis was only based on the pre-service mathematics teachers' responses. The results of this stage and the discussion is presented in Chapter Four.

In Stage 2, the research task was given to 19 pre-service mathematics teachers who were present for the written task. The research task consists of 5 questions. The skills and knowledge covered in the second stage consisted of a simplification of logarithm by applying the rules, solving the logarithmic equation in linear form, solving the logarithmic equation in quadratic form, solving logarithm equation that involves exponent, proving logarithmic equations and sketching logarithmic functions. Pre-service mathematics teachers' understanding of these concepts was explored. In order to identify pre-service mathematics teachers' written work as it was collected, each pre-service mathematics teacher was allocated a pseudo-name. The pseudo-names are also used in the audiotape transcriptions of the interview process. When all the task was completed and marked by the researcher, pre-service mathematics teachers' responses were categorized. The categories are shown in detail in the analysis of Chapter Four. For each category, a sample of pre-service mathematics teachers was selected and interviewed. This was done in order to clarify some of their responses and to explore their knowledge of logarithm. These were semi-structured allowing the researcher to probe further for more clarity where necessary.

3.9 Data collection procedures

Qualitative research methods involve the systematic collection, organisation, and interpretation of textual material derived from talk or observation (Ndlovu, 2016). They are used in the exploration of meanings of social phenomena as experienced by individuals themselves, in their natural context (Guetterman, 2015). In qualitative research, the researcher is the primary instrument for data collection and analysis (Merriam, 2002). In this study, the researcher used ideas generated from the literature to design the research task as one of the data collection methods. By its nature, qualitative studies use a variety of methods, such as interviews, observations, documents, etc., to gather data. In the data collection process, the decision as to which strategy to use is determined by the question of the study (Merriam, 2002). As a qualitative study, this study uses tasks and interviews as data collection methods, and each method used to respond to a particular question of the study as stated in the table below. This was done with the aim of enhancing the validity of the findings.

Table 3.1. Data collection procedure

Research questions	Research instruments	Participants under study
1. What do pre-service mathematics teachers know, about logarithm?	• Designed assessment task • Semi-structured interview schedule	• Pre-service Mathematics education teachers
2. What are the difficulties pre-service mathematics teachers encountered with logarithm?	• Designed assessment task • Semi-structured interview schedule	• Pre-service Mathematics education teachers
3. How do pre-service mathematics teachers conceptualize logarithm?	• Designed assessment task • Semi-structured interview schedule	• Pre-service Mathematics education teachers

3.10 Stages of data collection

In this section, the stages of data collection were discussed in detail. Stage one was the distribution and collection of the questionnaires. Stage two was the administration of the research task and the last stage was an interview with the selected participants.

3.10.1 Stage 1: Questionnaire

The questionnaire is the most widely used technique for obtaining information from participants, for many reasons and objectives (Kamgar & Navvabpour, 2017). A questionnaire is relatively economical, has the same question for all participants and can ensure anonymity. For the purpose of this research, the questionnaire was used to choose my participants from the entire group of undergraduate mathematics education students and to understand the attitude of the pre-service mathematics teacher towards the concept of the logarithm. Questionnaires can use statements or questions, but in all cases, the participants are responding to something written for a specific purpose (Kamgar & Navvabpour, 2017).

3.10.2 Stage 2: Structured research task

Structured research sheets or task model the way in which meaningful mathematics teaching could be planned with the aim of simultaneously addressing the cognitive and affective domains when students solve problems (Nievelstein, Van Gog, Van Dijck, & Boshuizen, 2013). Logarithm problems require pre-service mathematics teachers to apply the algorithms and manipulation skills they have learnt in high school algebra sections or at the first-year undergraduate level, but in a more critical way, showing their conceptual understanding of the learnt concept. The use of a structured research task can generate the required data that a researcher could use to understand how well the pre-service mathematics teachers know logarithm. Structured research task is the best sources of data collection since these give directions to learners on answering questions.

a. Key ideas targeted by the problems set

The problem sets were designed to provide experience with examples that could be used to motivate the learning of key ideas concerning logarithm. The problems focused mainly on different aspects of the logarithm. For example, the pre-service mathematics teachers were asked to simplify logarithm expression, solve a logarithmic equation that involves the knowledge of the laws of the logarithm, application of quadratic equation knowledge and the use of k-method. It equally includes the sketch of the graph of logarithmic functions (see Appendix). The researcher task comprised of five questions and each question covering certain aspects of these concepts.

3.10.3 Stage 3: Semi-structured interviews

According to Kendall (2014), qualitative interviews may be used either as the primary strategy for data collection, or in conjunction with observation, document analysis, or other techniques. Qualitative interviewing utilizes open-ended questions that allow for individual variations. Patton (1990) classified qualitative interviewing in three types namely: 1) informal, conversational interviews; 2) semi-structured interviews; and 3) standardized, open-ended interviews. Similarly, Cohen et al. (2011) group and discuss four main kinds of interviews, namely: the structured interview; the unstructured interview; the non-directive interview; and the focused interview. According to Lewis (2015), unstructured interviews provide greater breadth, with the main goal of understanding the phenomena. Using unstructured interviews allows the interviewer to probe where needed (Cohen et al., 2011).

This study is a qualitative study which uses an interpretive paradigm, and it sees humans as not just manipulative objects or data sources, but rather regards knowledge as generated between two humans through conversations (Cohen et al., 2011). Therefore, interviews are suitable as data collection method, because this study aimed to understand how pre-service mathematics teacher conceptualize logarithm and the difficulties they encounter. According to Sorsa, Kiikkala, and Åstedt-Kurki (2015), interviews are a good data collection tool for finding out what a person knows. In this study, it was important to discover how pre-service mathematics teachers interpret tasks, which thus led to the way in which they solve logarithm problems. According to Cohen et al. (2011), interviews enable participants to discuss their interpretation of the world, and to express how they regard the situations from their own point of view. This also is stated by Haahr, Norlyk, and Hall (2014), who notes that interviews allow us to enter into another person's perspective. Interviews are an important part of the research, as they provide the opportunity for the researcher to probe and to gather data, which could not have been obtained in other ways (Galvin, 2015). The use of interviews in this study provided the opportunity to gain a deeper understanding of why pre-service mathematics teachers have various difficulties in solving logarithm problem, which in some cases, were not explicit from their responses to the tasks.

After analyzing the research task, the in-depth task-based semi-structured interviews were conducted with eight participants so as to gain more clarity on pre-service mathematics teachers' thoughts about their solutions. These semi-structured interviews offered a versatile way of collecting data (King, Horrocks, & Brooks, 2018), as they raised key questions and allowed the researcher to enjoy some natural conversation with the pre-service mathematics teachers. The rationale for the interviews followed the overall aims of the study to understand how pre-service mathematics conceptualize logarithm and the difficulties they encounter. Therefore, the interviews were used as a means to gather feedback. To cater to participants' withdrawal, ten participants were selected for an interview, but only eight pre-service mathematics teachers availed themselves for the interview. The two pre-service mathematics teachers who withdrew did originally give consent to be interviewed and be audio recorded, but later indicated their unwillingness to take part in the interviewing process, citing several reasons.

The interviews were conducted for three weeks between the months of September and October at the University. This was due to the fact that conducting these interviews depended on the availability of the participants. To elicit pre-service mathematics teachers' understanding, open-ended questions were used. This allowed the researcher to probe further for an in-depth understanding if the pre-service mathematics teachers have a good knowledge of logarithm and the difficulties they encounter while solving the logarithm problems. It allowed the participants to express themselves freely, and to add or change whatever they wanted to. It also gave them another chance to relook at their responses and check if their understanding then is still the same, or if it has been improved. Before the commencement of the interview, participants were made aware that the interview would be audio-recorded and asked if they have any objection to this. Although they had given consent, it was important to remind them so that they would be made aware of how the process would unfold and develop a sense of trust. According to Lewis (2015), establishing rapport with the participants is of importance during the interview process.

The decision as to whether one relies on written notes or recording device appears to be largely a matter of personal preference. Since interviews provide a rich and detailed explanation of how pre-service mathematics teachers conceptualized concepts, it is vital that each and every detail of the interview is captured. In this study, it was of importance to gain clarity on pre-service mathematics teachers' explanations – therefore it was important to use an audio recorder to capture everything the participants were saying. This provided an in-depth understanding of how they used their experiences, and previous knowledge to solve logarithm problems. After conducting the interviews, they were transcribed, and data were analyzed inductively. In this study, participants were allowed to write down their explanation or to make sketches if they wished to do so or to redo their solutions. This was done to supplement the verbal data recorded.

Although interviews are considered to be an important part of the research, there are limitations, for example, interviews are lengthy and require more time. Thorne (2016) has highlighted the issue of bias on the side of the interviewer, by influencing the respondent responses. Using the semi-structured interview can produce data that were less systematic and comprehensive (Cohen et al., 2011). Since this study requires an in-depth understanding, it was important to spend some time with the participants. In order to ensure that the time spent was used to generate rich data, some of the questions were prepared ahead of time. Having those questions to begin with, allowed for more probing during the interview.

Although the issue of bias could not be totally eliminated, in this study the purpose was to know if the pre-service mathematics teachers have a good knowledge of logarithm and the difficulties they encounter with logarithm, and so it was important that the researcher attentively listened to pre-service mathematics teachers' explanations, without interfering. The questions were short and straight to the point, so as to avoid misunderstanding. In formulating the questions, the researcher took great care in sequencing the questions moving from general or broad to specific or narrow. Pre-service mathematics teachers were allowed to explore as they liked, but I always referred back to the questions to check if the key areas had been explored and responses to it had been given. This helped with maintaining control of the interviews, without interfering with their responses. The flexibility of the interviews also allowed pre-service mathematics teachers to provide more input, which probably was not said in their responses to the research task.

3.11 Trustworthiness and credibility of the study

In qualitative research the concepts credibility, dependability and transferability have been used to describe various aspects of trustworthiness. According to Bertram and Christiansen (2014), quality research within the interpretive research paradigm can be ensured by addressing the issues of credibility, construct validity, trustworthiness, transferability and confirmability. Credibility refers to confidence in how well the data and processes of analysis address the intended focus of the research (Polit Denise & Hungler Bernadette, 1999). According to Cohen et al. (2011) "reliability in qualitative research can be regarded as between what researchers record as data and what actually occurs in the natural setting that is being researched" (p. 149). Creswell et al. (2007) argued that reliability can be addressed in several ways in qualitative research, such as obtaining detailed field notes, as well as employing good quality recording materials for ease of recording and transcribing. These are the means to uncover participants' perspectives of the phenomena under study. The first question concerning credibility arises when deciding about the focus of the study, selection of context, participants and approach to gathering data. Choosing participants with various experiences increase the possibility of shedding light on the research question from a variety of aspects (Lewis, 2015). Credibility seeks to ensure that the research measures or tests what it is intended. The credibility of research findings also deals with how well no relevant data have been inadvertently or systematically excluded or irrelevant data included.

To ensure the credibility of this research findings, I used representative quotations from the transcribed text and seek agreement among co-researchers, experts and participants. I equally developed an early familiarity with my participants before the first data collection dialogues took place. This was achieved via preliminary visitation to the participants themselves during their tutorial classes. I also told the participants that the findings from the research will be discussed with them and this was to help ensure honesty in participants when contributing data. In particular, each participant was given an opportunity to refuse to participate in the research. So, the data collection sessions involve only those who are genuinely willing to take part and prepared to offer data freely. My participants were encouraged to be frank from the outset of each session, with the researcher aiming to establish a rapport in the opening moments. Finally, I had frequent debriefing sessions with my supervisor. Through discussion, my attention was drawn to flaws in the proposed course of action. The meetings also provided a sounding board to test my developing interpretations and helped me to recognize my own biases and preferences.

3.12 Ethical issues

According to Bertram and Christiansen (2014), studies that involved human beings, ethical considerations were seen as crucial. To ensure that all ethical issues were appropriately addressed, a letter outlining the nature, process and purpose of the study was given to the Dean school of education, seeking permission to conduct the study (see Appendix E). Letters of informed consent were given to all the participants to read and sign (see Appendix F). In the letter, it was clearly stated that participation was voluntary and that participants could withdraw anytime they wanted to, only needing to inform the researcher if they wished to do so. Participants can become aware of their rights as participants when they read and sign the statement (Palinkas et al., 2015). Before the commencement of the study, the researcher clearly explained and emphasized such issues to the participants. This was done to ensure that participants understand that they are under no obligations to take part in this study. At all times during the process of data collection, pre-service mathematics teachers were ensured that the data collected would only be used for the purpose of the study. All the participants in the study were promised confidentiality and anonymity. The nature, process and purpose of the study were outlined to all the participants. To protect the identity of the participants, pseudo-names were used, and participants were ensured that all their details would be kept away from the public. The pre-service mathematics teachers were further assured that the information would be kept safely in the university, and would not be shared with anyone, except for the purposes of the study. They were also invited to ask questions to seek clarity on any issue or any uncertainty they were experiencing during the course of the study. Before the researcher could commence with the study, it was necessary to seek ethical clearance from the university research office, which was granted, under ethical clearance number HSS/0347/018M. Also, the permission for conducting this study in the institution was granted by the Registrar. This was granted after a summary of the proposal was presented to the institution's research committee.

3.13 A methodological limitation of the study

As a case study, the sample use is quite small, using a group of 19 pre-service mathematics teachers out of 231, and only in one university, therefore the findings cannot be generalized to other contexts. Even so, it is hoped that the findings would be informative enough to the mathematics community regarding what pre-service mathematics teachers know about logarithm in the South African context. The first set of data was collected during tutorials where the researcher visited tutorial classes after taking permission from the concerned lecturer. Thereafter, from the response of the pre-service mathematics teachers from the questionnaire, the researcher invited those pre-service mathematics teachers who indicated that they are willing to participate further in the research. To conduct interviews, a neutral venue was used, and pre-service mathematics teachers were allowed to speak in English since this is the commonly spoken languages at this university.

3.14 Conclusion

This chapter thus serves as an overview of how this study was conducted, with respect to methods and procedures. The chapter started with a list of the critical research questions and a discussion of the interpretive research paradigm used in this study. This discussion was followed by a discussion of the research design, methodology and methods adopted. As can be expected, the research methodology served as a guideline and point of reference for the study, with respect to data collection and procedures followed. The data collection process, together with the research instruments, were discussed at length. Once all the data were collected and interviews transcribed, the data analysis process commenced. Issues of credibility and reliability under terms such as trustworthiness of the study were equally discussed. In the next chapter, the researcher presents the findings of the stages, and thereafter, the analysis of the data is discussed in detail.

CHAPTER FOUR

4. DATA PRESENTATION AND ANALYSIS

4.1 Introduction

In the previous chapter, the research design, methodology and stages of data collection methods used in this study were discussed. A clear and detailed description of the data collection process was provided. In this chapter, the presentation and analysis of the data collected will be discussed. To gather the required data for the study, a qualitative method was used. Data was collected through a questionnaire, research task and interview. The research task was designed to give insight into pre-service mathematics teachers' knowledge of logarithm. This was guided by the belief that having a good knowledge of mathematical concept leads to improved instructional methods and curriculum development. The tasks chosen were those that the researcher identified as suitable for allowing pre-service mathematics teachers to show whether they have a good knowledge of logarithm. The research tasks were administered to 19 pre-service mathematics teachers, who were registered as undergraduate students offering a major in mathematics. During this research task, pre-service mathematics teachers were asked to work individually in answering the activity and were given 45 minutes to complete the tasks. Thereafter, data were analyzed in stages so as to assess their performance on their knowledge of logarithm.

4.2 Stage 1: Presentation and analysis of the questionnaire

This study attempts to explore pre-service mathematics teachers' knowledge of logarithm. In this stage, I present the data collection from the questionnaire. The questionnaire consists of 6 questions with their respective options to choose from. Statistical Package for the Social Sciences (SPSS[4]) was used to analyze the responses of 231 pre-service mathematics teachers who returned their questionnaires. There were two important reasons for using the questionnaire in this study. The first reason was to determine the potential participant for the

[4] SPSS (Statistical Package for Social Sciences) is an integrated computer programme that enables the user to read data from questionnaire survey and other sources, to manipulate them in various ways and to produce a wide range of statistical analyses (both descriptive and inferential statistics) and reports, together with documentation.

study and the second was to know the attitude of preservice mathematics teachers towards logarithm.

Table 4.1. The analysis of the questionnaire on participation

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Yes	158	68.4	68.4	68.4
	No	73	31.6	31.6	100.0
	Total	231	100.0	100.0	

The above table shows the response of the pre-service mathematics teachers when they were asked if they will participate in the research. From the table, one can observe that 158 out of 231 (68.4%) pre-service mathematics teachers who returned their questionnaire said that they want to participate in the research. Out of these 158 pre-service mathematics teachers, only 19 (12%) of them participated in the research task. This shows that 139 of 158 (88%) pre-service mathematics teachers who said that they wanted to participate do not have a positive attitude towards logarithm.

Table 4.2. The analysis of the questionnaire on the participants that like logarithm

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Yes	142	61.5	61.5	61.5
	No	25	10.8	10.8	72.3
	Maybe	64	27.7	27.7	100.0
	Total	231	100.0	100.0	

The above table shows the response of the pre-service mathematics teachers when they were asked if they like logarithm. From Table 4.2, 142 out of 231 (61.5%) of pre-service mathematics teachers who returned their questionnaire said that they like logarithm while 64 of 231(27.7%) pre-service mathematics teachers cannot say for sure if they like logarithm or not. This shows that the majority pre-service mathematics teachers have a positive attitude towards logarithm.

Table 4.3. The analysis of the questionnaire on the participants that can teach logarithm

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	Yes	127	55.0	55.0	55.0
	No	17	7.4	7.4	62.3
	Maybe	87	37.7	37.7	100.0
	Total	231	100.0	100.0	

The above table shows the response of the pre-service mathematics teachers when they were asked if they can teach logarithm after their graduation. From Table 4.3, 127 out of 231 (55%) of pre-service mathematics teachers who returned their questionnaire said that they can teach logarithm after their graduation as a qualified teacher while 87 of 231(37.7%) pre-service mathematics teachers are not certain whether they can teach logarithm or not. This shows that most pre-service mathematics teachers have a positive attitude and confidence in the teaching and learning of logarithm.

4.3 Stage 2: Presentation of items from the research task

This study aims at exploring pre-service mathematics teachers' knowledge of logarithm. The test instrument consisted of five questions which will help to identify if pre-service mathematics teachers' have a good knowledge of logarithm, the difficulties they have with logarithm and how they conceptualize logarithm. In the research task, there are seven questions that were analyzed. Question 1 contained a substitution of a variable, which required pre-service mathematics teachers to show their understanding of simplification of the logarithm. This question equally tests the pre-service mathematics teachers understanding of logarithm of a number as mostly an irrational number. Question 2 has three sub-questions, which covers the knowledge of solving a logarithmic equation involving linear equation, quadratic equation and exponential equations. The pre-service mathematics teachers were required to display their procedural fluency. Question 3 required the pre-service mathematics teachers to prove the logarithmic equation. Here the pre-service mathematics teachers need to apply the rules of the logarithm to prove the right-hand side is equal to the left-hand side of the equation. The fourth question was of a higher order, because at this stage, pre-service mathematics teachers were expected to apply their problem-solving skills, show an understanding of the relationship between concepts, and apply their knowledge and procedures in solving the problem. Question 5 focuses on the sketching of a logarithmic function. The pre-service mathematics teachers' responses for each of the questions and the extracts from their responses are presented below.

The marks were allocated as a means to group the responses, but the analysis was based on an individual's procedural and conceptual fluency. Each category in each particular question was discussed. The purpose of administering the research task was explained to the pre-service mathematics teachers prior to the commencement of the data collection process.

4.3.1 Question 1: Simplification of logarithmic expression

This question focused on exploring pre-service teachers' knowledge of simplifying logarithm expression. Question 1 is presented below which involves determining if the pre-service mathematics teachers was able to simplify a given logarithmic expression to obtain a particular logarithm which a variable is assigned to. This will them to simply the whole expression in terms of those variables.

Question 1

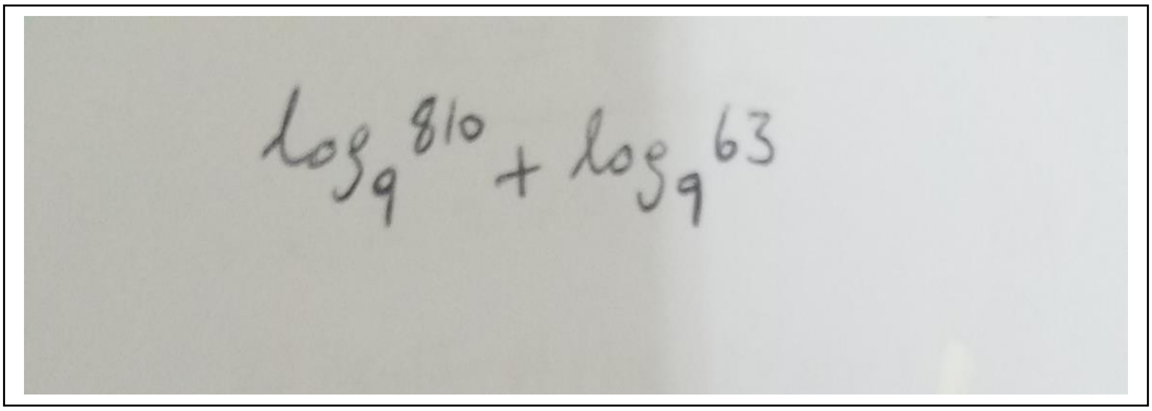
If $log_9 7 = A$ and $log_9 10 = B$, find the $log_9 810 + log_9 63$ in terms of A and B?

In Table 4.1 the allocation of scores for Question 1 is displayed.

Table 4.4. The allocation of response categories for Question 1

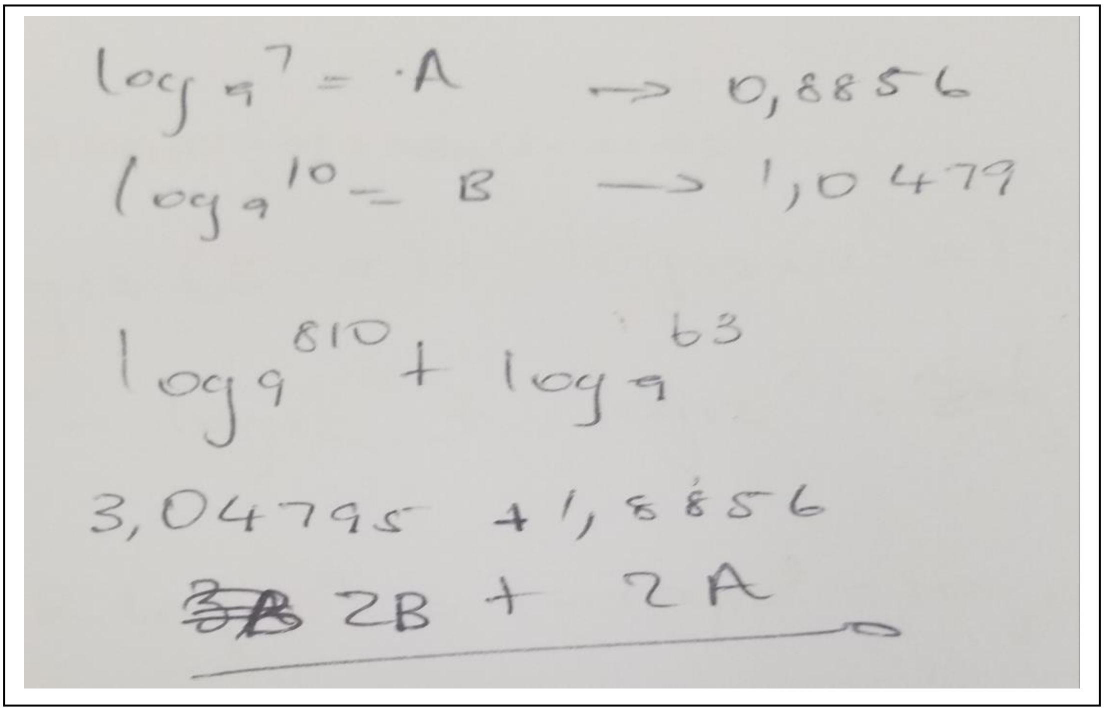
Categories	1	2	3	4	5
Indicator	Not answered or incorrect solution.	Expanding 810 and 63 as a product of 9 and a number.	Applying logarithmic rule for multiplication.	Correct substitution of the variables.	Simplifying and writing down the correct answer.
Number of responses	15	4	1	4	1

Three pre-service mathematics teachers out of the 15 of 19 (78.8%) them in category 1 did not write anything in this question. The rest of the pre-service mathematics teachers could not simplify 63 and 810 as a product of 7, 9 and 10. This shows that they don't have prior knowledge that will lead them to the simplification of the logarithmic expression. Some of their responses were shown in Extract 1 and 2. The responses of 3 out of 4 pre-service mathematics teachers in Category 2 indicated that they had no idea of the rules of the logarithm as indicated in Extract 3, 4 and 5. The common errors that pre-service mathematics teachers made ranged from writing 63 as a product of 7 and 9 to applying the suitable logarithm law to help them in simplifying the expression. In expanding 810 as a product of 9 and 10, one of the pre-service mathematics teachers in category 2 used 90 and 9, and then later expanded 90 as a product of 10 and 9 while the rest expanded 810 as a product of 10 and 81 but failed to apply the correct laws of the logarithm. In category 4, the same set of pre-service mathematics teachers in category 2 was able to make a correct substitute of the variables even though their application of laws of logarithm were incorrect.



Extract 1: Zee's written response for Question 1

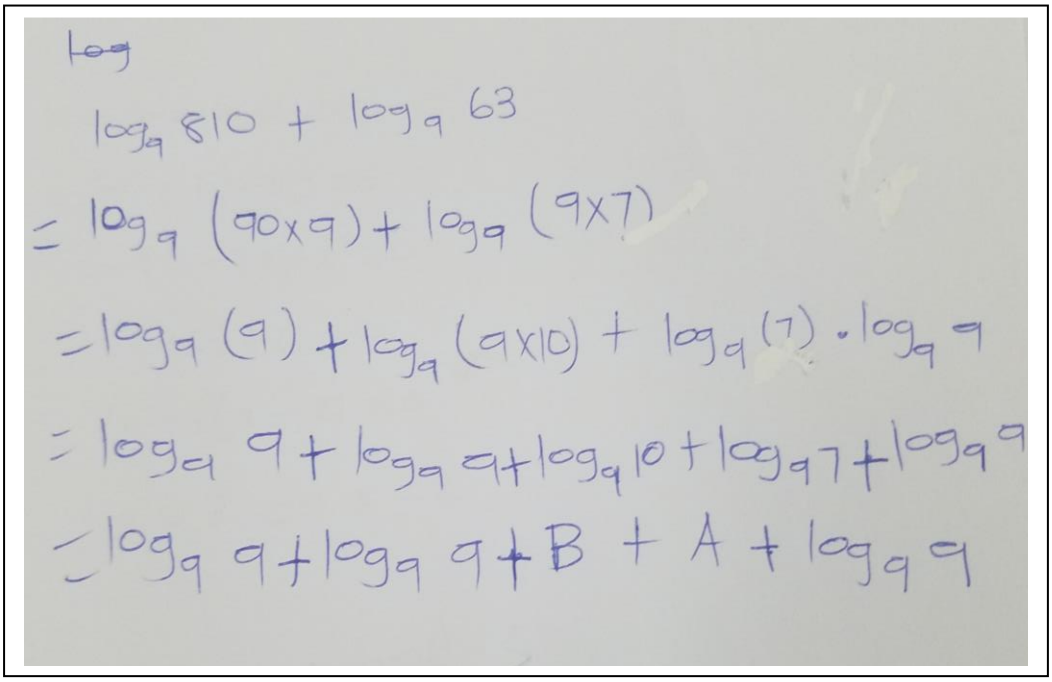
Zee is one of the fifteen pre-service mathematics teachers in Category 1 who do not have an idea of where to start simplifying the logarithmic expression. This is as a result of not having prior knowledge in dealing with problems like this. Looking at her solution, it seemed to have difficulties in understanding the concept of logarithm and having the necessary prior knowledge that is required to simplify the problem.



Extract 2: Aphi's written response for Question 1

Aphi is also one of the fifteen pre-service mathematics teachers in Category 1 providing an incorrect simplification to the problem. He could not correctly expand 810 as the product of 9 and 10, or 63 as a product of 9 and 7 rather, he made use of a calculator to obtain the value of $log_9 810 + \ log_9 63$. Looking at line 4 of his solution, one can tell the value of the expression from there, but the question was to simplify in terms of A and B. The fifth line of his solution shows that he understood that the final answer to the question should be in terms of A and B.

The pre-service mathematics teachers' responses in Category 2 revealed that they had made the necessary mental constructions, as they provided correct and complete set in expanding 810 as a product of 9 and 10 and equally 63 as a product of 9 and 7. They provided the correct step but failed to apply the laws or logarithm correctly (see Extract 3, and 4).

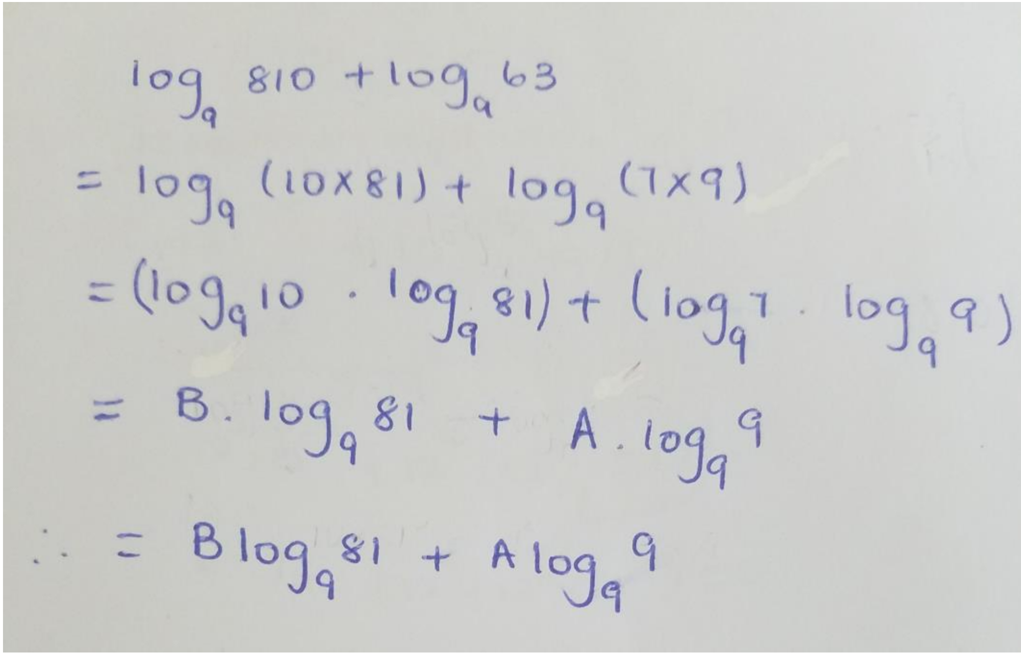


Extract 3: Amo's written response for Question 1

Amo is one of the four pre-service mathematics teachers in Category 2 applying an incorrect logarithmic law in the simplification of the logarithmic expression. Looking at his expansion of 810, he started by expressing it as 90×9 and then further expanded 90 as 9×10. This is to show that at this instance, Amo does not know that 81 is the square of 9. She could not correctly apply the law of the logarithm in the last part of step 3, indicating the underlying difficulties with the understanding of the laws of the logarithm. In her forth step, she equated $log_9 7 \times log_9 9$ to be equal with $log_9 7 + log_9 9$. This shows inconsistency in the application of the logarithm laws. And not knowing that $log_9 9 = 1$, shows that there is a barrier in understanding the laws of the logarithm. Note, that she correctly substituted the variables for both $log_9 7 = A$ and $log_9 10 = B$.

Patu and Efe used the same approach to simplify the expression. Looking at Patu's solution in Extract 4 below, you will realize that he expanded 810 as the product of 10 and 81 and then expands 63 as a product of 7 and 9. He could not apply the laws of logarithm correctly in step 3 which shows a lack of knowledge about the laws of the logarithm. In his case, $log_9(7 \times 9) = log_9 7 \times log_9 9$ and $log_9(10 \times 81) = log_9 10 \times log_9 81$. Note that both Patu and Efe were able to make a correct substitution of $log_9 7 = A$ and $log_9 10 = B$. According to Jojo (2011), students at intra-stage could solve some problems by simply applying memorized rules, and in some cases, could not remember correctly.

All the pre-service mathematics teachers in Category 2 were equally able to substitute the value of $log_9 7$ and $log_9 10$ as A and B respectively. Only one pre-service mathematics teacher out of the 19 of them was able to expand 810 as a product of 9 and 10, expand 63 as a product of 9 and 7, apply the correct law of logarithm and substitute correctly to arrive at correct simplification.



Extract 4: Patu's written response for Question 1

4.3.2 Question 2: Solving the logarithmic equation

Question 2 involves determining if the pre-service mathematics teachers will be able to solve a given logarithmic equation. This question is subdivided into three sections: solving a logarithmic equation that involves linear equation, solving logarithmic equations that involve quadratic equation and solving the logarithmic equation in exponential form.

Question 2.1

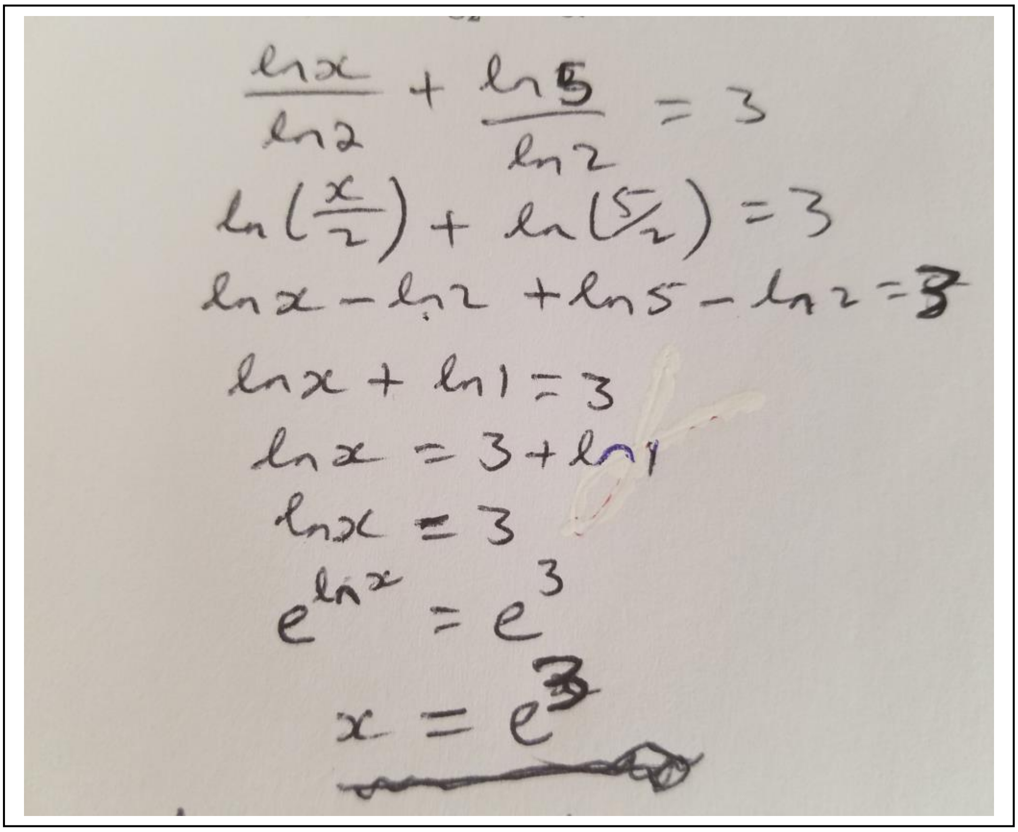
Solve for x: $log_2 x + log_2 5 = 3$.

In Table 4.2 the allocation of scores for Question 2.1 is displayed.

Table 4.5. The allocation of response categories for Question 2.1

Categories	1	2	3	4
Indicator	Not answered or incorrect solution.	Apply logarithm law for multiplication	Convert the log into the exponential form	Solving and writing down the correct answer.
Number of responses	15	3	4	3

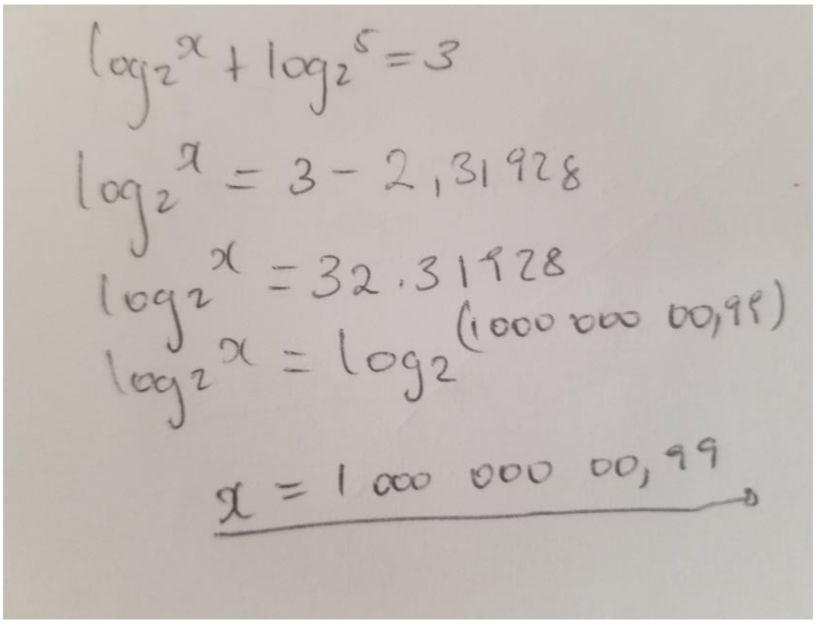
In determining whether the pre-service mathematics education teachers can solve a logarithmic equation that involves a simple linear equation, those of them in Category 1 could not apply the logarithm rule for multiplication thereby missed the total point. One of the pre-service mathematics teachers in Category 1, Zee, applied the change of base formula using natural logarithm instead (see Extract 5).



Extract 5: Zee’s written response for Question 2.1

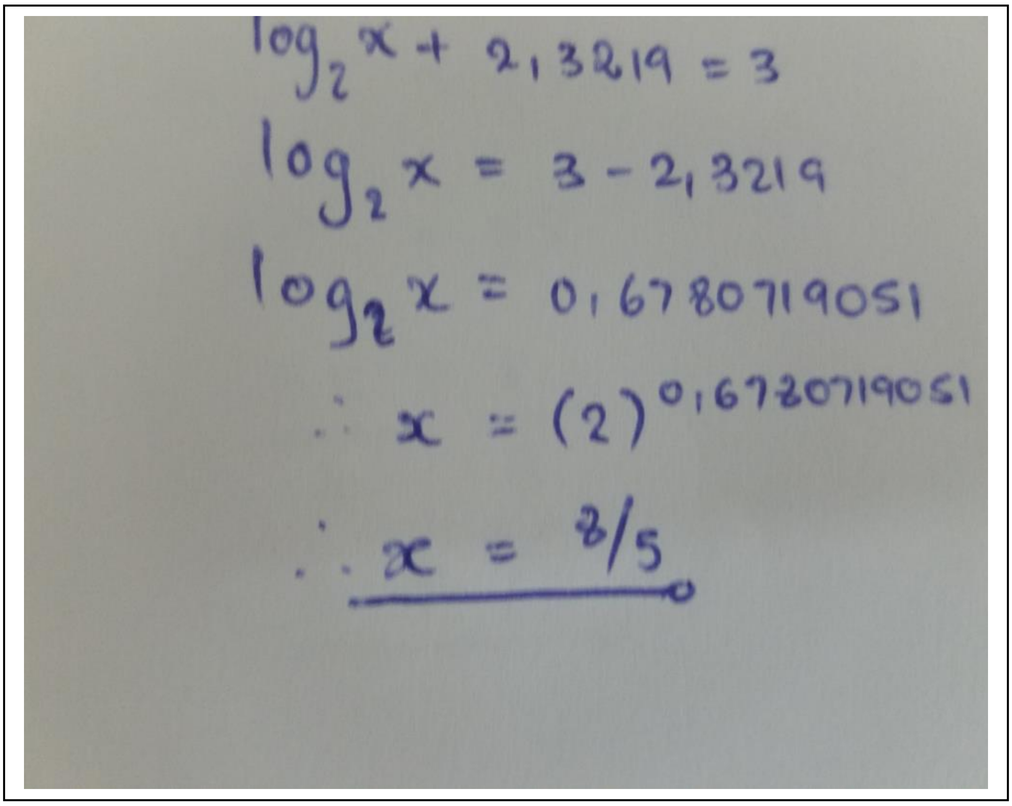
It seemed that Zee is confused with the difference between a logarithm and natural logarithm. She tried to apply the change of the base law but failed to apply the technique correctly. Her response to Item 2.1 indicated that she does not understand the difference between a logarithm and natural logarithm. She might have had an idea that there is a natural logarithm but did not understand how to apply it. Looking at her second step, you will realise that she did not know the logarithm law for the division. Her step three shows that she had an idea about the division rule but unable to understand how to apply it. What was evident here was that she failed to apply the logarithm rules correctly, which was a result of poor conceptualisation of the concept of the logarithm. As a result, she was not able to solve the question correctly due to her inconsistency with the procedures. According to Matz, (as cited in Siyepu, 2013), such errors persist due to surface level procedures, where an individual acquires knowledge by rote, without engaging with its meaning, which is what Zee appears to have done.

Aphi, on the other hand, understood that $\log_2 5$ is a constant and he used his calculator to convert $\log_2 5$ to a decimal number. Aphi understood how to solve a simple linear equation if you look at his step 2 in Extract 6, but he failed to change the logarithmic equation into the exponential equation. This shows that he lacks the prior knowledge for the conversion of the logarithmic equation to the exponential equation and vice versa.



Extract 6: Aphi's written response for Question 2.1

The pre-service mathematics teachers in Category 3 were able to apply the logarithm law of multiplication expect for Mpho who used a different method. She converted $\log_2 5$ to a decimal number and subtracted it from both sides of the equations (See Extract 7). This could be that she does not have a full understanding of the laws of the logarithm.



Extract 7: Mpho's written response for Question 2.1

The 3 pre-service mathematics teachers in Category 4 were able to solve the question correctly. These pre-service mathematics teachers demonstrated the understanding of the concepts and applied the procedures of solving a logarithmic equation that involves linear equations.

Question 2.2

Solve for x: $log_{12}(3 - x) + log_{12}(2 - x) = 1$.

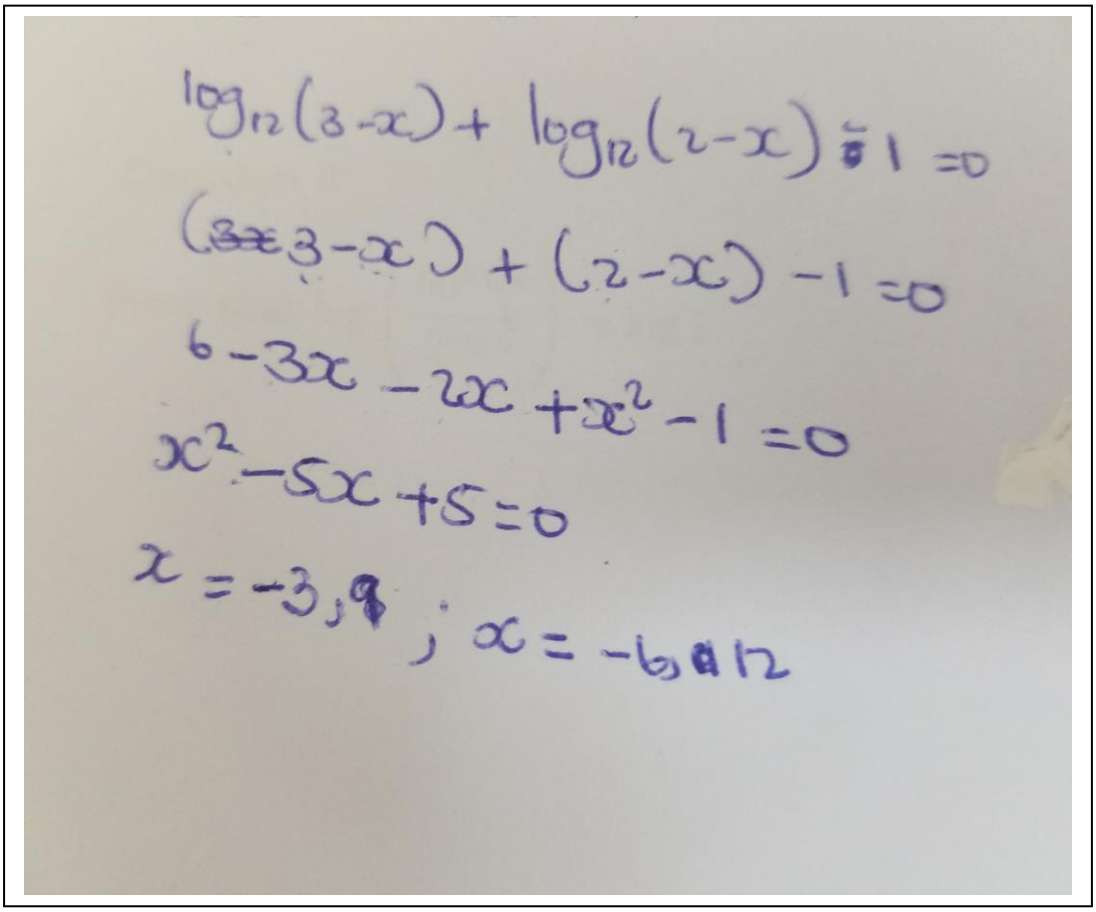
This question focuses on exploring pre-service mathematics teacher's understanding of logarithmic equations that involve quadratic equation and to whether they understand the restrictions for the value(s) of x which is not a solution.

In Table 4.3 the allocation of scores for Question 2.2 is displayed.

Table 4.6. The allocation of scores for Question 2.2

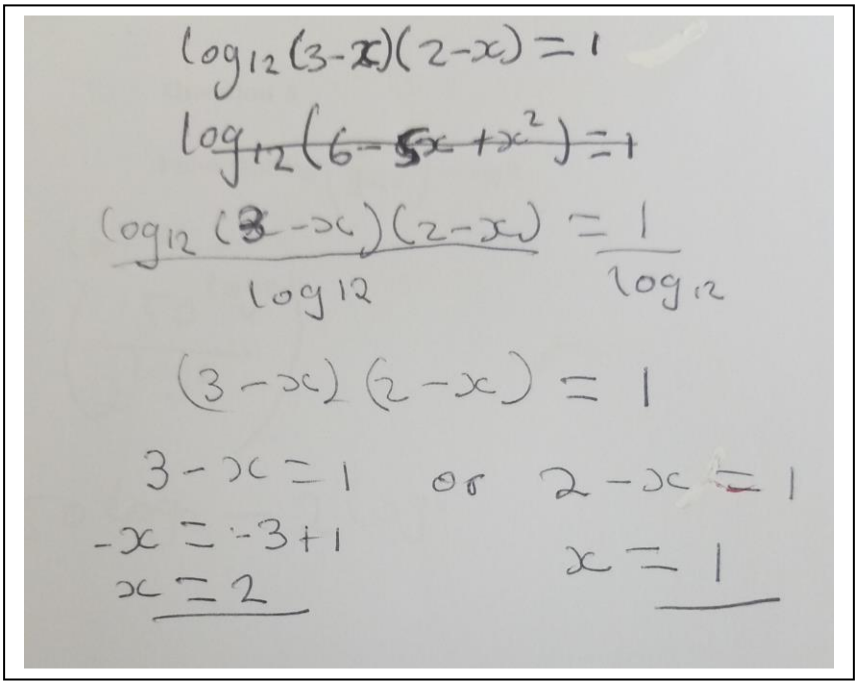
Categories	1	2	3	4	5
Indicator	Not answered or incorrect solution.	Apply logarithm law for multiplication	Convert the log into the exponential form	Solving the quadratic equation correctly.	Check for the restrictions for the values of x
Number of responses	16	3	2	2	0

Five out of 16 pre-service mathematics teachers in category 1 did not write anything in this question. The rest of them could not apply the multiplicative law of logarithm. This shows a lack of prior knowledge which will lead them to the solution of the logarithmic equation. Some of their responses were shown in Extract 8.



Extract 8: Efe’s written response for Question 2.2

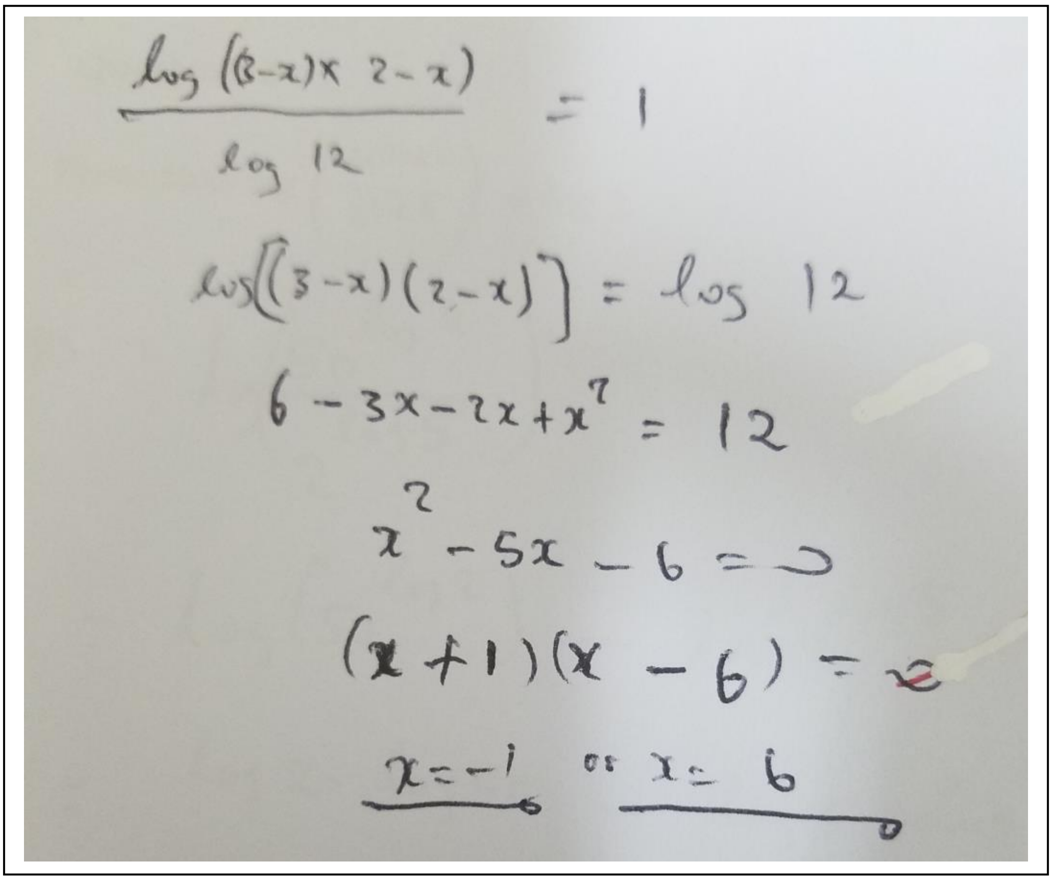
In solving the problem in Question 2.2, pre-service mathematics teachers in Category 2 displayed mathematically inaccuracy. Inaccuracies in mathematics mostly arose when pre-service mathematics teachers failed to carry out manipulations or algorithms, though they understood the concept. Pre-service mathematics teachers made procedural errors indicating a lack of algorithm skills. What transpired here was that the three pre-service mathematics teachers' in Category 3 knew the procedure to use but lacked the technique to carry out the procedures effectively (see Extract 9).



Extract 9: Ino's written response for Question 2.2

Looking at Ino's written response, certain mathematical errors were evident. In the first step, he was able to apply multiplicative law of logarithm but could not proceed to the conversion of the logarithm to the exponent. In his second step, he made a procedural error as he divided both sides of the equation by log_{12} which means that he sees log_{12} as a coefficient in the right-hand side of the equation that needed to be rid of. This is what (Dubinsky, 2002) indicated when discussing students' difficulties with linear algebra concepts.

Pre-service mathematics teachers in Category 4 also made some errors, but very few because they were not able to show that one of the solutions from the two solutions is not applicable. They displayed an understanding of operational rules of logarithm and have a sound knowledge of solving a quadratic equation, but they failed to check for the restrictions of the values of x. They provided a complete and correct indication that they had suitable prior knowledge necessary for developing a conceptual understanding of the concept. This can be seen in Zik's solution (Extract 10).



Extract 10: Zik's written response for Question 2.2

Zik's response revealed that she had made all the necessary mental constructions as she correctly applied the procedure for solving this logarithmic equation, indicating that she has constructed the procedural knowledge for solving quadratic equations. Her response in this item revealed that she could not understand that there are some restrictions as to the values of x which will satisfy the equation.

Question 2.3

Solve for x: $27^{log_3 x} = 8$.

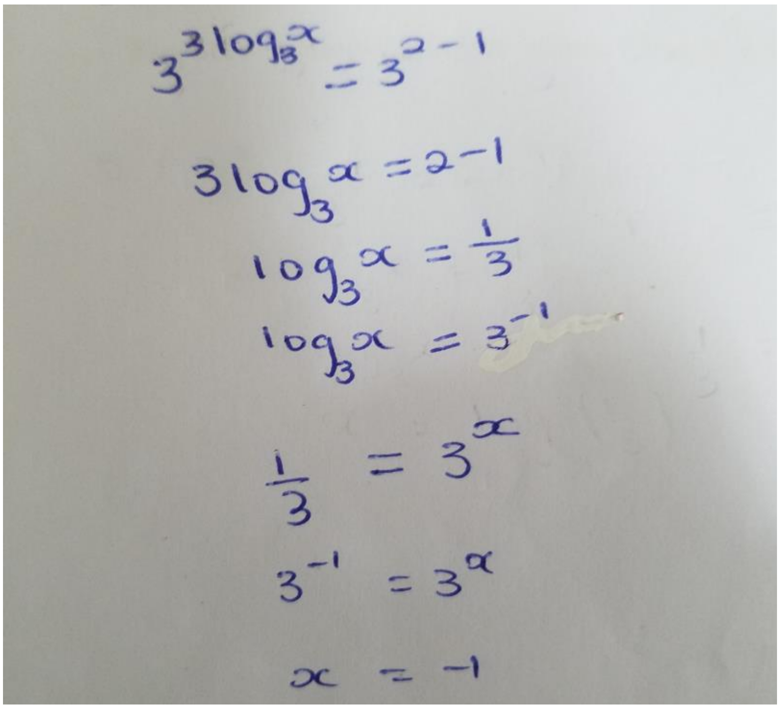
This question focuses on exploring the pre-service mathematics teacher's understanding of logarithmic equations that involve exponents.

In Table 4.4 the allocation of scores for Question 2.3 is displayed.

Table 4.7. the allocation of scores for Question 2.3

Categories	1	2	3	4	5
Indicator	Not answered or incorrect solution.	Introducing logarithm to both sides of the equation.	Application of logarithm power law.	Solving the equation correctly.	Using exponent to obtain the correct answer.
Number of responses	18	1	1	1	1

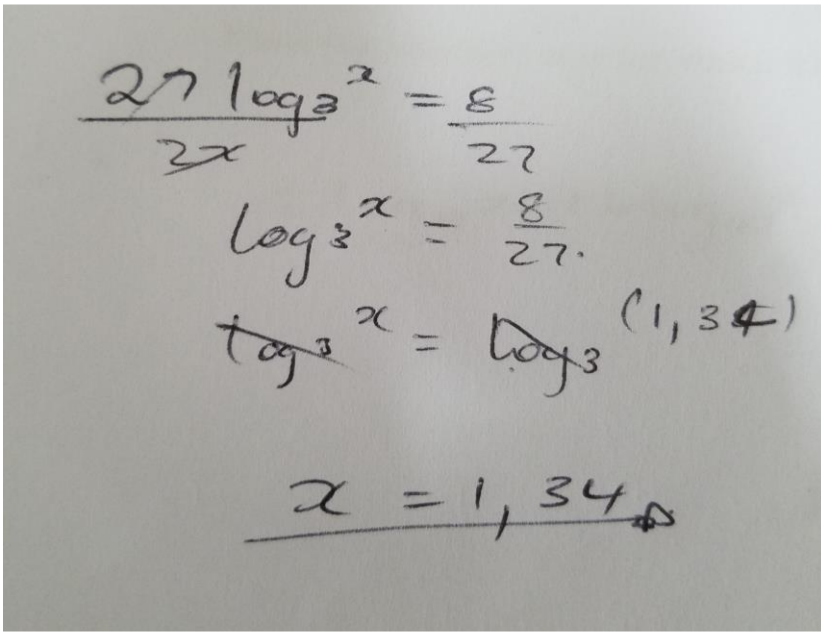
Nine pre-service mathematics teachers out of 18 of them in category 1 did not write anything in this question. The rest of the pre-service mathematics teachers do not have an understanding of what they should do to solve the equation. This shows a lack of prior knowledge of applying logarithm in the exponential equation which will help them to solve the equation. Some of their responses were shown in Extract 11 and 12.



Extract 11: Patu's written response for Question 2.3

Seems Patu knows that he can equate the exponents if the base is the same in an exponential equation. Looking at his solution, you will realise that he did not have prior knowledge of how to solve this kind of logarithmic equation. From his first step, he tried to put the bases as a product 3, but 8 can not necessarily be put as a product of 3 since $8 \neq 3^{2-1}$. Even though he made a computational error at that step, looking at line 4 and 5 of his solution, you will realize that he lacks the basic knowledge of converting logarithm to the exponent.

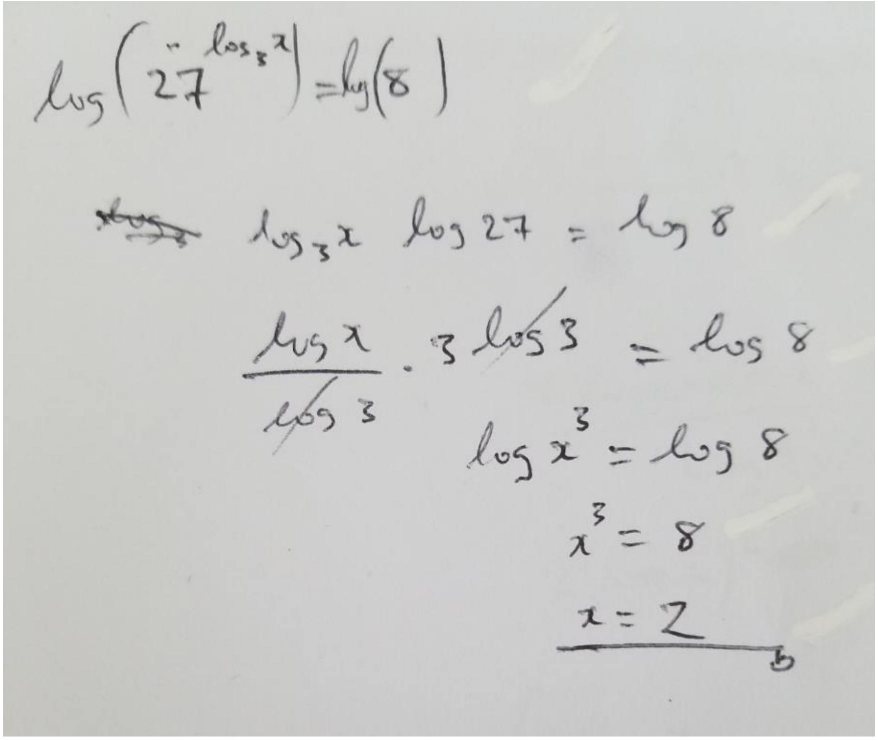
Aphi on the other hand showed that he had no prior knowledge on the calculations that involve exponent which became a problem for him to apply the knowledge in solving this logarithm problem. From his response (see Extract 12), he treated 27 as the coefficient of $\log_3 x$ not as the base of $\log_3 x$. He equally showed a lack of knowledge of the conversion from logarithm to exponent from what he solved in line 3.



Extract 12: Aphi's written response for Question 2.3

Most of the pre-service mathematics teachers in this category made a similar mistake to that of Patu. For the sake of keeping 8 as a multiple of 3, some of them were expressing 8 as 2^{-3}, $\sqrt[3]{2}$, $3^2 - 3^0$ and so on with which they were not able to proceed correctly to the next level.

It was only Zik who was able to start by introducing logarithm to both sides of the equations. She made the necessary mental constructions. Zik's response made it to Category 5 since she provided a complete solution for the question. Her responses indicated that she understood the relationship between logarithm and concepts. The way she applied the change of base in line 3 shows that she has a good knowledge of the laws of logarithm and how to apply them. In Extract 13, we observed that she had both the conceptual and procedural knowledge required to solve the problem. She demonstrated a clear understanding of the concept, as she displayed a clear understanding of the relationship between exponents and logarithm.



Extract 13: Zik's written response for Question 2.3

4.3.3 Question 3: Prove of logarithmic equations

This question was aimed at exploring pre-service mathematics teachers' knowledge of the laws of the logarithm. Question 3 was intended to provide insight into whether the pre-service mathematics teachers had the conceptual understanding of the laws of the logarithm, and whether they can apply them in proof concerning logarithm.

Question 3

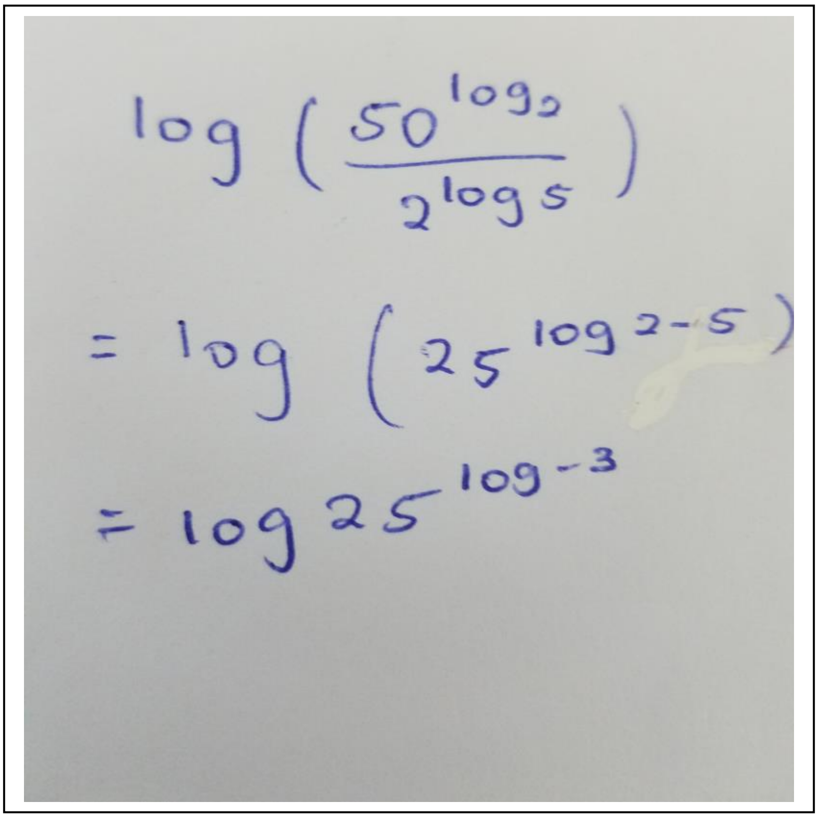
Prove that $\log\left(\frac{50^{log\,2}}{2^{log\,5}}\right) = log\ 2.$

In Table 4.5 the allocation of scores for Question 3 is displayed.

Table 4.8. The allocation of response categories for Question 3

Categories	1	2	3	4	5
Indicator	Not answered or incorrect solution.	Application of logarithm quotient law.	Application of logarithm power law.	Factorising the expression.	Solving to get the right-hand side of the equation.
Number of responses	17	2	1	1	1

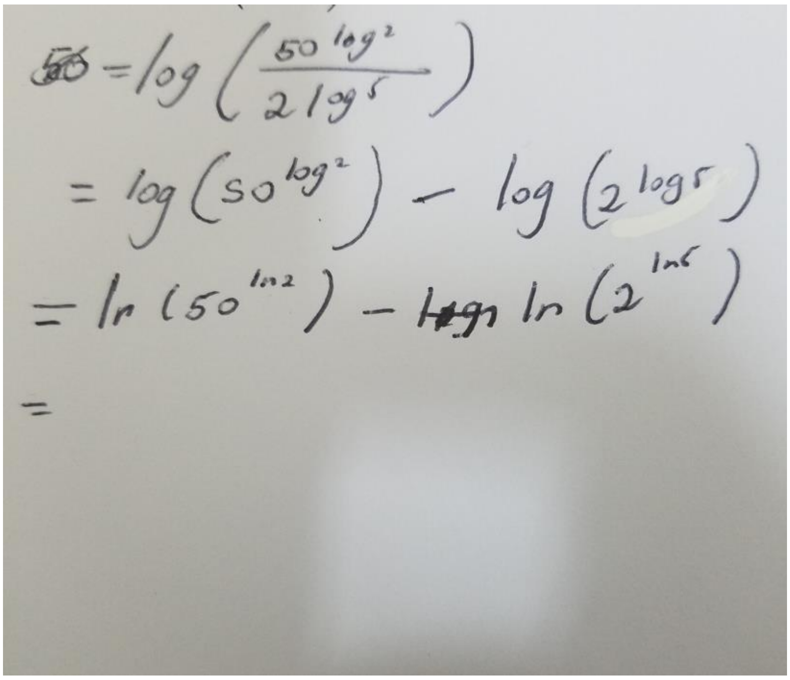
In answering this question, pre-service mathematics teachers' responses in Category 1 revealed that they failed to interpret the question correctly, since 7 of 19 (36.8%) could not provide any answer, and the rest just solved the question incorrectly. This could be due to the fact that they do not have prior knowledge about the laws of the logarithm. Some pre-service mathematics teachers divided the base 50 by 2 (see Extract 14).



Extract 14: Patu's written response for Question 3

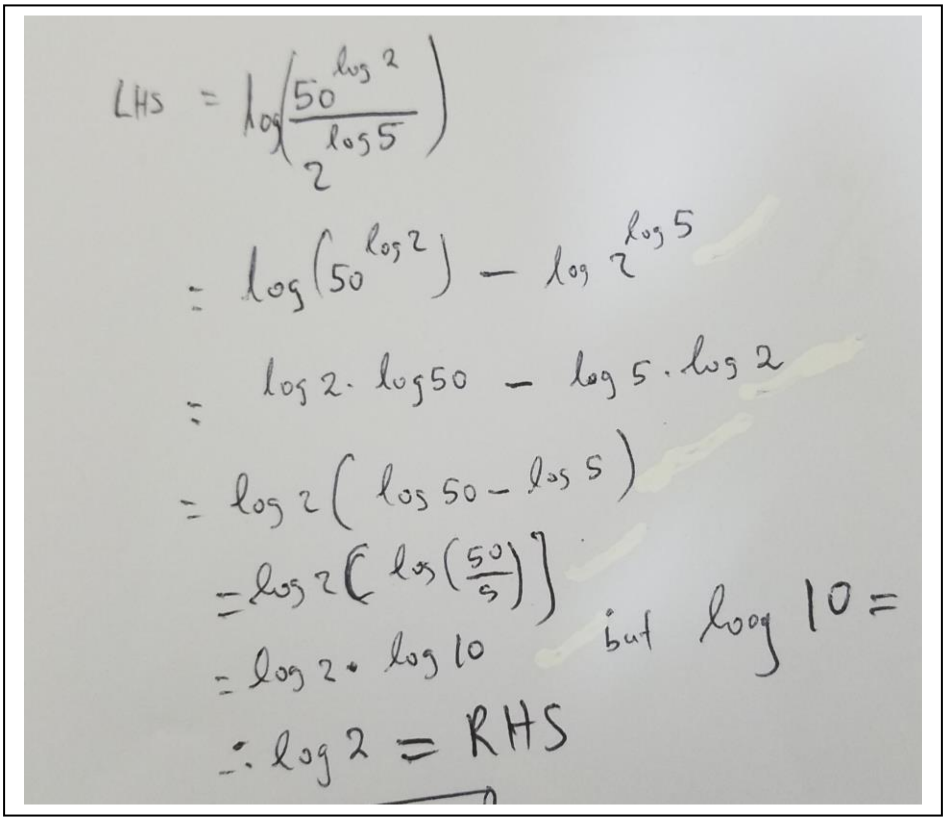
Looking at Patu’s response to question number 3, he treated 50 and 2 as the coefficient of log 2 and log 5 respectively which is why he got 25 in line 2. This clearly shows the lack of prior knowledge of exponent and logarithm. In line 2 it, show that he had the action conception of quotient law of exponent but does not know where to apply that.

The two pre-service mathematics teachers in Category 2 applied the logarithm quotient rule, but one of them could not proceed correctly from there. What she did was to change the logarithm to natural logarithm as shown in Extract 15. In line 3 of her work, it is clear that she lacks the procedural knowledge which she needs to continue to prove the expression.



Extract 15: Maza’s written response for Question 3

Only one of the pre-service mathematics teachers was able to make it until Category 5. She has the conceptual knowledge of the laws of logarithm and equally the procedural knowledge of where and when to apply the laws of the logarithm (see Extract 16). Zik’s response revealed that she had cognitively constructed the structure of proving logarithmic equations and from that, could apply the necessary laws which will help her prove the equation. Her response indicated that she could carry out the procedures not just for the application of logarithm laws, but to yield understanding on when to factorize so that it becomes easier for further simplifications. In line 5, she was able to equally apply the quotient law of logarithm in the revert order which shows a clear understanding of the proof.



$$LHS = \log\left(\frac{50^{\log 2}}{2^{\log 5}}\right)$$

$$= \log\left(50^{\log 2}\right) - \log 2^{\log 5}$$

$$= \log 2 \cdot \log 50 - \log 5 \cdot \log 2$$

$$= \log 2\left(\log 50 - \log 5\right)$$

$$= \log 2\left[\log\left(\frac{50}{5}\right)\right]$$

$$= \log 2 \cdot \log 10 \quad \text{but } \log 10 =$$

$$\therefore \log 2 = RHS$$

Extract 16: Zik's written response for Question 3

4.3.4 Question 4: Using K-method to solve a logarithmic equation involving a change of base.

This question was aimed at exploring pre-service mathematics teachers' knowledge of the change of the base law of logarithm. Question 4 was intended to provide insight into whether the pre-service mathematics teachers had the conceptual understanding of the change of the base of the law of logarithm, and whether they can use K-method to solve the equation.

Question 4

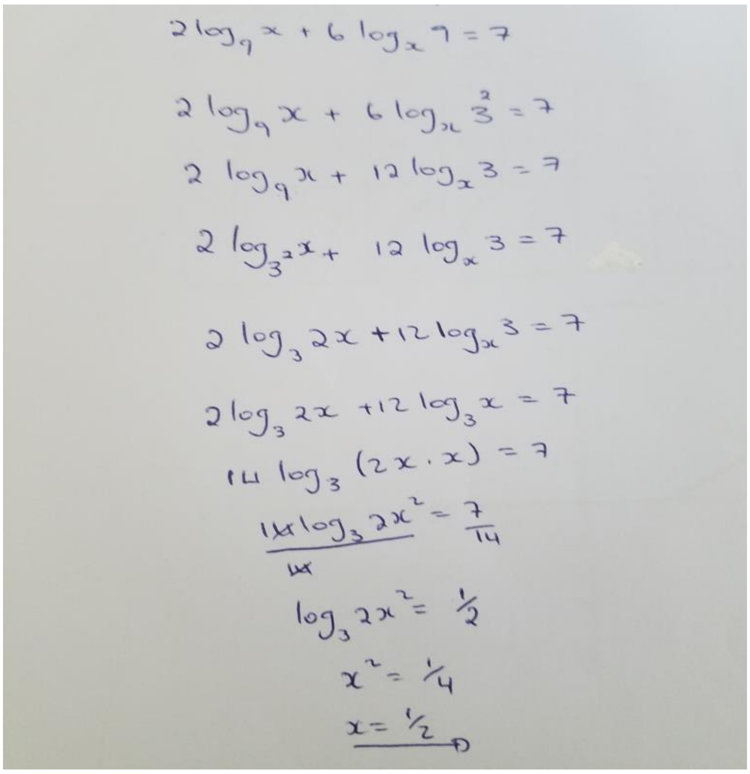
Find the value(s) of x for which: $2\log_9 x + 6\log_x 9 = 7$.

In Table 4.6 the allocation of scores for Question 4 is displayed.

Table 4.9. The allocation of response categories for Question 4

Categories	1	2	3	4	5
Indicator	Not answered or incorrect solution.	Application change of base law of logarithm.	Forming a quadratic equation using K-method.	Solving the quadratic equation.	Check for the restrictions for the values of x.
Number of responses	16	3	0	2	1

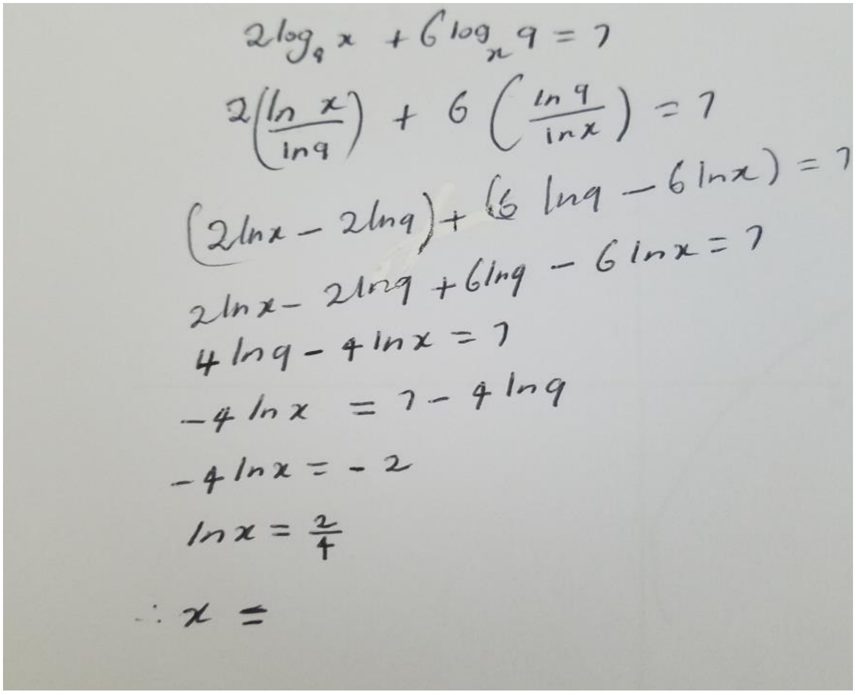
Pre-service mathematics teachers' responses in Category 1 revealed that they failed to interpret the question correctly, since 4 pre-service mathematics teachers could not write anything on the space provided for the response, and the rest just solved the question incorrectly. This shows that they are not familiar with the change of base law of logarithm or it could be due to the fact that they do not have prior knowledge about the laws of the logarithm. Some pre-service mathematics teachers who attempted the question tried to simplify the problem in the wrong way (see Extract 17).



Extract 17: Iwe's written response for Question 4

Looking at Iwe's response to question number 4, she tried to keep the equation in base 3. In line 3 of her solution, she applied the power law of logarithm properly which show that she has an idea about the laws of logarithm but do no when and where to apply each law. One can argue that she does not have prior knowledge of the change of the base law of logarithm.

The three pre-service mathematics teachers in Category 2 applied the logarithm change of base law, but two of them could not proceed correctly from there. What they did was to change the logarithm to natural logarithm as shown in Extract 18. In line 3 of Maza's response, she thinks she is applying the logarithm quotient law. It is clear that she lacks the procedural knowledge which she needs to continue to solve the equation.


$2\log_9 x + 6\log_x 9 = 7$

$2\left(\frac{\ln x}{\ln 9}\right) + 6\left(\frac{\ln 9}{\ln x}\right) = 7$

$(2\ln x - 2\ln 9) + (6\ln 9 - 6\ln x) = 7$

$2\ln x - 2\ln 9 + 6\ln 9 - 6\ln x = 7$

$4\ln 9 - 4\ln x = 7$

$-4\ln x = 7 - 4\ln 9$

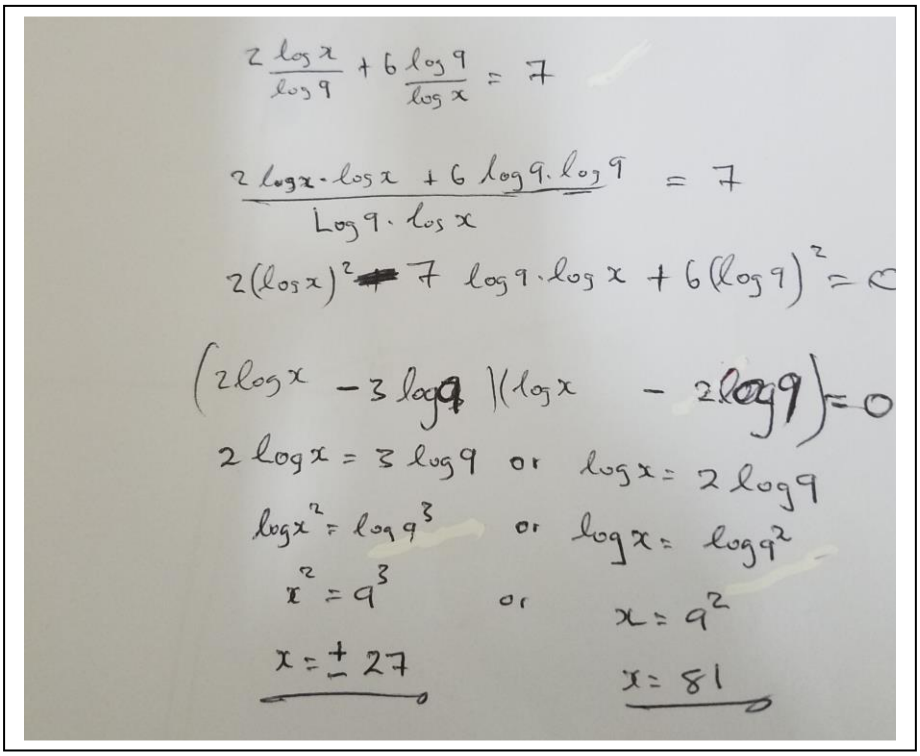
$-4\ln x = -2$

$\ln x = \frac{2}{4}$

$\therefore x =$

Extract 18: Maza's written response for Question 4

Zik is the only pre-service mathematics teacher out of the 19 of them who were able to make it until Category 5. She has the conceptual knowledge of the change of base of the law of logarithm and equally the procedural knowledge of how and when to apply the law (see Extract 19). Zik's response revealed that she had prior knowledge of the change of the base law of logarithm. She was able to form and solve the quadratic equation without the use of K-method. Even though she was not able to check for the restrictions for the correct values of, but it shows that she has a good understanding of the change of base law of logarithm and can equally apply it well. It is observed that none of the pre-service mathematics teachers were able to apply the K-method in solving this equation.



Extract 19: Zik's written response for Question 4

4.3.5 Question 5: Sketch of the graph of the logarithmic function.

This question was aimed at exploring pre-service mathematics teachers' knowledge of the sketch of a logarithmic function. This question was intended to provide insight into whether the pre-service mathematics teachers know how to sketch the graph of log function without plotting it as an inverse of an exponential function.

Question 5

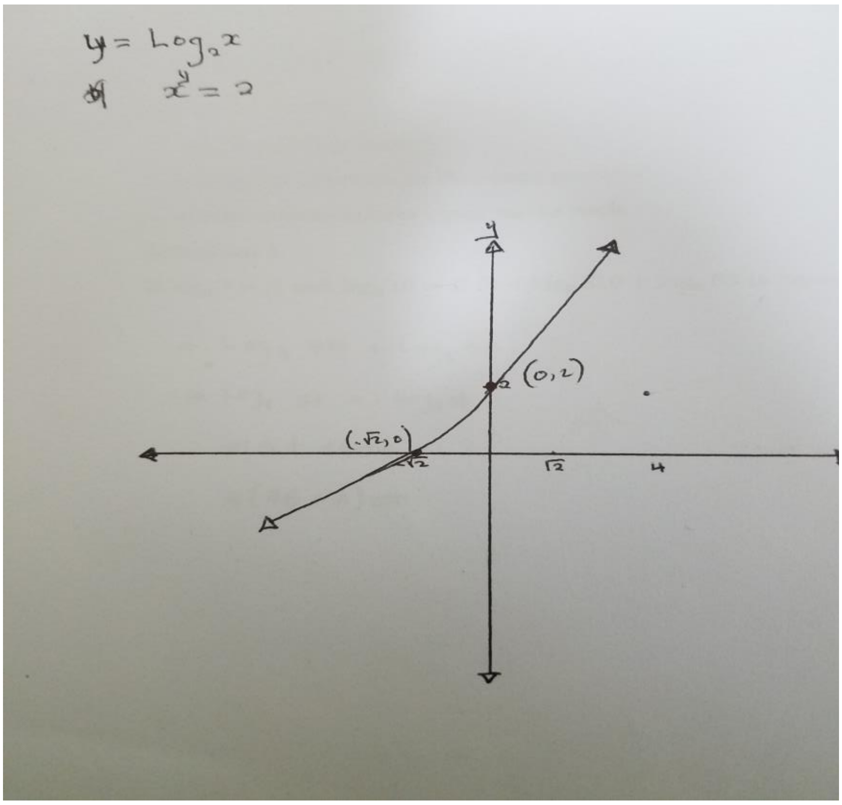
Sketch the graph of the function $y = \log_2 x$ indicating the intercepts and the point where $y = 2$.

In Table 4.7 the allocation of scores for Question 5 is displayed.

Table 4.10. The allocation of response categories for Question 5

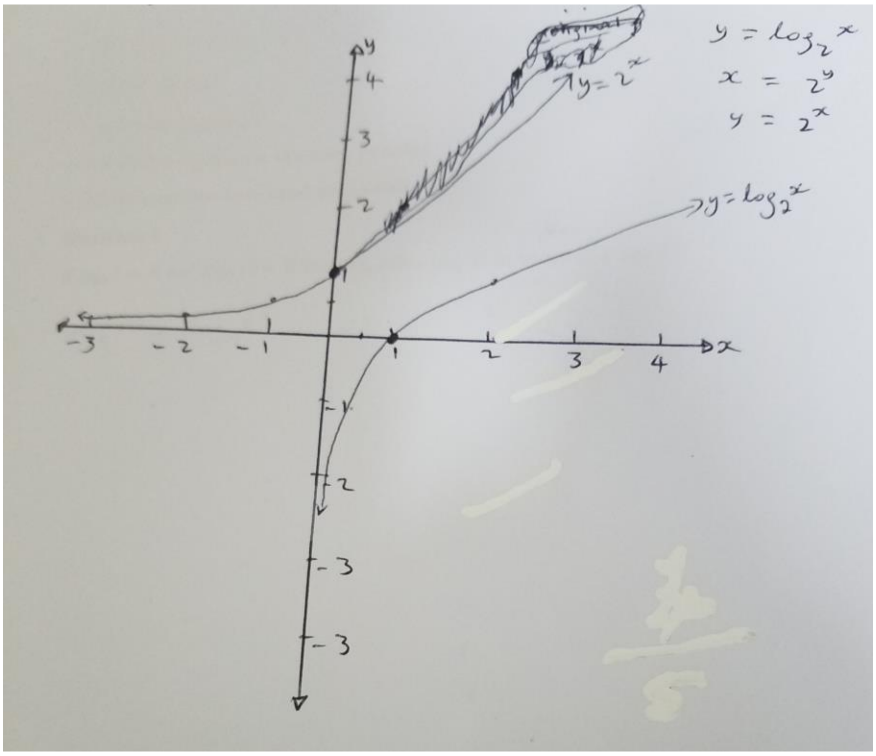
Categories	1	2	3	4
Indicator	Not answered or incorrect solution.	Correct x intercept.	Correct shape of the function.	The point where $y = 2$.
Number of responses	8	10	4	6

Pre-service mathematics teachers' responses in Category 1 revealed that they cannot sketch the graph of the logarithmic function. Two pre-service mathematics teachers' out of the eight of them drew an incorrect graph while the rest drew the x and y-axis. Looking at the solution given by Oke in his graph (see Extract 20), you will realize that his conversion from logarithm form to exponential form is incorrect. This shows that he does not have prior knowledge of how to plot a logarithmic graph.



Extract 20: Oke's graph for Question 5

Only 4 pre-service mathematics teachers out of the 10 (21%) of them who got the correct x-intercept were able to draw the shape of the logarithmic function correctly. This shows that the 6 other pre-service mathematics teachers were able to calculate the x-intercept of the logarithmic function but does not have an idea about the shape of the graph. Among these four pre-service mathematics teachers that drew the graph correctly, 3 of them drew the graph as an inverse of the exponential function and the other one uses the table method. This can be shown in Zee's graph (Extract 21). Here she converted the logarithmic function to exponential function. Then she plotted the exponential function and then plots the inverse of the exponential function as the logarithmic function. This shows that she cannot be able to plot the logarithm function except as the reflection of the exponential function at the point $y = x$.



Extract 21: Zee's graph for Question 5

4.3.6 Summary of the stage

Below are the tables that summarise the responses from the participants in the research task. Each question in the research task has five categories.

Table 4.11. Summary of participants' response to each question

Questions \ Categories	1	2	3	4	5
1	15	4	1	4	1
2.1	15	3	4	3	0
2.2	16	3	2	2	0
2.3	18	1	1	1	1
3	17	2	1	1	1
4	16	3	0	2	1
5	8	10	4	6	

From the above table, more that eighty-four (78.95%) of the pre-service mathematics teacher who participated in the research task did not respond to the question or solve the question wrongly and only a little bit above 5% of them got the solution correctly.

Table 4.12. Participants' performance on each question

Participants	Q1 5	Q2.1 5	Q2.2 5	Q2.3 5	Q3 5	Q4 5	Q5 5	Total 35	% 100
1	0	5	3	5	5	4	5	27	77.1
2	0	0	0	1	-	-	0	1	2.9
3	2	0	0	0	0	0	2	4	11.4
4	0	0	-	0	-	-	1	1	2.9
5	0	0	0	0	0	0	0	0	0.0
6	0	0	0	0	0	0	1	1	2.9
7	0	4	0	0	0	0	5	9	25.7
8	5	1	1	0	0	0	3	10	28.6
9	0	0	0	0	0	0	2	1	2.9
10	1	0	0	0	-	0	0	1	2.9
11	1	0	0	0	0	0	0	1	2.9
12	-	0	1	0	0	0	0	1	2.9
13	0	5	5	0	-	-	-	10	28.6
14	0	0	0	0	0	0	3	3	8.6
15	0	0	0	-	1	0	3	4	11.4

16	-	0	0	-	0	0	3	3	8.6
17	0	0	0	0	0	0	2	2	5.7
18	0	0	0	0	-	0	0	0	0.0
19	0	0	0	0	-	-	-	0	0.0
Average	0.47	0.79	0.53	0.32	0.32	0.21	1.58	4.2	11.9

Each question in the research task has 5 marks. The data was gathered from the written responses which provided useful insight into the nature of the knowledge that the pre-service mathematics teachers had with logarithm. Pre-service mathematics teachers' responses from Table 4.12 presented above revealed that most of them do not know much about logarithm in general. For pre-service mathematics to have a good knowledge of logarithm, he/she should get at least 3 out of 5 (60%) marks for each question or score at least 60 % in total. The response also provided useful insight into some of the errors that pre-service mathematics teachers made while solving the problems. The data gathered from Table 4.12 indicated that the average performance of the pre-service mathematics teachers who participated in the research task is 11.9%. This indicated that pre-service mathematics teachers mainly know the name "logarithm" and could not carry out correct procedures that required the knowledge of laws of the logarithm. This means that they mainly possessed factual knowledge of the logarithm concepts. This study supports the findings of (Chua & Wood, 2005), that pre-service mathematics teachers' lack of prior knowledge may contribute greatly to their difficulties with the learning of logarithm.

4.4 Stage 3: Analysis of written responses and interviews

In this section, the analysis of pre-service mathematics teachers' responses to the research task and the transcription of their interviews on selected tasks, based on their written responses to the task, are presented. The structure of the tasks was specifically designed to address what the pre-service mathematics teachers know about logarithm, their difficulties about logarithm and how they conceptualised logarithm. Pre-service mathematics teachers' responses to the task were categorised and some of them were selected for the interview to provide clarity regarding their responses and to verify how well they know logarithm, based on their written responses. The selected participants were asked various questions, with the aim to extract information on how knowledgeable they are with the logarithm and to discover where they are having difficulties with logarithm. For this study, it was important to detect whether the knowledge they have led to a factual understanding or conceptual understanding of the logarithm and if the pre-service teachers could recognise and apply the required procedures appropriately in the given tasks. This section reports on the analysis of pre-service mathematics teachers' responses (taken from both research task and interviews) revealed how good they know logarithm.

4.4.1 The structure and analysis of the research task

The research task that consists of five questions with question number 2 having three sub-questions, was administered to 19 pre-service mathematics teachers. These questions address the following skills and knowledge: Question 1 focused on pre-service mathematics teachers' understanding about logarithm of a number as mostly an irrational number. Question 2 focused on the pre-service mathematics teachers' knowledge of solving a logarithmic equation involving linear equation, quadratic equation and exponential equations. Question 3 focused on how pre-service mathematics teachers can apply the rules of the logarithm to prove the right-hand side is equal to the left-hand side of a logarithmic equation. Question 4 focused on how pre-service mathematics teachers' can apply their problem-solving skills involving the use of K-method, show an understanding of the relationship between concepts, and apply their knowledge and procedures in solving the problem that requires the knowledge of logarithm. Question 5 focused on the sketching of logarithmic function, not as an inverse of an exponential function. The purpose of administering the research task was explained to the pre-service mathematics teachers before the commencement of the task. The pre-service mathematics teachers were assured that their identity would not be revealed in any way.

All the seven questions were coded (scored) using a 3-point rubric scale (see Table 4.13). Several benefits of using scoring rubrics in performance assessments have been proposed, such as increased consistency of scoring, the possibility to facilitate valid judgment of complex competencies, and promotion of learning (Becker, 2016). Pre-service mathematics teachers under Score 1 are those who did not respond to the question and those who got the question wrong. Pre-service mathematics teachers under Score 2 are those who have the idea of what is required of them to do but could not solve the question to arrive at the solution while Pre-service mathematics teachers under Score 3 were those who could apply all the necessary knowledge and procedure to arrive at the correct answer. The SPSS will also be used in analyzing different questions in the research task for more clarity.

Table 4.13. The scoring codes

Scores	Description of knowledge	Behavior
1	Show no prior knowledge.	No written response/ incorrect response
2	Have conceptual knowledge.	Apply the correct law of logarithm but could not proceed to solve.
3	Have procedural knowledge.	Apply the correct law and was able to solve but could get to the correct answer.

Once the scripts were analyzed and the categories identified, one or two pre-service mathematics teachers were selected in each category for an interview. The interviews were conducted in order to verify what has transpired in the pre-service mathematics teachers' written responses, as well as to clarify their responses where it was not clear how they found their solution.

4.4.2 The structure and analysis of the interview

The semi-structured interviews of 40 minutes long were conducted by the researcher, with each of the eight participants selected from 19 pre-service mathematics teachers' who participated in the written task. Based on what their responses from the research task revealed about having a good knowledge of logarithm and the difficulties they encounter; an interview schedule was prepared by the researcher. The purpose of the interview was explained to each participant before the commencement of the interview. At all times, participants were assured of their anonymity and pseudonyms were used. In ensuring that every aspect of the interview was captured, the interviews were audio recorded. Although the interview questions were set before the interview commenced, probing questions were used to elicit more information about how participants constructed their knowledge and to ascertain their understanding of logarithm concepts. The probing questions were extensively used because it was of importance in this study to clearly elicit how the pre-service mathematics teachers conceptualized logarithm and difficulties they encountered. Pre-service mathematics teachers'' difficulties and misconceptions that emerged from their responses in the research task and during the interviews were analyzed, with the aim of understanding the barriers that might have caused them not to have a good knowledge of logarithm. Some of the questions used during the interviews aimed to find out what they know about logarithm and the difficulties they encounter with logarithm.

4.4.3 Analysis and discussion of written responses and interviews

The objective of the task administered to pre-service mathematics teachers was (1) what prior knowledge they have about logarithm; (2) to understand the difficulties and the misconceptions that pre-service mathematics teachers' display, which becomes a barrier in having a good knowledge of logarithm; (3) to explore the application of procedures in solving problems related to logarithm. The objective of the interviews was to: (1) get clarity on the written responses; (2) to identify how knowledgeable pre-service mathematics teachers are with logarithm. During the interviews, pre-service mathematics teachers were requested to explain not just their solution, but how best to solve the question in order to capture how knowledgeable they are with logarithm. Their explanations expressed in any vernacular language were then translated by the researcher into English. The pre-service mathematics teachers were asked to respond to the following issues: a) justifying their responses to particular questions in the task; b) looking at the strategies used in solving different questions; and c) examining their general understanding of logarithm. Different questions based on the categories discovered on their responses to the task were asked in order to elicit pre-service mathematics teachers' knowledge of these concepts. The analysis of pre-service mathematics teachers' responses to the tasks, followed by the interview extract, is presented hereafter.

4.4.3.1 Simplification of logarithmic expression

Question 1 was analyzed to explore pre-service mathematics teachers' knowledge of simplification of a logarithmic expression. It equally involves the substitution of a variable in place of a logarithm. This question was designed to provide insight into whether the pre-service mathematics teachers had prior knowledge and good knowledge concerning logarithmic expressions.

Question 1

If $log_9 7 = A$ and $log_9 10 = B$, find the $log_9 810 + log_9 63$ in terms of A and B?

The allocation of scores for Question 1 is displayed in Table 4.14 below.

Table 4.14. The allocation of scores for Question 1

Score	1	2	3
Indicator	Not answered or incorrect solution.	Application of correct logarithm law.	Simplifying and writing down the correct answer
Number of students	15	4	1

The analysis of Question 1 using the SPSS is displayed in the table below.

Table 4.15. Analysis of Question 1

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	S0	13	68.4	68.4	68.4
	S1	2	10.5	10.5	78.9
	S2	1	5.3	5.3	84.2
	S5	1	5.3	5.3	89.5
	EMPTY	2	10.5	10.5	100.0
	Total	19	100.0	100.0	

Out of 19 pre-service mathematics teachers who participated in this research task, 15 (78.9%) gave incorrect responses or did not respond to this question. This implied that they could not clearly simplify the logarithm expression. This may also mean that they do not understand the question. These pre-service mathematics teachers do not have a good understanding of logarithm. The pre-service mathematics teacher in Category 3 provided the correct procedure for the simplification of the logarithmic expression. Three pre-service mathematics teachers out of the eighteen (15.8%) who did not get the correct response were able to expand 810 as a product of 10 and 9 and then 63 as a product of 7 and 9. This is shown in Extract 4.

Patu must have had prior knowledge on how to expand numbers, but his inability to apply the correct logarithm law became a problem as shown in Extract 4. This shows that he does not have prior knowledge of logarithm. An interview with him indicated the following:

Researcher: In your solution to Question 1 you were able to expand 810 and 63 correctly, but failed to proceed with the rest of the solution correctly, why is that?

Patu: *I did not know that there are laws of logarithm which I should apply.*

Researcher: Okay but how did you get to your third step?

Patu: [silent]. *Eeeeem, here I was just solving it mathematically. Like I said, I didn't realize that I should apply any log rules. I was just opening the brackets by multiplying 10 and 81 with log base 9. And after that I replaced log 10 base 9 with B.*

The above responses revealed that he has no knowledge of the laws of the logarithm. He was just simplifying the expression without considering that the logarithm in the question makes it different from simplifying a linear expression. In trying to understand why he does not has any knowledge of the laws of the logarithm, the interview continued as follows.

Researcher: Okay Patu, can you tell me when you were first introduced to logarithm?

Patu*: I was first introduced to logarithm when I was in high school.*

Researcher: Okay. And how was the introduction?

Patu: *Eeehh…the logarithm introduction was quite a little bit of confusing. (Okay). Eehh, my teacher has mentioned that this logarithm is not much examinable. (Okay). So, we have no time to dwell on it. (Alright). So, we didn't do much of the logarithm chapter. (Okay). My teacher just told us how to convert the exponent to log and we did some examples. There was nothing much he said about log.*

Researcher: So, which means that, because it wasn't examinable, he didn't waste much time on it.

Patu: *Yes.*

His explanation shows that he has the knowledge for simplifying linear expressions. He did not have prior knowledge of the laws of the logarithm. What is outward though is that he was not exposed properly to the logarithm. His conception of logarithm was equally poor since his teacher did not dwell more about logarithm since it is not examinable.

4.4.3.2 Solving a logarithmic equation

Test the pre-service mathematics teachers' knowledge about solving logarithm equation was divided into three questions

Question 2.1

Solve for x: $log_2 x + log_2 5 = 3$.

The allocation of scores for Question 2.1 is displayed in the table below.

Table 4.16. The allocation of scores for Question 2.1

Score	1	2	3
Indicator	Not answered or incorrect solution.	Application of correct logarithm law.	Solving and writing down the correct answer
Number of students	15	4	2

The analysis of Question 2.1 using the SPSS is displayed in the table below.

Table 4.17. Analysis of Question 2.1

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	S0	15	78.9	78.9	78.9
	S1	1	5.3	5.3	84.2
	S4	1	5.3	5.3	89.5
	S5	2	10.5	10.5	100.0
	Total	19	100.0	100.0	

In this question, 78.9% of the pre-service mathematics teachers did not answer the question, indicating a clear lack of knowledge of solving linear logarithmic equations. The understanding of the reason why they could not be able to solve this question will be better clarified through interviews, but their responses revealed that they do not have a good knowledge of logarithm. The two pre-service mathematics teachers who provided correct and complete responses in this question proved to have good and prior knowledge of solving linear logarithmic equations. Having looked at Aphi's response to Extract 6, an interview with Aphi revealed the following.

Researcher: What is your understanding of the logarithmic equation, especially when it involves a simple linear equation?

Aphi: *I am not sure what the difference is with the logarithm, but I know how to solve simple linear equation* (Okay). *Eeeeeem, solving simple linear equation involves finding the unknown and to do that, you collect like terms.*

Researcher: Looking at your solution, can you explain to me what you did?

Aphi*: [silent], you see, I put log 5 base 2 in my calculator and it gave me a number. So, I take the number to the other side of the equation* (okay). *Eeeeem, aibo, I don't know how I got 32,31928. Eish, am not sure of what I did here. I have forgotten how I did this thing.*

Researcher: Okay. Do you know how to convert from logarithm to exponent and exponent to logarithm?

Aphi: *I don't think so, but I remember we did something like that in high school.*

Aphi was trying to explain his solution to Question 2.1 (See Extract 6). Although he provided the correct response on how to solve linear equations, he failed to solve the one that involves logarithm. This shows that he has prior knowledge on how to solve a linear equation, but he does not have a good knowledge of logarithm. Research has shown that students' previous knowledge plays a vital role in the construction of new knowledge (Ansah, 2016). However, if the previously learnt knowledge has not been conceptually formed, these could become a barrier in the pre-service mathematics teachers attempt to construct new knowledge.

To encourage him to think deeply about changing from logarithm to exponent and exponent to logarithm, more questions were asked.

Researcher: Given that $x = 2^y$, can you write y in terms of x?

Aphi: *I am not too sure of the answer, but y will be log something of x. I will be lying if I tell you I remember this but if I study it again, I will be able to answer the question.*

Researcher: Given that $x = 2^y$, and I said that $y = \log_2 x$ will you agree with me?

Aphi*: [Silent] I think you are right but I can't say for sure.*

This confirms that he had no knowledge of logarithm since he cannot say for certainty how to change exponent to logarithm and vice versa. Changing exponents to logarithm and logarithm to an exponent is considered to be the first thing one is exposed to while introducing logarithm.

The two pre-service mathematics teachers provided a correct response indicating a clear understanding of how to solve a linear logarithmic equation. They correctly solved for *x,* indicating that they had constructed a coherent understanding of the multiplicative law of logarithm and could apply them accordingly.

Question 2.2

Solve for x: $log_{12}(3 - x) + log_{12}(2 - x) = 1$.

The allocation of scores for Question 2.2 is displayed in the table below.

Table 4.18. The allocation of scores for Question 2.2

Score	1	2	3
Indicator	Not answered or incorrect solution.	Application of correct logarithm law, converting and solving.	Checking for restrictions and writing down the correct answer.
Number of students	16	3	0

The analysis of Question 2.2 using the SPSS is displayed in the table below.

Table 4.19. Analysis of Question 2.2

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	S0	14	73.7	73.7	73.7
	S1	2	10.5	10.5	84.2
	S3	1	5.3	5.3	89.5
	S5	1	5.3	5.3	94.7
	EMPTY	1	5.3	5.3	100.0
	Total	19	100.0	100.0	

In solving the problem in Question 2.2, pre-service mathematics teachers in Score 2 displayed mathematical inaccuracy. Inaccuracies in mathematics mostly arose when pre-service mathematics teachers failed to carry out manipulations or algorithms, though they understood the concept. Pre-service mathematics teachers made procedural errors indicating a lack of algorithm skills. What transpired here was that the three pre-service mathematics teachers in Score 2 knew the procedure to use but lacked the technique to carry out the procedures effectively (see Extract 10). They may have successfully made a link between a logarithm and quadratic equation, and therefore were able to perform the required operation accurately. But they failed to provide an accurate answer to the question because of their inability to check for the restrictions. The researcher, while interviewing Zik on why she thinks the values of x she got satisfied the equation, she said that it did not occur to her to check for the restrictions. She said, "I assume that the x values are correct since it was mathematical".

Likewise, 15 out of 19 (79%) pre-service mathematics teachers failed to solve this question correctly while 5.5% of them did not respond to the question. This shows that the majority of pre-service mathematics teachers do not have a good knowledge of logarithm.

Question 2.3

Solve for x: $27^{log_3 x} = 8$.

The allocation of scores for Question 2.3 is displayed in the table below.

Table 4.20. The allocation of scores for Question 2.3

Score	1	2	3
Indicator	Not answered or incorrect solution.	Introduction of logarithm and application of correct logarithm law.	Solving the equation and writing down the correct answer.
Number of students	17	2	1

The analysis of Question 2.3 using the SPSS is displayed in the table below.

Table 4.21. Analysis of Question 2.3

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	S0	15	78.9	78.9	78.9
	S1	1	5.3	5.3	84.2
	S5	1	5.3	5.3	89.5
	EMPTY	2	10.5	10.5	100.0
	Total	19	100.0	100.0	

It was important to know whether pre-service mathematics teachers could apply the knowledge of logarithm to solve a problem. Also, it was important to know whether pre-service mathematics teachers would associate the knowledge of logarithm in solving the exponential problem. The response of pre-service mathematics teachers in Score 1 indicated they had made no prior knowledge on how to deal with this type of problem. It is observed that 2 out of 19 (10.5%) pre-service mathematics teachers in Score 1 did not write any response on that question, while 15 of 19 (78.9%) pre-service mathematics teachers provided incorrect responses, indicating an inability to apply the correct procedure. These pre-service mathematics teachers have no knowledge of the laws of the logarithm, and demonstrated the poor interpretation of the concepts, and as a result, they applied inappropriate procedures (see Patu's response in Extract 11). Patu's written response indicated that he had failed to grasp the concept and could not interpret the problem appropriately. Also, in trying to manipulate rules, he consistently made systematic errors which indicated the lack of both conceptual and procedural understanding of solving the exponential problem. The interview with Patu revealed the following.

Researcher: In your response to Question 2.3, why did you think that $8 = 3^{2-1}$?

Patu: *Eeeeeeem, [silent] is it not? I am not sure what I did here. I guessed I pressed it in the calculator*.

Researcher: What were you trying to achieve when you change the base to 3?

Patu*: [silent], I know that when the exponents have the same base, then exponents can be equal to each other. So, I was trying to make the base the same so that I can start solving the exponents* (Okay). *Eeeeee, you see, that is why I put 3 log x base 5 to be equal to 2 minus 1. Eish, am not sure of what I did here. I don't think the rest of it is wrong.*

Researcher: Okay. Do you know how to convert from logarithm to exponent and exponent to logarithm?

Patu: *I have forgotten how to do that.*

The pre-service mathematics teacher in Score 3 revealed that she had made the necessary mental constructions, as she provided correct and complete responses for the solution of question 2.3. Her response shows that she has a prior and good knowledge on how to solve the logarithmic exponential equations equation.

4.4.3.3 Proof of logarithmic equations

This question was aimed at exploring pre-service mathematics teachers' knowledge of the laws of logarithm and the ability to apply the procedures accurately when proving logarithmic equations. This question was intended to provide insight into whether the pre-service mathematics teachers had the conceptual understanding of the laws of the logarithm, and whether they can apply them in proof concerning logarithm.

Question 3

This question required participants to prove that the left-hand side of the equation is equal to the right-hand side, the question was as follows;

Prove that $\log\left(\frac{50^{log\,2}}{2^{log\,5}}\right) = log\ 2$.

The allocation of scores of this question is displayed in the table below.

Table 4.22. The allocation of scores for Question 3

Score	1	2	3
Indicator	Not answered or incorrect solution.	Application of correct logarithm law.	Solving to get the right-hand side of the equation.
Number of students	17	2	1

The analysis of Question 3 using the SPSS is displayed in the table below.

Table 4.23. Analysis of Question 3

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	S0	11	57.9	57.9	57.9
	S1	1	5.3	5.3	63.2
	S5	1	5.3	5.3	68.4
	EMPTY	6	31.6	31.6	100.0
	Total	19	100.0	100.0	

Table 4.23 revealed that 17 out of 19 (89.5%) pre-service mathematics teachers could not solve this problem correctly while 31.6% of them could not attempt the question. The response of one of the two pre-service mathematics teacher in score 2 indicated that she has prior knowledge about the laws of the logarithm. She understands the question correctly, and applied the appropriate laws correctly, but had difficulty with solving the equations, which indicated the lack of computation skills. This meant that she has knowledge of the laws of logarithm but made some procedural errors and can be seen in Extract 15. An interview with Maza reveals the following:

Researcher: What comes to your mind when you are looking at the question?

Maza: *I know that I have to apply the quotient law of log. So, the right-hand side will be log 50 to power log 2 minus log 2 to power log 5.*

Researcher: So why did you change the logarithm to ln?

Maza: *I will be lying if I say that I know why but I was thinking that by changing it to ln, I will be able to see it differently and solve it.*

Researcher: Okay, but now do you know the difference between ehh, natural logarithm and logarithm

Maza: *Rational logarithms and logarithms. Rational isn't supposed to like fractions?*

Researcher: No, no, no… Natural. Natural as in nature. The one that you wrote "ln".

Maza: *(surprised) ln*

Researcher: Yeah, ln is natural logarithm

Maza: *I didn't know that ln was a natural logarithm, but I know it is related to logarithm.*

Researcher: (Exclaims) Seriously! Wow. Oh okay. So, if you don't know there's no way you will know the difference actually.

Maza: *Yes.*

Researcher: Oh Okay. So why didn't you proceed to complete the solution?

Maza: *Uuuuuu, I do not remember the next law which I can use to complete the proof. It was long I did this part of mathematics.*

Maza's inability to proceed from that level (see Extract 13) could be attributed to what Matz (as cited in Siyepu, 2013) referred to as surface level procedure. Based on her response in the interview, the researcher was able to conclude that Maza had memorized some of the laws of the logarithm, and that as a result, could not apply the next appropriate law to solve the problem.

4.4.3.4 Using K-method to solve a logarithmic equation involving a change of base.

This question was aimed at exploring pre-service mathematics teachers' knowledge of the change of the base law of logarithm. Question 4 was intended to provide insight into whether the pre-service mathematics teachers had the conceptual understanding of the change of the base law of logarithm, and whether the pre-service mathematics teachers can apply K-method to solve the equation.

Question 4

This question required participants to solve for x: The question was as follows;

Find the value(s) of x for which: $2\log_9 x + 6\log_x 9 = 7$.

The allocation of scores of this Question is displayed in the table below.

Table 4.24. The allocation of scores for Question 4

Score	1	2	3
Indicator	Not answered or incorrect solution.	Application of change of base law and forming quadratic equations.	Solving the quadratic equation and writing down the correct answers.
Number of students	16	2	1

The analysis of Question 4 using the SPSS is displayed in the table below.

Table 4.25. Analysis of Question 4

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	S0	14	73.7	73.7	73.7
	S4	1	5.3	5.3	78.9
	EMPTY	4	21.1	21.1	100.0
	Total	19	100.0	100.0	

In answering this item, pre-service mathematics teachers' responses in Score 1 revealed that they failed to interpret the question correctly, since 4 of 19 (21.1%) pre-service mathematics teachers could not provide any answer, and 14 of 19 (73.7%) just answered the question wrongly. This could be due to the fact that they were required to think of an appropriate law of logarithm which will be applied to make the equation simpler. This meant that they do not have good or prior knowledge of the change of base law of logarithm. Twenty-five percent of the pre-service mathematics teachers in Score 1 showed that they had an idea about some laws of logarithm but cannot apply the correct law to the question. This is shown in Extract 17, the response of Iwe to Question 4. Her interview revealed the following:

Researcher: Iwe, you did very well in your question number one which shows that you have the basic knowledge of logarithm. When you saw this question, what comes to your mind as a means of solving the problem?

Iwe: *I will be lying if I tell you that I remember what I had in mind when I saw the question. I am not too sure.*

Researcher: Okay. Can you explain to me what you wrote as the solution to this question?

Iwe: *Emmmmmm, I was kinda trying to the numbers to base 3 that was why I changed 9 to 3 power 2. And if you look at my third step, you will realize that I applied power law to obtain 12 log 3 base x.*

Researcher: With the knowledge of logarithm you have so far, looking at your step 4, do you think that 2 log x base 3 square is equal to 2 log 2x base 3?

Iwe: *No, it is not.*

Researcher: Okay, so why did you write that?

Iwe: *You know, to be honest, I don't know how to solve this question. I was just writing down what comes to my head since I don't want to leave any question vacant.*

Researcher: Now look at this solution [handed her the memorandum to the question], what do you think about the solution?

Iwe*: [checking the memorandum] oh, I forgot this rule of logarithm, eeeeemmmmm, [silent], change of base. Yes. Oh yeah, I think the solution is right here. I have seen that the solution is so straight forward. I could not have imagined to solve this question this way.*

This shows that Iwe does not have a good knowledge of logarithm since she could not remember the right law of logarithm to apply to the question. The pre-service mathematics teachers in Score 2 were able to apply the change of base law of logarithm. They did not use K-method to form the quadratic equation and in turn, could not solve the problem correctly to get the desired response.

4.4.3.5 Sketching the graph of the logarithmic function.

The analysis of Question 5 was aimed at exploring pre-service mathematics teachers' knowledge of the sketch of a logarithmic function. This question was intended to provide insight into whether the pre-service mathematics teachers know how to sketch the graph of log function without plotting as an inverse of an exponential function.

Question 5

Sketch the graph of the function $\mathrm{y} = \log_2 x$ indicating the intercepts and the point where $\mathrm{y} = 2$.

The allocation of scores for Question 5 is displayed in the table below.

Table 4.26. The allocation of scores for Question 5

Score	1	2	3
Indicator	Not answered or incorrect plot.	The incomplete shape of the graph with critical points indicated.	Plotting the graph correctly.
Number of students	8	7	4

The analysis of Question 5 using the SPSS is displayed in the table below.

Table 4.27. Analysis of Question 5

		Frequency	Percent	Valid Percent	Cumulative Percent
Valid	S0	6	31.6	31.6	31.6
	S1	2	10.5	10.5	42.1
	S2	3	15.8	15.8	57.9
	S3	4	21.1	21.1	78.9
	S5	2	10.5	10.5	89.5
	EMPTY	2	10.5	10.5	100.0
	Total	19	100.0	100.0	

Table 4.27 above revealed that 6 of the 19 (31.6%) pre-service teachers who answered this question could not plot the graph correctly while 2 (10.5%) of them did not answer the question at all. This is because they do not have prior knowledge of how to plot the logarithm graph. Pre-service mathematics teachers in Score 2 have the prior knowledge on how to plot the graph but lack procedural fluency to complete the solution and plot the graph completely. Two of the 19 (10.5%) pre-service mathematics teachers who plotted the graph completely did so as an inverse of an exponential function and one of them use the table method (See Extract 20 and Extract 21). The interview with them reveals the following:

Researcher: Alright. Now, can you explain to me if you can be able to sketch the logarithm function, Question 5?

Zee: *I can simply make, make the equation of that log, I can make it back to the equation of x = y something. (Okay) After the log have been introduced. (Okay) Then I'll then proceed with finding it, it's inverse of the function*

Researcher: Okay. So, so, so, what you are trying to say is that first, if you are given a log function, what you will do first is to change it to exponent?

Zee: *Yes to change to exponent*

Researcher: Okay. So, after changing to exponent then you solve for the exponent, then you convert it back to log?

Zee: *Yes*

Zee has confirmed that she can only plot the logarithmic function only as an inverse of an exponential function. The next interview reveals how Iwe plotted her graph.

Researcher: Can you explain to me how you sketched this your perfect graph?

Iwe: *(Laughs) perfect graph?*

Researcher: Yes, your graph looks so perfect. Can you explain to me how you sketched the graph?

Iwe: *Alright. So, to sketch the graph of function y log x base 2. So, you know that the log graph, if we look at it as functions, are the inverses of the exponents, exponential functions.* (Okay) *So, if it was an exponential function, we would have a graph which would pass at y equal to 1 and x equal to 0. But because this is a logarithm, which means it would reflect (yeah) at y equals to x.* (Okay) *This is why, this is how I came about to find this graph.*

Researcher: Okay. So, what you did first is to change that log to exponent?

Iwe: *to exponent, yes*

Researcher: And then from exponent, you now get your log as the inverse of the exponent. *(Yes)* And then you clean off your exponent graph?

Iwe: *Yes. Yes*

Researcher: Okay. Okay. Alright. That's, that's, that's okay. Alright…

Iwe: *Because the range become… on the exponential graph, the domain of the exponential graph becomes the range… (of the logarithm) of the log and the range of the exponential become the domain of the log.*

Researcher: Okay. So, now, is there any other way you can sketch this graph apart from this method, because when I was looking at your graph, I didn't know you even used exponent actually? Yeah, is there any other way you can sketch this graph without involving exponent.

Iwe: *Yes*

Researcher: Okay, how will you do that?

Iwe: *Eehh… another way is to use* calculator. (Okay) *Another way is to find ehh… what is this? Let log x be equal to zero, the y intercept*

Researcher: Okay the intercepts and ehh the critical points?

Iwe: *yes …and use it*

All the pre-service mathematics teachers could not plot the graph of a logarithmic function as a logarithm function rather they did so as an inverse of exponential function or by use of table method, as Zik noted thus:" *I decided to use table method and then, what I know with log is that when x is negative which means it is undefined or it has no solution. Meaning that x should be greater than zero. And then that will mean that I have to start from 1 going upwards and where there is x I will replace it with numbers like 1, 2, 3, and that is how I did it".*

4.4.4 Summary of Stage 3

In this stage, the analysis and presentation of pre-service mathematics teachers' interviews was presented. It can be seen from the interviews that most of the pre-service mathematics teachers could not clearly simplify the logarithm expression nor solve logarithmic equations correctly. It is worth noting that majority of pre-service mathematics teachers might have the knowledge of the laws of logarithm but could not apply it properly neither could they link their previous knowledge nor use K-method to solve the logarithmic problem.

4.4.5 Conclusion

In this chapter, the data analysis of pre-service mathematics teachers' responses to the written task was presented and analyzed. Pre-service mathematics teachers were prompted through guiding questions to explain the knowledge they have relating to some logarithm concepts. The interviews conducted with various participants aimed to clarify some responses, understanding their knowledge of logarithm and helping the participants learn to interrogate what they write, which in turn helped in identifying the difficulties the pre-service mathematics teachers have with logarithm. This chapter provides an explanation of whether pre-service mathematics teachers have a good knowledge of logarithm and the difficulties they experienced when solving problems that require the prior knowledge of logarithm. The analysis presented from pre-service mathematics teachers' responses and interviews served to explore the conceptual and procedural knowledge of pre-service mathematics teachers towards logarithm. The next chapter concludes the study by discussing these findings in response to the main questions, recommendations and limitations of the study.

CHAPTER FIVE

5. FINDINGS, DISCUSSIONS AND RECOMMENDATION

5.1 Introduction

This study is a contribution to research in undergraduate mathematics education, focusing on pre-service mathematics teachers' knowledge of logarithm. The study was aimed to describe pre-service mathematics teachers' knowledge of logarithm in one of the universities in the province of KwaZulu-Natal, South Africa. Knowing the importance of logarithm in problem solving, research on pre-service and in-service teachers' knowledge of logarithmic concept is rather slim in South Africa. This study is an attempt to fill this gap and it is guided by the belief that a teacher with good content knowledge in his/her field of study will improve his/her pedagogical content knowledge thus improve pre-service mathematics teachers' achievement through meaningful teaching and learning practices.

In Chapter Four the themes that were uncovered when analysing the research task were discussed. These themes focused on whether pre-service mathematics teachers have a good knowledge of logarithm and what difficulties they encounter while solving problems involving logarithm. In this chapter, a summary of the study, synthesising the themes that emerged from chapter four are presented. This chapter begins by exploring what it means for pre-service mathematics teacher to have a good knowledge logarithm, followed by the difficulties that pre-service mathematics teachers experienced when solving problems related to logarithm and lastly, investigate the way in which pre-service mathematics teachers conceptualize logarithm. This followed with the researcher's summary of the study, recommendations and limitations. It concludes by making a suggestion for further research in logarithm in the context of South Africa.

The analysis and the subsequent results are based largely on pre-service mathematics teachers' responses to the research task and from transcribed interviews conducted with 8 participants from the class of 19 pre- service mathematics teachers. The interview helped the researcher together with pre-service mathematics teachers' responses to the research task to make certain inferences about whether they have good knowledge of logarithm, the difficulties they encounter while solving logarithm problems and how they conceptualize logarithm. The written responses were clarified through the interviews. Detailed results for each of these analyses are organised according to the relevant mathematical concepts and are found in chapters four. This chapter presents a synthesis of the findings that transpired in pre-service mathematics teachers' responses. The main questions that this study aimed to answer were:

1. What does pre-service mathematics teacher know about logarithm?
2. What are the difficulties pre-service mathematics teachers encountered with logarithm?
3. How do pre-service mathematics teachers conceptualize logarithm?

Below the researcher presents the main findings of the study addressing each of the above research questions. The main aim of the study was to explore pre-service mathematics teachers' knowledge of logarithm. Under logarithm, pre-service mathematics teachers' knowledge of simplification of a logarithmic expression, solving a logarithmic equation involving linear, quadratic and exponential equation, proof of logarithmic equations, the use of K-method in solving a logarithmic equation involving a change of base and sketching of the logarithmic graph were covered. For the purpose of this study, the research task was administered with each question covering certain concepts as described in Chapter four. To explore whether pre-service mathematics teachers have a good knowledge of logarithm and the difficulties they experienced, it was important to analyze their responses to each question. This revealed their level of knowledge of concepts covered in those questions. In the next sections, the researcher will present the findings as they relate to each of the three research questions presented above.

5.2 Knowledge of pre-service mathematics teachers about logarithm.

One of the objectives of this study was to answer what pre-service mathematics teacher know about logarithm. This study is not about giving statistical comparisons of pre-service mathematics teachers' responses, but it aims to reveal whether they have good knowledge of logarithm. The theoretical framework that underpinned the study provided one way of revealing the knowledge pre-service mathematics teachers have about logarithm. Evidence from chapter four revealed that above 85% of pre-service mathematics teachers who participated in this study only have the factual knowledge of logarithm. The responses revealed that many pre-service mathematics teachers could not attempt problems that involve logarithm or do not know how to solve problems that involve logarithm. This was mainly observed from the research task that was administered to them. For example, in Table 4.11, the results of the 19 pre-service mathematics teachers who responded to the research task revealed that 84% were not able to attempt the questions or solve the question wrongly. Also, this showed that they had not made the necessary mental construction showing they did not have good knowledge of logarithm. As it was explained in chapter two, the factual knowledge is about the knowledge that can be declared, through words and symbol systems of all kind. The results from the research task suggested that most pre-service mathematics teachers have not constructed the necessary mental constructions, meaning that they do not have good knowledge on how to simplify, solve, proof or sketch graph of logarithmic equations.

Although some pre-service mathematics teachers carried out related procedures, it seemed that when dealing with problems involving multi-steps which required the internalization of procedures, most pre-service mathematics teachers were experiencing difficulties. Evidence from question 2.1 in chapter four revealed that only 15.8% of the pre-service mathematics teachers represented their knowledge in the manner described as having conceptual knowledge of the laws of the logarithm. These pre-service mathematics teachers were able to recognize the particular law of logarithm which they can apply to be able to solve the problems. Furthermore, based on the data from interviews, there is evidence that many pre-service teachers could not remember how the laws of logarithm work, but they know the laws. Iwe said:

> *I know the multiplicative law of logarithm, but I was not sure how to proceed from there after I applied it. Eeeem, (*silent*) it became a bit confusing from here (*question 2.2*) that was why I couldn't solve the question completely.*

This meant that some pre-service mathematics teachers could apply the multiplicative law of logarithm but does not have the procedural knowledge to proceed to solve the problem. The results showed that some pre-service teachers have the conceptual knowledge of solving a logarithmic equation.

As mentioned in the above paragraph 18 of the 19 (94.7%) pre-service mathematics teachers experienced difficulties when performing multi steps computation. It seemed that as the problem required them to carry more procedures and explain their solution they struggled to solve such problems. Evidence in question 4 showed that many of them did not have the procedural knowledge to solve or prove the logarithmic equations. In question 4 (the use of K-method to solve change of base problem) only 1 of the 19 (5.3%) pre-service mathematics teachers seemed to have the knowledge of the change of the base and interiorized the procedures of solving the problem even though the use of K-method was not applied. Similarly, in question 3, there were only 1 of the 19 (5.3%) pre-service mathematics teachers were able to apply the quotient rule of logarithm and know the procedure to solve the problem correctly. The findings in question 3 and 4 reveal that for the conceptual knowledge of the application of basic laws of the logarithm, only one pre-service mathematics teachers could do that, and also have the procedural knowledge to complete the question. Moreover, it shows that the majority (94.7%) of pre-service mathematics teachers do not have good knowledge of logarithm. Regarding the solving of logarithmic equations, the findings are consistent with findings in the literature that students have several misconceptions with solving logarithmic problems (Chua & Wood, 2005; C. Weber, 2013). Also, students develop conceptual knowledge by making mental constructions of mathematical objects and processes (Dubinsky, 2002).

Sketching the graph logarithmic function was also part of the question considered to help in determining how knowledgeable pre-service mathematics teachers are. To be able to sketch the logarithm graph, pre-service mathematics teachers to know some characteristics required to sketch the graph. Evidence from chapter four question 5 (sketching the graph of a logarithmic function) revealed that only 31.6% of pre-service mathematics teachers who wrote that the research task was able to sketch the graph. Among these pre-service mathematics teachers, none of them sketched the graph as a logarithm graph. Four out of the 19 pre-service mathematics teachers converted the logarithmic function to exponential function, sketch the exponential function and then sketch the inverse of the exponential function as the logarithmic function and 2 of 19 pre-service mathematics teachers use table method. This showed that none of the pre-service mathematics teachers have the knowledge of how to sketch a logarithmic function without the use of the table method or converting it to exponential function first. In this regard Iwe stated:

> *Alright. So, to sketch the graph of function y log x base 2. So, you know that the log graph, if we look at it as functions, are the inverses of the exponents, exponential functions.* (Okay) *So, if it was an exponential function, we would have a graph which would pass at y equal to 1 and x equal to 0. But because this is a logarithm, which means it would reflect (yeah) at y equals to x.* (Okay) *This is why, this is how I came about to find this graph.*

The findings of this study suggest that the majority of the pre-service mathematics teachers cannot sketch logarithm function unless they use the table method or sketch it as the inverse of the exponent. The pre-service mathematics teachers in Category 2 (see table 4.10) were able to find the x-intercept for the function which means that they know that x-intercept is needed for one to be able to sketch a logarithmic function. Aphi stated that "*I just put y=0 and convert the log to exponent so that I can be able to find x*. Although four pre-service mathematics teachers out of the ten had an incorrect answer for question 5, their responses showed that they have prior knowledge of sketching of logarithmic functions.

The findings in this study showed that pre-service mathematics teachers do not have good knowledge of logarithm. Most of them do not know the logarithm laws while some of them who know the laws were unable to apply them correctly. For question 1, pre-service mathematics teachers who were able to apply the required logarithm law failed to evaluate $\log_9 9$. In question 2.2, pre-service mathematics teachers who struggled to simplify the logarithmic equation involving the quadratic equation failed to check for the restrictions of the value of x. Similarly, in question 3 pre-service mathematics teachers who applied the logarithm quotient law to the question could not proceed because the forgot to use the logarithm power law to keep the equation in a linear form so as to make it easier to solve. In summary, the majority (94.7%) of pre-service mathematics teachers do not have good knowledge of logarithm.

5.3 Difficulties in solving problems involving logarithm.

From the results presented in Chapter four, it was evident that 94.8% of the questions pre-service mathematics teachers who participated in this study were experiencing difficulties in solving the problems in the research task most especially in the change of base question and the use of K-method. Literature has shown some of these difficulties relating to mistakes in manipulating logarithmic expressions and difficulties in understanding the meaning of the logarithmic concept. The literature has been silent when it comes to pre-service mathematics teachers' difficulties relating to solutions to problems involving logarithm. There are some occasional observations, in a non-exhaustive fashion and without any theoretical grounding around students' difficulties in solving problems involving logarithm. These studies have been done internationally and they look at these difficulties separately. This study explores pre-service mathematics teachers' knowledge of logarithm. While literature emphasises on the importance of identifying students' difficulties in solving problems involving logarithms (Fermsjö, 2014) to improve instructional methods, it also vital to understand the reasons that led to pre-service mathematics teachers' having difficulties in solving problems involving logarithm. To address the issue relating to pre-service mathematics teachers' difficulties in solving problems involving logarithm, the following research question "What are the difficulties pre-service mathematics teachers encountered with logarithm?" was posed.

In the previous section, the knowledge of pre-service mathematics teachers was discussed. In this section, pre-service mathematics teachers' difficulties in solving problems involving logarithm are discussed. These will be discussed under two distinct sub-sections, (1) lack of background knowledge and (2) misconceptions of logarithm concepts.

5.3.1 Lack of background content knowledge

It was assumed that when dealing with logarithmic concepts, pre-service mathematics teachers should be able to generalize their knowledge of arithmetic and school algebra to formulate new knowledge. The results of this study show many difficulties that pre-service mathematics teachers experienced, like failing to manipulate numbers, emanate from the lack of basic algebra. For example, in Chapter four, Patu tried to explain his solution in question one where he did not know that 81 is the same as 9^2. This shows that he has not developed the ability to apply basic number manipulations to help in solving problems. This was also evident in question 2 and 3 of chapter four. More than 80% of pre-service mathematics teachers could not provide the correct solutions to the problem because of lack of background knowledge such as (1) ability to apply the correct logarithm law and (2) to carry out computation involving numbers, indicating the lack of basic algebra schema and prior knowledge. In question 2.2 of Chapter four, it was evident that some pre-service teachers who recognized that they should apply the multiplicative law of logarithm to the problem were able to carry out computation effectively. When Zik was asked to explain the strategy she used to solve question 2.2 she said:

> "*since I know that I have to apply multiplicative law to the left-hand side, it helped me to see the solution easier. That was why I used change of base and cross multiplication to arrive at quadratic equation*".

This suggested that she had made the connection between the laws of logarithm and quadratic equations which helped her in solving the problem even though she did not check for the restrictions of the values of her answer. The findings showed that when pre-service mathematics teachers had developed the schema of basic concepts or have prior knowledge, they are more likely to get the correct solutions. Similarly, in the same question, some pre-service mathematics teachers were unable to generalize their school knowledge of solving a quadratic equation. When Patu was asked to compare his solution to the correct solution to the question, he said:

> "*wow, I do not think this question can actually lead to quadratic equation. I looked at it in a way that I will be dealing with exponents*".

As Biggs (2011) observes, students who mainly have the surface understanding of mathematical concepts would have barriers in conceptualizing the learnt concepts. The results revealed that for pre-service mathematics teachers to gain a proper understanding of logarithm concepts they needed to have at least a basic knowledge of algebra.

5.3.1 Misconceptions of logarithm concept

In search of the reasons why pre-service mathematics teachers have difficulties in solving problems involving logarithm, the results indicated pre-service mathematics teachers' misconception of logarithm concepts was one of the reasons. Also, the misconceptions the pre-service mathematics teachers had in other related concepts impacted in their understanding of logarithm concepts. In elaborating and synthesizing the results of this study, the researcher identified certain misconceptions that seemed to cause pre-service mathematics teachers' difficulties in solving problems involving logarithm. The evidence from the results revealed that preservice mathematics teachers tend to over-generalize rules. In question 1, chapter four, Patu was confused with the multiplicative law of logarithm. He treated $\log_9(7 \times 9)$ as $\log_9 7\, .\, \log_9 9$ (see Extract 4). He tried to solve this in the same way as the expansion of algebraic expression where you multiply each term in the bracket with the term outside the bracket. In an interview, he stated that he didn't remember that logarithm have rules. This shows that the lack of schema arithmetic algebra impacted negatively in the understanding of the application of laws of the logarithm in solving problems involving logarithm. These findings pointed out that a good understanding of elementary algebra is really important for pre-service mathematics teachers to learn concepts related to the logarithm. Another misconception was that pre-service mathematics teachers become absorbed on the laws rather than understanding how the laws can be applied to a problem. In several cases in this study when pre-service mathematics teachers were asked to explain the concept, they stated a rule. Zik was asked to explain how she could start proving one part of an equation involving logarithm when one side is equal to the other side. She said, "*since I know the laws, I will just use them*". When she was asked to explain how she can use laws to solve question number 3, she was not able to do that. This indicated that knowing the laws does not necessarily mean one understands the concepts or how to apply the laws. These findings are consistent with other studies as it could be argued from the results of the study that these misconceptions were mainly caused by lack of background knowledge (Dubinsky, 2002) as well as a misunderstanding of the previous concepts which are related to matrix algebra (Tall, 2004). Tall (2004) emphasized the previous knowledge learnt can either have positive or negative effects on the constructions of knowledge of the new concepts.

5.4 Conceptualization of logarithm concept

To explain how pre-service mathematics teachers, conceptualize logarithm, the question "How do pre-service mathematics teachers conceptualize logarithm?" was asked.

For this study, a framework based on the framework for research and curriculum development was conducted. From the researcher's point of view, constructivism theorem provided an excellent starting point for making sure that the concepts were constructed carefully and presented from many angles in the research task and interview. Furthermore, this constructivism proved to be a valuable theorem in analyzing how pre-service mathematics teachers conceptualize logarithm.

The findings of this study revealed that the majority of the pre-service mathematics teachers were not properly introduced to the logarithm. They do not have good prior knowledge before logarithm was introduced to them. What was most prevalent was that in all the questions where they are required to apply prior knowledge, they seem to have difficulties with that. In question 1, 18 of 19 (94.7%) pre-service mathematics teachers' responses showed that there is no prior knowledge linked to the simplification of the logarithm, meaning that most pre-service mathematics teachers do not have prior knowledge on how to simplify logarithmic expressions. However, 21.1 pre-service mathematics teachers' responses proved that they have prior knowledge of which helped them in simplifying the logarithmic expression. In their response the interview when they were asked how logarithm was introduced to them, Aphi said:

> *Eeehh…the logarithm introduction was quite a little bit of confusing. Eehh, my teacher has mentioned that this logarithm is not much examinable. So, we have no time to dwell on it. So, we don't much like the logarithm chapter".*

Iwe said

> "*Eeehhh… I think it was introduced to us as a relation to the exponent. Yeah, because we were dealing with exponents, then the reflection of exponential graphs. (Okay) Then afterwards we were introduced to log as they are related to exponents.*

Efe said:

> *Most of our classes were passive. When you were told what to do, how to do it, to remember the rules. Just for, for purposes of writing tests and writing exams. It wasn't like how are being taught now. That you allow the learner to participate. We were just passively taught. These are the rules, this is how you use them, here is an example and let's do it together. They'll now give you some to do on your own. But it was mostly let's do it do it together, let's do it together.*

Zee also said:

> *Well what I remembered was that he just came into the class and gave us an assessment and he told us to answer whatever we know, then after that he came back and give us some corrections about what we did in the assignment and then he started talking.*

This shows that mostly when logarithm was introduced, it was not linked to any prior knowledge which will help the pre-service mathematics teachers to better conceptualize logarithm.

In summary, the results showed that in relation to logarithmic concepts, most pre-service mathematics teachers do not have a link from their previous knowledge. Their mental constructions were mainly of factual knowledge. Therefore, this means that for many of the pre-service mathematics teachers their conceptual understanding of the concepts is still at the developmental stage. It could be said that they have constructed a procedural understanding of the concepts but as indicated that is not enough for them to understand the relationship between concepts. The results of this study showed that in the learning of logarithmic concepts the framework used as indicated by literature proved to be true. This was evident since pre-service mathematics teachers who had not constructed the meaning of the laws of logarithm had difficulty in applying the laws to solve a problem. Pre-service mathematics teachers can guess the answer to a problem while they might have not reached the procedural knowledge of the concept which means they have not conceptualised the concepts. Tall (2004) maintains that "There are many occasions when individuals do not summarize a given process into a thinkable object and instead carry out procedures in a routinized way based on repetition of the learned operation." (p. 30)

5.5 Summary of the study

In this study, the researcher has explored pre-service mathematics teachers' knowledge of logarithm. It also examines the difficulties preservice mathematics teachers encounter while solving logarithmic problems and how they conceptualized logarithm. Some pre-service mathematics teachers responded well to the research task in terms of completing the problems. However, the majority were unable to provide a correct and complete response to all the research task especially those questions where they supposed to show procedural knowledge. It was clear from the findings that pre-service mathematics teachers do not have a good knowledge of logarithm and they have difficulties while dealing with problems involving logarithm. This problem might be caused by the way logarithm was introduced to them. The use of constructivism theory to study the conceptual steps of logarithm proved to be useful since, in this study, the focus was on exploring pre-service mathematics teacher's knowledge of logarithm. Constructivism theory has proved to be useful in these cases as a way a web of concepts can be constructed. What was most prevalent was that constructivism theory provides a relevant framework and lens to understand the development of conceptual understanding of some mathematical topics especially in abstract algebra by pre-service mathematics teachers. Therefore, it could lead to the design of more effective instructional methods in the teaching of logarithm at any level.

5.6 Recommendations

The recommendations derived from this study are structured under the following headings: (1) pedagogical instructions, (2) re-examining the content of logarithm.

5.6.1 Pedagogical instructions

Part of the rationale of this study was to bring new knowledge to the teaching of the logarithm. As Dubinsky (2002) suggested, before pedagogical strategies are considered, the concepts that give students difficulties in linear algebra need to be analyzed epistemologically. The researcher has observed that pre-service mathematics teachers mostly had difficulties with conceptualizing logarithmic concepts. According to Tziritas (2011), students need to perform mathematical tasks, discuss their results and listen to fellow students and lecturer. On the other side, the lecturer needs to provide a theoretical analysis modelling the epistemology of the concepts in which the specific mental constructions that a learner might make in order to develop his/ her understanding are described. This provides an opportunity for an inter-play between teaching and learning since both lecturers and pre-service mathematics teachers are constantly evaluating the knowledge learnt and knowledge provided. Teaching for meaning goes beyond solving routinized problems, it requires pre-service mathematics teachers to be part and parcel of the learning activities which will instil in them the skills and knowledge to explore meaning and reasoning.

As part of pedagogical considerations, this study explores how pre-service mathematics teachers conceptualize logarithm. This is hoped to result in instructional treatment that would guide pre-service mathematics teachers to make necessary mental constructions relevant to logarithm and leads to improvement of their understanding of relevant concepts. The researcher makes two suggestions about the teaching of the logarithm. These are based on what transpired in the results of the study. First, it is important that pre-service mathematics teachers have a sufficient view of the logarithm concepts not only as concrete concepts but also the abstract nature of it. Therefore, the teaching of logarithm should involve problems that encourage pre-service mathematics teachers to explain their thinking strategies. Teaching should not only focus on solving problems, but students should be provided with opportunities to talk about their solution. Secondly, it is insufficient to only examine the mental constructions that pre-service mathematics teachers make. It is also important to analyse those mental constructions that pre-service mathematics teachers could not make and the possible reasons that cause them to fail to make those mental constructions. Therefore, the teaching would then focus on addressing those challenges. It is therefore recommended that lecturers in the mathematics discipline try to design teaching material that targets the development of conceptual understanding of the concepts by helping pre-service mathematics teachers make the necessary mental constructions of the learnt concept.

5.6.2 Re-examining the content of logarithm

Given the esoteric nature of logarithms, it seems clear that there is a need to devise different instructional programs in an attempt to alleviate pre-service mathematics teachers' misconceptions and the belief that mathematics is a rigid system of polished formalism. Today's curriculum presents logarithms as a simple exponent relationship; however, the topic of logarithms is more complex than this, and it has a long and rich history of work and improvements. Knowing how to solve simple logarithmic equations is not enough but pre-service mathematics teachers need to see its application to other concepts such as sequence and series, calculus as well as the application of logarithm to real life. Usiskin (2015) pointed out for students to understand the mathematics they need to see its application to real life. It is the researcher's view, based on the findings of this study that pre-service mathematics teachers need to start to engage with abstract algebra as early as in their first year of study. Abstract algebra could be a setting in which pre-service mathematics teachers develop a deep sense of the nature and role of definitions and proofs in mathematics (Wasserman, 2016). Therefore, if we hope for our secondary school learners to develop the sense of mathematical reasoning, then at the outset the same idea needs to be instilled in the teachers. The better place to start is with pre-service mathematics teachers, especially in their first year. It is insufficient for pre-service mathematics teachers to only have a concrete view of concepts such as laws of logarithm and its application in solving problems. Therefore, pre-service mathematics teachers need to develop a sufficient sense of dealing with more abstract concepts in order to do justice in the teaching of these concepts at the school level.

5.7 Limitations of the study and suggestion for further exploration

This study has some limitations. First, this was a small-scale study with 19 pre-service mathematics teachers in one university, so the results could not be generalized to the other universities. We are aware that variables differ from one setting to the other and from one discipline to the other. Second, the issue of Tutor/ researcher might have impacted in the way pre-service mathematics teachers presented their responses in the research task and also in the interviews. Pre-service mathematics teachers might have tried to get as much information as they could from the textbook in order to produce a correct answer. Also, the pre-service mathematics teachers might not have spoken freely during the interview since they might have felt they must present their answers in a particular way. However, the researcher did address some aspects of biasness by interviewing selected participants who volunteered to take part in the interview. Also, by allowing them to speak freely during the interview session. Moreover, participants were encouraged to ask the researcher anything they wanted to know relating to the study.

This study explored pre-service mathematics teachers' knowledge of logarithm in one of the universities in KwaZulu-Natal. It would be interesting to explore pre-service mathematics teachers' knowledge using other frameworks and compare the analysis of results. Furthermore, the research can also be carried out in other universities in South Africa, to compare the findings. This will improve instructional methods and develop a deeper understanding of logarithmic concepts among pre-service mathematics teachers. The study can also be carried out concerning student skills and belief about logarithm in the South African context.

5.8 Conclusion

This study has explored pre-service teachers' knowledge of logarithm in one particular university in Kwa-Zulu Natal Province. The entire report was structured into five chapters as presented previously. This concluding chapter commenced with the researcher revisiting the aims of this study. A summary of the research study followed. Within this summary key aspects directly related to each critical research question were discussed. The overall aim of the study was to explore pre-service mathematics teachers' knowledge of logarithm. The overarching questions guiding the study was what pre-service mathematics teacher know about logarithm, what difficulties they encounter and how they conceptualized logarithm. Based on data collected in this study, findings show that pre-service mathematics teachers do not have a good knowledge of logarithm. This was evident in the responses on the task in which the overall performance was 11.9% with the highest score being 77.1% and the lowest score being 0%. Furthermore, the study also found that pre-service teachers also have difficulties in solving problems involving logarithm. This was evident in both in the responses to the task particularly the one involving the use of K-method and also through the interviews. It is therefore concluded that there is a great need to bridge a gap between expected and acquired school mathematics knowledge for the pre-service mathematics teachers during their training at the university.

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APPENDICES

Appendix A: Research Questionnaire

Kindly answer the following questions. Tick where it applies to you. Please note that all your responses will be treated with confidentiality.

Gender;

Male ☐

Female ☐

After my Matriculation Mathematics examination, I got between;

40 – 49 ☐

50 – 59 ☐

60 – 69 ☐

70 – 79 ☐

80 and above ☐

I did EDMA110;

Yes ☐

No ☐

I like logarithm

Yes ☐

No ☐

Maybe ☐

I can teach logarithm well after my graduation

Yes ☐

No ☐

Maybe ☐

I will like to participate further in this study

Yes ☐

No ☐

If Yes, please provide us with your;

Contact number: ____

Email address: ____

THANK YOU FOR YOUR TIME.

Appendix B: Logarithm Assessment Task

Time: 45 minutes

Student Number:

INSTRUCTIONS

1. Answer all questions.
2. Answer the question in the space provided.
3. All the questions have equal marks each.

Question 1

If $\log_9 7 = A$ and $\log_9 10 = B$ find $\log_9 810 + \log_9 63$ in terms of A and B?

Question 2

Solve for x in the following, where $x \in R$.

2.1 $\log_2 x + \log_2 5 = 3$.

2.2 $\log_{12}(3 - x) + \log_{12}(2 - x) = 1$.

2.3 $27^{\log_3 x} = 8$.

Question 3

Prove that $\log\left(\frac{50^{\log 2}}{2^{\log 5}}\right) = \log 2$.

Question 4

Find the value(s) of x for which $2\log_9 x + 6\log_x 9 = 7$.

Question 5

Sketch the graph of the function $y = \log_2 x$ indicating the intercepts and the point where y = 2.

Appendix C: Solutions to the research task

Question 1

$= \log_9(81 \times 10) + \log_9(9 \times 7)$

$= \log_9(81) + \log_9(10) + \log_9(9) + \log_9(7)$

$= \log_9(9^2) + \log_9(10) + 1 + \log_9(7)$

$= 2 + \log_9(10) + 1 + \log_9(7)$

$= 3 + A + B$

Question 2

2.1 $\log_2(x \times 5) = 3$

$\log_2 5x = 3$

$\therefore 5x = 2^3$

$5x = 8$

$x = \frac{8}{5}$

2.2 $\log_{12}(3 - x)(2 - x) = 1$

$(3 - x)(2 - x) = 12^1$

$6 - 5x + x^2 = 12$

$x^2 - 5x - 6 = 0$

$(x - 6)(x + 1) = 0$

$x = 6 \; and \; x = -1$

2.3 $\log 27^{\log_3 x} = \log 8$

$\log_3 x \log 27 = \log 8$

$\log_3 x = \frac{\log 8}{\log 27}$

$\log_3 x = \frac{3 \log 2}{3 \log 3}$

$\log_3 x = \frac{\log 2}{\log 3}$

$x = 3^{\frac{\log 2}{\log 3}} \rightarrow x = 2$

Question 3

$= \log 50^{\log 2} - \log 2^{\log 5}$

$= \log 2 \log 50 - \log 5 \log 2$

$= \log 2\,(\log 50 - \log 5)$

$= \log 2 \left(\log \frac{50}{5}\right)$

$= \log 2 (\log 10)$

$= \log 2\,(1)$

$= \log 2$

Question 4

$\frac{2}{\log_x 9} + 6 \log_x 9 = 7$

$let\ k = \log_x 9$

$\frac{2}{k} + 6k = 7$

$6k^2 - 7k + 2 = 0$

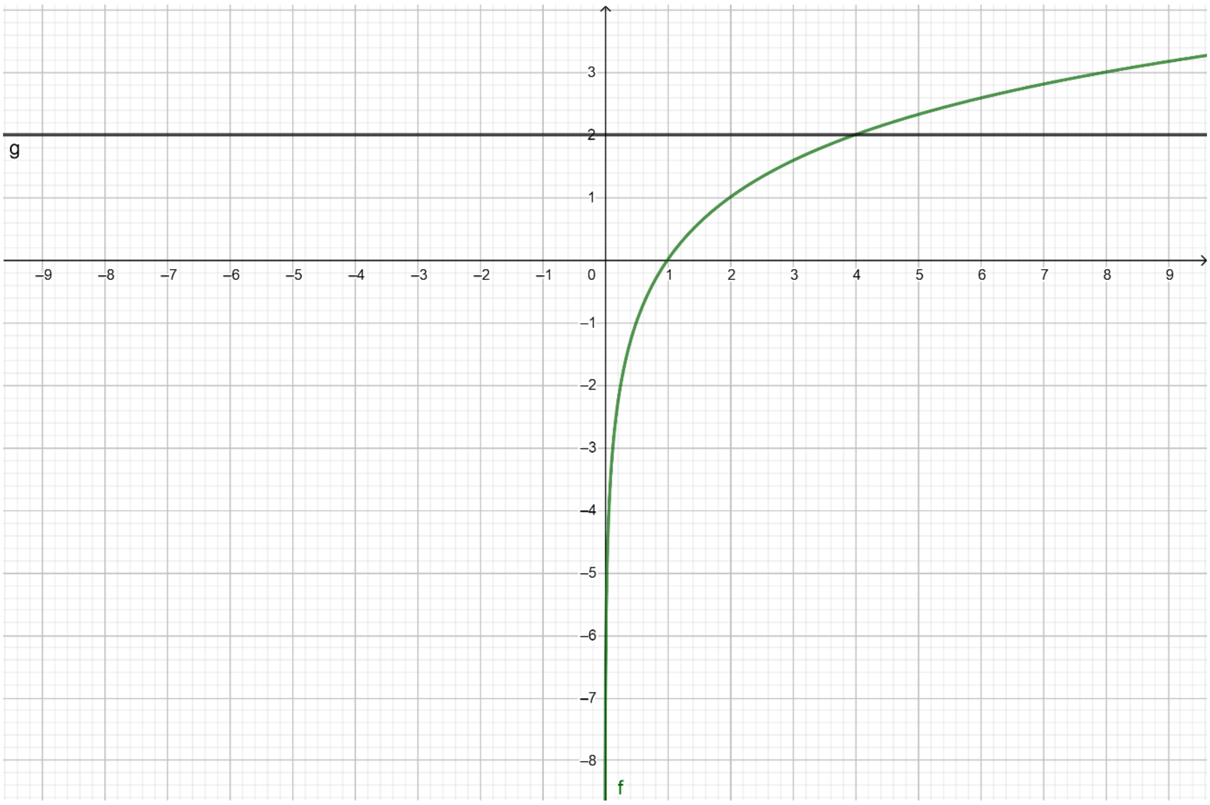
$(3k - 2)(2k - 1) = 0$

$k = \frac{2}{3}\ and\ k = \frac{1}{2}$

$\log_x 9 = \frac{2}{3}\ \therefore x^{\frac{2}{3}} = 9 \rightarrow x = 27$

$\log_x 9 = \frac{1}{2} \therefore x^{\frac{1}{2}} = 9 \rightarrow x = 3$

Question 5



Appendix D: Semi-Structured Interview Schedule

Code:

Researcher	
Interviewee	
Date	
Time	

Questions:

1. Can you tell me when first you were introduced to logarithm?
2. How was logarithm introduced to you?
3. Did you have any prior knowledge that links to logarithm? What is it?
4. Do you know about the history of logarithm?
5. What experience do you have with logarithm?
6. Looking at your task, can you explain to me how you solve the questions?
7. Can you be able to apply logarithm in solving other problems? Can you give me an example?
8. Can you be able to prove logarithm properties?
9. Looking at the solutions of the questions, can you identify where you went wrong in your solution? Why so?
10. Can you explain to me how you sketch the graph of the logarithm function of question 5?
11. From your response in the questionnaire, you said you like logarithm and you can teach it, can you explain more why you said this?

Appendix E: Letter of Permission to the Registrar

1406 Nedbank Plaza,
Scottsville,
3201 Pietermaritzburg.

izuchukwuokoyeogbalu@yahoo.com

The Registrar
University of KwaZulu-Natal
16 April 2018

Dear Sir,

Letter of Permission

My name is Mr. Izuchukwu Okoye-Ogbalu. I am a master's student studying at the University of KwaZulu-Natal, Edgewood Campus. The research I wish to conduct for my master's dissertation involves the exploration of pre-service mathematics teachers' knowledge of logarithm in mathematics'. The research focuses on explaining what constitute a good knowledge of Logarithm or what difficulties pre-service mathematics teachers' encounter with logarithm.

This letter serves as a formal request to ask for your kind permission to conduct this research with the undergraduate mathematics education students registered are the University of KwaZulu-Natal, Edgewood campus. I believe that the undergraduate mathematics education students will be the best choice because they are neither a novice nor an expert with the concept of logarithm. I am most interested in engaging with 10 students which I will sample using a questionnaire from the undergraduate students.

I would like to begin the data collection process in May 2018. The programme of data collection begins by selecting the students and then proceeds to an interview. The interview will take place between myself and the 10 selected students.

Please feel free to contact me if you have any queries. Alternatively, you may wish to contact my supervisor, Dr. Themba Mthethwa on 031 260 2634, if you would like a reference or other information.

Thanks for your anticipated consideration and I hope to hear from you soon.

Yours sincerely,

Mr. I.R. Okoye-Ogbalu

DECLARATION

I………………………………………………………………………… (Registrar's name) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent/do not consent to allow the student to carry out the research project.

SIGNATURE OF REGISTRAR DATE

……………………………………… …………………

Appendix F: Letter of Permission to the Dean

1406 Nedbank Plaza,
Scottsville,
3201 Pietermaritzburg.
izuchukwuokoyeogbalu@yahoo.com

The Dean,
College of Humanities
University of KwaZulu-Natal
16 April 2018

Dear Sir,

Letter of Permission

My name is Mr. Izuchukwu Okoye-Ogbalu. I am a master's student studying at the University of KwaZulu-Natal, Edgewood Campus. The research I wish to conduct for my master's dissertation involves the exploration of pre-service mathematics teachers' knowledge of logarithm in mathematics'. The research focuses on explaining what constitute a good knowledge of Logarithm or what difficulties pre-service mathematics teachers' encounter with logarithm.

This letter serves as a formal request to ask for your kind permission to conduct this research with the undergraduate mathematics education students registered are the University of KwaZulu-Natal, Edgewood campus. I believe that the undergraduate mathematics education students will be the best choice because they are neither a novice nor an expert with the concept of logarithm. I am most interested in engaging with 10 students which I will sample using a questionnaire from the undergraduate students.

I would like to begin the data collection process in May 2018. The programme of data collection begins by selecting the students and then proceeds to an interview. The interview will take place between myself and the 10 selected students.

Please feel free to contact me if you have any queries. Alternatively, you may wish to contact my supervisor Dr. Themba Mthethwa on 031 260 2634, if you would like a reference or other information.

Thanks for your anticipated consideration and I hope to hear from you soon.

Yours sincerely,

Mr. I.R. Okoye-Ogbalu

DECLARATION

I……………………………………………………………………………… (Dean's name) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent/do not consent to allow the student to carry out the research project.

SIGNATURE OF DEAN DATE

……………………………………… …………………

Appendix G: Informed Consent Letter

School of Education

College of Humanities

Edgewood Campus

University of KwaZulu-Natal

Dear Participant

INFORMED CONSENT LETTER

My name is Mr Izuchukwu Okoye-Ogbalu and I am a Master of Education candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in exploring pre-service teacher's knowledge of logarithm in mathematics. To gather the information, I am interested in asking you some questions.

Please note that:

- Your confidentiality is guaranteed as your inputs will not be attributed to you in person but reported only as a population member opinion.
- The task will last for 45 minutes.
- The interview may last for about 45 minutes to 1 hour.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- You have a choice to participate, not participate or stop participating in the research. You will not be penalized for taking such an action.
- Your involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are willing to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

Equipment	**Willing**	**Not willing**

Audio equipment		

I can be contacted at:

Email: izuchukwuokoyeogbalu@yahoo.com

Cell: 0817815707

My supervisor is Dr. Themba Mthethwa who is located at the School of Education, Edgewood campus of the University of KwaZulu-Natal.

Contact details: email: mthethwat@ukzn.ac.za Phone number: +27312602634.

You may also contact the Research Office through:

Ms P Ximba (HSSREC Research Office)

Tel: 031 260 3587

Email: ximbap@ukzn.ac.za)

Thank you for your contribution to this research.

DECLARATION

I……………………………………………………………………………… (full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent/do not consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT DATE

………………………………………… …………………

SIGNATURE OF PARENT (If participant is a minor) DATE

…………………………………………… …………………..

Appendix H: Approval Letter from the Dean (Gate Keeper)

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

2 May 2018

Mr Izuchukwu Rapuluchukwu Okoye-Ogbalu (SN 214572802)
School of Education
College of Humanities
Edgewood Campus
UKZN
Email: izuchukwuokoyeogbalu@yahoo.com mthethwat@ukzn.ac.za

Dear Mr Okoye-Ogbalu

RE: PERMISSION TO CONDUCT RESEARCH

Gatekeeper's permission is hereby granted for you to conduct research at the University of KwaZulu-Natal (UKZN), towards your postgraduate degree, provided Ethical clearance has been obtained. We note the title of your research project is:

"Exploring pre-service mathematics teachers' knowledge of logarithm in one of the universities in KwaZulu-Natal."

It is noted that you will be constituting your sample by handing out questionnaires, and/or conducting interviews with second year undergraduate mathematics education students on the Edgewood campus.

Please ensure that the following appears on your notice/questionnaire:

- Ethical clearance number;
- Research title and details of the research, the researcher and the supervisor;
- Consent form is attached to the notice/questionnaire and to be signed by user before he/she fills in questionnaire;
- gatekeepers approval by the Registrar.

You are not authorized to contact staff and students using 'Microsoft Outlook' address book. Identity numbers and email addresses of individuals are not a matter of public record and are protected according to Section 14 of the South African Constitution, as well as the Protection of Public Information Act. For the release of such information over to yourself for research purposes, the University of KwaZulu-Natal will need express consent from the relevant data subjects

Data collected must be treated with due confidentiality and anonymity.

Yours sincerely

MR SS MOKOENA
REGISTRAR

Office of the Registrar
Postal Address: Private Bag X54001, Durban, South Africa
Telephone: +27 (0) 31 260 8005/2206 **Facsimile:** +27 (0) 31 260 7824/2204 **Email:** registrar@ukzn.ac.za
Website: www.ukzn.ac.za

1910 - 2010
100 YEARS OF ACADEMIC EXCELLENCE

Founding Campuses: Edgewood, Howard College, Medical School, Pietermaritzburg, Westville

Appendix I: Ethical Clearance Certificate

UNIVERSITY OF KWAZULU-NATAL ™
INYUVESI YAKWAZULU-NATALI

29 May 2018

Mr Izuchukwu Rapuluchukwu Okoye-Ogbalu (214572802)
Scool of Education
Edgewood Campus

Dear Mr Okoye-Ogbalu,

Protocol reference number: HSS/0347/018M
Project Title: Exploring pre-service Mathematics teachers' knowledge of Logarithm in one of the universities in KwaZulu-Natal

Approval Notification – Expedited Application

In response to your application received 19 April 2018, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and the protocol has been granted **FULL APPROVAL.**

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment /modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

....
pp **Professor Shenuka Singh (Chair)**

/ms

Cc Supervisor: Dr Themba Mthethwa
Cc Academic Leader Research: Dr SB Khoza
Cc School Administrator: Ms Tyzer Khumalo

Humanities & Social Sciences Research Ethics Committee
Professor Shenuka Singh (Chair)
Westville Campus, Govan Mbeki Building
Postal Address: Private Bag X54001, Durban 4000
Telephone: +27 (0) 31 260 3587/8350/4557 **Facsimile:** +27 (0) 31 260 4609 **Email:** ximbap@ukzn.ac.za / snymanm@ukzn.ac.za / mohunp@ukzn.ac.za
Website: www.ukzn.ac.za

Appendix J: Turnitin Report

Exploring pre-service mathematics teachers' knowledge of logarithm in one of the universities in KwaZulu-Natal

ORIGINALITY REPORT

8%	3%	3%	2%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Submitted to Texas A&M University, College Station Student Paper	1%
2	Weber, Christof. "Making Logarithms Accessible – Operational and Structural Basic Models for Logarithms", Journal für Mathematik-Didaktik, 2016. Publication	1%
3	Submitted to University of Wales Swansea Student Paper	<1%
4	Submitted to Cardiff University Student Paper	<1%
5	Wheijen Chang. "Impact of constructivist teaching on students' beliefs about teaching and learning in introductory physics", Canadian Journal of Science Mathematics and Technology Education, 01/2005 Publication	<1%
	files.eric.ed.gov	