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Non-Newtonian channel flow – exact solutions

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Abstract

In this short communication exact solutions are obtained for a range of non-Newtonian flows between stationary parallel plates. The pressure driven flow of fluids with a variational viscosity that adheres to the Carreau governing relationship are considered. Solutions are obtained for both shear-thinning (viscosity decreasing with increasing shear-rate) and shear-thickening (viscosity increasing with increasing shear-rate) flows. A discussion is presented regarding the requirements for such analytical solutions to exist. The dependence of the flow rate on the channel half width and the governing non-Newtonian parameters is also considered. Non-Newtonian; Exact Solutions; Plane Poiseuille Flow

1 Introduction

It is not unreasonable to assume that any introductory undergraduate course in fluid mechanics will consider the problem of plane Poiseuille flow. The steady, incompressible, two-dimensional, pressure driven, viscous flow between parallel plates is often the first problem that students will encounter. The simplicity of the flow regime allows one to derive an analytical expression for the unidirectional flow using only a basic understanding of integration techniques and the no-slip boundary condition. This simple problem can then be extended to consider a variety of more sophisticated flow configurations. The problem can also be extended to consider *non-normal* fluids, those being fluids that are not necessarily incompressible or Newtonian in nature. However, in these cases, one often finds that what was previously a very simple problem quickly becomes too complex to solve via analytical means and instead numerical solutions methods must be employed. There are, of course, exceptions to this rule. For instance, if one considers the flow of a non-Newtonian fluid with viscosity governed by the power-law relationship, in Cartesian coordinates x and y, it is not too onerous to show that the velocity distribution between two parallel plates located at $y = -h$, and $y = h$, is given by

$$u(y) = \frac{n}{n+1}\left|\frac{G}{m}\right|^{1/n}[h^{(n+1)/n} - |y|^{(n+1)/n}],$$

where u is the streamwise velocity, n is the power-law index, the pressure gradient is $G = -\mathrm{d}p/\mathrm{d}x$ and m is the consistency coefficient. The classical Newtonian solution $u(y) = G(h^2 - y^2)/2\mu_c$, is returned in the case when $n = 1$ and $m = \mu_c =$ constant. As is always the case with problems involving the power-law viscosity relationship the model breaks down when the shear-rate is notionally equal to zero. In this example the breakdown occurs at the point when $y = 0$, at this point, according to this model, the fluid viscosity is either infinitely large (shear-thinning flows) or vanishingly small (shear-thickening flows). These results invalidate the use of the power-law model for generalised Newtonian channel flow. It is, however, worth noting that this solution can be used in scenarios when the shear-rate is regulated within the channel. A good example of this would be the plane Poiseuille flow of a visco-plastic Herschel-Bulkley fluid. In regions where the shear-rate is above some prescribed limit the power-law solution is valid and can be used to describe the shear-thinning (or shear-thickening) properties of the fluid alongside its yielding properties.

The Carreau viscosity model (Carreau, 1972) does not suffer from the same shortcomings as the power-law model and is able to accurately describe the flow of both shear-thinning and shear-thickening fluids for all shear rates (Lashgari *et al.*, 2012). As noted by Nouar & Frigaard (2009), when considering this model, explicit solutions for the velocity profile within a channel formed by parallel plates exist only in a number of limited cases. The focus of this article will be on the instances when these analytical solutions do exist and the form that these solutions take.

There are a limited number of instances when analytical solutions exist to non-Newtonian flow problems. Because of this, the literature concerning these types of problems is somewhat limited. Ferrás *et al.* (2012) obtain analytical solutions to Couette–Poiseuille flow problems for a range of generalised Newtonian fluids whilst considering the addition of wall slip. Solutions are obtained for fluids with viscosity adhering to the power-law, Sisko, Herschel-Bulkley and Robertson-Stiff models for a variety of different slip-laws. The authors find that analytic and semi-analytic solutions can be obtained for a variety of different slip models for both Newtonian and non-Newtonian flows. They also note that exact solutions to the Sisko fluid problem are only realisable for certain values of the fluid index. Oliveira & Pinho (1999) have demonstrated that analytical solutions are achievable for fully developed channel and pipe flows for visco-elastic fluids modelled by either the linear or exponential forms of the Phan-Thien–Tanner (PTT) constitutive viscosity law. Their solutions reveal that the wall shear stress of a PTT fluid is substantially smaller than the corresponding value for a Newtonian or upperconvected Maxwell fluid. Fetecau *et al.* (2009) utilise the concept of fractional calculus and Fourier and Laplace transforms in order to determine analytical solutions for the the flow of Oldroyd-B fluids induced by a constantly accelerating plate, providing a rare example of exact, unsteady solutions to a visco-elastic flow problem. A similar methodology is also presented by Jamil & Fetecau (2010) to obtain exact solutions to rotating generalised Burgers' flow problems in cylindrical domains. It is interesting to note that the literature regarding exact solutions to non-Newtonian flow problems appears to be largely dominated by the flow of visco-elastic fluids. One study of note that considers the Carreau fluid model is that of Peralta *et al.* (2017). The authors consider analytical solutions to the problem of the free-draining flow of a Carreau fluid on a vertical plate. They obtain expressions for the volumetric flow rate, per unit width, in terms of Gauss' hypergeometric function. As is highlighted in §3 of this study, volumetric flow rates, per unit depth, for specific values of the fluid index, are also expressed in terms of this power series function.

This article is organised as follows; in the next section the problem is formulated in a manner that makes it amenable to analytical study. In §3 results are presented for a range of shear-thinning and shear-thickening fluid indices. These results are obtained via a number of different hyperbolic transformations that serve to significantly simplify the process of integrating the shear-rate function. In the final section the results are briefly discussed as are extensions to the framework introduced here.

2 Formulation

Consider the incompressible, two-dimensional flow of a non-Newtonian fluid between two stationary parallel plates located at $y = -h$, and $y = +h$. The flow is driven by the pressure gradient $G = -\mathrm{d}p/\mathrm{d}x$.

The velocity distribution between the plates is determined from the following governing equation

$$-G = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}. \tag{1}$$

The stress tensor $\boldsymbol{\tau}$, is defined like so

$$\boldsymbol{\tau} = \mu(\dot{\gamma})\dot{\boldsymbol{\gamma}},$$

where $\dot{\boldsymbol{\gamma}} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}$ is the rate of strain tensor. Its magnitude, the shear-rate, is determined from the second invariant of the rate of strain tensor, $\dot{\gamma} = \sqrt{\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}/2}$. The non-Newtonian viscosity function, $\mu(\dot{\gamma})$, that will be considered herein, is a function of the shear-rate only.

Given the relative simplicity of this problem (1) can be reduced to the following ordinary differential equation

$$-G = \frac{\mathrm{d}}{\mathrm{d}y}\left[\mu(\dot{\gamma})\frac{\mathrm{d}u}{\mathrm{d}y}\right]. \tag{2}$$

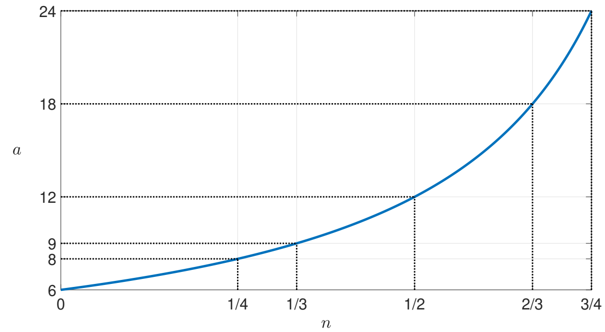

Figure 1: Plot of a against n for a range of shear-thinning indices.

The above is solved subject to the following conditions

$$\dot{\gamma}(y=0) = u(y=+h) = 0. \tag{3}$$

Given the symmetric nature of the problem, these conditions ensure that the criterion of no-slip at the lower boundary is satisfied; $u(y=-h)=0$. For this problem the shear-rate, $\dot{\gamma}$, is identically equal to the absolute value of the velocity gradient, $\dot{\gamma} \equiv |\mathrm{d}u/\mathrm{d}y|$.

The viscosity model that will be considered herein is the Carreau fluid model (Carreau, 1972). In this case the viscosity function takes the form

$$\mu(\dot{\gamma}) = \mu_\infty + (\mu_0 - \mu_\infty)[1 + (\lambda\dot{\gamma})^2]^{(n-1)/2}, \tag{4}$$

where μ_∞ is the infinite shear-rate viscosity, μ_0 is the zero shear-rate viscosity, λ is the characteristic time constant and n is the fluid index. A Newtonian viscosity is recovered when either $n=1$, $\lambda=0$, or $\mu_\infty = \mu_0$. The fluid is said to be shear-thinning when $n<1$, and is shear-thickening when $n>1$. Irrespective of the values of n or λ, a Newtonian viscosity is always recovered at zero shear-rate. It will be instructive, in what follows, to consider the behaviour of this model for both shear-thinning ($n<1$) and shear-thickening ($n>1$) flows in the limit as $\dot{\gamma} \to \infty$. In shear-thinning cases a constant viscosity (μ_∞) is recovered in the limit of large shear-rate. However, the behaviour at infinity is altered for shear-thickening flows, one finds that $\mu(\dot{\gamma} \to \infty) \to (\mu_0 - \mu_\infty)(\lambda\dot{\gamma})^{n-1}$. Therefore, in order to be able to predict a viscosity increase with increasing shear-rate it is a necessary requirement of the Carreau fluid model, even for shear-thickening flows, that $\mu_0 > \mu_\infty$.

It should be noted that the benchmark Newtonian solution is $u(y) = G(h^2 - y^2)/2\mu_c$. It is also worth noting, given the analysis in the forthcoming section, that the volumetric flow rate, per unit depth, for flows of this nature, is defined in the following manner

$$Q = \int_{-h}^{+h} u(y)\,\mathrm{d}y.$$

Given the above, then $Q = 2h^3G/3\mu_c$, in the Newtonian limit. This result will prove useful as a point of reference in §3.

3 Results

In the following section solutions of (2) subject to (3) are obtained for a range of both shear-thinning and shear-thickening values of the fluid index. These two cases are considered in turn. In both cases it transpires that hyperbolic transformations prove useful when integrating the shear-rate function to arrive at closed-form solutions for the variation of fluid velocity within the channel.

3.1 Shear-thinning flows

In reality, for many shear-thinning fluids, the infinite shear-rate viscosity is much smaller than the zero shear-rate viscosity (Bird *et al.*, 1977). Given that $\mu_\infty \ll \mu_0$, the variation of viscosity across the channel can be approximated like so

$$\hat{\mu}(\dot{\gamma}) = \mu_0[1 + n(\lambda\dot{\gamma})^2][1 + (\lambda\dot{\gamma})^2]^{(n-3)/2}. \tag{5}$$

This expression can be used to simplify (2) in the following manner

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = \frac{G}{\hat{\mu}}. \tag{6}$$

Integrating (6) and applying the shear-rate condition one then has that

$$-Gy = \mu_0[1 + (\lambda\dot{\gamma})^2]^{(n-1)/2}\frac{\mathrm{d}u}{\mathrm{d}y} \iff \dot{\gamma} = \frac{G|y|}{\mu_0[1 + (\lambda\dot{\gamma})^2]^{(n-1)/2}}.$$

Given that the fluid index will always be less than unity, the right-hand side of the above can be rewritten in the following manner

$$(\mu_0\dot{\gamma})^{a/3} = (G|y|)^{a/3}[1 + (\lambda\dot{\gamma})^2],$$

where $a = 6/(1-n)$. It is therefore possible to determine polynomial expressions for the shear-rate in the cases when $a \in \mathbb{N}$. Analytical expressions for the streamwise velocity can only exist in the cases when this condition holds. There are a range of shear-thinning fluid indices that ensure that the constant a is indeed a positive integer, a selection of these are highlighted graphically in Figure 1. In addition to this condition, a full parameter search reveals that for shear-thinning flows analytical solutions for the shear-rate function are obtainable for the six cases highlighted in Figure 1, those being $a = 6$, 8, 9, 12, 18, and 24.
Although it is non-physical in nature, the simplest case to consider is the case when $a = 6$, which is equivalent to a flow with zero fluid index. In this case it is trivial to show that the shear-rate is given by

$$\dot{\gamma}(y) = \frac{G|y|}{[\mu_0^2 - (\lambda G y)^2]^{1/2}}.$$

Therefore the solution for u that satisfies the no-slip condition is determined to be

$$u(y) = \frac{[\mu_0^2 - (\lambda G y)^2]^{1/2} - [\mu_0^2 - (\lambda G h)^2]^{1/2}}{\lambda^2 G}. \tag{7}$$

In the case when $n = 0$, the expression for the variation of the viscosity across the channel has a simple form, $\hat{\mu}(y) = \mu_0^{-2}[\mu_0^2 - (\lambda G y)^2]^{3/2}$. A zero wall viscosity is predicted in the case when $\mu_0 = \lambda G h$, therefore, for this case only, real solutions are obtained only in the instances when $0 < \lambda < \mu_0/(Gh)$. One also finds that a simple expression for the volumetric flow rate can be inferred from the above

$$Q = \int_{-h}^{+h} u(y)\,\mathrm{d}y = \frac{\mu_0^2 \arcsin(\lambda G h/\mu_0) - (\lambda G h)[\mu_0^2 - (\lambda G h)^2]^{1/2}}{\lambda^3 G^2}.$$

As is to be expected in the limit as $\lambda \to 0$, the Newtonian solution for Q is recovered.
The cases when $a = 8$ and $a = 24$, which are equivalent to flows with fluid index equal to one-quarter and three-quarters, respectively, will be considered together towards the end of this subsection. In both instances an expression for the shear-rate can be obtained from the a modified quartic equation. Because of the similarity between the two cases it is logical to address them in a generic fashion. The next case to consider is then the case when $a = 9$, which is equivalent to a flow with fluid index equal to one-third. A real solution for the shear-rate is obtained from a cubic equation, giving

$$\dot{\gamma}(y) = \frac{G|y|}{3\mu_0^3}\left[(\lambda G y)^2 + (p_y^+)^2 + \frac{(\lambda G y)^4}{(p_y^+)^2}\right],$$

where

$$p_k^{\pm} = \left\{ \frac{\pm 3\sqrt{3}\mu_0^3 + [27\mu_0^6 + 4(\lambda G k)^6]^{1/2}}{2} \right\}^{1/3}.$$

An expression for the variation of viscosity across the channel can then be directly inferred from (5). The solution for u is obtained from the sum of the following integrals

$$\begin{aligned}
\mathcal{I}_1 &= -\frac{\lambda^2 G^3}{3\mu_0^3} \int y^3 \, \mathrm{d}y = -\frac{\lambda^2 G^3 y^4}{12\mu_0^3} + c_1, \\
\mathcal{I}_2 &= -\frac{G}{3\mu_0^3} \int y (p_y^+)^2 \, \mathrm{d}y, \\
\mathcal{I}_3 &= -\frac{\lambda^4 G^5}{3\mu_0^3} \int \frac{y^5}{(p_y^+)^2} \, \mathrm{d}y.
\end{aligned}$$

The integrals $\mathcal{I}_2$ and $\mathcal{I}_3$ can be computed via the following hyperbolic transformation

$$y = \frac{\mu_0}{\lambda G} \left[\frac{3\sqrt{3}\sinh(z)}{2} \right]^{1/3}.$$

Resulting in the following

$$\begin{aligned}
\mathcal{I}_2 &= -\frac{\mu_0}{2\sqrt[3]{2}\lambda^2 G} \int [\sinh(z)]^{-1/3} \cosh(z) [1 + \cosh(z)]^{2/3} \, \mathrm{d}z \\
&= -\frac{3\mu_0 [\sinh(z)]^{2/3} [3 + \cosh(z)] [1 + \cosh(z)]^{-1/3}}{8\sqrt[3]{2}\lambda^2 G} + c_2, \\
\mathcal{I}_3 &= -\frac{\mu_0}{2\sqrt[3]{2}\lambda^2 G} \int \sinh(z) \cosh(z) [1 + \cosh(z)]^{-2/3} \, \mathrm{d}z \\
&= +\frac{3\mu_0 [3 - \cosh(z)] [1 + \cosh(z)]^{1/3}}{8\sqrt[3]{2}\lambda^2 G} + c_3.
\end{aligned}$$

Inverting the transformation gives

$$\mathcal{I}_2 = -\frac{G\phi_y^+ y^2}{24\mu_0^3 p_y^+} + c_2, \qquad \mathcal{I}_3 = +\frac{G\phi_y^- p_y^+}{24\mu_0^3 \lambda^2} + c_3,$$

where $\phi_k^{\pm} = 9\sqrt{3}\mu_0^3 \pm [27\mu_0^6 + 4(\lambda G k)^6]^{1/2}$. The constants c_i, are determined such that the solution for u satisfies the condition of no-slip at the walls. The solution for u is then determined to be

$$u(y) = \frac{G}{24\mu_0^3} \left[2(\lambda G)^2 (h^4 - y^4) + \frac{\phi_h^+ h^2}{p_h^+} - \frac{\phi_y^+ y^2}{p_y^+} + \frac{(\phi_y^- p_y^+ - \phi_h^- p_h^+)}{(\lambda G)^2} \right]. \tag{8}$$

Solutions for a range of non-zero λ values are overleaf in Figure 2. There is a special case to consider when $\lambda = \sqrt[6]{2}\sqrt{3}\mu_0/(Gh)$. In this case $\phi_h^- = 0$, and $\phi_h^+ = 18\sqrt{3}$, resulting in a considerably simplified expression for the fluid velocity

$$u(Y) = \frac{\sqrt[3]{2} G h^2}{4\mu_0} \left[\frac{3}{\sqrt[3]{4}} + 1 - Y^4 + \frac{(3 - \sqrt{1 + 8Y^6})(1 + \sqrt{1 + 8Y^6})^{2/3} - 2Y^2 (3 + \sqrt{1 + 8Y^6})}{4(1 + \sqrt{1 + 8Y^6})^{1/3}} \right],$$

where $Y = y/h$.

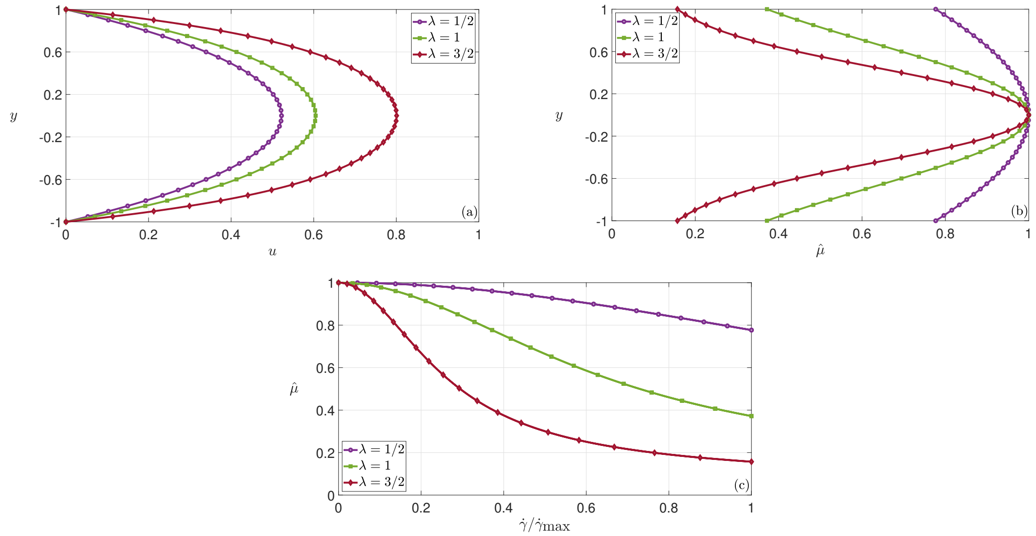

Figure 2: Plots of the analytical solutions for the variation of velocity (a) and viscosity (b) across the channel for flows with $n = 1/3$, for a range of dimensional λ values. A plot of the variation of viscosity with shear-rate (c) has also been included, where $\dot{\gamma}_{\text{max}} = u'(-h) = u'(+h)$. In this case the channel half width, pressure gradient and zero shear-rate viscosity have all been set equal to 1.

Somewhat surprisingly, given the relative complexity of the generalised solution for u (8), an analytical solution for Q is obtainable

$$
\begin{aligned}
Q = \int_{-h}^{+h} u(y)\,\mathrm{d}y &= \frac{hG}{12\mu_0^3}\left[2(\lambda G)^2h^4 + \frac{\phi_h^+ h^2}{p_h^+} - \frac{\phi_h^- p_h^+}{(\lambda G)^2}\right] - \frac{G(\lambda G)^2h^5}{30\mu_0^3} \\
&\quad + \frac{\sqrt[3]{2}\sqrt{3}\mu_0^2}{8\lambda^3G^2}\int_0^{\alpha}\cosh(z)[3-\cosh(z)]\left[\frac{1+\cosh(z)}{\sinh^2(z)}\right]^{1/3}\mathrm{d}z \\
&\quad - \frac{\sqrt[3]{2}\sqrt{3}\mu_0^2}{8\lambda^3G^2}\int_0^{\alpha}\cosh(z)[3+\cosh(z)]\left[\frac{1}{1+\cosh(z)}\right]^{1/3}\mathrm{d}z \\
&= \frac{2G(\lambda G)^2h^5}{15\mu_0^3} + \frac{3\sqrt{\pi}\mu_0^2}{20\lambda^3G^2}\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})} + \frac{Gh^3}{15p_h^-}\left[2\left(\frac{p_h^-}{\mu_0}\right)^3 - 3\sqrt{3} - \mathbb{R}(N_2)\right] \\
&\qquad + \frac{Gh^3}{15p_h^+}\left[2\left(\frac{p_h^+}{\mu_0}\right)^3 + 3\sqrt{3} - \mathbb{R}(N_6)\right].
\end{aligned}
$$

Where the upper limit of the integrals is given by $\alpha = \operatorname{arcsinh}\{[\sqrt[3]{2}\lambda Gh/(\sqrt{3}\mu_0)]^3\}$, and

$$
N_j = \left\{\frac{3\sqrt{3}\mu_0^3 - [27\mu_0^6 + 4(\lambda Gh)^6]^{1/2}}{162\sqrt{3}\mu_0^3}\right\}^{-j/12} {}_2F_1\left(\frac{1}{j},\frac{j+3}{3j};\frac{j+1}{j};\left(\frac{p_h^+}{\sqrt{3}\mu_0}\right)^3\right).
$$

Here ${}_2F_1(a,b;c;x)$ is Gauss' hypergeometric function as defined by Abramowitz & Stegun (1972). Thus, in this case, $Q = \mathcal{O}(h^5)$, with correction terms obtained from the evaluation of the hyperbolic integrals. Of note is the lone term involving the gamma function. It does not depend on the channel half width and for flows in thin channels with moderate values of the characteristic time constants this term dominates. It appears from the evaluation of the first hyperbolic integral at the lower limit when $z = 0$. At first sight this term may appear to be non-physical in nature, predicting a non-zero flow rate in the limit as $h \to 0$. However, as the channel half width vanishes the contribution from the integrals reduces to zero and $Q \to 0$, as $h \to 0$, as required. It should be noted that one need only take the real part of the function N_j for the cases when $\lambda > \sqrt[6]{2}\sqrt{3}\mu_0/(Gh)$.

The analysis of the case when $a = 12$ ($n = 1/2$) is somewhat simpler than its predecessor. The expression for the shear-rate is determined to be

$$\dot{\gamma}(y) = \frac{G|y|}{\mu_0^2}\left\{\frac{(\lambda G y)^2 + [4\mu_0^4 + (\lambda G y)^4]^{1/2}}{2}\right\}^{1/2}.$$

As before, the solution for u is then obtained from the integral of the velocity gradient, this calculation can be simplified via a slightly different hyperbolic transformation $y = \mu_0\sqrt{2\sinh(z)}/(\lambda G)$. One then finds that

$$\begin{aligned}\mathcal{J} = \int \frac{\mathrm{d}u}{\mathrm{d}y}\,\mathrm{d}y &= -\frac{\mu_0}{\lambda^2 G}\int \cosh(z)\sqrt{\sinh(z)+\cosh(z)}\,\mathrm{d}z \\ &= -\frac{2\mu_0[2\sinh(z)-\cosh(z)]\sqrt{\sinh(z)+\cosh(z)}}{3\lambda^2 G} + c.\end{aligned}$$

Inverting the transformation gives the following

$$\mathcal{J} = -\frac{[(\lambda G y)^2 + \chi_y^-](\chi_y^+)^{1/2}}{3\sqrt{2}\mu_0^2\lambda^2 G} + c,$$

where $\chi_k^{\pm} = (\lambda G k)^2 \pm [4\mu_0^4 + (\lambda G k)^4]^{1/2}$. Once again the constant c is determined such that the solution for u that satisfies the no-slip condition

$$u(y) = \frac{[(\lambda G h)^2 + \chi_h^-](\chi_h^+)^{1/2} - [(\lambda G y)^2 + \chi_y^-](\chi_y^+)^{1/2}}{3\sqrt{2}\mu_0^2\lambda^2 G}. \tag{9}$$

This solution is considerably simplified when $\lambda = \sqrt{2}\mu_0/(\sqrt[4]{3}Gh)$. In this case, $(\lambda G h)^2 + \chi_h^- = 0$, therefore

$$u(Y) = \frac{Gh^2(\sqrt{3+Y^4} - 2Y^2)(Y^2 + \sqrt{3+Y^4})^{1/2}}{3\sqrt[4]{3}\mu_0}.$$

The calculation for the volumetric flow rate is again simplified when compared to the case when $a = 9$, resulting in the following

$$\begin{aligned}Q = \int_{-h}^{+h} u(y)\,\mathrm{d}y &= \frac{2h[(\lambda G h)^2 + \chi_h^-](\chi_h^+)^{1/2}}{3\sqrt{2}\mu_0^2\lambda^2 G} \\ &\quad - \frac{2\sqrt{2}\mu_0^2}{3\lambda^3 G^2}\int_0^{\beta}\cosh(z)[2\sinh(z)-\cosh(z)]\left[\frac{\sinh(z)+\cosh(z)}{\sinh(z)}\right]^{1/2}\mathrm{d}z \\ &= \frac{1}{2\lambda^3 G^2}\left\{\mu_0^2\ln(\kappa + \sqrt{1+\kappa^2}) - \frac{\kappa[2\mu_0^4 - (\lambda G h)^2\chi_h^+]}{\sqrt{2}[2\mu_0^4 + (\lambda G h)^2\chi_h^+]^{1/2}}\right\}.\end{aligned}$$

Where the upper limit of the integral is given by $\beta = \operatorname{arcsinh}\{[\lambda G h/(\sqrt{2}\mu_0)]^2\}$, and

$$\kappa = \frac{\lambda G h}{\mu_0^2}\left(\frac{\chi_h^+}{2}\right)^{1/2}.$$

In this case the correction term is logarithmic, and $Q = \mathcal{O}(h^4)$. Unlike the previous case when $n = 1/3$, all terms are dependent on the channel half width h.

The next shear-thinning case to consider is the case when $a = 18$, which is equivalent to a flow with fluid index equal to two-thirds. By writing $\dot{\delta} = \dot{\gamma}^2$, it is possible to solve a cubic equation for $\dot{\delta}$ from which an expression for $\dot{\gamma}$ can be inferred. A real solution for the shear-rate is then determined to be

$$\dot{\gamma}(y) = \frac{G|y|}{\sqrt[4]{3\mu_0^6}}\mathbb{R}\left[q_y + \frac{(\lambda G y)^2}{q_y}\right]^{1/2},$$

where

$$q_k = \left\{ \frac{3\sqrt{3}\mu_0^3 + [27\mu_0^6 - 4(\lambda G k)^6]^{1/2}}{2} \right\}^{1/3}.$$

It should be noted that one need only take the real part of the expression for the shear-rate for the cases when $\lambda > \sqrt{3}\mu_0/(\sqrt[3]{2}Gh)$. As before the solution for u is obtained from the integral of the velocity gradient, this calculation can be simplified via yet another hyperbolic transformation

$$y = \frac{\mu_0}{\lambda G}\left[\frac{3\sqrt{3}\,\mathrm{sech}(3z)}{2}\right]^{1/3}.$$

One then finds that

$$\begin{aligned}\mathcal{K} = \int \frac{\mathrm{d}u}{\mathrm{d}y}\,\mathrm{d}y &= \frac{3\mu_0}{\sqrt[3]{2}\lambda^2 G}\int \frac{\sinh(3z)\sqrt{\cosh(z)}}{[\cosh(3z)]^{11/6}}\,\mathrm{d}z \\ &= \frac{3\mu_0}{5\sqrt[3]{2}\lambda^2 G}\frac{[2\cosh(2z) - 3]\sqrt{\cosh(z)}}{[\cosh(3z)]^{5/6}} + c.\end{aligned}$$

Inverting the transformation gives the following

$$\mathcal{K} = \frac{\sqrt{\psi_y}[\psi_y^2 - 5(\lambda G y)^2]}{5\sqrt[4]{3\mu_0^6}\lambda^2 G} + c,$$

where

$$\psi_k = \mathbb{R}\left[q_k + \frac{(\lambda G k)^2}{q_k}\right].$$

Therefore the solution for u that satisfies the no-slip boundary condition is determined to be

$$u(y) = \frac{\sqrt{\psi_y}[\psi_y^2 - 5(\lambda G y)^2] - \sqrt{\psi_h}[\psi_h^2 - 5(\lambda G h)^2]}{5\sqrt[4]{3\mu_0^6}\lambda^2 G} \tag{10}$$

As noted previously there is special case to consider when $\lambda = \sqrt{3}\mu_0/(\sqrt[3]{2}Gh)$. In this case $\psi_h = 2\lambda G h$. The expression for u can therefore be considerably simplified

$$u(Y) = \frac{Gh^2}{5\sqrt[6]{2}\mu_0}\left\{\sqrt{2} + \frac{[(1+\sqrt{1-Y^6})^{4/3} + Y^4 - 3Y^2(1+\sqrt{1-Y^6})^{2/3}][(1+\sqrt{1-Y^6})^{2/3} + Y^2]^{1/2}}{(1+\sqrt{1-Y^6})^{5/6}}\right\}.$$

Given the form of (10) it is again possible to determine an analytical expression for the volumetric flow rate per unit depth, this calculation is a little more involved than its predecessors and is outlined below

$$\begin{aligned}Q = \int_{-h}^{+h} u(y)\,\mathrm{d}y = &-\frac{2h\sqrt{\psi_h}[\psi_h^2 - 5(\lambda G h)^2]}{5\sqrt[4]{3\mu_0^6}\lambda^2 G} \\ &+ \frac{3\sqrt[3]{2}\sqrt{3}\mu_0^2}{5\lambda^3 G^2}\int_\gamma^\infty \frac{\sinh(3z)[2\cosh(2z) - 3]\sqrt{\cosh(z)}}{[\cosh(3z)]^{13/6}}\,\mathrm{d}z \\ = &\frac{h\sqrt{\psi_h}}{7\sqrt[4]{3\mu_0^6}\lambda^2 G}\left\{\frac{27\mu_0^6}{(\lambda G h \psi_h)^2}\frac{\psi_h^2 + (\lambda G h)^2}{[\psi_h^2 - 3(\lambda G h)^2]} - 2[\psi_h^2 - 5(\lambda G h)^2]\right\} \\ &+ \frac{9\sqrt[3]{3}\mu_0^2}{56\lambda^3 G^2}\zeta\left(\frac{\psi_h}{\lambda G h}\right)^{4/3},\end{aligned}$$

where

$$\gamma = \frac{1}{3}\,\text{arcsech}\left[\frac{2}{3\sqrt{3}}\left(\frac{\lambda G h}{\mu_0}\right)^3\right],$$

and

$$\zeta = \mathbb{I}\left\{{}_2F_1\left[\frac{1}{6},\frac{2}{3};\frac{5}{3};\frac{1}{3}\left(\frac{\psi_h}{\lambda G h}\right)^2\right]\right\} - \sqrt{3}\mathbb{R}\left\{{}_2F_1\left[\frac{1}{6},\frac{2}{3};\frac{5}{3};\frac{1}{3}\left(\frac{\psi_h}{\lambda G h}\right)^2\right]\right\}.$$

In this case, to leading order, one observes that $Q = \mathcal{O}(h^{7/2})$. It is also interesting to note that the hypergeometric term plays a significant role in minimising the overall value of the constant Q.
As noted previously, the final two cases, when $a = 8$ ($n = 1/4$) and $a = 24$ ($n = 3/4$), will be considered together. By writing $\dot{\delta} = \dot{\gamma}^2$, it is possible to solve a quartic equation for $\dot{\delta}$ from which an expression for $\dot{\gamma}$ can be inferred. In both cases an expression for the shear-rate function can be determined from a solution of a polynomial of the form $\dot{\delta}^4 + A\dot{\delta}^3 + B\dot{\delta}^2 + C\dot{\delta} + D = 0$. Where the factors A, B, C and D differ depending on the case in question. In the interest of brevity, these multiplicative factors are given in the Appendix. The required solution from the quartic equation noted above is then determine to be

$$\begin{aligned}\dot{\delta} = -\frac{A}{4} + \frac{1}{4}\left[\frac{4\mathcal{D}^2 + (3A^2 - 8B)\mathcal{D} + 4\Delta_2}{3\mathcal{D}}\right]^{1/2} \\ + \frac{1}{4}\left\{\Delta_3\left[\frac{3\mathcal{D}}{4\mathcal{D}^2 + (3A^2 - 8B)\mathcal{D} + 4\Delta_2}\right]^{1/2} - \left[\frac{4\mathcal{D}^2 - 2(3A^2 - 8B)\mathcal{D} + 4\Delta_2}{3\mathcal{D}}\right]\right\}^{1/2},\end{aligned}$$

where

$$\mathcal{D} = \left[\frac{\Delta_1 - (\Delta_1^2 - 4\Delta_2^3)^{1/2}}{2}\right]^{1/3},$$

with

$$\begin{aligned}\Delta_1 &= 2B^3 + 27A^2D - 9ABC + 27C^2 - 72BD,\\ \Delta_2 &= B^2 - 3AC + 12D,\\ \Delta_3 &= -2A^3 + 8AB - 16C.\end{aligned}$$

The shear-rate function is then given by the positive square root of this expression for $\dot{\delta}$. Therefore it remains possible to determine an analytic expression for $\dot{\gamma}$ in terms of μ_0, λ, G and y in these more complex cases. Given the relationship between the velocity gradient and the shear-rate function it is also possible to directly integrate (6) with respect to $\dot{\gamma}$, subject to (3), to arrive at a closed-form solution for the fluid velocity as a function of the shear-rate only

$$u(\dot{\gamma}) = \frac{\mu_0\{[1 - n(\lambda\dot{\gamma})^2][1 + (\lambda\dot{\gamma})^2]^{(n-1)/2} - [1 - n(\lambda s)^2][1 + (\lambda s)^2]^{(n-1)/2}\}}{\lambda^2 G(n+1)}, \tag{11}$$

where $s = \dot{\gamma}(h)$. Integrating (11) once more with respect to $\dot{\gamma}$, results in the following expression for the volumetric flow rate per unit depth

$$Q = \frac{2\mu_0^2 s\{(1 - n^2)H_2 + n^2(2H_1 - H_0) - [1 - n(\lambda s)^2][1 + (\lambda s)^2]^{(n-1)}\}}{\lambda^2 G^2(n+1)}, \tag{12}$$

where

$$H_K = {}_2F_1\left(\frac{1}{2}, K-n; \frac{3}{2}; -(\lambda s)^2\right).$$

Given that in each case an analytic expression for $\dot{\gamma}$ is obtainable, it is then possible to write down solutions for u and Q that depend only on the constants μ_0, λ, G and h, and in the case of the fluid velocity, also the wall-normal coordinate y. To improve readability these solutions are not given explicitly stated here.

This short analysis reveals that it is possible to arrive at closed form solutions for the cases when the shear-rate function is determined from an eighth-order polynomial expression. However, unlike the cases when $n = 1/3$, $1/2$, or $2/3$, the relative complexity of these solutions makes determining special cases, when the expression for the fluid velocity can be reduced to a combination of simpler radical functions, a very difficult prospect indeed. Nevertheless, it remains possible to determine the leading order dependence of the volumetric flow rate on the channel half wide. In the case when $n = 1/4$, to leading order $Q \sim h^6$, whereas $Q \sim h^{10/3}$, when $n = 3/4$.

3.2 Shear-thickening flows

Recall the discussion in §2 that in order to be able to predict shear-thickening behaviour it is a necessary requirement of the Carreau fluid model that $\mu_0 > \mu_\infty$. In the first instance, it will be assumed that the zero shear-rate viscosity is larger than the infinite shear-rate viscosity but that they are of approximately the same order, i.e., the assumption that $\mu_\infty \ll \mu_0$ no longer holds. Integrating (2) and applying the shear-rate condition one then finds that

$$-Gy = \{\mu_\infty + (\mu_0 - \mu_\infty)[1 + (\lambda\dot{\gamma})^2]^{(n-1)/2}\}\frac{\mathrm{d}u}{\mathrm{d}y} \iff \dot{\gamma} = \frac{G|y|}{\mu_\infty + (\mu_0 - \mu_\infty)[1 + (\lambda\dot{\gamma})^2]^{(n-1)/2}}.$$

Given that in shear-thickening cases the fluid index will always be greater than unity, the right-hand side of the above can be rewritten in the following manner

$$[(\mu_0 - \mu_\infty)\dot{\gamma}]^{b/3}[1 + (\lambda\dot{\gamma})^2] = (G|y| - \mu_\infty\dot{\gamma})^{b/3},$$

where $b = 6/(n-1)$. Given the relative complexity of this expression it proves fruitful only to consider the case when $b = 3$, which is equivalent to a highly shear-thickening flow with fluid index equal to three. In the case when $n = 3$, one obtains the following simple expression for the shear-rate

$$\dot{\gamma} = \sqrt{\frac{\mu_0}{3(\mu_0 - \mu_\infty)}}\frac{1}{\lambda}\left(r_y - \frac{1}{r_y}\right),$$

where

$$r_k = \frac{1}{\sqrt{\mu_0}}\left\{\frac{3\sqrt{3(\mu_0 - \mu_\infty)}\lambda G|k| + [4\mu_0^3 + 27(\mu_0 - \mu_\infty)(\lambda G k)^2]^{1/2}}{2}\right\}^{1/3}.$$

As with the shear-thinning analysis the solution for u is then obtained from the integral of the velocity gradient, this calculation requires the use of yet another hyperbolic transformation

$$y = \sqrt{\frac{4\mu_0}{27(\mu_0 - \mu_\infty)}}\frac{\mu_0}{\lambda G}\sinh(3z).$$

As before, this transformation serves to significantly simplify the integration process

$$\mathcal{L} = \int \frac{\mathrm{d}u}{\mathrm{d}y}\,\mathrm{d}y = -\frac{4\mu_0^2}{3(\mu_0 - \mu_\infty)\lambda^2 G}\int \sinh(z)\cosh(3z)\,\mathrm{d}z = \frac{4\mu_0^2\cosh^2(z) - \cosh(4z)}{6(\mu_0 - \mu_\infty)\lambda^2 G} + c.$$

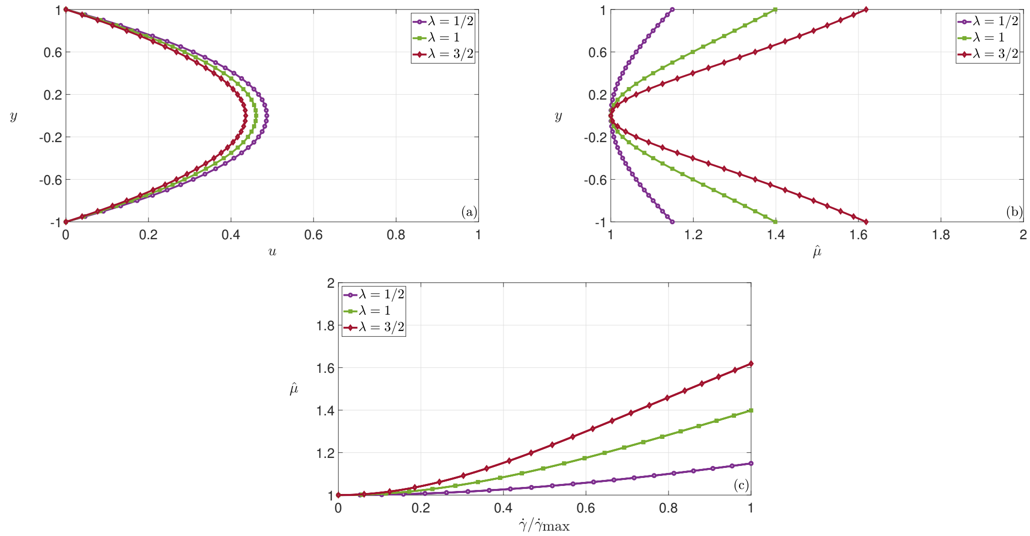

Figure 3: Plots of the analytical solutions for the variation of velocity (a) and viscosity (b) across the channel for flows with $n = 3/2$, for a range of dimensional λ values. A plot of the variation of viscosity with shear-rate (c) has also been included, where $\dot{\gamma}_{\max} = u'(-h) = u'(+h)$. In this case the channel half width, pressure gradient and zero shear-rate viscosity have all been set equal to 1.

Inverting the transformation gives the following

$$\mathcal{L} = \frac{\mu_0^2}{12(\mu_0 - \mu_\infty)\lambda^2 G}\left[4 + 2\left(r_y^2 + \frac{1}{r_y^2}\right) - \left(r_y^4 + \frac{1}{r_y^4}\right)\right] + c.$$

Therefore the solution for u that satisfies the no-slip boundary conditions is determined to be

$$u(y) = \frac{\mu_0^2}{12(\mu_0 - \mu_\infty)\lambda^2 G}\left[2\left(r_y^2 - r_h^2 + \frac{1}{r_y^2} - \frac{1}{r_h^2}\right) - \left(r_y^4 - r_h^4 + \frac{1}{r_y^4} - \frac{1}{r_h^4}\right)\right]. \tag{13}$$

It should be noted that this expression is valid for $0 \leq \mu_\infty < \mu_0$. Exact solutions for other shear-thickening flows are only obtainable if one continues to assume that $\mu_\infty \ll \mu_0$. In this case analytical solutions exist in the instances when $n = 4, 2, 3/2$, and $4/3$. In the interest of brevity the remaining analysis is excluded. The solution methodology for the cases when $n = 2$ and $n = 3/2$, is very similar to that which has gone before. In the case when the fluid index is equal to two, the variation of velocity across the channel has the form

$$u(y) = \frac{\sqrt{\mu_0}\{\mu_0 + [\mu_0^2 + 4(\lambda G y)^2]^{1/2}\}^{1/2} - \sqrt{\mu_0}\{\mu_0 + [\mu_0^2 + 4(\lambda G h)^2]^{1/2}\}^{1/2}}{2\sqrt{2}\lambda^2 G} - \frac{\mu_0}{12\lambda^2 G}\left(s_y^3 - s_h^3 + \frac{1}{s_y^3} - \frac{1}{s_h^3}\right), \tag{14}$$

where

$$s_k = \left\{\frac{2\lambda G|k| + [\mu_0^2 + 4(\lambda G k)^2]^{1/2}}{\mu_0}\right\}^{1/2}.$$

Whilst in the case when $n = 3/2$, the following is obtained

$$u(y) = \frac{\mu_0}{5\sqrt[4]{3}\lambda^2 G}\mathbb{R}\left\{\left[3 - \left(t_y^2 + \frac{1}{t_y^2}\right)\right]\left(t_y + \frac{1}{t_y}\right)^{1/2} - \left[3 - \left(t_h^2 + \frac{1}{t_h^2}\right)\right]\left(t_h + \frac{1}{t_h}\right)^{1/2}\right\}, \tag{15}$$

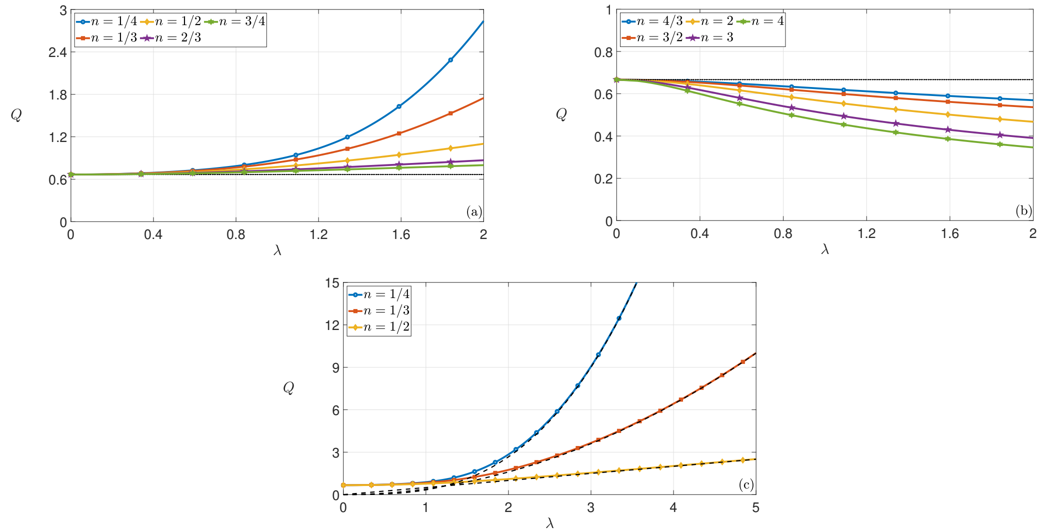

Figure 4: In (a) & (b) the volumetric flow rate per unit depth is plotted as a function of λ for each non-zero value of the fluid index that admits an analytical solution. The Newtonian solution is provided as a point of reference and is given by the dotted black line. To ensure consistency between the results the infinite shear-rate viscosity μ_∞, is set to zero. In (c) a selection of shear-thinning results are compared to the large λ approximate solutions (dashed black lines). In this case the channel half width, pressure gradient and zero shear-rate viscosity have all been set equal to 1.

where

$$t_k = \frac{1}{\sqrt[3]{\mu_0^2}} \left\{ \frac{3\sqrt{3}(\lambda G k)^2 + [27(\lambda G k)^4 - 4\mu_0^4]^{1/2}}{2} \right\}^{1/3}.$$

It should be noted that the real part of the above expression must be considered only for the cases when $\lambda < \sqrt{2}\sqrt[4]{3}\mu_0/(3Gh)$. Solutions for u for a range of non-zero λ values are presented in Figure 3.

The cases when $n = 4$ and $n = 4/3$ prove to be very similar to the shear-thinning cases when $n = 1/4$ and $n = 3/4$. The shear-rate function is again determined form an eighth-order polynomial expression that can be recast as a quartic equation. The factors that multiply each of the terms within this equation are noted alongside their shear-thinning equivalents in the Appendix. Expressions for the fluid velocity and volumetric flow rate per unit depth can then be directly inferred from (11) and (12), respectively.

Analytical solutions for Q are obtainable in all the shear-thickening cases mentioned, those not owing from the quartic equation analysis can be found in the Appendix. The variation of Q with λ for different values of the fluid index, and a fixed value of the channel width, is presented in Figure 4.

4 Discussion and Conclusions

The analysis presented here details one methodology for obtaining exact solutions to the problem of the pressure driven flow, between parallel plates, of a fluid with viscosity that is governed by the Carreau relationship. Excluding the non-physical case (when $n = 0$), explicit solutions can be obtained in ten cases, five shear-thinning cases and five shear-thickening cases. In each of these cases one finds that it is possible to isolate, from a given polynomial, an expression for the shear-rate function. Expressions for the fluid velocity and volumteric flow rate per unit depth can then be inferred from once and twice integrating the shear-rate function, respectively. Given the complexity of the prescribed viscosity law it is somewhat surprising to find that in each of the cases considered, the solutions for the fluid velocity are composed only of simple radical functions. The solutions for the volumetric flow rate are a little more convoluted and, in some instances, are expressed in terms of Gauss' hypergeometric function.

In the limit as $\lambda \to 0$, each of the solutions tend to the familiar Newtonian results

$$\dot{\gamma}(y) = \frac{G|y|}{\mu_c}, \quad u(y) = \frac{G(h^2 - y^2)}{2\mu_c}, \quad Q = \frac{2h^3 G}{3\mu_c}.$$

Conversely, as $\lambda \to \infty$, the solutions can be approximated in the follow manner

$$\dot{\gamma}(y) \to \Omega|y|^{1/n}, \quad u(y) \to \frac{n\Omega[h^{(n+1)/n} - |y|^{(n+1)/n}]}{(n+1)}, \quad Q \to \frac{2n\Omega h^{(2n+1)/n}}{(2n+1)},$$

where $\Omega = \lambda^{(1-n)/n} G^{1/n} / \mu_0^{1/n}$. The agreement between these approximate solutions for larger λ values has been highlighted in Figure 4. Given that the channel flow of non-Newtonian fluids are commonplace in MEMS (Micro-electromechanical Systems) it is proves instructive to know exactly how the flow rate depends on the channel half width and also the measurable non-Newtonian parameters. The exact solutions presented here provide an insight as to how to control Q for certain fixed values of the fluid index. As highlighted in Figure 4 (c), there is a reasonable discrepancy between the approximate and exact solutions for the volumetric flow rate for shear-thinning flows when $0 < \lambda \lesssim 2$. As noted by Bird *et al.* (1977), these values of the characteristic time constant are well with the range of what one may expect to measure experimentally. In the context of MEMS, where channel half widths are often of the order of micrometers in length, it is imperative that quantities such as the volumetric flow rate can be predicted with a high level of precision. Exact analytical solutions such as those presented here are able to do just this and, equally importantly, are able to offer insights as to how variables such as the pressure gradient can be tuned to achieve desired experimental outputs. For example, given the flow of a shear-thinning fluid with fluid index equal to one-half and a physically relevant value of the characteristic time constant ($\lambda = \text{arcsinh}(1)/\sqrt{2} = 0.6232\,\text{s}$), one finds that if the pressure gradient is set such that $G = \mu_0/(\sqrt[4]{2}\lambda h)\,\text{kg}\,\text{m}^{-2}\text{s}^{-2}$, then $Q = h^2\,\text{m}^2$. This result proves to be a significant departure from the approximate solution that would suggest that $Q \sim h^2/(2\sqrt{2}\lambda) \simeq 0.5673 h^2\,\text{m}^2$.
This study could certainly be extended to consider Couette-Poiseuille flows, i.e. flows where either the upper or lower boundary (or both) are moving with some prescribed velocity. This article demonstrates that exact analytical results for non-Newtonian flow problems are achievable in simple geometries and hopefully provides some insight as to the methodology one may wish to adopt when seeking to find such solutions.

5 Appendix

Expressions for the shear-rate function are obtained in the cases when $n = 1/4$, $3/4$, $4/3$, and 4, from the following quartic equation

$$\dot{\delta}^4 + A\dot{\delta}^3 + B\dot{\delta}^2 + C\dot{\delta} + D = 0, \tag{16}$$

where $\dot{\delta} = \dot{\gamma}^2$, and the multiplicative factors (A, B, C, D), change depending on the case in question. These factors are outlined in Table 1 for the four cases considered.

Table 1: The factors of the quartic equation (16) for the four flow cases considered.

n	A	B	C	D
1/4	$-\lambda^6(Gy/\mu_0)^8$	$-3\lambda^4(Gy/\mu_0)^8$	$-3\lambda^2(Gy/\mu_0)^8$	$-(Gy/\mu_0)^8$
3/4	0	0	$-\lambda^2(Gy/\mu_0)^8$	$-(Gy/\mu_0)^8$
4	$3\lambda^{-2}$	$3\lambda^{-4}$	λ^{-6}	$-\lambda^{-6}(Gy/\mu_0)^2$
4/3	λ^{-2}	0	0	$-\lambda^{-2}(Gy/\mu_0)^6$

The shear-rate function is then determined to be the positive square root of the resulting expression for $\dot{\delta}$, as noted in §3.1.

Below are the three expressions for the volumetric flow rate per unit depth for the cases when $n = 3$, $n = 2$, and $n = 3/2$, respectively.

$$
\begin{aligned}
n = 3: \qquad Q &= \frac{h}{6(\mu_0 - \mu_\infty)} \frac{\mu_0^2}{\lambda^2 G}\left[\left(r_h^4 + \frac{1}{r_h^4}\right) - 2\left(r_h^2 + \frac{1}{r_h^2}\right)\right] \\
&+ \sqrt{\frac{\mu_0^3}{27(\mu_0 - \mu_\infty)^3}} \frac{\mu_0^2}{\lambda^3 G^2}\left[\frac{1}{2}\left(r_h - \frac{1}{r_h}\right) + \frac{1}{5}\left(r_h^5 - \frac{1}{r_h^5}\right) - \frac{1}{14}\left(r_h^7 - \frac{1}{r_h^7}\right)\right], \\
n = 2: \qquad Q &= \frac{\sqrt{\mu_0} h}{3\sqrt{2}\lambda^2 G} \frac{\{\mu_0 - [\mu_0^2 + 4(\lambda G h)^2]^{1/2}\}}{\{\mu_0 + [\mu_0^2 + 4(\lambda G h)^2]^{1/2}\}^{1/2}} \\
&+ \frac{\mu_0 h}{6\lambda^2 G}\left(s_h^3 + \frac{1}{s_h^3}\right) - \frac{\mu_0^2}{60\lambda^3 G^2}\left[5\left(s_h - \frac{1}{s_h}\right) + \left(s_h^5 - \frac{1}{s_h^5}\right)\right], \\
n = 3/2: \qquad Q &= \frac{\mu_0^2}{40\lambda^3 G^2} \mathbb{R}\left\{\left[9\left(t_h + \frac{1}{t_h}\right) - 6\sqrt{3}\left(\frac{\lambda G h}{\mu_0}\right)^2\right]\left(t_h^2 + \frac{1}{t_h^2} - 1\right)^{1/2}\right. \\
&\left. + 5\ln\left[t_h + \frac{1}{t_h} + \left(t_h^2 + \frac{1}{t_h^2} - 1\right)^{1/2}\right]\right\} \\
&- \frac{\ln(3)\mu_0^2}{16\lambda^3 G^2} - \frac{2\mu_0 h}{5\sqrt[4]{3}\lambda^2 G} \mathbb{R}\left\{\left[3 - \left(t_h^2 + \frac{1}{t_h^2}\right)\right]\left(t_h + \frac{1}{t_h}\right)^{1/2}\right\},
\end{aligned}
$$

where the constants r_h, s_h and t_h are as defined in §3.

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