Existence and Stability for a Non-Local Isoperimetric Model of Charged Liquid Drops

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Abstract

We consider a variational problem related to the shape of charged liquid drops at equilibrium. We show that this problem never admits local minimizers with respect to L^1 perturbations preserving the volume. However, we prove that the ball is stable under small $C^{1,1}$ perturbations when the charge is small enough.

1. Introduction

In this paper we study an isoperimetric variational problem where the perimeter, which is local and attractive, competes with the Riesz potential energy, which is non-local and repulsive. More precisely, we denote

$$\mathcal{I}_\alpha(E) := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^\alpha} \; : \; \mu(E) = 1 \right\},$$

where $\alpha \in (0, d)$ and E is a compact subset of $\mathbb{R}^d$ and consider the functional

$$\mathcal{F}_{\alpha,Q}(E) \; := \; P(E) \; + \; Q^2 \mathcal{I}_\alpha(E) \tag{1.1}$$

where $Q > 0$ is a parameter and where $P(E)$ denotes the perimeter of E (which corresponds to $\mathcal{H}^{d-1}(\partial E)$ if E has smooth boundary, see [4]). We are, in particular, interested in the questions of the existence and characterization of stable sets under volume preserving perturbations. It turns out that the answer to these questions depends crucially on the regularity of the allowed perturbations. In fact, we prove that on the one hand, there are no local (or global) minimizers of (1.1) under volume constraint in the L^1 or even Hausdorff topology. This implies that there are no sets which are stable under such perturbations. On the other hand, we prove that for small enough charge Q, the ball is stable under small $C^{1,1}$ perturbations. This comes as a by-product of the global minimality of such a ball in the class of "regular enough" sets.

1.1. Description of the Model

For $\alpha = d - 2$, $\mathcal{I}_\alpha(E)$ corresponds to the Coulombic interaction energy and the functional (1.1) can be thought as modeling the equilibrium shape of a charged droplet for which surface tension and electric forces compete. Such charged droplets have received considerable attention since the seminal work of Lord Rayleigh [38] and are by now widely used in applications such as electrospray ionization, fuel injection and ink jet printing. Starting with the pioneering experiments of Zeleny [42], the following scenario emerged. For small charge, a spheric drop remains stable but when the charge overcomes a critical threshold Q_c, which depends on the volume of the drop and on the characteristic constants of the liquid (surface tension and dielectric constant), a symmetry breaking occurs. Typically, the drop deforms and quickly develops conical shaped singularities, ejecting a very thin liquid jet [11,20,41].

This jet carries very little mass but a large portion of the charge. This type of behavior has since been observed in more detail and in various experimental setups (see for instance [3,16]). We place particular emphasis on [1,15], where the disintegration of an evaporating drop is observed, since a model very similar to (1.1) has been proposed in [17,39] to explain these experiments. We should stress the fact that the study of the unstable regime, which is still very poorly understood both experimentally and mathematically (see for instance [20,22,36]), is far outside the scope of this paper. We focus instead on the rather simple variational model (1.1) which hopefully captures, at least for small charges, most of the characteristics of the system. However, the unconditional (in term of Q) non-linear instability of the ball that we obtain in contrast with numerical and experimental observations indicates that something is still missing in this model. A challenging question is identifying the relevant physical effect which stabilizes a charged drop.

In some applications such as electrowetting [37] it is more natural to impose the *electric potential* V_0 (see Definition 2.9) on the boundary of E rather than the total charge Q. In that case the energy of a drop E takes the form

$$P(E) - V_0^2 C_2(E), \tag{1.2}$$

where for a set $E \subset \mathbb{R}^d$ with $d \geq 3$,

$$C_2(E) := \min\left\{\int_{\mathbb{R}^d} |\nabla u|^2\, dx : u \in H_0^1(\mathbb{R}^d),\ u \geq 1 \text{ on } E\right\},$$

is the capacitary functional. Notice that since $\mathcal{I}_{d-2}(E) = C_2(E)^{-1}$ for compact sets (see Remark 2.5), the functionals (1.1) and (1.2) are qualitatively similar. The analogy is in fact deeper since both functionals give rise to the same Euler-Lagrange equation.

1.2. Main Results of the Paper

The first main result of the paper is that, when $\alpha \in (0, d-1)$, for every given charge and volume, quite surprisingly the functional $\mathcal{F}_{\alpha,Q}$ has no minimizer among subsets of $\mathbb{R}^d$ of this given volume. Indeed, it is more convenient to spread the excess charge into little drops far away from each other. Such result is contained in the following theorem:

Theorem 1.1. *For every $\alpha \in (0, d-1)$, there holds*

$$\inf_{|E|=m} \mathcal{F}_{\alpha,Q}(E) = \left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} P(B).$$

Ultimately, this comes from the fact that the perimeter is defined up to sets of Lebesgue measure zero while the Riesz potential energy is defined up to sets of zero capacity. This phenomenon is further illustrated when considering the problem among sets which are contained in a fixed bounded domain Ω. In this case we prove that the isoperimetric problem and the charge minimizing problem completely decouple.

Theorem 1.2. *Let Ω be a compact subset of $\mathbb{R}^d$ with smooth boundary, and let $0 < m < |\Omega|$. Let E_0 be a solution of the constrained isoperimetric problem*

$$\min \{P(E) : E \subset \Omega,\ |E| = m\}. \tag{1.3}$$

Then, for $\alpha \in (0, d-1)$ and $Q > 0$ we have

$$\inf_{|E|=m,\ E\subset\Omega} \mathcal{F}_{\alpha,Q}(E) = P(E_0) + Q^2 \mathcal{I}_\alpha(\Omega). \tag{1.4}$$

As a by-product of our analysis we also get that $\mathcal{F}_{\alpha,Q}$ does not have local minimizers with respect to the L^1 or even Hausdorff topology:

Theorem 1.3. *For any $\alpha \in (0, d-1)$ and $Q > 0$, the functional $\mathcal{F}_{\alpha,Q}$ does not admit local volume-constrained minimizers with respect to the L^1 or the Hausdorff topology.*

Let us stress the fact that Theorem 1.3 asserts, in particular, that there is *never* non-linear stability of the ball. However, we should also notice that the competitors that we construct and which are made of infinitely small droplets, are very singular. It would be interesting to better understand the mechanism preventing the formation of such micro drops.

One possible explanation is that global (or even local) L^1 minimizers are not the right objects to consider. One should instead look for stable configurations under smoother deformations. These are typically local minimizers for a stronger topology. It is then reasonable to look for minimizers of $\mathcal{F}_{\alpha,Q}$ in some smaller class of sets with some extra regularity conditions. The class that we take into consideration, and denote by $\mathcal{K}_\delta$, is that of sets which admit at every point of their boundary an internal and an external tangent ball of a fixed radius δ (namely, the δ-ball condition, see Definition 2.18). We denote by $\mathcal{K}_\delta^{co}$ the class of connected sets of $\mathcal{K}_\delta$. The purpose of introducing such a class is to prove the stability of the ball with respect to $C^{1,1}$ perturbations. There are indeed two main (mathematical) advantages of working in $\mathcal{K}_\delta$. The first, is that it ensures density estimates on the sets. These estimates are usually the most basic regularity results available for minimizers of minimal surfaces types of problems (see [26,27,30,33]). Thanks to the constructions of Theorem 1.1, we see that in our problem there is no hope to get such estimates without imposing them *a priori*. The second advantage is that, at least in the Coulombic case $\alpha = d-2$, for every set $E \in \mathcal{K}_\delta$, the minimizing measure for $\mathcal{I}_\alpha(E)$ is a uniformly bounded measure on ∂E (see the end of Section 2). We use in a crucial way this L^∞ control on the charge density in the analysis of the stability of the ball. Our second main result is then:

Theorem 1.4. *Let $d \geq 3$ and $\alpha = d-2$. Then for any $\delta > 0$ and $m \geq \omega_d \delta^d$, there exists a charge $\bar{Q}\left(\frac{\delta}{m^{1/d}}\right) > 0$, such that if*

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} \leq \bar{Q}\left(\frac{\delta}{m^{1/d}}\right)$$

the ball is stable for problem (1.6) under volume preserving perturbations with $C^{1,1}$ norm less than δ.

This extends a previous result of M.A. Fontelos and A. Friedman [20], which asserts the stability with respect to $C^{2,\alpha}$ perturbations. These authors also gave a detailed analysis of the linear stability. We remark that our proof of the stability of the ball is quite different from the one in [20], and is inspired by the proofs in [10,21,30]. In particular it lies between linear and non-linear stability since it follows from the following three theorems asserting that for small charge Q, the ball is the unique minimizer in the class $\mathcal{K}_\delta$.

The first result is an existence theorem in the class $\mathcal{K}_\delta^{co}$.

Theorem 1.5. *For all $Q \geq 0$ problem*

$$\min \left\{ \mathcal{F}_{\alpha,Q}(E) : |E| = m, \ E \in \mathcal{K}_\delta^{co} \right\}, \tag{1.5}$$

has a solution.

To avoid the strong hypothesis on the connectedness of the competitors, it is necessary to impose a bound from above on the charge Q.

Theorem 1.6. *There exists a constant $Q_0 = Q_0(\alpha, d)$ such that, for every $\delta > 0$, $m \geq \omega_d \delta^d$ and*

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} \leq Q_0 \frac{\delta^d}{m},$$

problem

$$\min \left\{ \mathcal{F}_{\alpha,Q}(E) : |E| = m, \ E \in \mathcal{K}_\delta \right\}, \tag{1.6}$$

has a solution.

It is worth remarking that the main ingredient of the proof of Theorem 1.6 is the isoperimetric inequality in quantitative form (see [9,19,23]). Finally, using delicate estimates on the Riesz potential energy $\mathcal{I}_\alpha(E)$ for small perturbations of the ball, we are able to prove the following stability theorem in the Coulombic case.

Theorem 1.7. *Let $d \geq 3$ and $\alpha = d-2$. Then for any $\delta > 0$ and $m \geq \omega_d \delta^d$, there exists a charge $\bar{Q}\left(\frac{\delta}{m^{1/d}}\right) > 0$, such that if*

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} \leq \bar{Q}\left(\frac{\delta}{m^{1/d}}\right)$$

the ball is the unique minimizer of problem (1.6).

It would be interesting to understand if our stability result could be extended both to the case $\alpha \neq d-2$ and maybe more interestingly to a weaker class of perturbations such as, for instance, small Lipschitz ones.

Let us point out that for $\alpha \leq d-2$, the optimal measure for the Riesz potential concentrates on the boundary of the sets whereas for $\alpha > d-2$ it has support on the whole set (see Lemma 2.15). Therefore, for $\alpha > d-2$, it makes also sense to consider the functional

$$\mathcal{G}_{\alpha,Q}(E) = P(E) + Q^2 \mathcal{I}_\alpha(\partial E)$$

for which we can prove similar results to the ones described above.

Let us close this introduction by comparing our results with the analysis in [8, 10, 28–30, 33] of the non-local isoperimetric problem, known as the sharp interface Ohta-Kawasaki model,

$$\min_{|E|=m} P(E) + \int_{E\times E} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^\alpha}, \tag{1.7}$$

which is motivated by the theory of diblock copolymers and the stability of atomic nuclei. The authors show that there exist two (possibly equal) critical volumes $0 < m_1(\alpha) \leq m_2(\alpha)$ such that minimizers exist if $m \leq m_1$, while there are no minimizers if $m > m_2$. Moreover, the minimizers are balls when $\alpha < d-1$ and the volume is sufficiently small. These results have been generalized to non-local perimeters in [18] (see also [13]). A crucial difference between our model and the Ohta-Kawasaki model is that in the latter, the non-local term is Lipschitz with respect to the measure of the symmetric difference between sets (see for instance [10, Prop. 2.1]). Hence, on small scales, the perimeter dominates the non-local part of the energy. This implies in particular that minimizers enjoy the same regularity properties as minimal surfaces. In our case, it is quite the contrary since on small scales, the functional $\mathcal{I}_\alpha$ dominates the perimeter. As already pointed out above, this prevents *a priori* the hope of getting any regularity result for stable configurations. Let us notice that the same type of existence/non-existence issues in variational models where the perimeter competes against a non-local energy have recently been addressed in other models. For instance, in [6] the authors study a model related to epitaxial growth where the non-local part forces compactness whereas the perimeter part favors spreading.

The paper is organized as follows. In Section 2 we recall and prove some properties of the Riesz potentials $\mathcal{I}_\alpha$. In Section 3, we prove the non-existence of minimizers for the functional $\mathcal{F}_{\alpha,Q}$ (in particular we prove Theorems 1.1 and 1.3). In Section 4, we study this existence issue, that is we prove Theorems 1.5 and

1.6, before proving in Section 5 the stability of the ball (Theorem 1.7). Finally, in Section 6, we extend our results to the logarithmic potential energy

$$I_{\log}(E) := \inf\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} \log\left(\frac{1}{|x-y|}\right) \mathrm{d}\mu(x)\mathrm{d}\mu(y) \,:\, \mu(E)=1\right\}.$$

2. The Riesz Potential Energy

In this section we recall some results regarding the Riesz potential energy (see Definition 2.1 below). Most of the material presented here comes from [31].

In the following, given an open set $\Omega \subset \mathbb{R}^d$, we denote by $\mathcal{M}(\Omega)$ the set of all Borel measures with support in Ω. For $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B_r(x)$ the open ball of radius r centered in x and simply by B the unit ball and by $\omega_d = |B|$ its Lebesgue measure. For $k \in [0, d]$, we will denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure.

Definition 2.1. Let $d \geq 2$ and $\alpha > 0$. Given $\mu,\ \nu \in \mathcal{M}(\mathbb{R}^d)$, we define the *interaction energy* (or *potential energy*) between μ and ν by

$$\mathcal{I}_\alpha(\mu,\nu) := \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\mathrm{d}\mu(x)\, d\nu(y)}{|x-y|^\alpha} \in [0,+\infty].$$

When $\mu = \nu$, we simply write $\mathcal{I}_\alpha(\mu) := \mathcal{I}_\alpha(\mu,\mu)$. When the measures are absolutely continuous with respect to the Lebesgue measure, that is $\mu = f\mathcal{H}^d \llcorner E$ and $\nu = g\mathcal{H}^d \llcorner E$ for some set E and functions f and g, we denote $\mathcal{I}_\alpha(\mu,\nu) = \mathcal{I}^E_\alpha(f,g)$ (and when $f = g$ we denote it by $\mathcal{I}^E_\alpha(f)$). Similarly, when $\mu = f\mathcal{H}^{d-1} \llcorner \partial E$ and $\nu = g\mathcal{H}^{d-1} \llcorner \partial E$ we write $\mathcal{I}_\alpha(\mu,\nu) = \mathcal{I}^{\partial E}_\alpha(f,g)$ (and when $f = g$ we denote it by $\mathcal{I}^{\partial E}_\alpha(f)$).

The following proposition can be found in [31, (1.4.5)].

Proposition 2.2. *The functional $\mathcal{I}_\alpha$ is lower semicontinous for the weak* convergence of measures.*

We can then define the Riesz potential energy of a set.

Definition 2.3. Let $d \geq 2$ and $\alpha > 0$ then for every Borel set A we define the Riesz potential energy of A by

$$\mathcal{I}_\alpha(A) := \inf\Big\{\mathcal{I}_\alpha(\mu) \,:\, \mu \in \mathcal{M}(\mathbb{R}^d),\ \mu(A) = 1 \ \ . \tag{2.1}$$

Remark 2.4. Notice that, if we change μ in $Q\mu$ for a given charge $Q > 0$, then for any Borel set $A \subset \mathbb{R}^d$, it holds

$$Q^2\mathcal{I}_\alpha(A) := \inf\Big\{\mathcal{I}_\alpha(\mu) \,:\, \mu \in \mathcal{M}(\mathbb{R}^d),\ \mu(A) = Q \ \ .$$

Notice also that, for all $\lambda > 0$, there holds

$$\mathcal{I}_\alpha(\lambda A) = \lambda^{-\alpha}\mathcal{I}_\alpha(A). \tag{2.2}$$

Remark 2.5. An important notion related to $\mathcal{I}_\alpha(A)$ is the so-called α-capacity [31,32,34]

$$C_{d-\alpha}(A) := \frac{1}{\mathcal{I}_\alpha(A)}.$$

For $\alpha = d-2$ and K compact, we have the following representation of the capacity [32]:

$$C_2(K) = \inf\left\{\int_{\mathbb{R}^d} |\nabla f|^2 \ :\ f \in C_c^1(\mathbb{R}^d),\ f \geq 0,\ f \geq 1 \text{ on } K\right\}.$$

We stress however, for the sake of completeness, that there are other notions of capacity in the literature (see for instance the discussion in [32, Section 11.15]).

Remark 2.6. It is well known that the ball minimizes the perimeter under volume constraint. On the other hand in [7] it was proven that if $\alpha > d-2$, then the ball maximizes the Riesz Potential $\mathcal{I}_\alpha$ among compact sets of given volume.

The proof of the following result is given in [31, p. 131 and 132].

Lemma 2.7. *If A is a compact set, the infimum in (2.1) is achieved.*

Remark 2.8. When the set A is unbounded, there does not always exist an optimal measure μ, that is the infimum in (2.1) is not achieved. Indeed, it is possible to construct a set E of finite volume with $\mathcal{I}_\alpha(E) = 0$. To this aim, consider $\alpha \in (0, d-1)$, $\gamma \in (\frac{1}{d-1}, +\infty)$ and the set $E = \{(x, x') \in \mathbb{R} \times \mathbb{R}^{d-1} \ :\ |x'| \leq 1 \text{ and } |x'| \leq \frac{1}{|x|^\gamma}\}$. The set E has finite volume and taking N balls of radius $r = N^{-\beta}$ inside E, at mutual distance $\ell = N^{\frac{\beta}{\gamma}-1}$, with charge $1/N$ distributed uniformly on each ball, we have

$$\mathcal{I}_\alpha(E) \leq C\left(N^{\alpha\beta-1} + N^{(1-\frac{\beta}{\gamma})\alpha}\right)$$

for some $C > 0$, so that $\mathcal{I}_\alpha(E) = 0$ if $\frac{1}{d-1} < \gamma < \beta < \frac{1}{\alpha}$. Similarly, if $d > 2$ and $\alpha < d-2$, taking $\gamma > \frac{1}{d-2}$ one can even construct a set with finite perimeter for which the same property holds.

Definition 2.9. Given a non-negative Radon measure μ on $\mathbb{R}^d$ and $\alpha \in (0, d)$, we define the potential function

$$v_\alpha^\mu(x) := \int_{\mathbb{R}^d} \frac{\mathrm{d}\mu(y)}{|x-y|^\alpha} = \mu * k_\alpha(x)$$

where $k_\alpha(x) = |x|^{-\alpha}$. We will sometime drop the dependence of μ and α in the definition of v_α^μ and we will refer to it as the *potential*.

Definition 2.10. We say that two functions u and v are equal α-quasi everywhere (briefly $u = v$ α-q.e.) if they coincide up to a set of α-capacity 0.

The Euler-Lagrange equation of $\mathcal{I}_\alpha(A)$ reads as follows:

Lemma 2.11. *Let A be a compact set and let μ be a minimizer for $\mathcal{I}_\alpha(A)$ then $v^\mu = \mathcal{I}_\alpha(A)$ α-q.e. on* spt(μ)*, and $v^\mu \geq \mathcal{I}_\alpha(A)$ α-q.e. on A. Moreover, the following equation holds in the distributional sense*

$$(-\Delta)^{\frac{d-\alpha}{2}} v^\mu = c(\alpha, d)\,\mu, \tag{2.3}$$

where $(-\Delta)^s$ denotes the fractional Laplacian (see [14]). *In particular,*

$$(-\Delta)^{\frac{d-\alpha}{2}} v^\mu = 0 \qquad on\ \mathbb{R}^d \backslash A.$$

Proof. The first assertions on v^μ follow from [31, Theorem 2.6 and p. 137] (see also [24] where these conditions were first derived).

Equation (2.3) can be directly verified by means of the Fourier Transform, namely

$$\widehat{(-\Delta)^{\frac{d-\alpha}{2}} v^\mu}(\xi) = |\xi|^{d-\alpha}\widehat{\mu * k_\alpha}(\xi) = c(\alpha, d)\,\mu(\xi),$$

where we used the fact [31, Equation (1.1.1)]

$$\widehat{k}_\alpha(\xi) = c(\alpha, d)\,k_{d-\alpha}(\xi) \qquad \text{with} \quad c(\alpha, d) := \pi^{\alpha - \frac{d}{2}}\,\frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}.$$

□

We recall another important result which will be exploited in Section 4. We refer to [31, Theorem 1.15] (see also [32, Corollary 5.10]) for its proof.

Theorem 2.12. *For any signed measure μ and for any $\alpha \in (0, d)$, there holds*

$$\mathcal{I}_\alpha(\mu) = \int_{\mathbb{R}^d} \left(v^\mu_{\alpha/2}(x)\right)^2 \,\mathrm{d}x$$

and therefore,

$$\mathcal{I}_\alpha(\mu) \geq 0.$$

Moreover equality holds if and only if $\mu = 0$.

Remark 2.13. A consequence of Theorem 2.12 is that the functional $\mathcal{I}_\alpha(\cdot, \cdot)$ is a *positive*, bilinear operator on the product space of Radon measures on $\mathbb{R}^d$, $\mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d)$. In particular it satisfies the Cauchy-Schwarz inequality

$$\mathcal{I}_\alpha(\mu, \nu) \leq \mathcal{I}_\alpha(\mu)^{1/2}\mathcal{I}_\alpha(\nu)^{1/2}. \tag{2.4}$$

The following uniqueness result can be found in [31, page 133].

Lemma 2.14. *For every compact set A the measure minimizing $\mathcal{I}_\alpha(A)$ is unique.*

In the next lemma, we recall some properties of the support of the optimal measures.

Lemma 2.15. *Let $\alpha \in (0, d-1)$. For every open bounded set E, the minimizer μ of $\mathcal{I}_\alpha(E)$ satisfies:*

(i) *If $\alpha \le d-2$ then* $\operatorname{spt}(\mu) \subset \partial E$. *In particular $\mathcal{I}_\alpha(E) = \mathcal{I}_\alpha(\partial E)$.*
(ii) *If $\alpha > d-2$ then* $\operatorname{spt}(\mu) = \overline{E}$.

Moreover, when $\alpha \ge d-2$, $v_\alpha^\mu = \mathcal{I}_\alpha(E)$ on $\overline{E}$.

Proof. The case $\alpha \le d-2$ can be found in [31, page 162]. If $\alpha > d-2$, by [31, Theorem 2.6 and page 137], we know that $v_\alpha^\mu = \mathcal{I}_\alpha(E)$ α-q.e. on $\overline{E}$ and $v_\alpha^\mu \le \mathcal{I}_\alpha(E)$ on $\mathbb{R}^d$. Moreover, outside of $\operatorname{spt}(\mu)$, v_α^μ is smooth and $\Delta v_\alpha^\mu > 0$. Assume that there exists $x \in E \backslash \operatorname{spt}(\mu)$. Then there exists an open ball $B_r(x) \subset E \backslash \operatorname{spt}(\mu)$, but this is impossible since this would imply $v_\alpha^\mu = \mathcal{I}_\alpha(E)$ in $B_r(x)$ and hence $\Delta v_\alpha^\mu = 0$ in $B_r(x)$, contradicting $\Delta v_\alpha^\mu > 0$. The last claim of the lemma follows by the fact that v_α^μ is, in this case, a regular function on E which is α-q.e. equal to $\mathcal{I}_\alpha(E)$. □

We now prove a density result which is an adaptation of [31, Theorem 1.11 and Lemma 1.2].

Proposition 2.16. *Let E be a smooth connected closed set of $\mathbb{R}^d$, then for every $\alpha \in (0, d)$,*

$$\mathcal{I}_\alpha(E) = \inf\left\{ \mathcal{I}_\alpha^E(\mu) \ : \ \mu = f \mathrm{d}x, \ f \in L^\infty(E), \ \int_E f \ \mathrm{d}x = 1 \right\}.$$

Proof. By Definition 2.3 and Lemma 2.7 the proof reduces to the approximation of $I_\alpha(\mu)$ for a given measure μ supported on E and such that $\mu(E) = 1$. Let μ be such that $\mu(E) = 1$, $\operatorname{spt}(\mu) \subset E$ and $\mathcal{I}_\alpha(\mu) < +\infty$ then for $\varepsilon > 0$ consider the measure $\mu_\varepsilon \, \mathrm{d}x$ defined as

$$\mathrm{d}\mu_\varepsilon(x) = \left(\int_{B_\varepsilon(x)\cap E} \frac{\mathrm{d}\mu(y)}{|E \cap B_\varepsilon(y)|} \right) d\mathcal{H}^d \llcorner E.$$

Notice that by definition $\operatorname{spt}(\mu_\varepsilon) \subseteq E$. Moreover we have, by the Fubini Theorem,

$$\begin{aligned} \mu_\varepsilon(E) = \int_E \mu_\varepsilon(x)\,\mathrm{d}x &= \int_E \int_E \frac{\chi_{B_\varepsilon(x)}(y)\,\mathrm{d}\mu(y)}{|B_\varepsilon(y) \cap E|}\,\mathrm{d}x \\ &= \int_E \int_E \chi_{B_\varepsilon(y)}(x)\,\mathrm{d}x \frac{\mathrm{d}\mu(y)}{|B_\varepsilon(y) \cap E|} = \int_E \mathrm{d}\mu(y) = 1. \end{aligned}$$

Since $\|\mu_\varepsilon\|_{L^\infty(E)} \le (\min_{x\in E} |B_\varepsilon(x) \cap E|)^{-1} \le (C\varepsilon^d)^{-1}$, we only have to prove that $\mathcal{I}_\alpha^E(\mu_\varepsilon) \to \mathcal{I}_\alpha(\mu)$. By Theorem 2.12 we have

$$\mathcal{I}_\alpha^E(\mu_\varepsilon) = \int_{\mathbb{R}^d} \left(v_{\alpha/2}^{\mu_\varepsilon}(x) \right)^2 \mathrm{d}x.$$

Let us show that for all $x \in \mathbb{R}^d$,

$$v_{\alpha/2}^{\mu_\varepsilon}(x) \le C v_{\alpha/2}^\mu(x) \quad \text{and} \quad \lim_{\varepsilon\to 0} v_{\alpha/2}^{\mu_\varepsilon}(x) = v_{\alpha/2}^\mu(x)$$

from which we can conclude by means of the Dominated Convergence Theorem. Denoting by χ_A the characteristic function of the set A, we have, for any $x \in \mathbb{R}^d$,

$$\begin{aligned} v^{\mu_\varepsilon}_{\alpha/2}(x) &= \int_E \int_E \frac{1}{|B_\varepsilon(y) \cap E|} \chi_{B_\varepsilon(y)}(z) \frac{\mathrm{d}\mu(z)}{|x-y|^{\alpha/2}} \,\mathrm{d}y \\ &= \int_E \left(\int_{B_\varepsilon(z)\cap E} \frac{1}{|B_\varepsilon(y)\cap E|} \frac{|x-z|^{\alpha/2}}{|x-y|^{\alpha/2}} \,\mathrm{d}y \right) \frac{\mathrm{d}\mu(z)}{|x-z|^{\alpha/2}} \\ &\le \int_E \left(\frac{C}{\varepsilon^d} \int_{B_\varepsilon(z)} \frac{|x-z|^{\alpha/2}}{|x-y|^{\alpha/2}} \,\mathrm{d}y \right) \frac{\mathrm{d}\mu(z)}{|x-z|^{\alpha/2}}. \end{aligned} \tag{2.5}$$

Moreover it is possible to prove that the function

$$(x, z, \varepsilon) \mapsto \varepsilon^{-d} \int_{B_\varepsilon(z)} \frac{|x-z|^{\frac{\alpha}{2}}}{|x-y|^{\frac{\alpha}{2}}} \,\mathrm{d}y \tag{2.6}$$

is uniformly bounded in (x, z, ε) (see [31, Theorem 1.11]) so that $v^{\mu_\varepsilon}_{\alpha/2}(x) \le C v^{\mu}_{\alpha/2}(x)$ for a suitable constant $C > 0$. Consider now a point $x \in \mathbb{R}^d$ such that $v^{\mu}_{\alpha/2}(x) < +\infty$. Then for every $\delta > 0$ there is a ball $B_\eta(x)$ such that $v^{\mu'}_{\alpha/2} < \delta$ where $\mu' = \mu \llcorner B_\eta(x)$. By the previous computations, we know that $v^{(\mu')_\varepsilon}_{\alpha/2}(x) \le C\delta$. Moreover, $\lim_{\varepsilon\to 0} v^{(\mu-\mu')_\varepsilon}_{\alpha/2}(x) = v^{\mu-\mu'}_{\alpha/2}(x)$. Indeed, denoting for simplicity $\nu := \mu - \mu'$, we have that

$$\begin{aligned} v^{\nu_\varepsilon}_{\alpha/2}(x) &= \int_E \frac{d\nu_\varepsilon(y)}{|x-y|^{\alpha/2}} = \int_E \int_E \frac{\chi_{B_\varepsilon(y)}(z)\, d\nu(z)}{|B_\varepsilon(z)\cap E|} \frac{\mathrm{d}y}{|x-y|^{\alpha/2}} \\ &= \int_E \int_E \frac{\chi_{B_\varepsilon(z)}(y)\, d\nu(z)}{|B_\varepsilon(z)\cap E|} \frac{\mathrm{d}y}{|x-y|^{\alpha/2}} \\ &= \int_E \left(\frac{1}{|B_\varepsilon(z)\cap E|} \int_{E\cap B_\varepsilon(z)} \frac{\mathrm{d}y}{|x-y|^{\alpha/2}} \right) d\nu(z). \end{aligned}$$

From this the claim follows since the last quantity inside the parentheses uniformly converges to the function $|x-z|^{-\alpha/2}$ on every compact set which does not contain x, and since $\mathrm{spt}(\nu) = \mathrm{spt}(\mu - \mu') \subset B(x, \eta)^c$.

Furthermore, we have that $v^{(\mu-\mu')_\varepsilon}_{\alpha/2} = v^{\mu_\varepsilon}_{\alpha/2} - v^{\mu'_\varepsilon}_{\alpha/2}$. Thus we get

$$\begin{aligned} v^{\mu}_{\alpha/2}(x) &= v^{\mu'}_{\alpha/2}(x) + v^{\mu-\mu'}_{\alpha/2}(x) \le \delta + \lim_{\varepsilon\to 0} v^{(\mu-\mu')_\varepsilon}_{\alpha/2}(x) \\ &\le (1+C)\delta + \varliminf_{\varepsilon\to 0} v^{\mu_\varepsilon}_{\alpha/2}(x) \le (1+C)\delta + \varlimsup_{\varepsilon\to 0} v^{\mu_\varepsilon}_{\alpha/2}(x) \\ &\le (1+C)\delta + \varlimsup_{\varepsilon\to 0} v^{\mu'_\varepsilon}_{\alpha/2}(x) + \varlimsup_{\varepsilon\to 0} v^{(\mu-\mu')_\varepsilon}_{\alpha/2}(x) \\ &\le 2(1+C)\delta + v^{\mu}_{\alpha/2}(x) \end{aligned} \tag{2.7}$$

so that letting $\delta \to 0$ we get that $\lim_{\varepsilon\to 0} v^{\mu_\varepsilon}_{\alpha/2}(x) = v^{\mu}_{\alpha/2}(x)$ as claimed. □

For the unit ball, since the problem is invariant by rotations, it is not hard to compute the exact minimizer of $\mathcal{I}_\alpha(B)$ or $\mathcal{I}_\alpha(\partial B)$, see [31, Chapter II.13].

Lemma 2.17. *The uniform measure on the sphere* ∂B

$$d\mathcal{U}_B = \frac{1}{P(B)} d\mathcal{H}^{d-1} \llcorner \partial B$$

is the unique optimizer for $\mathcal{I}_\alpha(\partial B)$. *For* $d > \alpha > d-2$, *the measure*

$$d\tilde{\mathcal{U}}_B = \frac{C_\alpha}{(1-|x|^2)^{\frac{\alpha}{2}}} d\mathcal{H}^d \llcorner B$$

is the unique optimizer for $\mathcal{I}_\alpha(B)$ *(where* C_α *is a suitable renormalization constant).*

Definition 2.18. Given $\delta > 0$, we say that E satisfies the *internal* δ*-ball condition* if for any $x \in \partial E$ there is a ball of radius δ contained in E and tangent to ∂E in x. Analogously, E satisfies the *external* δ*-ball condition* if for any $x \in \partial E$ there is a ball of radius δ contained in E^c. Finally, if E satisfies both the internal and the external δ-ball condition we shall say that it satisfies the δ*-ball condition*.

We remark that the sets which satisfy the δ-ball condition have $C^{1,1}$ boundary with principal curvatures bounded from above by $1/\delta$, see [12]. We denote by $\mathcal{K}_\delta$ the class of all the closed sets which satisfy the δ-ball condition and by $\mathcal{K}_\delta^{co}$ the subset of $\mathcal{K}_\delta$ composed of connected sets.

Remark 2.19. An equivalent formulation of Definition 2.18 is requiring that $d_E \in C^{1,1}(\{|d_E| < \delta\})$, where

$$d_E(x) = \begin{cases} \operatorname{dist}(x, \partial E) & \text{if } x \notin E \\ -\operatorname{dist}(x, \partial E) & \text{if } x \in E \end{cases}$$

is the signed distance function from ∂E. See for instance [12].

Lemma 2.20. *Let* $\delta > 0$, *then every set* $E \in \mathcal{K}_\delta^{co}$ *with* $|E| = m$ *satisfies*

$$\operatorname{diam}(E) \le \sqrt{d}\, 2^{d+2} \frac{m}{\omega_d} \delta^{1-d}.$$

Proof. Consider the tiling of $\mathbb{R}^d$ given by $[0, 2\delta)^d + 2\delta\mathbb{Z}^d$ and for $k \in \mathbb{Z}^d$ let $C_k = [0, 2\delta)^d + 2\delta k$. For every $k \in \mathbb{Z}^d$ such that $C_k \cap E \neq \emptyset$, let $B_\delta(x_k)$ be a ball of radius δ such that $B_\delta(x_k) \subset E$ and $B_\delta(x_k) \cap C_k \neq \emptyset$. The existence of such a ball is guaranteed by the δ-ball condition. Any such ball can intersect at most 2^d cubes C_j so that

$$\sharp\{k \in \mathbb{Z}^d \,:\, E \cap C_k \neq \emptyset\} = \frac{1}{|B_\delta|} \sum_{k: C_k \cap E \neq \emptyset} |B_\delta(x_k)| \le \frac{2^d}{|B_\delta|} |E|,$$

where $\sharp A$ is the cardinality of the set A. The fact that E is connected implies that, up to translation, $E \subset [0, 4\delta \frac{2^d}{|B_\delta|} m]^d$. Thus we can conclude that

$$\operatorname{diam}(E) \le \operatorname{diam}\left(\left[0, 4\delta \frac{2^d}{|B_\delta|} m\right]^d\right) = \sqrt{d}\, 2^{d+2} \frac{m}{\omega_d} \delta^{1-d}.$$

□

Remark 2.21. As already pointed out in the introduction, in some sense the δ-ball condition is the analog of the famous density estimates for a problem in which the perimeter term is dominant, see [26]. Since, in the problems we are going to consider, both the perimeter and the Riesz potential energy are of the same order, there is *a priori* no hope of getting such density estimates from the minimality. It is a classical feature that for connected sets, these density estimates provide a bound on the diameter [27].

Proposition 2.22. *Let $d \geq 3$, $\alpha = d-2$, $\delta > 0$ and $E \subset \mathbb{R}^d$ be a compact set which satisfies the δ-ball condition. Then the optimal measure μ for $\mathcal{I}_\alpha(E) = \mathcal{I}_\alpha(\partial E)$ can be written as $\mu = f\mathcal{H}^{d-1} \llcorner \partial E$ with $\|f\|_{L^\infty(\partial E)} \leq \mathcal{I}_\alpha(E)(d-2)\delta^{-1}$.*

Proof. By Lemma 2.15 we know that the optimizer μ is concentrated on ∂E. Denote by $v = v^\mu_{d-2}$ the potential related to μ on E. By Lemma 2.15, we know that $v = \mathcal{I}_\alpha(E)$ on E, and that $-\Delta v = \mu$. By classical elliptic regularity (see for instance [25, Cor. 8.36]), v is regular in $\mathbb{R}^d \backslash E$, and $C^{1,\beta}$ up to the boundary of E. Consider now a point $x \in \partial E$ and let $y \in E$ such that the ball $B_\delta(y)$ is contained in E and is tangent to ∂E in x. The existence of such a y is guaranteed by the δ-ball condition satisfied by E. Let u be a solution of

$$\Delta u = 0 \quad \text{in } B^c_\delta(y); \qquad u = v(x) = \mathcal{I}_\alpha(E) \quad \text{on } \partial B_\delta(y).$$

Notice that $u(z) = \frac{\mathcal{I}_\alpha(E)\delta^{d-2}}{|z-y|^{d-2}}$ out of $B_\delta(y)$. By the maximum principle for harmonic functions, $u \leq \mathcal{I}_\alpha(E)$ on ∂E. Thus, again by the maximum principle, applied to $u - v$, we get that $v \geq u$ on $\mathbb{R}^d \backslash E$. Since $u(x) = v(x)$,

$$|\nabla v(x)| \leq |\nabla u(x)| = \mathcal{I}_\alpha(E)(d-2)\delta^{-1}. \tag{2.8}$$

Let us prove that $\mu = |\nabla v|\mathcal{H}^{d-1} \llcorner \partial E$. For this, let $x \in \partial E$ and $r > 0$ and consider a test function $\varphi \in C^\infty_c(\mathbb{R}^d)$. Then we have

$$\begin{aligned}\int_{\partial E} \varphi \mathrm{d}\mu &= -\int_{\mathbb{R}^d} \varphi \Delta v = \int_{\mathbb{R}^d} \langle \nabla\varphi, \nabla v\rangle \,\mathrm{d}y \\ &= \int_{E^c} \langle \nabla\varphi, \nabla v\rangle \,\mathrm{d}y = \int_{\partial E} \varphi \langle \nabla v, \nu^E\rangle d\mathcal{H}^{d-1}\end{aligned} \tag{2.9}$$

where ν^E is the external normal to E. Since v is constant on ∂E, its tangential derivative is zero. Thus, since $v < \mathcal{I}_\alpha(E)$ on $\mathbb{R}^d \backslash \overline{E}$ we have that $\langle \nabla v, \nu^E\rangle \geq 0$. Therefore, $\langle \nabla v, \nu^E\rangle = |\nabla v|$ on ∂E. Hence, by (2.9) we conclude that for every test function φ,

$$\int_{\partial E} \varphi \mathrm{d}\mu = \int_{\partial E} \varphi |\nabla v| d\mathcal{H}^{d-1},$$

which is equivalent to the claim $\mu = |\nabla v|\mathcal{H}^{d-1} \llcorner \partial E$. □

3. Non-Existence of Minimizers

Definition 3.1. Let $d \geq 2$ and $\alpha > 0$. For every $Q > 0$ and every open set $E \subset \mathbb{R}^d$ we define the functionals

$$\mathcal{F}_{\alpha,Q}(E) := P(E) + Q^2 \mathcal{I}_\alpha(E) \tag{3.1}$$

and

$$\mathcal{G}_{\alpha,Q}(E) := P(E) + Q^2 \mathcal{I}_\alpha(\partial E). \tag{3.2}$$

Notice that by Lemma 2.15, for $\alpha \in (0, d-2]$ the functionals $\mathcal{F}_{\alpha,Q}$ and $\mathcal{G}_{\alpha,Q}$ coincide. Notice also that $\mathcal{F}_{\alpha,Q}(E) \equiv +\infty$ if $\alpha \geq d$, and $\mathcal{G}_{\alpha,Q}(E) \equiv +\infty$ if $\alpha \geq d-1$.

In this section we consider a closed, connected, regular set $\Omega \subset \mathbb{R}^d$ (not necessarily bounded) of measure $|\Omega| > m$ and address the following problems:

$$\inf_{|E|=m,\, E\subset\Omega} \mathcal{F}_{\alpha,Q}(E) \tag{3.3}$$

and

$$\inf_{|E|=m,\, E\subset\Omega} \mathcal{G}_{\alpha,Q}(E), \tag{3.4}$$

where the (implicit) parameter α belongs to $(0, d)$.

Theorem 3.2. *For every $\alpha \in (0, d-1)$, there holds*

$$\inf_{|E|=m} \mathcal{F}_{\alpha,Q}(E) = \inf_{|E|=m} \mathcal{G}_{\alpha,Q}(E) = \min_{|E|=m} P(E) = \left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} P(B).$$

In particular, problems (3.3) and (3.4) do not admit minimizers when $\Omega = \mathbb{R}^d$.

Proof. Let $N \in \mathbb{N}$ and consider a number β which will be fixed later on. Consider N balls of radius $r_N = N^{-\beta}$ which we can consider mutually infinitely far away (since sending them away leaves unchanged the perimeter and decreases the potential interaction energy), and put on each of these balls a charge $\frac{1}{N}$. Let $V_N = N r_N^d \omega_d$ be their total volume and consider the set E to be given by the union of these balls with a (non-charged) ball of volume $m - V_N$. If we choose $\beta \in (1/(d-1), 1/\alpha)$, then we get

$$\lim_{N\to+\infty} N r_N^{d-1} = 0 \quad \text{and} \quad \lim_{N\to+\infty} \frac{1}{N}\frac{1}{r_N^\alpha} = 0, \tag{3.5}$$

which implies that $V_N \to 0$ and

$$\left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} P(B) \leq P(E) + Q^2 \mathcal{I}_\alpha(E) \leq \left(\frac{m - V_N}{\omega_d}\right)^{\frac{d-1}{d}} P(B) + C\left(N r_N^{d-1} + \frac{Q^2}{N}\frac{1}{r_N^\alpha}\right).$$

Since the right-hand side converges to $\left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} P(B)$, as N tends to $+\infty$, the claim follows. □

The following result follows directly from the construction made in the previous theorem.

Corollary 3.3. *Let* $\alpha \in (0, d-1)$ *and* $m > 0$*. For every* $0 < \delta < (m/\omega_d)^{\frac{1}{d}}$ *there exists a charge* $Q_\delta = Q_\delta(\alpha, m)$ *such that* $Q_\delta \to 0$ *as* $\delta \to 0$*, and the ball of volume m is not the minimizer of* $\mathcal{F}_{\alpha,Q}$*, among sets in* $\mathcal{K}_\delta$ *with volume m and charge* $Q > Q_\delta$*.*

We now consider the case of bounded Ω where the situation is more involved.

Theorem 3.4. *Let* Ω *be a compact subset of* $\mathbb{R}^d$ *with smooth boundary, and let* $0 < m < |\Omega|$*. Let* E_0 *be a solution of the constrained isoperimetric problem*

$$\min\{P(E) : E \subset \Omega,\ |E| = m\}. \tag{3.6}$$

Then, for $\alpha \in (0, d-1)$ *and* $Q > 0$ *we have*

$$\inf_{|E|=m,\ E\subset\Omega} \mathcal{F}_{\alpha,Q}(E) = \inf_{|E|=m,\ E\subset\Omega} \mathcal{G}_{\alpha,Q}(E) = P(E_0) + Q^2\mathcal{I}_\alpha(\Omega). \tag{3.7}$$

Proof. We divide the proof into three steps.

Step 1. For $\varepsilon > 0$ and $f \in L^\infty(\Omega)$, with $f \geq 0$ and $\displaystyle\int_\Omega f\,\mathrm{d}x = 1$, we shall construct a measure $\tilde{\mu}_\varepsilon$ with $\mathrm{spt}(\tilde{\mu}_\varepsilon) \subset \Omega$, $\tilde{\mu}_\varepsilon(\Omega) = 1$, satisfying

$$P(\mathrm{spt}(\tilde{\mu}_\varepsilon)) \leq \varepsilon \tag{3.8}$$

and

$$\mathcal{I}_\alpha(\tilde{\mu}_\varepsilon) \leq \mathcal{I}_\alpha^\Omega(f) + \varepsilon. \tag{3.9}$$

Let $\delta > \lambda > 0$ be small parameters to be fixed later and consider the tiling of the space given by $[0,\lambda)^d + \lambda\mathbb{Z}^d$. For every $k \in \mathbb{Z}^d$ such that $(\lambda k + [0,\lambda)^d) \cap \Omega \neq \emptyset$, we let $C_k = \lambda k + [0,\lambda)^d$ and denote by x_k be the center of C_k. Notice that the number N of such squares C_k is bounded by $C(\Omega)\lambda^{-d}$. Letting $f_k := \displaystyle\int_{C_k} f\,\mathrm{d}x$, it holds that

$$\begin{aligned}
\sum_{|x_k - x_j| \geq 2\delta} \frac{f_k f_j}{|x_k - x_j|^\alpha} &= \sum_{|x_k - x_j| \geq 2\delta} \int_{C_k \times C_j} \frac{f(x)f(y)}{|x-y|^\alpha}\,\frac{|x-y|^\alpha}{|x_k - x_j|^\alpha}\mathrm{d}x\,\mathrm{d}y \\
&\leq \sum_{|x_k - x_j| \geq 2\delta} \int_{C_k \times C_j} \frac{f(x)f(y)}{|x-y|^\alpha}\,\frac{\left(|x_k - x_j| + 2\lambda\right)^\alpha}{|x_k - x_j|^\alpha}\mathrm{d}x\,\mathrm{d}y \\
&\leq \sum_{|x_k - x_j| \geq 2\delta} \int_{C_k \times C_j} \frac{f(x)f(y)}{|x-y|^\alpha}\left(1 + C(\alpha)\frac{\lambda}{\delta}\right)\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{3.10}$$

where we used the fact that

$$\sum_{|x_k - x_j| \geq 2\delta} \int_{C_k \times C_j} \frac{f(x)f(y)}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y \leq \int_{\Omega\times\Omega} \frac{f(x)f(y)}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y = \mathcal{I}_\alpha^\Omega(f) < \infty.$$

Let now $r = (\lambda/2)^\beta$, with $\beta > 1$. If $\mathrm{dist}(x_k, \mathbb{R}^d \setminus \Omega) \le r$, we replace the point x_k with a point $\tilde{x}_k \in C_{j(k)}$, with $|\tilde{x}_k - x_{j(k)}| \ge \lambda/4$, where $C_{j(k)} \subset \Omega$ is a cube adjacent to C_k. For simplicity of notation, we still denote $\tilde{x}_k$ by x_k. We consider N balls of radius r centered at the points x_k, and we set

$$\tilde{\mu}_\varepsilon := \sum_k \frac{f_k}{\mathcal{H}^{d-1}(\partial B_r)} \chi_{\partial B_r(x_k)}.$$

Notice that such measures are suitable competitors in the definition of both the minima appearing in the definition of $\mathcal{F}_{\alpha,Q}$ and $\mathcal{G}_{\alpha,Q}$. By construction it holds that $\mathrm{spt}(\tilde{\mu}_\varepsilon) \subset \Omega$ and $\tilde{\mu}_\varepsilon(\Omega) = \int_\Omega f \,\mathrm{d}x = 1$. We have

$$\begin{aligned}
\mathcal{I}_\alpha(\tilde{\mu}_\varepsilon) &= \sum_{j,k} \frac{f_k f_j}{\mathcal{H}^{d-1}(B_r)^2} \int_{\partial B_r(x_j)\times \partial B_r(x_k)} \frac{d\mathcal{H}^{d-1}(x)\, d\mathcal{H}^{d-1}(y)}{|x-y|^\alpha} \\
&= \sum_k \frac{f_k^2}{\mathcal{H}^{d-1}(B_r)^2} \int_{\partial B_r(x_k)\times \partial B_r(x_k)} \frac{d\mathcal{H}^{d-1}(x)\, d\mathcal{H}^{d-1}(y)}{|x-y|^\alpha} \\
&\quad + \sum_{|x_j - x_k| < 2\delta,\, k\neq j} \frac{f_k f_j}{\mathcal{H}^{d-1}(B_r)^2} \int_{\partial B_r(x_j)\times \partial B_r(x_k)} \frac{d\mathcal{H}^{d-1}(x)\, d\mathcal{H}^{d-1}(y)}{|x-y|^\alpha} \\
&\quad + \sum_{|x_j - x_k| \ge 2\delta} \frac{f_k f_j}{\mathcal{H}^{d-1}(B_r)^2} \int_{\partial B_r(x_j)\times \partial B_r(x_k)} \frac{d\mathcal{H}^{d-1}(x)\, d\mathcal{H}^{d-1}(y)}{|x-y|^\alpha} \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Moreover we have that

$$I_1 \le CN \|f\|^2_{L^\infty(\Omega)} |C_k|^2 \frac{1}{r^\alpha} \le C\|f\|^2_{L^\infty(\Omega)} \lambda^{d-\alpha\beta}, \tag{3.11}$$

and

$$I_2 \le C\delta^d N^2 \|f\|^2_{L^\infty(\Omega)} |C_k|^2 \frac{1}{\lambda^\alpha} \le C\|f\|^2_{L^\infty(\Omega)} \frac{\delta^d}{\lambda^\alpha}. \tag{3.12}$$

Eventually, from (3.10) it follows that

$$\begin{aligned}
I_3 &= \sum_{|x_j - x_k| \ge 2\delta} \frac{f_k f_j}{|x_k - x_j|^\alpha} \frac{1}{\mathcal{H}^{d-1}(B_r)^2} \\
&\qquad \int_{\partial B_r(x_j)\times \partial B_r(x_k)} \frac{|x_k - x_j|^\alpha}{|x-y|^\alpha}\, d\mathcal{H}^{d-1}(x)\, d\mathcal{H}^{d-1}(y) \\
&\le \sum_{|x_k - x_j| \ge 2\delta} \frac{f_k f_j}{|x_k - x_j|^\alpha} \left(1 + C(\alpha)\frac{r}{\delta}\right. \\
&\le \mathcal{I}^\Omega_\alpha(f) \left(1 + C(\alpha)\frac{\lambda}{\delta}\right) \left(1 + C(\alpha)\frac{r}{\delta}\right. \\
&\le \mathcal{I}^\Omega_\alpha(f) + C(\alpha)\mathcal{I}^\Omega_\alpha(f)\frac{\lambda}{\delta}.
\end{aligned} \tag{3.13}$$

Letting $\lambda = \delta^{\gamma}$, from (3.11), (3.12), (3.13) we then get

$$\begin{aligned}\mathcal{I}_\alpha(\tilde{\mu}_\varepsilon) = I_1 + I_2 + I_3 &\leq \mathcal{I}_\alpha^{\Omega}(f) + C(\alpha)\mathcal{I}_\alpha^{\Omega}(f)\delta^{\gamma-1} \\ &\quad + C\|f\|^2_{L^\infty(\Omega)}\left(\delta^{\gamma(d-\alpha\beta)} + \delta^{d-\alpha\gamma}\right. .\end{aligned}$$

Choosing $1 < \beta < d/\alpha$ and $1 < \gamma < d/\alpha$, for δ small enough, we obtain (3.9).

We now show that (3.8) also holds. To this end, we notice that

$$\mathcal{H}^{d-1}(\mathrm{spt}(\tilde{\mu}_\varepsilon)) \leq CNr^{d-1} = CN\lambda^{\beta(d-1)} = C\lambda^{\beta(d-1)-d} \tag{3.14}$$

so that, for λ small enough, (3.8) follows from (3.14) by letting $d/\alpha > \beta > d/(d-1)$, the choice of which is allowed since $\alpha < d-1$.

Step 2. Let now E_0 be a solution of the constrained isoperimetric problem (3.6), and let

$$E_\varepsilon := \left(E_0 \cup \bigcup_k B_r(x_k)\right) \setminus B_\eta, \qquad \mu_\varepsilon := \frac{\tilde{\mu}_\varepsilon \llcorner E_\varepsilon}{1 - \tilde{\mu}_\varepsilon(B_\eta)},$$

where $B_\eta \subset E_0$ is a ball such that $|E_\varepsilon| = m$. Notice that $\mathrm{spt}(\mu_\varepsilon) \subset E_\varepsilon$ and $\mu_\varepsilon(E_\varepsilon) = 1$. Since

$$|B_\eta| = \left|E_0 \cup \bigcup_k B_r(x_k)\right| - |E_\varepsilon| \leq \left|\bigcup_k B_r(x_k)\right|,$$

by (3.14) we have

$$|B_\eta|^{\frac{d-1}{d}} \leq \left|\bigcup_k B_r(x_k)\right|^{\frac{d-1}{d}} \leq CP\left(\bigcup_k B_r(x_k)\right) \leq C\lambda^{\beta(d-1)-d},$$

so that $\eta \leq C\lambda^{\beta - \frac{d}{d-1}}$. In particular, recalling (3.9), for λ sufficiently small the measure μ_ε satisfies

$$\mathcal{I}_\alpha(\mu_\varepsilon) \leq \mathcal{I}_\alpha(\tilde{\mu}_\varepsilon) + \varepsilon \leq \mathcal{I}_\alpha^{\Omega}(f) + 2\varepsilon. \tag{3.15}$$

From (3.15) we then get

$$\overline{\lim_{\varepsilon\to 0}}\, P(E_\varepsilon) + Q^2\mathcal{I}_\alpha(\mu_\varepsilon) = P(E_0) + Q^2\mathcal{I}_\alpha^{\Omega}(f). \tag{3.16}$$

Step 3. By Proposition 2.16 we can find a function $f \in L^\infty(\Omega)$ such that $\int_\Omega f\,\mathrm{d}x = 1$ and $\mathcal{I}_\alpha^{\Omega}(f) \leq \mathcal{I}_\alpha(\Omega) + \varepsilon$. Thus (3.7) follows by (3.16) and a diagonal argument. □

Thanks to Theorem 3.4 we are able to prove:

Theorem 3.5. *For any $\alpha \in (0, d-1)$ and $Q > 0$, the functional $\mathcal{F}_{\alpha,Q}$ does not admit local volume-constrained minimizers with respect to the L^1 or the Hausdorff topology.*

Proof. Let K be a compact set, and let Ω_ε, for $\varepsilon > 0$, be a family of open sets with smooth boundary, such that $K \subset \Omega_\varepsilon$ for any $\varepsilon > 0$, and $\Omega_\varepsilon \to K$ as $\varepsilon \to 0$ in the Hausdorff topology (in particular $|\Omega_\varepsilon \setminus K| \to 0$ as $\varepsilon \to 0$). By Theorem 3.4, it is enough to show that $\mathcal{I}_\alpha(\Omega_\varepsilon) < \mathcal{I}_\alpha(K)$ for any $\varepsilon > 0$ (with strict inequality), which follows directly from Lemma 2.15. □

Remark 3.6. Notice that when $\alpha \in (d-2, d-1)$, Problem (3.4) relaxes to its "natural" domain, in the sense that the infimum is $P(E_0) + Q^2\mathcal{I}_\alpha(\Omega)$ and not $P(E_0) + Q^2\mathcal{I}_\alpha(\partial\Omega)$ as one might expect.

Remark 3.7. Notice also that as soon as Ω contains a ball of volume m then the solution of the isoperimetric problem (3.6) is a ball.

Remark 3.8. In the statement of Theorem 3.4 it is possible to replace $P(E)$ by the relative perimeter $P(E;\Omega)$ (see for instance [4]) almost without changing the proof. In other words, under the hypotheses of Theorem 3.4 we have that

$$\begin{aligned}\inf_{|E|=m,\, E\subset\Omega} P(E;\Omega) + Q^2\mathcal{I}_\alpha(E) &= \inf_{|E|=m,\, E\subset\Omega} P(E;\Omega) + Q^2\mathcal{I}_\alpha(\partial E)\\ &= P(E_\Omega;\Omega) + Q^2\mathcal{I}_\alpha(\Omega),\end{aligned} \tag{3.17}$$

E_Ω being a solution of the relative isoperimetric problem

$$\min_{E\subset\Omega,|E|=m} P(E;\Omega).$$

Remark 3.9. An interpretation of Theorem 3.4 is that Problem (3.7) decouples into the isoperimetric problem (3.6) and the *charge-minimizing* problem (2.1), which are minimized separately. This is essentially due to the fact that the perimeter is defined up to a set of zero Lebesgue measure, while the Riesz potential energy is defined up to a set of zero capacity [31, Chapter 2].

A consequence of this is that the minimum problem

$$\min \ \mathcal{F}_{\alpha,Q}(E) : |E| = m,\ E \subset A$$

has in general no solution.

Remark 3.10. For $\alpha \in [d-1, d)$, it seems difficult to construct a sequence of open sets with vanishing perimeter but of positive capacity. This is due to the fact that sets of positive α-capacity have Hausdorff measure at least α (see [34]). As a consequence, the infimum of (3.7) should be strictly larger than $P(E_0)$. In order to study the question of the existence or non-existence of minimizers, one would need to extend the definition of $\mathcal{F}_{\alpha,Q}$ to sets which are not open. There are mainly two possibilities to do this. The first is to let for every Borel set E

$$\mathcal{F}_{\alpha,Q}(E) := P(E) + Q^2\mathcal{I}_\alpha(E)$$

where $P(E)$ now denotes the total variation of χ_E (see [4]). It is easy to see that the problem is still ill posed in this class. Indeed, for every set E, it is possible to consider a set F of positive α-capacity but of Lebesgue measure zero so that $\mathcal{F}_{\alpha,Q}(E \cup$

$F) < \mathcal{F}_{\alpha,Q}(E)$. The second possibility would be to consider the relaxation of the functional $\mathcal{F}_{\alpha,Q}$ defined on open sets for a suitable topology. Because of the previous discussion, we see that the L^1 topology, for which the perimeter has good compactness and lower semicontinuity properties, is not the right one. The Hausdorff topology might be more adapted to this situation. Unfortunately, the resulting functional seems hard to identify.

Remark 3.11. When considering a bounded domain A it is also interesting to study the Riesz potential associated with the Green kernel G_A, with Dirichlet or Neumann boundary conditions. Since

$$G_A(x, y) = k_{d-2}(|x-y|) + h(x, y)$$

with h harmonic in A (see [31, Chapter 1.3], [10]), Theorem 3.4 can be easily extended to that case.

4. Existence of Minimizers Under Some Regularity Conditions

In the previous section we have seen that we cannot hope to get existence for Problem (3.3) without some further assumptions on the class of minimization. In this section we investigate the existence of minimizers in the classes $\mathcal{K}_\delta$ and $\mathcal{K}_\delta^{co}$, defined in Definition 2.18. More precisely, we consider the following problems:

$$\min\left\{\mathcal{F}_{\alpha,Q}(E): \; |E| = m, \; E \in \mathcal{K}_\delta^{co}\right\}, \tag{4.1}$$

$$\min\left\{\mathcal{G}_{\alpha,Q}(E): \; |E| = m, \; E \in \mathcal{K}_\delta^{co} \quad ,\right. \tag{4.2}$$

$$\min\left\{\mathcal{F}_{\alpha,Q}(E): \; |E| = m, \; E \in \mathcal{K}_\delta\right\}, \tag{4.3}$$

$$\min \;\; \mathcal{G}_{\alpha,Q}(E): \; |E| = m, \; E \in \mathcal{K}_\delta \quad . \tag{4.4}$$

Notice that, up to rescaling, we can always assume that $|E| = \omega_d$. Indeed, if we let $\tilde{E} := \left(\frac{\omega_d}{m}\right)^{1/d} E$, so that $|\tilde{E}| = \omega_d$, from (2.2) we get

$$\mathcal{F}_{\alpha,Q}(E) = \mathcal{F}_{\alpha,Q}\left(\left(\frac{m}{\omega_d}\right)^{1/d} \tilde{E}\right) = \left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} \mathcal{F}_{\alpha,\left(\frac{\omega_d}{m}\right)^{\frac{d-1+\alpha}{2d}} Q}(\tilde{E}) \tag{4.5}$$

$$\mathcal{G}_{\alpha,Q}(E) = \mathcal{G}_{\alpha,Q}\left(\left(\frac{m}{\omega_d}\right)^{1/d} \tilde{E}\right) = \left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} \mathcal{G}_{\alpha,\left(\frac{\omega_d}{m}\right)^{\frac{d-1+\alpha}{2d}} Q}(\tilde{E}). \tag{4.6}$$

Definition 4.1. For any set E with $|E| = \omega_d$, we let $\delta P(E) := P(E) - P(B) \geq 0$ be the *isoperimetric deficit* of E.

Theorem 4.2. *For all $Q \geq 0$ problem (4.1) and (4.2) have a solution.*

Proof. Let us focus on (4.1) since the proof of the existence for (4.2) is very similar. Let $E_n \in \mathcal{K}_\delta^{co}$ be a minimizing sequence, with $|E_n| = \omega_d$. And let μ_n be the corresponding optimal measures for $\mathcal{I}_\alpha(E_n)$. Since $P(E_n) + Q^2\mathcal{I}_\alpha(E_n) \leq P(B) + Q^2\mathcal{I}_\alpha(B)$, we have that

$$\delta P(E_n) \leq Q^2\mathcal{I}_\alpha(B),$$

therefore $P(E_n)$ is uniformly bounded. By Lemma 2.20, the sets E_n are also uniformly bounded so that by the compactness criterion for functions of bounded variation (see for instance [4]), there exists a subsequence converging in L^1 to some set E with $|E| = m$. Similarly, up to subsequence, μ_n is weakly* converging to some probability measure μ.

Let us prove that E_n converges to E also in the Kuratowski convergence, or equivalently, in the Hausdorff metric (see for instance [5]). Namely we have to check the following two conditions:

$$(i)\, x_n \to x, \quad x_n \in E_n \;\Rightarrow\; x \in E;$$
$$(ii)\, x \in E \;\Rightarrow\; \exists x_n \in E_n \text{ such that } x_n \to x.$$

The second condition is an easy consequence of the L^1-convergence. To prove the first one, we notice that by the internal δ-ball condition, up to choosing a radius r small enough there exists a constant $c = c(d, \delta) > 0$ such that $|B(x_n, r) \cap E_n| \geq cr^d$ which implies, together with the L^1-convergence, that a limit point x must be in $\overline{E}$. Similarly one can also prove the Hausdorff convergence of ∂E_n to ∂E. Since the family $\mathcal{K}_\delta^{co}$ is stable under Hausdorff convergence, we get $E \in \mathcal{K}_\delta^{co}$.

Recalling that P is lower semicontinuous under L^1 convergence, and that $\mathcal{I}_\alpha(\mu)$ is lower semicontinuous under weak*-convergence (for the kernel is a positive function, and thus $\mathcal{I}_\alpha(\cdot)$ is the supremum of continuous functional over $\mathcal{M}$), we have

$$\varliminf_{n\to+\infty} P(E_n) + Q^2\mathcal{I}_\alpha(\mu_n) \geq P(E) + Q^2\mathcal{I}_\alpha(\mu).$$

By the Hausdorff convergence of E_n, there also holds $\operatorname{spt}(\mu) \subset E$, which concludes the proof. □

Thanks to the quantitative isoperimetric inequality [23], we can also prove existence for small charges of minimizers even without assuming *a priori* the connectedness. This is reminiscent of [10,29,30].

Theorem 4.3. *There exists a constant $Q_0 = Q_0(\alpha, d)$ such that, for every $\delta > 0$, $m \geq \omega_d \delta^d$ and*

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} \leq Q_0 \frac{\delta^d}{m},$$

problems (4.3) and (4.4) have a solution.

Proof. We only consider (4.3), since the proof of (4.4) is identical. Assume first that $m = \omega_d$.

As noticed in Theorem 1.5, for every minimizing sequence $E_n \in \mathcal{K}_\delta$, with $|E_n| = \omega_d$, we can assume that there holds

$$\delta P(E_n) \leq Q^2\mathcal{I}_\alpha(B).$$

Thus, up to translating the sets E_n, by the quantitative isoperimetric inequality [23] we can assume that

$$|B \Delta E_n|^2 \leq C(d)\, \delta P(E_n) \leq C(d) Q^2 \mathcal{I}_\alpha(B)$$

so that $|E_n \cap B^c| \leq CQ$. Since every connected component of $E_n \in \mathcal{K}_\delta$ has volume of at least $|B_\delta| = \omega_d \delta^d$, for $Q \leq c(\alpha, d)\delta^d$ the set E_n must be connected. The existence of minimizers then follows as in Theorem 1.5.

The case of a general volume m can be obtain by rescaling from (4.5). □

It is natural to expect that, for a charge Q large enough, it is more favorable to have two connected components rather than one, which would lead to non-existence of minimizers in $\mathcal{K}_\delta$. Let us prove that it is indeed the case, at least for small enough α. We start with the following lemma.

Lemma 4.4. *Let $\alpha > 0$ and let E be a compact set then*

$$\mathcal{I}_\alpha(E) \geq \frac{1}{\operatorname{diam}(E)^\alpha}.$$

In particular,

$$\inf_{|E|=\omega_d, E\in\mathcal{K}_\delta^{co}} \mathcal{F}_{\alpha,Q}(E) \geq \left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} P(B) + \left(\sqrt{d}\, 2^{d+2}\right)^{-\alpha} Q^2 \delta^{(d-1)\alpha}, \quad (4.7)$$

and

$$\inf_{|E|=\omega_d, E\in\mathcal{K}_\delta^{co}} \mathcal{F}_{\alpha,Q}(E) \geq \left(\frac{m}{\omega_d}\right)^{\frac{d-1}{d}} P(B) + \left(\sqrt{d}\, 2^{d+2}\right)^{-\alpha} Q^2 \delta^{(d-1)\alpha}. \quad (4.8)$$

Proof. Let μ be any positive measure with support in $\overline{E}$ such that $\mu(E) = 1$ then

$$\mathcal{I}_\alpha(E) \geq \int_{E\times E} \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^\alpha} \geq \int_{E\times E} \frac{d\mu(x)\mathrm{d}\mu(y)}{\operatorname{diam}(E)^\alpha} = \frac{1}{\operatorname{diam}(E)^\alpha}.$$

By Lemma 2.20 and thanks to the isoperimetric inequality, we get (4.7) and (4.8). □

We can now prove a non-existence result in $\mathcal{K}_\delta$.

Theorem 4.5. *For all $\alpha < 1$ there exist $c_0 = c_0(\alpha) > 0$ and $Q_0 = Q_0(\alpha) > 0$ such that, for every $\delta > 0$, $m \geq c_0\delta^d$, and*

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} > Q_0 \left(\frac{m}{\delta^d}\right)^{\frac{d\alpha+1-\alpha}{2d}}$$

problems (4.3) and (4.4) do not have a solution.

Proof. We only discuss problem (4.3), since the non-existence result for problem (4.4) follows analogously.

As in Theorem 1.6 we first consider the case $m = \omega_d$, so that $\delta \leq 1$. If there exists a minimizer then the optimal measure μ is necessarily contained in a connected component of the minimizer. From (4.8) it then follows that the energy of the minimizer is greater than

$$P(B) + \left(\sqrt{d}\, 2^{d+2}\right)^{-\alpha} \delta^{(d-1)\alpha} Q^2, \quad (4.9)$$

which bounds from below the energy of any set in $\mathcal{K}_\delta^{co}$ with volume ω_d. Hence, in order to prove the non-existence, it is enough to construct a competitor $E \in \mathcal{K}_\delta$ with energy less than (4.9).

Consider the set E given by N (which we suppose to be an integer) balls of radius δ, equally charged. Up to increasing their mutual distances, we can suppose that the Riesz potential energy of E is made only of the self interaction of each ball with itself. Since $N = \delta^{-1}$ we then have

$$P(E) + Q^2\mathcal{I}_\alpha(E) = N\delta^{d-1}P(B) + \frac{Q^2}{N}\mathcal{I}_\alpha(B_\delta) = \frac{1}{\delta}P(B) + \mathcal{I}_\alpha(B)\delta^{d-\alpha}Q^2. \tag{4.10}$$

Notice that, if $d-\alpha > (d-1)\alpha$, that is if $\alpha < 1$, there exists $\delta_0 = \delta_0(\alpha)$ such that for all $\delta \le \delta_0$ there holds

$$\mathcal{I}_\alpha(B)\,\delta^{d-\alpha} \le \frac{1}{2}\left(\sqrt{d}\,2^{d+2}\right)^{-\alpha}\delta^{(d-1)\alpha}.$$

With this condition in force, from (4.10) we get

$$P(E) + Q^2\mathcal{I}_\alpha(E) < P(B) + \left(\sqrt{d}\,2^{d+2}\right)^{-\alpha} Q^2\delta^{(d-1)\alpha},$$

for

$$Q > \sqrt{2P(B)}\left(\sqrt{d}\,2^{d+2}\right)^{\frac{\alpha}{2}}\frac{1}{\delta^{\frac{d\alpha+1-\alpha}{2}}}.$$

The general case can be obtain by rescaling from (4.5). □

Remark 4.6. If $\alpha < \frac{d-1}{d}$, we can improve the previous estimate on Q by considering a construction similar to the one of Theorem 3.2. Indeed, for $\beta \in (d\alpha, d-1)$, taking $N := \delta^{-\beta}$ charged balls of radius δ and a non charged ball of volume $m - \omega_d N\delta^d$, we find a contradiction if

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} > \widetilde{Q}_0(\alpha)\left(\frac{m}{\delta^d}\right)^{\frac{\beta-(1-\alpha)(d-1)}{2d}}.$$

Notice that, if $\alpha < \frac{d-1}{2d-1}$, we can choose β such that the exponent $\frac{\beta-(1-\alpha)(d-1)}{2d}$ is negative.

Remark 4.7. We expect that the non-existence result in Theorem 4.5 also holds for $\alpha \ge 1$, but we where unable to show this, as the class $\mathcal{K}_\delta$ is fairly rigid which makes the construction of competitors quite delicate.

5. Minimality of the Ball

In this section we prove that, in the harmonic case $\alpha = d-2$, the ball is a minimizer for Problem (4.3) (for $\Omega = \mathbb{R}^d$) among sets in the family of the *nearly spherical sets* belonging to $\mathcal{K}_\delta^{co}$ introduced in Definition 2.18, that is, the sets which are a small $W^{1,\infty}$ perturbation of the ball and that satisfy the δ-ball condition.

Consider a set E such that $|E| = \omega_d$, and such that ∂E can be written as a graph over ∂B. In polar coordinates we have

$$E = \quad R(x)x \ : \ R(x) = 1 + \varphi(x), \ x \in \partial B \quad .$$

The condition $|E| = \omega_d$ then becomes

$$\int_{\partial B} \Big((1 + \varphi(x))^d - 1 \quad d\mathcal{H}^{d-1}(x) = 0$$

which implies that if $\|\varphi\|_{L^\infty(\partial B)}$ is small enough, then

$$\int_{\partial B} \varphi d\mathcal{H}^{d-1} = O\left(\|\varphi\|^2_{L^2(\partial B)} \quad . \tag{5.1}$$

Letting

$$\bar{\varphi} := \frac{1}{|\partial B|} \int_{\partial B} \varphi d\mathcal{H}^{d-1},$$

the Poincaré Inequality gives

$$\begin{aligned}
\int_{\partial B} |\nabla \varphi|^2 d\mathcal{H}^{d-1} &\geq C \int_{\partial B} |\varphi - \bar{\varphi}|^2 d\mathcal{H}^{d-1} \\
&= C(d) \int_{\partial B} \varphi^2 \mathcal{H}^{d-1} - \frac{C(d)}{d\omega_d} \left(\int_{\partial B} \varphi d\mathcal{H}^{d-1}\right)^2 \\
&= C(d) \int_{\partial B} \varphi^2 d\mathcal{H}^{d-1} - \frac{C}{4d\omega_d} \left(\int_{\partial B} \varphi^2 d\mathcal{H}^{d-1}\right)^2 \\
&\geq \frac{3}{4} C(d) \int_{\partial B} \varphi^2 d\mathcal{H}^{d-1}
\end{aligned} \tag{5.2}$$

as soon as

$$\int_{\partial B} \varphi^2 d\mathcal{H}^{d-1} \leq d\omega_d. \tag{5.3}$$

Up to translation, we can also assume that the barycenter of E is 0. This implies that

$$\left|\int_{\partial B} x\varphi(x) d\mathcal{H}^{d-1}(x)\right| \ = \ O\left(\|\varphi\|^2_{L^2(\partial B)} \quad . \tag{5.4}$$

Lemma 5.1. *Suppose that $\varphi : \partial B \to \mathbb{R}^d$ parametrizes ∂E and $\|\varphi\|_{L^\infty(\partial B)}$ is small enough so that (5.3) is satisfied. Assume also that the barycenter of E is in* 0. *Then,*

$$\delta P(E) \geq c_0 \int_{\partial B} |\nabla \varphi|^2 d\mathcal{H}^{d-1} \geq c_1 \int_{\partial B} |\varphi|^2 d\mathcal{H}^{d-1} = \frac{c_1}{2} \left|\int_{\partial B} \varphi d\mathcal{H}^{d-1}\right|. \tag{5.5}$$

Proof. We refer to [21] for the proof of the first inequality. The second inequality is (5.2), while the third one follows from (5.1). □

A consequence of Lemma 5.1 is the following corollary.

Corollary 5.2. *Suppose that ∂E is parametrized on ∂B by a function φ which satisfies the hypothesis of Lemma 5.1. Then there exists a positive constant $C = C(\alpha, d)$ such that*

$$|\mathcal{I}_\alpha^{\partial B}(\varphi)| \leq C\, \delta P(E), \tag{5.6}$$

and, for any positive constant λ,

$$|\mathcal{I}_\alpha^{\partial B}(\lambda, \varphi)| \leq C\lambda\, \delta P(E). \tag{5.7}$$

Proof. Inequality (5.7) is an immediate consequence of (5.5). Concerning the first one we have, by the Hölder inequality and the Fubini Theorem,

$$\begin{aligned}
\mathcal{I}_\alpha^{\partial B}(\varphi) &= \int_{\partial B \times \partial B} \frac{\varphi(x)\varphi(y)}{|x-y|^\alpha}\, d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\
&\leq \left(\int_{\partial B \times \partial B} \frac{\varphi(x)^2}{|x-y|^\alpha}\, d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \right)^{1/2} \\
&\quad \left(\int_{\partial B \times \partial B} \frac{\varphi(y)^2}{|x-y|^\alpha}\, d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \right)^{1/2} \\
&= C \int_{\partial B} \varphi(x)^2\, d\mathcal{H}^{d-1}(x).
\end{aligned}$$

So (5.6) follows again from (5.5). □

We will use the following technical lemma.

Lemma 5.3. *Let $E = \left\{ R(x)x \,:\, R(x) = 1 + \varphi(x),\ x \in \partial B \right\}$ and let $g \in L^\infty(\partial B)$, then there exists $\varepsilon_0(\alpha, d)$ and a constant $C = C(\alpha, d) > 0$ such that if $\|\varphi\|_{W^{1,\infty}(\partial B)} \leq \varepsilon_0 \leq 1$,*

$$\begin{aligned}
&\left| \int_{\partial B \times \partial B} \left(\frac{1}{|R(x) - R(y)|^\alpha} \right.\right. \\
&\qquad \left.\left. - \frac{(1 - \frac{\alpha}{2}\varphi(x))(1 - \frac{\alpha}{2}\varphi(y))}{|x-y|^\alpha} \right) g(x)g(y) d\mathcal{H}^{d-1}(x)\, d\mathcal{H}^{d-1}(y) \right| \\
&\leq C(\alpha, d)(1 + \varepsilon_0) \|g\|^2_{L^\infty(\partial B)}\, \delta P(E).
\end{aligned} \tag{5.8}$$

Proof. First, notice that since $|x| = |y| = 1$ we have

$$|R(x)x - R(y)y|^2 = |x-y|^2 \left(1 + \varphi(x) + \varphi(y) + \varphi(x)\varphi(y) + \psi(x, y)\right) \tag{5.9}$$

where $\psi(x, y) = \frac{(\varphi(x) - \varphi(y))^2}{|x-y|^2}$. Hence, for any $x, y \in \partial B$ there holds,

$$\begin{aligned}
&|R(x)x - R(y)y|^{-\alpha} \\
&= \frac{(1 - \frac{\alpha}{2}\varphi(x))(1 - \frac{\alpha}{2}\varphi(y)) + \frac{\alpha(4-\alpha)}{4}\varphi(x)\varphi(y) - \frac{\alpha}{2}(\psi(x, y) + \eta(x, y))}{|x-y|^\alpha}
\end{aligned} \tag{5.10}$$

where

$$0 \le \eta(x,y) \le C\left(\varphi^2(x)+\varphi^2(y)+\psi^2(x,y)\right).$$

By (5.10) we get

$$\begin{aligned}
&\int_{\partial B\times\partial B}\left(\frac{1}{|R(x)-R(y)|^\alpha}-\frac{(1-\frac{\alpha}{2}\varphi(x))(1-\frac{\alpha}{2}\varphi(y))}{|x-y|^\alpha}\right)\\
&\quad g(x)g(y)d\mathcal{H}^{d-1}(x)\,d\mathcal{H}^{d-1}(y)\\
&\quad=\frac{\alpha(4-\alpha)}{4}\int_{\partial B\times\partial B}\frac{\varphi(x)\varphi(y)}{|x-y|^\alpha}g(x)g(y)\,d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)\\
&\qquad-\frac{\alpha}{2}\int_{\partial B\times\partial B}\frac{\psi(x,y)+\eta(x,y)}{|x-y|^\alpha}g(x)g(y)\,d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y).
\end{aligned}\tag{5.11}$$

By Corollary 5.2 we get

$$\int_{\partial B\times\partial B}\frac{\varphi(x)\varphi(y)}{|x-y|^\alpha}\,d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)=\mathcal{I}_\alpha^{\partial B}(\varphi)\le C\delta P(E).$$

Furthermore, we have

$$0\le\psi(x,y)\le\|\nabla\varphi\|^2_{L^\infty(\partial B)}\le\varepsilon_0,$$

and

$$\begin{aligned}
\int_{\partial B\times\partial B}\frac{\varphi(x)^2\,d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)}{|x-y|^\alpha}&=\int_{\partial B}\frac{d\mathcal{H}^{d-1}(y)}{|x-y|^\alpha}\int_{\partial B}\varphi(x)^2\,d\mathcal{H}^{d-1}(x)\\
&\le c(\alpha,d)\varepsilon_0^2,
\end{aligned}$$

for a suitable constant $c(\alpha,d)$. Therefore, since $\eta(x,y)\le C\big(\varphi^2(x)+\varphi^2(y)$ $+\psi(x,y)\big)$, to prove (5.8) we only have to check that

$$\int_{\partial B\times\partial B}\frac{\psi(x,y)}{|x-y|^\alpha}\,d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)\le C\delta P(E).$$

To this end, consider x,y in ∂B and denote by $\Gamma_{x,y}$ the geodesic going from x to y and by $\ell(x,y)$ the geodesic distance between x and y (that is the length of $\Gamma_{x,y}$). Notice that on ∂B, the euclidean distance and ℓ are equivalent so that it is enough to prove

$$\int_{\partial B\times\partial B}\ell(x,y)^{-(\alpha+2)}(\varphi(x)-\varphi(y))^2\,d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)\le C\delta P(E).$$

We have

$$\begin{aligned}
&\int_{\partial B\times\partial B}\ell(x,y)^{-(\alpha+2)}(\varphi(x)-\varphi(y))^2\\
&\quad\le c(d)\int_{\partial B\times\partial B}\ell(x,y)^{-(\alpha+1)}\int_{\Gamma_{x,y}}|\nabla\varphi|^2(z)dz\,d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)
\end{aligned}$$

$$
\begin{aligned}
&\leq c(d)\int_{\partial B}\int_0^{2\pi} t^{-(\alpha+1)}t^{d-1}\left(\int_{\{\ell(x,z)\leq t\}}|\nabla\varphi|^2(z)d\mathcal{H}^{d-1}(z)\right)\mathrm{d}t\,d\mathcal{H}^{d-1}(x)\\
&= c(d)\int_0^{2\pi} t^{(d-1)-(\alpha+1)}\left(\int_{\partial B}\int_{\{\ell(x,z)\leq t\}}|\nabla\varphi|^2(z)d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(z)\right)\mathrm{d}t\\
&= c(d)\mathcal{H}^{d-2}(\mathbb{S}^{d-2})\int_0^{2\pi} t^{(d-1)-\alpha}\left(\int_{\partial B}|\nabla\varphi|^2(z)d\mathcal{H}^{d-1}(z)\right)\mathrm{d}t\\
&= c(d)\mathcal{H}^{d-2}(\mathbb{S}^{d-2})\int_0^{2\pi} t^{(d-1)-\alpha}\,\mathrm{d}t\left(\int_{\partial B}|\nabla\varphi|^2(z)d\mathcal{H}^{d-1}(z)\right)\\
&\leq C\delta P(E)
\end{aligned}
$$

where $\mathbb{S}^{d-2}$ is the $(d-2)$-dimensional sphere and where we used the fact that $\alpha < d-1$ together with (5.5). □

Before we prove our main stability estimates, we recall a classical interpolation inequality.

Lemma 5.4. *For every $0 \leq p < q < r < +\infty$, there exists a constant $C(r, p, q)$ such that for every $\varphi \in H^r(\mathbb{R}^d)$, there holds*

$$
\|\varphi\|_{H^q(\mathbb{R}^d)} \leq C\left(\|\varphi\|_{H^r(\mathbb{R}^d)}\right)^{\frac{r-q}{r-p}}\left(\|\varphi\|_{H^p(\mathbb{R}^d)}\right)^{\frac{q-p}{r-p}}, \tag{5.12}
$$

where we adopted the notation $\|u\|_{H^p(\mathbb{R}^d)} := \||\xi|^p\hat{u}\|_{L^2(\mathbb{R}^d)}$ and $H^p(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d) : \|u\|_{H^p} < +\infty\}$, $\hat{u}$ being the Fourier transform of the function u.

Proof. Let $\varphi \in H^r(\mathbb{R}^d)$ and $\lambda > 0$, then we have

$$
\begin{aligned}
\|\varphi\|^2_{H^q(\mathbb{R}^d)} = \int_{\mathbb{R}^d}|\hat{\varphi}|^2|\xi|^{2q}d\xi &= \int_{|\xi|\leq\lambda}|\hat{\varphi}|^2|\xi|^{2p}|\xi|^{2(q-p)}d\xi\\
&\quad+\int_{|\xi|\geq\lambda}|\hat{\varphi}|^2|\xi|^{2r}|\xi|^{2(q-r)}d\xi\\
&\leq \lambda^{2(q-p)}\|\varphi\|^2_{H^p(\mathbb{R}^d)} + \lambda^{-2(r-q)}\|\varphi\|^2_{H^r(\mathbb{R}^d)}.
\end{aligned}
$$

An optimization in λ yields (5.12). □

Proposition 5.5. *Let $\alpha \in [d-2, d-1)$, $f \in L^\infty(\partial E)$ and*

$$
\partial E = \{R(x)x \,:\, R(x) = 1+\varphi(x),\ x \in \partial B\}.
$$

Then there exist $\varepsilon_0(\alpha) > 0$ and $C = C(\alpha) > 0$ such that if $\|\varphi\|_{W^{1,\infty}(\partial B)} \leq \varepsilon_0$ then

$$
\mathcal{I}_\alpha^{\partial E}(f) - \mathcal{I}_\alpha^{\partial B}(\bar{f}) \geq -C\|f\|^2_{L^\infty(\partial E)}\delta P(E), \tag{5.13}
$$

where $\bar{f} := \dfrac{1}{P(E)}\int_{\partial E} f d\mathcal{H}^{d-1}$.

Proof. We have

$$\begin{aligned} \mathcal{I}_\alpha^{\partial E}(f) &= \int_{\partial E\times\partial E} \frac{f(x)f(y)}{|x-y|^\alpha}\, d\mathcal{H}^{d-1}(x)\, d\mathcal{H}^{d-1}(y) \\ &= \int_{\partial B\times\partial B} \frac{g(x)g(y)}{|R(x)-R(y)|^\alpha}\, d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \end{aligned} \tag{5.14}$$

where we set

$$g(x) = f(R(x)x)R(x)^{d-2}\sqrt{R(x)^2 + |\nabla R(x)|^2}.$$

To choose ε_0 small enough, we can suppose that

$$\|g\|_{L^\infty(\partial B)} \le 2\|f\|_{L^\infty(\partial E)}. \tag{5.15}$$

Let $\bar{g} := \dfrac{1}{P(B)}\displaystyle\int_{\partial B} g d\mathcal{H}^{d-1} = \dfrac{P(E)}{P(B)}\bar{f}$. Then we have

$$\mathcal{I}_\alpha^{\partial E}(f) - \mathcal{I}_\alpha^{\partial B}(\overline{f}) = \mathcal{I}_\alpha^{\partial E}(f) - \mathcal{I}_\alpha^{\partial B}(\overline{g}) + \mathcal{I}_\alpha^{\partial B}(\overline{g}) - \mathcal{I}_\alpha^{\partial B}(\overline{f}).$$

Focusing on the last two terms in the previous equality we have

$$\begin{aligned} \left|\mathcal{I}_\alpha^{\partial B}(\overline{g}) - \mathcal{I}_\alpha^{\partial B}(\overline{f})\right| &= \mathcal{I}_\alpha^{\partial B}(\overline{f})\left|1 - \left(\frac{P(E)}{P(B)}\right)^2\right| \\ &= C\bar{f}^2\frac{P(E)+P(B)}{P(B)^2}|P(E)-P(B)| \\ &\le C(\alpha,d)\|f\|^2_{L^\infty(\partial E)}\delta P(E). \end{aligned}$$

Therefore, to prove (5.13) we only need to show that

$$\mathcal{I}_\alpha^{\partial E}(f) \ge \mathcal{I}_\alpha^{\partial B}(\bar{g}) - \|g\|^2_{L^\infty(\partial B)}\,\delta P(E). \tag{5.16}$$

Formula (5.14) together with Lemma 5.3 imply

$$\mathcal{I}_\alpha^{\partial E}(f) = \mathcal{I}_\alpha^{\partial B}\left(g(1 - \frac{\alpha}{2}\varphi)\right. \; + \mathcal{R}(g,\varphi)$$

with

$$|\mathcal{R}(g,\varphi)| \le c\|g\|^2_{L^\infty(\partial E)}\,\delta P(E),$$

so that

$$\mathcal{I}_\alpha^{\partial E}(f) \ge \mathcal{I}_\alpha^{\partial B}\left(g(1 - \frac{\alpha}{2}\varphi)\right. \; - c\|g\|^2_{L^\infty(\partial E)}\,\delta P(E). \tag{5.17}$$

We need to estimate $\mathcal{I}_\alpha^{\partial B}(g(1-\alpha/2)\varphi)$. By the bilinearity of $\mathcal{I}_\alpha^{\partial B}$ we have that

$$\begin{aligned}
&\mathcal{I}_\alpha^{\partial B}(g(1-\frac{\alpha}{2}\varphi)) = \mathcal{I}_\alpha^{\partial B}(g(1-\frac{\alpha}{2}\varphi), g(1-\frac{\alpha}{2}\varphi)) \\
&= \mathcal{I}_\alpha^{\partial B}(g,g) - \alpha\mathcal{I}_\alpha^{\partial B}(g, g\varphi) + \frac{\alpha^2}{4}\mathcal{I}_\alpha^{\partial B}(g\varphi, g\varphi) \\
&= \mathcal{I}_\alpha^{\partial B}(\bar{g},\bar{g}) + \mathcal{I}_\alpha^{\partial B}(g-\bar{g}, g-\bar{g}) - \alpha\mathcal{I}_\alpha^{\partial B}(g-\bar{g}, g\varphi) - \alpha\mathcal{I}_\alpha^{\partial B}(\bar{g}, g\varphi) \\
&\quad + \frac{\alpha^2}{4}\mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi, \bar{g}\varphi) + \frac{\alpha^2}{2}\mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi, (g-\bar{g})\varphi) + \frac{\alpha^2}{4}\mathcal{I}_\alpha^{\partial B}((g-\bar{g})\varphi, (g-\bar{g})\varphi) \\
&= \mathcal{I}_\alpha^{\partial B}(\bar{g}) + \mathcal{I}_\alpha^{\partial B}(g-\bar{g}) + \frac{\alpha^2}{4}\mathcal{I}_\alpha^{\partial B}((g-\bar{g})\varphi) - \alpha\mathcal{I}_\alpha^{\partial B}(g-\bar{g}, (g-\bar{g})\varphi) \\
&\quad - \alpha\mathcal{I}_\alpha^{\partial B}(\bar{g}, (g-\bar{g})\varphi) - \alpha\mathcal{I}_\alpha^{\partial B}(g-\bar{g}, \bar{g}\varphi) + \frac{\alpha^2}{2}\mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi, (g-\bar{g})\varphi) \\
&\quad - \alpha\mathcal{I}_\alpha^{\partial B}(\bar{g}, \bar{g}\varphi) + \frac{\alpha^2}{4}\mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi).
\end{aligned} \tag{5.18}$$

Thanks to (5.7), the last two terms in the right hand side of (5.18) satisfy

$$-\mathcal{I}_\alpha^{\partial B}(\bar{g}, \bar{g}\varphi) + \frac{\alpha}{4}\mathcal{I}_\alpha^{\partial B}(\bar{g}\varphi) \geq -c\bar{g}^2\,\delta P(E). \tag{5.19}$$

By the Cauchy-Schwarz inequality (2.4) and Young's inequality, we get that for every function h_1 and h_2 and for any $\varepsilon > 0$,

$$\mathcal{I}_\alpha^{\partial B}(h_1, h_2) \leq \mathcal{I}_\alpha^{\partial B}(h_1)^{\frac{1}{2}}\mathcal{I}_\alpha^{\partial B}(h_2)^{\frac{1}{2}} \leq \varepsilon\mathcal{I}_\alpha^{\partial B}(h_1) + \frac{1}{4\varepsilon}\mathcal{I}_\alpha^{\partial B}(h_2). \tag{5.20}$$

In particular, applying such an inequality to the functions $h_1 = g - \bar{g}$ and $h_2 = (g-\bar{g})\varphi$ in the fourth term in the right hand side of (5.18), and then to $h_1 = g - \bar{g}$ and $h_2 = \bar{g}\varphi$ in the sixth term, and exploiting (5.19), we obtain the existence of a positive constant C such that

$$\begin{aligned}
&\mathcal{I}_\alpha^{\partial B}(g(1-\frac{\alpha}{2}\varphi)) - \mathcal{I}_\alpha^{\partial B}(\bar{g}) \\
&\geq C\left(\frac{1}{2}\mathcal{I}_\alpha^{\partial B}(g-\bar{g}) - \mathcal{I}_\alpha^{\partial B}(\bar{g}, (g-\bar{g})\varphi) - \mathcal{I}_\alpha^{\partial B}((g-\bar{g})\varphi) - \bar{g}^2\,\delta P(E)\right).
\end{aligned} \tag{5.21}$$

Again, by Lemma 5.1, we have that

$$-\mathcal{I}_\alpha^{\partial B}((g-\bar{g})\varphi) \geq -\|g\|^2_{L^\infty(\partial B)}\mathcal{I}_\alpha^{\partial B}(\varphi) \geq -C\|g\|^2_{L^\infty(\partial B)}\,\delta P(E).$$

Let us show that the term $\mathcal{I}_\alpha^{\partial B}(\bar{g}, (g-\bar{g})\varphi)$ can be estimated by the term $\mathcal{I}_\alpha^{\partial B}(g-\bar{g})$.

Let $\widetilde{\varphi} : \mathbb{R}^d \to \mathbb{R}$ be a regular extension of φ, and let $\widetilde{g} = (g - \bar{g}) d\mathcal{H}^{d-1} \llcorner \partial B$. By a Fourier transform we get

$$\begin{aligned}
\mathcal{I}_\alpha^{\partial B}(\bar{g}, (g-\bar{g})\varphi) &= \int_{\partial B} \frac{d\mathcal{H}^{d-1}(x)}{|x-y|^\alpha} \bar{g} \int_{\partial B} (g-\bar{g})\, d\mathcal{H}^{d-1}(y)\varphi = c(\alpha, d)\bar{g} \int_{\mathbb{R}^d} \widehat{\widetilde{\varphi}}\widehat{\widetilde{g}} \\
&\le \bar{g} \left(\int_{\mathbb{R}^d} \widehat{\widetilde{\varphi}}^2 |\xi|^{d-\alpha} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \frac{\widehat{\widetilde{g}}^2}{|\xi|^{d-\alpha}} \right)^{\frac{1}{2}} \\
&= \bar{g} \|\widetilde{\varphi}\|_{H^{\frac{d-\alpha}{2}}(\mathbb{R}^d)} \mathcal{I}_\alpha^{\partial B}(g-\bar{g}, g-\bar{g})^{\frac{1}{2}} \\
&\le C(d)\bar{g}\|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)} \mathcal{I}_\alpha^{\partial B}(g-\bar{g})^{\frac{1}{2}}.
\end{aligned}$$

We now observe that, if

$$\mathcal{I}_\alpha^{\partial B}(\bar{g}, (g-\bar{g})\varphi) \le \frac{1}{2} \mathcal{I}_\alpha^{\partial B}(g-\bar{g}), \tag{5.22}$$

then we would get

$$\mathcal{I}_\alpha^{\partial B}(g(1 - \frac{\alpha}{2}\varphi)) - \mathcal{I}_\alpha^{\partial B}(\bar{g}) \ge -C\|\bar{g}\|^2_{L^\infty(\partial B)}\, \delta P(E),$$

which would imply (5.16) and so the claim of the proposition. On the other hand if (5.22) does not hold, then, considering again a regular extension $\tilde{\varphi} : \mathbb{R}^d \to \mathbb{R}$ of φ, we have

$$\mathcal{I}_\alpha^{\partial B}(g-\bar{g}) < C(d)\bar{g}\|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)} \mathcal{I}_\alpha^{\partial B}(g-\bar{g})^{\frac{1}{2}},$$

which implies that

$$\mathcal{I}_\alpha^{\partial B}(g-\bar{g})^{\frac{1}{2}} < C\bar{g}\|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)},$$

so that

$$\mathcal{I}_\alpha^{\partial B}(\bar{g}, (g-\bar{g})\varphi) \le C\bar{g}\|\varphi\|_{H^{\frac{d-\alpha}{2}}(\partial B)} \mathcal{I}_\alpha^{\partial B}(g-\bar{g})^{\frac{1}{2}} \le C\bar{g}^2\|\varphi\|^2_{H^{\frac{d-\alpha}{2}}(\partial B)}.$$

If $\frac{d-\alpha}{2} \le 1$ then using (5.12) with $p = 0$, $q = \frac{d-\alpha}{2}$ and $r = 1$, in order to once again regularly extend φ on $\mathbb{R}^d$, we obtain

$$\begin{aligned}
\|\varphi\|^2_{H^{\frac{d-\alpha}{2}}(\partial B)} &\le c_0 \left(\|\varphi\|^2_{H^1(\partial B)} \right)^{1-\frac{d-\alpha}{2}} \left(\|\varphi\|^2_{L^2(\partial B)} \right)^{\frac{d-\alpha}{2}} \\
&\le c_1 \left(\|\varphi\|^2_{H^1(\partial B)} + \|\varphi\|^2_{L^2(\partial B)} \right) \le C\delta P(E),
\end{aligned}$$

which concludes the proof. □

Theorem 5.6. *Let $d \geq 3$ and $\alpha = d - 2$. Then for any $\delta > 0$ and $m \geq \omega_d \delta^d$, there exists a charge $\bar{Q}\left(\frac{\delta}{m^{1/d}}\right) > 0$, such that if*

$$\frac{Q}{m^{\frac{d-1+\alpha}{2d}}} \leq \bar{Q}\left(\frac{\delta}{m^{1/d}}\right)$$

the ball is the unique minimizer of problem (4.3).

Proof. As a rescaling we can assume $m = \omega_d$. By Theorem 1.6, there exists $C > 0$ such that problem (4.3) admits a minimizer E_Q for every $Q \in (0, C\delta^{\frac{d}{2}})$. Since $|E_Q \Delta B|^2 \leq C\delta P(E_Q) \leq Q^2 \mathcal{I}_\alpha(B)$, E_Q converges to B in L^1 when $Q \to 0$. As in Theorem 1.5, there is also convergence in the Hausdorff sense of E_Q and ∂E_Q thanks to the δ-ball condition. Again, by the δ-ball condition and the Hausdorff convergence of the boundaries, for Q small enough, ∂E_Q is a graph over ∂B of some $C^{1,1}$ function with $C^{1,1}$ norm bounded by $2/\delta$. From this we see that if $\partial E_Q = \{(1 + \varphi_Q(x))x \ : \ x \in \partial B\}$ then $\|\varphi_Q\|_{W^{1,\infty}(\partial B)}$ is converging to 0. We can thus assume that φ_Q satisfies the hypotheses of Proposition 5.5.

Let $\mu = f d\mathcal{H}^{d-1} \llcorner \partial E_Q$ be the minimizer of $\mathcal{I}_\alpha(E_Q)$. Since $\mathcal{I}_\alpha(E_Q) \leq P(B) + Q^2 \mathcal{I}_\alpha(B)$, by Proposition 2.22, $\|f\|_{L^\infty(\partial E)} \leq (d-2)\delta^{-1}(P(B) + Q^2 \mathcal{I}_\alpha(B))$. Let $\bar{f} := \dfrac{1}{P(E_Q)} = \dfrac{1}{P(E_Q)} \int_{\partial E_Q} f d\mathcal{H}^{d-1}$. By Lemma 2.17 we know that the optimal measure for $\mathcal{I}_\alpha(B)$ is given by $\frac{\mathcal{H}^{d-1} \llcorner \partial B}{P(B)}$. By the minimality of E_Q we then have

$$\begin{aligned}
\delta P(E_Q) &= P(E_Q) - P(B) \leq Q^2(\mathcal{I}_\alpha(B) - \mathcal{I}_\alpha(E_Q)) \\
&= Q^2 \left(\mathcal{I}_\alpha^{\partial B}(\bar{f}) - \mathcal{I}_\alpha^{\partial E_Q}(f) + \mathcal{I}_\alpha^{\partial B}(1/P(B)) - \mathcal{I}_\alpha^{\partial B}(1/P(E_Q))\right) .
\end{aligned}$$

A simple computation shows that

$$\mathcal{I}_\alpha^{\partial B}(1/P(B)) - \mathcal{I}_\alpha^{\partial B}(1/P(E_Q)) \leq C^2\, \delta P(E_Q)$$

for a suitable positive constant $C = C(\alpha, d)$. Hence, by Proposition 5.5 we have that

$$\delta P(E_Q) \leq C Q^2\, \delta P(E_Q)(1 + \|f\|^2_{L^\infty(\partial E_Q)}) \leq C Q^2\, \delta P(E_Q),$$

which implies $\delta P(E_Q) = 0$ that is $E_Q = B$, for Q small enough. □

Remark 5.7. We recall that a counterpart of Theorem 1.7 holds as well. Indeed, in [20] it was proven that if Q overcomes a certain threshold, any radial set (and in particular the ball) is unstable under small $C^{2,\beta}$ perturbations.

Remark 5.8. The previous proof of stability does not apply to the case $\alpha > d - 2$. Indeed, this proof relies on L^∞ bounds for the optimal measure μ for $\mathcal{I}_\alpha$ which we are not able to obtain in that case. For the very same reason, our approach seems not to work if we replace the class $\mathcal{K}_\delta$ by the class of convex sets. In fact, for a set with Lipschitz boundary, the optimal measure is not expected to be in L^∞. In particular, if E is convex, then its optimal measure blows-up at every non-regular point of ∂E, as shown in Example 6.5.

Remark 5.9. Notice that as an aside of the previous analysis we obtained a stability result of Fuglede type (see [21]). Indeed collecting the results of Lemma 5.3, Proposition 5.5 and the proof of Theorem 5.6, we showed that if E is a $C^{1,1}$ small perturbation of the ball B, with $|E| = |B|$, then the following quantitative inequality holds true

$$P(E) - P(B) \geq C\left(\mathcal{I}_\alpha(B) - \mathcal{I}_\alpha(E)\right),$$

where the constant $C > 0$ is independent of E.

6. The Logarithmic Potential Energy

In this section we investigate the same types of questions for the logarithmic potential which are given by $-\log(|x|)$. This potential naturally arises in two dimensions where it corresponds to the Coulomb interaction. Let then

$$\mathcal{I}_{\log}(E) := \min_{\mu(\overline{E})=1} \int_{\mathbb{R}^d\times\mathbb{R}^d} -\log(|x-y|)\mathrm{d}\mu(x)\mathrm{d}\mu(y), \tag{6.1}$$

and consider the problem

$$\min_{|E|=m} P(E) + Q^2\mathcal{I}_{\log}(E). \tag{6.2}$$

In analogy to the notation adopted for the Riesz potential we define, for any Borel functions f and g, the following quantity

$$\mathcal{I}_{\log}{}^{\partial E}(f,g) := \int_{\partial E\times\partial E} -\log(|x-y|)\,f(x)\,g(y) d\mathcal{H}^{d-1}(x)\,d\mathcal{H}^{d-1}(y).$$

We list below some important properties of $\mathcal{I}_{\log}$ without proof, since they are analogous to those given in Section 2 for the Riesz potential. We refer to [31,40] for comprehensive guides on the logarithmic potential.

Proposition 6.1. *The following properties hold:*

(i) *for every compact set E, there exists a unique optimal measure μ for $\mathcal{I}_{\log}(E)$ which is concentrated on the boundary of E;*

(ii) *for every Borel measure μ it holds*

$$\mathcal{I}_{\log}(\mu) = \int_{\mathbb{R}^d} \left(v^\mu_{d/2}(x)\right)^2 \mathrm{d}x \geq 0$$

where

$$v^\mu_{d/2}(x) = \int_{\mathbb{R}^d} -\log|x-y|\,\mathrm{d}\mu(y);$$

(iii) *for every smooth set E, if μ is the optimal measure for $\mathcal{I}_{\log}$, then the equality $\int_{\partial E} -\log(|x-y|)\mathrm{d}\mu(y) = \mathcal{I}_{\log}(E)$ holds for every $x \in \partial E$. Moreover the optimal measure for the ball is the uniform measure;*

(iv) *if* $d = 2$, *then for every bounded set* E *satisfying the* δ*-ball condition, the optimal measure is given by some measure* $\mu = f\mathcal{H}^{d-1}\llcorner\partial E$ *with* $\|f\|_{L^\infty(\partial E)} \le \frac{\mathcal{I}_{\log}(E)}{|\log(\delta)|}$.

In this setting, since the potential can be negative, the picture is slightly different from that related to the Riesz energy. Indeed, we have the following Theorem.

Theorem 6.2. *The following statements hold true:*

(i) $\inf_{|E|=m} P(E) + Q^2\mathcal{I}_{\log}(E) = -\infty$.
(ii) *for any* $\delta > 0$, *if* $m > 2\omega_d\delta^d$ *then* $\inf_{|E|=m, E\in\mathcal{K}_\delta} P(E) + Q^2\mathcal{I}_{\log}(E) = -\infty$,
(iii) *for every* $Q > 0$ *and every* $m > \omega_d\delta^d$, *there exists a minimizer of*

$$\min_{|E|=m, E\in\mathcal{K}_\delta^{co}} P(E) + Q^2\mathcal{I}_{\log}(E),$$

(iv) *for every bounded smooth domain* Ω,

$$\inf_{|E|=m, E\subset\Omega} P(E) + Q^2\mathcal{I}_{\log}(E) = \min_{|E|=m, E\subset\Omega} P(E) + Q^2\mathcal{I}_{\log}(\Omega).$$

Proof. Statement (ii) implies (i) while (iii) can be proven exactly as in Theorem 1.5 and (iv) as Theorem 3.4. To prove (ii) we set $E_n = B_\delta(x_1^n) \cup B_\delta(x_2^n)$ and notice that if $\mathrm{dist}(x_1^n, x_2^n)$ goes to infinity, then $\mathcal{I}_{\log}(E_n) \to -\infty$ as $n \to +\infty$. □

Since $\mathcal{I}_{\log}(\lambda E) = \mathcal{I}_{\log}(E) - \log(\lambda)$ for every $\lambda > 0$, without loss of generality we shall assume that $m = |B_{1/2}| = \pi/4$ in Problem (6.2). The following result is the counterpart of Proposition 5.5.

Proposition 6.3. *Let* $d = 2$, $E = \{R(x)x : R(x) = 1 + \varphi(x),\ x \in \partial B_{1/2}\}$ *and let* $f \in L^\infty(\partial E)$ *then there exists* ε_0 *and a constant* $C = C(\alpha) > 0$ *such that if* $\|\varphi\|_{W^{1,\infty}(\partial B_{1/2})} \le \varepsilon_0$. *Then*

$$\mathcal{I}_{\log}^{\partial E}(f) - \mathcal{I}_{\log}^{\partial B_{1/2}}(\bar{f}) \ge -C\|f\|^2_{L^\infty(\partial E)}\,\delta P(E),$$

where $\bar{f} := \dfrac{1}{P(E)}\displaystyle\int_{\partial E} f\,d\mathcal{H}^1$.

Proof. Notice that since $E \subset B$, the logarithmic potential is positive. As in the proof of Proposition 5.5, we have

$$\mathcal{I}_{\log}^{\partial E}(f) = \int_{\partial B_{1/2}\times\partial B_{1/2}} -\log(|R(x) - R(y)|)g(x)g(y)d\mathcal{H}^1(x)d\mathcal{H}^1(y),$$

where $g(x) = f(R(x)x)\sqrt{R(x)^2 + |\nabla R(x)|^2}$. Reminding that from (5.9), we have

$$|R(x)x - R(y)y| = |x - y|\left(1 + \varphi(x) + \varphi(y) + \varphi(x)\varphi(y) + \psi(x, y)\right)^{1/2},$$

where, $\psi(x,y)=\frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}$, we see that

$$\begin{aligned}\mathcal{I}_{\log}^{\partial E}(f)=&\int_{\partial B_{1/2}\times\partial B_{1/2}}-\log(|x-y|)\,g(x)g(y)\,d\mathcal{H}^1(x)d\mathcal{H}^1(y)\\&+\frac{1}{2}\int_{\partial B_{1/2}\times\partial B_{1/2}}-\log(1+\varphi(x)+\varphi(y)+\varphi(x)\varphi(y)+\psi(x,y))\\&g(x)g(y)\,d\mathcal{H}^1(x)d\mathcal{H}^1(y).\end{aligned}$$

As in Proposition 5.5, letting $\bar{g}:=\dfrac{1}{P(B_{1/2})}\displaystyle\int_{\partial B_{1/2}}g\,d\mathcal{H}^1$, we have

$$\begin{aligned}\mathcal{I}_{\log}^{\partial B_{1/2}}(g)&=\int_{\partial B_{1/2}\times\partial B_{1/2}}-\log(|x-y|)\,g(x)g(y)\,d\mathcal{H}^1(x)d\mathcal{H}^1(y)\\&=\mathcal{I}_{\log}^{\partial B_{1/2}}(\bar{g})+\mathcal{I}_{\log}^{\partial B_{1/2}}(g-\bar{g})\end{aligned}$$

and

$$\mathcal{I}_{\log}^{\partial B_{1/2}}(\bar{g})-\mathcal{I}_{\log}^{\partial B}(\bar{f})\le C\|f\|^2_{L^\infty(\partial E)}\,\delta P(E).$$

Using that for $|t|\le 1$, $|\log(1+t)-t|\le\frac{t^2}{2}$, we see that

$$\begin{aligned}&\int_{\partial B_{1/2}\times\partial B_{1/2}}-\log(1+\varphi(x)+\varphi(y)+\varphi(x)\varphi(y)\\&\quad+\psi(x,y))\,g(x)g(y)\,d\mathcal{H}^1(x)d\mathcal{H}^1(y)\\&=-\int_{\partial B_{1/2}\times\partial B_{1/2}}(\varphi(x)+\varphi(y)+\varphi(x)\varphi(y)+\psi(x,y)\\&\quad+\eta(x,y))\;g(x)g(y)\,d\mathcal{H}^1(x)d\mathcal{H}^1(y)\end{aligned}$$

where the function $\eta(x,y)$ is well controlled. As in Lemma 5.3,

$$\int_{\partial B_{1/2}\times\partial B_{1/2}}\varphi(x)\varphi(y)g(x)g(y)\,d\mathcal{H}^1(x)d\mathcal{H}^1(y)\le C\|g\|^2_{L^\infty(\partial B_{1/2})}\,\delta P(E)$$

and

$$\int_{\partial B_{1/2}\times\partial B_{1/2}}\psi(x,y)\,g(x)g(y)\,d\mathcal{H}^1(x)d\mathcal{H}^1(y)\le C\left(\int_0^{2\pi}t\;\,\mathrm{d}t\right)\delta P(E).$$

Since

$$\begin{aligned}\int_{\partial B_{1/2}\times\partial B_{1/2}}\varphi(x)g(x)g(y)d\mathcal{H}^1(x)d\mathcal{H}^1(y)=\bar{g}&\int_{\partial B_{1/2}}\varphi(x)\,(g(x)-\bar{g})\,d\mathcal{H}^1(x)\\&+\bar{g}^2P(B_{1/2})\int_{\partial B_{1/2}}\varphi(x)d\mathcal{H}^1(x),\end{aligned}$$

and since $\int_{\partial B_{1/2}} \varphi(x) d\mathcal{H}^1(x) \le C\delta P(E)$, we are left to prove that

$$\mathcal{I}_{\log}^{\partial B_{1/2}}(g - \bar{g}) - \bar{g} \int_{\partial B_{1/2}} \varphi(x)\,(g(x) - \bar{g})\,d\mathcal{H}^1(x) \ge C\bar{g}^2 \delta P(E). \tag{6.3}$$

As in the proof of Proposition 5.5, we use the Fourier transform to assert that for some regular extension $\tilde{\varphi}$ of φ and for $\widetilde{g} := (g - \bar{g})\mathcal{H}^1 \llcorner \partial B_{1/2}$,

$$\begin{aligned}\int_{\partial B_{1/2}} \varphi(x)\,(g(x) - \bar{g})\,d\mathcal{H}^1(x) &\le \left(\int_{\mathbb{R}^2} \widetilde{\varphi}^2 |\xi|^2\,d\xi\right)^{1/2} \left(\int_{\mathbb{R}^2} \widetilde{g}^2 |\xi|^{-2}\,d\xi\right)^{1/2} \\ &\le C\|\varphi\|_{H^1} \mathcal{I}_{\log}^{\partial B_{1/2}}(g - \bar{g})\end{aligned}$$

from which (6.3) follows arguing exactly as in the last part of the proof of Proposition 5.5. □

Arguing as in the proof of Theorem 1.7, we get the following result.

Corollary 6.4. *Let $d = 2$ then for any $\delta > 0$ and $m > 0$, there exists a $\bar{Q}\left(\frac{\delta}{\sqrt{m}}\right) > 0$ such that, if $\frac{Q}{m^{1/4}} < \bar{Q}\left(\frac{\delta}{\sqrt{m}}\right)$, the ball is the unique minimizer of problem (6.2) among the sets in $\mathcal{K}_\delta$ with charge Q.*

Example 6.5. In this example we show that if the boundary of a convex set is non-regular at a point x, then the optimal measure for K is not bounded at x. For simplicity we offer the example just in dimensions 2 and 3. It is not difficult to extend such an example to any dimension. Let us start with the case $d = 2$. Let $K \subset \mathbb{R}^2$ be a compact convex set and let μ be the optimal measure for K in the sense of (2.1). Suppose that $x \in \partial K$ is not a regular point, that is the tangent cone of K at x spans an $\gamma < \pi$. Let us denote such a cone by C. Up to a rotation and a translation of K we can suppose that $x = 0$ and that C takes the form

$$C = \{(x, y) : 0 \ge y \ge \tan(\gamma)x\}.$$

Let, as usual, v be the potential of K with respect to the logarithmic kernel, so that, in particular

$$\begin{cases} -\Delta v = 0 & \text{on } \mathbb{R}^2 \backslash K \\ v = c & \text{on } \partial K. \end{cases}$$

Let us consider the function u which, in polar coordinates takes the form

$$u(r, \theta) = r^{\frac{\pi}{2\pi - \gamma}} \sin\left(\frac{\pi}{2\pi - \gamma}\theta\right).$$

Then we can construct the barrier function u_ε as follows:

$$u_\varepsilon = c - \varepsilon u,$$

where ε is a positive parameter that will be fixed later. Notice that u_ε is an harmonic function on $\mathbb{R}^2 \setminus C$ which is constantly equal to c on ∂C. Since v is a continuous function, we can choose a radius $R > 0$ such that $v > c/2$ on $B(0, R) \cap (\mathbb{R}^2 \setminus C)$. By imposing

$$u_\varepsilon > v \qquad \text{on } \partial B(0, R) \cap (\mathbb{R}^2 \setminus C),$$

that is,

$$\varepsilon < \frac{c}{2 \max_{\theta \in [0, 2\pi - \gamma]} u(R, \theta)},$$

we get, by the comparison principle between harmonic functions, that $u_\varepsilon \geq v$ on $(\mathbb{R}^2 \setminus C) \cap B(0, R)$. Since $v(0) = u(0) = c$, this entails that

$$\lim_{y \to 0, y \notin K} |\nabla v(y)| \geq |\nabla u(0)|.$$

Moreover we have $|\nabla u(\rho, \theta)| = C(\gamma) \rho^{\frac{\pi}{2\pi - \gamma} - 1}$ which is finite in 0 only if $\gamma \geq \pi$. We conclude thanks to Proposition 2.22 that $\mu = |\nabla v| \mathcal{H}^1 \llcorner \partial K$ holds.

To deal with the case $d = 3$ we simply notice that if ∂K is not regular at a point $x \in \partial K$, where K is now a convex set contained in $\mathbb{R}^3$, then there exist two tangent planes intersecting at x which divide $\mathbb{R}^3$ into two conical components of the form $C' = C \times \mathbb{R}$, and $\mathbb{R}^3 \setminus C'$, C being a cone of $\mathbb{R}^2$, and such that $K \subseteq C'$. Thus, by considering the function which in cylindric coordinates takes the form

$$u(\rho, \theta, z) = r^{\frac{\pi}{2\pi - \gamma}} \sin\left(\frac{\pi}{2\pi - \gamma} \theta\right),$$

and as before, $u_\varepsilon = c - \varepsilon u$, since such a function is harmonic in $\mathbb{R}^3 \setminus C'$ and equals v on x, we can repeat an analogous argument to that performed in the two dimensional case to show that $\infty = |\nabla u_\varepsilon(x)| \leq |\nabla v(x)|$, v being the (Coulombic) potential of the set K.

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