

$\alpha\beta$ -STATISTICAL CONVERGENCE ON TIME SCALES

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Abstract. In this paper, we introduce the concepts of $\alpha\beta$ -statistical convergence and strong $\alpha\beta$ -Cesàro summability of delta measurable functions on an arbitrary time scale. Then some inclusion relations and results about these new concepts are presented. We will also investigate the relationship between statistical convergence and $\alpha\beta$ -statistical convergence on a time scale.

Keywords: statistical convergence, time scale, delta measurable functions, Cesàro summable.

1. Introduction

The idea of statistical convergence for sequences of real and complex numbers was introduced by Fast [14] and Steinhaus [15] independently in the same year (1951) as follows. Let $K \subseteq \mathbb{N}$, the set of natural numbers and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n n^{-1} |K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n . A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero, i.e., for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim x = L$. It is known that every convergent sequence is statistically convergent, but not conversely. For example, suppose that the sequence $x = (x_k)$ defined by $x_k = \sqrt{k}$ if k is square and $x_k = 0$ otherwise. It is clear that the sequence $x = (x_k)$ is statistically convergent to 0 but it is not convergent. Over the years, generalizations and applications of this notion have been investigated by various researchers [2, 8, 10, 11, 13, 16, 17, 18, 19, 20, 21, 24, 26, 29].

Aktuglu [13] introduced $\alpha\beta$ -statistical convergence as follows. Let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying the following conditions:

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- P_1 : α and β are both non – decreasing,
 P_2 : $\beta(n) \geq \alpha(n)$,
 P_3 : $\beta(n) - \alpha(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Let Λ denote the set of pairs (α, β) satisfying P_1, P_2 and P_3 .

For each pair $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leq 1$ and $K \subset \mathbb{N}$, we define

$$\delta^{\alpha, \beta}(K, \gamma) = \lim_{n \rightarrow \infty} \frac{|K \cap P_n^{\alpha, \beta}|}{(\beta(n) - \alpha(n) + 1)^\gamma}$$

where $P_n^{\alpha, \beta}$ is the closed interval $[\alpha(n), \beta(n)]$ and $|S|$ represents the cardinality of S .

Definition 1.1. [13] A sequence $x = (x_n)$ is said to be $\alpha\beta$ –statistically convergent of order γ to L , if for every $\varepsilon > 0$

$$\delta^{\alpha, \beta}(\{k : |x_k - L| \geq \varepsilon\}, \gamma) = \lim_{n \rightarrow \infty} \frac{|\{k \in P_n^{\alpha, \beta} : |x_k - L| \geq \varepsilon\}|}{(\beta(n) - \alpha(n) + 1)^\gamma} = 0,$$

which is denoted by $st_{\alpha\beta}^\gamma - \lim x_n = L$. For $\gamma = 1$, we say that x is $\alpha\beta$ –statistically convergent to L , and this is denoted by $st_{\alpha\beta} - \lim x_n = L$.

The purpose of our study is to introduce the concept of $\alpha\beta$ –statistical convergence on an arbitrary time scale.

A time scale \mathbb{T} is an arbitrary non-empty closed subset of the real numbers \mathbb{R} with the subspace topology inherited from the standard topology of \mathbb{R} . The theory of time scales was introduced by Hilger in his Ph. D. thesis supervised by Auldbach in 1988 (see [3, 27]), in order to unify continuous and discrete analysis. Since this theory is applicable to any field in which dynamic processes can be described with discrete or continuous models and is also effective in modeling some real life problems, it has a tremendous potential for applications and has recently received much attention, see [1, 12, 22, 23, 28]. In addition, statistical convergence is applied to time scales by various researchers in literature. For instance, Seyyidoglu and Tan [25] defined some new notions such as Δ –convergence and Δ –Cauchy, by using Δ –density. Turan and Duman introduced the concepts of density, statistical convergence and lacunary statistical convergence of delta measurable real-valued functions defined on time scales in [5] and [6], respectively. Also, in [7], they obtained a Tauberian condition for statistical convergence, and established a relationship between statistical convergence and lacunary statistical convergence on time scales. Altin, Koyunbakan and Yilmaz [30] gave the notions of m – and (λ, m) –uniform density of a set and m – and (λ, m) –uniform statistical convergence on an arbitrary time scales. Furthermore, λ –statistical convergence on time scales was defined by Yilmaz, Altin and Koyunbakan [9]. Recently, Sozbir and Altundag [4] introduced the concepts of weighted statistical convergence and $[\bar{N}, p]_{\mathbb{T}}$ –summability of delta measurable functions on time scales, and investigated their relations. We here recall some concepts and notations about the theory of time scales.

The forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ can be defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

for $t \in \mathbb{T}$. And the graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ can be defined by $\mu(t) = \sigma(t) - t$. In this definition we put $\inf \emptyset = \sup \mathbb{T}$, where \emptyset is an empty set. A closed interval in a time scale \mathbb{T} is given by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. Open intervals or half-open intervals are defined accordingly.

Let F_1 denote the family of all left closed and right open intervals of \mathbb{T} of the form $[a, b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t < b\}$ with $a, b \in \mathbb{T}$ and $a \leq b$. The interval $[a, a)$ is understood as the empty set. F_1 is a semiring of subsets of \mathbb{T} . Let $m_1 : F_1 \rightarrow [0, \infty)$ be a set function on F_1 such that $m_1([a, b)_{\mathbb{T}}) = b - a$. Then, it is known that m_1 is a countably additive measure on F_1 . Now, the Caratheodory extension of the set function m_1 associated with family F_1 is said to be the Lebesgue Δ -measure on \mathbb{T} is denoted by μ_{Δ} . In this case, it is known that if $a \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, then the single point set $\{a\}$ is Δ -measurable and $\mu_{\Delta}(\{a\}) = \sigma(a) - a$. If $a, b \in \mathbb{T}$ and $a \leq b$, then $\mu_{\Delta}([a, b)_{\mathbb{T}}) = b - a$ and $\mu_{\Delta}((a, b)_{\mathbb{T}}) = b - \sigma(a)$. If $a, b \in \mathbb{T} \setminus \{\max \mathbb{T}\}$ and $a \leq b$, then $\mu_{\Delta}((a, b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$ and $\mu_{\Delta}([a, b]_{\mathbb{T}}) = \sigma(b) - a$ (see [12]).

We should note that throughout the paper, we consider that \mathbb{T} is a time scale satisfying $\inf \mathbb{T} = t_0 > 0$ and $\sup \mathbb{T} = \infty$. Turan and Duman [5] introduced the concepts of density, statistical convergence and strong p -Cesàro summability of measurable real valued functions defined on time scales in the following way.

Definition 1.2. [5] Let Ω be a Δ -measurable subset of \mathbb{T} . Then, for $t \in \mathbb{T}$, we define the set $\Omega(t)$ by

$$\Omega(t) = \{s \in [t_0, t]_{\mathbb{T}} : s \in \Omega\}.$$

In this case, we define the density of Ω on \mathbb{T} , denoted by $\delta_{\mathbb{T}}(\Omega)$, as follows:

$$\delta_{\mathbb{T}}(\Omega) = \lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(\Omega(t))}{\mu_{\Delta}([t_0, t]_{\mathbb{T}})}$$

provided that the above limit exists.

Definition 1.3. [5] Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a Δ -measurable function. We say that f is statistically convergent on \mathbb{T} to a number L , if for every $\varepsilon > 0$

$$\delta_{\mathbb{T}}(\{t \in \mathbb{T} : |f(t) - L| \geq \varepsilon\}) = 0$$

holds, i.e., for every $\varepsilon > 0$,

$$\lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(\{s \in [t_0, t]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([t_0, t]_{\mathbb{T}})} = 0,$$

which is denoted by $st_{\mathbb{T}} - \lim_{t \rightarrow \infty} f(t) = L$.

Definition 1.4. [5] Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a Δ -measurable function and $0 < p < \infty$. We say that f is strongly p -Cesàro summable on the time scale \mathbb{T} to a number L , if there exists some $L \in \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{\mu_{\Delta}([t_0, t]_{\mathbb{T}})} \int_{[t_0, t]_{\mathbb{T}}} |f(s) - L|^p \Delta s = 0.$$

2. Main Results

In this section, we will begin by introducing the new concepts of $\alpha\beta$ -statistical convergence and strong $\alpha\beta$ -Cesàro summability on an arbitrary time scale, which are our main definitions, and we establish some relations about these notions. We also examine the relationship between statistical convergence and $\alpha\beta$ -statistical convergence on a time scale.

Now let $\alpha, \beta : \mathbb{T} \rightarrow \mathbb{R}^+$ be two functions satisfying the following conditions:

T_1 : α and β are both non - decreasing,

T_2 : $\sigma(\beta(t)) > \alpha(t) \geq t_0$ for all $t \in \mathbb{T}$,

T_3 : $\sigma(\beta(t)) - \alpha(t) \rightarrow \infty$ as $t \rightarrow \infty$.

And let $\Lambda_{\mathbb{T}}$ denote the set of pairs (α, β) satisfying T_1, T_2 and T_3 .

Definition 2.1. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a Δ -measurable function and $(\alpha, \beta) \in \Lambda_{\mathbb{T}}$. Then, f is said to be $\alpha\beta$ -statistically convergent to $L \in \mathbb{R}$ on a time scale \mathbb{T} , if for every $\varepsilon > 0$

$$\lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} = 0,$$

which is denoted by $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$.

This definition includes the following special cases:

i) If we take $\alpha(t) = t_0$ and $\beta(t) = t$ for all $t \in \mathbb{T}$, then $\alpha\beta$ -statistical convergence is reduced to statistical convergence on a time scale introduced in [5].

ii) Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$. For $\mathbb{T} = \mathbb{N}$, if we choose $\alpha(t) = t - \lambda_t + t_0$ and $\beta(t) = t$, then $\alpha\beta$ -statistical convergence on a time scale is reduced to λ -statistical convergence introduced in [24].

Remark 2.1. Let $\theta = (k_r)$ be an increasing sequence of non-negative integers with $k_0 = 0$ and $\sigma(k_r) - \sigma(k_{r-1}) \rightarrow \infty$ as $r \rightarrow \infty$, which means that θ is a lacunary sequence with respect to \mathbb{T} . If we take $\mathbb{T} = \mathbb{N}$, $\alpha(t) = k_{t-1} + 1$ and $\beta(t) = k_t$, then $\alpha\beta$ -statistical convergence on \mathbb{T} gives us the concept of lacunary statistical convergence introduced in [19]. However, for an arbitrary time scale \mathbb{T} , this is not clear, and we leave it as an open problem.

Proposition 2.1. *If $f : \mathbb{T} \rightarrow \mathbb{R}$ is $\alpha\beta$ -statistically convergent, then its limit is unique.*

Proposition 2.2. *If $f, g : \mathbb{T} \rightarrow \mathbb{R}$ with $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L_1$ and $st_{\mathbb{T}-\alpha\beta} - \lim g(t) = L_2$, then we have the following:*

- i) $st_{\mathbb{T}-\alpha\beta} - \lim (f(t) + g(t)) = L_1 + L_2$,*
- ii) $st_{\mathbb{T}-\alpha\beta} - \lim (cf(t)) = cL_1$ for any $c \in \mathbb{R}$.*

Definition 2.2. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a Δ -measurable function and $(\alpha, \beta) \in \Lambda_{\mathbb{T}}$. Then, one says f is said to be strongly $\alpha\beta$ -Cesàro summable on a time scale \mathbb{T} , if there exists some $L \in \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} \frac{1}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}} |f(s) - L| \Delta s = 0.$$

Theorem 2.1. *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be Δ -measurable function and $L \in \mathbb{R}$. Then we have the following:*

- i) If f is strongly $\alpha\beta$ -Cesàro summable to L , then $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$, but not conversely.*
- ii) If $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$ and f is a bounded function, then f is strongly $\alpha\beta$ -Cesàro summable to L .*

Proof. i) Let f is strongly $\alpha\beta$ -Cesàro summable to L . Then, for every $\varepsilon > 0$, we can write that

$$\begin{aligned} \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}} |f(s) - L| \Delta s &\geq \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}: |f(s) - L| \geq \varepsilon} |f(s) - L| \Delta s \\ &\geq \varepsilon \mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\}), \end{aligned}$$

which implies that $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$.

To prove the converse, define a function f in each intervals $[\alpha(t), \beta(t)]_{\mathbb{T}}$ by

$$f(s) = \begin{cases} 1, & \text{if } s \in [\alpha(t), \alpha(t) + 1]_{\mathbb{T}}, \\ 2, & \text{if } s \in [\alpha(t) + 1, \alpha(t) + 2]_{\mathbb{T}}, \\ \vdots & \\ \lfloor \sqrt{u_t} \rfloor, & \text{if } s \in [\alpha(t) + \lfloor \sqrt{u_t} \rfloor - 1, \alpha(t) + \lfloor \sqrt{u_t} \rfloor]_{\mathbb{T}}, \\ 0, & \text{otherwise,} \end{cases}$$

where $u(t) = \sigma(\beta(t)) - \alpha(t)$.

Then, for every $\varepsilon > 0$, we observe that

$$\begin{aligned} \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} &= \frac{\mu_{\Delta}([\alpha(t), \alpha(t) + \lfloor \sqrt{u_t} \rfloor]_{\mathbb{T}})}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \\ &= \frac{\lfloor \sqrt{u_t} \rfloor}{u_t} \rightarrow 0 \quad (\text{as } t \rightarrow \infty). \end{aligned}$$

Thus, $st_{\mathbb{T}-\alpha\beta} - \lim_{t \rightarrow \infty} f(t) = 0$.

On the other hand,

$$\begin{aligned}
& \frac{1}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}} |f(s)| \Delta s \\
&= \frac{1}{\sigma(\beta(t)) - \alpha(t)} \sum_{m=1}^{[\sqrt{u_t}]} m \mu_{\Delta}([\alpha(t) + m - 1, \alpha(t) + m]_{\mathbb{T}}) \\
&= \frac{1}{u_t} \sum_{m=1}^{[\sqrt{u_t}]} m \\
&= \frac{1+2+\dots+[\sqrt{u_t}]}{u_t} \\
&= \frac{([\sqrt{u_t}]([\sqrt{u_t}]+1))/2}{u_t} \rightarrow \frac{1}{2} \neq 0 \quad (\text{as } t \rightarrow \infty).
\end{aligned}$$

Hence, we obtain that f is not strongly $\alpha\beta$ -Cesàro summable to 0. This completes the proof.

ii) Let f be bounded and $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$. Then, there exists a positive number M such that $|f(t)| \leq M$ for all $t \in \mathbb{T}$, and for a given $\varepsilon > 0$, we also have

$$\lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} = 0.$$

Then, we can easily see that

$$\begin{aligned}
& \frac{1}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}} |f(s) - L| \Delta s \\
&= \frac{1}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}: |f(s) - L| \geq \varepsilon} |f(s) - L| \Delta s \\
&\quad + \frac{1}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}: |f(s) - L| < \varepsilon} |f(s) - L| \Delta s \\
&\leq \frac{M+|L|}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}: |f(s) - L| \geq \varepsilon} \Delta s + \frac{\varepsilon}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}} \Delta s \\
&= (M + |L|) \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} + \varepsilon.
\end{aligned}$$

Letting $t \rightarrow \infty$ on the both sides of the last inequality, since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{t \rightarrow \infty} \frac{1}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \int_{[\alpha(t), \beta(t)]_{\mathbb{T}}} |f(s) - L| \Delta s = 0.$$

So, the proof is completed. \square

Theorem 2.2. *If $\liminf_{t \rightarrow \infty} \frac{\sigma(\beta(t))}{\alpha(t)} > 1$, then $st_{\mathbb{T}} - \lim f(t) = L$ implies $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$.*

Proof. Suppose that $\liminf_{t \rightarrow \infty} \frac{\sigma(\beta(t))}{\alpha(t)} > 1$. Then, for sufficiently large t , there exists $\delta > 0$ such that $\frac{\sigma(\beta(t))}{\alpha(t)} \geq 1 + \delta$, and hence $\frac{\sigma(\beta(t)) - \alpha(t)}{\sigma(\beta(t))} \geq \frac{\delta}{1 + \delta}$. For a given $\varepsilon > 0$, we have

$$\begin{aligned} & \frac{\mu_{\Delta}(\{s \in [t_0, \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([t_0, \beta(t)]_{\mathbb{T}})} \\ & \geq \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\sigma(\beta(t)) - t_0} \\ & \geq \frac{\sigma(\beta(t)) - \alpha(t)}{\sigma(\beta(t))} \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\sigma(\beta(t)) - \alpha(t)} \\ & \geq \frac{\delta}{1 + \delta} \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\sigma(\beta(t)) - \alpha(t)}. \end{aligned}$$

Letting $t \rightarrow \infty$ on the both sides of the last inequality and also using the $st_{\mathbb{T}} - \lim f(t) = L$, we get

$$\lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} = 0.$$

This completes the proof of the theorem. \square

Theorem 2.3. *If $\lim_{t \rightarrow \infty} \frac{\alpha(t) - t_0}{\sigma(\beta(t)) - t_0} = 0$, then $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$ implies $st_{\mathbb{T}} - \lim f(t) = L$.*

Proof. Assume that $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$. Then, for every $\varepsilon > 0$, we may write

$$\begin{aligned} & \frac{\mu_{\Delta}(\{s \in [t_0, \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([t_0, \beta(t)]_{\mathbb{T}})} \\ & = \frac{\mu_{\Delta}(\{s \in [t_0, \alpha(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([t_0, \beta(t)]_{\mathbb{T}})} + \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([t_0, \beta(t)]_{\mathbb{T}})} \\ & = \frac{\mu_{\Delta}(\{s \in [t_0, \alpha(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\sigma(\beta(t)) - t_0} + \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\sigma(\beta(t)) - t_0} \\ & \leq \frac{\alpha(t) - t_0}{\sigma(\beta(t)) - t_0} + \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\sigma(\beta(t)) - \alpha(t)}. \end{aligned}$$

Taking limit as $t \rightarrow \infty$ on the both sides of last inequality and using the condition of $\lim_{t \rightarrow \infty} \frac{\alpha(t) - t_0}{\sigma(\beta(t)) - t_0} = 0$, we have

$$\lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(\{s \in [t_0, \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([t_0, \beta(t)]_{\mathbb{T}})} = 0,$$

which completes the proof. \square

Now, let $(\alpha, \beta) \in \Lambda_{\mathbb{T}}$ and $(\alpha', \beta') \in \Lambda_{\mathbb{T}}$. In the following theorem $\alpha\beta$ -statistical convergence and $\alpha'\beta'$ -statistical convergence are compared under the restriction

$$\alpha(t) \leq \alpha'(t) < \beta'(t) \leq \beta(t)$$

for all $t \in \mathbb{T}$. Under these conditions above, we have the following theorem:

Theorem 2.4. *If $\lim_{t \rightarrow \infty} \frac{\sigma(\beta'(t)) - \alpha'(t)}{\sigma(\beta(t)) - \alpha(t)} > 0$, then $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$ implies $st_{\mathbb{T}-\alpha'\beta'} - \lim f(t) = L$.*

Proof. Suppose that $\lim_{t \rightarrow \infty} \frac{\sigma(\beta'(t)) - \alpha'(t)}{\sigma(\beta(t)) - \alpha(t)} > 0$ and $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$. We have the inclusion

$$\{s \in [\alpha'(t), \beta'(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\} \subseteq \{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\}$$

for every $\varepsilon > 0$, and hence

$$\begin{aligned} \mu_{\Delta}(\{s \in [\alpha'(t), \beta'(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\}) \\ \leq \mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\}). \end{aligned}$$

So, we may write that

$$\begin{aligned} \frac{\mu_{\Delta}(\{s \in [\alpha(t), \beta(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \\ \geq \frac{\mu_{\Delta}(\{s \in [\alpha'(t), \beta'(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\mu_{\Delta}([\alpha(t), \beta(t)]_{\mathbb{T}})} \\ = \frac{\sigma(\beta'(t)) - \alpha'(t)}{\sigma(\beta(t)) - \alpha(t)} \frac{\mu_{\Delta}(\{s \in [\alpha'(t), \beta'(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\sigma(\beta'(t)) - \alpha'(t)}. \end{aligned}$$

Since $st_{\mathbb{T}-\alpha\beta} - \lim f(t) = L$, taking limit as $t \rightarrow \infty$ on the both sides of last inequality, we get

$$\lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(\{s \in [\alpha'(t), \beta'(t)]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon\})}{\sigma(\beta'(t)) - \alpha'(t)} = 0,$$

which means $st_{\mathbb{T}-\alpha'\beta'} - \lim f(t) = L$. Hence, the proof is completed. \square

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