EXTRACTA MATHEMATICAE Vol. **28**, Núm. 2, 169 – 195 (2013)

On Extension of Multi-parametric Local Semigroups of Isometric Operators and some Applications

R. Bruzual [1,*], M. Domínguez [2,*], A. Pérez [3]

[1] *Escuela de Matemática, Fac. Ciencias, Universidad Central de Venezuela*
Apartado Postal 47686*, Caracas* 1041*-A, Venezuela*
ramonbruzual.ucv@gmail.com , ramon.bruzual@ciens.ucv.ve

[2] *Escuela de Matemática, Fac. Ciencias, Universidad Central de Venezuela*
Apartado Postal 47159*, Caracas* 1041*-A, Venezuela*
marisela.dominguez@ciens.ucv.ve , dominguez.math@gmail.com

[3] *Universidad Nacional Experimental Politécnica "Antonio José de Sucre" and Universidad Central de Venezuela*
UNEXPO, Vice-Rectorado "Luis Caballero Mejías", Caracas, Venezuela
andres.perez@ciens.ucv.ve , anperez1302@gmail.com

Presented by Alfonso Montes — Received August 5, 2013

Abstract: It is shown that, under certain uniqueness conditions, a strongly continuous n-parametric local semigroup of isometric operators on a Hilbert space, can be extended to a strongly continuous n-parametric group of unitary operators on a larger Hilbert space. As an application a proof of the G.I. Èskin multi-dimensional version of the M.GK̇reĭn extension theorem, for positive definite functions, is given.

Key words: symmetric operator, self-adjoint operator, semigroup of operators, unitary operator, Cayley transform, positive definite function.

AMS *Subject Class.* (2010): 47D03, 43A35, 42A82.

1. Introduction

The notion of uni-parametric local semigroups of operators appears in several problems of mathematical analysis. In particular local semigroups of isometric operators appear in some problems on Fourier representation of positive definite functions of a real variable, where the unitary extensions of the semigroup provide solutions of the problem. To such problems belongs the classical theorem of M.G. Kreĭn [13], which asserts that every continuous positive definite function defined on an interval $I \subset \mathbb{R}$ can be extended to a continuous positive definite function on $\mathbb{R}$.

* First and second authors were supported in part by the CDCH of the Universidad Central de Venezuela. Grant PI-03-8623-2013/2

The study of uni-parametric local semigroups of isometries in Pontrjagyn spaces was started by Grossman and Langer in [11], who proved the existence of unitary extensions of such semigroups and derived from this result a generalization of Kreĭn's theorem for κ-indefinite functions. In the case of Hilbert spaces such study was developed independently by the first author in [4], for the more general case of local semigroups of contractions, giving several applications to generalized Toeplitz kernels as well as a simple proof of the Kreĭn extension theorem.

It is known that the Kreĭn theorem may fail in the two-dimensional case, i.e., not every continuous positive definite function defined on a rectangle extends to a positive definite function in the whole plane (see the paper of W. Rudin [18]). However A. Devinatz [8] proved that such an extension exists if the positive definite function satisfies some additional conditions, which were later relaxed by G.I. Èskin [10], for more information see the book of Iu. Berezanski [3].

The notion of n-parametric local semigroups of isometries can be defined in a natural way and a n-tuple of infinitesimal generators can be attached to it. F. Peláez [16] gave a necessary and sufficient condition for a bi-parametric local semigroup in a Hilbert space to extend to a two parameter unitary group and, with this result and some results of Devinatz, obtained a new proof of the above mentioned theorem of Devinatz. The bi-parametric case was studied later in [5], where an extension result for bi-parametric local semigroups of isometric operators was obtained, as an application of this result a new proof of the case $n = 2$ of the Èskin extension result was given. A result about commutative self-adjoint extensions of a pair of symmetric operators given by A. Koranyi in [12] was an important tool in that paper. The main results of [5] were generalized to the κ-indefinite case in [6], by the first and the second author.

In the present paper we discuss the problem of unitary extensions for n-parametric local semigroups of isometric operators on Hilbert spaces, with a different approach to that used in [5]. The paper is organized as follows: In Section 2 we give some preliminary results and definitions, in Section 3 we obtain some results about commutative unitary extensions which are necessary tools in this paper, in Section 4 we prove our extension result for n-parametric local semigroups of isometric operators (see Theorem 15) and finally, in Section 5 we give a new proof of the Èskin result (see Theorem 23).

2. Preliminaries

As usual if $(\mathcal{H}, \langle\ ,\ \rangle_{\mathcal{H}})$ is a Hilbert space, $\|\ \|_{\mathcal{H}}$ will denote the norm on $\mathcal{H}$ and $L(\mathcal{H})$ will denote the space of the bounded linear transforms on $\mathcal{H}$. If T is a linear operator $\mathcal{D}(T)$ and $\mathcal{R}(T)$ will denote the domain and the rank of T respectively and, if T is closable, the closure of T will be denoted by $\overline{T}$. Also by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ we will denote the sets of natural, integers, real and complex numbers.

2.1. Symmetric and self-adjoint operators

Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{D}(A) \to \mathcal{H}$ be a densely defined linear operator. The operator A is said to be *symmetric* if

$$\langle Af, g\rangle_{\mathcal{H}} = \langle f, Ag\rangle_{\mathcal{H}} \qquad \text{for all } f, g \in \mathcal{D}(A)\,.$$

Symmetric operator satisfies $A \subset A^*$. If $A = A^*$ the operator is called *self-adjoint*.

Following the usual terminology about unbounded self-adjoint operators we give the following definition, for more details see [17, page 271].

Definition 1. Two self-adjoint operators are said to *commute* if their spectral measures commute.

Remark 2. Given a pair of symmetric operators A and B, the problem of finding conditions for the existence of a pair of commuting self-adjoint operators $\widetilde{A}$ and $\widetilde{B}$ extending A and B respectively is not easy. It is important to recall the surprising counterexample given by E. Nelson, which shows how difficult is to deal with unbounded operators [15]: *There exist two symmetric operators A and B on a Hilbert space $\mathcal{H}$ having a common invariant domain $\mathcal{D}$ such that for all real a and b, $aA + bB$ is essentially self-adjoint and such that for all $x \in \mathcal{D}$ $ABx = BAx$, but such that the spectral measures of $\overline{A}$ and $\overline{B}$ do not commute.* See [17, Chapter VIII, Section 5] for additional comments.

An operator A is called *skew-symmetric* if iA is symmetric and it is called *skew-adjoint* if iA is self-adjoint. Symmetric and skew-symmetric operators are closable. A symmetric operator is called *essentially self-adjoint* if its closure is self-adjoint.

2.2. Cayley transform

Let $\mathcal{H}$ be a Hilbert space and let $A : \mathcal{D}(A) \to \mathcal{H}$ be a symmetric operator, then it holds that

$$\|Af + if\|_{\mathcal{H}}^2 = \|Af\|_{\mathcal{H}}^2 + \|f\|_{\mathcal{H}}^2 = \|Af - if\|_{\mathcal{H}}^2 \qquad \text{for all } f \in \mathcal{D}(A)\,.$$

Therefore the operator $T_A : \mathcal{R}(A + iI) \to \mathcal{R}(A - iI)$ defined by

$$T_A(Af + if) = Af - if \qquad \text{for } f \in \mathcal{D}(A)$$

is isometric. The operator T_A is called the *Cayley transform of* A.

The *deficiency indexes* of the operator A are defined by

$$d_+(A) = \dim\ker(A^* + iI) = \dim\big(\mathcal{R}(A - iI)\big)^{\perp}$$

and

$$d_-(A) = \dim\ker(A^* - iI) = \dim\big(\mathcal{R}(A + iI)\big)^{\perp}.$$

The following properties of the Cayley transform will be used in this paper (for details see [9, 19]).

1. Let A be a symmetric operator, then:
   (a) T_A is a partial isometric operator, with domain $\mathcal{R}(A+iI)$ and rank $\mathcal{R}(A - iI)$;
   (b) T_A is closed if and only if A is closed;
   (c) $\mathcal{R}(I - T_A) = \mathcal{D}(A)$, $I - T_A$ is one to one (that is, 1 is not an eigenvalue of T_A), and A can be reconstructed from T_A by the formula
   $$A = i(I + T_A)(I - T_A)^{-1};$$
   (d) T_A is unitary if and only if A is self-adjoint;
   (e) A is essentially self-adjoint if and only if $d_+(A) = d_-(A) = 0$.
2. If T is a partial isometric operator such that 1 is not an eigenvalue of T, then T is the Cayley transform of a symmetric operator on $\mathcal{H}$.
3. If A and B are symmetric operators on $\mathcal{H}$, then $A \subset B$ if and only if $T_A \subset T_B$.
4. If A is a symmetric operator and $V \in L(\mathcal{H})$ then the following conditions are equivalent

(a) $V\mathcal{D}(A) \subset \mathcal{D}(A)$ and $VA = AV|_{\mathcal{D}(A)}$;
(b) $V\mathcal{R}(A+iI) \subset \mathcal{R}(A+iI)$ and $VT_A = T_AV|_{\mathcal{R}(A+iI)}$.

In other words: V commutes with A if and only if V commutes with T_A.

5. If A is a skew-symmetric operator then the Cayley transform of iA is given by
$$T_{iA}(Af+f) = Af - f \quad \text{for } f \in \mathcal{D}(A)\,.$$

2.3. Multi-parametric unitary groups

The following n-parametric version of the Stone theorem will be used (see, for example, [17, Theorem VIII.13]).

Suppose that $A_1, \cdots, A_n$ are self-adjoint operators on the Hilbert space $\mathcal{H}$. Then the following conditions are equivalent:

(a) the operators $A_1, \cdots, A_n$ commute;

(b) the Cayley transforms operators $T_{A_1}, \cdots, T_{A_n}$ commute;

(c) the n-tuple $(A_1, \cdots, A_n)$ generates a strongly continuous group of unitary operators on $L(\mathcal{H})$. This group is given by
$$U(x_1, \ldots, x_n) = e^{iA_1x_1} \cdots e^{iA_nx_n}$$
for $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

2.4. Uni-parametric local semigroups of isometric operators

According to the definition given in [4], if a is a positive real number and $\mathcal{H}$ is a Hilbert space, a *uni-parametric local semigroup of isometric operators* is a family $(S(x), \mathcal{H}(x))_{x\in[0,a)}$ such that:

(i) for each $x \in [0,a)$ we have that $\mathcal{H}(x)$ is a closed subspace of $\mathcal{H}$ and $\mathcal{H}(0) = \mathcal{H}$;

(ii) for each $x \in [0,a)$, $S(x) : \mathcal{H}(x) \to \mathcal{H}$ is a linear isometry and $S(0) = I_{\mathcal{H}}$;

(iii) $\mathcal{H}(z) \subset \mathcal{H}(x)$ if $x, z \in [0,a)$ and $x \le z$;

(iv) if $x, z \in [0,a)$ and $x+z \in [0,a)$ then
$$S(z)\mathcal{H}(x+z) \subset \mathcal{H}(x)$$
and
$$S(x+z)h = S(x)S(z)h$$
for all $h \in \mathcal{H}(x+z)$;

(v) $\displaystyle\bigcup_{\substack{z>x \\ z\in[0,a)}} \mathcal{H}(z)$ is dense in $\mathcal{H}(x)$ for every $x \in [0,a)$.

The local semigroup is said to be *strongly continuous* if for all $r \in [0,a)$ and $f \in \mathcal{H}(r)$ the function $x \mapsto S(x)f$, from $[0,r]$ to $\mathcal{H}$, is continuous.

The *infinitesimal generator* of the semigroup if defined by

$$Ah = \lim_{t\to 0^+} \frac{S(t)h-h}{t} \qquad \text{for } h \in \mathcal{D}(A)$$

where

$$\mathcal{D}(A) = \left\{ h \in \bigcup_{r\in(0,a)} \mathcal{H}(r) \, : \, \lim_{t\to 0^+} \frac{S(t)h-h}{t} \text{ exists} \right\}.$$

Remark 3. For the strongly continuous case, it can be proved (see [4] for details) that $\mathcal{D}(A)$ is dense in $\mathcal{H}$, $A : \mathcal{D}(A) \to \mathcal{H}$ is a skew-symmetric operator and if $\tilde{A}$ is a skew-adjoint extension of A to a larger Hilbert space then

$$S(r) = e^{r\tilde{A}}|_{\mathcal{H}(r)} \qquad \text{for all } r \in [0,a)\,.$$

It also holds that the local semigroup $(S(r),\mathcal{H}(r))_{r\in[0,a)}$ has a unique unitary extension to the same Hilbert space $\mathcal{H}$ if and only if A is essentially skew-adjoint.

3. Some results about commutative unitary extensions

In this section we prove some results about commutative unitary extensions of a group of unitary operators and a partial isometric operator, which are necessary for our main result. The following two propositions will be useful.

Proposition 4. *Let $\mathcal{H}$ be a Hilbert space.*

(a) *If $C \in L(\mathcal{H})$ is a contraction such that 1 is not an eigenvalue of C, then 1 is not an eigenvalue of the minimal unitary dilation of C.*

(b) *If $\mathcal{D}$ is a closed subspace of $\mathcal{H}$ and $T : \mathcal{D} \to \mathcal{H}$ is a partial isometric operator such that 1 is not an eigenvalue of T, then 1 is not an eigenvalue of the contraction operator $TP^{\mathcal{H}}_{\mathcal{D}}$.*

Proof. (a) Let $\mathcal{G} = \bigoplus_{-\infty}^{+\infty} \mathcal{H}$, according to the construction given in [20, Chapter 1, Section 5] the minimal unitary dilation of C is the restriction to a suitable subspace of $\mathcal{G}$ of the unitary operator $U : \mathcal{G} \to \mathcal{G}$ defined by

$$\begin{aligned} (Ug)_{-1} &= D_C h_0 - C^* h_1 \,, \\ (Ug)_0 &= C h_0 + D_{C^*} h_1 \,, \\ (Ug)_j &= h_{j+1} \qquad (j \neq 0, -1) \,, \end{aligned}$$

for $g = (h_n)_{n=-\infty}^{+\infty} \in \mathcal{G}$, where

$$D_C = (I - C^*C)^{1/2} \,, \qquad D_{C^*} = (I - CC^*)^{1/2}.$$

If $g = (h_n)_{n=-\infty}^{+\infty} \in \mathcal{G}$ and $Ug = g$ it must hold that

$$\begin{aligned} h_{-1} &= D_C h_0 - C^* h_1 \,, \\ h_0 &= C h_0 + D_{C^*} h_1 \,, \\ h_j &= h_{j+1} \qquad (j \neq 0, -1) \,. \end{aligned}$$

From $h_j = h_{j+1}$ if $j \neq 0, -1$ we obtain that $h_j = 0$ if $j \geq 1$ or $j \leq -2$, so we have that $h_0 = C h_0$. Since 1 is not an eigenvalue of C we have that $h_0 = 0$ and finally

$$h_{-1} = D_C h_0 - C^* h_1 = 0 \,.$$

Therefore $g = 0$, so 1 is not an eigenvalue of U.

(b) Suppose that $h \in \mathcal{H}$ and that $TP_{\mathcal{D}}^{\mathcal{H}} h = h$, then

$$\|P_{\mathcal{D}}^{\mathcal{H}} h\|_{\mathcal{H}}^2 + \|P_{\mathcal{D}^\perp}^{\mathcal{H}} h\|_{\mathcal{H}}^2 \;=\; \|h\|_{\mathcal{H}}^2 \;=\; \|TP_{\mathcal{D}}^{\mathcal{H}} h\|_{\mathcal{H}}^2 \;\leq\; \|P_{\mathcal{D}}^{\mathcal{H}} h\|_{\mathcal{H}}^2 \,,$$

so we have that $P_{\mathcal{D}^\perp}^{\mathcal{H}} h = 0$, thus $h \in \mathcal{D}$ and $Th = h$. Therefore $h = 0$. ∎

Proposition 5. *Let $\mathcal{H}$ be a Hilbert space and let Γ be an abelian group. Suppose that:*

(a) *$\mathcal{D}(T)$ is a closed subspace of $\mathcal{H}$ and $T : \mathcal{D}(T) \to \mathcal{H}$ is a partial isometric operator;*

(b) *$(V(\gamma))_{\gamma \in \Gamma} \subset L(\mathcal{H})$ is a unitary representation of Γ on $L(\mathcal{H})$;*

(c) *$V(\gamma)\mathcal{D}(T) \subset \mathcal{D}(T)$ for all $\gamma \in \Gamma$;*

(d) *for all* $\gamma \in \Gamma$ *and* $h \in \mathcal{D}(T)$,

$$V(\gamma)Th = TV(\gamma)h\,.$$

If $\widetilde{T} \in L(\mathcal{F})$ *is the minimal unitary dilation of* $TP_{\mathcal{D}}^{\mathcal{H}}$, *then*

$$\left\langle \widetilde{T}^j V(\gamma)h, \widetilde{T}^m V(\gamma)h \right\rangle_{\mathcal{F}} = \left\langle \widetilde{T}^j h, \widetilde{T}^m h \right\rangle_{\mathcal{F}} \qquad \textit{for all } j, m \in \mathbb{Z}\,, \ \ \gamma \in \Gamma\,.$$

Proof. Since $(V(\gamma))_{\gamma\in\Gamma}$ is a unitary group we have that $V(\gamma)\mathcal{D}(T) = \mathcal{D}(T)$ for all $\gamma \in \Gamma$, so the operators $V(\gamma)$ and $P_{\mathcal{D}(T)}^{\mathcal{H}}$ commute.

For $h \in \mathcal{H}$ and $\gamma \in \Gamma$ we have that

$$V(\gamma)\left(TP_{\mathcal{D}(T)}^{\mathcal{H}}\right)h \;=\; TV(\gamma)P_{\mathcal{D}(T)}^{\mathcal{H}}h \;=\; \left(TP_{\mathcal{D}(T)}^{\mathcal{H}}\right)V(\gamma)h\,.$$

Let $\gamma \in \Gamma$ and $j, m \in \mathbb{Z}$ such that $j \geq m$, then

$$\begin{aligned}
\left\langle \widetilde{T}^j V(\gamma)h, \widetilde{T}^m V(\gamma)h \right\rangle_{\mathcal{F}} &= \left\langle \widetilde{T}^{j-m} V(\gamma)h, V(\gamma)h \right\rangle_{\mathcal{F}} \\
&= \left\langle \left(TP_{\mathcal{D}(T)}^{\mathcal{H}}\right)^{j-m} V(\gamma)h, V(\gamma)h \right\rangle_{\mathcal{H}} \\
&= \left\langle V(\gamma)\left(TP_{\mathcal{D}(T)}^{\mathcal{H}}\right)^{j-m} h, V(\gamma)h \right\rangle_{\mathcal{H}} \\
&= \left\langle \left(TP_{\mathcal{D}(T)}^{\mathcal{H}}\right)^{j-m} h, h \right\rangle_{\mathcal{H}} \\
&= \left\langle \widetilde{T}^{j-m} h, h \right\rangle_{\mathcal{F}} \\
&= \left\langle \widetilde{T}^j h, \widetilde{T}^m h \right\rangle_{\mathcal{F}}\,.
\end{aligned}$$

The case $j < m$ is analogous. ∎

Theorem 6. *Let* $\mathcal{H}$ *be a Hilbert space and let* Γ *be an abelian group. Suppose that:*

(a) $\mathcal{D}(T)$ *is a closed subspace of* $\mathcal{H}$ *and* $T : \mathcal{D}(T) \to \mathcal{H}$ *is a partial isometric operator such that* 1 *is not an eigenvalue of* T.

(b) $(V(\gamma))_{\gamma\in\Gamma} \subset L(\mathcal{H})$ *is a unitary representation of* Γ *on* $L(\mathcal{H})$.

(c) $V(\gamma)\mathcal{D}(T) \subset \mathcal{D}(T)$ *for all* $\gamma \in \Gamma$.

(d) *for all* $\gamma \in \Gamma$ *and* $h \in \mathcal{D}(T)$,

$$V(\gamma)Th = TV(\gamma)h\,.$$

Then there exist a Hilbert space $\mathcal{F}$ *containing* $\mathcal{H}$ *as a closed subspace, a unitary operator* $\widetilde{T} \in L(\mathcal{F})$ *and a unitary representation* $(\widetilde{V}(\gamma))_{\gamma\in\Gamma} \subset L(\mathcal{F})$ *such that:*

(i) $\widetilde{V}(\gamma)\widetilde{T} = \widetilde{T}\widetilde{V}(\gamma)$ *for all* $\gamma \in \Gamma$;

(ii) $\widetilde{T}|_{\mathcal{D}(T)} = T$;

(iii) 1 *is not an eigenvalue of* $\widetilde{T}$;

(iv) $\widetilde{V}(\gamma)|_{\mathcal{H}} = V(\gamma)$ *for all* $\gamma \in \Gamma$.

If Γ *is a topological group and* $(V(\gamma))_{\gamma\in\Gamma}$ *is strongly continuous then* $(\widetilde{V}(\gamma))_{\gamma\in\Gamma}$ *can be chosen to be strongly continuous.*

Proof. Let $\widetilde{T} \in L(\mathcal{F})$ be the minimal unitary dilation of the contraction operator $TP^{\mathcal{H}}_{\mathcal{D}}$. We have that $\widetilde{T}$ extends T and, from Proposition 4, it follows that 1 is not an eigenvalue of $\widetilde{T}$.

The space $\mathcal{F}$ is given by

$$\mathcal{F} = \bigvee_{n=-\infty}^{+\infty} \widetilde{T}^n\mathcal{H}\,.$$

If $f = \sum_{j=-N}^{N} a_j\widetilde{T}^j h_j \in \mathcal{F}$ $(a_j \in \mathbb{C}, h_j \in \mathcal{H})$, from Proposition 5 it follows that

$$\Big\|\sum_{j=-N}^{N} a_j\widetilde{T}^j V(\gamma)h_j\Big\|_{\mathcal{F}} = \Big\|\sum_{j=-N}^{N} a_j\widetilde{T}^j h_j\Big\|_{\mathcal{F}} \qquad \text{for all } \gamma \in \Gamma\,.$$

Since the set of the functions of the form $\sum_{j=-N}^{N} a_j\widetilde{T}^j h_j$ $(a_j \in \mathbb{C}, h_j \in \mathcal{H})$ is dense in $\mathcal{F}$, for each $\gamma \in \Gamma$ we have a unitary operator $\widetilde{V}(\gamma) : \mathcal{G} \to \mathcal{G}$ which satisfies

$$\widetilde{V}(\gamma)\left(\sum_{j=-N}^{N} a_j\widetilde{T}^j h_j\right) = \sum_{j=-N}^{N} a_j\widetilde{T}^j V(\gamma)h_j\,.$$

From the definition of $\widetilde{V}(\gamma)$ it follows that $\widetilde{V}(\gamma)\widetilde{T} = \widetilde{T}\widetilde{V}(\gamma)$, $\widetilde{V}(\gamma)|_{\mathcal{H}} = V(\gamma)$ for all $\gamma \in \Gamma$ and that $(\widetilde{V}(\gamma))_{\gamma\in\Gamma}$ is a group of unitary operators.

The last part about continuity also follows from the definition of $\widetilde{V}(\gamma)$. ∎

Remarks 7. (a) This theorem can also be deduced from [12, Lemma 2], but the part that refers to the eigenvalue is easily obtained following the construction given here. Another proof can be obtained following the idea of the proof of part (iii) of [1, Theorem 1, p. 330].

(b) Corollary 9 can be deduced from a result slightly different than Theorem 6, but in the context of indefinite metric spaces, ([7, Theorem 3.6]). Its proof uses a modified Cayley transform, mostly considered in indefinite metric spaces. Also, for the particular case $\Gamma = \mathbb{R}$ see [14, Theorem 2.5].

COROLLARY 8. *Let $\mathcal{H}$ be a Hilbert space and let Γ be an abelian group. Suppose that:*

(a) *$\mathcal{D}(A)$ is a dense linear manifold of $\mathcal{H}$ and $A : \mathcal{D}(A) \to \mathcal{H}$ is a symmetric operator;*

(b) *$(V(\gamma))_{\gamma\in\Gamma} \subset L(\mathcal{H})$ is a unitary representation of Γ on $L(\mathcal{H})$;*

(c) *$V(\gamma)\mathcal{D}(A) \subset \mathcal{D}(A)$ for all $\gamma \in \Gamma$;*

(d) *for all $\gamma \in \Gamma$ and $h \in \mathcal{D}(A)$,*

$$V(\gamma)Ah = AV(\gamma)h\,.$$

Then there exist a Hilbert space $\mathcal{F}$ containing $\mathcal{H}$ as a closed subspace, a self-adjoint operator $\widetilde{A}$ defined on a dense linear manifold $\mathcal{D}(\widetilde{A})$ of $\mathcal{F}$ and a unitary representation $(\widetilde{V}(\gamma))_{\gamma\in\Gamma} \subset L(\mathcal{F})$ such that:

(i) *$\widetilde{V}(\gamma)\mathcal{D}(\widetilde{A}) \subset \mathcal{D}(\widetilde{A})$ for all $\gamma \in \Gamma$;*

(ii) *$\widetilde{V}(\gamma)\widetilde{A}h = \widetilde{A}\widetilde{V}(\gamma)h$ for all $\gamma \in \Gamma$ and $h \in \mathcal{D}(\widetilde{A})$;*

(iii) *$\widetilde{A}|_{\mathcal{D}(A)} = A$;*

(iv) *$\widetilde{V}(\gamma)|_{\mathcal{H}} = V(\gamma)$ for all $\gamma \in \Gamma$.*

If A is essentially self-adjoint, then the domain of the extensions can be the same Hilbert space $\mathcal{H}$. That is we can take $\mathcal{F} = \mathcal{H}$.

If Γ is a topological group and $(V(\gamma))_{\gamma\in\Gamma}$ is strongly continuous then $(\widetilde{V}(\gamma))_{\gamma\in\Gamma}$ can be chosen to be strongly continuous.

Proof. From the construction of $\overline{A}$ if follows that for all $\gamma \in \Gamma$, $V(\gamma)\mathcal{D}(\overline{A}) \subset \mathcal{D}(\overline{A})$ and

$$V(\gamma)\overline{A}h = \overline{A}V(\gamma)h \qquad \text{if } h \in \mathcal{D}(\overline{A})\,.$$

So we have that the Cayley transform $T_{\overline{A}}$, of $\overline{A}$, and $(\widetilde{V}(\gamma))_{\gamma\in\Gamma}$ satisfy the conditions of Theorem 6. Therefore there exist a Hilbert space $\mathcal{F}$ containing $\mathcal{H}$ as a closed subspace, a unitary operator $\widetilde{T} \in L(\mathcal{F})$ and a unitary representation $(\widetilde{V}(\gamma))_{\gamma\in\Gamma} \subset L(\mathcal{F})$ such that $\widetilde{V}(\gamma)\widetilde{T} = \widetilde{T}\widetilde{V}(\gamma)$ for all $\gamma \in \Gamma$, $\widetilde{T}|_{\mathcal{D}(T_{\overline{A}})} = T_{\overline{A}}$, 1 is not an eigenvalue of $\widetilde{T}$ and $\widetilde{V}(\gamma)|_{\mathcal{H}} = V(\gamma)$ for all $\gamma \in \Gamma$.

Taking $\widetilde{A}$ as the inverse Cayley transform of $\widetilde{T}$ we obtain (i), (ii), (iii) and (iv).

If A is essentially self-adjoint we can take $\widetilde{T} = T_{\overline{A}}$ and $\widetilde{V} = V$.

The last part about continuity follows from the last part of Theorem 6. ∎

Corollary 9. *Let $\mathcal{H}$ be a Hilbert space and let Γ be an abelian group. Suppose that:*

(a) $\big(W(t), \mathcal{H}(t)\big)_{t\in[0,a)}$ *is a strongly continuous uni-parametric local semigroup of isometric operators on $\mathcal{H}$;*

(b) $(V(\gamma))_{\gamma\in\Gamma} \subset L(\mathcal{H})$ *is a unitary representation of Γ on $L(\mathcal{H})$;*

(c) $V(\gamma)\mathcal{H}(t) \subset \mathcal{H}(t)$ *for all $\gamma \in \Gamma$ and $t \in [0, a)$;*

(d) *for all $\gamma \in \Gamma$, $t \in [0, a)$ and $h \in \mathcal{H}(t)$,*

$$V(\gamma)W(t)h = W(t)V(\gamma)h\,.$$

Then there exist a Hilbert space $\mathcal{F}$ containing $\mathcal{H}$ as a closed subspace, a strongly continuous group of unitary operators $(\widetilde{W}(t))_{t\in\mathbb{R}} \subset L(\mathcal{F})$ and a unitary representation $(\widetilde{V}(\gamma))_{\gamma\in\Gamma} \subset L(\mathcal{F})$ such that:

(i) $\widetilde{V}(\gamma)\widetilde{W}(t)f = \widetilde{W}(t)\widetilde{V}(\gamma)f$ *for all $\gamma \in \Gamma$, $t \in \mathbb{R}$ and $h \in \mathcal{F}$;*

(ii) $\widetilde{W}(t)|_{\mathcal{H}(t)} = W(t)$ *for all $t \in [0, a)$;*

(iii) $\widetilde{V}(\gamma)|_{\mathcal{H}} = V(\gamma)$ *for all $\gamma \in \Gamma$.*

If the local semigroup $(W(t), \mathcal{H}(t))_{t\in[0,a)}$ has a unique unitary extension on the Hilbert space $\mathcal{H}$, then the domain of the unitary extensions can be the same Hilbert space $\mathcal{H}$. That is we can take $\mathcal{F} = \mathcal{H}$.

If Γ is a topological group and $(V(\gamma))_{\gamma\in\Gamma}$ is strongly continuous then $(\widetilde{V}(\gamma))_{\gamma\in\Gamma}$ can be chosen to be strongly continuous.

Proof. Let A be the infinitesimal generator of the local semigroup $(W(t), \mathcal{H}(t))_{t\in[0,a)}$, then A is a skew-symmetric operator with domain

$$\mathcal{D}(A) = \left\{ h \in \bigcup_{r\in(0,a)} \mathcal{H}(r) \,:\, \lim_{t\to 0^+} \frac{W(t)h - h}{t} \ \text{ exists} \right\}.$$

For $\gamma \in \Gamma$, $r \in (0,a)$ and $h \in \mathcal{H}_r$ we have that

$$V(\gamma)\left(\frac{W(t)h-h}{t}\right) = \frac{W(t)V(\gamma)h - V(\gamma)h}{t} \qquad \text{for } t \in (0,r)\,,$$

so $V(\gamma)\mathcal{D}(A) \subset \mathcal{D}(A)$ and $V(\gamma)Ah = AV(\gamma)h$ for all $\gamma \in \Gamma$ and $h \in \mathcal{D}(A)$.

Therefore, from Corollary 8, we have that there exist a Hilbert space $\mathcal{F}$ containing $\mathcal{H}$ as a closed subspace, a skew-adjoint operator $\widetilde{A}$ defined on a dense linear manifold $\mathcal{D}(\widetilde{A})$ of $\mathcal{F}$ and a unitary representation $(\widetilde{V}(\gamma))_{\gamma\in\Gamma} \subset L(\mathcal{F})$ such that $\widetilde{V}(\gamma)\mathcal{D}(\widetilde{A}) \subset \mathcal{D}(\widetilde{A})$ for all $\gamma \in \Gamma$, $\widetilde{V}(\gamma)\widetilde{A}h = \widetilde{A}\widetilde{V}(\gamma)h$ for all $\gamma \in \Gamma$ and $h \in \mathcal{D}(\widetilde{A})$, $\widetilde{A}|_{\mathcal{D}(A)} = A$ and $\widetilde{V}(\gamma)|_{\mathcal{H}} = V(\gamma)$ for all $\gamma \in \Gamma$. Taking $\widetilde{W}(t) = e^{At}$ we obtain (i), (ii) and (iii).

The last part follows from the last part of Corollary 8. ∎

4. Multi-parametric local semigroups of isometric operators

Let n be a positive integer and let $\overrightarrow{x} = (x_1,\dots,x_n)$ and $\overrightarrow{z} = (z_1,\dots,z_n)$ be points of $\mathbb{R}^n$. We will say that $\overrightarrow{x} < \overrightarrow{z}$ if $x_j < z_j$ for $j = 1,\dots,n$.

Suppose that $\overrightarrow{a} = (a_1,\dots,a_n) \in \mathbb{R}^n$ and $a_j > 0$, for $j = 1,\dots,n$. Let $Q = [0,a_1) \times \cdots \times [0,a_n)$.

By $\overrightarrow{e_j}$ we will denote the vector $(0,\dots,0,1,0,\dots,0)$ where the number 1 is in the place corresponding to j.

Definition 10. Let $\mathcal{H}$ be a Hilbert space. A *n-parametric local semigroup of isometric operators* is a family $\big(S(\overrightarrow{x}),\mathcal{H}(\overrightarrow{x})\big)_{\overrightarrow{x}\in Q}$ such that:

(i) for each $\overrightarrow{x} \in Q$ we have that $\mathcal{H}(\overrightarrow{x})$ is a closed subspace of $\mathcal{H}$ and $\mathcal{H}(\overrightarrow{0}) = \mathcal{H}$;

(ii) for each $\overrightarrow{x} \in Q$, $S(\overrightarrow{x}) : \mathcal{H}(\overrightarrow{x}) \to \mathcal{H}$ is a linear isometry and $S(\overrightarrow{0}) = I_{\mathcal{H}}$;

(iii) $\mathcal{H}(\overrightarrow{z}) \subset \mathcal{H}(\overrightarrow{x})$ if $\overrightarrow{x}, \overrightarrow{z} \in Q$ and $\overrightarrow{x} \leq \overrightarrow{z}$;

(iv) if $\overrightarrow{x}, \overrightarrow{z} \in Q$ and $\overrightarrow{x} + \overrightarrow{z} \in Q$ then

$$S(\overrightarrow{z})\mathcal{H}(\overrightarrow{x} + \overrightarrow{z}) \subset \mathcal{H}(\overrightarrow{x})$$

and

$$S(\overrightarrow{x} + \overrightarrow{z})h = S(\overrightarrow{x})S(\overrightarrow{z})h$$

for all $h \in \mathcal{H}(\overrightarrow{x} + \overrightarrow{z})$;

(v) $\bigcup_{\substack{\overrightarrow{z}>\overrightarrow{x} \\ \overrightarrow{z}\in Q}} \mathcal{H}(\overrightarrow{z})$ is dense in $\mathcal{H}(\overrightarrow{x})$ for every $\overrightarrow{x} \in Q$.

The local semigroup is said to be *strongly continuous* if for all $\overrightarrow{r} = (r_1, \dots, r_n) \in Q$ and $f \in \mathcal{H}(\overrightarrow{r})$ the function $\overrightarrow{x} \mapsto S(\overrightarrow{x})f$, from $[0, r_1] \times \dots \times [0, r_n]$ to $\mathcal{H}$, is continuous.

Remark 11. Note that if $\big(S(\overrightarrow{x}), \mathcal{H}(\overrightarrow{x})\big)_{\overrightarrow{x}\in Q}$ is a strongly continuous n-parametric local semigroup of isometric operators, then for each $j \in \{1, \dots, n\}$ the family $\big(S(t\overrightarrow{e_j}), \mathcal{H}(t\overrightarrow{e_j})\big)_{t\in[0,a_j)}$ is a strongly continuous uni-parametric local semigroup of isometric operators. So, if we denote by $A^{(j)}$ the infinitesimal generator of this semigroup, we have that the n-parametric local semigroup can be extended to a strongly continuous unitary group, with parameter in $\mathbb{R}^n$, on a larger Hilbert space if and only if the operators $iA^{(1)}, \dots, iA^{(n)}$ have commuting self-adjoint extensions to a larger Hilbert space.

We also have that for $j, m \in \{1, \dots, n\}$, $j \neq m$ and $x_m \in [0, a_m)$ the family

$$\Big(S(t\,\overrightarrow{e_j})|_{\mathcal{H}(t\,\overrightarrow{e_j}+x_m\,\overrightarrow{e_m})}, \mathcal{H}(t\,\overrightarrow{e_j} + x_m\,\overrightarrow{e_m})\Big)_{t\in[0,a_j)}$$

is a uni-parametric local semigroup of isometric operators on $\mathcal{H}(x_m\,\overrightarrow{e_m})$. For simplicity, when we refer to this local semigroup, we will use $S(t\,\overrightarrow{e_j})$ instead of $S(t\,\overrightarrow{e_j})|_{\mathcal{H}(t\,\overrightarrow{e_j}+x_m\,\overrightarrow{e_m})}$.

4.1. The bi-parametric case

Suppose that $\big(S(x,y), \mathcal{H}(x,y)\big)_{(x,y)\in[0,a)\times[0,b)}$ is a strongly continuous bi-parametric local semigroup of isometric operators on the Hilbert space $\mathcal{H}$.

For $x \in [0, a)$, B_x will denote the infinitesimal generator of the uni-parametric local semigroup of isometric operators $\big(S(0,y), \mathcal{H}(x,y)\big)_{y\in[0,b)} \subset L(\mathcal{H}(x,0))$, and for $y \in [0, b)$, A_y will denote the infinitesimal generator of the uni-parametric local semigroup of isometric operators $\big(S(x,0), \mathcal{H}(x,y)\big)_{x\in[0,a)} \subset L(\mathcal{H}(0,y))$.

Proposition 12. *With the same notation as before it holds that*

$$S(0,y)\mathcal{D}(A_y) \subset \mathcal{D}(A_0)$$

and

$$S(0,y)A_y f = S(0,y)A_0 f = A_0 S(0,y) f \qquad \text{for all } f \in \mathcal{D}(A_y)\,.$$

Proof. Let $f \in \mathcal{D}(A_y)$, then $f \in \bigcup_{x\in[0,a)} \mathcal{H}(x,y)$ and

$$\lim_{t\to 0^+} \frac{S(t,0)f - f}{t}$$

exists. So we have that $S(0,y)f \in \bigcup_{x\in[0,a)} \mathcal{H}(x,0)$ and

$$\begin{aligned} S(0,y)\left(\frac{S(t,0)f - f}{t}\right) &= \frac{S(t,y)f - S(0,y)f}{t} \\ &= \frac{S(t,0)S(0,y)f - S(0,y)f}{t}\,, \end{aligned}$$

for t positive and small enough. Taking limit as $t \to 0^+$ we obtain the result. ∎

Theorem 13. *Let $\big(S(x,y), \mathcal{H}(x,y)\big)_{(x,y)\in[0,a)\times[0,b)}$ be a strongly continuous bi-parametric local semigroup of isometric operators on the Hilbert space $\mathcal{H}$. Suppose that for every $y \in [0,b)$, the uni-parametric local semigroup of isometric operators $\big(S(x,0), \mathcal{H}(x,y)\big)_{x\in[0,a)}$, has a unique extension to a strongly continuous group of unitary operators on the Hilbert space $\mathcal{H}(0,y)$. Then:*

(i) *For each $y \in [0,b)$ it holds that $T_{\overline{iA_0}}\mathcal{H}(0,y) \subset \mathcal{H}(0,y)$ and*

$$T_{\overline{iA_0}}\, S(0,y) = S(0,y)\, T_{\overline{iA_0}}|_{\mathcal{H}(0,y)}\,.$$

(ii) *The local semigroup $\big(S(x,y), \mathcal{H}(x,y)\big)_{(x,y)\in[0,a)\times[0,b)}$ can be extended to a strongly continuous group of unitary operators $(U(x,y))_{(x,y)\in\mathbb{R}^2}$ on a larger Hilbert space.*

(iii) *If we also suppose that for every $x \in [0,a)$, the uni-parametric local semigroup of isometric operators $\big(S(0,y), \mathcal{H}(x,y)\big)_{y\in[0,b)}$, has a unique extension to a strongly continuous group of unitary operators on the Hilbert space $\mathcal{H}(x,0)$, then*

(1) *the local semigroup $\big(S(x,y), \mathcal{H}(x,y)\big)_{(x,y)\in[0,a)\times[0,b)}$ has a unique extension to a strongly continuous group of unitary operators $(U(x,y))_{(x,y)\in\mathbb{R}^2}$ on $L(\mathcal{H})$:*

(2) *the unitary operators $T_{\overline{iA_0}}$ and $T_{\overline{iB_0}}$ commute, that is*

$$T_{\overline{iA_0}}\, T_{\overline{iB_0}} = T_{\overline{iB_0}}\, T_{\overline{iA_0}}\,.$$

Proof. (i) It holds that the operator A_0 extends A_y and from the hypothesis it follows that iA_0 and iA_y are essentially self-adjoint operators on the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}(0,y)$ respectively. So we have that $T_{\overline{iA_0}}$ is a unitary operator on $\mathcal{H}$, which extends the unitary operator $T_{\overline{iA_y}}$ on $\mathcal{H}(0,y)$. Therefore $T_{\overline{iA_0}}\mathcal{H}(0,y) \subset \mathcal{H}(0,y)$.

Since $\mathcal{R}(I + A_y)$ is dense in $\mathcal{H}(0,y)$ we only need to show that

$$T_{\overline{iA_0}}\, S(0,y)f = S(0,y)\, T_{\overline{iA_0}} f$$

for $f \in \mathcal{R}(I + A_y)$.

Let $f \in \mathcal{R}(I + A_y)$, then there exists $g \in \mathcal{D}(A_y)$ such that $f = (I + A_y)g$.

From Proposition 12 it follows that $S(0,y)(I + A_y)g = (I + A_0)S(0,y)g$, so we have

$$\begin{aligned} T_{\overline{iA_0}}\, S(0,y)f &= \big(\overline{A_0} - I\big)\big(I + \overline{A_0}\big)^{-1} S(0,y)\big(I + A_y\big)g \\ &= \big(\overline{A_0} - I\big)S(0,y)g \\ &= S(0,y)\big(\overline{A_0} - I\big)g \\ &= S(0,y)T_{\overline{iA_0}} f\,. \end{aligned}$$

(ii) For $x \in \mathbb{R}$ let $V(x)$ the unitary operator defined by $V(x) = e^{\overline{A_0}x}$, then $(V(x))_{x\in\mathbb{R}}$ is a strongly continuous group of unitary operators. From (i) it follows that, $V(x)\,\mathcal{H}(0,y) \subset \mathcal{H}(0,y)$ and

$$V(x)S(0,y) = S(0,y)V(x)|_{\mathcal{H}(0,y)}$$

for all $x \in \mathbb{R}$ and $y \in [0,b)$.

So, from Corollary 9 it follows that there exist a Hilbert space $\mathcal{F}$ containing $\mathcal{H}$ as a closed subspace, a strongly continuous group of unitary operators $(\widetilde{W}(y))_{y\in\mathbb{R}} \subset L(\mathcal{F})$ and a unitary representation $(\widetilde{V}(x))_{x\in\mathbb{R}} \subset L(\mathcal{F})$ such that

$$\begin{aligned} \widetilde{V}(x)\widetilde{W}(y)f = \widetilde{W}(y)\widetilde{V}(x)f \qquad &\text{for all } x, y \in \mathbb{R} \text{ and } f \in \mathcal{F}\,, \\ \widetilde{W}(y)|_{\mathcal{H}(0,y)} = S(0,y) \qquad &\text{for all } y \in [0,b)\,, \\ \widetilde{V}(x)|_{\mathcal{H}} = V(x) \qquad &\text{for all } x \in \mathbb{R}\,. \end{aligned}$$

Taking

$$U(x,y) = \widetilde{V}(x)\widetilde{W}(y)$$

we obtain the desired result.

(iii) From (i) and Corollary 9 it follows that we can take $\mathcal{F} = \mathcal{H}$ in the last construction, so (1) follows. To prove the uniqueness note that if we have that $U(x,y) = e^{Ax}e^{By}$, where iA and iB are self-adjoint operators on $\mathcal{H}$, then these operators must be self-adjoint extensions of iA_0 and iB_0 respectively. Since iA_0 and iB_0 are essentially self-adjoint operators the uniqueness result follows.

Finally, since $U(x,y) = e^{\overline{A_0}x}e^{\overline{B_0}y}$ is a unitary group, we have that $T_{\overline{A_0}}$ and $T_{\overline{B_0}}$ commute. ∎

Remark 14. Another proof of (ii) was given in [5].

4.2. The multi-parametric case

Theorem 15. *Let $a_1, \dots, a_n, b$ be positive real numbers, $Q = [0,a_1)\times\cdots\times[0,a_n)$ and let $\big(S(\overrightarrow{x}, y), \mathcal{H}(\overrightarrow{x}, y)\big)_{(\overrightarrow{x},y)\in Q\times[0,b)}$ be a $(n+1)$-parametric strongly continuous local semigroup of isometric operators on the Hilbert space $\mathcal{H}$.*

Suppose that:

(a) *For each pair $j, m \in \{1,\dots,n\}$ such that $j \neq m$ and $x_m \in [0,a_m)$ each of the uni-parametric local semigroup of isometric operators*
$$\Big(S(t\,\overrightarrow{e_j}, 0), \mathcal{H}(t\,\overrightarrow{e_j} + x_m\,\overrightarrow{e_m}, 0)\Big)_{t\in[0,a_j)}$$
has a unique unitary extension to a strongly continuous group of unitary operators on the Hilbert space $\mathcal{H}(x_m\,\overrightarrow{e_m}, 0)$.

(b) *For each $y \in [0,b)$ and $j \in \{1,\dots,n\}$, each of the uni-parametric local semigroup of isometric operators*
$$\Big(S(t\,\overrightarrow{e_j}, 0), \mathcal{H}(t\,\overrightarrow{e_j}, y)\Big)_{t\in[0,a_j)}$$
has a unique extension to a strongly continuous group of unitary operators on the Hilbert space $\mathcal{H}(\overrightarrow{0}, y)$.

Then there exist a Hilbert space $\mathcal{F}$ containing $\mathcal{H}$ as a closed subspace and a strongly continuous group of unitary operators $(U(\overrightarrow{x}, y))_{(\overrightarrow{x},y)\in\mathbb{R}^{n+1}}$ on $L(\mathcal{F})$ such that
$$U(\overrightarrow{x}, y)|_{\mathcal{H}(\overrightarrow{x},y)} = S(\overrightarrow{x}, y)\,.$$

Proof. For $j \in \{1, \dots, n\}$ let $A_0^{(j)}$ denote the infinitesimal generator of the uni-parametric local semigroup $\big(S(t\overrightarrow{e_j}, 0), \mathcal{H}(t\overrightarrow{e_j}, 0)\big)_{t\in[0,a_j)}$, then the operators $iA_0^{(j)}$ are essentially self-adjoint.

From (iii) of Theorem 13, considering the bi-parametric local semigroup of isometric operators

$$\Big(S(t\overrightarrow{e_j} + r\overrightarrow{e_m}, 0), \mathcal{H}(t\overrightarrow{e_j} + r\overrightarrow{e_m}, 0)\Big)_{(t,r)\in[0,a_j)\times[0,a_m)},$$

we obtain that, for $j, m \in \{1, \dots, n\}$, the unitary operators $T_{\overline{iA_0^{(j)}}}$ and $T_{\overline{iA_0^{(m)}}}$ commute. So $(i\overline{A_1}, \cdots, i\overline{A_n})$ generates a strongly continuous group of unitary operators on $L(\mathcal{H})$.

Also from (i) of Theorem 13, considering the bi-parametric local semigroup of isometric operators $\big(S(t\overrightarrow{e_j}, y), \mathcal{H}(t\overrightarrow{e_j}, y)\big)_{(t,y)\in[0,a_j)\times[0,b)}$, we obtain that, for $j \in \{1, \dots, n\}$ and $y \in [0, b)$,

$$T_{\overline{iA_0^{(j)}}}\,\mathcal{H}(0, y) \subset \mathcal{H}(0, y)$$

and

$$T_{\overline{iA_0^{(j)}}}\,S(0, y) = S(0, y)\,T_{\overline{iA_0^{(j)}}}\,|_{\mathcal{H}(0,y)}\,.$$

Therefore if we consider the strongly unitary group of operators on $L(\mathcal{H})$, with parameter on $\mathbb{R}^n$ defined by

$$V(\overrightarrow{x}) = e^{\overline{A_0^{(1)}}x_1} \cdots e^{\overline{A_0^{(n)}}x_n}\,,$$

we obtain that, for $\overrightarrow{x} \in \mathbb{R}^n$ and $y \in [0, b)$, $V(\overrightarrow{x})\,\mathcal{H}(0, y) \subset \mathcal{H}(0, y)$ and

$$V(\overrightarrow{x})\,S(0, y) = S(0, y)\,V(\overrightarrow{x})\,|_{\mathcal{H}(0,y)}\,,$$

so the result follows from Corollary 9. ∎

5. Extension of positive definite functions on a multi-dimensional box

In this section some results and definitions given in [4, 5], for the one and the two parameters case, are extended to the multi-parametric case, see also [8].

Let $a_1, \dots, a_n \in \mathbb{R}$ such that $a_j > 0$, for $j = 1, \dots, n$. Let $Q = [0, a_1) \times \cdots \times [0, a_n)$ and let $R = (-a_1, a_1) \times \cdots \times (-a_n, a_n)$, so $R - R = 2R = (-2a_1, 2a_1) \times \cdots \times (-2a_n, 2a_n)$.

Definition 16. A function $k : R - R \to \mathbb{C}$ is *positive definite* if for each $N \in \mathbb{N}$, $\overrightarrow{x_1}, \ldots, \overrightarrow{x_N} \in R$ and $c_1, \ldots, c_N \in \mathbb{C}$ it holds that

$$\sum_{p,q=1}^{N} c_p \, \overline{c_q} \, k \, (\overrightarrow{x_p} - \overrightarrow{x_q}) \geq 0 \, .$$

Throughout this section Q and R will be as before and $k : R - R \to \mathbb{C}$ will be a continuous positive definite function.

5.1. The reproducing kernel Hilbert space associated to a positive definite function

Let $K : R \times R \to \mathbb{C}$ be the kernel defined by

$$K \, (\overrightarrow{x}, \overrightarrow{z}) = k(\overrightarrow{x} - \overrightarrow{z}) \, .$$

Then K is a continuous positive definite kernel. The reproducing kernel Hilbert space associated to K (see [2]) is constructed as follows.

For $\overrightarrow{z} \in R$ let $K_{\overrightarrow{z}} : R \to \mathbb{C}$ be the function defined by

$$K_{\overrightarrow{z}}(\overrightarrow{x}) = K(\overrightarrow{x}, \overrightarrow{z})$$

and let $\mathcal{E}$ be the linear space defined by

$$\mathcal{E} = \left\{ u : R \to \mathbb{C} \, : \, u = \sum_{p=1}^{N} \alpha_p K_{\overrightarrow{z_p}} \, , \; N \in \mathbb{N} \, , \; \alpha_p \in \mathbb{C} \, , \; \overrightarrow{z_p} \in R \right\} .$$

The elements of $\mathcal{E}$ are continuous functions. If

$$u = \sum_{p=1}^{N} \alpha_p K_{\overrightarrow{z_p}} \qquad \text{and} \qquad v = \sum_{q=1}^{M} \beta_q K_{\overrightarrow{x_q}}$$

are elements of $\mathcal{E}$, we define

$$\langle u, v \rangle_{\mathcal{E}} = \sum_{p=1}^{N} \sum_{q=1}^{M} \alpha_p \, \overline{\beta_q} \, K(\overrightarrow{x_q}, \overrightarrow{z_p}) \, .$$

Then $\langle \, , \, \rangle_{\mathcal{E}}$ is a positive semi-definite sesquilinear form on $\mathcal{E}$ and

$$u \, (\overrightarrow{x}) = \left\langle u, K_{\overrightarrow{x}} \right\rangle_{\mathcal{E}} \qquad \text{for } u \in \mathcal{E} \text{ and } \overrightarrow{x} \in R \, ,$$

so we have

$$|u(\overrightarrow{x})| \leq \|u\|_{\mathcal{E}} \, \|K_{\overrightarrow{x}}\|_{\mathcal{E}} = \|u\|_{\mathcal{E}} \, \left(K\big(\overrightarrow{0}\,,\overrightarrow{0}\big)\right)^{1/2}$$

for all $u \in \mathcal{E}$ and $\overrightarrow{x} \in R$. Let $\mathcal{H}$ be the completion of $\mathcal{E}$. Then the elements of $\mathcal{H}$ are continuous functions, convergence in $\mathcal{H}$ implies uniform convergence and it also holds that

$$\varphi(\overrightarrow{x}) = \big\langle \varphi, K_{\overrightarrow{x}} \big\rangle_{\mathcal{H}} \qquad \text{for } \varphi \in \mathcal{H} \text{ and } \overrightarrow{x} \in R\,.$$

5.2. The n-parametric local semigroup associated to a positive definite function

For $\overrightarrow{x} \in Q$, let $\mathcal{E}(\overrightarrow{x})$ be the linear space defined by

$$\mathcal{E}(\overrightarrow{x}) = \left\{ u : R \to \mathbb{C} \,:\, u = \sum_{p=1}^{N} \alpha_p K_{\overrightarrow{z_p}}\,,\ N \in \mathbb{N}\,,\ \alpha_p \in \mathbb{C}\,,\ \overrightarrow{z_p}, \overrightarrow{z_p} + \overrightarrow{x} \in R \right\}.$$

If $\overrightarrow{x} \in Q$ and $u = \sum_{p=1}^{N} \alpha_p K_{\overrightarrow{z_p}} \in \mathcal{E}(\overrightarrow{x})$, we define $S(\overrightarrow{x}) : \mathcal{E}(\overrightarrow{x}) \to \mathcal{E}$ by

$$S(\overrightarrow{x})u = \sum_{p=1}^{N} \alpha_p K_{\overrightarrow{z_p} + \overrightarrow{x}}\,.$$

Note that $S(\overrightarrow{x})\varphi(\overrightarrow{\omega}) = \varphi(\overrightarrow{\omega} - \overrightarrow{x})$.

We have that $S(\overrightarrow{x})$ is a linear operator and, for $u, v \in \mathcal{E}(\overrightarrow{x})$ it holds that

$$\big\langle S(\overrightarrow{x})u, S(\overrightarrow{x})v \big\rangle_{\mathcal{E}} = \langle u, v \rangle_{\mathcal{E}}\,.$$

If $\mathcal{H}(\overrightarrow{x})$ is the closure of $\mathcal{E}(\overrightarrow{x})$ in $\mathcal{H}$, then $S(\overrightarrow{x})$ can be extended to a linear isometric operator from $\mathcal{H}(\overrightarrow{x})$ into $\mathcal{H}$. If this extension is denoted by $S(\overrightarrow{x})$ too, it is easy to verify that $\big(S(\overrightarrow{x}), \mathcal{H}(\overrightarrow{x})\big)_{\overrightarrow{x} \in Q}$ is an n-parametric local semigroup of isometric operators on the Hilbert space $\mathcal{H}$. Also, from the continuity of k follows the strong continuity of the local semigroup.

Proposition 17. *The function k can be extended to a continuous positive definite function on $\mathbb{R}^n$ if and only if the local semigroup, $\big(S(\overrightarrow{x}), \mathcal{H}(\overrightarrow{x})\big)_{\overrightarrow{x} \in Q}$, can be extended to a strongly continuous group of unitary operators, on a larger Hilbert space.*

Proof. ($\Rightarrow$) If the function k can be extended to a continuous positive definite function $\widetilde{k}$ on $\mathbb{R}^n$, following the previous construction with $\widetilde{k}$ instead of k, we will obtain a strongly continuous group of unitary operators which extends $\big(S(\overrightarrow{x}), \mathcal{H}(\overrightarrow{x})\big)_{\overrightarrow{x} \in Q}$.

($\Leftarrow$) Suppose that $\big(S(\overrightarrow{x}), \mathcal{H}(\overrightarrow{x})\big)_{\overrightarrow{x} \in Q}$ can be extended to a strongly continuous group of unitary operators $(U(\overrightarrow{x}))_{\overrightarrow{x} \in \mathbb{R}^n}$ on a larger Hilbert space $\mathcal{F}$.

Suppose that $\overrightarrow{x} \in R$.

If $\overrightarrow{x} \in Q$ and $\overrightarrow{\omega} \in R$ is such that $\overrightarrow{\omega} + \overrightarrow{x} \in R$, then

$$U(\overrightarrow{x})K_{\overrightarrow{\omega}} = S(\overrightarrow{x})K_{\overrightarrow{\omega}} = K_{\overrightarrow{\omega}+\overrightarrow{x}}\,.$$

If $-\overrightarrow{x} \in Q$ then $U(-\overrightarrow{x})K_{\overrightarrow{x}} = S(-\overrightarrow{x})K_{\overrightarrow{x}} = K_{\overrightarrow{0}}$, so

$$U(\overrightarrow{x})K_{\overrightarrow{0}} = K_{\overrightarrow{x}}\,.$$

In the general case $\overrightarrow{x} = \overrightarrow{x_1} + \overrightarrow{x_2}$, where $\overrightarrow{x_1}, -\overrightarrow{x_2} \in Q$, so we have that

$$\begin{aligned} U(\overrightarrow{x})K_{\overrightarrow{0}} &= U(\overrightarrow{x_1})\,U(\overrightarrow{x_2})\,K_{\overrightarrow{0}} \\ &= U(\overrightarrow{x_1})\,K_{\overrightarrow{x_2}} \;=\; K_{\overrightarrow{x_1}+\overrightarrow{x_2}} \;=\; K_{\overrightarrow{x}}\,. \end{aligned}$$

If $\overrightarrow{x}, \overrightarrow{z} \in R$ then

$$\begin{aligned} k\big(\overrightarrow{z} - \overrightarrow{x}\big) &= \big\langle K_{\overrightarrow{z}}, K_{\overrightarrow{x}} \big\rangle_{\mathcal{H}} \\ &= \Big\langle U(\overrightarrow{z})\,K_{\overrightarrow{0}}, U(\overrightarrow{x})\,K_{\overrightarrow{0}} \Big\rangle_{\mathcal{F}} \\ &= \Big\langle U(\overrightarrow{z} - \overrightarrow{x})\,K_{\overrightarrow{0}}, K_{\overrightarrow{0}} \Big\rangle_{\mathcal{F}}\,, \end{aligned}$$

therefore $k(\overrightarrow{\omega}) = \big\langle U(\overrightarrow{\omega})\,K_{\overrightarrow{0}}, K_{\overrightarrow{0}} \big\rangle_{\mathcal{F}}$ for $\overrightarrow{\omega} \in R - R$. Taking

$$\widetilde{k}(\overrightarrow{\omega}) = \Big\langle U(\overrightarrow{\omega})\,K_{\overrightarrow{0}}, K_{\overrightarrow{0}} \Big\rangle_{\mathcal{F}}$$

for $\overrightarrow{\omega} \in \mathbb{R}^n$, we obtain a strongly continuous positive definite extension of k. ∎

It will be necessary to give a characterization of the infinitesimal generators of the uni-parametric local semigroups associated to $\big(S(\overrightarrow{x}), \mathcal{H}(\overrightarrow{x})\big)_{\overrightarrow{x} \in Q}$.

For $j, m \in \{1, \ldots, n\}$, $j \neq m$, and $\zeta_m \in [0, a_m)$, let $A^{(j)}_{\zeta_m}$ be the infinitesimal generator of the uni-parametric local semigroup

$$\left(S(t\overrightarrow{e_j}), \mathcal{H}(t\overrightarrow{e_j} + \zeta_m \overrightarrow{e_m})\right)_{t\in[0,a_j)} \subset L(\mathcal{H}(\zeta_m \overrightarrow{e_m}))\,,$$

and let $D^{(j)}_{\zeta_m}$ be the linear operator with domain

$$\mathcal{D}\left(D^{(j)}_{\zeta_m}\right) = \left\{\varphi \in \mathcal{H}(\zeta_m \overrightarrow{e_m}) \ : \begin{array}{c} \frac{\partial\varphi}{\partial x_j} \text{ exists, and } \frac{\partial\varphi}{\partial x_j} = \psi \\ \text{for some } \psi \in \mathcal{H}(\zeta_m \overrightarrow{e_m}) \end{array}\right\}$$

defined by

$$D^{(j)}_{\zeta_m}\varphi = \frac{\partial\varphi}{\partial x_j}\,.$$

Since convergence in $\mathcal{H}$ implies uniform convergence, we have that $D^{(j)}_{\zeta_m}$ is a closed operator.

PROPOSITION 18. *Let $\overrightarrow{x} \in R$ such that $x_m < a_m - \zeta_m$ and let $r_o > 0$ such that $\overrightarrow{x} + r\overrightarrow{e_j} \in R$ for $|r| < r_o$. For $r \in (-r_o, r_o)$ consider the element $\varphi^{(j)}_{r,\overrightarrow{x}} \in \mathcal{H}(\zeta_m \overrightarrow{e_m})$ defined by the Riemann integral*

$$\varphi^{(j)}_{r,\overrightarrow{x}} = \frac{1}{r}\int_0^r K_{\lambda\overrightarrow{e_j}+\overrightarrow{x}}\,\mathrm{d}\lambda\,.$$

Then $\varphi^{(j)}_{r,\overrightarrow{x}} \in \mathcal{D}\left(A^{(j)}_{\zeta_m}\right)$ and

$$A^{(j)}_{\zeta_m}\varphi^{(j)}_{r,\overrightarrow{x}} = \frac{1}{r}\left(K_{r\overrightarrow{e_j}+\overrightarrow{x}} - K_{\overrightarrow{x}}\right)\,.$$

Proof. For $t \in (0, +\infty)$ small we have

$$\begin{aligned}
\frac{S(t\overrightarrow{e_j})\varphi^{(j)}_{r,\overrightarrow{x}} - \varphi^{(j)}_{r,\overrightarrow{x}}}{t} &= \frac{1}{tr}\left(\int_0^r K_{\lambda\overrightarrow{e_j}+\overrightarrow{x}+t\overrightarrow{e_j}}\,\mathrm{d}\lambda - \int_0^r K_{\lambda\overrightarrow{e_j}+\overrightarrow{x}}\,\mathrm{d}\lambda\right) \\
&= \frac{1}{tr}\left(\int_t^{r+t} K_{\lambda\overrightarrow{e_j}+\overrightarrow{x}}\,\mathrm{d}\lambda - \int_0^r K_{\lambda\overrightarrow{e_j}+\overrightarrow{x}}\,\mathrm{d}\lambda\right) \\
&= \frac{1}{tr}\left(\int_r^{r+t} K_{\lambda\overrightarrow{e_j}+\overrightarrow{x}}\,\mathrm{d}\lambda - \int_0^t K_{\lambda\overrightarrow{e_j}+\overrightarrow{x}}\,\mathrm{d}\lambda\right)\,.
\end{aligned}$$

Taking limit as $t \to 0^+$ we obtain the result. ∎

Proposition 19. *Let $r_o \in (0, a_j)$, for $\varphi \in \mathcal{H}(r_o\overrightarrow{e_j} + \zeta_m\overrightarrow{e_m})$ and $r \in (0, r_o)$ let*

$$M_r^{(j)}\varphi = \frac{1}{r}\int_0^r S(\lambda\overrightarrow{e_j})\varphi \,\mathrm{d}\lambda\,.$$

Then $M_r^{(j)}\varphi \in \mathcal{D}\left(D_{\zeta_m}^{(j)}\right)^*$ and

$$\left(D_{\zeta_m}^{(j)}\right)^* M_r^{(j)}\varphi = \frac{1}{r}\big(S(r\overrightarrow{e_j})\varphi - \varphi\big) \qquad \text{for } \ 0 < r < r_o\,.$$

Proof. Let $\psi \in \mathcal{D}(D_{\zeta_m}^{(j)})$ and let $\overrightarrow{z} \in Q$ such that $z_j \in [0, a_j - r_o)$ and $z_m \in [0, a_m - \zeta_m)$. Then

$$\begin{aligned}
\left\langle D_{\zeta_m}^{(j)}\psi, M_r^{(j)}K_{\overrightarrow{z}}\right\rangle_{\mathcal{H}(\zeta_m\overrightarrow{e_m})} &= \frac{1}{r}\int_0^r \left\langle D_{\zeta_m}^{(j)}\psi, K_{\overrightarrow{z}+\lambda\overrightarrow{e_j}}\right\rangle_{\mathcal{H}(\zeta_m\overrightarrow{e_m})} \mathrm{d}\lambda \\
&= \frac{1}{r}\int_0^r \frac{\partial\psi}{\partial x_j}(\overrightarrow{z} + \lambda\overrightarrow{e_j})\,\mathrm{d}\lambda \\
&= \frac{1}{r}\Big(\psi(\overrightarrow{z} + r\overrightarrow{e_j}) - \psi(\overrightarrow{z})\Big)\,.
\end{aligned}$$

Thus $M_r^{(j)}K_{\overrightarrow{z}} \in \mathcal{D}\left(D_{\zeta_m}^{(j)}\right)^*$ and

$$\left(D_{\zeta_m}^{(j)}\right)^* M_r^{(j)}K_{\overrightarrow{z}} \;=\; \frac{1}{r}\big(K_{\overrightarrow{z}+r\overrightarrow{e_j}} - K_{\overrightarrow{z}}\big) \;=\; \frac{1}{r}\big(S(r\overrightarrow{e_j})K_{\overrightarrow{z}} - K_{\overrightarrow{z}}\big)\,.$$

From this last equality it follows that

$$\left(D_{\zeta_m}^{(j)}\right)^* M_r^{(j)}u = \frac{1}{r}\big(S(r\overrightarrow{e_j})u - u\big)$$

for $0 < r < r_o$ and $u \in \mathcal{E}(r_o\overrightarrow{e_j} + \zeta_m\overrightarrow{e_m})$.

Since $\left(D_{\zeta_m}^{(j)}\right)^*$ is a closed operator and the function $u \mapsto M_r^{(j)}u$ is continuous we obtain that for $\varphi \in \mathcal{H}(r_o\overrightarrow{e_j} + \zeta_m\overrightarrow{e_m})$ and $r \in (0, r_o)$, $M_r^{(j)}\varphi \in \mathcal{D}\left(D_{\zeta_m}^{(j)}\right)^*$ and

$$\left(D_{\zeta_m}^{(j)}\right)^* M_r^{(j)}\varphi = \frac{1}{r}\big(S(r\overrightarrow{e_j})\varphi - \varphi\big) \qquad \text{for } \ 0 < r < r_o\,.$$

∎

Lemma 20. *It holds that*

$$\left(A_{\zeta_m}^{(j)}\right)^* = D_{\zeta_m}^{(j)}\,.$$

Proof. The proof will be done in three steps.

Step 1: $\left(A^{(j)}_{\zeta_m}\right)^* \subset D^{(j)}_{\zeta_m}$.

Suppose that $\varphi \in \mathcal{D}\left(\left(A^{(j)}_{\zeta_m}\right)^*\right)$. Let $\overrightarrow{x} \in R$ such that $x_m < a_m - \zeta_m$ and let $\varphi^{(j)}_{r,\overrightarrow{x}}$ as in Proposition 18, then

$$\begin{aligned}\left\langle \left(A^{(j)}_{\zeta_m}\right)^* \varphi, \varphi_{r,\overrightarrow{x}} \right\rangle_{\mathcal{H}} &= \left\langle \varphi, A^{(j)}_{\zeta_m}\varphi_{r,\overrightarrow{x}} \right\rangle_{\mathcal{H}} \\ &= \left\langle \varphi, \tfrac{1}{r}\left(K_{r\overrightarrow{e_j}+\overrightarrow{x}} - K_{\overrightarrow{x}}\right)\right\rangle_{\mathcal{H}} \\ &= \tfrac{1}{r}\Big(\varphi(r\overrightarrow{e_j} + \overrightarrow{x}) - \varphi(\overrightarrow{x})\Big).\end{aligned}$$

Since $\lim_{r\to 0}\varphi_{r,\overrightarrow{x}} = K_{\overrightarrow{x}}$, we obtain that $\frac{\partial\varphi}{\partial x_j}(\overrightarrow{x})$ exists and

$$\frac{\partial\varphi}{\partial x_j}(\overrightarrow{x}) = \left(\left(A^{(j)}_{\zeta_m}\right)^* \varphi\right)(\overrightarrow{x}).$$

Step 2: $A^{(j)}_{\zeta_m} \subset \left(D^{(j)}_{\zeta_m}\right)^*$.

Let $\varphi \in \mathcal{D}(A^{(j)}_{\zeta_m})$. Then $\varphi \in \mathcal{H}(r_o\overrightarrow{e_j} + \zeta_m\overrightarrow{e_m})$ for some $r_o > 0$ and

$$\lim_{t\to 0^+}\frac{S(t\overrightarrow{e_j})\varphi - \varphi}{t} \quad \text{exists}.$$

Let $r_n \subset (0, r_o)$ such that $r_n \to 0$ as $n \to \infty$.

From Proposition 19 it follows that

$$\left(D^{(j)}_{\zeta_m}\right)^* M_{r_n}\varphi = \frac{S(r_n\overrightarrow{e_j})\varphi - \varphi}{r_n} \to A^{(j)}_{\zeta_m}\varphi \qquad \text{as } n \to \infty.$$

Also

$$M_{r_n}\varphi \to \varphi \qquad \text{as } n \to \infty.$$

Since $\left(D^{(j)}_{\zeta_m}\right)^*$ is closed, we obtain

$$A^{(j)}_{\zeta_m}\varphi = \left(D^{(j)}_{\zeta_m}\right)^*\varphi.$$

Step 3: $\left(A^{(j)}_{\zeta_m}\right)^* = D^{(j)}_{\zeta_m}$.

From Step 1 we have that $\left(A^{(j)}_{\zeta_m}\right)^* \subset D^{(j)}_{\zeta_m}$ and form Step 2 we have that $A^{(j)}_{\zeta_m} \subset \left(D^{(j)}_{\zeta_m}\right)^*$.

Since $D^{(j)}_{\zeta_m}$ is a closed operator we have that $D^{(j)}_{\zeta_m} = \overline{D^{(j)}_{\zeta_m}} = \left(D^{(j)}_{\zeta_m}\right)^{**}$, therefore

$$D^{(j)}_{\zeta_m} = \left(D^{(j)}_{\zeta_m}\right)^{**} \subset \left(A^{(j)}_{\zeta_m}\right)^* \subset D^{(j)}_{\zeta_m}.$$

∎

From this lemma it follows that $\left(iA^{(j)}_{\zeta_m}\right)^* = -iD^{(j)}_{\zeta_m}$, so the deficiency indexes of the operator $iA^{(j)}_{\zeta_m}$ are

$$d_+\left(iA^{(j)}_{\zeta_m}\right) = \dim\ker\left(D^{(j)}_{\zeta_m} + I\right),$$

$$d_-\left(iA^{(j)}_{\zeta_m}\right) = \dim\ker\left(D^{(j)}_{\zeta_m} - I\right).$$

For $j = 1, \ldots, m$ let $k^{(j)} : (-2a_j, 2a_j) \to \mathbb{C}$ be the function defined by

$$k^{(j)}(t) = k(t\overrightarrow{e_j}).$$

Then $k^{(j)}$ is a positive definite function, also to $k^{(j)}$ corresponds a uni-parametric local semigroup of isometric operators $\left(S^{(j)}(t), \mathcal{H}^{(j)}(t)\right)_{t\in[0,a_j)}$ on the reproducing kernel Hilbert space $\mathcal{H}^{(j)}$ corresponding to $k^{(j)}$.

Remark 21. From Lemma 20 it follows that the adjoint of the infinitesimal generator $A^{(j)}$, of $\left(S^{(j)}(t), \mathcal{H}^{(j)}(t)\right)_{t\in[0,a_j)}$, is the derivative operator. This result corresponds with a particular case of [4, Theorem 6]. From [4, Propositions 2 and Propositions 3] the following two result follow:

(i) The deficiency indexes of $A^{(j)}$ are equal, its possible values are 0 and 1.

(ii) If the function $k^{(j)}$ has only one continuous positive definite extension to the real line, then the functions ξ_1 and ξ_2 defined by $\xi_1(t) = e^t$ and $\xi_2(t) = e^{-t}$ are not elements of $\mathcal{H}^{(j)}$.

LEMMA 22. *If the function $k^{(j)}$ has only one continuous positive definite extension to the real line, then for each $m \in \{1, \ldots, n\}$ such that $j \neq m$ and $\zeta_m \in [0, a_m)$, each of the uni-parametric local semigroup of isometric operators*

$$\left(S(t\,\overrightarrow{e_j}, 0), \mathcal{H}(t\,\overrightarrow{e_j} + \zeta_m\,\overrightarrow{e_m}, 0)\right)_{t\in[0,a_j)}$$

has a unique unitary extension to a strongly continuous group of unitary operators on the Hilbert space $\mathcal{H}(\zeta_m\,\overrightarrow{e_m}, 0)$.

Proof. It is enough to show that the operator $iA^{(j)}_{\zeta_m}$ has deficiency indexes equal to 0.

Let $K^{(j)} : (-a_j, a_j) \times (-a_j, a_j) \to \mathbb{C}$ be the kernel defined by

$$K^{(j)}(r,t) = k^{(j)}(r-t)\,.$$

For a point $(x_1^o, \dots, x_{j-1}^o, x_{j+1}^o, \dots, x_n^o)$ such that $x_m^o \in (-a_m, a_m)$ consider the set R_o of the points $\overrightarrow{x} \in R$ such that $x_1 = x_1^o, \dots, x_{j-1} = x_{j-1}^o, x_{j+1} = x_{j+1}^o, \dots, x_n = x_n^o$. Then $K^{(j)}$ is the restriction of K to $R_o \times R_o$. So, according to the theorem of [2, page 351], the elements of $\mathcal{H}^{(j)}$ are the restriction to any R_o set of the functions of $\mathcal{H}$.

Suppose that $d_+\left(iA^{(j)}_{\zeta_m}\right) = \dim\ker\left(D^{(j)}_{\zeta_m} + I\right)$ is not 0, then a non trivial function of the form

$$\varphi(x_1, \dots, x_n) = \gamma(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)\, e^{-x_j}$$

must be an element of $\mathcal{H}(\zeta_m \overrightarrow{e_m}, 0)$.

Consider $(x_1^o, \dots, x_{j-1}^o, x_{j+1}^o, \dots, x_n^o)$ such that

$$c_o = \gamma(x_1^o, \dots, x_{j-1}^o, x_{j+1}^o, \dots, x_n^o) \neq 0\,,$$

then the restriction of φ to the set R_o is the function $\xi(x_j) = c_o e^{-x_j}$, so we must have that the function $\xi_2(t) = e^{-t}$ is an element of $\mathcal{H}^{(j)}$, which contradicts affirmation (ii) in Remark 21.

In the same way it is proved that $d_-\left(iA^{(j)}_{\zeta_m}\right) = 0$. ∎

5.3. An extension result

As an application of Theorem 15 we give a new proof of the following extension result due to G.I. Èskin [10].

We will suppose that $a_1, \dots, a_n, b$ are positive real numbers and, for our next result, we will consider

$$R = (-a_1, a_1) \times \cdots \times (-a_n, a_n) \times (-b, b)\,.$$

THEOREM 23. *Let $k : R - R \to \mathbb{C}$ be a continuous positive definite function. Suppose that, for $j = 1, \dots, n$ each one of the functions $k^{(j)} : (-2a_j, 2a_j) \to \mathbb{C}$ defined by $k^{(j)}(t) = k(t\overrightarrow{e_j})$, has a unique continuous positive definite extension to the real line. Then k can be extended to a continuous positive definite function on $\mathbb{R}^{n+1}$.*

Proof. Let $Q = [0, a_1) \times \cdots \times [0, a_n) \times [0, b)$ and let

$$\Big(S(\overrightarrow{x}, y), \mathcal{H}(\overrightarrow{x}, y)\Big)_{(\overrightarrow{x}, y) \in Q \times [0,b)}$$

be the $(n+1)$-parametric strongly continuous local semigroup of isometric operators associated to k.

From Lemma 22 it follows that $\big(S(\overrightarrow{x}, y), \mathcal{H}(\overrightarrow{x}, y)\big)_{(\overrightarrow{x}, y) \in Q \times [0,b)}$ satisfies the conditions of Theorem 15, so the local semigroup can be extended to a strongly continuous group on a larger Hilbert space, so from Proposition 17 the result follows. ∎

References

[1] R. Arocena, On the extension problem for a class of translation invariant positive forms, *J. Operator Theory* **21** (2) (1989), 323 – 347.

[2] N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68** (1950), 337 – 404.

[3] Iu. Berezanski, "Expansions in Eigenfunctions of Selfadjoint Operators" (Russian) Kiev: Naukova Dumka 1965 (Translated from the Russian by R. Bolstein, J.M. Danskin, J. Rovnyak and L. Shulman, Translations of Mathematical Monographs, Vol. 17, American Mathematical Society, Providence, R.I., 1968).

[4] R. Bruzual, Local semigroups of contractions and some applications to Fourier representation theorems, *Integral Equations Operator Theory* **10** (1987), 780 – 801.

[5] R. Bruzual, Unitary extensions of two-parameters local semigroups of isometric operators and the Kreĭn extension theorem, *Integral Equations Operator Theory* **17** (1993), 301 – 321.

[6] R. Bruzual, M. Domínguez, On extensions of indefinite functions defined on a rectangle, *Complex Anal. Oper. Theory* **5** (2011), 985 – 1001.

[7] R. Bruzual, S. Marcantognini, The Kreĭn-Langer problem for Hilbert space operator valued functions on the band, *Integral Equations Operator Theory* **34** (1999), 396 – 413.

[8] A. Devinatz, On the extensions of positivedefinite functions, *Acta Math.* **102** (1-2) (1959), 109 – 134.

[9] N. Dunford, J. Schwartz, "Linear Operators. Part II: Spectral Theory, Self Adjoint Operators in Hilbert Spaces" (With the assistance of William G. Bade and Robert G. Bartle), Interscience Publishers John Wiley & Sons, New York-London, 1963.

[10] G.I. Èskin, A sufficient condition for the solvability of a multi-dimensional problem of moments, *Dokl. Akad. Nauk SSSR* **113** (1960), 540 – 543 (Russian).

[11] M. Grossmann, H. Langer, Über indexerhaltende Erweiterungen eines hermiteschen Operators im Pontrjaginraum, *Math. Nachr.* **64** (1974), 289–317.

[12] A. Korányi, On some classes of analytic functions of several variables, *Trans. Amer. Math. Soc.* **101** (1961), 520–554.

[13] M.G. Kreĭn, Sur le problème du prolongement des fonctions hermitiennes positives et continues, *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **26** (1940), 17–22 (French).

[14] S. Marcantognini, M. Morán, Commuting unitary Hilbert space extensions of a pair of Kreĭn space isometries and the Kreĭn-Langer problem in the band, *J. Funct. Anal.* **138** (1996), 379–409.

[15] E. Nelson, Analytic vectors, *Ann. of Math. (2)* **70** (1959), 572–615.

[16] F. Peláez, "Semigrupos Locales Biparamétricos de Isometrías, Extensiones Autoadjuntas de Parejas de Operadores Simétricos y Aplicaciones", Publicaciones Matemáticas del Uruguay, 1991.

[17] M. Reed, B. Simon, "Methods of Modern Mathematical Physics. I. Functional Analysis. Second edition", Academic Press, Inc., New York, 1980.

[18] W. Rudin, The extension problem for positive definite functions, *Illinois J. Math.* **7** (1963), 532–539.

[19] W. Rudin, "Functional Analysis. Second edition", McGraw-Hill, Inc., New York, 1991.

[20] B. Sz.-Nagy, C. Foias, "Harmonic Analysis of Operators on Hilbert Space", North-Holland Publishing Co., Amsterdam-London, 1970.