A DECOMPOSITION OF CALDERÓN–ZYGMUND TYPE AND SOME OBSERVATIONS ON DIFFERENTIATION OF INTEGRALS ON THE INFINITE-DIMENSIONAL TORUS

EMILIO FERNÁNDEZ AND LUZ RONCAL

ABSTRACT. In this note we will show a Calderón–Zygmund decomposition associated with a function $f \in L^1(\mathbb{T}^\omega)$. The idea relies on an adaptation of a more general result by J. L. Rubio de Francia in the setting of locally compact groups. Some related results about differentiation of integrals on the infinite-dimensional torus are also discussed.

1. INTRODUCTION

In [28], José L. Rubio de Francia (JLR) showed a result on differentiation of integrals in the context of a locally compact group G, that contained a decomposition of Calderón–Zygmund type [9, Ch. I, Lemma 1] under certain conditions. In view of the publication date, his study could be a contemporary of the one by Edwards and Gaudry in [13, Ch. 2]. In this note we revisite and adapt those results by JLR to the case of the (compact, abelian) group $\mathbb{T}^\omega$ (the infinite torus) defined below. Indeed, our goals are the following:

(1) Present a decomposition of Calderón–Zygmund (CZ) type in $\mathbb{T}^\omega$, devised by JLR.
(2) Observe some issues on differentiation of integrals in $\mathbb{T}^\omega$.

Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ be the one-dimensional torus, identified naturally with the interval $[0,1)$ of the real line through the group isomorphism $e^{2\pi i t} \longleftrightarrow t$. We can also identify $\mathbb{T}$ with the additive quotient group $\mathbb{R}/\mathbb{Z}$. We denote by $\mathbb{T}^\omega$ the compact group formed by the product of countably infinite many copies of $\mathbb{T}$ (*complete direct sum*, [30, B7.]). We will call it briefly the *infinite torus*. The operation in the group $\mathbb{T}^\omega$ is the sum (mod 1) of real sequences, with identity element $\bar{0} := (0, 0, \ldots)$.

For a fixed $n \in \mathbb{N}$ we can write the infinite torus, of points $x = (x_1, x_2, \ldots)$, as the cartesian product $\mathbb{T}^n \times \mathbb{T}^{n,\omega}$ of an n-dimensional torus $\mathbb{T}^n$ of points $x_{(n} = (x_1, \ldots, x_n)$ times a copy, but denoted $\mathbb{T}^{n,\omega}$, of the infinite torus itself, of points $x^{(n} = (x_{n+1}, x_{n+2}, \ldots)$.

We denote by m, or dx, the Haar measure (translation invariant) on $\mathbb{T}^\omega$, normalized such that $m(\mathbb{T}^\omega) = 1$. This measure coincides [19, §22] with the measure product of countably infinite many copies of the Lebesgue measure $|\cdot|$ on $\mathbb{T}$, so the basic m-measurable sets are the so-called *intervals*, i.e. subsets of $\mathbb{T}^\omega$ of the form

$I = \prod_{j \in \mathbb{N}} I_j$, where I_j is an interval of $\mathbb{T}$ for each j, and $\exists\, N \in \mathbb{N}$ such that $I_j = \mathbb{T}$ for all $j > N$. The measure of the interval I is then $m(I) = \int_{\mathbb{T}^\omega} \chi_I(x)\, dx = \prod_{j=1}^{N} |I_j|$.

The space $\mathbb{T}^\omega$ is metrizable. For instance, the function

$$(1) \qquad\qquad d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n} \quad (x, y \in \mathbb{T}^\omega)$$

defines a metric in $\mathbb{T}^\omega$ [31, p. 157]. We write $\delta(S) := \sup_{x,y \in S} d(x,y)$ for the diameter of the set $S \subset \mathbb{T}^\omega$.

The σ-algebra $\mathcal{B}$ of Borel sets in $\mathbb{T}^\omega$, which is the smallest σ-algebra containing the open intervals, coincides with the least σ-algebra containing the open balls with respect to the metric (1) [33, II §2.4].

The study of Harmonic Analysis on the infinite torus finds a motivation, on one hand, because it constitutes a logical extension of the n-dimensional setting in which estimates have to be obtained independent of the dimension n. On the other hand, $\{e^{2\pi i x_k} : k = 1, 2, \ldots\}$ is a system of independent random variables uniformly distributed in the complex unit circumference (i.e., of a complexified version of Rademacher's functions), whose natural completion in L^2 is the trigonometric system on $\mathbb{T}^\omega$. Then, the Fourier series of infinitely many variables turn out to be the complex analogue of the Walsh series. Fourier series in $\mathbb{T}^\omega$ also have connection with the Dirichlet series [6] and with Prediction Theory [18]. All these issues were already pointed out by JLR in [29] (see also references therein), where he studied pointwise and norm convergence of Fourier series of infinite variables, although the proofs are just sketched.

There is considerable interest on the infinite torus from the point of view of Potential Theory, see [1, 2, 3, 4, 5]. Apart from this, problems of approximation theory on $\mathbb{T}^\omega$ have been analyzed for instance in [26].

The JLR decomposition of Calderón–Zygmund type in $\mathbb{T}^\omega$ will be shown in Section 2 (see Subsection 2.2), and the issues related to differentiation of integrals in $\mathbb{T}^\omega$ are contained in Section 3. To be precise, we will look at three differentiation bases. First, the *Rubio de Francia restricted basis* $\mathcal{R}_0$, which is the family associated to the Calderón–Zygmund decomposition in Section 2 and that differentiates $L^1(\mathbb{T}^\omega)$, see Corollary 16. Second, the *Rubio de Francia basis* $\mathcal{R}$, which arises naturally in the light of the general results by JLR in [28, Thm. 8]. For such a basis several questions remain open concerning differentiation and the associated maximal function, see Subsection 3.1. Finally, we present a negative result of differentiation on $\mathbb{T}^\omega$ relative to the so called *extended Rubio de Francia basis* $\mathcal{R}^*$, see Subsection 3.3.

2. A CZ DECOMPOSITION IN $\mathbb{T}^\omega$

For completeness we will recall some concepts from Probability Theory used later in this section (see for instance [32, p. 89–94], [13, Ch. 5], [33, 2.7]).

Definition 1. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and $\mathcal{B}$ be a σ-algebra contained in $\mathcal{A}$. The *conditional expectation of f given $\mathcal{B}$* is the (unique μ-a.e.) $\mathcal{B}$-measurable function $E^{\mathcal{B}} f$ (the notation is that of [25]), such that

$$(2) \qquad \int_B f \, d\mu = \int_B (E^{\mathcal{B}} f) \, d\mu \quad \forall B \in \mathcal{B}.$$

E.g., suppose that $\{B_n\}_{n=1}^\infty$ is a countable division of X in $\mathcal{A}$-measurable sets of positive measure, and consider the least σ-algebra $\mathcal{B}$ which contains those sets (we write $\mathcal{B} := \sigma(\{B_n\})$). Then,

$$(3) \qquad E^{\mathcal{B}} f(x) = \sum_n f_{B_n} \chi_{B_n}(x)$$

(where $f_B := \frac{1}{\mu(B)} \int_B f \, d\mu$ and χ_S denotes the characteristic function of the set S), since the function $s(x)$ at the right hand side of (3) is $\mathcal{B}$-measurable and $\int_{B_n} s \, d\mu = f_{B_n} \mu(B_n) = \int_{B_n} f \, d\mu$ holds.

Property 2. If $\mathcal{B} \subset \mathcal{C}$ are sub-σ-algebras of $\mathcal{A}$, then $E^{\mathcal{B}}(E^{\mathcal{C}} f) = E^{\mathcal{B}} f$ a.e.

Definition 3. Let $(X, \mathcal{A}, m)$ be a finite measure space and let

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \subset \cdots$$

be an increasing sequence of sub-σ-algebras of $\mathcal{A}$.

A sequence of functions $\{f_n\}_{n \in \mathbb{N}} \subset L^1(m)$ such that, for each $n \geq 1$ the function f_n is $\mathcal{B}_n$-measurable and $E^{\mathcal{B}_n} f_{n+1} = f_n$ (a.e.), is called a *martingale*.

For instance, for every $f \in L^p(\mu)$ ($1 \leq p \leq \infty$) the sequence $f_n := E^{\mathcal{B}_n} f$ ($n \in \mathbb{N}$) is a martingale, since $E^{\mathcal{B}_n} f_{n+1} = E^{\mathcal{B}_n}(E^{\mathcal{B}_{n+1}} f) = E^{\mathcal{B}_n} f$ a.e., according to Property 2. Moreover the following holds:

Theorem 4. (i) *The maximal operator, associated to $\{\mathcal{B}_n\}$, defined on $L^1(\mu)$ by $E^* f(x) := \sup_n |f_n(x)|$, where $f_n = E^{\mathcal{B}_n} f$ ($n \in \mathbb{N}$), is weak $(1,1)$ (Doob's inequality [11, VII, Thm. 3.2]), and strong (p,p), $1 < p \leq \infty$.*

(ii) *Furthermore, (f_n) converges almost everywhere. Actually,*

$$\lim_{n \to \infty} f_n(x) = (E^{\mathcal{B}} f)(x) \ \mu\text{-a.e.,}$$

where $\mathcal{B} = \sigma\left(\bigcup_{n=1}^\infty \mathcal{B}_n\right)$.

2.1. **JLR on decomposition of CZ type in locally compact groups.** Let G be a locally compact group with identity e and Haar measure (left invariant) m, and H be a discrete subgroup of G. We will first give a definition and a lemma.

Definition 5. ([28, Section 1].) An open subset V of G is called a *fundamental domain* (FD) for the quotient group G/H if these two conditions hold:

(1) $VV^{-1} \cap H = \{e\}$ (or what is the same, the restriction $\pi|_V$ of the canonical projection $\pi \colon G \to G/H$ is $1-1$).

(2) The complement of VH in G is a locally null set.[1]

For example, the open interval $(0, 1)$ is a FD for $\mathbb{R}/\mathbb{Z}$. For each $n \in \mathbb{N}$, the interval $\left(0, \frac{1}{n}\right)$ is a FD for $\mathbb{T}/R_n$, where $R_n := \left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}\right\}$ is the subgroup of the nth roots of unity in $\mathbb{T}$.

Lemma 6. ([28, Lemma 2].) Assume that G contains a sequence of discrete subgroups

$$H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots \subset G$$

---

[1]Cf. [19, (20.11) Definition].

such that each G/H_n is compact, and write $k_n := \operatorname{order}(H_{n+1}/H_n)$. Then, there is a sequence of open sets

$$V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots,$$

such that V_n is a FD for G/H_n, and each V_n is, except for a null set, the disjoint union of k_n translates of V_{n+1} by elements of H_{n+1}.

The main result of JLR on decomposition of CZ type in this context, is the following:

Theorem 7. ([28, Thm. 8, into the first part of the proof].) *Assume additionally that $\cup_n H_n$ is dense in G and that*

(4) $$\sup_n k_n = k < \infty.$$

Then, for each $f \in L^1(G)$ and $a > \|f\|_1$ there is a disjoint sequence of open sets S_j belonging to the family $\{tV_n : t \in H_n, \, n \in \mathbb{N}\}$, such that $|f(x)| \leq a$ a.e. outside $A = \cup_j S_j$, $m(A) \leq C\|f\|_1/a$ for a constant C independent from f and a, and $a \leq |f|_{S_j} \leq ka$ ($j = 1, 2, \ldots$).

2.2. **A decomposition of CZ type in $\mathbb{T}^\omega$.**

Our subsequent Theorem 9 will show the original proof of JLR Theorem 7 in case of the group (compact, abelian) $G = \mathbb{T}^\omega$. Before its statement we must establish a suitable sequence of subgroups (that JLR did actually teach us in the aforementioned personal communication), which we present below. The decomposition of CZ type in $\mathbb{T}^\omega$ will turn out to be associated with a certain family $\mathcal{R}_0$ (see (6)) of "dyadic intervals".

Definitions 8. ([24, VII.43].) A *net*[2] in $\mathbb{T}^\omega$ is a countable class of disjoint measurable sets whose union is $\mathbb{T}^\omega$ except for a set of null measure. Let $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ be a sequence of nets. The sequence is called *monotonic* if for each positive integer n, every set of $\mathcal{M}_{n+1}$ is a subset of some set of $\mathcal{M}_n$. In this case, for almost all x there exists, for each $n \in \mathbb{N}$, an unique set $I_x^{(n)} \in \mathcal{M}_n$ such that $x \in I_x^{(n)}$.

Remember that $R_k := \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\}$, $k \in \mathbb{N}$. In the following table we see the first terms of the increasing sequence of subgroups $H_m \subset \mathbb{T}^\omega$ proposed by JLR, as well as some of the first terms of the associated decreasing sequence of FD:

m	H_m	V_m
1^2	$H_1 = R_2 \times \{\overline{0}^{(1}\}$	$V_1 = (0, \frac{1}{2}) \times \mathbb{T}^{1,\omega}$
1^2+1	$H_2 = R_2 \times R_2 \times \{\overline{0}^{(2}\}$	$V_2 = (0, \frac{1}{2})^2 \times \mathbb{T}^{2,\omega}$
	$H_3 = R_4 \times R_2 \times \{\overline{0}^{(2}\}$	
2^2	$H_4 = R_4 \times R_4 \times \{\overline{0}^{(2}\}$	$V_4 = (0, \frac{1}{4})^2 \times \mathbb{T}^{2,\omega}$
	$H_5 = R_4 \times R_4 \times R_2 \times \{\overline{0}^{(3}\}$	
2^2+2	$H_6 = R_4 \times R_4 \times R_4 \times \{\overline{0}^{(3}\}$	$V_6 = (0, \frac{1}{4})^3 \times \mathbb{T}^{3,\omega}$
	$H_7 = R_8 \times R_4 \times R_4 \times \{\overline{0}^{(3}\}$	
	$H_8 = R_8 \times R_8 \times R_4 \times \{\overline{0}^{(3}\}$	$V_8 = (0, \frac{1}{8})^2 \times (0, \frac{1}{4}) \times \mathbb{T}^{3,\omega}$
3^2	$H_9 = R_8 \times R_8 \times R_8 \times \{\overline{0}^{(3}\}$	$V_9 = (0, \frac{1}{8})^3 \times \mathbb{T}^{3,\omega}$
	$H_{10} = R_8 \times R_8 \times R_8 \times R_2 \times \{\overline{0}^{(4}\}$	
. . . .	. . . .	. . . .

---

[2] Do not confuse it with net in the sense [23, Ch. 2] of directed set or generalized sequence. The concept of sequence of nets, originally in the euclidean space, is due to de la Vallée Poussin [34, 10.67]. Saks [31, p. 153] generalizes it to measure metric spaces. See also [20, §6].

Actually, after $H_1 = R_2 \times \{\bar{0}^{(1)}\}$ we define, for each $n \geq 1$

$$H_{n^2+j} = \widetilde{H}_{n^2+j} \times \{\bar{0}^{(n+1)}\}, \quad (1 \leq j \leq 2n+1),$$

and

$$\widetilde{H}_{n^2+j} := \begin{cases} R_{2^n} \times \overset{(n)}{\cdots} \times R_{2^n} \times R_{2^j} & \text{if } j \in \{1, \ldots, n\}, \\[2mm] R_{2^{n+1}} \times \overset{(j-n)}{\cdots} \times R_{2^{n+1}} \times R_{2^n} \times \overset{(2n+1-j)}{\cdots} \times R_{2^n} & \\[1mm] & \text{if } j \in \{n+1, \ldots, 2n+1\}. \end{cases}$$

For each positive integer m, H_m is a discrete (finite, of order 2^m) subgroup of $\mathbb{T}^\omega$, $H_m \subset H_{m+1}$, and $\text{order}(H_{m+1}/H_m) = 2$ for all $m \geq 1$. Moreover, each $\mathbb{T}^\omega/H_m$ is compact, because $\mathbb{T}^\omega$ is, and the union $\bigcup_m H_m$ is a dense subset of $\mathbb{T}^\omega$ as is easily checked.
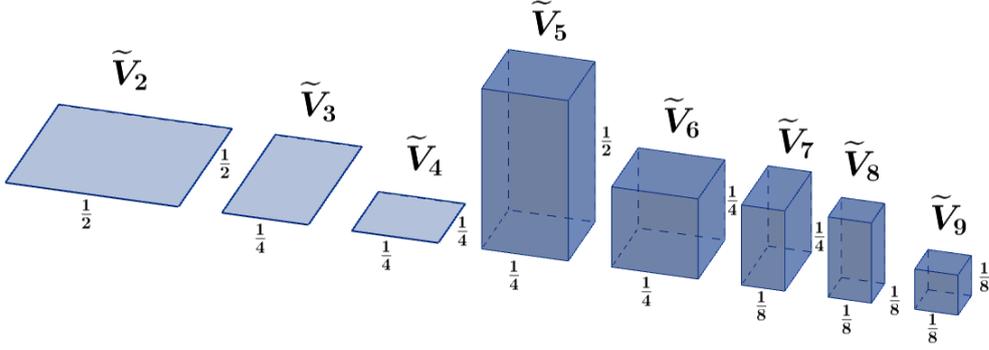

FIGURE 1. First members of the sequence $\{\widetilde{V}_m\}$. E.g., $\widetilde{V}_7 = (0, \frac{1}{8}) \times (0, \frac{1}{4})^2$. Two translations of V_{m+1} by elements of H_{m+1} cover (a.e.) V_m.

The associated decreasing sequence of open sets $\{V_m\}$, where for each $m \geq 1$, V_m is a FD for $\mathbb{T}^\omega/H_m$, is defined by $V_1 = (0, \frac{1}{2}) \times \mathbb{T}^{1,\omega}$ and, for each $n \geq 1$,

$$(5) \qquad\qquad V_{n^2+j} = \widetilde{V}_{n^2+j} \times \mathbb{T}^{n+1,\omega} \quad (1 \leq j \leq 2n+1),$$

with

$$\widetilde{V}_{n^2+j} := \begin{cases} (0, \frac{1}{2^n})^n \times (0, \frac{1}{2^j}) & \text{if } j \in \{1, \ldots, n\}, \\[2mm] (0, \frac{1}{2^{n+1}})^{j-n} \times (0, \frac{1}{2^n})^{2n+1-j} & \text{if } j \in \{n+1, \ldots, 2n+1\} \end{cases}$$

(see Figure 1).

We can consider the (finite) net $\mathcal{N}_m := \{t + V_m \colon t \in H_m\}$ for each $m \in \mathbb{N}$. The sequence $\{\mathcal{N}_m\}_{m \in \mathbb{N}}$ is monotonic. Our final family $\mathcal{R}_0$ of dyadic intervals in $\mathbb{T}^\omega$ is the union of this monotonic sequence of nets,

$$(6) \qquad\qquad \mathcal{R}_0 := \bigcup_m \mathcal{N}_m = \{t + V_m \colon m \in \mathbb{N}, t \in H_m\}.$$

The following announced result holds. The collection $\{I_j\}$ will be a *Calderón-Zygmund decomposition* of intervals of $\mathcal{R}_0$ associated with the function f at level a.

Theorem 9 (José L. Rubio de Francia). *Let $f \in L^1(\mathbb{T}^\omega)$ and $a > \|f\|_1$. There exists a closed set F_a and an open set $\Omega_a = \mathbb{T}^\omega \setminus F_a$ such that*

  (i) $|f(x)| \leq a$ *for almost all $x \in F_a$.*

(ii) *The set Ω_a is the union of a sequence $\{I_j\}_{j=1}^{\infty}$ of pairwise disjoint intervals of the family $\mathcal{R}_0$ such that $a \leq |f|_{I_j} \leq 2a$ for all I_j.*

(iii) *It is verified $m(\Omega_a) \leq \frac{\|f\|_1}{a}$.*

Proof. (See also [12, Thm. 2.10 and 2.11].)

For each $m \in \mathbb{N}$, let $\mathcal{B}_m := \sigma(\mathcal{N}_m)$. Consider as well the trivial σ-algebra $\mathcal{B}_0 = \{\emptyset, \mathbb{T}^\omega\}$ (which is generated by the open set $V_0 := \mathbb{T}^\omega = \bar{0} + \mathbb{T}^\omega$ and thus has the above general form if we also consider the trivial subgroup $H_0 = \{\bar{0}\}$). We have

$$\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \mathcal{B}_m \subset \mathcal{B}_{m+1} \subset \cdots.$$

Define, for $m = 0, 1, 2, \ldots$

(7)
$$f_m(x) := \sum_{t \in H_m} |f|_{t+V_m} \cdot \chi_{t+V_m}(x).$$

The function f_m is $\mathcal{B}_m$-measurable for each m, since it is constant on each interval $t + V_m$ component of the net $\mathcal{N}_m$. Moreover, $f_m \geq 0$ and

$$\int_{\mathbb{T}^\omega} f_m = \sum_{t \in H_m} |f|_{t+V_m} \cdot m(V_m) = \sum_{t \in H_m} \int_{t+V_m} |f| = \int_{\mathbb{T}^\omega} |f| = \|f\|_1 < \infty$$

by hypothesis (we have used that $\mathbb{T}^\omega \setminus (H_m + V_m)$ is a null set, since V_m is a FD for $\mathbb{T}^\omega/H_m$ for each m). If we compare (7) with (3) we will see that f_m is the conditional expectation of $|f|$ related to the σ-algebra $\mathcal{B}_m$.

By applying Theorem 4(ii) we deduce that $f_m(x) \to (E^{\mathcal{B}}|f|)(x)$ a.e. as $m \to \infty$, $\mathcal{B}$ being the least σ-algebra containing the union $\bigcup_{m=0}^{\infty} \mathcal{B}_m$. In our case, the set $\bigcup_m H_m$ is dense in $\mathbb{T}^\omega$, and thus the class $\bigcup_{m=0}^{\infty} \mathcal{B}_m$ is the class of the open sets in $\mathbb{T}^\omega$. Then $\mathcal{B}$ is the σ-algebra of Borel sets in $\mathbb{T}^\omega$. Therefore the operator $E^{\mathcal{B}}$ of conditional expectation with respect to the σ-algebra $\mathcal{B}$ is the identity, and so, for all $g \in L^1(\mathbb{T}^\omega)$ it is verified that $E^{\mathcal{B}}g = g$ (a.e.). We conclude that

(8)
$$\lim_{m \to \infty} f_m(x) = |f(x)| \quad \text{a.e. in } \mathbb{T}^\omega.$$

Let

$$f^*(x) = \sup_{m \in \mathbb{N}} f_m(x).$$

If $x \in F_a := \{x \colon f^*(x) \leq a\}$, we will have $f_m(x) \leq a$ for all m, and an application of (8) yields $|f(x)| \leq a$, so part (i) is proven.

The set $\Omega_a = \mathbb{T}^\omega \setminus F_a$ from part (ii) is defined as $\Omega_a = \{x \colon f^*(x) > a\}$, and the weak type $(1,1)$ inequality (Theorem 4(i)) for the maximal operator[3] $E^* \colon f \mapsto f^*$ gives

(9)
$$m(\Omega_a) \leq \frac{A}{a} \|f\|_1,$$

where A is a constant independent of f and a.

Finally, we have supposed that $\|f\|_1 < a$, thus $f_0(x) \leq a$ holds for all x, and we can recognise the set Ω_a as the disjoint union of the sets

$$\Omega_a^{(n)} = \{x \colon f_i(x) \leq a < f_n(x), \ 0 \leq i \leq n-1\}, \quad n = 1, 2, \ldots.$$

For each $n \geq 1$, the set $\Omega_a^{(n)}$ is obviously $\mathcal{B}_n$-measurable, therefore it is the disjoint union of intervals of the form $t + V_n$ with $t \in H_n$. If $I_j^{(n)}$ is one of these intervals

---

[3]According to Definitions 10 below, the operator E^* would be denoted as $M^{\mathcal{R}_0}$.

we have, on one hand,

$$(10) \qquad \frac{1}{m\big(I_j^{(n)}\big)} \int_{I_j^{(n)}} |f(x)|\,dx = \frac{1}{m\big(I_j^{(n)}\big)} \int_{I_j^{(n)}} \big(E^{\mathcal{B}_n}|f|\big)(x)\,dx$$

$$= \frac{1}{m\big(I_j^{(n)}\big)} \int_{I_j^{(n)}} f_n(x)\,dx \geq a$$

because $f_n(x) > a \ \forall x \in I_j^{(n)} \subset \Omega_a^{(k)}$. On the other hand, $I_j^{(n)}$ is contained in an interval of the form $s + V_{n-1}$ $(s \in H_{n-1})$ which is not contained in $\Omega_a^{(n-1)}$ (we make the agreement that $\Omega_a^{(0)} = \emptyset$), so it is $f_{n-1}(x) \leq a$ for all $x \in s + V_{n-1}$. By using also that $m\big(I_j^{(n)}\big) = m(V_n) = \frac{1}{2}m(V_{n-1})$, we have

$$\frac{1}{m\big(I_j^{(n)}\big)} \int_{I_j^{(n)}} |f(x)|\,dx \leq \frac{2}{m(V_{n-1})} \int_{s+V_{n-1}} |f(x)|\,dx$$

$$= \frac{2}{m(V_{n-1})} \int_{s+V_{n-1}} \big(E^{\mathcal{B}_{n-1}}|f|\big)(x)\,dx$$

$$= \frac{2}{m(V_{n-1})} \int_{s+V_{n-1}} f_{n-1}(x)\,dx \leq 2a,$$

which finishes the proof of (ii). Moreover, from (10) it follows that $A = 1$ in (9). $\quad\square$

Remarks.

(1) It is well known that the standard use of the CZ decomposition of the open set $\Omega_a = \cup_j I_j$ involves [9, Ch. I, proof of Lemma 2], also in $\mathbb{T}^\omega$, a decomposition of the function f, at each level a, in the sum of

$$g(x) := f(x)\chi_{F_a}(x) + \sum_j f_{I_j}\chi_{I_j}(x) \quad \text{and} \quad b(x) := f(x) - g(x)$$

($f = g + b$, g and b *good* and *bad* (level a)-parts of f), verifying properties like the following:

$$|g(x)| \leq 2a \quad \text{(a.e.)}, \qquad \int_{I_j} b(x)\,dx = 0 \quad \text{and} \quad |b|_{I_j} \leq 4a \ \text{ for all } j, \text{ etc.,}$$

(see [14, 5.3.8]).

(2) We have seen that JLR [28] uses Theorem 4 in his proof. The a.e. convergence part (ii) of this theorem plays the role which is played by the differentiation theorem (DT) in a standard proof of the classic result of this type. But here this is not just a style option, we believe, because DT is not a priori assured. These issues are the main content of the next section.

3. On differentiation of integrals in $\mathbb{T}^\omega$

We will start establishing the concepts of differentiation basis and differentiation of integrals adapted to the infinite torus space, which we will adopt in this section.

Definitions 10. ([7, Section 6.1], [15, Ch. 2].) For every $y \in \mathbb{T}^\omega$ let $\mathcal{B}(y)$ be a collection of measurable sets of positive measure that contain (or whose topological closures contain) the point y.

If $\{S_n\}_n \subset \mathcal{B}(y)$ and $\delta(S_n) \to 0$, we say that the sequence S_n "contracts to" y, and write $S_n \Rightarrow y$.

Suppose that there exists at least a sequence $\{S_n\} \subset \mathcal{B}(y)$ such that $S_n \Rightarrow y$.

Let $\mathcal{B} := \bigcup_{y \in \mathbb{T}^\omega} \mathcal{B}(y)$, and suppose that $\mathcal{B}$ covers (a.e.) $\mathbb{T}^\omega$. We call $(\mathcal{B}, \Rightarrow)$ a *differentiation basis*[4] (DB).

EXAMPLES (The names are ours):

$$\mathcal{R}_0 := \{t + V_m \colon m \in \mathbb{N},\, t \in H_m\} \quad (\textit{restricted Rubio de Francia basis}),$$
$$\mathcal{R} := \{y + V_m \colon m \in \mathbb{N},\, y \in \mathbb{T}^\omega\} \quad (\textit{Rubio de Francia basis}),$$
$$\mathcal{J} := \{J \subset \mathbb{T}^\omega \colon J \text{ is an interval}\} \quad (\textit{Jessen basis}),\, [21, 22].$$

Let $(\mathcal{B}, \Rightarrow)$ be a differentiation basis in $\mathbb{T}^\omega$. Given $f \in L^1(\mathbb{T}^\omega)$, we define the *upper and lower derivative* of $\int f$ with respect to $\mathcal{B}$ (and the Haar measure m) in the point $x \in \mathbb{T}^\omega$ by (without loss of generality we assume here that f is a real function)

$$\overline{D}\big(\textstyle\int f, x\big) = \sup_{\substack{\{B_n\} \subset \mathcal{B} \\ B_n \Rightarrow x}} \big\{\limsup_n f_{B_n}\big\} \quad \text{and} \quad \underline{D}\big(\textstyle\int f, x\big) = \inf_{\substack{\{B_n\} \subset \mathcal{B} \\ B_n \Rightarrow x}} \big\{\liminf_n f_{B_n}\big\},$$

respectively. When

$$(11) \qquad\qquad \overline{D}\Big(\textstyle\int f, x\Big) = \underline{D}\Big(\textstyle\int f, x\Big) = f(x) \quad \text{a.e.}$$

holds, we write $D\big(\int f, x\big) = f(x)$ and say that the basis $\mathcal{B}$ *differentiates* $\int f$ and that the *derivative* of $\int f$ is f. A necessary condition for (11) is that

$$\lim_{n \in \mathbb{N}} f_{B_n} = f(x) \quad \text{a.e.}$$

holds, for every sequence $\{B_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ such that $B_n \Rightarrow x$.

When (11) is satisfied for all $f \in L^\infty$ (resp. $f \in L^1(\mathbb{T}^\omega)$), we say that $\mathcal{B}$ *differentiates* $L^\infty(\mathbb{T}^\omega)$ (resp. $L^1(\mathbb{T}^\omega)$). Note that $L^\infty(\mathbb{T}^\omega) \subset L^1(\mathbb{T}^\omega)$ and thus, if the basis $\mathcal{B}$ *does not* differentiate $L^\infty(\mathbb{T}^\omega)$, then also does not differentiate $L^1(\mathbb{T}^\omega)$.

Let $\mathcal{B}$ be a DB in $\mathbb{T}^\omega$. Suppose that the *maximal operator* associated with $\mathcal{B}$ given by

$$M^\mathcal{B} f(x) = \sup_{x \in B \in \mathcal{B}} |f|_B, \quad f \in L^1(m),$$

is well defined (i.e., we suppose that for each $f \in L^1(m)$, $M^\mathcal{B} f$ is measurable). The following result (due to de Guzmán and Welland) holds, its proof is standard.

Theorem 11. ([16, Thm. 1.1(a)].) *If the operator $M^\mathcal{B}$ is of weak type $(1,1)$, then the basis $\mathcal{B}$ does differentiate $L^1(\mathbb{T}^\omega)$.*

3.1. **Differentiation Theorem on locally compact groups. The basis $\mathcal{R}$ in $\mathbb{T}^\omega$.**

Theorem 12. ([28, Thm. 8].) *With the hypothesis of* Theorem 7*, let $\mathcal{R}$ be the family formed by all sets of the form yV_n with $y \in G$, $n = 1, 2, \dots$. If*

$$(12) \qquad\qquad \sup_n \frac{m(V_n V_n^{-1} V_n)}{m(V_n)} < \infty$$

holds, then $M^\mathcal{R}$ is weak type $(1,1)$ and strong (p,p) for $1 < p \leq \infty$.

As an immediate consequence JLR gives the following result, which establishes a sufficient condition for the basis $\mathcal{R}$ to differentiate $L^1_{\mathrm{loc}}(G)$ (in this setting, the notion of contraction of a sequence $(S_n) \subset \mathcal{R}$ to a point involves $m(S_n) \to 0$).

---

[4]If every $B \in \mathcal{B}$ is an open set and if $x \in B \in \mathcal{B}$ then $B \in \mathcal{B}(x)$, $\mathcal{B}$ is called a *Busemann-Feller basis*.

Corollary 13. ([28, Corol. 5].) Suppose, in addition to the hypothesis of Theorem 12, that $V_n \subset U_n$ for a basis $\{U_n\}_{n \in \mathbb{N}}$ of neighbourhoods of e. Then:

$$\lim_{\substack{x \in R \in \mathcal{R} \\ m(R) \to 0}} f_R = f(x) \quad (\text{a.e.})$$

for any locally integrable function f.

In the case in which G is the compact group $\mathbb{T}^\omega$ and $\mathcal{R}$ is our Rubio de Francia basis, the condition (12) does not hold, because e.g. for $n \geq 1$,

$$V_{n^2} = \left(0, \frac{1}{2^n}\right)^n \times \mathbb{T}^{n,\omega}$$

and

$$V_{n^2} - V_{n^2} + V_{n^2} = \left(\left[0, \frac{1}{2^{n-1}}\right) \cup \left(\frac{2^n - 1}{2^n}, 1\right)\right)^n \times \mathbb{T}^{n,\omega},$$

so that

$$\frac{m(V_{n^2} - V_{n^2} + V_{n^2})}{m(V_{n^2})} = \frac{(3/2^n)^n}{(1/2^n)^n} = 3^n, \quad \text{and} \quad \sup_n \frac{m(V_n - V_n + V_n)}{m(V_n)} = \infty.$$

Therefore, we can not guarantee (in principle) the result of Theorem 12 for the Rubio de Francia basis $\mathcal{R}$. The additional sufficient condition of the Corollary 13 is satisfied because, for instance, the family $\{V_n - V_n\}_{n \in \mathbb{N}}$ is a basis of (symmetric) neighbourhoods of 0 in $\mathbb{T}^\omega$. On the Rubio de Francia basis $\mathcal{R}$ the following questions remain open:

- Does the converse of Theorem 11 for the basis $\mathcal{R}$ in $\mathbb{T}^\omega$ hold? (in de Guzmán and Welland theorem in $\mathbb{R}^n$ [16, Thm. 1.1(b)] the BD $\mathcal{B}$ is required to be homothecy invariant).
- Is the operator $M^{\mathcal{R}}$ weak type (1,1)?[5]
- Does $\mathcal{R}$ differentiate $L^\infty(\mathbb{T}^\omega)$?

Customizing Jessen's proof for the basis $\mathcal{J}$, we will prove below (Subsection 3.3) that a certain basis $\mathcal{R}^*$ slightly wider than $\mathcal{R}$ does not differentiate $L^\infty(\mathbb{T}^\omega)$.

3.2. Bases $\mathcal{R}_0$ and $\mathcal{J}$.

Definition 14. ([31, p. 153], [24, VII.43].) Let $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ be a monotonic sequence of nets in $\mathbb{T}^\omega$, and $\mathcal{M} := \bigcup_n \mathcal{M}_n$. For each $y \in \mathbb{T}^\omega$ and each k, write $I_y^{(k)}$ for the unique element of the net $\mathcal{M}_k$ which contains y. We say that the sequence is *indefinitely fine* if for each $x \in \mathbb{T}^\omega$ and each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\delta\big(I_x^{(n_0)}\big) < \varepsilon$.

Then, $I_x^{(n)} \Rightarrow x$, and $\mathcal{M}$ is a differentiation basis. The following result holds.

Theorem 15. ([24, 43.7], cf. also [20, §9], [31, 15.7].) *If $\{\mathcal{M}_n\}_{n \in \mathbb{N}}$ is a monotonic sequence of nets indefinitely fine, then the basis $\mathcal{M}$ differentiates $L^1(\mathbb{T}^\omega)$.*

At the end of [28], JLR pointed out, in the setting of the locally compact group G, that if $\mathcal{R}$ is defined to be only consisting of the sets tV_n ($t \in H_n$), $n = 1, 2, \ldots$, then Theorem 12 is valid without assumptions (4) and (12). In order to corroborate this statement with an example, we provide an immediate consequence of Theorem 15. It is, on the other hand, an immediate consequence of Theorem 11, because the maximal operator $M^{\mathcal{R}_0}$ is of weak type (1, 1).

Corollary 16. The basis $\mathcal{R}_0 = \{t + V_m : m \in \mathbb{N}, t \in H_m\}$ does differentiate $L^1(\mathbb{T}^\omega)$.

---

[5]We owe this question to Sheldy J. Ombrosi.

Proof. The basis $\mathcal{R}_0$ is the union, for $m \in \mathbb{N}$, of the monotonic sequence of nets $\{\mathcal{N}_m\}$, where $\mathcal{N}_m = \{t + V_m : t \in H_m\}$. This sequence is indefinitely fine, because if $I \in \mathcal{N}_m$ and $(n-1)^2 < m \leq n^2$ $(n \geq 2)$, it is easily seen that

$$\delta(I) \leq \sum_{j=1}^{n-1} \frac{1}{2^{n-1+j}} + \frac{1}{2^{n+1}} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} < \frac{7}{2^{n+1}}.$$

$\square$

Any subfamily of a basis that differentiates L^1 and which is in turn a basis of differentiation, also differentiates L^1. In particular, the subfamily of cubic intervals of the base $\mathcal{R}_0$ that Saks already considered [31, p. 158] (see [7, p. 28])

$$\mathcal{S} := \bigcup_{m=1}^{\infty} \mathcal{S}_m \quad \text{where} \quad \mathcal{S}_m := \{t + V_{m^2} : t \in H_{m^2}\},$$

also differentiates $L^1(\mathbb{T}^\omega)$.

The question (posed by A. Zygmund, see [21, p. 55]) about the differentiation of integrals in $L^1(\mathbb{T}^\omega)$ with respect to this basis $\mathcal{J}$ of all the intervals of $\mathbb{T}^\omega$ was answered negatively around 1950 by Jessen [21, 22]. The counterexample proposed by Jessen refers to the characteristic function of certain measurable set of positive measure, so in fact he proves that the basis $\mathcal{J}$ *does not* differentiate even $L^\infty(\mathbb{T}^\omega)$. It is indeed a curious phenomenon, since the basis formed by the *intervals* (i.e., the parallelepipeds of edges parallel to the coordinate axes) of $\mathbb{T}^n$ does differentiate $L^\infty(\mathbb{T}^n)$ for all $n \in \mathbb{N}$ [15, p. 74].

3.3. A negative result of differentiation on $\mathbb{T}^\omega$: The basis $\mathcal{R}^*$.

We begin with questions of nomenclature and notation to briefly represent some dyadic sets in $\mathbb{T}^\omega$.

Definitions 17. For $m, q \in \mathbb{N}$, $m \geq 2$, and $q \leq m$, write $\widetilde{\square}_{m,q} := \left(0, \frac{1}{2^m}\right)^q$ and

$$\square_{m,q} := \widetilde{\square}_{m,q} \times \mathbb{T}^{q,\omega}.$$

Call $\square_{m,q}$ the (m,q)-*cube*. E.g., $V_{m^2} = \square_{m,m} = \widetilde{\square}_{m,q} \times \left(0, \frac{1}{2^m}\right)^{m-q} \times \mathbb{T}^{m,\omega}$, for all $m \geq q$.

Consider in $\mathbb{T}^\omega$, for $j \in \mathbb{N}$, the translation τ_j^m which adds $\frac{1}{2^m}$ to the coordinate x_j. Define the sets (we call them *sacks* of the corresponding cubes)

$$S(\square_{m,q}) := \square_{m,q} \cup \left(\cup_{j=1}^{q} \tau_j^m(\square_{m,q})\right)$$

and, for $y \in \mathbb{T}^\omega$, $S(y + \square_{m,q}) = y + S(\square_{m,q})$.

On the other hand, write $W_{m,r} := \square_{m,m} \cup \tau_r^m(\square_{m,m})$ $(1 \leq r \leq m)$. Call $W_{m,r}$ a *double* (m,m)-*cube*. We have $W_{m,r} = \widetilde{W}_{m,r} \times \mathbb{T}^{m,\omega}$, where

$$\widetilde{W}_{m,r} := \prod_{j=1}^{m} (1 + \delta_{rj}) \cdot \left(0, \frac{1}{2^m}\right) \qquad \text{(Kronecker's delta)}.$$
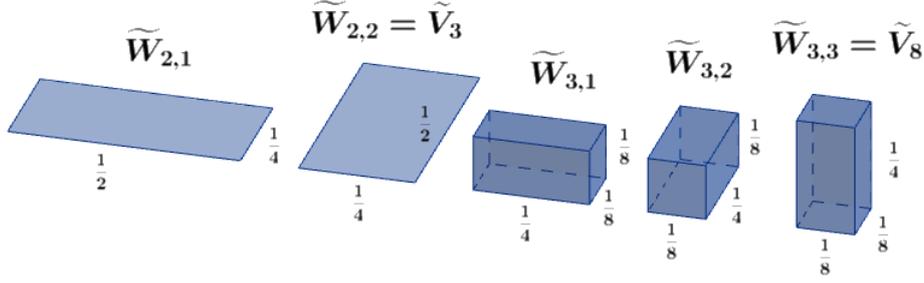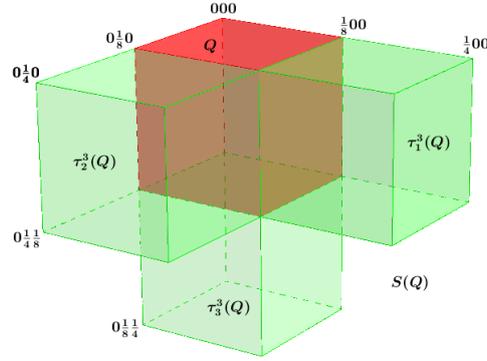
E.g., $W_{m,m} = V_{m^2-1}$, but $W_{m,j} \notin \mathcal{R}$ when $1 \leq j \leq m-1$ (see Figure 2).

We define the *extended Rubio de Francia basis* to be the collection

$$(13) \qquad \mathcal{R}^* := \mathcal{R} \cup \{y + W_{m,r} : y \in \mathbb{T}^\omega, \ m \in \mathbb{N}, \ m \geq 2, \ 1 \leq r \leq m\}.$$

Lemma 18. Let $Q \in \{y + \square_{m,q} : y \in \mathbb{T}^\omega\}$. For each point $x \in S(Q)$ there is an interval $I_x \in \mathcal{R}^*$ such that

$$\frac{m(I_x \cap Q)}{m(I_x)} \geq \frac{1}{2}.$$

FIGURE 2. First members of the family $\widetilde{W}_{m,r}$.

FIGURE 3. The sack of the cube $Q = \square_{3,3}$.

Proof. First consider any case in which $y = \bar{0}$, $Q = \square_{m,q}$. Then, if $x \in Q$ we can take the interval $I_x^0 = \widetilde{\square}_{m,q} \times \left(0, \frac{1}{2^m}\right)^{m-q} \times \mathbb{T}^{m,\omega} = V_{m^2} \in \mathcal{R}$. We have $I_x^0 \subset Q$, and

$$\frac{m(I_x^0 \cap Q)}{m(I_x^0)} = 1.$$

Otherwise, if $x \in \tau_r(Q)$ for any r, $1 \leq r \leq q$, define (δ_{rj} is Kronecker's delta)

$$D_{m,r}^q := \prod_{j=1}^q (1 + \delta_{rj}) \cdot \left(0, \frac{1}{2^m}\right) \subset \mathbb{T}^q.$$

Then, let us take $I_x^0 = D_{m,r}^q \times \left(0, \frac{1}{2^m}\right)^{m-q} \times \mathbb{T}^{m,\omega}$. We have $I_x^{(0)} = W_{m,r} \in \mathcal{R}^*$, $I_x^0 \cap Q = \widetilde{\square}_{m,q} \times \left(0, \frac{1}{2^m}\right)^{m-q} \times \mathbb{T}^{m,\omega} = \square_{m,m}$, and

$$\frac{m(I_x^0 \cap Q)}{m(I_x^0)} = \frac{m(\square_{m,m})}{m(W_{m,r})} = \frac{1}{2}.$$

In a general case in which $y \neq \bar{0}$, take $I_x = y + I_x^0$. The lemma follows. $\square$

Lemma 19. *Let $n \geq 2$ be a fixed integer. It is possible to find in $\mathbb{T}^\omega$ an enumerable family of pairwise disjoint intervals $\{Q_\alpha\}_{\alpha \in A_n}$, $Q_\alpha = y(\alpha) + \square_{m(\alpha),n}$ ($y(\alpha) \in \mathbb{T}^\omega$, $m(\alpha) \geq n$), with the sets $S(Q_\alpha)$ also pairwise disjoint, and:*

(1) If $C_n := \bigcup_\alpha Q_\alpha$, and $N_n := \mathbb{T}^\omega \setminus \left(\cup_\alpha S(Q_\alpha)\right)$, then $m(C_n) = \frac{1}{n+1}$, and $m(N_n) = 0$.

(2) For every $x \notin N_n$ there exists an interval $I_x^n \in \mathcal{R}^*$ (whose first n edges are $\leq 1/2^{n-1}$; in fact, I_x^n is a translate either of a cube V_{n^2}, or of a double cube $W_{n^2-1,r}$) such that
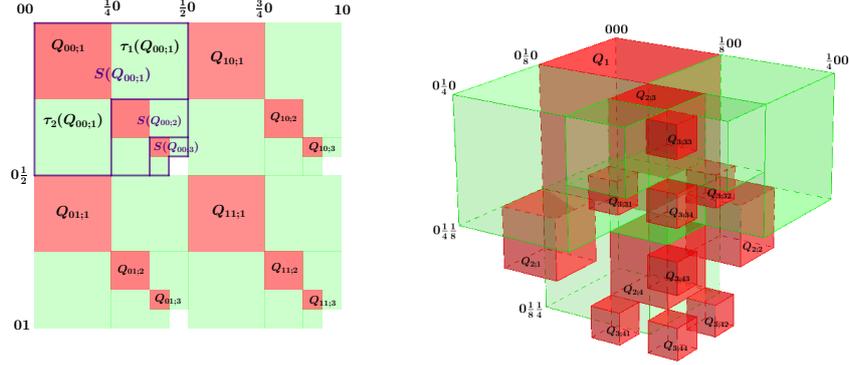$$\frac{m(I_x^n \cap C_n)}{m(I_x^n)} \geq \frac{1}{2}.$$

FIGURE 4. The construction in Lemma 19, $n = 2$ and 3.

Proof. (In what follows, we write $\boldsymbol{t_1 \ldots, t_n \overline{0}}$ for the point $(t_1, \ldots, t_n, 0^{(n)}) \in \mathbb{T}^\omega$.) In general, for each $n \geq 2$, we consider the division of $\mathbb{T}^\omega$ into $2^{n(n-1)}$ open cubes (call them 0-*cells*) of edge $1/2^{n-1}$ by the "hyperplanes"
$$x_i = \frac{j}{2^{n-1}} \qquad (i = 1, \ldots, n; \ j = 0, 1, \ldots, 2^{n-1} - 1).$$
The 0-cells have the form
$$I_{i_1 \ldots i_n}^{(n)} = \tfrac{\boldsymbol{i_1}}{\boldsymbol{2^{n-1}}} \cdots \tfrac{\boldsymbol{i_n}}{\boldsymbol{2^{n-1}}} \overline{\boldsymbol{0}} + \left(0, \tfrac{1}{2^{n-1}}\right)^n \times \mathbb{T}^{n,\omega},$$
where $(i_1, \ldots, i_n) \in \{0, 1, \ldots, 2^{n-1} - 1\}^n$.

Each one of the 0-cells is firstly subdivided in 2^n open cubic intervals of edge length $1/2^n$ (1-*cells*). As an example, the 1-cells of the $I_{0 \ldots 0}^{(n)}$ 0-cell have the form
$$\tfrac{\boldsymbol{j_1}}{\boldsymbol{2^n}} \cdots \tfrac{\boldsymbol{j_n}}{\boldsymbol{2^n}} \overline{\boldsymbol{0}} + \square_{n,n},$$
where $(j_1, \ldots, j_n) \in \{0, 1\}^n$.

Among these 1-cells, we call $\boldsymbol{0 \ldots 0 \overline{0}} + \square_{n,n}$ the *principal 1-cell* and denote it by $Q_{0 \ldots 0;1}^{(n)}$ (abbreviated by $Q_1^{(n)}$). The sack of $Q_1^{(n)}$ is
$$S(Q_1^{(n)}) = Q_1^{(n)} \cup \left(\bigcup_{j=1}^n \left(\tfrac{\boldsymbol{\delta_{1j}}}{\boldsymbol{2^n}} \cdots \tfrac{\boldsymbol{\delta_{nj}}}{\boldsymbol{2^n}} \overline{\boldsymbol{0}} + Q_1^{(n)} \right) \right) \quad \text{(Kronecker's } \delta_{ij}\text{)},$$
and we have $m(S(Q_1^{(n)})) = (n+1) \cdot m(Q_1^{(n)})$.

In the 0-cell $I_{0 \ldots 0}^{(n)}$, apart from the $(n+1)$ 1-cells which form the sack $S(Q_1^{(n)})$, there remain other $2^n - (n+1)$ 1-cells. In each one of these is carried out a subdivision into 2^n open cubes of edge $1/2^{n+1}$ (2-*cells*), one of which is the principal 2-cell $Q_{2;\beta}^{(n)} = \boldsymbol{y_\beta} + \square_{n+1,n}$, with an appropriate $\boldsymbol{y_\beta} \in \mathbb{T}^\omega$. There are $2^n - (n+1)$ principal 2-cells for each principal 1-cell. Now, we forget the sacks of the principal 2-cells, and in all the remaining 2-cells we proceed inductively.

The family $\{Q_\alpha^{(n)}\}_{\alpha \in A_n}$ of the statement of the Lemma is formed by all the principal k-cells ($k \geq 1$) in this construction. The indicial set A_n can be defined

explicitly (look at Figure 4), but this is not essential, and for clarity of exposition we will avoid doing it. On the other hand, from the inductive definition it is immediate that

$$Q_\alpha^{(n)} \cap Q_\beta^{(n)} = \emptyset \quad \text{and} \quad S(Q_\alpha^{(n)}) \cap S(Q_\beta^{(n)}) = \emptyset \quad \text{for all } \alpha \neq \beta \ (n \in \mathbb{N}, \ \alpha, \beta \in A_n).$$

(1) Let now $C_n := \bigcup_{\alpha \in A_n} Q_\alpha^{(n)}$. Then,

$$m(C_n) = \sum_{\alpha \in A_n} m(Q_\alpha^{(n)}) = \frac{2^{n(n-1)}}{2^{n^2}} \sum_{k=0}^\infty \Big(\frac{2^n - (n+1)}{2^n} \Big)^k = \frac{1}{n+1},$$

and

$$m\Big(\bigcup_{\alpha \in A_n} S(Q_\alpha^{(n)}) \Big) = \sum_{\alpha \in A_n} m(S(Q_\alpha^{(n)})) = (n+1) \sum_{\alpha \in A_n} m(Q_\alpha^{(n)}) = (n+1)m(C_n) = 1,$$

from which, denoting $N_n = \mathbb{T}^\omega \setminus \Big(\bigcup_{\alpha \in A_n} S(Q_\alpha^{(n)}) \Big)$, we have $m(N_n) = 0$.

(2) Let $x \notin N_n$. Then, there exists $\alpha_0 \in A_n$ such that $x \in S(Q_{\alpha_0}^{(n)})$. Applying Lemma 18, there exists an interval $I_x \in \mathcal{R}^*$ (in fact, either a cube of edge not greater than $1/2^n$, or a double cube) such that

$$\frac{m(I_x \cap Q_{\alpha_0}^{(n)})}{m(I_x)} = \frac{m(I_x \cap C_n)}{m(I_x)} \geq \frac{1}{2},$$

as required. $\qquad\square$

Theorem 20. *The Rubio de Francia extended basis $\mathcal{R}^*$ does not differentiate $L^\infty(\mathbb{T}^\omega)$.*

Proof. (This argumentation is taken from Jessen [22].)

Choose an increasing sequence of positive integers $(n_p)_{p=1}^\infty$ ($n_1 \geq 2$) such that $\sum_p 1/(n_p + 1) \leq 3/4$. Then, the union $C := \bigcup_{p=1}^\infty C_{n_p}$ is a measurable set whose measure satisfies $0 < \frac{1}{n_1+1} \leq m(C) \leq 3/4$, and the union $N := \bigcup_{p=1}^\infty N_{n_p}$ is a null measurable set, since for each p, the set N_{n_p} is measurable and $m(N_{n_p}) = 0$.

If $x \notin N$, then $x \in \bigcup_\alpha S(Q_\alpha^{(n_p)})$ for every p, and thus there exists a sequence of indexes $(\alpha_p)_{p=1}^\infty$ such that $x \in S(Q_{\alpha_p}^{(n_p)})$ for each p. Then, applying Lemma 19(2), for each p there exists an interval $I_x^{(p)} \in \mathcal{R}^*$ such that $\delta(I_x^{(p)}) \leq \delta(W_{n_p,1}) < 3/2^{n_p}$ (consequently these intervals $I_x^{(p)}$ form a sequence of $\mathcal{R}^*$ contracting to the point x), and

$$\frac{m(C \cap I_x^{(p)})}{m(I_x^{(p)})} \geq \frac{1}{2}.$$

Consider the characteristic function χ_C. For all $x \in \mathbb{T}^\omega \setminus N$ (i.e., a.e. in $\mathbb{T}^\omega$), we have

$$\limsup_{p \to \infty} \frac{1}{m(I_x^{(p)})} \int_{I_x^{(p)}} \chi_C(y) \, dy = \limsup_{p \to \infty} \frac{m(C \cap I_x^{(p)})}{m(I_x^{(p)})} \geq \frac{1}{2},$$

which immediately implies that $\overline{D}\big(\int \chi_C, x\big) \geq \frac{1}{2}$ for almost all $x \in \mathbb{T}^\omega$. But $\chi_C(x) = 0 < \frac{1}{2}$ for all $x \notin C$, a set of measure $\geq 1/4$. It follows that $\mathcal{R}^*$ does not differentiate $\int \chi_C$. $\qquad\square$

Remarks.

(1) The proof of Theorem 20 in fact shows that the subfamily extracted from $\mathcal{R}^*$ which is formed by the cubes $\{y + V_{m^2} : y \in \mathbb{T}^\omega, \ m \geq 2\}$ and the double cubes $\{y + W_{m,r} : y \in \mathbb{T}^\omega, \ m \geq 2, \ 1 \leq r \leq m\}$ (this subfamily is not contained in the Rubio de Francia basis $\mathcal{R}$) does not differentiate $L^\infty(\mathbb{T}^\omega)$.

(2) The question whether the DB formed only by the cubes $\{y + V_{m^2} \colon y \in \mathbb{T}^{\omega}, m \geq 2\}$ does differentiate $L^{\infty}(\mathbb{T}^{\omega})$ (with our notion of contraction of a sequence to a point) remains open for us at the moment.

(3) Dieudonné [10] also proves that the basis of intervals in $\mathbb{T}^{\omega}$ (in fact, the subfamily of cubic intervals in $[0,1]^{\omega}$), does not differentiate $L^{\infty}(\mathbb{T}^{\omega})$ (see [7, p. 28]). But Dieudonné works with the notion of contraction to a point for generalized sequences in the Moore-Smith sense $\{S_{\alpha}\}_{\alpha \in D} \subset \mathcal{B}(y)$, being D a directed set [23, p. 81-86], as we explain next:

Let $\mathcal{F}$ be the set of finite subsets of $\mathbb{N}$. For each $J \in \mathcal{F}$ we consider

$$\mathbb{T}^{\omega} = \mathbb{T}^{J} \times \mathbb{T}^{J,\omega}$$

in such a way that, if $x \in \mathbb{T}^{\omega}$, $x = (x_J, x_{J'})$ with $x_J \in \mathbb{T}^{J}$ and $x_{J'} \in \mathbb{T}^{J,\omega}$. Dieudonné deals with the DB $\mathcal{D} = \bigcup_{x \in \mathbb{T}^{\omega}} \mathcal{D}(x)$ where $\mathcal{D}(x)$ is the net (according to the set $\mathbb{N} \times \mathcal{F}$ directed by the order relation $(n_1, J_1) \leq (n_2, J_2)$ if and only if $n_1 \leq n_2$ and $J_1 \subseteq J_2$) that consists of the cubic intervals

$$(14) \qquad V_{n,J}(x) = \widetilde{V}_{n,J}(x_J) \times \mathbb{T}^{J,\omega}, \quad (n \in \mathbb{N};\ J \in \mathcal{F}),$$

where $\widetilde{V}_{n,J}(x_J) \subset \mathbb{T}^{J}$ is the cube of center x_J and side $1/n$.

Dieudonné defines a measurable set for whose characteristic function f, the means $f_{V_{n,J}(x)}$ cannot converge a.e. to $f(x)$ according to the directed set $\mathbb{N} \times \mathcal{F}$.

(4) A differentiation basis $\mathcal{B}$ satisfies the *density property* if $\mathcal{B}$ differentiates χ_E for each measurable set E, i.e. for almost every $x \in \mathbb{T}^{\omega}$ we have, if $\{I_k\}$ is any arbitrary sequence of $\mathcal{B}(x)$ contracting to x,

$$\lim_{k \to \infty} \frac{m(E \cap I_k)}{m(I_k)} = \chi_E(x)$$

([8, p. 227], [17, p. 30], [15, III.1]). From our proof of Theorem 20 it follows that the basis $\mathcal{R}^*$ does not satisfies the density property.

In fact, it holds the following result (which for the space $\mathbb{R}^n$ can be found, for instance, in [15, III, Thm. 1.4]): *The basis $\mathcal{B}$ differentiates $L^{\infty}(\mathbb{T}^{\omega})$ if and only if satisfies the density property* [27, Num. 11, C⇔D].

References

[1] BENDIKOV, A. D., *Potential Theory on Infinite-dimensional Abelian Groups*. Walter de Gruyter, 1995.

[2] BENDIKOV, A., COULHON, T., AND SALOFF-COSTE, L. Ultracontractivity and embedding into L^{∞}. *Math. Ann.* **337** (2007), no. 4, 817–853.

[3] BENDIKOV, A. AND SALOFF-COSTE, L. Spaces of smooth functions and distributions on infinite-dimensional compact groups. *J. Funct. Anal.* **218** (2005), no. 1, 168–218.

[4] BENDIKOV, A. AND SALOFF-COSTE, L. Hypoelliptic bi-invariant Laplacians on infinite dimensional compact groups. *Canad. J. Math.* **58** (2006), no. 4, 691–725.

[5] BERG, C., Potential theory on the infinite dimensional torus. *Invent. Math.* **32** (1976), no. 1, 49–100.

[6] BOHR, H., Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen. *Nachr. Ges. Wiss. Göttingen Math.-Phys. Klasse* 1913, Heft 4, 441–488.

[7] BRUCKNER, A. M., Differentiation of Integrals. *The American Mathematical Monthly*, Vol. 78, No. 9 Part 2, Nov., 1971 i-iv+1–52.

[8] BUSEMANN, H. AND FELLER, W., Zur Differentiation der Lebesgueschen Integrale. *Fund. Math.* **22**, 1934, 226–256.

[9] CALDERÓN, A. P. AND ZYGMUND, A., On the existence of certain singular integrals. *Acta Mathematica* **88**, (1952), 85–139.

[10] DIEUDONNÉ, J., Sur un théorème de Jessen. *Fund. Mat.*, **37** (1950), 242–248.

[11] DOOB, J. L., *Stochastic Processes*. John Wiley & Sons, Inc., 1953.

[12] DUOANDIKOETXEA, J., *Fourier Analysis*, translated and revised by David Cruz-Uribe, SFO. Graduate Studies in Mathematics, Vol. 29. American Mathematical Society, 2001.

[13] EDWARDS, R. E. AND GAUDRY, G. I., *Littlewood-Paley and Multiplier Theory.* Springer, 1977.

[14] GRAFAKOS, L., *Classical Fourier Analysis*, 3rd ed. Springer, 2014.

[15] DE GUZMÁN, M., *Differentiation of Integrals in $\mathbb{R}^n$*. Lecture Notes in Mathematics 481, Springer, 1975.

[16] DE GUZMÁN, M. AND WELLAND G. W., On the differentiation of integrals. *Rev. Un. Mat. Argentina*, **25** (1971), 253–276.

[17] HAYES, C. A. AND PAUC, C. Y., *Derivation and Martingales.* Ergebnisse der Mathematik und ihrer Grenzgebiete Band 49, Springer, 1970.

[18] HELSON, H. AND LOWDENSLAGER, D., Prediction theory and Fourier series in several variables. *Acta Mathematica* **99**, 1958, 165–202.

[19] HEWITT, E. AND STROMBERG, K., *Real and Abstract Analysis*, 3rd ed. Springer, 1975.

[20] JESSEN, B., The theory of integration in a space of an infinite number of dimensions. *Acta Mathematica* **63**, 1934, 249–323.

[21] JESSEN, B., A remark on strong differentiation in a space of an infinite number of dimensions. *Mat. Tidsskr. B.* (1950), 54–57.

[22] JESSEN, B., On strong differentiation. *Mat. Tidsskr. B.* (1952), 90–91.

[23] KELLEY, J. L., *General Topology.* D. van Norstrand Co. Inc., 1955).

[24] MUNROE, M. E., *Introduction to Measure and Integration.* Addison-Wesley Pub. Co., Inc., 1953.

[25] NEVEU, J., *Martingales à temps discret.* Masson et Cie, 1972.

[26] PLATONOV, S. S., On some problems of the theory of the approximation of functions on an infinite-dimensional torus: analogues of Jackson theorems. (Russian) *Algebra i Analiz* **26** (2014), no. 6, 99–120; translation in *St. Petersburg Math. J.* **26** (2015), no. 6, 933–947.

[27] DE POSSEL, R., Sur la dérivation abstraite des fonctions d'ensemble. *J. Math. Pures Appl.*, **15** (1936), 391–409. Source Bibliothèque nationale de France.

[28] RUBIO DE FRANCIA, J. L., Nets of subgroups in locally compact groups. *Comment. Math. Prace Mat.* **20** (1977/78), no. 2, 453–466.

[29] RUBIO DE FRANCIA, J. L., Convergencia de series de Fourier de infinitas variables. Proceedings of the seventh Spanish-Portuguese Conference on Mathematics, Part II (Sant Feliu de Guíxols, 1980). *Publ. Sec. Mat. Univ. Autónoma Barcelona* No. 21 (1980), 237–241.

[30] RUDIN, W., *Fourier Analysis on Groups.* Wiley, New York, 1962.

[31] SAKS, S., *Theory of the Integral*, 2nd revised edition. Monografie Matematyczne, Vol. VII, Warsaw 1937.

[32] STEIN, E. M., *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory.* Annals of Mathematics Studies, No. 63. Princeton Univ. Press and University of Tokyo Press, 1970.

[33] SHIRYAEV, A. N., *Probability*, 2nd edition. Springer, 1992.

[34] DE LA VALLÉE POUSSIN, C., Sur l'Intégrale de Lebesgue. *Trans. Amer. Math. Soc.*, **16** (1915), 435–501.

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, C/ MADRE DE DIOS, 53, 26006 LOGROÑO, SPAIN.

   E-mail address: `emfernan@unirioja.es`

BCAM – BASQUE CENTER FOR APPLIED MATHEMATICS, 48009 BILBAO, SPAIN AND IKERBASQUE, BASQUE FOUNDATION FOR SCIENCE, 48011 BILBAO, SPAIN

   E-mail address: `lroncal@bcamath.org`