



Fractional Ostrowski type inequalities for functions whose derivatives are s -preinvex

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Abstract: In this paper, we establish a new integral identity, and then we derive some new fractional Ostrowski type inequalities for functions whose derivatives are s -preinvex.

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1. INTRODUCTION

In 1938, A.M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

THEOREM 1. ([21]) *Let $I \subseteq \mathbb{R}$ be an interval. Let $f : I \rightarrow \mathbb{R}$, be a differentiable mapping in the interior I° of I , and $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq M (b-a) \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right]. \quad (1.1)$$

The above inequality has attracted many researchers, various generalizations, refinements, extensions and variants have appeared in the literature see for instance [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 24, 25], and references therein.



Set [25], gave the following fractional Ostrowski inequality for differentiable s -convex functions

$$\begin{aligned} & \left| \left(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{b-a} ((I_{x^-}^\alpha f)(a) + (I_{x^+}^\alpha f)(b)) \right| \\ & \leq \frac{M}{b-a} \left(1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \right) \left(\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha+s+1} \right) \end{aligned}$$

and

$$\begin{aligned} & \left| \left(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right) f(x) - \frac{\Gamma(\alpha+1)}{b-a} ((I_{x^-}^\alpha f)(a) + (I_{x^+}^\alpha f)(b)) \right| \\ & \leq \frac{M}{(1+p\alpha)^{\frac{1}{p}}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \left(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a} \right). \end{aligned}$$

Kirmaci et al. [4], presented some results connected with inequality (1.1)

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|).$$

Zhu et al. [27], established the following fractional midpoint inequality

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [(J_{a^+}^\alpha f)(b) + (J_{b^-}^\alpha f)(a)] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{4(1+\alpha)} (|f'(a)| + |f'(b)|) \left(\alpha + 3 - \frac{1}{2^{\alpha-1}} \right). \end{aligned}$$

Motivated by the above results, in this paper, we establish a new integral identity, and then we derive some new fractional Ostrowski type inequalities for functions whose derivatives are s -preinvex.

2. PRELIMINARIES

In this sections we recall some definitions and lemmas

DEFINITION 1. ([23]) A function $f : I \rightarrow \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

DEFINITION 2. ([1]) A nonnegative function $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense for some fixed $s \in (0, 1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

DEFINITION 3. ([18]) A set $K \subseteq \mathbb{R}^n$ is said to be an invex with respect to the bifunction $\eta : K \times K \rightarrow \mathbb{R}^n$, if for all $x, y \in K$, we have

$$x + t\eta(y, x) \in K.$$

In what follows we assume that $K \subseteq \mathbb{R}$ be an invex set with respect to the bifunction $\eta : K \times K \rightarrow \mathbb{R}$.

DEFINITION 4. ([26]) A function $f : K \rightarrow \mathbb{R}$ is said to be preinvex with respect to η , if

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y)$$

holds for all $x, y \in K$ and all $t \in [0, 1]$.

DEFINITION 5. ([5]) A nonnegative function $f : K \subset [0, \infty) \rightarrow \mathbb{R}$ is said to be s -preinvex in the second sense with respect to η for some fixed $s \in (0, 1]$, if

$$f(x + t\eta(y, x)) \leq (1 - t)^s f(x) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0, 1]$.

DEFINITION 6. ([2, 3, 17]) Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals $I_{a^+}^\alpha f$ and $I_{b^-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t) dt, \quad x > a,$$

$$I_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t) dt, \quad b > x,$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$, is the Gamma function and $I_{a^+}^0 f(x) = I_{b^-}^0 f(x) = f(x)$.

We also recall that the beta function for any complex numbers and non-positive integers ρ, τ such that $\text{Re}(\rho) > 0$ and $\text{Re}(\tau) > 0$ is defined by

$$B(\rho, \tau) = \int_0^1 \theta^{\rho-1} (1 - \theta)^{\tau-1} d\theta = \frac{\Gamma(\rho)\Gamma(\tau)}{\Gamma(\rho + \tau)}.$$

The incomplete beta function is given by

$$B_t(\rho, \tau) = \int_0^t \theta^{\rho-1} (1 - \theta)^{\tau-1} d\theta, \quad 0 < t < 1.$$

LEMMA 1. ([22]) For any $0 \leq a < b$ in \mathbb{R} and fixed $p \geq 1$, we have

$$(b - a)^p \leq b^p - a^p.$$

3. MAIN RESULTS

In what follows, in order to simplify and lighten the writing, we note in all the proofs $\eta(b, a)$ by c .

LEMMA 2. Let $f : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a differentiable mapping on $(a, a + \eta(b, a))$ with $\eta(b, a) > 0$, and assume that $f' \in L([a, a + \eta(b, a)])$. Then the following equality holds

$$\begin{aligned} f(x) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} & \left((I_{a^+}^\alpha f)(a + \eta(b, a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \\ & = \frac{\eta(b,a)}{2} \left(\int_0^1 k f'(a + t\eta(b, a)) dt \right. \\ & \quad \left. + \int_0^1 (t^\alpha - (1-t)^\alpha) f'(a + t\eta(b, a)) dt \right), \end{aligned} \quad (3.1)$$

where

$$k = \begin{cases} 1 & \text{if } 0 \leq t < \frac{x-a}{\eta(b,a)}, \\ -1 & \text{if } \frac{x-a}{\eta(b,a)} \leq t < 1. \end{cases} \quad (3.2)$$

Proof. Let $c = \eta(b, a)$. And let

$$\begin{aligned} I & = \int_0^1 k f'(a + tc) dt + \int_0^1 (t^\alpha - (1-t)^\alpha) f'(a + tc) dt \\ & = I_1 + I_2, \end{aligned} \quad (3.3)$$

where

$$I_1 = \int_0^1 k f'(a + tc) dt, \quad (3.4)$$

$$I_2 = \int_0^1 (t^\alpha - (1-t)^\alpha) f'(a + tc) dt, \quad (3.5)$$

and k is defined by (3.2).

Clearly,

$$\begin{aligned} I_1 & = \int_0^1 k f'(a + tc) dt = \int_0^{\frac{x-a}{c}} f'(a + tc) dt - \int_{\frac{x-a}{c}}^1 f'(a + tc) dt \\ & = \frac{2}{c} f(x) - \frac{1}{c} [f(a) + f(a + c)]. \end{aligned} \quad (3.6)$$

Now, by integration by parts, I_2 gives

$$\begin{aligned}
 I_2 &= \frac{1}{c} f(a+c) + \frac{1}{c} f(a) \\
 &\quad - \frac{\alpha}{c} \left(\int_0^1 t^{\alpha-1} f(a+tc) dt + \int_0^1 (1-t)^{\alpha-1} f(a+tc) dt \right) \\
 &= \frac{1}{c} f(a+c) + \frac{1}{c} f(a) \\
 &\quad - \frac{\alpha}{c^{\alpha+1}} \left(\int_a^{a+c} (u-a)^{\alpha-1} f(u) du + \int_a^{a+c} (c+a-u)^{\alpha-1} f(u) du \right) \\
 &= \frac{1}{c} f(a+c) + \frac{1}{c} f(a) - \frac{\Gamma(\alpha+1)}{c^{\alpha+1}} \left((I_{a^+}^\alpha f)(a+c) + (I_{(a+c)^-}^\alpha f)(a) \right).
 \end{aligned} \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.3). Multiplying the resulting equality by $\frac{c}{2}$, and then replacing c by $\eta(a, b)$, we get the desired result. ■

THEOREM 2. *Let $f : [a, a + \eta(b, a)] \rightarrow \mathbb{R}$ be a positive function on $[a, b]$ with $\eta(b, a) > 0$ and $f \in L[a, b]$. If $|f'|$ is s -preinvex function with $s \in (0, 1]$, then the following fractional inequality holds*

$$\begin{aligned}
 &\left| f(x) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a + \eta(b, a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\
 &\leq \frac{\eta(b,a)}{2} \left(|f'(a)| + |f'(b)| \right) \\
 &\quad \times \left(\frac{1}{s+1} + \frac{1 - (\frac{1}{2})^{\alpha+s}}{\alpha+s+1} + B_{\frac{1}{2}}(s+1, \alpha+1) - B_{\frac{1}{2}}(\alpha+1, s+1) \right).
 \end{aligned} \tag{3.8}$$

where $B_{\frac{1}{2}}(\cdot, \cdot)$ is the incomplete beta function.

Proof. Let $c = \eta(a, b)$. From Lemma 2, and properties of modulus, we have

$$\begin{aligned}
 &\left| f(x) - \frac{\Gamma(\alpha+1)}{2c^\alpha} \left((I_{a^+}^\alpha f)(a+c) + (I_{(a+c)^-}^\alpha f)(a) \right) \right| \\
 &\leq \frac{c}{2} \left[\int_0^{\frac{x-a}{c}} |f'(a+tc)| dt + \int_{\frac{x-a}{c}}^1 |f'(a+tc)| dt \right. \\
 &\quad \left. + \int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) |f'(a+tc)| dt + \int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha) |f'(a+tc)| dt \right].
 \end{aligned}$$

Using the s -preinvexity of $|f'|$, the above inequality gives

$$\begin{aligned}
& \left| f(x) - \frac{\Gamma(\alpha+1)}{2c^\alpha} \left((I_{a^+}^\alpha f)(a+c) + (I_{(a+c)^-}^\alpha f)(a) \right) \right| \\
& \leq \frac{c}{2} \left(\int_0^{\frac{x-a}{c}} ((1-t)^s |f'(a)| + t^s |f'(b)|) dt \right. \\
& \quad + \int_{\frac{x-a}{c}}^1 ((1-t)^s |f'(a)| + t^s |f'(b)|) dt \\
& \quad + \int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) ((1-t)^s |f'(a)| + t^s |f'(b)|) dt \\
& \quad \left. + \int_{\frac{1}{2}}^1 ((1-t)^\alpha - t^\alpha) ((1-t)^s |f'(a)| + t^s |f'(b)|) dt \right) \\
& = \frac{c}{2} \left[\left(\frac{(1-(1-\frac{x-a}{c})^{s+1}) |f'(a)| + (\frac{x-a}{c})^{s+1} |f'(b)|}{s+1} \right) \right. \\
& \quad + \left(\frac{(1-\frac{x-a}{c})^{s+1} |f'(a)| + (1-(\frac{x-a}{c})^{s+1}) |f'(b)|}{s+1} \right) \\
& \quad + |f'(a)| \left(\int_0^{\frac{1}{2}} ((1-t)^{\alpha+s} - t^\alpha (1-t)^s) dt \right. \\
& \quad \quad \left. + \int_{\frac{1}{2}}^1 (t^\alpha (1-t)^s - (1-t)^{\alpha+s}) dt \right) \\
& \quad \left. + |f'(b)| \left(\int_0^{\frac{1}{2}} (t^s (1-t)^\alpha - t^{\alpha+s}) dt + \int_{\frac{1}{2}}^1 (t^{\alpha+s} - t^s (1-t)^\alpha) dt \right) \right] \\
& = \frac{c}{2} (|f'(a)| + |f'(b)|) \\
& \quad \times \left(\frac{1}{s+1} + \int_0^{\frac{1}{2}} ((1-t)^{\alpha+s} - t^\alpha (1-t)^s) dt + \int_0^{\frac{1}{2}} (t^s (1-t)^\alpha - t^{\alpha+s}) dt \right) \\
& = \frac{c}{2} (|f'(a)| + |f'(b)|) \\
& \quad \times \left(\frac{1}{s+1} + \frac{1-(\frac{1}{2})^{\alpha+s}}{\alpha+s+1} + B_{\frac{1}{2}}(s+1, \alpha+1) - B_{\frac{1}{2}}(\alpha+1, s+1) \right).
\end{aligned}$$

Replacing c by $\eta(b, a)$ in the above inequality, we get the desired result. The proof is completed. \blacksquare

COROLLARY 1. In Theorem 2 if we choose $x = \frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality

$$\begin{aligned} & \left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a + \eta(b,a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{\eta(b,a)}{2} (|f'(a)| + |f'(b)|) \\ & \quad \times \left(\frac{1}{s+1} + \frac{1 - (\frac{1}{2})^{\alpha+s}}{\alpha+s+1} + B_{\frac{1}{2}}(s+1, \alpha+1) - B_{\frac{1}{2}}(\alpha+1, s+1) \right). \end{aligned}$$

Moreover if we take $\eta(b,a) = b - a$, we obtain

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left((I_{a^+}^\alpha f)(b) + (I_{b^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{b-a}{2} (|f'(a)| + |f'(b)|) \\ & \quad \times \left(\frac{1}{s+1} + \frac{1 - (\frac{1}{2})^{\alpha+s}}{\alpha+s+1} + B_{\frac{1}{2}}(s+1, \alpha+1) - B_{\frac{1}{2}}(\alpha+1, s+1) \right). \end{aligned}$$

COROLLARY 2. In Theorem 2 if we put $s = 1$, we obtain

$$\begin{aligned} & \left| f(x) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a + \eta(b,a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{\eta(b,a)}{4(\alpha+1)} (|f'(a)| + |f'(b)|) \left(\alpha + 3 - \left(\frac{1}{2}\right)^{\alpha-1} \right). \end{aligned}$$

Moreover if we choose $x = \frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality

$$\begin{aligned} & \left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a + \eta(b,a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{\eta(b,a)}{4(\alpha+1)} (|f'(a)| + |f'(b)|) \frac{1}{2(\alpha+1)} \left(\alpha + 3 - \left(\frac{1}{2}\right)^{\alpha-1} \right). \end{aligned}$$

Remark 1. Theorem 2 will be reduced to Theorem 2.3 from [27], if we choose $\eta(b,a) = b - a$, $x = \frac{a+b}{2}$ and $s = 1$.

THEOREM 3. Let $f : [a, b] \subset [0, \infty) \rightarrow \mathbb{R}$ be a positive function on $[a, b]$ with $a < b$ and $f \in L[a, b]$. If $|f'|^q$ is s -preinvex function, where $s \in (0, 1]$ and

$q \geq 1$, then the following fractional inequality holds

$$\begin{aligned}
& \left| f(x) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a + \eta(b,a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\
& \leq \frac{\eta(b,a)}{2} \left[\left(\frac{\left(1 - \left(1 - \frac{x-a}{\eta(b,a)}\right)^{s+1}\right) |f'(a)|^q + \left(\frac{x-a}{\eta(b,a)}\right)^{s+1} |f'(b)|^q d}{s+1} \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\frac{\left(1 - \frac{x-a}{\eta(b,a)}\right)^{s+1} |f'(a)|^q + \left(1 - \left(\frac{x-a}{\eta(b,a)}\right)^{s+1}\right) |f'(b)|^q d}{s+1} \right)^{\frac{1}{q}} \quad (3.9) \\
& \quad + \left(\frac{2^\alpha - 1}{(\alpha+1)2^\alpha} \right)^{1-\frac{1}{q}} \left((\mu_{\alpha,s}^1 |f'(a)|^q + \mu_{\alpha,s}^2 |f'(b)|^q)^{\frac{1}{q}} \right. \\
& \quad \left. \left. + (\mu_{\alpha,s}^2 |f'(a)|^q + \mu_{\alpha,s}^1 |f'(b)|^q)^{\frac{1}{q}} \right) \right],
\end{aligned}$$

where

$$\mu_{\alpha,s}^1 = \frac{1 - \left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1} - B_{\frac{1}{2}}(\alpha+1, s+1) \quad (3.10)$$

and

$$\mu_{\alpha,s}^2 = B_{\frac{1}{2}}(s+1, \alpha+1) - \frac{\left(\frac{1}{2}\right)^{\alpha+s+1}}{\alpha+s+1}. \quad (3.11)$$

Proof. Let $c = \eta(b, a)$. From Lemma 2, properties of modulus, and power mean inequality, we have

$$\begin{aligned}
& \left| f(x) - \frac{\Gamma(\alpha+1)}{2c^\alpha} \left((I_{a^+}^\alpha f)(a+c) + (I_{(a+c)^-}^\alpha f)(a) \right) \right| \\
& \leq \frac{c}{2} \left[\left(\int_0^{\frac{x-a}{c}} |f'(a+tc)|^q dt \right)^{\frac{1}{q}} + \left(\int_{\frac{x-a}{c}}^1 |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) dt \right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \\
& \quad \left. + \left(\int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha) dt \right)^{1-\frac{1}{q}} \left(\int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha) |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Since $|f'|^q$ is s -preinvex function, we deduce

$$\begin{aligned}
 & \left| f(x) - \frac{\Gamma(\alpha+1)}{2c^\alpha} \left((I_{a^+}^\alpha f)(a+c) + (I_{(a+c)^-}^\alpha f)(a) \right) \right| \\
 & \leq \frac{c}{2} \left[\left(\int_0^{\frac{x-a}{c}} (1-t)^s |f'(a)|^q + t^s |f'(b)|^q dt \right)^{\frac{1}{q}} \right. \\
 & \quad + \left(\int_{\frac{x-a}{c}}^1 (1-t)^s |f'(a)|^q + t^s |f'(b)|^q dt \right)^{\frac{1}{q}} \\
 & \quad + \left. \left(\frac{1 - (\frac{1}{2})^\alpha}{\alpha+1} \right)^{1-\frac{1}{q}} \left\{ \left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) ((1-t)^s |f'(a)|^q + t^s |f'(b)|^q) dt \right)^{\frac{1}{q}} \right. \right. \\
 & \quad \quad \left. \left. + \left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha) (t^s |f'(a)|^q + (1-t)^s |f'(b)|^q) dt \right)^{\frac{1}{q}} \right\} \right] \\
 & = \frac{c}{2} \left[\left(\frac{(1 - (1 - \frac{x-a}{c})^{s+1}) |f'(a)|^q + (\frac{x-a}{c})^{s+1} |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right. \\
 & \quad + \left(\frac{(1 - \frac{x-a}{c})^{s+1} |f'(a)|^q + (1 - (\frac{x-a}{c})^{s+1}) |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \\
 & \quad + \left. \left(\frac{1 - (\frac{1}{2})^\alpha}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\left\{ \left(\frac{1 - (\frac{1}{2})^{\alpha+s+1}}{\alpha+s+1} - B_{\frac{1}{2}}(\alpha+1, s+1) \right) |f'(a)|^q \right. \right. \right. \\
 & \quad \quad \left. \left. + \left(B_{\frac{1}{2}}(s+1, \alpha+1) - \frac{(\frac{1}{2})^{\alpha+s+1}}{\alpha+s+1} \right) |f'(b)|^q \right\}^{\frac{1}{q}} \right. \\
 & \quad \quad \left. + \left\{ \left(B_{\frac{1}{2}}(s+1, \alpha+1) - \frac{(\frac{1}{2})^{\alpha+s+1}}{\alpha+s+1} \right) |f'(a)|^q \right. \right. \\
 & \quad \quad \left. \left. + \left(\frac{1 - (\frac{1}{2})^{\alpha+s+1}}{\alpha+s+1} - B_{\frac{1}{2}}(\alpha+1, s+1) \right) |f'(b)|^q \right\}^{\frac{1}{q}} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{c}{2} \left[\left(\frac{\left(1 - \left(1 - \frac{x-a}{c}\right)^{s+1}\right) |f'(a)|^q + \left(\frac{x-a}{c}\right)^{s+1} |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\frac{\left(-\left(1 - \frac{x-a}{c}\right)^{s+1}\right) |f'(a)|^q + \left(1 - \left(\frac{x-a}{c}\right)^{s+1}\right) |f'(b)|^q}{s+1} \right)^{\frac{1}{q}} \\
&\quad + \left(\frac{2^\alpha - 1}{(\alpha+1)2^\alpha} \right)^{1-\frac{1}{q}} \left\{ \left(\mu_{\alpha,s}^1 |f'(a)|^q + \mu_{\alpha,s}^2 |f'(b)|^q \right)^{\frac{1}{q}} \right. \\
&\quad \quad \left. \left. + \left(\mu_{\alpha,s}^2 |f'(a)|^q + \mu_{\alpha,s}^1 |f'(b)|^q \right)^{\frac{1}{q}} \right\} \right],
\end{aligned}$$

where $\mu_{\alpha,s}^1$ and $\mu_{\alpha,s}^2$ are defined as in (3.10) and (3.11) respectively. By replacing c by $\eta(b, a)$ in the above inequality, we get the desired result. The prove is completed. ■

COROLLARY 3. *In Theorem 3 if we choose $x = \frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality*

$$\begin{aligned}
&\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a + \eta(b, a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\
&\leq \frac{\eta(b,a)}{2} \left[\left(\frac{\left(1 - \left(\frac{1}{2}\right)^{s+1}\right) |f'(a)|^q + \left(\frac{1}{2}\right)^{s+1} |f'(b)|^q d}{s+1} \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\frac{\left(\left(\frac{1}{2}\right)\right)^{s+1} |f'(a)|^q + \left(1 - \left(\frac{1}{2}\right)^{s+1}\right) |f'(b)|^q d}{s+1} \right)^{\frac{1}{q}} \\
&\quad + \left(\frac{2^\alpha - 1}{(\alpha+1)2^\alpha} \right)^{1-\frac{1}{q}} \left\{ \left(\mu_{\alpha,s}^1 |f'(a)|^q + \mu_{\alpha,s}^2 |f'(b)|^q \right)^{\frac{1}{q}} \right. \\
&\quad \quad \left. \left. + \left(\mu_{\alpha,s}^2 |f'(a)|^q + \mu_{\alpha,s}^1 |f'(b)|^q \right)^{\frac{1}{q}} \right\} \right].
\end{aligned}$$

Moreover if we take $\eta(b, a) = b - a$, we obtain

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left((I_{a^+}^\alpha f)(b) + (I_{b^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{b-a}{2} \left[\left(\frac{\left(1 - \left(\frac{1}{2}\right)^{s+1}\right) |f'(a)|^q + \left(\frac{1}{2}\right)^{s+1} |f'(b)|^q d}{s+1} \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\frac{\left(\left(\frac{1}{2}\right)^{s+1} |f'(a)|^q + \left(1 - \left(\frac{1}{2}\right)^{s+1}\right) |f'(b)|^q d\right)}{s+1} \right)^{\frac{1}{q}} \\ & \quad + \left(\frac{2^\alpha - 1}{(\alpha+1)2^\alpha} \right)^{1-\frac{1}{q}} \left\{ \left(\mu_{\alpha,s}^1 |f'(a)|^q + \mu_{\alpha,s}^2 |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. \left. + \left(\mu_{\alpha,s}^2 |f'(a)|^q + \mu_{\alpha,s}^1 |f'(b)|^q \right)^{\frac{1}{q}} \right\} \right]. \end{aligned}$$

COROLLARY 4. In Theorem 3 if we put $s = 1$, we obtain

$$\begin{aligned} & \left| f(x) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a + \eta(b,a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{\eta(b,a)}{2} \left[\left(\frac{\left(1 - \left(1 - \frac{x-a}{\eta(b,a)}\right)^2\right) |f'(a)|^q + \left(\frac{x-a}{\eta(b,a)}\right)^2 |f'(b)|^q d}{2} \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\frac{\left(1 - \frac{x-a}{\eta(b,a)}\right)^2 |f'(a)|^q + \left(1 - \left(\frac{x-a}{\eta(b,a)}\right)^2\right) |f'(b)|^q d}{2} \right)^{\frac{1}{q}} \\ & \quad + \left(\frac{2^\alpha - 1}{(\alpha+1)2^\alpha} \right)^{1-\frac{1}{q}} \left\{ \left(\mu_{\alpha,s}^1 |f'(a)|^q + \mu_{\alpha,s}^2 |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. \left. + \left(\mu_{\alpha,s}^2 |f'(a)|^q + \mu_{\alpha,s}^1 |f'(b)|^q \right)^{\frac{1}{q}} \right\} \right], \end{aligned}$$

where

$$\mu_{\alpha,1}^1 = \frac{1}{\alpha+2} - \frac{1}{\alpha+1} \left(\frac{1}{2}\right)^{\alpha+1} \tag{3.12}$$

and

$$\mu_{\alpha,1}^2 = \frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{\alpha+1} \left(\frac{1}{2}\right)^{\alpha+1}. \tag{3.13}$$

Moreover if we choose $x = \frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality

$$\begin{aligned} & \left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a+\eta(b,a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{\eta(b,a)}{2} \left[\left(\frac{3|f'(a)|^q + |f'(b)|^q d}{8} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q d}{8} \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\frac{2^\alpha - 1}{(\alpha+1)2^\alpha} \right)^{1-\frac{1}{q}} \left\{ \left(\mu_{\alpha,1}^1 |f'(a)|^q + \mu_{\alpha,1}^2 |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. \left. + \left(\mu_{\alpha,1}^2 |f'(a)|^q + \mu_{\alpha,1}^1 |f'(b)|^q \right)^{\frac{1}{q}} \right\} \right]. \end{aligned}$$

Additionally if we take $\eta(b,a) = b-a$, we obtain

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left((I_{a^+}^\alpha f)(b) + (I_{b^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{b-a}{2} \left[\left(\frac{3|f'(a)|^q + |f'(b)|^q d}{8} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q d}{8} \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\frac{2^\alpha - 1}{(\alpha+1)2^\alpha} \right)^{1-\frac{1}{q}} \left\{ \left(\mu_{\alpha,1}^1 |f'(a)|^q + \mu_{\alpha,1}^2 |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. \left. + \left(\mu_{\alpha,1}^2 |f'(a)|^q + \mu_{\alpha,1}^1 |f'(b)|^q \right)^{\frac{1}{q}} \right\} \right], \end{aligned}$$

where $\mu_{\alpha,1}^1$ and $\mu_{\alpha,1}^2$ are defined as in (3.12) and (3.13) respectively.

THEOREM 4. Let $f : [a, b] \subset [0, \infty) \rightarrow \mathbb{R}$ be a positive function on $[a, b]$ with $a < b$ and $f \in L[a, b]$. If $|f'|^q$ is s -preinvex function, where $s \in (0, 1]$, and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following fractional inequality holds

$$\begin{aligned} & \left| f(x) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a+\eta(b,a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{\eta(b,a)}{2(s+1)^{\frac{1}{q}}} \left[\left(\left(1 - \left(1 - \frac{x-a}{\eta(b,a)} \right)^{s+1} \right) |f'(a)|^q + \left(\frac{x-a}{\eta(b,a)} \right)^{s+1} |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\left(1 - \frac{x-a}{\eta(b,a)} \right)^{s+1} |f'(a)|^q + \left(1 - \left(\frac{x-a}{\eta(b,a)} \right)^{s+1} \right) |f'(b)|^q \right)^{\frac{1}{q}} \right] \end{aligned}$$

$$+ \left(\frac{1 - (\frac{1}{2})^{\alpha p}}{\alpha p + 1} \right)^{\frac{1}{p}} \times \left[\left(\frac{(2^{s+1} - 1)|f'(a)|^q + |f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + (2^{s+1} - 1)|f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} \right].$$

Proof. Let $c = \eta(b, a)$. From Lemma 2, properties of modulus, Hölder’s inequality, and Lemma 1, we have

$$\begin{aligned} & \left| f(x) - \frac{\Gamma(\alpha+1)}{2c^\alpha} \left((I_{a^+}^\alpha f)(a+c) + (I_{(a+c)^-}^\alpha f)(a) \right) \right| \\ & \leq \frac{c}{2} \left[\left(\int_0^{\frac{x-a}{c}} |f'(a+tc)|^q dt \right)^{\frac{1}{q}} + \left(\int_{\frac{x-a}{c}}^1 |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\int_0^{\frac{1}{2}} ((1-t)^\alpha - t^\alpha)^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 (t^\alpha - (1-t)^\alpha)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \right] \\ & \leq \frac{c}{2} \left[\left(\int_0^{\frac{x-a}{c}} |f'(a+tc)|^q dt \right)^{\frac{1}{q}} + \left(\int_{\frac{x-a}{c}}^1 |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\int_0^{\frac{1}{2}} ((1-t)^{\alpha p} - t^{\alpha p}) dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 (t^{\alpha p} - (1-t)^{\alpha p}) dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \right] \\ & = \frac{c}{2} \left[\left(\int_0^{\frac{x-a}{c}} |f'(a+tc)|^q dt \right)^{\frac{1}{q}} + \left(\int_{\frac{x-a}{c}}^1 |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad + \left(\frac{1 - (\frac{1}{2})^{\alpha p}}{\alpha p + 1} \right)^{\frac{1}{p}} \\ & \quad \left. \times \left(\left(\int_0^{\frac{1}{2}} |f'(a+tc)|^q dt \right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |f'(a+tc)|^q dt \right)^{\frac{1}{q}} \right) \right]. \end{aligned}$$

Since $|f'|^q$ is s -preinvex function, we have

$$\begin{aligned}
& \left| f(x) - \frac{\Gamma(\alpha+1)}{2c^\alpha} \left((I_{a^+}^\alpha f)(a+c) + (I_{(a+ct)^-}^\alpha f)(a) \right) \right| \\
& \leq \frac{c}{2} \left[\left(\int_0^{\frac{x-a}{c}} ((1-t)^s |f'(a)|^q + t^s |f'(b)|^q) dt \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\int_{\frac{x-a}{c}}^1 ((1-t)^s |f'(a)|^q + t^s |f'(b)|^q) dt \right)^{\frac{1}{q}} \\
& \quad + \left. \left(\frac{1 - (\frac{1}{2})^{\alpha p}}{\alpha p + 1} \right)^{\frac{1}{p}} \left\{ \left(\int_0^{\frac{1}{2}} ((1-t)^s |f'(a)|^q + t^s |f'(b)|^q) dt \right)^{\frac{1}{q}} \right. \right. \\
& \quad \quad \left. \left. + \left(\int_{\frac{1}{2}}^1 ((1-t)^s |f'(a)|^q + t^s |f'(b)|^q) dt \right)^{\frac{1}{q}} \right\} \right] \\
& = \frac{c}{2(s+1)^{\frac{1}{q}}} \left[\left(\left(1 - \left(1 - \frac{x-a}{c} \right)^{s+1} \right) |f'(a)|^q + \left(\frac{x-a}{c} \right)^{s+1} |f'(b)|^q \right)^{\frac{1}{q}} \right. \\
& \quad + \left(\left(1 - \frac{x-a}{c} \right)^{s+1} |f'(a)|^q + \left(1 - \left(\frac{x-a}{c} \right)^{s+1} \right) |f'(b)|^q \right)^{\frac{1}{q}} \\
& \quad + \left. \left(\frac{1 - (\frac{1}{2})^{\alpha p}}{\alpha p + 1} \right)^{\frac{1}{p}} \right. \\
& \quad \quad \left. \times \left(\left(\frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} \right) \right].
\end{aligned}$$

Replacing c by $\eta(b, a)$ in the above inequality, we get the desired result. ■

COROLLARY 5. *In Theorem 4 if we choose $x = \frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a+\eta(b,a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right|$$

$$\begin{aligned} &\leq \frac{\eta(b,a)}{2(s+1)^{\frac{1}{q}}} \left(1 + \left(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p+1} \right)^{\frac{1}{p}} \right) \\ &\quad \times \left(\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} \right). \end{aligned}$$

Moreover if we take $\eta(b, a) = b - a$, we obtain

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left((I_{a^+}^\alpha f)(b) + (I_{b^-}^\alpha f)(a) \right) \right| \\ &\leq \frac{b-a}{2(s+1)^{\frac{1}{q}}} \left(1 + \left(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p+1} \right)^{\frac{1}{p}} \right) \\ &\quad \times \left(\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^{s+1}} \right)^{\frac{1}{q}} \right). \end{aligned}$$

COROLLARY 6. In Theorem 4 if we put $s = 1$, we obtain

$$\begin{aligned} &\left| f(x) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a + \eta(b, a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\ &\leq \frac{\eta(b,a)}{2^{1+\frac{1}{q}}} \left[\left(\left(1 - \left(1 - \frac{x-a}{\eta(b,a)} \right)^2 \right) |f'(a)|^q + \left(\frac{x-a}{\eta(b,a)} \right)^2 |f'(b)|^q \right)^{\frac{1}{q}} \right. \\ &\quad + \left(\left(1 - \frac{x-a}{\eta(b,a)} \right)^2 |f'(a)|^q + \left(1 - \left(\frac{x-a}{\eta(b,a)} \right)^2 \right) |f'(b)|^q \right)^{\frac{1}{q}} \\ &\quad \left. + \left(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p+1} \right)^{\frac{1}{p}} \left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q+3|f'(b)|^q}{4} \right)^{\frac{1}{q}} \right) \right]. \end{aligned}$$

Moreover if we choose $x = \frac{2a+\eta(b,a)}{2}$, we obtain the following fractional midpoint inequality

$$\begin{aligned} &\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)} \left((I_{a^+}^\alpha f)(a + \eta(b, a)) + (I_{(a+\eta(b,a))^-}^\alpha f)(a) \right) \right| \\ &\leq \frac{\eta(b,a)}{2^{1+\frac{1}{q}}} \left(1 + \left(\frac{1-(\frac{1}{2})^{\alpha p}}{\alpha p+1} \right)^{\frac{1}{p}} \right) \left(\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q+3|f'(b)|^q}{4} \right)^{\frac{1}{q}} \right). \end{aligned}$$

Additionally if we take $\eta(b, a) = b - a$, we obtain

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left((I_{a^+}^\alpha f)(b) + (I_{b^-}^\alpha f)(a) \right) \right| \\ \leq \frac{b-a}{2^{1+\frac{1}{q}}} \left(1 + \left(\frac{1 - (\frac{1}{2})^{\alpha p}}{\alpha p + 1} \right)^{\frac{1}{p}} \right) \left(\left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} \right).$$

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