

 EXTRACTA MATHEMATICAE Vol. 27, Núm. 2, 155–161 (2012)

## Superstability of Approximate Cosine Type Functions on the Monoid $\mathbb{R}^2$

B. BOUIKHALENE<sup>1</sup>, E. ELQORACHI<sup>2</sup>, A. CHARIFI<sup>1</sup>

<sup>1</sup>University Sultan Moulay Slimane, Laboratory LIRST,  
Polydisciplinary Faculty, Beni-Mellal, Morocco

<sup>2</sup>Ibn Zohr University, Faculty of Sciences, Agadir, Morocco  
*bbouikhalene@yahoo.fr, elqorachi@yahoo.fr, charifi2000@yahoo.fr*

Presented by Pier L. Papini

Received March 26, 2012

*Abstract:* In this paper, we study the superstability problem for the cosine type functional equation

$$f(x_1x_2, x_1y_2 + x_2y_1) + f(x_1x_2, y_1x_2 - x_1y_2) = 2f(x_1, y_1)f(x_2, y_2)$$

on the commutative monoid  $(\mathbb{R}^2, \times)$ . As a result we obtain cosine type functions satisfying the equation approximately.

*Key words:* Functional equation, cosine function, superstability, multiplicative function.

AMS *Subject Class.* (2010): 39B72, 39B22.

### 1. INTRODUCTION

In 1940, S. M. Ulam [17] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

**QUESTION 1.1.** Let  $(G_1, *)$  be a group and let  $(G_2, \diamond, d)$  be a metric group with the metric  $d$ . Given  $\epsilon > 0$ , does there exist  $\delta(\epsilon) > 0$  such that if a mapping  $h : G_1 \rightarrow G_2$  satisfies the inequality  $d(h(x * y), h(x) \diamond h(y)) < \delta$  for all  $x, y \in G_1$ , then there is a homomorphism  $H : G_1 \rightarrow G_2$  with  $d(h(x), H(x)) < \epsilon$  for all  $x_1 \in G_1$ ?

In 1941, Hyers [11] answered this question for the case where  $G_1$  and  $G_2$  are Banach spaces. In [2] and [15] Aoki and Th. M. Rassias respectively provided a generalization of Hyer's theorem which allows the Cauchy difference to be unbounded. The interested reader may refer to the book by Hyers, Isac,

Rassias [12] for an in depth account on the subject of stability of functional equations. In 1982, J.M. Rassias [14] solved the Ulam problem by involving a product of powers of norms. Since then, the stability problems of various functional equations has been investigated by many authors (see [9], [10]). In [4] and [7] Baker et al. and Bourgin respectively, introduced the notion that by now is frequently referred to as superstability or Baker's stability : if a function  $f$  satisfies the stability inequality  $|E_1(f) - E_2(f)| \leq \varepsilon$ , then either  $f$  is bounded or  $E_1(f) = E_2(f)$ . The superstability of d'Alembert's functional equation  $f(x+y) + f(x-y) = 2f(x)f(y)$  was investigated by Baker [5] and Cholewa [8]. Badora and Ger [3] proved its superstability under the condition  $|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varphi(x)$  or  $\varphi(y)$ . In a previous work, Bouikhalene et al [6] investigated the superstability of the cosine functional equation on the Heisenberg group.

Now, Let  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$  be the commutative monoid equipped with composition rule

$$(x_1, y_1)(x_2, y_2) := (x_1x_2, x_1y_2 + x_2y_1). \quad (1.1)$$

The map  $i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , given by  $i(x, y) = (x, -y)$  for any  $(x, y) \in \mathbb{R}^2$ , is an involution of  $\mathbb{R}^2$ , i.e.,  $i((x_1, y_1)(x_2, y_2)) = i(x_1, y_1)i(x_2, y_2)$  for any  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$  and  $i \circ i = id$  (the identity map). Consider the functional equation

$$f(x_1x_2, x_1y_2 + x_2y_1) + f(x_1x_2, y_1x_2 - x_1y_2) = 2f(x_1, y_1)f(x_2, y_2) \quad (1.2)$$

for  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ . By setting  $a = (x_1, y_1)$ ,  $b = (x_2, y_2)$  in (1.2) we obtain the cosine type functional equation

$$f(ab) + f(ai(b)) = 2f(a)f(b), \quad a, b \in \mathbb{R}^2 \quad (1.3)$$

on the commutative monoid  $\mathbb{R}^2$ . This equation has the same form as the cosine functional equation, also called d'Alembert's functional equation ([1], [13])

$$f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G, \quad (1.4)$$

on an abelian group  $G$ , except that the group inversion  $y \rightarrow -y$  is replaced by the involution  $i$ . We say that a function  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  is of approximate a cosine type function, if there is  $\delta > 0$  such that

$$|f(ab) + f(ai(b)) - 2f(a)f(b)| < \delta, \quad a, b \in \mathbb{R}^2. \quad (1.5)$$

In the case where  $\delta = 0$ ,  $f$  satisfies the functional equation (1.3). We call  $f$  a cosine type function on  $\mathbb{R}^2$ . The main purpose of this work is to prove the superstability problem of equation (1.2) in the commutative monoid  $\mathbb{R}^2$ .

2. SUPERSTABILITY OF EQUATION (1.2)

PROPOSITION 2.1. *let  $\varphi, \psi, \phi, \zeta : \mathbb{R} \rightarrow [0, +\infty[$  be functions and let  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  satisfies the functional inequality*

$$|f(ab) + f(ai(b)) - 2f(a)f(b)| \leq \min \{ \varphi(x_1), \psi(y_1), \phi(x_2), \zeta(y_2) \} \quad (2.1)$$

for any  $a = (x_1, y_1), b = (x_2, y_2) \in \mathbb{R}^2$ . Then  $m(x) = f(x, 0)$  for any  $x \in \mathbb{R}$ , is either bounded or multiplicative function from  $\mathbb{R}$  to  $\mathbb{C}$ . Furthermore  $f$  satisfies the following inequality

$$\left| f(a)^2 - \frac{1}{2}f(a^2) - \frac{1}{2}m(x^2) \right| \leq \frac{1}{2} \min \{ \varphi(x), \psi(y), \phi(x), \zeta(y) \} \quad (2.2)$$

for any  $a = (x, y) \in \mathbb{R}^2$ .

*Proof.* Setting  $a = (x, 0), b = (y, 0)$  in (2.1), we get

$$|f(x, 0)f(y, 0) - f(xy, 0)| \leq \frac{1}{2} \min \{ \varphi(x), \psi(0), \phi(y), \zeta(0) \}$$

for any  $x, y \in \mathbb{R}$ . According to [16] we get that  $m(x) = f(x, 0)$  for any  $x \in \mathbb{R}$  is either bounded or a multiplicative function from  $\mathbb{R}$  to  $\mathbb{C}$ . Once again, putting  $a = (x, y)$  in (2.1) we get that

$$|f(x^2, 2xy) + f(x^2, 0) - 2f(x, y)^2| \leq \min \{ \varphi(x), \psi(y), \phi(x), \zeta(y) \}$$

for any  $x, y \in \mathbb{R}$ . So that

$$\left| f(a)^2 - \frac{1}{2}f(a^2) - \frac{1}{2}m(x^2) \right| \leq \frac{1}{2} \min \{ \varphi(x), \psi(y), \phi(x), \zeta(y) \}$$

for any  $a = (x, y) \in \mathbb{R}^2$ . ■

PROPOSITION 2.2. *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  satisfies the functional inequality (2.1) and let  $F(y) = f(1, y)$  for any  $y \in \mathbb{R}$ . Then*

- i)  $F$  is either bounded, or
- ii)  $F$  satisfies the cosine functional equation

$$F(x + y) + F(x - y) = 2F(x)F(y), \quad x, y \in \mathbb{R}. \quad (2.3)$$

Further, in the latter case, there exists an exponential function  $\gamma : \mathbb{R} \rightarrow \mathbb{C}$  such that

$$F(x) = \frac{1}{2}(\gamma(x) + \gamma(-x))$$

for any  $x \in \mathbb{R}$ . ■

*Proof.* Let  $a = (1, x)$ ,  $b = (1, y)$  for any  $x, y \in \mathbb{R}$  in (2.1). By setting  $F(y) = f(1, y)$  for any  $y \in \mathbb{R}$  we get

$$|F(x+y) + F(x-y) - 2F(x)F(y)| \leq \min \{\varphi(1), \psi(x), \phi(1), \zeta(y)\}$$

for any  $x, y \in \mathbb{R}$ . According to ([3], [5]) it follows that  $F$  is either bounded or  $F$  is a cosine function. In view of ([1], [5], [13]) we get that there exists an exponential function  $\gamma : \mathbb{R} \rightarrow \mathbb{C}$  such that  $F(x) = \frac{1}{2}(\gamma(x) + \gamma(-x))$  for any  $x \in \mathbb{R}$ . ■

**PROPOSITION 2.3.** *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  satisfies the functional inequality (2.1). Then  $f$  is either bounded or  $f \circ i = f$ .*

*Proof.* Let  $P_f = \frac{f+f \circ i}{2}$ . Since  $f$  satisfies (2.1), we have

$$|P_f(ab) + P_f(ai(b)) - 2P_f(a)f(b)| \leq \min \{\varphi(x_1), \tilde{P}_\psi(y_1), \phi(x_2), \tilde{P}_\zeta(y_2)\}$$

for any  $a = (x_1, y_1)$ ,  $b = (x_2, y_2) \in \mathbb{R}^2$ , where  $\tilde{P}_\psi(x) = \frac{\psi(x) + \psi(-x)}{2}$  for any  $x \in \mathbb{R}$ . By using the same way as in [3] and [5] we get that  $f$  is either bounded or  $f$  satisfies the Wilson's type functional equation

$$P_f(ab) + P_f(ai(b)) = 2P_f(a)f(b), \quad a, b \in \mathbb{R}^2$$

on the commutative monoid  $\mathbb{R}^2$ . By small computations we get that  $f \circ i = f$ . ■

**PROPOSITION 2.4.** *Let  $\varphi, \psi, \phi, \zeta : \mathbb{R} \rightarrow [0, +\infty[$  be functions and let  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$ , with  $f(0, 0) \neq 0$ , satisfies the functional inequality (2.1). Then  $f$  is bounded and we have*

$$|f(a) - 1| \leq \frac{1}{2|f(0, 0)|} \min \{\varphi(x), \psi(y), \phi(0), \zeta(0)\} \quad (2.4)$$

for any  $a = (x, y) \in \mathbb{R}^2$ .

*Proof.* By letting  $b = (0, 0)$  in (2.1) we get

$$|2f(0, 0) - 2f(a)f(0, 0)| \leq \min \{ \varphi(x), \psi(y), \phi(0), \zeta(0) \}$$

for any  $a = (x, y) \in \mathbb{R}$ . So that we have

$$|2f(0, 0)||f(a) - 1| \leq \min \{ \varphi(x), \psi(y), \phi(0), \zeta(0) \}$$

for any  $a = (x, y) \in \mathbb{R}^2$ . ■

**THEOREM 2.5.** *Let  $\varphi, \psi, \phi, \zeta : \mathbb{R} \rightarrow [0, +\infty[$  be functions and let  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  satisfies the functional inequality (2.1). Then*

i)  $f$  is either bounded and

$$|f(0, y)^2 - f(0, 0)| \leq \frac{1}{2} \min \{ \varphi(0), \psi(y), \phi(0), \zeta(y) \} \quad (2.5)$$

for any  $y \in \mathbb{R}$  or

ii)  $f$  satisfies the functional inequality

$$\left| f(a) - m(x) \frac{\gamma(\frac{y}{x}) + \gamma(\frac{-y}{x})}{2} \right| \leq \frac{1}{2} \min \{ \varphi(x), \psi(0), \phi(1), \zeta(\frac{y}{x}) \} \quad (2.6)$$

for any  $a = (x, y) \in \mathbb{R}$  with  $x \neq 0$ , where  $m : \mathbb{R} \rightarrow \mathbb{C}$  is a multiplicative function and  $\gamma : \mathbb{R} \rightarrow \mathbb{C}$  is an exponential function.

*Proof.* i) Letting  $a = b = (0, y)$  in (2.1), we get

$$|f(0, y)^2 - f(0, 0)| \leq \frac{1}{2} \min \{ \varphi(0), \psi(y), \phi(0), \zeta(y) \}$$

for any  $y \in \mathbb{R}$ .

ii) Let  $f$  be unbounded. Hence by Propositions 2.1 and 2.2 we get that  $f(x, 0) = m(x)$  for any  $x \in \mathbb{R}$  is a multiplicative function from  $\mathbb{R}$  to  $\mathbb{C}$  and  $f(1, y) = F(y)$  for any  $y \in \mathbb{R}$  is a solution of the cosine functional equation (1.4). Therefore there exists an exponential function  $\gamma : \mathbb{R} \rightarrow \mathbb{C}$  such that  $f(1, y) = F(y) = \frac{\gamma(y) + \gamma(-y)}{2}$  for any  $y \in \mathbb{R}$ . By letting  $a = (x, 0), b = (1, \frac{y}{x})$ , with  $x \neq 0$ , in (2.1) we get the following inequality

$$|f(x, y) + f(x, -y) - 2f(x, 0)f(1, \frac{y}{x})| \leq \min \{ \varphi(x), \psi(0), \phi(1), \zeta(\frac{y}{x}) \} \quad (2.7)$$

for any  $x, y \in \mathbb{R}$  with  $x \neq 0$ . Therefore by Proposition 2.3 we get that  $f(x, y) = f \circ i(x, y) = f(x, -y)$  for any  $x, y \in \mathbb{R}$ . So that we get from (2.7) that

$$|f(x, y) - m(x)F(\frac{y}{x})| \leq \frac{1}{2} \min \{ \varphi(x), \psi(0), \phi(1), \zeta(\frac{y}{x}) \}$$

for any  $x, y \in \mathbb{R}$  with  $x \neq 0$ . ■

In the next corollary we let  $\varphi(x_1) = \psi(y_1) = \varphi(x_2) = \zeta(y_2) = \delta$  for any  $x_1, y_1, x_2, y_2 \in \mathbb{R}$ .

**COROLLARY 2.6.** *Let  $\delta > 0$  and let  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  satisfies the functional inequality*

$$|f(ab) + f(ai(b)) - 2f(a)f(b)| \leq \delta \quad (2.8)$$

for any  $a, b \in \mathbb{R}^2$ . Then

- i)  $f$  is bounded and there exists  $\eta \in \mathbb{C}^*$  such that  $|f(a) - \eta| \leq \frac{\delta}{2\eta}$  for any  $a = (x, y) \in \mathbb{R}^2$ , with  $x \neq 0$ . Furthermore  $|f(0, y) - \eta| \leq \frac{\delta}{2}$  for any  $y \in \mathbb{R}$  or
- ii)  $f$  is unbounded and there exist a multiplicative function  $m : \mathbb{R} \rightarrow \mathbb{C}$  and an exponential function  $\gamma : \mathbb{R} \rightarrow \mathbb{C}$  such that

$$\left| f(a) - m(x) \frac{\gamma(\frac{y}{x}) + \gamma(-\frac{y}{x})}{2} \right| \leq \frac{\delta}{2} \quad (2.9)$$

for any  $a = (x, y) \in \mathbb{R}^2$  with  $x \neq 0$ .

*Proof.* By using Proposition 2.4 and Theorem 2.5 with  $\eta = f(0, 0)$ . ■

In the next corollary we give the explicit formula of cosine type functions on  $\mathbb{R}^2$

**COROLLARY 2.7.** *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{C}$  be a cosine type function on  $\mathbb{R}^2$ . Then*

- i)  $f(x, y) = 1$  for any  $x, y \in \mathbb{R}$  or
- ii)

$$f(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{m(x)}{2} (\gamma(\frac{y}{x}) + \gamma(-\frac{y}{x})) & \text{if } x \neq 0, \end{cases}$$

for any  $x, y \in \mathbb{R}$ .

*Proof.* By letting  $\delta = 0$  in Corollary 2.6. ■

## REFERENCES

- [1] J. ACZÉL, J. DHOMBRES, “Functional Equations in Several Variables”, Cambridge University Press, Cambridge, 1989.
- [2] T. AOKI, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan* **2** (1950), 64-66.
- [3] R. BADORA, R. GER, On some trigonometric functional inequalities, in “Functional Equation-Results and Advances”, Kluwer Acad. Publ., Dordrecht, 2002, 3–15.
- [4] J. BAKER, J. LAWRENCE, F. ZORZITTO, The stability of the equation  $f(x + y) = f(x)f(y)$ , *Proc. Amer. Math. Soc.* **74** (2) (1979), 242–246.
- [5] J. BAKER, The stability of the cosine equation, *Proc. Amer. Math. Soc.* **80** (3) (1980), 411–416.
- [6] B. BOUIKHALENE, E. ELQORACHI, J.M. RASSIAS, The superstability of d’Alembert’s functional equation on the Heisenberg group *Appl. Math. Lett.* **23** (1) (2010), 105–109.
- [7] D.G. BOURGIN, Approximately isometric and multiplicative transformations on continuous function rings, *Duke. Math. J.* **16** (2) (1949), 385–397.
- [8] P.W. CHOLEWA, The stability of the sine equation, *Proc. Amer. Math. Soc.* **88** (4) (1983), 631–634.
- [9] Z. GAJDA, On stability of additive mappings, *Internat. J. Math. Math. Sci.* **14** (3) (1991), 431–434.
- [10] P. GĂVRUTA, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (3) (1994), 431–436.
- [11] D.H. HYERS, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222–224.
- [12] D.H. HYERS, G. ISAC, TH.M. RASSIAS, “Stability of Functional Equations in Several Variables”, Birkhäuser Boston Inc., Boston, MA, 1998.
- [13] PL. KANNAPPAN, The functional equation  $f(xy) + f(xy^{-1}) = 2f(x)f(y)$  for groups, *Proc. Amer. Math. Soc.* **19** (1) (1968), 69–74.
- [14] J.M. RASSIAS, On approximation of approximately linear mapping by linear mappings, *J. Funct. Anal.* **46** (1) (1982), 126–130.
- [15] TH.M. RASSIAS, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (2), (1978), 297–300.
- [16] L. SZÉKELYHIDI, On a theorem of Baker, Lawrence and Zorzitto, *Proc. Amer. Math. Soc.* **84** (1) (1982), 95–96.
- [17] S.M. ULAM, “A Collection of Mathematical Problems”, Interscience Publishers, New York-London, 1960.